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Physics 41N

Mechanics: Insights, Applications and Advances

Lecture 2: Dimensional Analysis – from Biology to Cosmology

In today's seminar, we will see how it is possible to deduce a great deal about the equations that describe the behaviour of a physical system through an analysis of dimensions – with some physical intuition thrown in. We will use this technique to determine the form of the equation describing the period of oscillation of a pendulum, up to a dimensionless constant. We will also use this technique to determine the form of the equations describing the walking frequency of animals and the angle by which a ray of light is bent when passing a massive object, both up to a dimensionless constant. We will also explore some of the limitations of dimensional analysis.

1 Variables, Dimensions and Units

First, let's start with dimensions. How many independent fundamental dimensions are there in mechanics? In mechanics we can almost always get away with just three fundamental dimensions: mass, length and time. We will use bold-face, upper-case letters to denote the dimensions of mass, length and time: **M**, **L** and **T**. Non-boldface, lower-case letters will denote the names of variables that have these dimensions; *e.g.,* mass m, distance d, or time t. In Table 1, we summarize the basic dimensions used in mechanics.

An example of a dimension that is not independent of mass, length and time is the dimension of acceleration. Acceleration has dimensions of length divided by time squared, $\mathbf{L}/\mathbf{T}^\mathbf{2}$. We will use square brackets [ ] around a variable to denote the dimensions of that variable:

$$[a] = \mathbf{L}/\mathbf{T}^\mathbf{2}.$$

Let's now look at the relation

$$F = ma,$$

Table 1: The basic dimensions used in mechanics.

Variable	Dimension	Sample Units
mass m	$\mathbf{M}$	kg, g, lbs
distance d	$\mathbf{L}$	m, cm, feet
time t	$\mathbf{T}$	s, years

which expresses the concept that in order to accelerate a mass m with acceleration a, you must provide a force F. F, m and a are physical variables. Each variable has well-defined dimensions, which we again denote with square brackets:

$$[m] = \mathbf{M}, \quad [a] = \mathbf{L/T^2}, \quad [F] = \mathbf{ML/T^2}.$$

The notion of a variable's dimensions should be distinguished from the variable itself. For example, the variable m can eventually be replaced by a number, with some chosen units such as kg or pounds. The dimension $\mathbf{M}$ identifies the physical character of the variable m, but has nothing to do with its magnitude.

An equation must always be dimensionally correct; that is, the dimensions on the left and right of the equal sign must be the same. We write this as follows for the equation $F = ma$:

$$[F] = [m][a]$$

or

$$[F] = \mathbf{ML/T^2}.$$

Why must the dimensions on the left and right match? Only if the dimensions match will the relation remain true, independent of the size of the units used to measure each quantitiy.

Note that units are distinct from dimensions – you do not need to choose particular units for any of the variables until you are about to substitute numerical values for the variables. The numerical value of each variable depends on the size of the chosen unit. For example, the numerical value of the variable m will be larger if the units used for m are grams rather than kilograms. Only if dimensions of mass appear to the same power on both sides of the equation will the equation be independent of the units chosen.

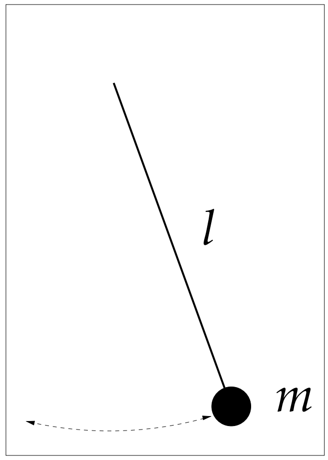

Figure 1: A simple pendulum.

2 An Example of Dimensional Analysis: Period of a Simple Pendulum

We will look at an example that illustrates the basic methods of dimensional analysis. We will find the form of the equation for the period of oscillation for a simple pendulum. That is, how much time does it take a bob of mass m, at the end of a (massless) string of length ℓ, to swing back and forth through one complete oscillation? (See Figure 1.) The basic steps are the following:

1. Make a list of all the physical variables and dimensional[1] fundamental constants on which the answer could depend.

2. Write down the dimensions of these quantities.

3. Demand that these quantities be combined in a functional relation such that the equation is dimensionally correct.

For the first step, you must always use physical intuition. On which physical variables might the period of oscillation of the pendulum depend? One could guess that the period T might depend on the mass m and the

[1]By "dimensional", I mean a constant that has dimensions. This is in contrast to dimensionless constants such as π, $1/2$, etc.

length ℓ. We will assume that the period can be expressed as a product of the variables m and ℓ, each raised to an unknown power:

$$T = k\ell^\alpha m^\beta,$$

where k is a dimensionless constant.[2] The method of dimensional analysis now consists of finding values for α and β that make the dimensions of the right-hand side equal to the dimensions of the left-hand side. The equation relating the dimensions does not involve the dimensionless constant k but it does involve α and β:

$$\mathbf{T} = \mathbf{L}^\alpha \mathbf{M}^\beta.$$

We have used the fact that the dimensionality of the period of oscillation is time ($\mathbf{T}$). We now write three equations that equate the exponents of $\mathbf{M}$, $\mathbf{L}$ and $\mathbf{T}$ on the two sides of the equation:

$$\text{Exponents of } \mathbf{M}: \quad 0 = \beta;$$

$$\text{Exponents of } \mathbf{L}: \quad 0 = \alpha;$$

$$\text{Exponents of } \mathbf{T}: \quad 1 = 0.$$

The last condition cannot be satisfied as would have been obvious from just looking at the equation. None of the variables on the right-hand side involve the dimensions of time, so we cannot balance dimensions.

We must be missing a physical variable or a physical *dimensional* constant on which the period of oscillation depends. We can imagine that the period of oscillation will be different on the moon where the acceleration due to gravity, denoted by g, is different. So we should have included g in the equation. Let's start again. We now assume

$$T = k\ell^\alpha m^\beta g^\gamma.$$

The equation relating dimensions is

$$\mathbf{T} = \mathbf{L}^\alpha \mathbf{M}^\beta (\mathbf{L}\mathbf{T}^{-2})^\gamma.$$

Equating the exponents of the basic dimensions $\mathbf{M}$, $\mathbf{L}$ and $\mathbf{T}$, we get

$$\text{Exponents of } \mathbf{M}: \quad 0 = \beta;$$

[2] α is the Greek letter *alpha* and β is the Greek letter *beta*. Below, we will also introduce the Greek letter γ (*gamma*).

$$\text{Exponents of } \mathbf{L}: \quad 0 = \alpha + \gamma;$$

$$\text{Exponents of } \mathbf{T}: \quad 1 = -2\gamma.$$

The first equation shows that contrary to our intuition, the mass of the bob is not involved in determining the period of oscillation. The effect of gravity is uniquely determined by the third equation, because gravity is the only variable on the right involving time: $\gamma = -1/2$. Substituting this into the second equation, we get $\alpha = 1/2$. Therefore, our equation for the period of oscillation becomes

$$T = k\sqrt{\frac{\ell}{g}}.$$

This method does not allow us to find the value of the constant k. There are two ways that we can find k. We can do a dynamical calculation from which it turns out that $k = 2\pi$. Or we can do what we did in the seminar: we used a weight swinging at the end of a string to determine the constant k for one system by measuring the length of the string, timing the period of the oscillation, and using the fact that $g = 9.8 \text{ m/s}^2$. Using the equation $k = T\sqrt{\frac{g}{\ell}}$, we found that $k \approx 6.28$. We then used the fact that the dimensionless numbers that appear in fundamental equations in physics are always "special" numbers like 2, 4, π, etc., to conclude that $k = 2\pi$. Therefore, the final equation for the period of oscillation is

$$T = 2\pi\sqrt{\frac{\ell}{g}}.$$

3 Modelling

A major application of dimensional analysis, which unfortunately we won't have time to discuss in any detail, is modelling – *i.e.,* determining which parameters or combination of parameters can be varied when modelling a system without changing the physical behavior of the system. You will encounter some important dimensionless combinations of parameters (such as the Reynold number) if you study fluid dynamics. For example, the Reynold number R determines whether flow is laminar or turbulent, and is given by

$$R = \frac{\rho D v}{\eta},$$

where ρ and η are the density and viscosity of the fluid, respectively, v is the velocity of the fluid and D is the characteristic length of the object around which the fluid is flowing. In using scale models of aircraft to study performance characteristics, the performance of the model and the aircraft are the same if the Reynold number is the same. Therefore, if the size of the aircraft is scaled down, the speed of the air in a test wind tunnel must increase by the same factor.

Incidently, NASA Ames at Moffett Field in Mountain View has one of the largest wind tunnels in the world, used for testing the design of airplane fuselages, wings, etc. NASA Ames also has a smaller wind tunnel that can produce air speeds in excess of Mach 2 (twice the speed of sound in air). NASA Ames is occasionally open for tours. Another place where you can see the use of scale models is in Sausalito, north of San Francisco. A 1.5-acre scale model of the San Francisco Bay was built by the Army Corp of Engineers after World War II to study the hydraulics of the Bay and to understand the effects of landfills and other human activity. It's still in use but it is open to the public for tours (http://www.spn.usace.army.mil/bmvc/). If you happen to go there, look at the spikes that stake many regions of the model Bay. They are there to alter the viscous behavior of the system in such a way that the data collected can be extrapolated to the real Bay.

4 Dimensions and Mathematical Formulae

We already discussed the fact that each term in a physical equation must have the same dimensions; otherwise, the validity of the equation will depend on the units being used to measure each quantitiy. This fact can sometimes be used to determine fundamental equations up to a dimensionless constant and can always be used to check whether an equation is blatantly incorrect. Note that in mathematics classes, dimensions are rarely, if ever, mentioned because the variables are usually not defined as physical quantities. In particular, in mathematics, you will see mathematical terms like $\sin x$, $\cos y$, and $\exp t$. In physics, the quantities x, y, and t have dimensions (e.g., length for x and y; time for t). In that case, expressions like $\sin x$, $\cos y$, and $\exp t$ are *not* valid because the arguments of the sine, cosine, and exponential functions must be dimensionless. Only if the arguments are dimensionless will the series expansions for each function be valid:

$$\sin \alpha = \alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - ...,$$

$$\cos \alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - ...,$$

and

$$\exp \alpha = 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + ...$$

Therefore, in physics, we encounter formulae with terms such as

$$\cos(2\pi f t), \quad \sin(2\pi x/\lambda), \quad \exp(t/\tau).$$

In each of these cases, the argument of the function is dimensionless. [Also note that the series expansions for the trigonometric functions will only be valid if the argument α is expressed in radians (which is dimensionless), not degrees.]

5 Dimensional Analysis for Biological Systems – An Example

The same method can be applied to many problems involving frequency. For example, consider the stepping frequency of walking mammals. It has long been argued that efficient walking is like an oscillating pendulum, as shown in Figure 2. In fact, walks, trots, canters – all 'efficient' gaits – are essentially free pendulums. The idea is that you add just enough energy to keep the oscillation going against frictional losses. (Think of 'pumping' a swing.)

Dimensional analysis will again tell us that the walking frequency f is independent of the mass of the mammal, but depends on the leg length ℓ as $1/\sqrt{\ell}$, for animals of similar morphology. (We are using our earlier result for the period of oscillation, T, and the fact that $f = 1/T$.) In Figure 3, the stepping frequency of wild African animals is shown as a function of shoulder height for walking, trotting and cantering. Since this is a log-log plot, the data should lie along a straight line:

$$f = k\sqrt{\frac{g}{\ell}} = \sqrt{\frac{gk^2}{\ell}},$$

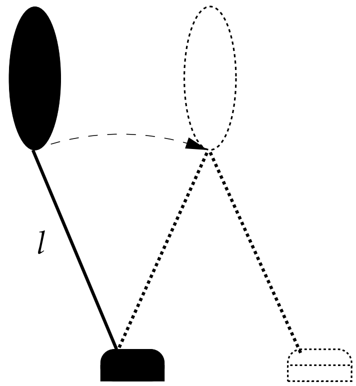

Figure 2: Physicist's view of an animal's gait.

$$\log f = \frac{1}{2}\log\frac{gk^2}{\ell} = \frac{1}{2}\log(gk^2) - \frac{1}{2}\log\ell.$$

Therefore, on a log-log plot of f versus ℓ, the data should lie on a line with slope $-1/2$. From Figure 3, we see that indeed the data are consistent with a line with slope $-1/2$ (check the slope).

Hill[3] studied the *maximum* stepping frequency that an animal can attain when running at top speed. This frequency is determined by the maximum stress that a tendon can safely carry without tearing and the moment of inertia of the leg. He found through dimensional analysis that the maximum stepping frequency depends on ℓ^{-1} and does not depend on gravity at all, unlike the natural walking frequency, which depends on $\sqrt{\frac{g}{\ell}}$. These expected behaviours are shown on a log-log plot in Figure 4. For the natural walking frequency, we plot two lines for two different values of g. What does this plot tell us? First, because the lines for maximum frequency and natural frequency cross at some point, we can conclude that there is a maximum size for an animal that is strong enough to run (or fly). Also, the line representing natural walking frequency is higher if gravity is greater, but the maximum stepping frequency is not affected by gravity. Therefore, the maximum size of an animal that is strong enough to run or fly at its natural frequency decreases if gravity increases. Over the past decade, searches for planets orbiting other stars have revealed tantalizing evidence ("wobble" in the positions of the

[3] A.V.Hill, *The dimensions of animals and their muscular dynamics,* Science Progress, 38, 209 (1950).

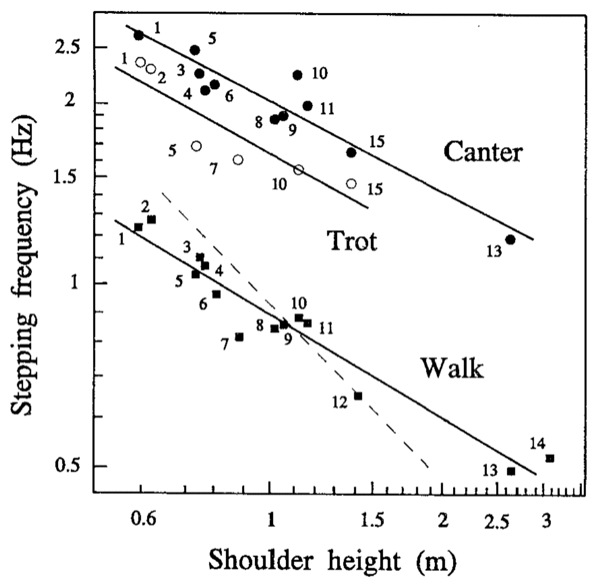

Fig 4.8 The stepping frequencies of wild African animals vary about with the -1/2 power of the shoulder height, when comparisons are made within any of the three gaits, walk, trot, and canter. This is the 'pendulum law' predicted by Alexander and Jayes (1983). A. V. Hill's 'strength limited' law, which requires the stepping frequency to vary inversely with the length, is not reconcilable with the data (dashed line). This '–1 power' law is expected to apply to maximum sprint speed, rather than to cruising speed (from Pennycuick 1975). 1, Thomson's gazelle; 2, warthog; 3, wildebeest (adult); 4, spotted hyaena; 5, Grant's gazelle; 6, impala; 7, lion; 8, hartebeest; 9, topi; 10, zebra; 11, wildebeest (calf); 12, black rhinoceros; 13, giraffe; 14, elephant; 15, buffalo.

Figure 3: From "Newton Rules Biology", by C.J.Pennycuick.

stars) for planets orbiting a significant fraction of the stars that have been studied. The planets tend to be large. Our argument says that for big planets whose gravity is stronger than ours on Earth, we may anticipate that the largest walking and flying animals will be smaller than those here on earth.

We can combine the above conclusion that the maximum stepping frequency is inversely proportional to the leg length ℓ with the observation that step size is proportional to ℓ to deduce that the maximum *speed* of an animal is independent of the leg length, or the size, of the animal, for animals of similar morphology. This is indeed true. A study was done to measure the top speeds of ten species of wild African ungulates. The animals ranged from gazelles to giraffes. The top speed was basically constant, decreasing slightly in the larger animals.

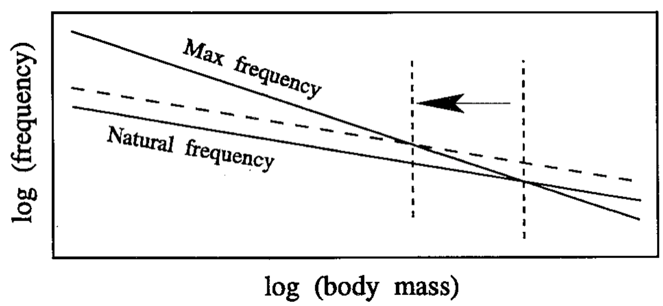

Fig 4.9 **Maximum stepping or flapping frequency, determined by Hill's argument, is not affected by an increase in the strength of gravity, but the natural frequency for cruising locomotion increases in proportion to $\sqrt{g}$ (dotted line). The maximum mass of an animal that is strong enough to run or fly at its natural frequency therefore decreases if gravity increases.**

Figure 4: From "Newton Rules Biology", by C.J.Pennycuick.

6 A Second Example: the Deflection of Light by Mass

We'll now use dimensional analysis to determine the form of the equation describing the deflection angle due to gravity for a light ray passing by a star (or other object) of mass m. First, let's define the angle θ as the angle between the directions of the ray of light when it is asymptotically far from the star (coming towards the star and going away from the star), as shown in Figure 5.

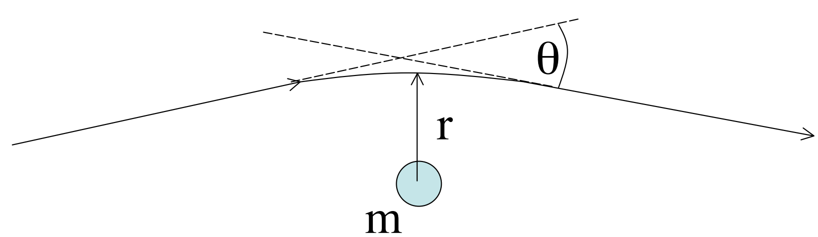

Figure 5: The bending of a single light ray by the mass m.

Note that when angles appear in an equation, they should always be expressed in radians.[4] An angle expressed in radians is dimensionless.[5] Therefore, the deflection angle θ is dimensionless.

On which physical variables might the deflection angle depend? Our physical intuition tells us that the angle should depend on the mass of the star m and on the distance of the ray of light from the star. Let's define r to be the distance of closest approach of the ray to the star as shown in the sketch above. If we proceed with our dimensional analysis at this point, we will find that there is no dimensionally consistent form for the equation expressing

[4]For example, the equation $\sin\theta \approx \theta$ is true for small θ *only* if θ is expressed in radians on the right-hand side.

[5]To see that an angle expressed in radians is dimensionless, note that many expressions for an angle involve only the ratio of two lengths. For example, the angle subtended by an arc length ℓ is given by $\theta = \ell/r$ where r is the radius of the circle. Because ℓ/r is dimensionless, θ must be dimensionless as well.

θ in terms of m and r, just as we found that the period of oscillation of a pendulum could not be expressed in terms of m and ℓ alone. So, again there must be a dimensional constant that we need to include. Since the deflection of light is due to gravity, we might suspect that the angle depends on the gravitational constant G.[6] What are the dimensions of G? Recall that the equation for the gravitational force between two massive objects of mass m_1 and m_2 a distance r apart is given by $F = G\frac{m_1 m_2}{r^2}$. Therefore,

$$[G] = [\frac{Fr^2}{m_1 m_2}] = \mathbf{M}^{-1}\mathbf{L}^3\mathbf{T}^{-2},$$

where we used $[F] = \mathbf{MLT}^{-2}$. Now let's try to find the equation for θ:

$$\theta = km^\alpha r^\beta G^\gamma.$$

The equation relating dimensions is

$$\mathbf{M}^0\mathbf{L}^0\mathbf{T}^0 = \mathbf{M}^\alpha\mathbf{L}^\beta(\mathbf{M}^{-1}\mathbf{L}^3\mathbf{T}^{-2})^\gamma.$$

Equating the exponents of the basic dimensions $\mathbf{M}$, $\mathbf{L}$ and $\mathbf{T}$, we get

$$\text{Exponents of } \mathbf{M}: \quad 0 = \alpha - \gamma;$$

$$\text{Exponents of } \mathbf{L}: \quad 0 = \beta + 3\gamma;$$

$$\text{Exponents of } \mathbf{T}: \quad 0 = -2\gamma.$$

But the last equation gives us $\gamma = 0$, the second one gives us $\beta = 0$ and the first one gives $\alpha = 0$! So, we must still be missing a physical variable or a dimensional constant. Which dimensional constant is most likely to be relevant for the case of the bending of light by gravity? How about the speed of light, c? Let's try it:

$$\theta = km^\alpha r^\beta G^\gamma c^\delta.$$

The equation relating dimensions is now

$$\mathbf{M}^0\mathbf{L}^0\mathbf{T}^0 = \mathbf{M}^\alpha\mathbf{L}^\beta(\mathbf{M}^{-1}\mathbf{L}^3\mathbf{T}^{-2})^\gamma(\mathbf{LT}^{-1})^\delta.$$

Equating the exponents of the basic dimensions $\mathbf{M}$, $\mathbf{L}$ and $\mathbf{T}$, we get

$$\text{Exponents of } \mathbf{M}: \quad 0 = \alpha - \gamma;$$

[6]We do not expect it to depend on g because, although g is related to G, g is particular to the gravitational force at the earth's surface.

$$\text{Exponents of } \mathbf{L}: \quad 0 = \beta + 3\gamma + \delta;$$

$$\text{Exponents of } \mathbf{T}: \quad 0 = -2\gamma - \delta.$$

So now we have three equations in four unknowns. The four exponents α, β, γ and δ are constrained but are not uniquely determined. Each of the three equations involves γ, so lets express the other three exponents in terms of γ. From the first equation, $\alpha = \gamma$. From the last equation, $\delta = -2\gamma$. And from the second equation, $\beta = -\delta - 3\gamma = 2\gamma - 3\gamma = -\gamma$. Therefore, the equation for the bending angle is of the form

$$\theta = km^{\gamma} r^{-\gamma} G^{\gamma} c^{-2\gamma} = k(\frac{mG}{rc^2})^{\gamma}.$$

Actually, there could be more than one term in the equation for θ, each with a different value of the exponent γ and the constant k, but each term must have the above form. In fact, there could be an infinite number of terms (an infinite series), in which case the right-hand side might be a function of $\frac{mG}{rc^2}$ that can be represented as a series expansion. So, we have not uniquely determined the form of the equation for θ but we can already draw some conclusions from the above equation. For example, we can see that the bending angle depends on the ratio m/r; if m and r are both changed by the same factor, the bending angle will be the same.

We can go further and restrict γ by using physical intuition. First, we expect that θ approaches zero as m becomes very small or as r becomes very large. If γ were negative, then θ would approach infinity as m became small or r became large. Therefore, γ must be a *positive* exponent: $\gamma > 0$.

To further restrict γ, we can try to apply physical intuition to the derivative of θ with respect to m or r. Since the ratio m/r appears in the equation for θ, let's consider the derivative with respect to $x = \frac{mG}{rc^2}$:

$$\frac{d\theta}{dx} = \gamma k x^{\gamma - 1}, \quad \gamma > 0.$$

Our physical intuition might tell us that in the limit of $x = mG/(rc^2)$ becoming very small, the change in θ with respect to a change in m/r should become small, but should not vanish. Therefore, we want the exponent of $\frac{mG}{rc^2}$ to be zero in the equation for $\frac{d\theta}{dx}$. So γ must equal 1 and the equation for θ, at least for small values of the dimensionless combination of variables $\frac{mG}{rc^2}$, is

$$\theta = k\frac{mG}{rc^2}.$$

I admit that this last argument is a bit of a stretch...

What about the dimensionless constant k? A survey of *all* the equations that you will learn this year in the Introductory Physics sequence will convince you that the *dimensionless* constants in physical equations are always of order 1. And the equation we just derived is no exception. It turns out that $\theta = k\frac{mG}{rc^2}$ with $k = 4$.

Now let's actually calculate the bending angle for two different cases: (a) light passing close to the edge of the sun, and (b) light from a distant quasar[7] passing close to the edge of a galaxy that is about 200,000 light years across and weighs about 10^{12} times as much as the sun (not an atypical galaxy). We will need the values of the dimensional constants in the equation:

$$c = 3.0 \times 10^8 \, \mathrm{m/s}, \quad G = 6.67 \times 10^{-11} \, \mathrm{m^3/(kg \cdot s^2)}.$$

6.1 Bending of Light by the Sun

Before we calculate the angle by which light is bent when passing close to the sun, let's review the historical significance of this phenomenon. One of the first tests of Einstein's theory of general relativity was a measurement of the bending of starlight as it passed by the edge of the sun. The difficulty with this measurement is that it is normally impossible to see a star when it is in line with the edge of the sun. It was necessary to wait for a solar eclipse so that the measurement could be made since we can see the stars during a solar eclipse! After Einstein first presented his ideas on light deflection, an expedition to the location of the next solar eclipse was prevented by war. A few years later, in 1919, another total solar eclipse ocurred and his theory was tested and verified. Now let's calculate the bending angle.

For a light ray passing near the edge of the sun, r is the radius of the sun ($r = 6.96 \times 10^8$ m) and m is the mass of the sun ($m = 1.99 \times 10^{30}$ kg). The bending angle is

$$\theta = \frac{4mG}{rc^2} \tag{1}$$

[7] Quasars are extremely bright objects that are very far away. QUASAR stands for Quasi-stellar Radio Sources because they were first observed in the radio band, in the early 1960's, but they have now been observed in many parts of the electromagnetic spectrum. Quasars give off huge amounts of electromagnetic radiation, about the same amount as 10 to 1000 typical galaxies! Quasars are candidates for black holes.

$$
\begin{aligned}
&= \frac{4 \times 1.99 \times 10^{30}\,\text{kg} \times 6.67 \times 10^{-11}\,\text{m}^3/(\text{kg}\cdot\text{s}^2)}{6.96 \times 10^{8}\,\text{m} \times (3.0 \times 10^{8}\,\text{m/s})^2} && (2)\\
&= 8.5 \times 10^{-6}\,\text{radians} && (3)\\
&= 8.5\,\text{microradians.} && (4)
\end{aligned}
$$

We can convert this into degrees by multiplying by $180°/\pi$ to get $\theta = 0.0005°$ or 5/10,000 of a degree. To get a sense of the size of this angle, let's compare it to the angular diameter of the sun $\Delta\phi$:

$$
\begin{aligned}
\Delta\phi &= \frac{\text{diameter of the sun}}{\text{distance from earth to sun}} && (5)\\
&= \frac{2 \times 6.96 \times 10^{8}\,\text{m}}{1.49 \times 10^{11}\,\text{m}} && (6)\\
&= 0.0093\,\text{radians} \times \frac{180°}{\pi} && (7)\\
&= 0.54° && (8)
\end{aligned}
$$

Therefore, the angular diameter of the sun is about half a degree.[8] So the angular shift of the starlight passing near the sun's edge is about 1/1000 of the angular diameter of the sun itself. This shift would certainly not be apparent to the naked eye. Precise measurements of images in a telescope are required to measure this shift relative to the angular position of other stars.

6.2 Bending of Light by a Galaxy

For light passing near the edge of a galaxy that is 200,000 light-years across and weighs about 10^{12} times as much as the sun, $r = 100,000$ light-years $\times 9.5 \times 10^{15}$ m/light-year $= 9.5 \times 10^{20}$ m and $m = 10^{12}\, m_{\text{sun}} = 10^{12} \times 1.99 \times 10^{30}$ kg $= 1.99 \times 10^{42}$ kg. The bending angle is

$$
\begin{aligned}
\theta &= \frac{4mG}{rc^2} && (9)\\
&= \frac{4 \times 1.99 \times 10^{42}\,\text{kg} \times 6.67 \times 10^{-11}\,\text{m}^3/(\text{kg}\cdot\text{s}^2)}{9.5 \times 10^{20}\,\text{m} \times (3.0 \times 10^{8}\,\text{m/s})^2} && (10)\\
&= 6.2 \times 10^{-6}\,\text{radians} \times \frac{180°}{\pi} && (11)\\
&= 0.0004°. && (12)
\end{aligned}
$$

[8] For a sense of scale, it is handy to note that a degree is approximately the angular width of your thumb held at arms length: $\delta\phi \approx 2$ cm/1 m $= 0.02$ radians $= 1.1°$.

The bending angle we just calculated for light bent by a galaxy is almost exactly the same as that for light bent when passing close to the sun. In both cases the bending angle is between 5 and 10 microradians. They are so similar because the bending angle scales as m/r and, although the galaxy weighs 10^{12} times as much as the sun, the light passes about 10^{12} times further from its center than from the center of the sun ($\frac{9.5\times10^{20}\text{m}}{6.96\times10^{8}\text{m}} = 1.4\times10^{12}$). Once we had already determined the bending angle for the sun, it would have been simpler for us to use this scaling to determine the bending angle for the galaxy by merely dividing the result for the sun by 1.4.

What does the image of an object such as a quasar look like if the light from the quasar is bent by an object such as a galaxy between the quasar and the earth? First, think about the case when the quasar, the galaxy and the earth are all in a line. As illustrated in Figure 6 below, the image of the quasar will look like a ring, called an Einstein Ring.

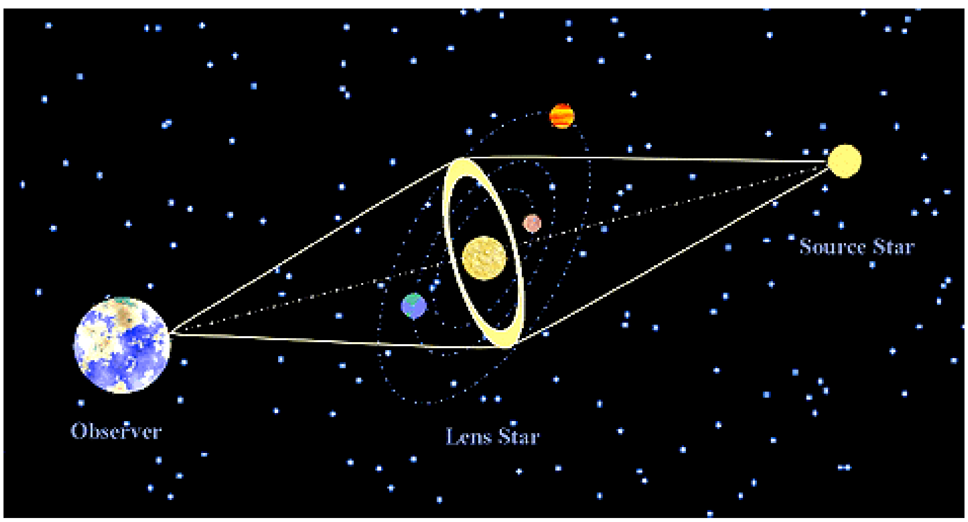

Figure 6: Light from a distant source is bent by an intermediate mass to form a ring-shaped image, called an Einstein Ring.

Figure 7 shows a plot of the intensity of radio waves for the first Einstein Ring ever discovered The angular radius of the ring is about 1 arcsecond or $(\frac{1}{60})\frac{1}{60}^{\circ} = \frac{1}{3600}^{\circ} = 0.0003°$, about the same as the deflection angle we calculated above.

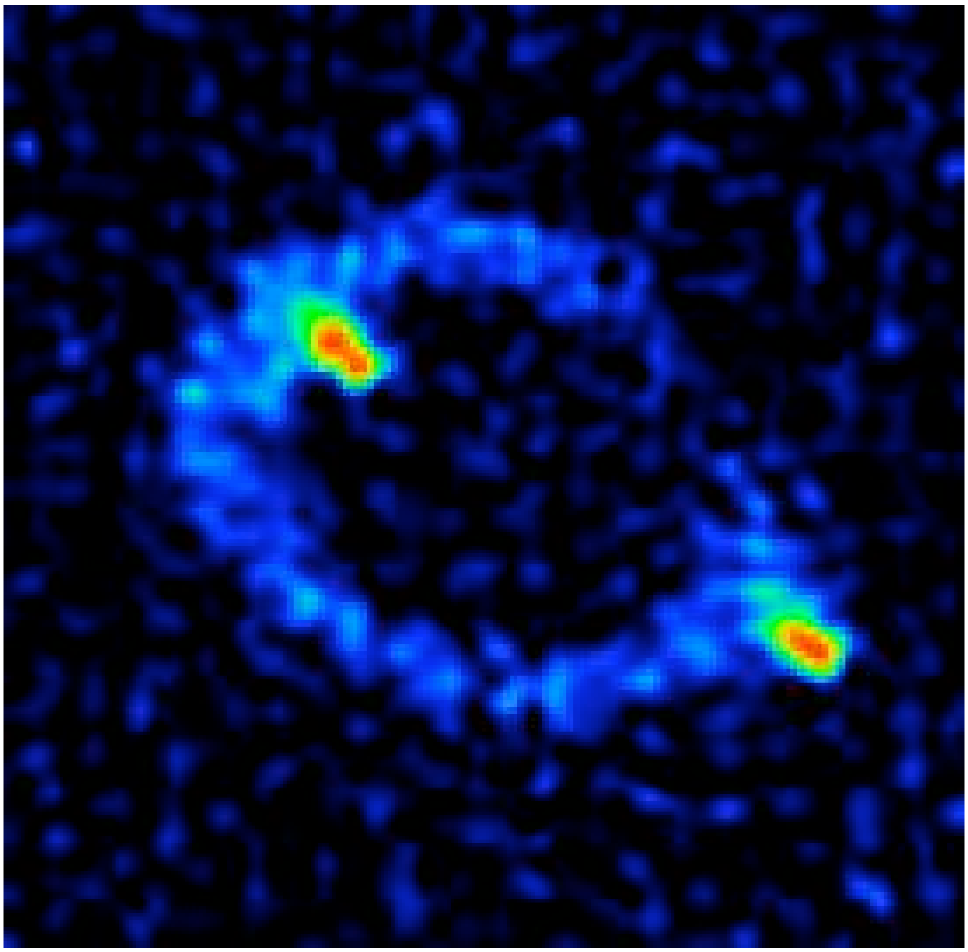

Figure 7: The first Einstein ring was observed with radio waves in 1987 by Jacqueline Hewitt from MIT.

If the mass doing the bending is not perfectly aligned with the observer and the source, the image will look like arcs or separated spots. The phenomenon is called graviational lensing. The first lensing candidates were discovered in 1979. As mentioned above, the first Einstein ring was discovered in 1987. The first evidence of giant arcs in the sky due to the lensing of background galaxies by foreground clusters of galaxies was presented in 1987 by R. Lynds and Vahe Petrosian of Stanford. Today, gravitational lensing is a major tool being used to understand the distribtution of regular matter and dark matter in the universe, and to address key questions in cosmology.

For more information about gravitational lensing and for images that show evidence of gravitational lensing, go to google and enter any of the following groups of words: "Einstein rings", "graviational lensing", "weak gravitational lensing".

We will return to gravitational lensing in a later seminar when we discuss evidence for Dark Matter in the Universe.