Digital PI Controller Equations

Probably the most common type of controller in industrial power electronics is the "PI" (Proportional - Integral) controller. In field oriented motor control, PI controllers are widely used for inner current control loops. In digital power supply control, it finds application in buck, boost, SEPIC, and many other power topologies.

In general, the controller may be designed to meet specifications expressed in either the time domain or the frequency domain. Time domain specifications typically constrain properties of the transient response, such as overshoot, settling time, and rise time. Frequency domain specification involves the selection of a single real zero. Either way, the result is two real numbers corresponding to the gains in the proportional and integral paths. More information on transient tuning using PI control can be found in the "Control Theory Fundamentals" seminar, and in chapter 3 of the accompanying book [1].

In this paper we will focus on the relationship between the gains of continuous time (analogue) and discrete time (digital) PI controllers. We begin by describing two common configurations of controller (series and parallel), both of which can be expressed in a simple "zero plus integrator" transfer function. We then transform this into discrete time form and compare the difference equation with those of practical series and parallel PI implementations. The objective is to find a pair of equations for each configuration which relate the discrete time proportional gains with those of the corresponding continuous time original.

1. Controller Configurations

The PI controller may be implemented in either of two configurations: series or parallel. The parallel configuration is shown below.

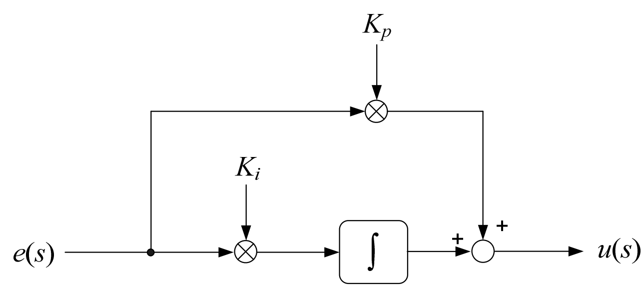

Figure 1

In this configuration, proportional and integral gains appear in parallel paths. Conceptually, the process of tuning the controller for transient response is straightforward: one adjusts each gain in turn, blending together different amounts of proportional and integral control action, until the desired specifications are met.

The parallel PI controller transfer function is (by inspection of Fig. 1)

$$F(s) = K_p + \frac{K_i}{s} = \frac{K_p s + K_i}{s} \tag{1}$$

An alternative, but related, configuration is the series configuration (shown below) in which the proportional gain appears in series with the controller. An attraction of this structure is that there is less inter-action between the two gains, slightly simplifying the tuning process. Note that the series configuration cannot be used in applications where zero proportional gain might be required.

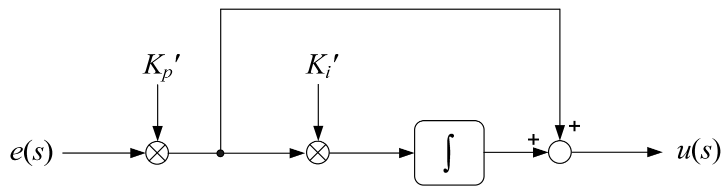

Figure 2

The transfer function of the series PI controller is (Fig. 2) is

$$F(s) = K_p'\left(1 + \frac{K_i'}{s}\right) = \frac{K_p's + K_p'K_i'}{s} \tag{2}$$

Comparing equations (1) and (2), we see the relationship between series and parallel controller gains is:

$$K_p = K_p'; \qquad K_i = K_p'K_i' \tag{3}$$

Consequently, once the P & I gains for one configuration have been found it is a simple matter to compute the gains for the other. In general, both transfer functions have the form of an integrator with a single real zero. Adopting a somewhat neutral notation, we can write either configuration in the form

$$F(s) = \frac{b_1 s + b_0}{s} \tag{4}$$

This form is the same as the "zero plus integrator" commonly used in power supply loop compensation, in which $b_1 = 1$ and b_0 is the zero frequency. We will now examine how the gains are related to the digital PI controller.

2. Discrete Transformation

There are several methods for converting a continuous time transfer function into equivalent discrete time form. Among them, the best known is probably the bi-linear, or "Tustin" transform. This method, named after the English mathematician whose work on non-linear systems led to its introduction, can be derived from a numerical approximation of the controller output (see chapter 4 in ref. [1]). The method involves replacement of each instance of 's' in the original transfer function with the following term involving 'z' and the sampling period T.

$$s \leftarrow= \frac{2}{T}\frac{z-1}{z+1}$$

Applying the substitution to equation (4), we have

$$F(z) = \frac{b_1 \frac{2}{T}\frac{z-1}{z+1} + b_0}{\frac{2}{T}\frac{z-1}{z+1}}$$

$$F(z) = \frac{2b_1(z-1) + Tb_0(z+1)}{2(z-1)}$$

After some re-arrangement, we can write the transformed equation in the form

$$F(z) = \frac{c_1 z + c_0}{z-1} \tag{6}$$

where the numerator coefficients are

$$c_1 = b_1 + \frac{T}{2} b_0; \qquad c_0 = -b_1 + \frac{T}{2} b_0$$

Tustin's method requires that the gains of the original and transformed systems be matched. This is usually done at $\omega = 0$, however the PI controller has infinite gain there since it contains an integrator. We could match the gains at a different frequency, however in this case it is probably easier to neglect the integrators and match the numerator gains in (4) and (6).

$$b_1 s + b_0 \big|_{s=0} = b_0$$

$$c_1 z + c_0 \big|_{z=1} = c_1 + c_0$$

Therefore the gain of the transformed equation (6) must be modified by

$$A = \frac{b_0}{c_1 + c_0}$$

which in this case turns out to be $1/T$.

$$F(z) = A \frac{c_1 z + c_0}{z - 1} \tag{7}$$

We now have a discrete time transfer function representing our PI controller. The corresponding difference equation is found by re-arrangement and application of the shifting theorem of the z transform [1].

$$(z-1)u(z) = A(c_1 z + c_0)e(z)$$

$$zu(z) = u(z) + zAc_1 e(z) + Ac_0 e(z)$$

$$u(k) = u(k-1) + Ac_1 e(k) + Ac_0 e(k-1) \tag{8}$$

3. Parallel Controller Gains

A reasonable question is to ask is: what proportional and integral gains do we need to apply in order for the discrete time version to behave similarly to the continuous time original? In the following, we will address this question to the parallel controller. The series configuration is dealt with in section 4. In order to proceed, we'll need the difference equation of the parallel discrete time PI controller. We can then find a relationship between the gains by matching coefficients.

The parallel form discrete time PI controller structure is shown below. To avoid confusion between the original P & I controller gains and those in the discrete time structure, we will refer to the latter as V_p and V_i respectively.

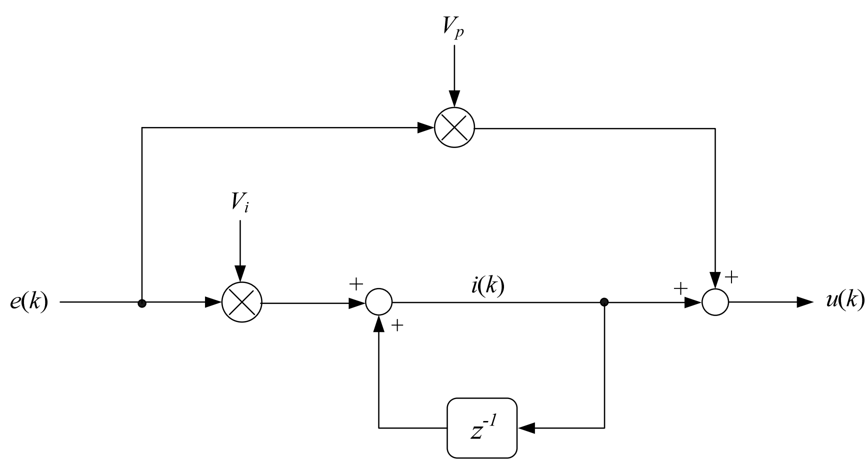

Figure 3

The difference equation can be found as follows. Notice that the discrete integrator introduces an internal variable "*i*" into the equation.

$$u(k) = V_p e(k) + V_i e(k) + i(k-1)$$

$$i(k) = u(k) - V_p e(k)$$

$$u(k) = V_p e(k) + V_i e(k) + u(k-1) - V_p e(k-1)$$

$$u(k) = u(k-1) + (V_p + V_i)e(k) - V_p e(k-1) \tag{9}$$

The relationship between the continuous and discrete time controller gains can be found by matching coefficients in (8) and (9).

$$V_p = -Ac_0 \tag{10}$$

$$V_p + V_i = Ac_1$$

$$V_i = A(c_1 + c_0) \tag{11}$$

Finally, substituting in (10) and (11) for c_1 & c_0, and then for b_1 & b_0, we find the required discrete time controller gains.

$$V_p = A\left(K_p - \frac{T}{2}K_i\right) \tag{12}$$

$$V_i = ATK_i \tag{13}$$

4. Series Controller Gains

If we are working with the series PI controller, we can correlate difference equations coefficients in a similar way. To avoid ambiguity, we will denote discrete time P & I gains W_p and W_i respectively.

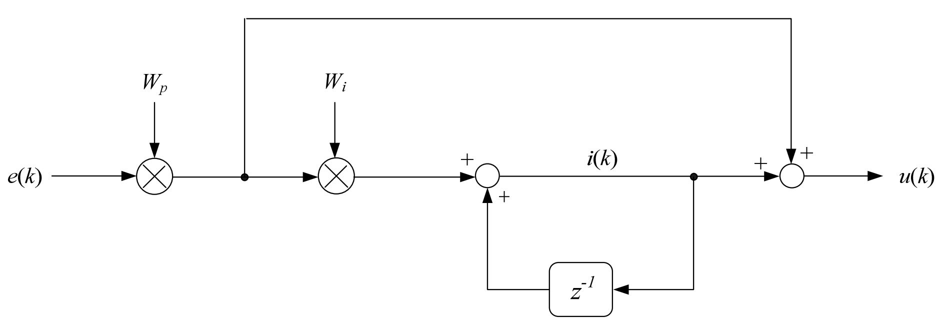

Figure 4

Proceeding as before, we have

$$u(k) = W_p e(k) + i(k)$$

$$i(k) = W_p W_i e(k) + i(k-1)$$

$$i(k-1) = u(k-1) - W_p e(k-1)$$

$$u(k) = W_p e(k) + W_p W_i e(k) + u(k-1) - W_p e(k-1)$$

$$u(k) = u(k-1) + W_p(1+W_i)e(k) - W_p e(k-1) \tag{14}$$

Comparing equations (14) and (8), we see

$$W_p = -Ac_0 \tag{15}$$

$$W_i = -\left(1 + \frac{c_1}{c_0}\right) \tag{16}$$

We can now substitute for c_1 & c_0, b_1 & b_0, and A, to express the discrete time series PI gains in terms of the continuous time PI gains.

$$W_p = \frac{1}{T} K_p' \left(1 - \frac{T}{2} K_i'\right) \tag{17}$$

$$W_i = \frac{TK_i'}{1 - \frac{T}{2} K_i'} \tag{18}$$

Equations (17) and (18) are related by

$$W_p = \frac{K_p'}{K_i'} W_i \tag{19}$$

Summary

Equations (12) & (13), and (17) & (18) allow us to compute equivalent digital PI controller gains from an analogue prototype. This has value when a designer wishes to substitute digital control action for an analogue PI controller. However, the designer must understand that digital control introduces a sampler (A/D converter) and a reconstruction block (typically PWM). Both involve scaling the input and output variables to match the digital numeric format. Furthermore, digital control introduces dynamic effects into the loop, principally in the form of phase lags, which are not present in the analogue system. In almost all power electronic control systems, phase lag is detrimental to control and must be accounted for carefully. Further information on these matters can be found in the following references.

References

[1] *Control Theory Fundamentals*, R. Poley, CreateSpace, 3rd Ed., 2015

[2] *Digital Control of Dynamic Systems*, Franklin, Powell, & Workman, 3rd. Ed., 1997