PISA 2022 MATHEMATICS FRAMEWORK (DRAFT)

November 2018

OECD

Table of contents

Tables

Figures

Introduction

1. The assessment of mathematics has particular significance for PISA 2022, as mathematics is again the major domain assessed. Although mathematics was assessed by PISA in 2000, 2003, 2006, 2009, 2012, 2015 and 2018, the domain was the main area of focus only in 2003 and 2012.

2. The return of mathematics as the major domain in PISA 2022 provides both the opportunity to continue to make comparisons in student performance over time, and to re-examine what should be assessed in light of changes that have occurred in the world, the field and in instructional policies and practices.

3. Each country has a vision of mathematical competence and organises their schooling to achieve it as an expected outcome. Mathematical competence historically encompassed performing basic arithmetic skills or operations, including adding, subtracting, multiplying, and dividing whole numbers, decimals, and fractions; computing percentages; and computing the area and volume of simple geometric shapes. In recent times, the digitisation of many aspects of life, the ubiquity of data for making personal decisions involving initially education and career planning, and, later in life, health and investments, as well as major societal challenges to address areas such as climate change, governmental debt, population growth, spread of pandemic diseases and the globalising economy, have reshaped what it means to be mathematically competent and to be well equipped to participate as a thoughtful, engaged, and reflective citizen in the 21st century.

4. The critical issues listed above as well as others that are facing societies throughout the world all have a quantitative component to them. Understanding them, as well as addressing them, at least in part, requires being mathematically literate and thinking mathematically. Such mathematical thinking in more and more complex contexts is not driven by the reproduction of the basic computational procedures mentioned earlier, but rather by reasoning[1] (both deductive and inductive). The important role of reasoning needs greater emphasis in our understanding of what it means for students to be mathematically literate. In addition to problem solving, this framework argues that mathematical literacy in the 21st century includes mathematical reasoning and some aspects of computational thinking.

5. Countries today face new opportunities and challenges in all areas of life, many of which stem from the rapid deployment of computers and devices like robots, smartphones and networked machines. For example, the vast majority of young adults and students who started university post 2015 have always considered phones to be mobile hand-held devices capable of sharing voice, texts, and images and accessing the internet – capabilities seen as science fiction by many of their parents and certainly by all of their grandparents (Beloit College, 2017[1]). The recognition of the growing contextual discontinuity between the last century and the future has prompted a discussion around the development of 21st century skills in students (Ananiadou and Claro, 2009[2]; Fadel, Bialik and Trilling, 2015[3]; National Research Council, 2012[4]; Reimers and Chung, 2016[5]).

[1] Throughout this framework, references to mathematical reasoning assume both mathematical (deductive) and statistical (inductive) type reasoning.

6. It is this discontinuity that also drives the need for education reform and the challenge of achieving it. Periodically, educators, policy makers, and other stakeholders revisit public education standards and policies. In the course of these deliberations new or revised responses to two general questions are generated: 1) What do students need to learn, and 2) Which students need to learn what? The most used argument in defence of mathematics education for all students is its usefulness in various practical situations. However, this argument alone gets weaker with time – a lot of simple activities have been automated. Not so long ago waiters in restaurants would multiply and add on paper to calculate the price to be paid. Today they just press buttons on hand-held devices. Not so long ago people used printed timetables to plan travel – it required a good understanding of the time axis and inequalities as well as interpreting complex two-way tables. Today we can just make a direct internet inquiry.

7. As to the question of "what to teach", many restrictive understandings arise from the way mathematics is conceived. Many people see mathematics as no more than a useful toolbox. A clear trace of this approach can be found in the school curricula of many countries. These are sometimes confined to a list of mathematics topics or procedures, with students asked to practice a selected few, in predictable (often test) situations. This perspective on mathematics is far too narrow for today's world. It overlooks key features of mathematics that are growing in importance. Notwithstanding the above remark, there are an increasing number of countries that emphasise reasoning and the importance of relevant contexts in their curricula. Perhaps these countries cab serve as helpful models to others.

8. Ultimately the answer to these questions is that every student should learn (and be given the opportunity to learn) to think mathematically, using mathematical reasoning (both deductive and inductive) in conjunction with a small set of fundamental mathematical concepts that support this reasoning and which themselves are not necessarily taught explicitly but are made manifest and reinforced throughout a student's learning experiences. This equips students with a conceptual framework through which to address the quantitative dimensions of life in the 21st century.

9. The PISA 2022 framework is designed to make the relevance of mathematics to 15-year-old students clearer and more explicit, while ensuring that the items developed remain set in meaningful and authentic contexts. The mathematical modelling cycle, used in earlier frameworks (e.g. OECD (2004[6]; 2013[7])) to describe the stages individuals go through in solving contextualised problems, remains a key feature of the PISA 2022 framework. It is used to help define the mathematical processes in which students engage as they solve problems – processes that together with mathematical reasoning (both deductive and inductive) will provide the primary reporting dimensions.

10. For PISA 2022, computer-based assessment of mathematics (CBAM) will be the primary mode of delivery for assessing mathematical literacy. However, paper-based assessment instruments will be provided for countries choosing not to test their students by computer. The framework has been updated to also reflect the change in delivery mode introduced in 2015, including a discussion of the considerations that should inform the development of the CBAM items as this will be the first major update to the mathematics framework since computer-based assessment was introduced in PISA.

11. The development of the PISA 2022 framework takes into account the expectation of OECD that there will be an increase in the participation in PISA of low- and middle-income countries. In particular the PISA 2022 framework recognises the need to increase the resolution of the PISA assessments at the lower end of the student

performance distribution by drawing from the PISA for Development (OECD, 2017[8]) framework when developing the assessment; the need to expand the performance scale at the lower end; the importance of capturing a wider range of social and economic contexts; and the anticipation of incorporating an assessment of out-of-school 14- to 16-year-olds.

12. The increasing and evolving role of computers and computing tools in both day-to-day life and in mathematical literacy problem solving contexts is reflected in the recognition in the PISA 2022 framework that students should possess and be able to demonstrate computational thinking skills as they apply to mathematics as part of their problem-solving practice. Computational thinking skills include pattern recognition, designing and using abstraction, pattern decomposition, determining which (if any) computing tools could be employed in analysing or solving a problem, and defining algorithms as part of a detailed solution. By foregrounding the importance of computational thinking as it applies to mathematics, the framework anticipates a reflection by participating countries on the role of computational thinking in mathematics curricula and pedagogy.

13. The PISA 2022 mathematics framework is organised into three major sections. The first section, 'Definition of Mathematical Literacy', explains the theoretical underpinnings of the PISA mathematics assessment, including the formal definition of the *mathematical literacy* construct. The second section, 'Organisation of the Domain', describes four aspects: a) mathematical reasoning and the three mathematical *processes* (of the modelling/problem solving cycle); b) the way mathematical *content* knowledge is organised in the PISA 2022 framework, and the content knowledge that is relevant to an assessment of 15-year-old students; c) the relationship between mathematical literacy and the so-called *21st Century skills*; and d) the *contexts* in which students will face mathematical challenges. The third section, 'Assessing Mathematical Literacy', outlines structural issues about the assessment, including a test blueprint and other technical information.

14. For the sake of ensuring the preservation of trend, the majority of the items in the PISA 2022 will be items that have been used in previous PISA assessments. A large collection of release items based on the previous framework can be found at http://www.oecd.org/pisa/test. **Annex A** provides seven illustrative items that attempt to illustrate the most important new elements of the 2022 framework.

15. The 2022 framework was written under the guidance of the 2022 mathematics expert group (MEG), a body appointed by the PISA contractor for the mathematics framework (RTI International), in consultation with the PISA Governing Board (PGB). The eight MEG members included mathematicians, statisticians, mathematics educators, and experts in assessment, technology, and education research from a range of countries. The MEG were further supported by an extended MEG (eMEG) group, made up of ten experts acting as peer reviewers of the framework version created by the MEG. The eMEG included experts with a range of mathematics expertise from differing countries. Additional reviews were undertaken by experts on behalf of the over 80 countries constituting the PISA Governing Board. RTI International, as contracted by the Organisation for Economic Co-operation and Development (OECD), conducted two further research efforts: a face validity validation survey amongst educators, universities and employers; and a cognitive laboratory with 15-year-olds in different countries to obtain student feedback on the sample items presented in the framework. The work of the PISA 2022 MEG builds on previous versions of the PISA Mathematics Framework and incorporates the recommendations of the Mathematics Strategic Advisory Group convened by OECD in 2017.

Definition of Mathematical Literacy

16. An understanding of mathematics is central to a young person's preparedness for participation in and contribution to modern society. A growing proportion of problems and situations encountered in daily life, including in professional contexts, require some level of understanding of mathematics before they can be properly understood and addressed. Mathematics is a critical tool for young people as they confront a wide range of issues and challenges in the various aspects of their lives.

17. It is therefore important to have an understanding of the degree to which young people emerging from school are adequately prepared to use mathematics to think about their lives, plan their futures, and reason about and solve meaningful problems related to a range of important issues in their lives. An assessment at age 15 provides countries with an early indication of how individuals may respond in later life to the diverse array of situations they will encounter that both involve mathematics and rely on mathematical reasoning (both deductive and inductive) and problem solving to make sense of.

18. As the basis for an international assessment of 15-year-old students, it is reasonable to ask: "What is important for citizens to know and be able to do in situations that involve mathematics?" More specifically, what does being mathematically competent mean for a 15-year-old, who may be emerging from school or preparing to pursue more specialised training for a career or university admission? It is important that the construct of mathematical literacy, which is used in this framework to denote *the capacity of individuals to reason mathematically and solve problems in a variety of 21st century contexts*, not be perceived as synonymous with minimal, or low-level, knowledge and skills. Rather, it is intended to describe the capacities of individuals to *reason mathematically and use mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena*. This conception of mathematical literacy recognises the importance of students developing a sound understanding of a range of mathematical concepts and processes and realising the benefits of being engaged in real-world explorations that are supported by that mathematics. The construct of mathematical literacy, as defined for PISA, strongly emphasises the need to develop students' capacity to use mathematics in context, and it is important that they have rich experiences in their mathematics classrooms to accomplish this. This is as true for those 15-year-old students who are close to the end of their formal mathematics training, students who will continue with the formal study of mathematics, as well as out of school 15-year-olds.

19. Mathematical literacy transcends age boundaries. For example, OECD's Programme for the International Assessment of Adult Competencies (PIAAC) defines numeracy as *the ability to access, use, interpret, and communicate mathematical information and ideas, in order to engage in and manage the mathematical demands of a range of situations in adult life.* The parallels between this definition for adults and the PISA 2022 definition of mathematical literacy for 15-year-olds are both marked and unsurprising.

20. The assessment of mathematical literacy for 15-year-olds must take into account relevant characteristics of these students; hence, there is a need to identify age-appropriate content, language and contexts. This framework distinguishes between broad categories of content that are important to mathematical literacy for individuals generally, and the specific content topics that are appropriate for 15-year-old students. Mathematical literacy

is not an attribute that an individual either has or does not have. Rather, mathematical literacy is an attribute that is on a continuum, with some individuals being more mathematically literate than others – and with the potential for growth always present.

21. For the purposes of PISA 2022, mathematical literacy is defined as follows:

> *Mathematical literacy is an individual's capacity to reason mathematically and to formulate, employ, and interpret mathematics to solve problems in a variety of real-world contexts. It includes concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to know the role that mathematics plays in the world and to make the well-founded judgments and decisions needed by constructive, engaged and reflective 21st century citizens.*

22. The PISA 2022 framework, when compared with the PISA 2003 and PISA 2012 frameworks, while appreciating and preserving the basic ideas of mathematical literacy developed there, acknowledges a number of shifts in the world of the student which in turn signal a shift on how to assess mathematical literacy in comparison to the approach used in previous frameworks. The trend is to move away from the need to perform basic calculations to a rapidly changing world driven by new technologies and trends in which citizens are creative and engaged, making judgements for themselves and the society in which they live.

23. As technology will play a growing role in the lives of students, the long-term trajectory of mathematical literacy should also encompass the synergistic and reciprocal relationship between mathematical thinking and computational thinking, introduced in (Wing 2006[2]) as "the way computer scientists think" and regarded as a thought process entailed in formulating problems and designing their solutions in a form that can be executed by a computer, a human, or a combination of both (Wing 2010[3]) (Cuny, Snyder and Wing, 2010[9]). The roles computational thinking play in mathematics include how specific mathematical topics interact with specific computing topics, and how mathematical reasoning complements computational thinking (Gadanidis, 2015[10]; Rambally, 2017[11]). For example, Pratt and Noss (2002[12]) discuss the use of a computational microworld for developing mathematical knowledge in the case of randomness and probability; Gadanidis et al. (2018[13]) propose an approach to engage young children with ideas of group theory, using a combination of hands-on and computational thinking tools. Hence, while mathematics education evolves in terms of the tools available and the potential ways to support students in exploring the powerful ideas of the discipline (Pei, Weintrop and Wilensky, 2018[14]), the thoughtful use of computational thinking tools and skill sets can deepen the learning of mathematics contents by creating effective learning conditions (Weintrop et al., 2016[15]). Moreover, computational thinking tools offer students a context in which they can reify abstract constructs (by exploring and engaging with maths concepts in a dynamic way) (Wing 2008[4]), as well as express ideas in new ways and interact with concepts through media and

[2] (Wing 2006) is J.Wing, *Computational Thinking*, Communications of the ACM, Vol. 49, No. 3, March 2006, pp. 33–35.

[3] J. Wing, *Computational Thinking – What and Why?*, The Magazine of Carnegie Mellon University's School of Computer Science, March 2011. The LINK, Research Notebook. https://www.cs.cmu.edu/link/research-notebook-computational-thinking-what-and-why.

[4] J. Wing, *Computational thinking and thinking about computing*, Philosophical Transactions of The Royal Society A, 366:3717-3725, 2008

new representational tools (Grover, 2018[16]; Niemelä et al., 2017[17]; Pei, Weintrop and Wilensky, 2018[14]; Resnick et al., 2009[18]).

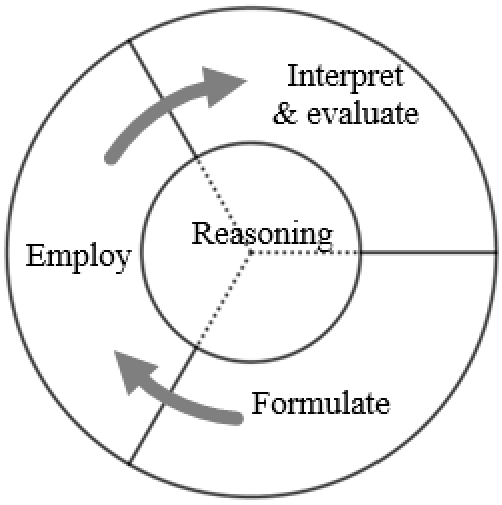
A View of Mathematically Literate Individuals in PISA 2022

24. The focus of the language in the definition of mathematical literacy is on active engagement with mathematics to solve real-world problems in a variety of contexts, and is intended to encompass mathematical reasoning (both deductive and inductive) and problem solving using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena.

25. It is important to note that the definition of mathematical literacy not only focuses on the use of mathematics to solve real-world problems, but also identifies mathematical reasoning as a core aspect of being mathematically literate. The contribution that the PISA 2022 framework makes is to highlight the centrality of mathematical reasoning both to the problem solving cycle and to mathematical literacy in general.

26. ***Figure 1*** depicts the relationship between mathematical reasoning (both deductive and inductive) and problem solving as reflected in the mathematical modelling cycle of both the PISA 2003 and PISA 2012 framework.

Figure 1. Mathematical literacy: the relationship between mathematical reasoning and the problem solving (modelling) cycle.



27. In order for students to be mathematically literate they must be able, first to use their mathematics content knowledge to recognise the mathematical nature of a situation (problem) especially those situations encountered in the real world and then to formulate it in mathematical terms. This transformation – from an ambiguous, messy, real-world situation to a well-defined mathematics problem – requires mathematical reasoning. Once the transformation is successfully made, the resulting mathematical problem needs to be solved using the mathematics concepts, algorithms and procedures taught in schools. However, it may require the making of strategic decisions about the selection of those tools and the order of their application – this is also a manifestation of mathematical reasoning. Finally, the PISA definition reminds us of the need for the student to evaluate the mathematical solution by interpreting the results within the original real-world situation. Additionally, students should also possess and be able to demonstrate computational

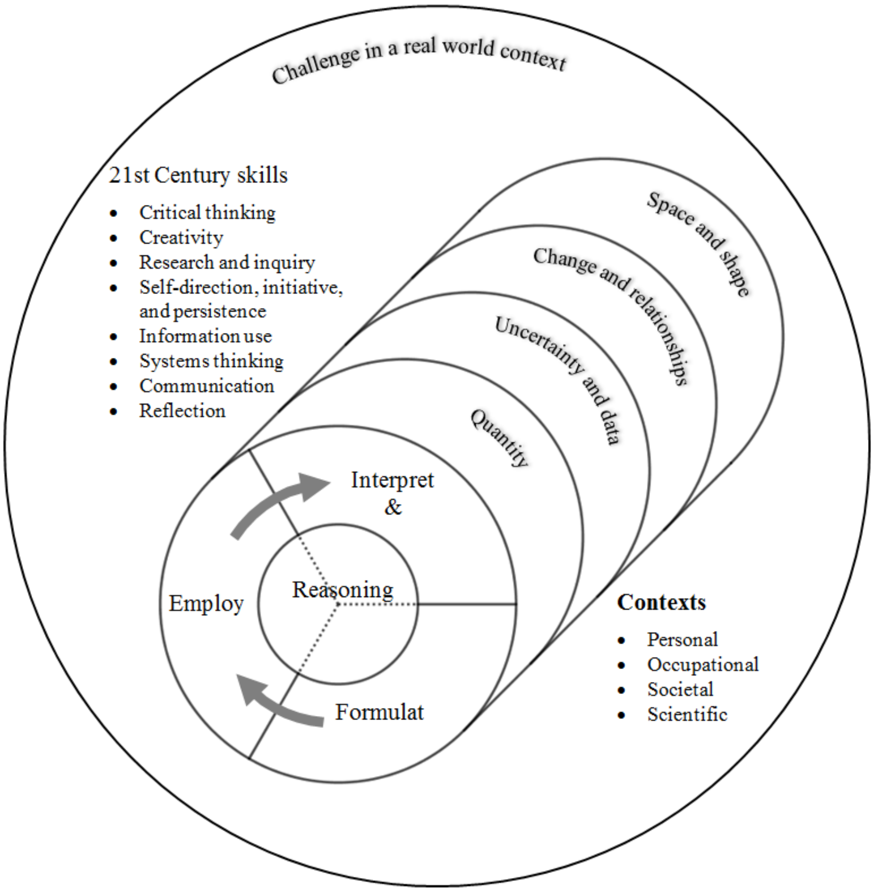
thinking skills as part of their problem-solving practice. These computational thinking skills which are applied in formulating, employing, evaluating and reasoning include pattern recognition, decomposition, determining which (if any) computing tools could be employed in the analysing or solving the problem, and defining algorithms as part of a detailed solution.

28. Although mathematical reasoning and solving real-world problems overlap, there is an aspect to mathematical reasoning which goes beyond solving practical problems. Mathematical reasoning is also a way of evaluating and making arguments, evaluating interpretations and inferences related to statements (e.g. in public policy debates etc.) and problem solutions that are, by their quantitative nature, best understood mathematically.

29. Mathematical literacy therefore comprises two related aspects: *mathematical reasoning* and *problem solving*. *Mathematical literacy* plays an important role in being able to use mathematics to *solve real-world problems*. In addition, mathematical reasoning (both deductive and inductive) also goes beyond solving real-world problems to include the making of informed judgements about that important family of societal issues which can be addressed mathematically. It also includes making judgements about the validity of information that bombards individuals by means of considering their quantitative and logical, implications. It is here where mathematical reasoning also contributes to the development of a select set of 21st century skills (discussed elsewhere in the framework).

30. The outer circle of Figure 2 shows that mathematical literacy takes place in the context of a challenge or problem that arises in the real world.

Figure 2. PISA 2022: the relationship between mathematical reasoning, the problem solving (modelling) cycle, mathematical contents, context and selected 21st century skills.



31. Figure 2 also depicts the relationship between mathematical literacy as depicted in Figure 1 and: the mathematical contents domains in which mathematical literacy is applied; the problem contexts and the selected 21st century skills that are both supportive of and developed through mathematical literacy.

32. These categories of mathematics content include: quantity, uncertainty and data, change and relationships, and space and shape. It is these categories of mathematics content knowledge which students must draw on to reason, to formulate the problem (by transforming the real world situation into a mathematical problem situation), to solve the mathematical problem once formulated, and to interpret and evaluate the solution determined.

33. As in the previous frameworks, the four context areas that PISA continues to use to define real-world situations are personal, occupational, societal and scientific. The context may be of a personal nature, involving problems or challenges that might confront an individual or one's family or peer group. The problem might instead be set in a societal context (focusing on one's community – whether it be local, national or global), an occupational context (centred on the world of work), or a scientific context (relating to the application of mathematics to the natural and technological world).

34. Included for the first time in the PISA 2022 framework (and depicted in Figure 2) are selected 21st century skills that mathematical literacy both relies on and develops. 21st century skills are discussed in greater detail in the next section of this framework. For now, it should be stressed that while contexts (personal, societal, occupational and scientific) influence the development of test items, there is no expectation that items will be deliberately developed to incorporate or address 21st century skills. Instead, the expectation is that by responding to the spirit of the framework and in line with the definition of mathematical literacy, the 21st century skills that have been identified will be incorporated in the items.

35. The language of the definition and the representation in Figure 1 and Figure 2 retain and integrate the notion of mathematical modelling, which has historically been a cornerstone of the PISA framework for mathematics e.g. (OECD, 2004[6]; OECD, 2013[7]). The modelling cycle (formulate, employ, interpret and evaluate) is a central aspect of the PISA conception of mathematically literate students; however, it is often not necessary to engage in every stage of the modelling cycle, especially in the context of an assessment (Galbraith, Henn and Niss, 2007[19]). It is often the case that significant parts of the mathematical modelling cycle have been undertaken by others, and the end user carries out some of the steps of the modelling cycle, but not all of them. For example, in some cases, mathematical representations, such as graphs or equations, are given that can be directly manipulated in order to answer some question or to draw some conclusion. In other cases, students may be using a computer simulation to explore the impact of variable change in a system or environment. For this reason, many PISA items involve only parts of the modelling cycle. In reality, the problem solver may also sometimes oscillate between the processes, returning to revisit earlier decisions and assumptions. Each of the processes may present considerable challenges, and several iterations around the whole cycle may be required.

36. In particular, the verbs 'formulate', 'employ' and 'interpret' point to the three processes in which students as active problem solvers will engage. Formulating situations mathematically involves applying mathematical reasoning (both deductive and inductive) in identifying opportunities to apply and use mathematics – seeing that mathematics can be applied to understand or resolve a particular problem or challenge presented. It includes being able to take a situation as presented and transform it into a form amenable to mathematical treatment, providing mathematical structure and representations, identifying variables and making simplifying assumptions to help solve the problem or meet the challenge. Employing mathematics involves applying mathematical reasoning while using mathematical concepts, procedures, facts and tools to derive a mathematical solution. It includes performing calculations, manipulating algebraic expressions and equations or other mathematical models, analysing information in a mathematical manner from mathematical diagrams and graphs, developing mathematical descriptions and explanations and using mathematical tools to solve problems. Interpreting mathematics involves reflecting upon mathematical solutions or results and interpreting them in the context of a problem or challenge. It involves applying mathematical reasoning to evaluate mathematical solutions in relation to the context of the problem and determining whether the results are reasonable and make sense in the situation; determining also what to highlight when explaining the solution.

37. Included for the first time in the PISA 2022 framework is an appreciation of the intersection between mathematical and computational thinking engendering a similar set of perspectives, thought processes and mental models that learners need to succeed in an increasingly technological world. A set of constituent practices positioned

under the computational thinking umbrella (namely abstraction, algorithmic thinking, automation, decomposition and generalisation) are also central to both mathematical reasoning and problem solving processes. The nature of computational thinking within mathematics is conceptualised as defining and elaborating mathematical knowledge that can be expressed by programming, allowing students to dynamically model mathematical concepts and relationships. A taxonomy of computational thinking practices geared specifically towards mathematics and science learning entails data practices, modelling and simulation practices, computational problem solving practices, and systems thinking practices (Weintrop et al., 2016[15]). The combination of mathematical and computational thinking not only becomes essential to effectively support the development of students' conceptual understanding of the mathematical domain, but also to develop their computational thinking concepts and skills, giving learners a more realistic view of how mathematics is practiced in the professional world and used in the real-world and, in turn, better prepares them for pursuing careers in related fields (Basu et al., 2016[20]; Benton et al., 2017[21]; Pei, Weintrop and Wilensky, 2018[14]; Beheshti et al., 2017[22]).

An Explicit Link to a Variety of Contexts for Problems in PISA 2022

38. The reference to 'a variety of real-world contexts' in the definition of mathematical literacy recognises that the 21st century citizen is a consumer of quantitative, sometimes statistical, arguments. The reference is intended as a way to link to the specific contexts that are described and exemplified more fully later in this framework. The specific contexts themselves are not so important, but the four categories selected for use here (personal, occupational, societal and scientific) reflect a wide range of situations in which individuals may meet mathematical opportunities. The definition also acknowledges that mathematical literacy helps individuals to recognise the role that mathematics plays in the world and to make the kinds of well-founded judgments and decisions required of constructive, engaged and reflective citizens faced with messages and arguments of the form: "a study found that on average...", "a survey shows a big drop in....", "certain scientists claim that population growth will outpace food production in *x* years …" etc.

A Visible Role for Mathematical Tools, including Technology in PISA 2022

39. The definition of mathematical literacy explicitly includes the use of mathematical tools. These tools include a variety of physical and digital equipment, software and calculation devices. Computer-based mathematical tools are in common use in workplaces of the 21st century, and will be increasingly more prevalent as the century progresses both in the workplace and in society generally. The nature of day-to-day and work-related problems and the demands on individuals to be able to employ mathematical reasoning (both deductive and inductive) in situations where computational tools are present has expanded with these new opportunities – creating enhanced expectations for mathematical literacy.

40. Since the 2015 cycle, computer-based assessment (CBA) has been the primary mode of testing, although an equivalent paper-based instrument is available for those countries who chose not to test their students by computer. The 2015 and 2018 mathematical literacy assessments did not exploit the opportunities that the computer provides.

41. Computer-Based Assessment of Mathematics (CBAM) will be the format of the mathematical literacy from 2022. Although the option of a paper based assessment

will remain for countries who want to continue in that way, the CBAM will exploit the opportunities of the CBAM. The opportunities that this transition creates are discussed in greater detail later in the framework.

Organisation of the Domain

42. The PISA mathematics framework defines the domain of mathematics for the PISA survey and describes an approach to the assessment of the mathematical literacy of 15-year-olds. That is, PISA assesses the extent to which 15-year-old students can reason mathematically and handle mathematics adeptly when confronted with situations and problems – the majority of which are presented in real-world contexts.

43. For purposes of the assessment, the PISA 2022 definition of mathematical literacy can be analysed in terms of three interrelated aspects (see Figure 2):

- Mathematical reasoning (both deductive and inductive) and problem solving (which includes the mathematical processes that describe what individuals do to connect the context of the problem with mathematics and thus solve the problem);
- The mathematical content that is targeted for use in the assessment items; and
- The contexts in which the assessment items are located coupled with selected[5] 21st century skills that support and are developed by mathematical literacy.

44. The following sections elaborate these aspects to support understanding and to provide guidance to the test developers. In highlighting these aspects of the domain, the PISA 2022 mathematics framework helps to ensure that assessment items developed for the survey reflect a range of mathematical reasoning and problem solving, content, and contexts and 21st century skills, so that, considered as a whole, the set of assessment items effectively operationalises what this framework defines as mathematical literacy. Several questions, based on the PISA 2022 definition of mathematical literacy lie behind the organisation of this section of the framework. They are:

- What do individuals engage in when reasoning mathematically and solving contextual mathematical problems?
- What mathematical content knowledge can we expect of individuals – and of 15-year-old students in particular?
- In what context is mathematical literacy able to be both observed and assessed and how do these interact with the identified 21st century skills?

Mathematical Reasoning and Problem Solving Processes

Mathematical reasoning

45. Mathematical reasoning (both deductive and inductive) involves evaluating situations, selecting strategies, drawing logical conclusions, developing and describing solutions, and recognising how those solutions can be applied. Students reason mathematically when they:

- Identify, recognise, organise, connect, and represent,
- Construct, abstract, evaluate, deduce, justify, explain, and defend; and

[5] The selected skills were recommended by the OECD Subject Advisory Group (SAG) (*PISA 2022 Mathematics: A Broadened Perspective* [EDU/PISA/GB(2017)17] by finding the union between generic 21st Century skills and related but subject-matter specific skills that are a natural part of the instruction related in the subject matter. The advisory group identified eight 21st Century skills for inclusion in the mathematics curriculum and, as such, in the PISA 2022 assessment framework. These skills are listed in paragraph 124.

- Interpret, make judgements, critique, refute, and qualify.

46. The ability to reason logically and to present arguments in honest and convincing ways is a skill that is becoming increasingly important in today's world. Mathematics is a science about well-defined objects and notions which can be analysed and transformed in different ways using 'mathematical reasoning' to obtain conclusions about which we are certain. Through mathematics, students learn that using appropriate reasoning they can reach results and conclusions which they can trust to be true. Further, those conclusions are logical and objective, and hence impartial, without any need for validation by an external authority. This kind of reasoning which is useful far beyond mathematics, can be learned and practiced most effectively within mathematics.

47. Two aspects of mathematical reasoning are especially important in today's world and in defining the PISA items. One is deduction from clear assumptions (deductive reasoning), which is a characteristic feature of mathematical process. The usefulness of this ability has already been stressed. The second important dimension is statistical and probabilistic (inductive) reasoning. At the logical level, there is these days frequent confusion in the minds of individuals between the possible and the probable, leading many to fall prey to conspiracy theories or fake news. From a technical perspective, today's world is increasingly complex and its multiple dimensions are represented by terabytes of data. Making sense of these data is one of the biggest challenges that humanity will face in the future. Our students should be familiarised with the nature of such data and making informed decisions in the context of variation and uncertainty.

48. Mathematical reasoning (both deductive and inductive), enabled by some key understandings that undergird school mathematics, is the core of mathematical literacy. Included among these key understandings are:

- Understanding quantity, number systems and their algebraic properties;
- Appreciating the power of abstraction and symbolic representation;
- Seeing mathematical structures and their regularities;
- Recognising functional relationships between quantities;
- Using mathematical modelling as a lens onto the real world (e.g. those arising in the physical, biological, social, economic, and behavioural sciences); and
- Understanding variation as the heart of statistics.

The description of each of these that follows provides an overview of the understanding and how it supports reasoning. While the descriptions may appear abstract, the intention is not for them to be treated in an abstract way in the PISA assessment. The message that the descriptions should convey is how these ideas surface throughout school mathematics and how, by reinforcing their occurrence in teaching we support students to realise how they can be applied in new and different contexts.

Understanding quantity, number systems and their algebraic properties

49. The basic notion of quantity may be the most pervasive and essential mathematical aspect of engaging with, and functioning in, the world (OECD, 2017, p. 18[23]). At the most basic level it deals with the useful ability to compare cardinalities of sets of objects. The ability to count usually involves rather small sets – in most languages, only a small subset of numbers have names. When we assess larger sets, we engage in more complex operations of estimating, rounding and applying orders of magnitude. Counting is very closely related to another fundamental operation of classifying things, where the ordinal aspect of numbers emerges. Quantification of attributes of objects (measurement),

relationships, situations and entities in the world is one of the most basic ways of conceptualising the surrounding world (OECD, 2017[23]).

50. Understanding quantity, number systems and their algebraic properties includes the basic concept of number, nested number systems (e.g., whole numbers to integers to rationals to reals), the arithmetic of numbers, and the algebraic properties that the systems enjoy. In particular, it is useful to understand how progressively more expansive systems of numbers enable the solution of progressively more complicated equations. This lays the foundation for enabling students to see more evidence of mathematics in the real world in as they learn more mathematics.

51. To use quantification efficiently, one has to be able to apply not just numbers, but the number systems. Numbers themselves are of limited relevance; what makes them into a powerful tool are the operations that we can perform with them. As such, a good understanding of the operations of numbers is the foundation of mathematical reasoning.

52. It is also important to understand matters of representation (as symbols involving numerals, as points on a number line, as geometric quantities, and by special symbols such as π) and how to move between them; the ways in which these representations are affected by number systems; the ways in which algebraic properties of these systems are relevant and matter for operating within the systems; and the significance of the additive and multiplicative identities, associativity, commutativity, and the distributive property of multiplication over addition. Algebraic principles undergird the place value system, allowing for economical expression of numbers and efficient approaches to operations on them. They are also central to number-line based operations with numbers, including work with additive inverses that are central to addition and subtraction of first integers, then rationals and finally reals.

53. The centrality of number as a key concept in all the other mathematical areas under consideration here and to mathematical reasoning itself, is undeniable. Students' grasp of the algebraic principles and properties first experienced through work with numbers is fundamental to their understanding of the concepts of secondary school algebra, along with their ability to become fluent in the manipulations of algebraic expressions necessary for solving equations, setting up models, graphing functions, and programing and making spreadsheet formulas. And in today's data-intensive world, facility with interpretation of patterns of numbers, comparison of patterns, and other numerical skills are evolving in importance.

54. *A broad understanding of quantity and number systems supports reasoning in the real-world applications of mathematics envisaged by this framework*.

Appreciating the power of abstraction and symbolic representation

55. The fundamental ideas of mathematics have arisen from human experience in the world and the need to provide coherence, order, and predictability to that experience. Many mathematical objects model reality, or at least reflect aspects of reality in some way. However, the essence of abstraction in mathematics is that it is a self-contained system, and mathematical objects derive their meaning from within that system. Abstraction involves deliberately and selectively attending to structural similarities between mathematical objects, and constructing relationships between those objects based on these similarities. In school mathematics, abstraction forms relationships between concrete objects, symbolic representations and operations including algorithms and mental models. This ability also plays a role in working with computational devices. The ability to create,

manipulate, and draw meaning in working with abstractions in technological contexts in an important computational thinking skill.

56. For example, children begin to develop the concept of "circle" by experiencing specific objects that lead them to an informal understanding of circles as being "roundish". They might draw circles to represent these objects, noticing similarities between the drawings to generalise about "roundness" even though the circles are of different sizes. "Circle" becomes an abstract mathematical object when students start to "use" circles as objects in their work and more formally when it is defined as the locus of points equidistant from a fixed point in a two-dimensional plane.

57. Students use representations – whether text-based, symbolic, graphical, numerical, geometric or in programming code – to organise and communicate their mathematical thinking. Representations enable us to present mathematical ideas in a succinct way which, in turn, lead to efficient algorithms. Representations are also a core element of mathematical modelling, allowing students to abstract a simplified or idealised formulation of a real world problem. Such structures are also important for interpreting and defining the behaviour of computational devices.

58. *Having an appreciation of abstraction and symbolic representation supports reasoning in the real-world applications of mathematics envisaged by this framework by allowing students to move from the specific details of a situation to the more general features and to describe these in an efficient way.*

Seeing mathematical structures and their regularities

59. When elementary students see: $5 + (3 + 8)$ some see a string of symbols indicating a computation to be performed in a certain order according to the rules of order of operations; others see a number added to the sum of two other numbers. The latter group are seeing structure; and because of that they don't need to be told about the order of operation, because if you want to add a number to a sum you first have to compute the sum.

60. Seeing structure continues to be important as students move to higher grades. A student who sees $f(x) = 5 + (x - 3)^2$ as saying that $f(x)$ is the sum of 5 and a square which is zero when $x = 3$ understands that the minimum of f is 5. This lays the foundation for functional thinking discussed in the next section.

61. Structure is intimately related to symbolic representation. The use of symbols is powerful, but only if they retain meaning for the symboliser, rather than becoming meaningless objects to be rearranged on a page. Seeing structure is a way of finding and remembering the meaning of an abstract representation. Such structures are also important for interpreting and defining the behaviour of computational devices. Being able to see structure is an important conceptual aid to procedural knowledge.

62. The examples above illustrate how seeing structure in abstract mathematical objects is a way of replacing parsing rules, which can be performed by a computer, with conceptual images of those objects that make their properties clear. An object held in the mind in such a way is subject to reasoning at a level that is higher than simple symbolic manipulation.

63. A robust sense of mathematical structure also supports modelling. When the objects under study are not abstract mathematical objects, but rather objects from the real world to be modelled by mathematics, then mathematical structure can guide the modelling. Students can also impose structure on non-mathematical objects in order to make them subject to mathematical analysis. An irregular shape can be approximated by simpler

shapes whose area is known. A geometric pattern can be understood by hypothesising translational, rotational, or reflectional transformations and symmetry and abstractly extending the pattern into all of space. Statistical analysis is often a matter of imposing a structure on a set of data, for example by assuming it comes from a normal distribution or supposing that one variable is a linear function of another, but measured with normally distributed error.

64. *Being able to see mathematical structures supports reasoning in the real-world applications of mathematics envisaged by this framework by allowing students to apply knowledge about situations or problems in one context to problems in another context that share a similar structure.*

Recognising functional relationships between quantities

65. Students in elementary school encounter problems where they must find specific quantities. For example, how fast do you have to drive to get from Tucson to Phoenix, a distance of 180 km, in 1 hour and 40 minutes? Such problems have a specific answer: to drive 180 km in 1 hour and 40 minutes you must drive at 108 km per hour.

66. At some point students start to consider situations where quantities are variable, that is, where they can take on a range of values. For example, what is the relation between the distance driven, d, in kilometres, and time spent driving, t, in hours, if you drive at a constant speed of 108 *km* per hour? Such questions introduce functional relationships. In this case the relationship, expressed by the equation $d = 108t$, is a proportional relationship, the fundamental example and perhaps the most important for general knowledge.

67. Relationships between quantities can be expressed with equations, graphs, tables, or verbal descriptions. An important step in learning is to extract from these the notion of a function itself, as an abstract object of which these are representations. The essential elements of the concept are a domain, from which inputs are selected, a codomain, in which outputs lie, and a process for producing outputs from inputs.

68. *Recognising the functional relationships between the variables in the real-world applications of mathematics envisaged by this framework supports reasoning by allowing students to focus on how the interdependence of and interaction between the variables impacts on the situation.*

Using mathematical modelling as a lens onto the real world

69. Models represent a conceptualisation of phenomena. Models are simplifications of reality that fore- ground certain features of a phenomenon while approximating or ignoring other features. As such, ‘‘all models are wrong, but some are useful’’ (Box and Draper, 1987, p. 424[24]). The usefulness of a model comes from its explanatory and/or predictive power (Weintrop et al., 2016[15]). Models are, in that sense, abstractions of reality. A model may present a conceptualisation that is understood to be an approximation or working hypothesis concerning the object phenomenon or it may be an intentional simplification. Mathematical models are formulated in mathematical language and use a wide variety of mathematical tools and results (e.g., from arithmetic, algebra, geometry, etc.). As such, they are used as ways of precisely defining the conceptualisation or theory of a phenomenon, for analysing and evaluating data (does the model fit the data?), and for making predictions. Models can be operated – that is, made to run over time or with varying inputs, thus producing a simulation. When this is done, it is possible to make predictions, study consequences, and evaluate the adequacy and accuracy of the models. Throughout

the modelling process cognisance needs to be taken of the real world parameters that impact on the model and the solutions developed using the model.

70. Computer-based (or computational) models provide the ability to test hypothesis, generate data, introduce randomness and so on. Mathematical literacy includes the ability to understand, evaluate and draw meaning from computational models.

71. *Using models in general and mathematical models in particular supports reasoning about the real-world applications of mathematics envisaged in this framework by encouraging students to focus on the most significant elements of the situations and in so doing to reduce the problem to its essence.*

Understanding variation as the heart of statistics

72. In statistics accounting for variability is one, if not the central, defining element around which the discipline is based. In today's world people often deal with these types of situations by merely ignoring the variation and as a result suggesting sweeping generalisations which are often misleading, if not wrong, and as a result very dangerous. Bias in the social science sense is usually created by not accounting for the sources and magnitudes of the variability in the trait under discussion.

73. Statistics is essentially about accounting for or modelling variation as measured by the variance or in the case of multiple variables the covariance matrix. This provides a probabilistic environment in which to understand various phenomena as well as to make critical decisions. Statistics is in many ways a search for patterns in a highly variable context: trying to find the single defining "truth" in the midst of a great deal of random noise. "Truth" is set in quotes as it is not the nature of truth that mathematics can deliver but an estimate of truth set in a probabilistic context, accompanied by an estimate of the error contained in the process. Ultimately, the decision maker is left with the dilemma of never knowing for certain what the truth is. The estimate that has been developed is, at best a range of possible values – the better the process, for example, the larger the sample of data, the narrower the range of possible values, although a range cannot be avoided. Some aspects of this have been present in previous PISA cycles, the growing significance contributes to the increased stress in this framework.

74. *Understanding variation as a central feature of statistics supports reasoning about the real-world applications of mathematics envisaged in this framework in that students are encouraged to engage with data based arguments with awareness of the limitations of the conclusions that can be drawn.*

Problem solving

75. The definition of mathematical literacy refers to an individual's capacity to formulate, employ, and interpret (and evaluate) mathematics. These three words, formulate, employ and interpret, provide a useful and meaningful structure for organising the mathematical processes that describe what individuals do to connect the context of a problem with the mathematics and solve the problem. Items in the 2022 PISA mathematics test will be assigned to either mathematical reasoning or one of three mathematical processes:

- Formulating situations mathematically;
- Employing mathematical concepts, facts, procedures and reasoning; and
- Interpreting, applying and evaluating mathematical outcomes.

76. It is important for both policy makers and those engaged more closely in the day-to-day education of students to know how effectively students are able to engage in each of these elements of the problem solving model/cycle. *Formulating* indicates how effectively students are able to recognise and identify opportunities to use mathematics in problem situations and then provide the necessary mathematical structure needed to formulate that contextualised problem in a mathematical form. *Employing* refers to how well students are able to perform computations and manipulations and apply the concepts and facts that they know to arrive at a mathematical solution to a problem formulated mathematically. *Interpreting* (and *evaluating*) relates to how effectively students are able to reflect upon mathematical solutions or conclusions, interpret them in the context of the real-world problem and determine whether the result(s) or conclusion(s) are reasonable and/or useful. Students' facility at applying mathematics to problems and situations is dependent on skills inherent in all three of these stages, and an understanding of students' effectiveness in each category can help inform both policy-level discussions and decisions being made closer to the classroom level.

77. Moreover, encouraging students to experience mathematical problem solving processes through computational thinking tools and practices encourage students to practice prediction, reflection and debugging skills (Brennan and Resnick, 2012[25]).

Formulating Situations Mathematically

78. The word *formulate* in the mathematical literacy definition refers to individuals being able to recognise and identify opportunities to use mathematics and then provide mathematical structure to a problem presented in some contextualised form. In the process of formulating situations mathematically, individuals determine where they can extract the essential mathematics to analyse, set up and solve the problem. They translate from a real-world setting to the domain of mathematics and provide the real-world problem with mathematical structure, representations and specificity. They reason about and make sense of constraints and assumptions in the problem. Specifically, this process of formulating situations mathematically includes activities such as the following:

- selecting an appropriate model from a list;[6]
- identifying the mathematical aspects of a problem situated in a real-world context and identifying the significant variables;
- recognising mathematical structure (including regularities, relationships, and patterns) in problems or situations;
- simplifying a situation or problem in order to make it amenable to mathematical analysis (for example by decomposing);
- identifying constraints and assumptions behind any mathematical modelling and simplifications gleaned from the context;
- representing a situation mathematically, using appropriate variables, symbols, diagrams, and standard models;
- representing a problem in a different way, including organising it according to mathematical concepts and making appropriate assumptions;
- understanding and explaining the relationships between the context-specific language of a problem and the symbolic and formal language needed to represent it mathematically;

[6] This activity is included in the list to foreground the need for the test items developers to include items that are accessible to students at the lower end of the performance scale.

- translating a problem into mathematical language or a representation;
- recognising aspects of a problem that correspond with known problems or mathematical concepts, facts or procedures;
- choosing among an array of and employing the most effective computing tool to portray a mathematical relationship inherent in a contextualised problem; and
- creating an ordered series of (step-by-step) instructions for solving problems.

Employing Mathematical Concepts, Facts, Procedures and Reasoning

79. The word *employ* in the mathematical literacy definition refers to individuals being able to apply mathematical concepts, facts, procedures, and reasoning to solve mathematically-formulated problems to obtain mathematical conclusions. In the process of employing mathematical concepts, facts, procedures and reasoning to solve problems, individuals perform the mathematical procedures needed to derive results and find a mathematical solution (e.g. performing arithmetic computations, solving equations, making logical deductions from mathematical assumptions, performing symbolic manipulations, extracting mathematical information from tables and graphs, representing and manipulating shapes in space, and analysing data). They work on a model of the problem situation, establish regularities, identify connections between mathematical entities, and create mathematical arguments. Specifically, this process of employing mathematical concepts, facts, procedures and reasoning includes activities such as:

- performing a simple calculation;[7] **
- drawing a simple conclusion; **
- selecting an appropriate strategy from a list; **
- devising and implementing strategies for finding mathematical solutions;
- using mathematical tools, including technology, to help find exact or approximate solutions;
- applying mathematical facts, rules, algorithms, and structures when finding solutions;
- manipulating numbers, graphical and statistical data and information, algebraic expressions and equations, and geometric representations;
- making mathematical diagrams, graphs, simulations, and constructions and extracting mathematical information from them;
- using and switching between different representations in the process of finding solutions;
- making generalisations and conjectures based on the results of applying mathematical procedures to find solutions;
- reflecting on mathematical arguments and explaining and justifying mathematical results; and
- evaluating the significance of observed (or proposed) patterns and regularities in data.

Interpreting, Applying and Evaluating Mathematical Outcomes

80. The word *interpret* (and *evaluate*) used in the mathematical literacy definition focuses on the ability of individuals to reflect upon mathematical solutions, results or conclusions and interpret them in the context of the real-life problem that initiated

[7] These activities (**) are included in the list to foreground the need for the test items developers to include items that are accessible to students at the lower end of the performance scale.

the process. This involves translating mathematical solutions or reasoning back into the context of the problem and determining whether the results are reasonable and make sense in the context of the problem. *Interpreting, applying and evaluating mathematical outcomes* encompasses both the ‘interpret’ and ‘evaluate’ elements of the mathematical modelling cycle. Individuals engaged in this process may be called upon to construct and communicate explanations and arguments in the context of the problem, reflecting on both the modelling process and its results. Specifically, this process of interpreting, applying and evaluating mathematical outcomes includes activities such as:

- interpreting information presented in graphical form and/or diagrams;[8] **
- evaluating a mathematical outcome in terms of the context; **
- interpreting a mathematical result back into the real-world context;
- evaluating the reasonableness of a mathematical solution in the context of a real-world problem;
- understanding how the real world impacts the outcomes and calculations of a mathematical procedure or model in order to make contextual judgments about how the results should be adjusted or applied;
- explaining why a mathematical result or conclusion does, or does not, make sense given the context of a problem;
- understanding the extent and limits of mathematical concepts and mathematical solutions;
- critiquing and identifying the limits of the model used to solve a problem; and
- using mathematical thinking and computational thinking to make predictions, to provide evidence for arguments, to test and compare proposed solutions.

Mathematical Content Knowledge

81. An understanding of mathematical content – and the ability to apply that knowledge to solving meaningful contextualised problems – is important for citizens in the modern world. That is, to reason mathematically and to solve problems and interpret situations in personal, occupational, societal and scientific contexts, there is a need to draw upon certain mathematical knowledge and understanding.

82. Since the goal of PISA is to assess mathematical literacy, an organisational structure for mathematical content knowledge is proposed that is based on mathematical phenomena that underlie broad classes of problems. Such an organisation for content is not new, as exemplified by two well-known publications: On the Shoulders of Giants: New Approaches to Numeracy (Steen, 1990[26]) and Mathematics: The Science of Patterns (Devlin, 1994[27]).

83. The following content categories (previously used in 2012) are again used in PISA 2022 to reflect both the mathematical phenomena that underlie broad classes of problems, the general structure of mathematics, and the major strands of typical school curricula. These four categories characterise the range of mathematical content that is central to the discipline and illustrate the broad areas of content used in the test items for

[8] These activities (**) are included in the list to foreground the need for the test items developers to include items that are accessible to students at the lower end of the performance scale

PISA 2022 (which will include PISA-D items to increase opportunities at the lower end of the performance spectrum):

- change and relationships
- space and shape
- quantity
- uncertainty and data

84. With these four categories, the mathematical domain can be organised in a way that ensures a spread of items across the domain and focuses on important mathematical phenomena, while at the same time, avoiding too granular a classification that would prevent the analysis of rich and challenging mathematical problems based on real situations.

85. While categorisation by content category is important for item development, selection and reporting of the assessment results, it is important to note that some items could potentially be classified in more than one content category.

86. National school mathematics curricula are typically organised around content strands (most commonly: numbers, algebra, functions, geometry, and data handling) and detailed topic lists help to define clear expectations. These curricula are designed to equip students with knowledge and skills that address these same underlying mathematical phenomena that organise the PISA content. The outcome is that the range of content arising from organising it in the way that PISA does is closely aligned with the content that is typically found in national mathematics curricula. This framework lists a range of content topics appropriate for assessing the mathematical literacy of 15-year-old students, based on analyses of national standards from eleven countries.

87. The broad mathematical content categories and the more specific content topics appropriate for 15-year-old students described in this section reflect the level and breadth of content that is eligible for inclusion in the PISA 2022 assessment. Descriptions of each content category and the relevance of each to reasoning and solving meaningful problems are provided, followed by more specific definitions of the kinds of content that are appropriate for inclusion in an assessment of mathematical literacy of 15-year-old students and out-of-school youth.

88. Four topics have been identified for special emphasis in the PISA 2022 assessment. These topics are not new to the mathematics content categories. Instead, these are topics within the existing content categories that deserve special emphasis. In the work of Mahajan et al. (“PISA Mathematics 2022”, (2016[28])) the four topics are presented not only as commonly encountered situations in adult life in general, but as the types of mathematics needed in the emerging new areas of the economy such as high-tech manufacturing etc. The four are: growth phenomena; geometric approximations; computer simulations; and conditional decision making. These topics should be approached in the test items in a way that is consistent with the experiences of 15-year-olds. Each topic is discussed with the discussion of the corresponding content category as follows:

- Growth phenomena (change and relationships)
- Geometric approximation (space and shape)
- Computer simulations (quantity)
- Conditional decision making (uncertainty and data)

Change and Relationships

89. The natural and designed worlds display a multitude of temporary and permanent relationships among objects and circumstances, where changes occur within systems of interrelated objects or in circumstances where the elements influence one another. In many cases these changes occur over time, and in other cases changes in one object or quantity are related to changes in another. Some of these situations involve discrete change; others change continuously. Some relationships are of a permanent, or invariant, nature. Being more literate about change and relationships involves understanding fundamental types of change and recognising when they occur in order to use suitable mathematical models to describe and predict change. Mathematically this means modelling the change and the relationships with appropriate functions and equations, as well as creating, interpreting and translating among symbolic and graphical representations of relationships.

90. Change and relationships is evident in such diverse settings as growth of organisms, music, seasonal change and cycles, weather patterns, employment levels and economic conditions. Aspects of the traditional mathematical content of functions and algebra, including algebraic expressions, equations and inequalities, tabular and graphical representations, are central in describing, modelling and interpreting change phenomena. Computational tools provide a means to visualise and interact with change and relationships. Recognising how and when a computational device can augment and complement mathematical concepts is an important computational thinking skill.

91. Representations of data and relationships described using statistics are also used to portray and interpret change and relationships, and a firm grounding in the basics of number and units is also essential to defining and interpreting change and relationships. Some interesting relationships arise from geometric measurement, such as the way that changes in perimeter of a family of shapes might relate to changes in area, or the relationships among lengths of the sides of triangles.

92. Growth phenomena: Understanding the dangers of flu pandemics and bacterial outbreaks, as well as the threat of climate change, demand that people think not only in terms of linear relationships but recognise that such phenomena need non-linear (often exponential but also other) models. Linear relationships are common and are easy to recognise and understand but to assume linearity can be dangerous. A good example of linearity and one probably used by everyone is estimating the distance travelled in various amounts of time while traveling at a given speed. Such an application provides a reasonable estimate as long as the speed stays relatively constant. But with flu epidemics, for example, such a linear approach would grossly underestimate the number of people sick in 5 days after the initial outbreak. Here is where a basic understanding of non-linear (including quadratic and exponential) growth and how rapidly infections can spread given that the rate of change increases from day to day is critical. The spread of the Zika infection is an important example of exponential growth; recognising it as such helped medical personnel to understand the inherent threat and the need for fast action.

93. Identifying growth phenomena as a focal point of the change and relationships content category is not to signal that there is an expectation that participating students should have studied the exponential function and certainly the items will not require knowledge of the exponential function. Instead, the expectation is that there will be items that expect students to (a) recognise that not all growth is linear, (b) that non-linear growth has particular and profound implications on how we understand certain situations, and (c) appreciate the intuitive meaning of “exponential growth” as an extremely rapid rate of growth, for example in the earthquake scale, every increase by 1 unit on the Richter scale

does not mean a proportional increase in its effect, but rather by 10, 100, and 1000 times etc.

Space and Shape

94. Space and shape encompasses a wide range of phenomena that are encountered everywhere in our visual and physical world: patterns, properties of objects, positions and orientations, representations of objects, decoding and encoding of visual information, navigation and dynamic interaction with real shapes as well as with representations, movement, displacement, and the ability to anticipate actions in space. Geometry serves as an essential foundation for space and shape, but the category extends beyond traditional geometry in content, meaning and method, drawing on elements of other mathematical areas such as spatial visualisation, measurement and algebra. For instance, shapes can change and a point can move along a locus, thus requiring function concepts. Measurement formulas are central in this area. The recognition, manipulation and interpretation of shapes in settings that call for tools ranging from dynamic geometry software to Global Positioning Systems (GPS), and to machine learning software are included in this content category.

95. PISA assumes that the understanding of a set of core concepts and skills is important to mathematical literacy relative to space and shape. Mathematical literacy in the area of space and shape involves a range of activities such as understanding perspective (for example in paintings), creating and reading maps, transforming shapes with and without technology, interpreting views of three-dimensional scenes from various perspectives and constructing representations of shapes.

96. Geometric approximations: Today’s world is full of shapes that do not follow typical patterns of evenness or symmetry. Because simple formulas do not deal with irregularity, it has become more difficult to understand what we see and find the area or volume of the resulting structures. For example, finding the needed amount of carpeting in a building in which the apartments have acute angles together with narrow curves demands a different approach than would be the case with a typically rectangular room.

97. Identifying geometric approximations as a focal point of the space and shape content category signals the need for students to be able use their understanding of traditional space and shape phenomena in a range of typical situations.

Quantity

98. The notion of quantity may be the most pervasive and essential mathematical aspect of engaging with, and functioning in, our world. It incorporates the quantification of attributes of objects, relationships, situations and entities in the world, understanding various representations of those quantifications and judging interpretations and arguments based on quantity. To engage with the quantification of the world involves understanding measurements, counts, magnitudes, units, indicators, relative size and numerical trends and patterns. Aspects of quantitative reasoning – such as number sense, multiple representations of numbers, elegance in computation, mental calculation, estimation and assessment of reasonableness of results – are the essence of mathematical literacy relative to quantity.

99. Quantification is a primary method for describing and measuring a vast set of attributes of aspects of the world. It allows for the modelling of situations, for the examination of change and relationships, for the description and manipulation of space and shape, for organising and interpreting data and for the measurement and assessment of

uncertainty. Thus mathematical literacy in the area of quantity applies knowledge of number and number operations in a wide variety of settings.

100. Computer simulations: Both in mathematics and statistics there are problems that are not so easily addressed because the required mathematics are complex or involve a large number of factors all operating in the same system or because of ethical issues relating to the impact on living beings or their environment. Increasingly in today's world such problems are being approached using computer simulations driven by algorithms. In the illustrative example *Savings Simulation* the student uses a computer simulation as a tool in decision making. The computer simulation does the calculations for the student, leaving the student to plan, predict and solve problems based on the variables that they can control.

101. Identifying computer simulations as a focal point of the quantity content category signals that in the context the Computer-Based Assessment of Mathematics (CBAM) of PISA being used from 2022, there are a broad category of complex problems including budgeting and planning that students can analyse in terms of the variables of the problem using computer simulations provided as part of the test item.

Uncertainty and Data

102. In science, technology and everyday life, variation and its associated uncertainty is a given. It is a phenomenon at the heart of the theory of probability and statistics. The uncertainty and data content category includes recognising the place of variation in the real world including, having a sense of the quantification of that variation, and acknowledging its uncertainty and error in related inferences. It also includes forming, interpreting and evaluating conclusions drawn in situations where uncertainty is present. The presentation and interpretation of data are key concepts in this category (Moore, 1997[29]).

103. Economic predictions, poll results, and weather forecasts all include measures of variation and uncertainty. There is variation in manufacturing processes, test scores and survey findings, and chance is fundamental to many recreational activities enjoyed by individuals. The traditional curricular areas of probability and statistics provide formal means of describing, modelling and interpreting a certain class of phenomena in which variation plays a central role, and for making corresponding stochastic inferences. In addition, knowledge of number and of aspects of algebra such as graphs and symbolic representation contribute to engaging in problem solving in this content category.

104. Conditional decision making: statistics provides a measure of the variation characteristic of much of what people encounter in their daily lives. That measure is the variance. When there is more than one variable, there is variation in each of the variables as well as co-variation characterising the relationships among the variables. These inter-relationships can often be represents in two-way tables that provide the basis for making conditional decisions (inferences). In a two-way table for two dichotomous variables (i.e. two variables with two possibilities each), there are four combinations. The two-way table (analysis of the situation) provides three types of percentages which, in turn, provide estimates of the corresponding probabilities. These include the probabilities of the four joint events, the two marginal, and the conditional probabilities which play the central role in what we have termed conditional decision making. The expectation for the PISA test items is that students will be able to read the relevant data from the table with a deep understanding for the meaning of the data that they are extracting.

105. In the illustrative example *Purchasing Decision* the student is presented with a summary of customer ratings for a product in an online store. Additionally, the student is provided with more a more detailed analysis of the reviews by the customers who provided 1- and 2-start ratings. This is effect sets up a two way table and the student is asked to demonstrate an understanding of the different probability estimates that the two-way table provides

106. Identifying conditional decisions making as a focal point of the uncertainty and data content category signals that students should be expected to appreciate how the formulation of the analysis in a model impacts the conclusions that can be dawn and that different assumptions/relationships may well result in different conclusions.

Content Topics for Guiding the Assessment of Mathematical Literacy of 15-year-old Students

107. To effectively understand and solve contextualised problems involving change and relationships; space and shape; quantity; and uncertainty and data requires drawing upon a variety of mathematical concepts, procedures, facts, and tools at an appropriate level of depth and sophistication. As an assessment of mathematical literacy, PISA strives to assess the levels and types of mathematics that are appropriate for 15-year-old students on a trajectory to become constructive, engaged and reflective 21st century citizens able to make well-founded judgments and decisions. It is also the case that PISA, while not designed or intended to be a curriculum-driven assessment, strives to reflect the mathematics that students have likely had the opportunity to learn by the time they are 15 years old.

108. In the development of the PISA 2012 mathematical literacy framework, with an eye toward developing an assessment that is both forward-thinking yet reflective of the mathematics that 15-year-old students have likely had the opportunity to learn, analyses were conducted of a sample of desired learning outcomes from eleven countries to determine both what is being taught to students in classrooms around the world and what countries deem realistic and important preparation for students as they approach entry into the workplace or admission into a higher education institution. Based on commonalities identified in these analyses, coupled with the judgment of mathematics experts, content deemed appropriate for inclusion in the assessment of mathematical literacy of 15-year-old students on PISA 2012, and continued for PISA 2022, is described below.

109. For PISA 2022 four additional focus topics have been added to the list. The resulting lists is intended to be illustrative of the content topics included in PISA 2022 and not an exhaustive listing:

- *Growth phenomena*: Different types of linear and non-linear growth
- *Geometric approximation*: Approximating the attributes and properties of irregular or unfamiliar shapes and objects by breaking these shapes and objects up into more familiar shapes and objects for which there are formulae and tools.
- *Computer simulations*: Exploring situations (that may include budgeting, planning, population distribution, disease spread, experimental probability, reaction time modelling etc.) in terms of the variables and the impact that these have on the outcome.
- *Conditional decision making*: Using basic principles of combinatorics and an understanding of interrelationships between variables to interpret situations and make predictions.

- *Functions*: The concept of function, emphasising but not limited to linear functions, their properties, and a variety of descriptions and representations of them. Commonly used representations are verbal, symbolic, tabular and graphical.
- *Algebraic expressions*: Verbal interpretation of and manipulation with algebraic expressions, involving numbers, symbols, arithmetic operations, powers and simple roots.
- *Equations and inequalities*: Linear and related equations and inequalities, simple second-degree equations, and analytic and non-analytic solution methods.
- *Co-ordinate systems*: Representation and description of data, position and relationships.
- *Relationships within and among geometrical objects in two and three dimensions*: Static relationships such as algebraic connections among elements of figures (e.g. the Pythagorean theorem as defining the relationship between the lengths of the sides of a right triangle), relative position, similarity and congruence, and dynamic relationships involving transformation and motion of objects, as well as correspondences between two- and three-dimensional objects.
- *Measurement*: Quantification of features of and among shapes and objects, such as angle measures, distance, length, perimeter, circumference, area and volume.
- *Numbers and units*: Concepts, representations of numbers and number systems (including converting between number systems), including properties of integer and rational numbers, as well as quantities and units referring to phenomena such as time, money, weight, temperature, distance, area and volume, and derived quantities and their numerical description.
- *Arithmetic operations*: The nature and properties of these operations and related notational conventions.
- *Percents, ratios and proportions*: Numerical description of relative magnitude and the application of proportions and proportional reasoning to solve problems.
- *Counting principles*: Simple combinations.
- *Estimation*: Purpose-driven approximation of quantities and numerical expressions, including significant digits and rounding.
- *Data collection, representation and interpretation*: Nature, genesis and collection of various types of data, and the different ways to analyse, represent and interpret them.
- *Data variability and its description*: Concepts such as variability, distribution and central tendency of data sets, and ways to describe and interpret these in quantitative and graphical terms.
- *Samples and sampling*: Concepts of sampling and sampling from data populations, including simple inferences based on properties of samples including accuracy and precision.
- *Chance and probability*: Notion of random events, random variation and its representation, chance and frequency of events, and basic aspects of the concept of probability and conditional probability.

Contexts for the assessment items and selected 21st century skills

110. The definition of mathematical literacy introduces two important considerations for the PISA assessment items. First, the definition makes it clear that mathematical literacy takes place in *real-world contexts*. Second, mathematical literacy *assists individuals to know the role that mathematics plays in the world and to make the well-founded judgments*

and decisions needed by constructive, engaged and reflective 21st century citizens. In this section we discuss how both real-world contexts and 21st century skills impact on item development.

111. The *real-world context* nature of mathematical literacy is not unproblematic for PISA. Real-world contexts involve information and that information is communicated using text. The quantitative and statistical information that flows in the world and reaches citizens is communicated through printed or spoken text, e.g. media articles, press releases, blogs, social networks, advertisements etc. This printed and spoken text is used to present messages or arguments that may or may not involve numbers and/or graphs. Text is the main tool for communicating context, and it follows that text comprehension is a fundamental and pre-requisite skill for success in mathematical literacy. The challenge this creates for PISA and item development is not insignificant. On the one hand the assessment must present socially meaningful quantitative messages using rich text, on the other hand the comparative nature of the assessment, the many languages it is translated into and the wide range of text comprehension levels among participating 15-year-olds places limits on the richness of the text that can realistically be used. This challenge is discussed further in the section on item development.

Contexts

112. An important aspect of mathematical literacy is that mathematics is used to solve a problem set in a context. The context is the aspect of an individual's world in which the problems are placed. The choice of appropriate mathematical strategies and representations is often dependent on the context in which a problem arises, and by implication there is the need to utilise knowledge of the real world context in developing the model. Being able to work within a context is widely appreciated to place additional demands on the problem solver (see Watson and Callingham, (2003[30]), for findings about statistics). For PISA, it is important that a wide variety of contexts are used. This offers the possibility of connecting with the broadest possible range of individual interests and with the range of situations in which individuals operate in the 21st century.

113. In light of the number of countries participating in PISA 2022 and with that an increasing range of participants from low- and middle-income countries as well as the possibility of out-of-school 15-year-olds, it is important that item developers take great care to ensure that the contexts used for items are accessible to a very broad range of participants. In this regard it is also important that the reading load of the items remains modest so that the items continue to assess mathematical literacy.

114. For purposes of the PISA 2022 mathematics framework, the four context categories of the PISA 2012 framework have been retained and are used to inform assessment item development. It should be noted that while these contexts are intended to inform item development, there is no expectation that there will be reporting against these contexts.

115. **Personal** – Problems classified in the personal context category focus on activities of one's self, one's family or one's peer group. The kinds of contexts that may be considered personal include (but are not limited to) those involving food preparation, shopping, games, personal health, personal transportation, recreation, sports, travel, personal scheduling and personal finance.

116. **Occupational** – Problems classified in the occupational context category are centred on the world of work. Items categorised as occupational may involve (but are not limited to) such things as measuring, costing and ordering materials for building,

payroll/accounting, quality control, scheduling/inventory, design/architecture and job-related decision making either with or without appropriate technology. Occupational contexts may relate to any level of the workforce, from unskilled work to the highest levels of professional work, although items in the PISA survey must be accessible to 15-year-old students.

117. **Societal** – Problems classified in the societal context category focus on one's community (whether local, national or global). They may involve (but are not limited to) such things as voting systems, public transport, government, public policies, demographics, advertising, health, entertainment, national statistics and economics. Although individuals are involved in all of these things in a personal way, in the societal context category, the focus of problems is on the community perspective.

118. **Scientific** – Problems classified in the scientific category relate to the application of mathematics to the natural world and issues and topics related to science and technology. Particular contexts might include (but are not limited to) such areas as weather or climate, ecology, medicine, space science, genetics, measurement and the world of mathematics itself. Items that are intra-mathematical, where all the elements involved belong in the world of mathematics, fall within the scientific context.

119. PISA assessment items are arranged in units that share stimulus material. It is therefore usually the case that all items in the same unit belong to the same context category. Exceptions do arise; for example, stimulus material may be examined from a personal point of view in one item and a societal point of view in another. When an item involves only mathematical constructs without reference to the contextual elements of the unit within which it is located, it is allocated to the context category of the unit. In the unusual case of a unit involving only mathematical constructs and being without reference to any context outside of mathematics, the unit is assigned to the scientific context category.

120. Using these context categories provides the basis for selecting a mix of item contexts and ensures that the assessment reflects a broad range of uses of mathematics, ranging from everyday personal uses to the scientific demands of global problems. Moreover, it is important that each context category be populated with assessment items having a broad range of item difficulties. Given that the major purpose of these context categories is to challenge students in a broad range of problem contexts, each category should contribute substantially to the measurement of mathematical literacy. It should not be the case that the difficulty level of assessment items representing one context category is systematically higher or lower than the difficulty level of assessment items in another category.

121. In identifying contexts that may be relevant, it is critical to keep in mind that a purpose of the assessment is to gauge the use of mathematical content knowledge and skills that students have acquired by age 15. Contexts for assessment items, therefore, are selected in light of relevance to students' interests and lives and the demands that will be placed upon them as they enter society as constructive, engaged and reflective citizens. National Project Managers from countries participating in the PISA survey are involved in judging the degree of such relevance.

21st Century skills

122. There is increased interest worldwide in what are called 21st century skills and their possible inclusion in educational systems. The OECD has put out a publication focusing

on such skills and has sponsored a research project entitled *The Future of Education and Skills: An OECD 2030 Framework* in which some 25 countries are involved in a cross-national study of curriculum including the incorporation of such skills. The project has as its central focus what the curriculum might look like in the future, focusing initially on mathematics and physical education.

123. Over the past 15 years or so a number of publications have sought to bring clarity to the discussion and consideration of 21st century skills. A summary of key reports and their conceptualisation of 21st century skills is provided in *PISA 2022 Mathematics: A Broadened Perspective* [EDU/PISA/GB(2017)17]. After careful analysis of these publications the authors recommended that a strong case can be made for the infusion of specific 21st century skills into specific disciplines. For example, it will become increasingly important to teach students at school how to make reasonable arguments with appropriate justification. The arguments they make should be mathematically rigorous, based on sound theory and strong enough to withstand criticism, and yet, whenever possible, avoid referring to authorities (e.g. 'it says so on the internet'). This is part of the fundamental competence to make independent judgements and take responsibility for them (OECD, 2005[31]). In the social context it is not enough to be right; one must be able and ready to present arguments and to defend them. Learning mathematics, with its clarity of contexts and strong emphasis on logical reasoning and rigour at the appropriate level, is a perfect opportunity to practice and develop the ability for this kind of argumentation.

124. Similarly, in the modern era, it is critical to equip students with tools that they can use to defend themselves from lies and inferences that purport to be based on mathematical reasoning. Quite often some fluency in logical reasoning is sufficient; a lie usually hides some hidden contradiction. The alertness of young minds towards possible contradictions can be developed most easily in good classes of mathematics.

125. Using the logic of finding the intersection between generic 21st century skills and related but subject-matter specific skills that are a natural part of the instruction related to that subject matter results in the following identified eight 21st century skills for inclusion in the PISA 2022 assessment framework. They are:

- Critical thinking
- Creativity
- Research and inquiry
- Self-direction, initiative, and persistence
- Information use
- Systems thinking
- Communication
- Reflection

Assessing Mathematical Literacy

126. This section outlines the approach taken to implement the elements of the framework described in previous sections into the PISA survey for 2022. This includes the structure of the mathematics component of the PISA survey, the desired distribution of score points for mathematical reasoning and the processes of problem solving; the distribution of score points by content area; a discussion on the range of item difficulties; the structure of the survey instrument; the role of the computer-based assessment of mathematics; the design of the assessment items; and the reporting of levels of mathematical proficiency.

Structure of the PISA 2022 Mathematics Assessment

127. In accordance with the definition of mathematical literacy, assessment items used in any instruments that are developed as part of the PISA survey are set within a context. Items involve the application of important mathematical concepts, knowledge, understandings and skills (mathematical content knowledge) at the appropriate level for 15-year-old students, as described earlier. The framework is used to guide the structure and content of the assessment, and it is important that the survey instrument include an appropriate balance of items reflecting the components of the mathematical literacy framework.

Desired Distribution of Score Points by Mathematical Reasoning and Problem solving process

128. Assessment items in the PISA 2022 mathematics survey can be assigned to either mathematical reasoning or one of three mathematical processes associated with mathematical problem solving. The goal in constructing the assessment is to achieve a balance that provides approximately equal weighting between the two processes that involve making a connection between the real world and the mathematical world (formulating and interpreting/evaluating) and mathematical reasoning and employing which call for students to be able to work on a mathematically formulated problem. While it is true that mathematical reasoning can be observed within the process of formulating, interpreting and employing items will only contribute to one domain.

Table 1. Approximate distribution of score points by domain for PISA 2022

		Percentage of score points in PISA 2022
Mathematical Reasoning		Approximately 25
Mathematical Problem Solving	Formulating Situations Mathematically	Approximately 25
	Employing Mathematical Concepts, Facts, Procedures and Reasoning	Approximately 25
	Interpreting, Applying and Evaluating Mathematical Outcomes	Approximately 25
TOTAL		**100**

129. It is important to note that items in each process category should have a range of difficulty and mathematical demand. This is further addressed in the table of demands for mathematical reasoning and each of the problem solving processes.

Desired Distribution of Score Points by Content Category

130. PISA mathematics items are selected to reflect the mathematical content knowledge described earlier in this framework. The trend items selected for PISA 2022 will be distributed across the four content categories, as shown in Table 2. The goal in constructing the survey is a distribution of items with respect to content category that provides as balanced a distribution of score points as possible, since all of these domains are important for constructive, engaged and reflective citizens.

Table 2. Approximate distribution of score points by content category for PISA 2022

Content category	Percentage of score points in PISA 2022
Change and Relationships	Approximately 25
Space and Shape	Approximately 25
Quantity	Approximately 25
Uncertainty and Data	Approximately 25
TOTAL	**100**

131. It is important to note that items in each content category should have a range of difficulty and mathematical demand.

A Range of Item Difficulties

132. The PISA 2022 mathematical literacy survey includes items with a wide range of difficulties, paralleling the range of abilities of 15-year-old students. It includes items that are challenging for the most able students and items that are suitable for the least able students assessed on mathematical literacy. From a psychometric perspective, a survey that is designed to measure a particular cohort of individuals is most effective and efficient when the difficulty of assessment items matches the ability of the measured subjects. Furthermore, the described proficiency scales that are used as a central part of the reporting of PISA outcomes can only include useful details for all students if the items from which the proficiency descriptions are drawn span the range of abilities described.

133. Table 3 describes the range of actions that are expected of students for mathematical reasoning and each of the problem solving processes. These lists describe the actions that the items will demand of students. For each category there are a number of

items marked with "**" to denote the actions that are expected of the students that will perform at levels 1a, 1b and 1c as well as level 2 of the proficiency scale. Item developers will need to ensure that there are sufficient items at the lower end of the performance scale to allow students at these levels to be able to show what they are capable of.

134. In order to gain useful information for the new lower levels, 1b and 1c, it is vital that context and language do not interfere with the mathematics being assessed. To this end, the context and language must be carefully considered. That said, the items must still be interesting to avoid the possibility that students will simply not attempt the items because it holds no interest.

135. The context for both 1b and 1c level items should be situations that students encounter on a daily basis. Examples of these contexts may include money, temperature, food, time, date, weight, size and distance. All items should be concrete and not abstract. The focus of the item should be mathematical only. The understanding of the context should not interfere with the performance of the item.

136. Equally important, it is to have all items formulated in the simplest possible terms. Sentences should be short and direct. Compound sentences, compound nouns and conditional sentences should be avoided. Vocabulary used in the items must be carefully examined to ensure that students will have a clear understanding of what is being required. In addition, special care will be given to ensure that no extra difficulty is added due to a heavy text load or by a context that is unfamiliar to students based on their cultural background.

137. Items designed for Level 1c should only ask for a single step or operation. However, it is important to note that a single step or operation is not limited to an arithmetical step. This step might be demonstrated by making a selection or identifying some information. Both mathematical reasoning and all of the problem solving processes should be used to measure the mathematical literacy capabilities of students at Levels 1b and 1c.

Table 3. Expected student actions for mathematical reasoning and each of the problem solving processes[9]

Reasoning
** Draw a simple conclusion
** Select an appropriate justification
** Explain why a mathematical result or conclusion does, or does not, make sense given the context of a problem
Represent a problem in a different way, including organising it according to mathematical concepts and making appropriate assumptions
Utilise definitions, rules and formal systems as well as employing algorithms and computational thinking
Explain and defend a justification for the identified or devised representation of a real-world situation
Explain or defend a justification for the processes and procedures or simulations used to determine a mathematical result or solution
Identify the limits of the model used to solve a problem
Understand definitions, rules and formal systems as well as employing algorithms and computational reasoning
Provide a justification for the identified or devised representation of a real-world situation
Provide a justification for the processes and procedures used to determine a mathematical result or solution
Reflect on mathematical arguments, explaining and justifying the mathematical result
Critique the limits of the model used to solve a problem
Interpret a mathematical result back into the real-world context in order to explain the meaning of the results
Explain the relationships between the context-specific language of a problem and the symbolic and formal language needed to represent it mathematically.
Reflect on mathematical arguments, explaining and justifying the mathematical result
Reflect on mathematical solutions and create explanations and arguments that support, refute or qualify a mathematical solution to a contextualised problem
Analyse similarities and differences between a computational model and the mathematical problem that it is modelling
Explain how a simple algorithm works and to detect and correct errors in algorithms and programs

Formulating	Employing	Interpreting
** Select a mathematical description or a representation that describes a problem	** Perform a simple calculation	**Interpret a mathematical result back into the real world context
** Identify the key variables in a model	** Select an appropriate strategy from a list	** Identify whether a mathematical result or conclusion does, or does not, make sense given the context of a problem
** Select a representation appropriate to the problem context	** Implement a given strategy to determine a mathematical solution	** Identify the limits of the model used to solve a problem
Read, decode and make sense of statements, questions, tasks, objects or images to create a model of the situation	** Make mathematical diagrams, graphs, constructions or computing artifacts	Use mathematical tools or computer simulations to ascertain the reasonableness of a mathematical solution and any limits and constraints on that solution, given the context of the problem
Recognise mathematical structure (including regularities, relationships, and patterns) in problems or situations	Understand and utilise constructs based on definitions, rules and formal systems including employing familiar algorithms	Interpret mathematical outcomes in a variety of formats in relation to a situation or use; compare or evaluate two or more representations in relation to a situation

[9] ***Table 3*** is a reformulation of the figure used in previous frameworks to link mathematical processes with mathematical capabilities. All of the examples and illustrations from that figure are included in this reformulation.

Formulating	Employing	Interpreting
Identify and describe the mathematical aspects of a real-world problem situation including identifying the significant variables	Develop mathematical diagrams, graphs, constructions or computing artifacts and extracting mathematical information from them	Use knowledge of how the real world impacts the outcomes and calculations of a mathematical procedure or model in order to make contextual judgments about how the results should be adjusted or applied
Simplify or decompose a situation or problem in order to make it amenable to mathematical analysis	Manipulate numbers, graphical and statistical data and information, algebraic expressions and equations, and geometric representations	Construct and communicate explanations and arguments in the context of the problem
Recognise aspects of a problem that correspond with known problems or mathematical concepts, facts or procedures	Articulate a solution, showing and/or summarising and presenting intermediate mathematical results	Recognise [demonstrate, interpret, explain] the extent and limits of mathematical concepts and mathematical solutions
Translate a problem into a standard mathematical representation or algorithm	Use mathematical tools, including technology, simulations and computational thinking, to help find exact or approximate solutions	Understand the relationship between the context of the problem and representation of the mathematical solution. Use this understanding to help interpret the solution in context and gauge the feasibility and possible limitations of the solution
Use mathematical tools (using appropriate variables, symbols, diagrams) to describe the mathematical structures and/or relationships in a problem	Make sense of, relate and use a variety of representations when interacting with a problem	
Apply mathematical tools and computing tool to portray mathematical relationships	Switch between different representations in the process of finding solutions	
Identify the constraints, assumptions simplifications in a mathematical model	Use a multi-step procedure leading to a mathematical solution, conclusion or generalisation	
	Use an understanding of the context to guide or expedite the mathematical solving process, e.g. working to a context-appropriate level of accuracy	
	Make generalisations based on the results of applying mathematical procedures to find solutions	

Computer-based Assessment of Mathematics

138. The main mode of delivery for PISA 2022 will be the computer-based assessment of mathematics (CBAM). The transition has been anticipated with both the 2015 and 2018 studies moving to computer-based delivery. In order to maintain trends across the studies, both the 2015 and 2018 assessments were computer neutral despite using a computer-based delivery mode. The transition to a full CBAM in 2022 provides a range of opportunities to develop the assessment of mathematical literacy to be better aligned with the evolving nature of mathematics in the modern world, while ensuring backward trends to previous cycles. These opportunities include new item formats (e.g. drag and drop); presenting students with real-world data (such as large, sortable datasets); creating mathematical models or simulations that students can explore by changing the variable values; curve fitting and using the best fit curve to make predictions. In addition to a wider range of question types and mathematical opportunities that the CBAM provides, it also allows for adaptive assessment.

139. The adaptive assessment capability of the CBAM, which was previously implemented in the PISA reading assessment, provides the opportunity of better describing what it is that students at both ends of the performance spectrum are able to do. By providing students with increasing individualised combinations of test units according to their responses and scores to the early units that they respond to, increasingly detailed information on the performance characteristics of students at both ends of the performance scale is generated.

140. Making use of enhancements offered by computer technology results in assessment items that are more engaging to students, more visually appealing, and easier to understand. For example, students may be presented with a moving stimulus, representations of three-dimensional objects that can be rotated or more flexible access to relevant information. New item formats, such as those calling for students to 'drag and drop' information or use 'hot spots' on an image, are designed to engage students, permit a wider range of response types and give a more rounded picture of mathematical literacy. A key challenge is to ensure that these items continue to assess *mathematical literacy* and that interference from domain irrelevant dimensions is kept to a minimum.

141. Investigations show that the mathematical demands of work increasingly occur in the presence of electronic technology so that mathematical literacy and computer use are melded together (Hoyles et al., 2002[32]). For employees at all levels of the workplace, there is now an interdependency between mathematical literacy and the use of computer technology. A key challenge is to distinguish the mathematical demands of a PISA computer-based item from demands unrelated to mathematical competence, such as the information and communications technology (ICT) demands of the item, and the presentation format. Solving PISA items on a computer rather than on paper moves PISA into the reality and the demands of the 21st Century.

142. Questions that seem well suited to the CBAM and the evolving nature of mathematical literacy include:

- Simulation in which a mathematical model has been established and students can change the variable values to explore the impact of the variables to create "an optimal solution".
- Fitting a curve (by selecting a curve from a limited set of curves provided) to a data set or a geometric image to determine the "best fit" and using the resulting best fit curve to determine the answer to a question about the situation.
- Budgeting situations (e.g. online store) in which the student must select combinations of products to meet achieve a range of objectives within a given budget.
- Purchase simulation in which the student selects from different loan and associates repayment options to purchase an item using a loan and meeting a budget. The challenge in the problem is to understand how the variables interact.
- Problems that include visual coding to achieve a given sequence of actions.

143. Notwithstanding the opportunities that the CBAM presents (described above), it is important that the CBAM remains focussed on assessing mathematical literacy and does not shift to assessing ICT skills. Similarly it is important that the simulations and other questions hinted at above do not become so "noisy" that the mathematical reasoning and problem solving processes are lost.

144. The CBAM must also retain some of the paper version features for example the ability to revisit items already attempted – although in the context of adaptive testing this will of necessity be limited to the unit on which the student is working.

Design of the PISA 2022 Mathematics Items

145. Three item format types are used to assess mathematical literacy in PISA 2022: open constructed-response, closed constructed-response and selected-response (multiple-choice) items.

- Open constructed-response items require a somewhat extended written response from a student. Such items also may ask the student to show the steps taken or to explain how the answer was reached. These items require trained experts to manually code student responses. To facilitate the adaptive assessment feature of the CBAM, it will be necessary to minimise the number of items that rely on trained experts to code the student responses.
- Closed constructed-response items provide a more structured setting for presenting problem solutions, and they produce a student response that can be easily judged to be either correct or incorrect. Often student responses to questions of this type can be coded automatically. The most frequently used closed constructed-responses are single numbers.
- Selected-response items require the choice of one or more responses from a number of response options. Responses to these questions can usually be automatically processed. About equal numbers of each of these item format types are being used to construct the survey instruments.

146. The PISA mathematics survey is composed of assessment *units* comprising written stimulus material and other information such as tables, charts, graphs or diagrams, plus one or more items that are linked to this common stimulus material. This format gives students the opportunity to become involved with a context or problem by responding to a series of related items.

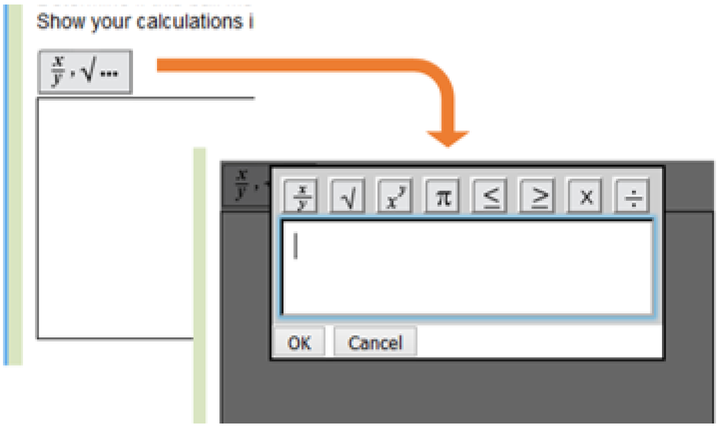
147. Items selected for inclusion in the PISA survey represent a broad range of difficulties, to match the wide ability range of students participating in the assessment. In addition, all the major categories of the assessment (the content categories; mathematical reasoning and problem solving process categories and the different context categories and 21st century skills) are represented, to the degree possible, with items of a wide range of difficulties. Item difficulties are established as one of a number of measurement properties in an extensive field trial prior to item selection for the main PISA survey. Items are selected for inclusion in the PISA survey instruments based on their fit with framework categories and their measurement properties.

148. In addition, the level of reading required to successfully engage with an item is considered very carefully in item development and selection. A goal in item development is to make the wording of items as simple and direct as possible. Care is also taken to avoid item contexts that would create a cultural bias, and all choices are checked with national teams. Translation of the items into many languages is conducted very carefully, with extensive back-translation and other protocols.

149. PISA 2022 will include a tool that will allow students to provide typed constructed-response answers and show their work as required for mathematical literacy. The tool allows students to enter both text and numbers. By clicking the appropriate button, students can enter a fraction, square root, or exponent. Additional symbols such as π and

greater/less than signs are available, as are operators such as multiplication and division signs. An example is shown in Figure 3 below.

Figure 3. Example of the PISA 2022 editor tool



150. The suite of tools available to students is also expected to include a basic scientific calculator. Operators to be included are addition, subtraction, multiplication and division, as well as square root, pi, parentheses, exponent, square, fraction (y/x), inverse (1/x) and the calculator will be programmed to respect the standard order of operations.

151. Students taking the assessment on paper can have access to a hand-held calculator, as approved for use by 15-year-old students in their respective school systems.

Item Scoring

152. Although the majority of the items are dichotomously scored (that is, responses are awarded either credit or no credit), the open constructed-response items can sometimes involve partial credit scoring, which allows responses to be assigned credit according to differing degrees of "correctness" of responses and or to the extent to which an item has been engaged with or not. It is anticipated that the need for partial credit scoring will be particularly significant for the mathematical reasoning items which will seldom involve the production of single number response but rather responses with one or more elements.

Reporting Proficiency in Mathematics

153. The outcomes of the PISA mathematics survey are reported in a number of ways. Estimates of overall mathematical proficiency are obtained for sampled students in each participating country, and a number of proficiency levels are defined. Descriptions of the degree of mathematical literacy typical of students in each level are also developed. For PISA 2022, the six proficiency levels reported for the overall PISA mathematics in previous cycles will be expanded as follows: Level 1 will be renamed Level 1a, and the table describing the proficiencies will be extended to include Levels 1b and 1c. These additional levels have been added to provide greater granularity of reporting in students performing at the lower end of the proficiency scale.

154. As well as the overall mathematics scale, additional described proficiency scales are developed after the Field Trial and are then reported. These additional scales are for mathematical reasoning and for the three processes of mathematical problem solving:

formulating situations mathematically; employing mathematical concepts, facts, procedures, and reasoning; and interpreting, applying and evaluating mathematical outcomes.

Mathematical Literacy and the Background Questionnaires

155. Since the first cycle of PISA, student and school context questionnaires have served two interrelated purposes in service of the broader goal of evaluating educational systems: first, the questionnaires provide a context through which to interpret the PISA results both within and between education systems. Second, the questionnaires aim to provide reliable and valid measurement of additional educational indicators, which can inform policy and research in their own right.

156. Since mathematical literacy is the major domain in the 2022 survey, the background questionnaires are expected to provide not only trend data for the constructs that continue to be assessed, but additionally to provide rich information on the innovations that are evident in the PISA 2022 mathematical literacy framework. In particular it is expected that mathematical literacy will feature prominently in the analysis of the domain-specific contextual constructs as well in a number of the different categories of policy focus that range from individual level variables such a demographics and social and emotional characteristics to school practices, policies and infrastructure (OECD, 2018[33]).

157. Two broad areas of students' attitudes towards mathematics that dispose them to productive engagement in mathematics were identified as being of potential interest as an adjunct to the PISA 2012 mathematics assessment. These are students' interest in mathematics and their willingness to engage in it. It is expected that these will continue to be a focus of the questionnaires in 2022.

158. Interest in mathematics has components related to present and future activity. Relevant questions focus on students' interest in mathematics at school, whether they see it as useful in real life as well as their intentions to undertake further study in mathematics and to participate in mathematics-oriented careers. There is international concern about this area, because in many participating countries there is a decline in the percentage of students who are choosing mathematics related future studies, whereas at the same time there is a growing need for graduates from these areas.

159. Students' willingness to do mathematics is concerned with the attitudes, emotions and self-related beliefs that dispose students to benefit, or prevent them from benefitting, from the mathematical literacy that they have achieved. Students who enjoy mathematical activity and feel confident to undertake it are more likely to use mathematics to think about the situations that they encounter in the various facets of their lives, inside and outside school. The constructs from the PISA survey that are relevant to this area include the emotions of enjoyment, confidence and (lack of) mathematics anxiety, and the self-related beliefs of self-concept and self-efficacy. An analysis of the subsequent progress of young Australians who scored poorly on PISA at age 15 found that those who "recognise the value of mathematics for their future success are more likely to achieve this success, and that includes being happy with many aspects of their personal lives as well as their futures and careers" (Hillman and Thomson, 2010, p. 31[34]). The study recommends that a focus on the practical applications of mathematics in everyday life may help improve the outlook for these low-achieving students.

160. The innovations evident in the PISA 2022 mathematics framework point to at least four areas in which the background questionnaires can provide rich data. These areas are: **mathematical reasoning**; **computational thinking** and the role of technology in both doing and teaching mathematics; the **four focal content areas**; and **21st century skills in the context of mathematics**.

Mathematical reasoning

161. The PISA 2022 mathematics framework foregrounds mathematical reasoning enabled by some key understandings that undergird school mathematics (understanding quantity, number systems and their algebraic properties; appreciating the power of abstraction and symbolic representation; seeing mathematical structures and their regularities; recognising functional relationships between quantities; using mathematical modelling as a lens onto the real world; and understanding variation as the heart of statistics).

162. The focus on reasoning has implications for the background questionnaires which should provide measures to understand students' opportunities to learn to reason mathematically and employ the key understandings that undergird school mathematics. In particular the questionnaires should establish the frequency with which students, for example:

- Identify, recognise, organise, connect, and represent;
- Construct, abstract, evaluate, deduce, justify, explain, and defend; and
- Interpret, make judgements, critique, refute, and qualify.

163. In addition to establishing the frequency of the opportunities (to learn) to reason, the questionnaires should get at what forms these opportunities take (verbal or written).

164. Finally, with respect to reasoning, the questionnaires should get a sense of the willingness of students to persist with tasks that involve reasoning.

165. In the case of teachers and teaching there is the need to better understand how they see the role of reasoning in mathematics in general and in their teaching and assessment practices in particular.

Computational thinking

166. Aspects of computational thinking form a rapidly evolving and growing dimension of both mathematics and mathematical literacy. The PISA 2022 mathematical literacy framework illustrates how computational thinking is both part of doing mathematics and impacting on doing mathematics. The *values and beliefs about learning* and *open-mindedness* modules of the background questionnaires can explore student's experience of the role of computational thinking in doing mathematics.

167. The PISA 2022 mathematical literacy framework draws attention to the different ways in which technology is both changing the world in which we live and changing what it means to engage in mathematics. Key questions for the background questionnaires include developing a deep understanding of first, how students' experiences of mathematics and doing mathematics are changing (if at all) and second, how classroom pedagogy is evolving due to the impact that technology is having on how students engages with mathematics and mathematical artefacts and on what it means to do mathematics. In the case of students, it is of interest to better understand how technology is impacting student performance which could be explored in the *task performance* module of the

questionnaire framework. The pedagogical issues could be explored in both the *learning time and curriculum* and *teaching practices* modules.

168. The focus on computational thinking and the role of technology in both doing and teaching mathematics has implications for the background questionnaires which should provide measures to better understand students' opportunities to learn in this regard. In particular the questionnaires should establish the frequency with which students, for example:

- Design or work with computer simulations and or computer models;
- Code or program both inside the mathematics classroom and outside it; and
- Are exposed to Computer Mathematics Systems (CSM) (including dynamic geometry software; spreadsheets; programming software (e.g. Logo and Scratch); graphing calculators; games etc.).

Four focal content areas

169. In recognition of the changing world the PISA 2022 Mathematics Framework has suggested that four content areas within the existing content framework receive special focus. These content areas are: growth phenomena (within change and relationships); geometric approximation (within space and shape); computer simulations (within quantity); and conditional decision making (within uncertainty and data). The focus on these content areas has implications for the background questionnaires which should provide measures to better understand students' opportunities to learn in this regard. In particular the questionnaires should establish the frequency with which students are exposed to this contents and the different forms that the opportunities take.

21st century skills in the context of mathematics

170. The PISA 2022 mathematical literacy framework introduces a particular set of 21st century skills both as an outcome of and focus for mathematics. The background questionnaires could productively examine both whether or not mathematics is contributing to the development of these skills and if teaching practices are focussing on them. In particular, the *learning time and curriculum* module could explore whether or not these skills appear in the enacted curriculum.

171. The results of the PISA 2022 survey will provide important information for educational policy makers in the participating countries about both the achievement-related and attitude-related outcomes of schooling. By combining information from the PISA assessment of mathematical literacy and the survey information on attitudes, emotions and beliefs that predispose students to use their mathematical literacy as well as the impact of the four developments described above, a more complete picture will emerge.

Summary

172. The PISA 2022 mathematical literacy framework while maintaining coherence with the previous mathematical literacy frameworks acknowledges that the world is ever changing and with it the demand for mathematically literate citizens to reason mathematically rather than reproducing mathematical techniques as routines,

173. The aim of PISA with regard to mathematical literacy is to develop indicators that show how effectively countries are preparing students to use mathematics in the everyday aspect of their personal, civic and professional lives, as constructive, engaged and reflective 21st century citizens. To achieve this, PISA has developed a definition of mathematical literacy and an assessment framework that reflects the important components of this definition.

174. The mathematics assessment items selected for inclusion in PISA 2022, based on this definition and framework, are intended to reflect a balance between mathematical reasoning, problem solving processes, mathematical content and contexts.

175. The assessment design will assure valid measurement of ability across the range of achievement extending to two levels below the previous PISA scale, while preserving the quality and content of the assessment.

176. The CBAM to be used from 2022 provides problems in a variety of item formats with varying degrees of built-in guidance and structure and a range of formats retaining throughout an emphasis on authentic problems that require students to reason and demonstrate their thinking.

References

Ananiadou, K. and M. Claro (2009), “21st Century Skills and Competences for New Millennium Learners in OECD Countries”, *OECD Education Working Papers*, No. 41, OECD Publishing, Paris, http://dx.doi.org/10.1787/218525261154. [2]

Basu, S. et al. (2016), “Identifying middle school students’ challenges in computational thinking-based science learning”, *Research and Practice in Technology Enhanced Learning*, Vol. 11/3, http://dx.doi.org/10.1186/s41039-016-0036-2. [20]

Beheshti, E. et al. (2017), *Computational Thinking in Practice: How STEM Professionals Use CT in Their Work*, Northwestern University, San Antonio, Texas, http://ccl.northwestern.edu/papers.shtml. [22]

Beloit College (2017), *The Beloit College Mindset List for the Class of 2022*, https://www.beloit.edu/mindset/2022/. [1]

Benton, L. et al. (2017), “Bridging Primary Programming and Mathematics: Some Findings of Design Research in England”, *Digital Experiences in Mathematics Education*, Vol. 3, pp. 115-138, http://dx.doi.org/10.1007/s40751-017-0028-x. [21]

Box, G. and N. Draper (1987), *Empirical Model-Building and Response Surfaces*, John Wiley. [24]

Brennan, K. and M. Resnick (2012), *New frameworks for studying and assessing the development of computational thinking*, https://web.media.mit.edu/~kbrennan/files/Brennan_Resnick_AERA2012_CT.pdf. [25]

Cuny, J., L. Snyder and J. Wing (2010), *Demystifying computational thinking for non-computer scientists*. [9]

Devlin, K. (1994), *Mathematics: The Science of Patterns : The Search for Order in Life, Mind, and the Universe*, W H Freeman & Co. [27]

Fadel, C., M. Bialik and B. Trilling (2015), *Four-Dimensional Education : The Competencies Learners Need to Succeed*, CreateSpace Independent Publishing Platform. [3]

Gadanidis, G. (2015), *Coding for Young Mathematicians*, Western University, London, Ontario, Canada, http://worldiscoveries.ca/technology/18155 (accessed on 05 April 2018). [10]

Gadanidis, G., E. Clements and C. Yiu (2018), “Group Theory, Computational Thinking, and Young Mathematicians”, *Mathematical Thinking and Learning*, Vol. 20/1, pp. 32-53, http://dx.doi.org/10.1080/10986065.2018.1403542. [13]

Galbraith, P., H. Henn and M. Niss (eds.) (2007), *Modelling and Applications in Mathematics Education*, Springer US, http://dx.doi.org/10.1007/978-0-387-29822-1. [19]

Grover, S. (2018), *The 5th 'C' of 21st Century Skills? Try Computational Thinking (Not Coding) | EdSurge News*, https://www.edsurge.com/news/2018-02-25-the-5th-c-of-21st-century-skills-try-computational-thinking-not-coding (accessed on 05 April 2018). [16]

Hillman, K. and S. Thomson (2010), *Against the odds: influences on the post-school success of 'low performers'.*, NCVER, Adelaide, Australia, https://www.ncver.edu.au/publications/publications/all-publications/against-the-odds-influences-on-the-post-school-success-of-low-performers# (accessed on 05 April 2018). [34]

Hoyles, C. et al. (2002), "Mathematical skills in the workplace: final report to the Science Technology and Mathematics Council", *Institute of Education, University of London; Science, Technology and Mathematics Council, London. (2002)*, http://discovery.ucl.ac.uk/1515581/ (accessed on 05 April 2018). [32]

Mahajan, S. et al. (2016), *PISA Mathematics in 2022*, Center for Curriculum Redesign, http://curriculumredesign.org/wp-content/uploads/Recommendations-for-PISA-Maths-2022-FINAL-EXTENDED-VERSION-WITH-EXAMPLES-CCR.pdf. [28]

Moore, D. (1997), "New Pedagogy and New Content: The Case of Statistics", *International Statistical Review*, Vol. 65/2, pp. 123-165, https://onlinelibrary.wiley.com/doi/epdf/10.1111/j.1751-5823.1997.tb00390.x. [29]

National Research Council (2012), *Education for Life and Work: Developing Transferable Knowledge and Skills in the 21st Century*, The National Academies Press, Washington D.C., https://www.nap.edu/resource/13398/dbasse_070895.pdf (accessed on 05 April 2018). [4]

Niemelä, P. et al. (2017), *Computational thinking as an emergent learning trajectory of mathematics*, ACM Press, New York, New York, USA, http://dx.doi.org/10.1145/3141880.3141885. [17]

OECD (2018), *PISA 2018 Assessment and Analytical Framework: Reading, Mathematics and Science*. [33]

OECD (2017), *PISA 2015 Assessment and Analytical Framework: Science, Reading, Mathematic, Financial Literacy and Collaborative Problem Solving*, OECD Publishing, Paris, http://dx.doi.org/10.1787/9789264281820-en. [23]

OECD (2017), *PISA for Development Assessment and Analytical Framework: Reading, Mathematics and Science, Preliminary Version*, OECD Publishing, Paris, http://www.oecd.org/about/publishing/corrigenda.htm. (accessed on 06 April 2018). [8]

OECD (2013), *PISA 2012 Assessment and Analytical Framework*, OECD Publishing, http://dx.doi.org/10.1787/9789264190511-en. [7]

OECD (2005), *The Definition and Selection of Key Competencies: Executive Summary*, OECD, Paris, http://www.oecd.org/pisa/35070367.pdf (accessed on 05 April 2018). [31]

OECD (2004), *The PISA 2003 Assessment Framework*, OECD Publishing, http://dx.doi.org/10.1787/9789264101739-en. [6]

Pei, C., D. Weintrop and U. Wilensky (2018), “Cultivating Computational Thinking Practices and Mathematical Habits of Mind in Lattice Land”, *Mathematical Thinking and Learning*, Vol. 20/1, pp. 75-89, http://dx.doi.org/10.1080/10986065.2018.1403543. [14]

Pratt, D. and R. Noss (2002), “The Microevolution of Mathematical Knowledge: The Case of Randomness”, *Journal of the Learning Sciences*, Vol. 11/4, pp. 453-488, http://dx.doi.org/10.1207/S15327809JLS1104_2. [12]

Rambally, G. (2017), *Applications of Computational Matrix Algebra*, AACE, Austin, Texas, https://www.learntechlib.org/p/177277/. [11]

Reimers, F. and C. Chung (2016), *Teaching and Learning for the Twenty-First Century : Educational Goals, Policies, and Curricula from Six Nations*, Harvard Education Press, Cambridge, MA. [5]

Resnick, M. et al. (2009), “Scratch: Programming for all”, *Communications of the ACM*, Vol. 52/11, p. 60, http://dx.doi.org/10.1145/1592761.1592779. [18]

Steen, L. (1990), *On the Shoulders of Giants*, National Academies Press, Washington, D.C., http://dx.doi.org/10.17226/1532. [26]

Watson, J. and R. Callingham (2003), “Statistical literacy: A Complex hierarchical construct”, *Statistics Education Research Journal*, Vol. 2, pp. 3-46. [30]

Weintrop, D. et al. (2016), “Defining Computational Thinking for Mathematics and Science Classrooms”, *Journal of Science Education and Technology*, Vol. 25/1, pp. 127-147, http://dx.doi.org/10.1007/s10956-015-9581-5. [15]

Weintrop, D. and U. Wilensky (2015), *To block or not to block, that is the question*, ACM Press, New York, New York, USA, http://dx.doi.org/10.1145/2771839.2771860. [35]

Annex A. Illustrative examples

177. The items included in this Annex illustrate some of the most important new elements of the framework. For the sake of ensuring the preservation of trend, the majority of the items in the PISA 2022 will be items that have been used in previous PISA assessments. A larger set of release items to illustrate the item pool can be found at http://www.oecd.org/pisa/test.

178. The items provided in this annex illustrate some of the following new elements:

- The assessment of mathematical reasoning as described in the framework;
- The four topics that have been identified for special emphasis in the PISA 2022 assessment, growth phenomena; geometric approximations; computer simulations; and conditional decision making;
- The range of item features that are possible on account of the Computer-Based Assessment of Mathematics (CBAM); and
- Computational thinking.

179. The seven illustrative items provided in this annex include:

- **SMARTPHONE USE**: This item illustrates:
    - CBAM capabilities in particular the use of spreadsheets with sorting and other capabilities.
- **THE BEAUTY OF POWERS**: This item illustrates:
    - A range of mathematics reasoning items from simple to more complex in a mathematical context; and
    - Hints at growth phenomena, although, in fairness, the context for this item is more focused on reasoning and pattern recognition than it is on growth.
- **ALWAYS SOMETIMES NEVER**: This item illustrates:
    - A range of reasoning items from simple to more complex including a range of question types from yes/no and multiple choice to open-ended items
- **TILING**: This item illustrates:
    - Reasoning and computational thinking; and
    - Geometric representations.
- **PURCHASING DECISION**: This item illustrates:
    - The application of conditional decision making.
- **NAVIGATION**: This item illustrates:
    - Reasoning in a geometric context; and
    - CBAM capabilities in items.
- **SAVINGS SIMULATION**: This item illustrates:
    - The use a computer simulation; and
    - Hints at growth in the context and impact of interest.

1. Smartphone Use

Smartphone use

Introduction

Read the introduction. Then click on the NEXT arrow.

SMARTPHONE USE

The spreadsheet shows the population (in millions) and the number of smartphone users (in millions) for a range of countries in Asia. The data has been sorted by country name.

Column A	Column B	Column C	Column D
Country	**Population (in millions)**	**Number of smartphone users (in millions)**	
Bangladesh	166.735	8.921	
Indonesia	266.357	67.57	
Japan	125.738	65.282	
Malaysia	31.571	20.98	
Pakistan	200.663	23.228	
Philippines	105.341	28.627	
Thailand	68.416	30.486	
Turkey	81.086	44.771	
Vietnam	96.357	29.043	

Smartphone use

Question 1/3

Refer to "Smartphone use" on the right. Click on a choice to answer the question.

Which operation on columns B and C will determine the correct values in Column D?

For each country:

- ◯ Divide the Column B value by the Column C value:

  B / C

- ◯ Divide the sum of the Column B and Column C values by the Column C value:

  (B + C) / C

- ◯ Divide the Column C value by the Column B value:

  C / B

- ◯ Divide the Column B value by the sum of the Column B and Column C values:

  B / (B + C)

SMARTPHONE USE

The spreadsheet shows the population (in millions) and the number of smartphone users (in millions) for a range of countries in Asia. The data has been sorted by country name.

Column A	Column B	Column C	Column D
Country	**Population (in millions)**	**Number of smartphone users (in millions)**	**Proportion of smartphone users**
Bangladesh	166.735	8.921	
Indonesia	266.357	67.57	
Japan	125.738	65.282	
Malaysia	31.571	20.98	
Pakistan	200.663	23.228	
Philippines	105.341	28.627	
Thailand	68.416	30.486	
Turkey	81.086	44.771	
Vietnam	96.357	29.043	

Smartphone use

Question 2/3

You can sort the data in the spreadsheet by selecting the sort button in the column header. The data will be sorted in ascending order.

Use the sort buttons help you evaluate each statement.

Click on either **True** or **False** for each of the following statements.

Statement	True	False
The country with the largest population also has the largest number of smartphone users.	○	○
The country with the fewest number of smartphone users also has the smallest population.	○	○
The country with the highest proportion of smartphone users also has the smallest population.	○	○
The country with the median proportion of smartphone users is also the country with the median number of smartphone users.	○	○

SMARTPHONE USE

The data for the proportion of smartphone users (expressed as a percentage) has been added to the spreadsheet in Column D.

Column A	Column B	Column C	Column D
Country	**Population (in millions)**	**Number of smartphone users (in millions)**	**Proportion of smartphone users**
Bangladesh	166.735	8.921	5%
Indonesia	266.357	67.57	25%
Japan	125.738	65.282	52%
Malaysia	31.571	20.98	38%
Pakistan	200.663	23.228	12%
Philippines	105.341	28.627	27%
Thailand	68.416	30.486	45%
Turkey	81.086	44.771	55%
Vietnam	96.357	29.043	30%

Smartphone use

Question 3/3

You can change the horizontal axis variable between the **Population (in millions)** and the **Minimum hourly wage (in Zeds)** for each country by selecting the corresponding tab.

By selecting the corresponding tabs study the different graphs and answer the question.

For which variable (population or minimum hourly wage) does the proportion of smartphone users in a country increase as the variable value increases?

◯ Population

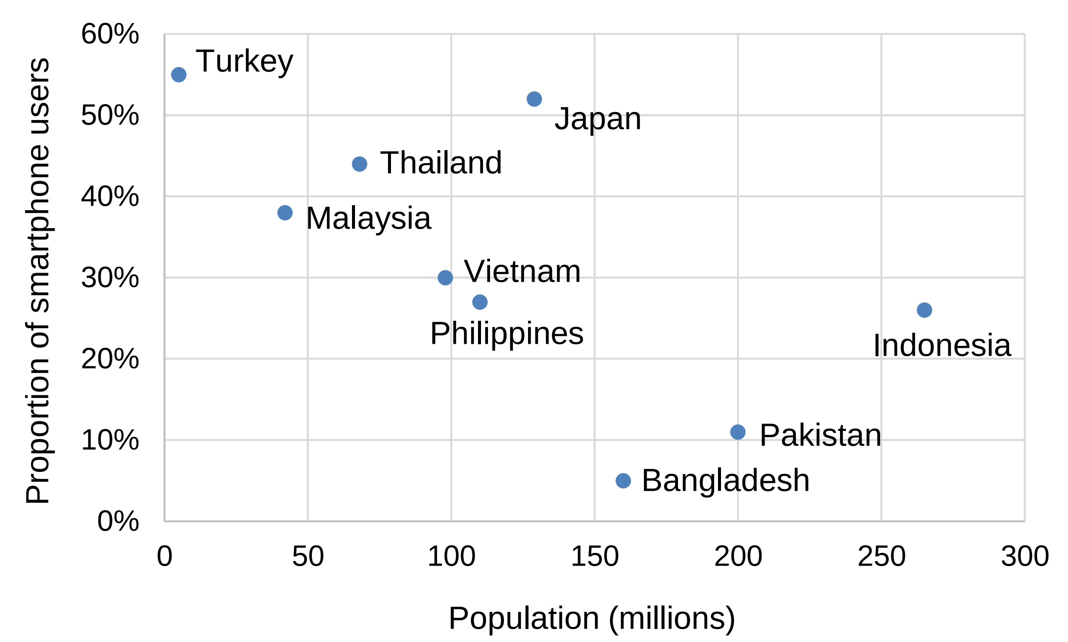
◯ Minimum hourly wage (Zeds)

Explain your reasoning:

SMARTPHONE USE

The graph plots the proportion of smartphone users per country in terms of either the **Population (in millions)** and the **Minimum hourly wage (in Zeds)** for each country.

Population | Hourly wage



Smartphone use

Question 3/3

You can change the horizontal axis variable between the **Population (in millions)** and the **Minimum hourly wage (in Zeds)** for each country by selecting the corresponding tab.

By selecting the corresponding tabs study the different graphs and answer the question.

For which variable (population or minimum hourly wage) does the proportion of smartphone users in a country increase as the variable value increases?

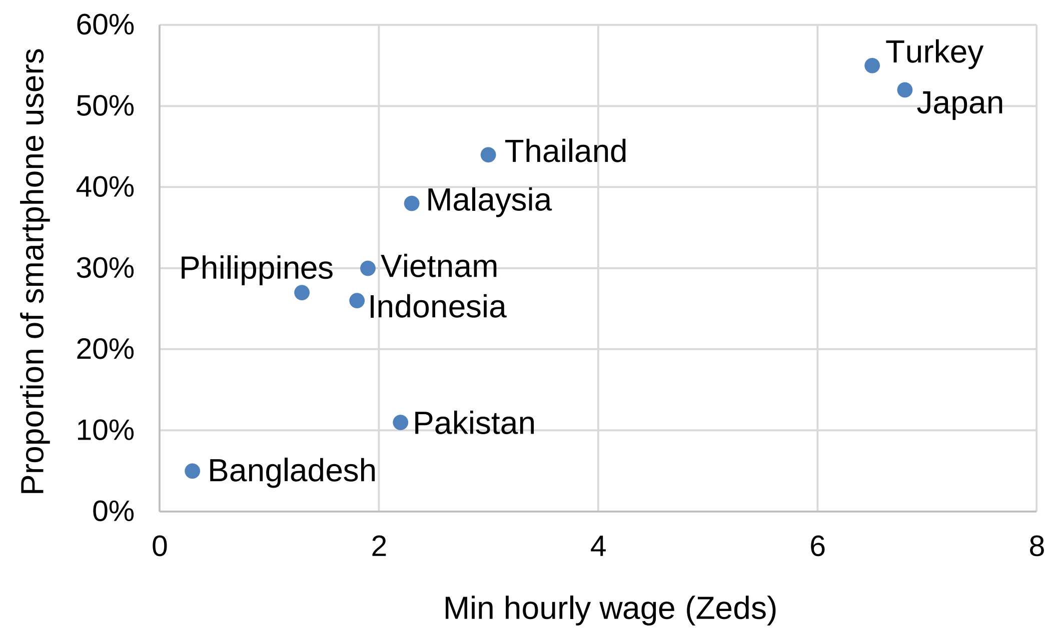
- ◯ Population
- ◯ Minimum hourly wage (Zeds)

Explain your reasoning:

SMARTPHONE USE

The graph plots the proportion of smartphone users per country in terms of either the **Population (in millions)** and the **Minimum hourly wage (in Zeds)** for each country.

Population | Hourly wage



2. Beauty of Powers

The beauty of powers

Introduction

Read the introduction. Then click on the NEXT arrow.

THE BEAUTY OF POWERS

When you perform repeated multiplication with the same number, you can use power notation to summarise what you are doing.

For example:

$$8 \times 8 \times 8 \times 8 = 8^4 \quad \text{(four 8s multiplied together)}$$

and

$$7 \times 7 \times 7 \times 7 \times 7 \times 7 = 7^6 \quad \text{(six 7s multiplied together)}$$

The beauty of powers

Question 1/3

*Refer to "The beauty of powers" on the right. Click on either **True** or **False** for each of the statements.*

Statement	True	False
The number 8^{16} is 8 times larger than the number 8^{15}	○	○
The number 8^{10} is 10 times larger than the number 8	○	○

THE BEAUTY OF POWERS

When you perform repeated multiplication with the same number, you can use power notation to summarise what you are doing.

For example:

$$8 \times 8 \times 8 \times 8 = 8^4 \quad \text{(four 8s multiplied together)}$$

and

$$7 \times 7 \times 7 \times 7 \times 7 \times 7 = 7^6 \quad \text{(six 7s multiplied together)}$$

The beauty of powers

Question 2/3

Refer to "The beauty of powers" on the right. Click on a choice to answer the question.

$$(-5)^{43} + (-1)^{43} + (5)^{43}$$

What is the value of the expression above?

- ○ –1
- ○ 1
- ○ 0
- ○ 5

THE BEAUTY OF POWERS

When you perform repeated multiplication with the same number, you can use power notation to summarise what you are doing.

For example:

$$8 \times 8 \times 8 \times 8 = 8^4 \quad \text{(four 8s multiplied together)}$$

and

$$7 \times 7 \times 7 \times 7 \times 7 \times 7 = 7^6 \quad \text{(six 7s multiplied together)}$$

PISA 2022

The beauty of powers

Question 3/3

Refer to "The beauty of powers" on the right. Click on a choice to answer the question.

What is the last digit of the number 7^{190}?

- ○ 1
- ○ 3
- ○ 7
- ○ 9

THE BEAUTY OF POWERS

The first nine powers of the number 7 are listed below.

Notice how fast they grow!

The last digits of these numbers follow a rule or pattern. Study the pattern to answer the question.

$$
\begin{aligned}
7^1 &= 7\\
7^2 &= 49\\
7^3 &= 343\\
7^4 &= 2\,401\\
7^5 &= 16\,807\\
7^6 &= 117\,649\\
7^7 &= 823\,543\\
7^8 &= 5\,764\,801\\
7^9 &= 40\,353\,607
\end{aligned}
$$

3. Always Sometimes Never

Always sometimes never

Introduction

Read the introduction. Then click on the NEXT arrow.

ALWAYS SOMETIMES NEVER

Statements that people make can generally be grouped into three different categories:

Statements that are **ALWAYS** true;
Statements that are **SOMETIMES** true; and
Statements that are **NEVER** true.

The statement:

"A number that is divisible by 4 is also divisible by 2"

is **ALWAYS** true because 2 is a factor of 4.

The statement:

"A number that is divisible by 9 is also divisible by 6"

is **SOMETIMES** true. For example, 36 is divisible by 9 and by 6, but 27 is divisible by 9, but not divisible by 6.

The statement:

"The sum of two odd numbers is odd"

is **NEVER** true because the sum of two odd numbers is always even.

Always sometimes never

Question 1/3

For each statement, indicate if it is **always true**, **sometimes true** or **never true**

Statement	Always True	Sometimes True	Never True
A 14-year old girl was at least once in her life half her current height.	○	○	○
A 14-year old girl is taller than a 10-year old girl.	○	○	○

ALWAYS SOMETIMES NEVER

Statements that people make can generally be grouped into three different categories:

Statements that are **ALWAYS** true;
Statements that are **SOMETIMES** true; and
Statements that are **NEVER** true.

The statement:

"A number that is divisible by 4 is also divisible by 2"

is **ALWAYS** true because 2 is a factor of 4.

The statement:

"A number that is divisible by 9 is also divisible by 6"

is **SOMETIMES** true. For example, 36 is divisible by 9 and by 6, but 27 is divisible by 9, but not divisible by 6.

The statement:

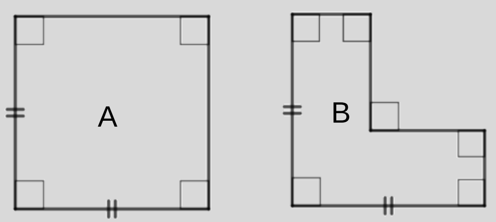
"The sum of two odd numbers is odd"

is **NEVER** true because the sum of two odd numbers is always even.

Always sometimes never

Question 2/3

For each statement, indicate if it is **always true**, **sometimes true** or **never true**

Statement	Always True	Sometimes True	Never True
When a whole number is multiplied by itself the answer is even.	○	○	○
Doubling a whole number produces an even number.	○	○	○
Halving an odd whole number produces a whole number	○	○	○
$3x + 1 = \dfrac{6x + 2}{2}$	○	○	○
 The perimeter of figure A is greater than the perimeter of figure B.	○	○	○
If a coin is flipped 50 times it will land heads up 25 times.	○	○	○

Always sometimes never

Question 3/3

Each of the following statement is **SOMETIMES TRUE**.

For each statement provide an example of when the statement is true and when the statement is not true.

Statement	Example of when the statement is true	Example of when the statement is not true
The person with the largest number of coins has the largest amount of money.	*Enter your example here*	*Enter your example here*
A – B = B – A	*Enter your example here*	*Enter your example here*
If you add the same number to the numerator (top) and the denominator (bottom) of a fraction, the fraction value increases.	*Enter your example here*	*Enter your example here*

4. Tiling

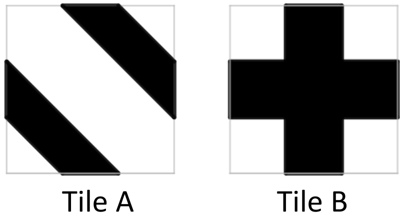
Tiling

Introduction

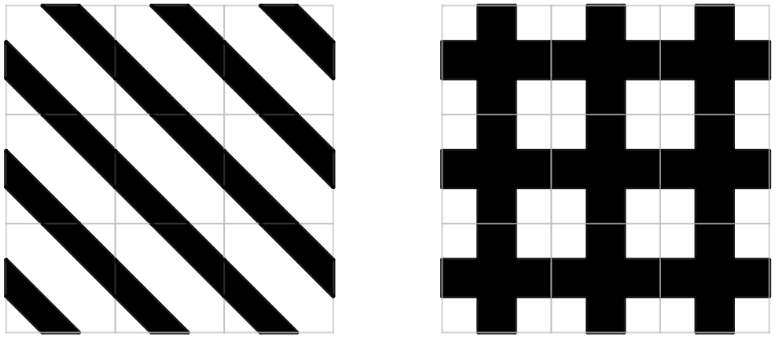
Read the introduction. Then click on the NEXT arrow

TILING

A tiler is tiling the floor. He has two different tiles that he can use, tile A and tile B.



Using only tile A he makes the left hand pattern below and using only tile B he makes the right hand pattern below.



Tiling

Question 1/5

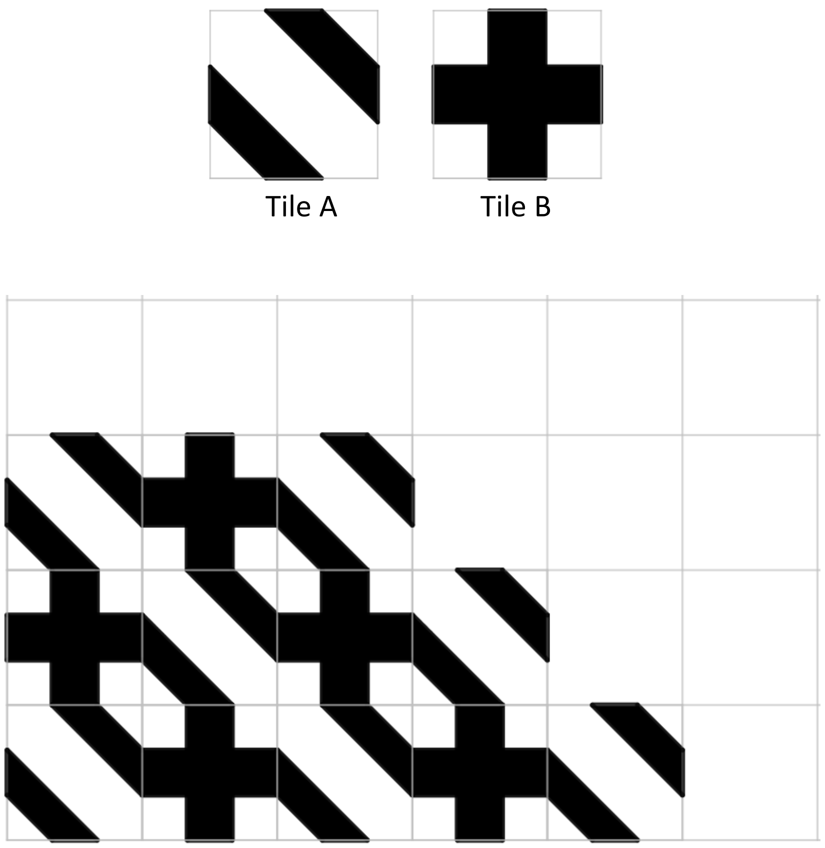
Refer to "tiling" on the right. Use drag-and-drop to complete the problem.

The tiling pattern on the right is created using a combination of the two tiles. The tiler continues to tile the floor by extending the pattern in the same way.

Study the pattern.

Use your mouse to drag and drop the tiles into position and finish tiling the rest of the floor using the same pattern.

TILING



PISA 2022

Tiling

Question 2/5

Refer to "tiling" on the right. Use drag-and-drop to complete the problem.

The tiler wants to make a set of instructions that he can give to people who want to make the same tiling pattern.

Drag and drop the elements into the spaces to complete the instructions that will produce the pattern on the right.

IF | THEN | ELSE | TILE A | TILE B

TILING INSTRUCTIONS

For row = 1 to 4

"First determine the left hand tile in the row"

IF the row is an odd numbered row

THEN the first tile is ____

ELSE the first tile is ____

"Complete the row by adding tiles"

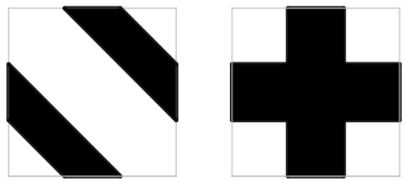
IF the previous tile is ____

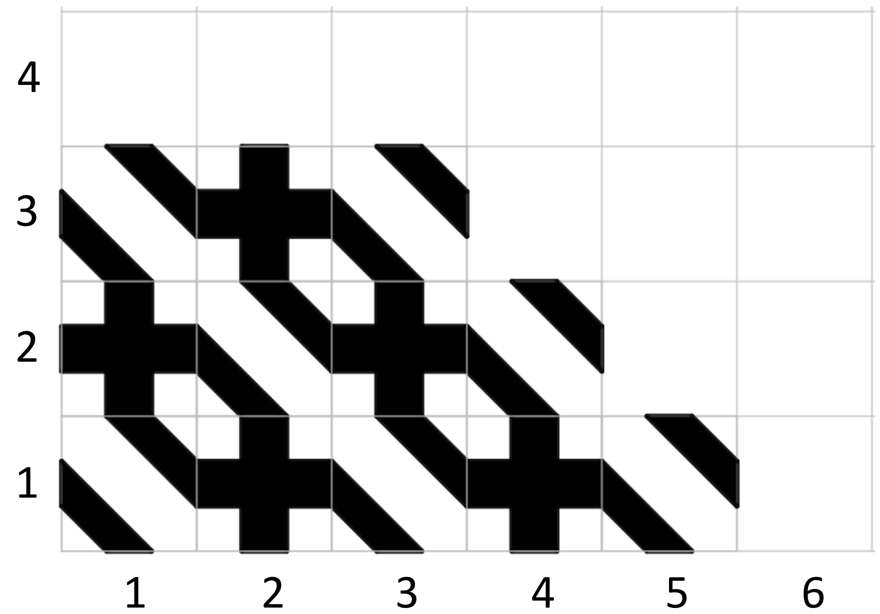
____ use ____

____ use ____

Next row

TILING





Tiling

Question 3/5

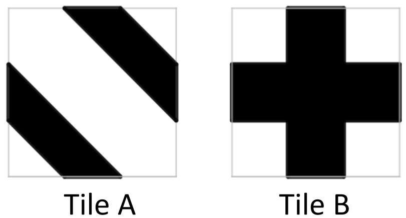
Refer to "tiling" on the right. Click on the choices to answer the question.

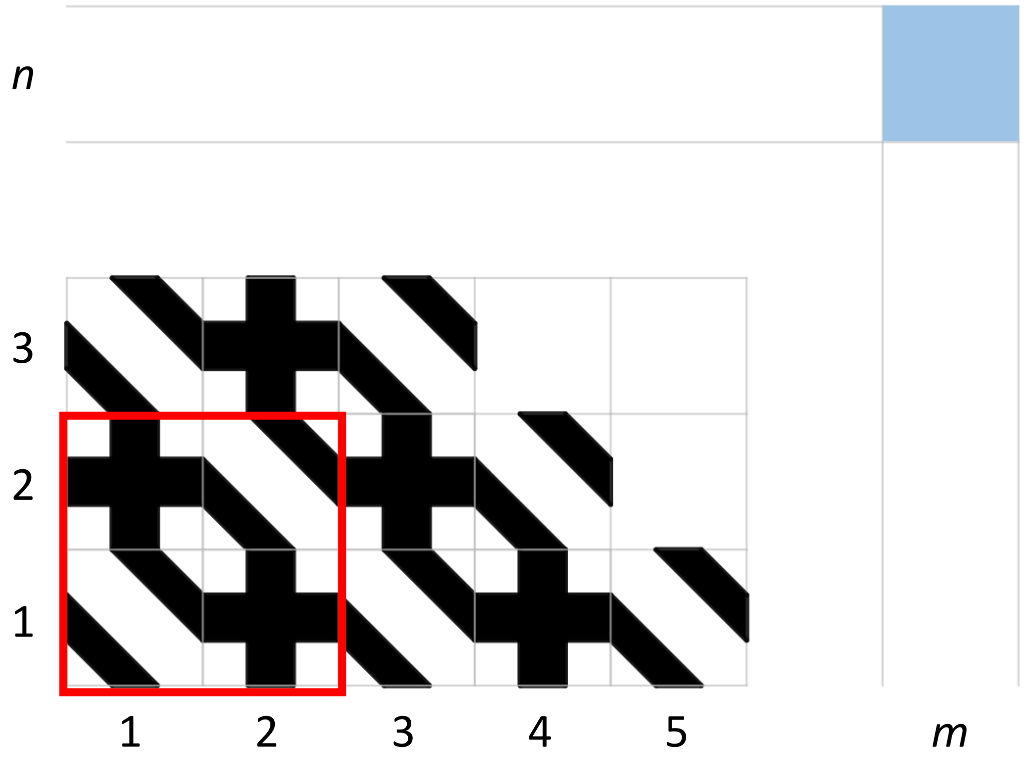
The tiler wants to be able to predict what tile will go in any position on the grid. For example, he wants to know what tile he will use in the marked position (m; n).

Study the tiling pattern and in particular the four tiles highlighted with a red border. Select **ALL** of the rules below that will correctly predict the tile that is needed for any grid position (m; n).

Rule	
If $m + n$ is odd use tile A, otherwise use tile B	
If $m + n$ is even use tile A, otherwise use tile B	
If $m \times n$ is odd use tile A, otherwise use tile B	
If $m \times n$ is even use tile A, otherwise use tile B	
If m is odd and n is odd use tile A, otherwise use tile B	
If m and n are both odd or both even use tile A, otherwise use tile B	

TILING





Tiling

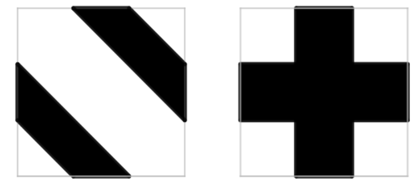
Discussion

Read the introduction

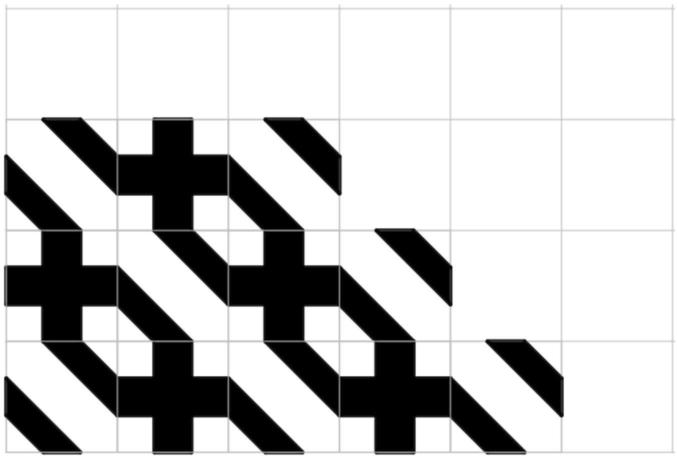
Another way of describing the pattern is to simply write the letters for each tile in the corresponding grid position.

Study the use of letters to record the tiling pattern. Then click on the NEXT arrow.

TILING



Tile A Tile B



A	B	A			
B	A	B	A		
A	B	A	B	A	

Tiling

Question 4/5

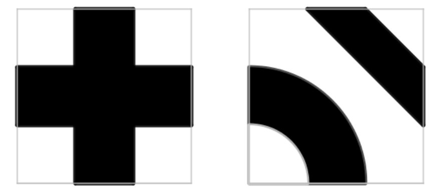
The tiling pattern on the right is created using a combination of two tiles: B and C. Ameer continues to tile the floor by extending the pattern in the same way.

Study the pattern.

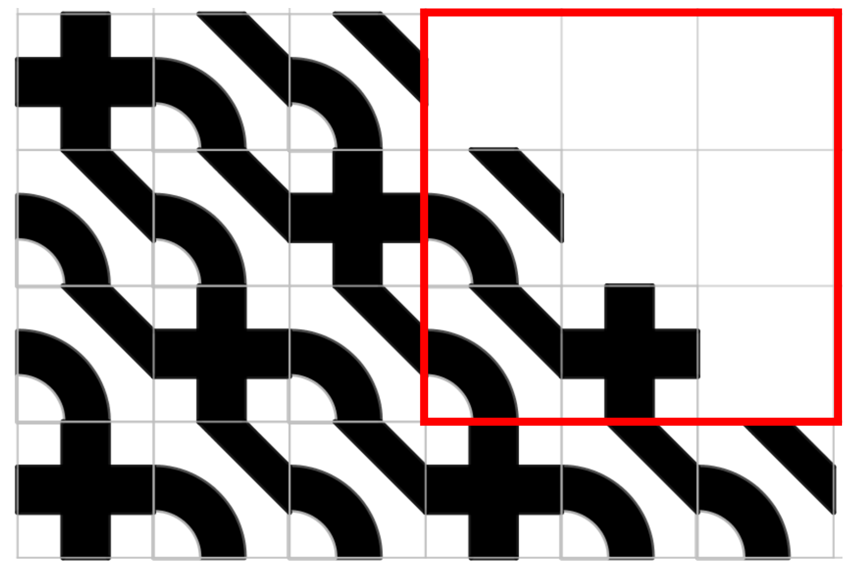
The red square on the grid below corresponds to the red square on the grid on the right. Use the letters B and C to record the tile that goes in each position of the red square.

			___	___	___
			___	___	___
			C	___	___

TILING



Tile B Tile C



Tiling

Question 5/5

The tiling pattern on the right is a section from the middle of a much larger area created using a combination of three tiles: A, B and C.

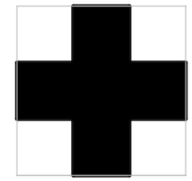
Study the pattern.

Which of the codes below describes a 3 x 3 unit of tiles that can be repeated to create the pattern on the right (select **ALL** that apply).

3 x 3 unit used to create the pattern	
A B C B A C B C A	○
B C A C A B A C B	○
A B C B C A B A C	○
A B C B C A C A B	○

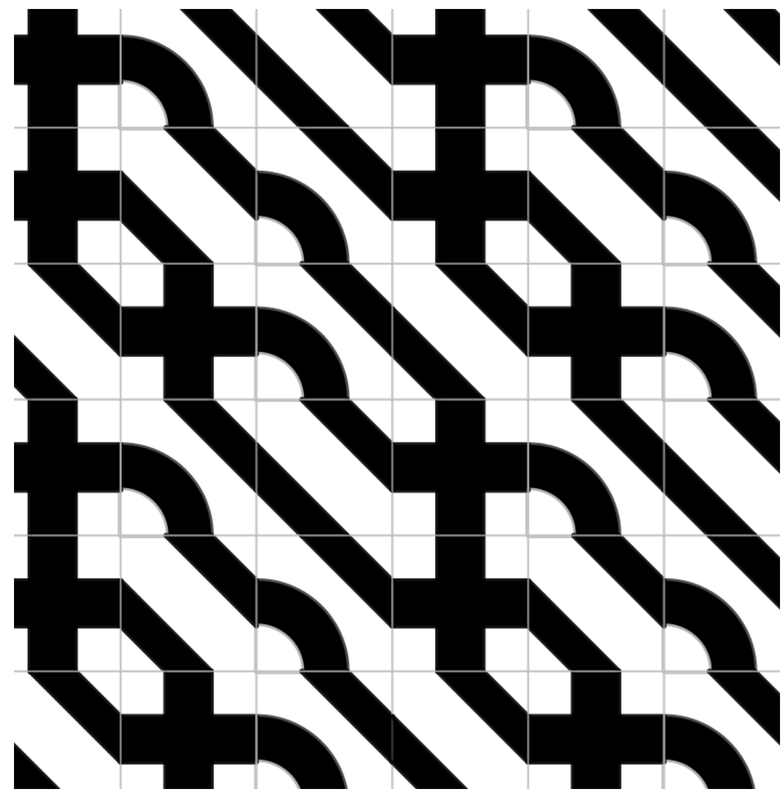
TILING

Tile A



Tile B

Tile C



5. Purchasing Decisions

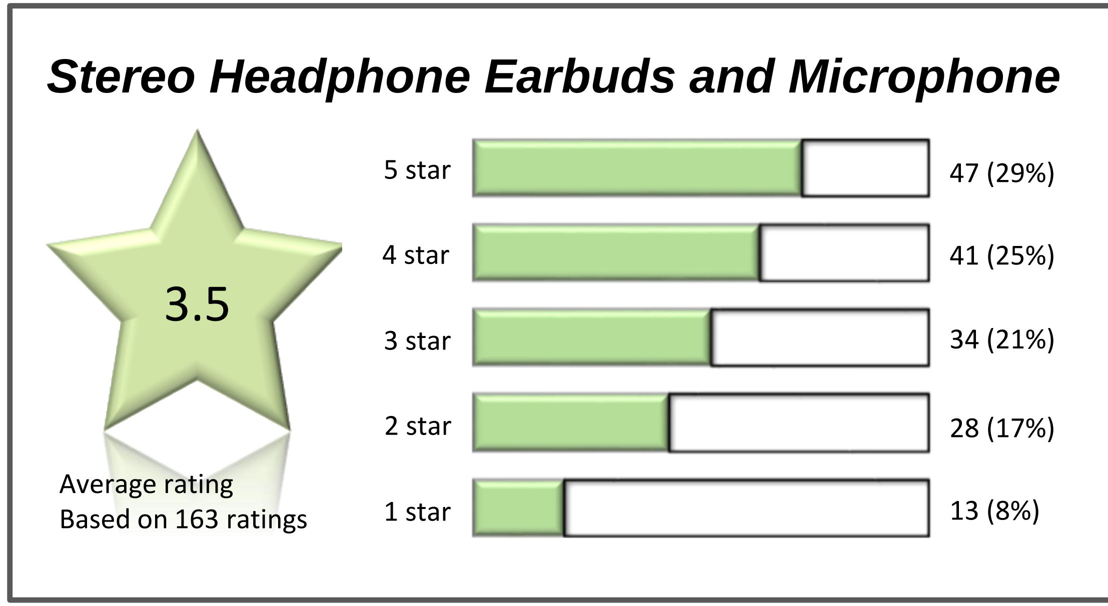
Purchasing decision

Introduction

Read the introduction. Then click on the NEXT arrow.

PURCHASING DECISION

Andrea is shopping online for a new pair of headphones. She has identified a pair that she likes. However, she notices that even though the total number of reviews is small, the product received many poor reviews: a total of 25% 1- and 2-star reviews.



Purchasing decision

Introduction continued

Read the extended introduction. Then click on the NEXT arrow.

PURCHASING DECISION

To help with her decision to buy the product or not, Andrea studied the comments for the 1- and 2-star reviews and noticed that some of the reviews have nothing to do with the quality or the functioning of the product.

She grouped the responses for the 1- and 2-star reviews and summarised her findings in the table.

REASON	Number
Headphones arrived late	13
Headphones did not arrive at all	4
Cable was damaged or missing	7
One or both earbuds were broken	4
Packaging was unattractive	5
Wrong rating (good review, bad rating)	8

PISA 2022

Purchasing decision

Question 1/2

Andrea looked through all the reviewers comments and noticed that only the 1- and 2-star reviewers made comments about poor quality or about the product arriving late or not at all.

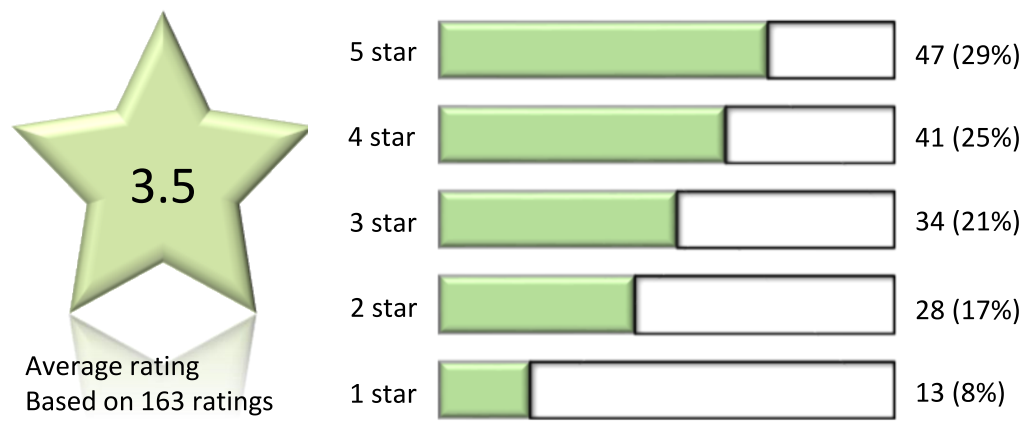
Use the information from the **Online reviews** tab and from the **Summary table** tab as well as the built in calculator to answer the questions.

Question	Response
What percentage of all of the reviews deal with poor quality of the product?	
What percentage of the 1- and 2-star reviews deal with the product arriving late or not at all?	

PURCHASING DECISION

Online reviews | Summary table

Stereo Headphone Earbuds and Microphone



Purchasing decision

Question 1/2

Andrea looked through all the reviewers comments and noticed that only the 1- and 2-star reviewers made comments about poor quality or about the product arriving late or not at all.

Use the information from the **Online reviews** tab and from the **Summary table** tab as well as the built in calculator to answer the questions.

Question	Response
What percentage of all of the reviews deal with poor quality of the product?	
What percentage of the 1- and 2-star reviews deal with the product arriving late or not at all?	

PURCHASING DECISION

Online reviews | Summary table

REASON	Number
Headphones arrived late	13
Headphones did not arrive at all	4
Cable was damaged or missing	7
One or both earbuds were broken	4
Packaging was unattractive	5
Wrong rating (good review, bad rating)	8

Purchasing decision

Question 2/2

Andrea looked through all the reviewers comments and noticed that only the 1- and 2-star reviewers made comments about poor quality or about the product arriving late or not at all.

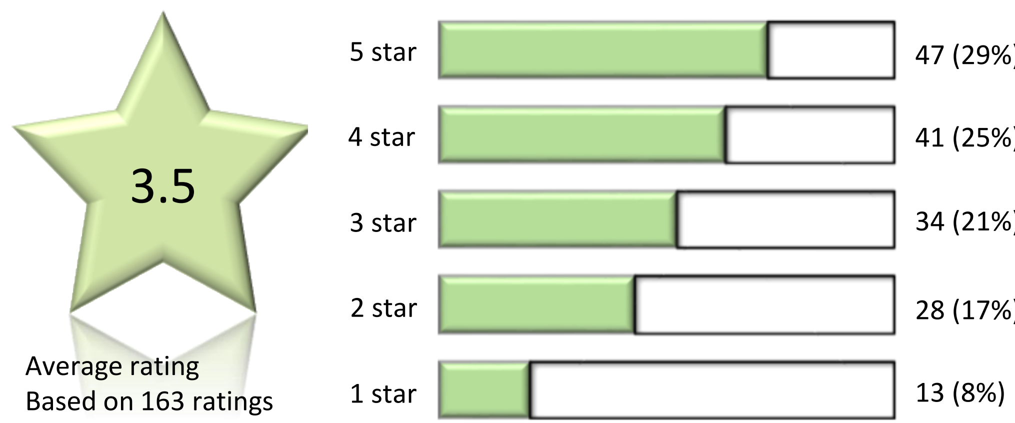
Use the information from the **Online reviews** tab and from the **Summary table** tab as well as the built in calculator to answer the question.

Question	Response
Andrea is concerned about the headphones arriving late or not at all. Based on the information in the **Online reviews** tab and the **Summary table**. How likely is it that the product will arrive late or not at all? **Express your answer as a fraction or percentage.**	

PURCHASING DECISION

Online reviews | Summary table

Stereo Headphone Earbuds and Microphone



Purchasing decision

Question 2/2

Andrea looked through all the reviewers comments and noticed that only the 1- and 2-star reviewers made comments about poor quality or about the product arriving late or not at all.

Use the information from the **Online reviews** tab and from the **Summary table** tab as well as the built in calculator to answer the question.

Question	Response
Andrea is concerned about the headphones arriving late or not at all. Based on the information in the **Online reviews** tab and the **Summary table**. How likely is it that the product will arrive late or not at all? **Express your answer as a fraction or percentage.**	

PURCHASING DECISION

Online reviews | Summary table

REASON	Number
Headphones arrived late	13
Headphones did not arrive at all	4
Cable was damaged or missing	7
One or both earbuds were broken	4
Packaging was unattractive	5
Wrong rating (good review, bad rating)	8

6. Navigation

Navigation

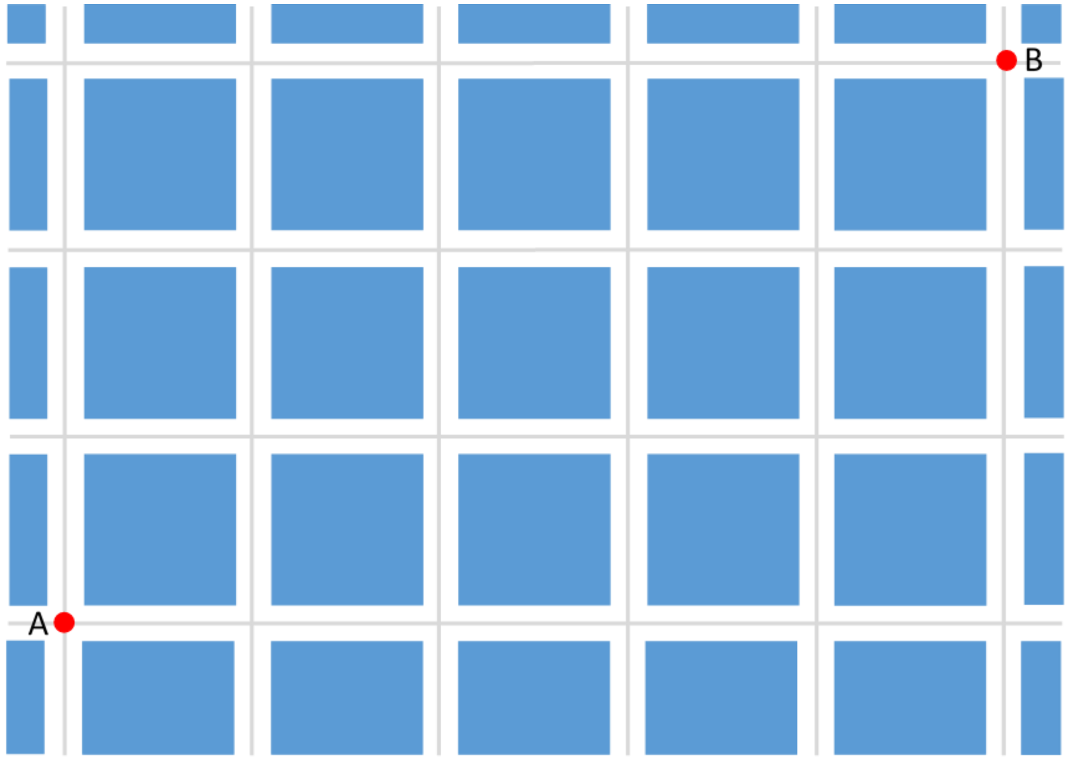
Introduction

Read the introduction. Then click on the NEXT arrow.

NAVIGATION

The shortest distance between two points is a straight line. It is, however not usually possible to navigate along a straight line in a town. Look at the map below. The grey lines are the roads and the square blue blocks are the buildings.

In this unit you will explore different strategies for planning a route from one point to another in this town.



Navigation

Introduction continued

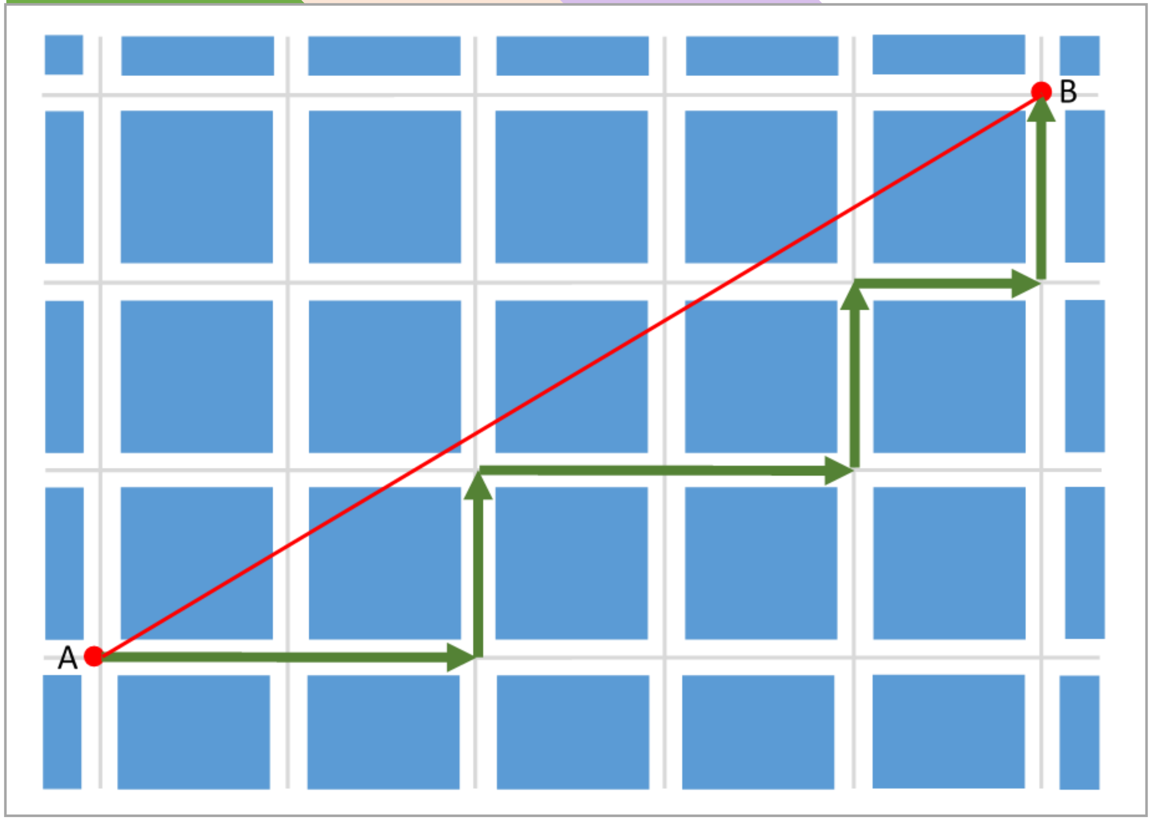
Read the introduction and select the different tabs to see the different routes. Then click on the NEXT arrow.

NAVIGATION

Ann, Bob and Corey have different ideas about how to determine the shortest route from A to B.

- Ann always moves right or up and stays below but as close as possible to the straight red line joining A and B (green line).
- Bob always moves right or up and tries to cross the straight red line joining A and B as often as possible (orange line).
- Corey always moves right or up and stays above but as close as possible to the straight red line joining A and B (purple line).

Ann's route | Bob's route | Corey's route



Navigation

Introduction continued

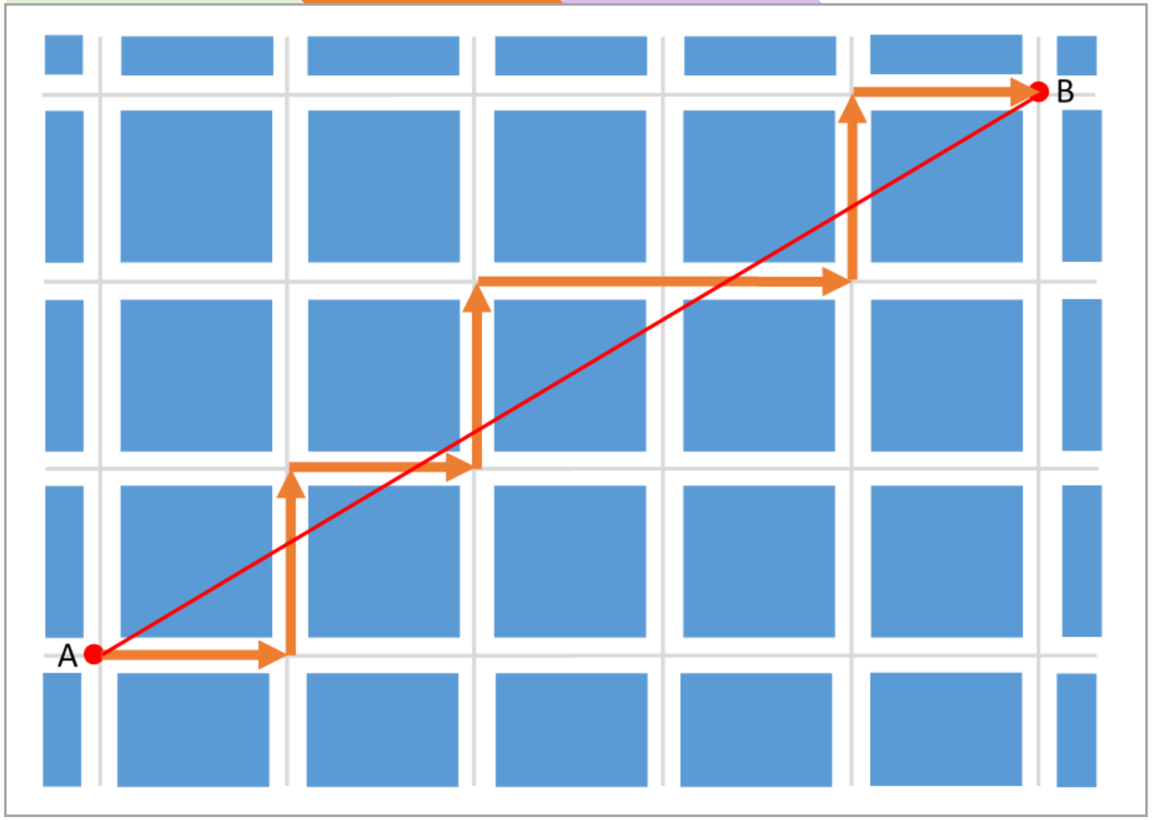
Read the introduction and select the different tabs to see the different routes. Then click on the NEXT arrow.

NAVIGATION

Ann, Bob and Corey have different ideas about how to determine the shortest route from A to B.

- Ann always moves right or up and stays below but as close as possible to the straight red line joining A and B (green line).
- Bob always moves right or up and tries to cross the straight red line joining A and B as often as possible (orange line).
- Corey always moves right or up and stays above but as close as possible to the straight red line joining A and B (purple line).

Ann's route | Bob's route | Corey's route



Navigation

Introduction continued

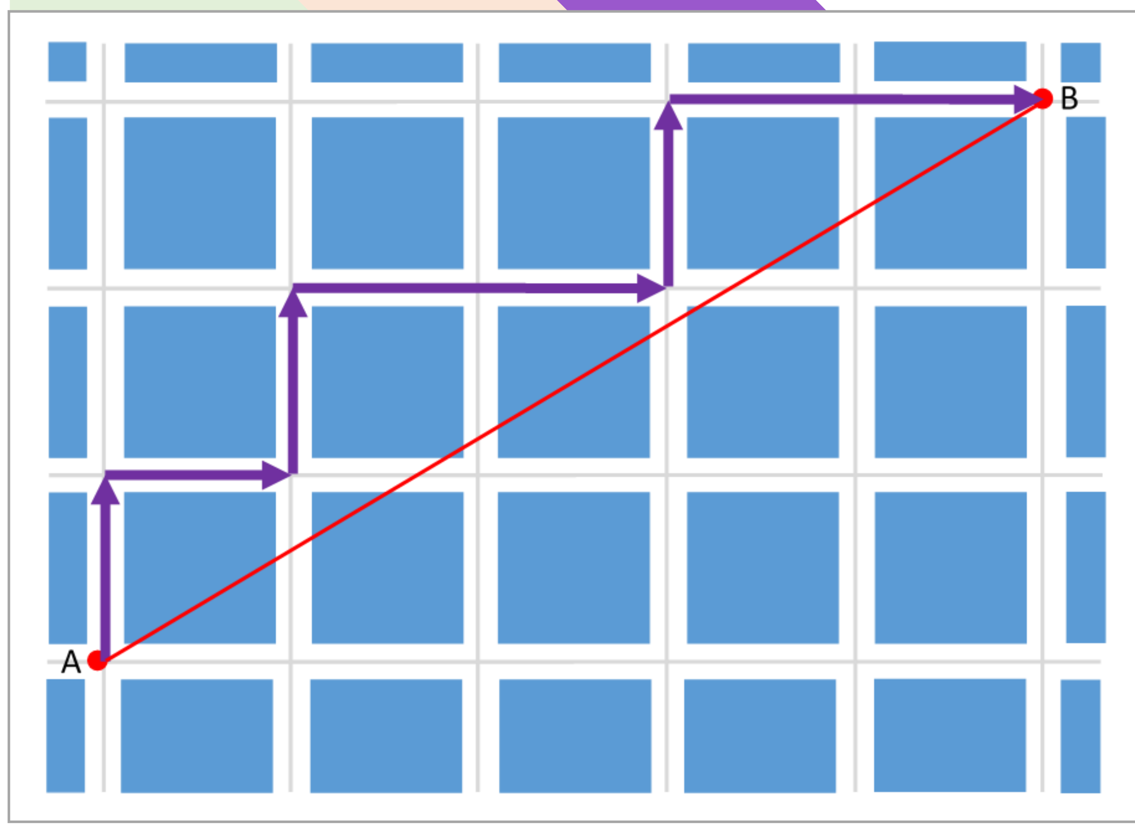
Read the introduction and select the different tabs to see the different routes. Then click on the NEXT arrow.

NAVIGATION

Ann, Bob and Corey have different ideas about how to determine the shortest route from A to B.

- Ann always moves right or up and stays below but as close as possible to the straight red line joining A and B (green line).
- Bob always moves right or up and tries to cross the straight red line joining A and B as often as possible (orange line).
- Corey always moves right or up and stays above but as close as possible to the straight red line joining A and B (purple line).

Ann's route | Bob's route | Corey's route



Navigation

Question 1/2

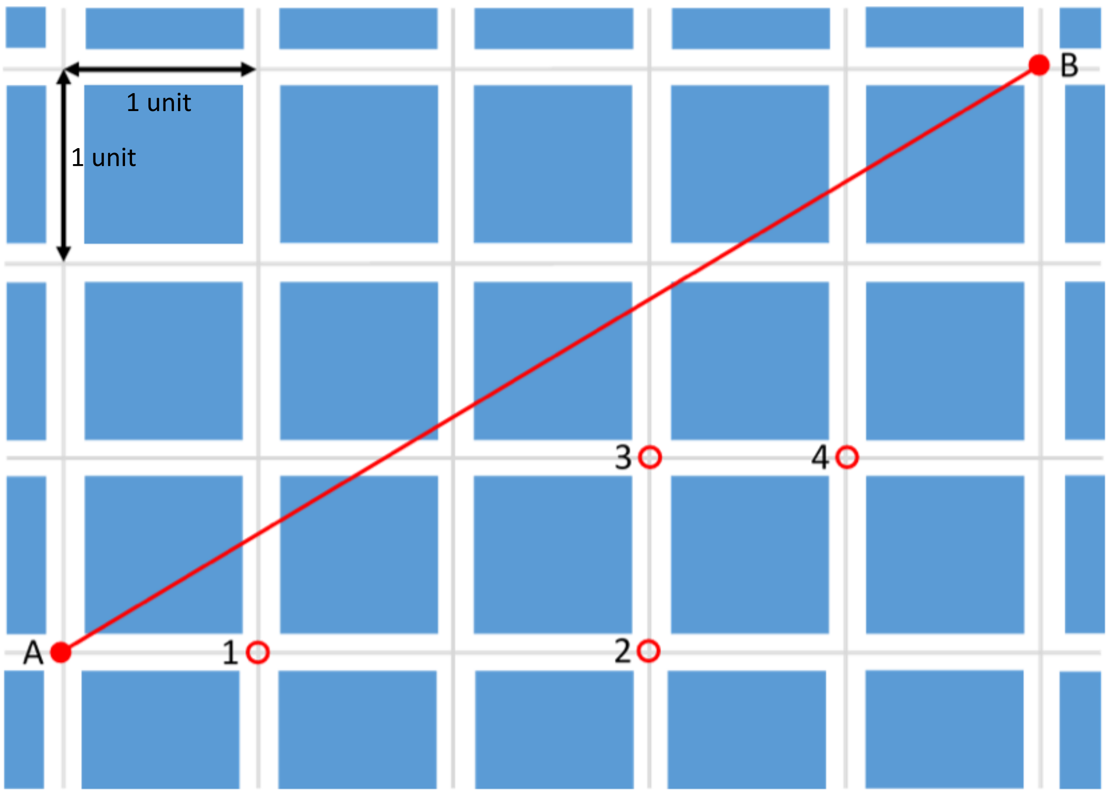
Use your mouse to move point A onto the different marked intersections of the roads – for each position of A, the route for each strategy for getting to B is shown and the distance recorded in the table.

You will notice that the irrespective of the starting position, Ann's route, Bob's route and Corey's route are all the same length for each route from A to B.

Explain why all three strategies produce routes that are equal in length.

Provide an explanation

NAVIGATION



Position of A	Distance from A to B (in units)		
	Ann's route	Bob's route	Corey's route
1			
2			
3			
4			

Navigation

Question 2/2

Three diagonal streets have been added to the map.

We know from the earlier work that without the diagonal streets the shortest route from point C to point B will be 7 units long.

Click on either **True** or **False** for each of the statements and provide a reason for your answer.

1. There exists a route from C to B that includes Diagonal 1 and is shorter than 7 units.

   ○ True
   ○ False

   Provide a reason for your answer

2 There exists a route from C to B that includes Diagonal 2 and is shorter than 7 units.

   ○ True
   ○ False

   Provide a reason for your answer

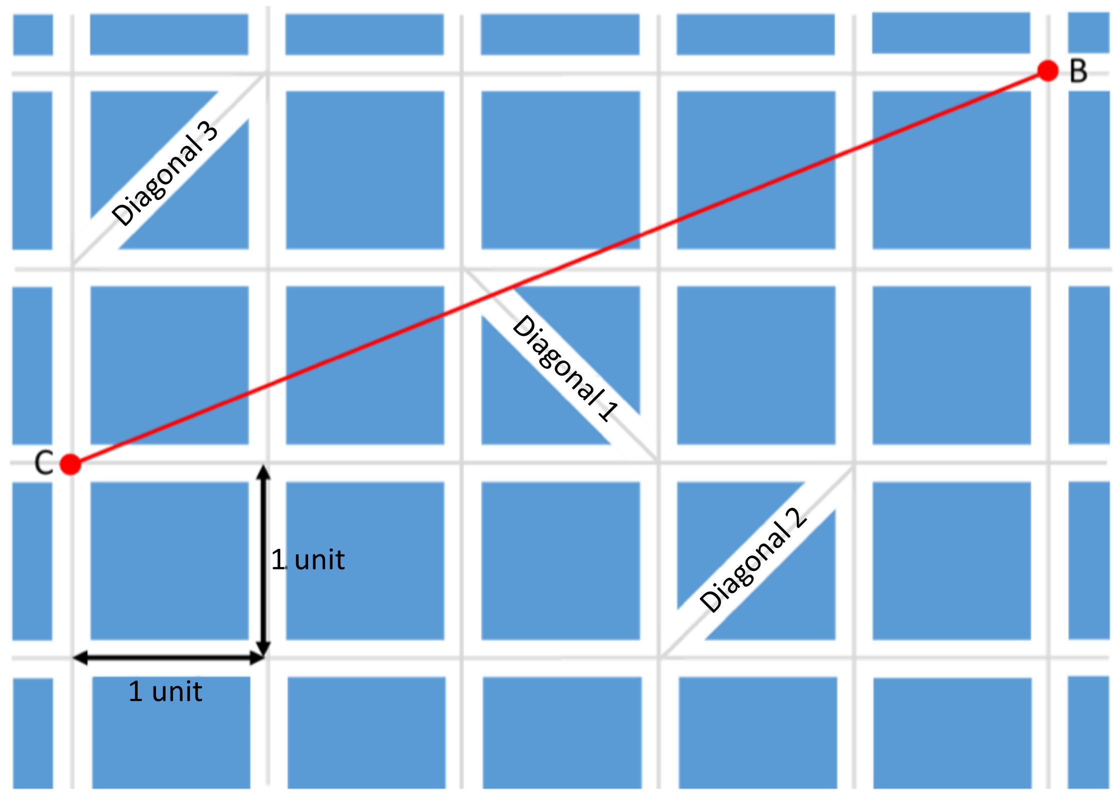
3. There exists a route from C to B that includes Diagonal 3 and is shorter than 7 units.

   ○ True
   ○ False

   Provide a reason for your answer

NAVIGATION

Three diagonal streets have been added to the map.



7. Savings

Savings simulation

Introduction

Read the introduction. Then click on the NEXT arrow.

SAVINGS SIMULATION

Sizwe and her parents are discussing how best to save money to support her expenses when she starts college. They have identified an online saving simulation application that allows them to explore different ways in which they can achieve the outcome they require.

The simulation considers four variables:

- **Monthly deposit**: the amount of money that the family deposits into the savings account every month;
- **Savings period**: the number of months for which the family makes a monthly deposit into the savings account;
- The **annual interest rate** that the savings account attracts; and
- **Total savings**: the total amount that will be saved at the end of the savings period.

The application allows the user to perform three simulations:

- **Total savings**: the total savings that will accumulate if the monthly deposit, interest rate and savings period are known;
- **Monthly deposit**: the monthly deposit that is needed to achieve a desired total savings over a given time period and interest rate; and
- **Savings period**: the total period (number of months) that is needed to achieve a desired total savings for a given monthly deposit and interest rate.

Savings simulation

Introduction

Using the simulator involves two steps:

1. *Selecting the what you want to simulate; and*
2. *Entering the values of the relevant variables.*

The simulator allows you to save the details for up to five simulations at a time.

Explore the way that the simulator works then click on the NEXT arrow.

SAVINGS SIMULATOR

Step 1: Select what you want to simulate: Select what you want to simulate:

Step 2: Complete the required information using the highlighted (red) sliders

Savings period: 48 Months

Monthly deposit: 82 Zeds

Annual interest rate: 12 % per year

Total saving: 5000 Zeds

Save the data

Clear the saved data

Simulation #	Savings Period (Months)	Monthly deposit (Zeds)	Annual Interest Rate (%)	Total amount saved (Zeds)
1				
2				
3				
4				
5				

Savings simulation

Introduction

Using the simulator involves two steps:

1. *Selecting the what you want to simulate; and*
2. *Entering the values of the relevant variables.*

The simulator allows you to save the details for up to five simulations at a time.

Explore the way that the simulator works then click on the NEXT arrow.

This screen does not appear in the unit. It is provided here to give the reader a sense of what the student will experience.

SAVINGS SIMULATOR

Step 1: Select what you want to simulate: The total amount you will save

Step 2: Complete the required information using the highlighted (red) sliders

Savings period: 48 Months

Monthly deposit: 40 Zeds

Annual interest rate: 10 % per year

Total saving: 2350 Zeds

Save the data

Clear the saved data

Simulation #	Savings Period (Months)	Monthly deposit (Zeds)	Annual Interest Rate (%)	Total amount saved (Zeds)
1	12	40	6	495
2	48	40	6	2165
3	12	40	10	505
4	48	40	10	2350
5				

Savings simulation

Introduction

Using the simulator involves two steps:

1. *Selecting the what you want to simulate; and*

2. *Entering the values of the relevant variables.*

The simulator allows you to save the details for up to five simulations at a time.

Explore the way that the simulator works then click on the NEXT arrow.

This screen does not appear in the unit. It is provided here to give the reader a sense of what the student will experience.

SAVINGS SIMULATOR

Step 1: Select what you want to simulate: The monthly payment you should make

Step 2: Complete the required information using the highlighted (red) sliders

Savings period: 48 Months

Monthly deposit: 82 Zeds

Annual interest rate: 12 % per year

Total saving: 5000 Zeds

Save the data | Clear the saved data

Simulation #	Savings Period (Months)	Monthly deposit (Zeds)	Annual Interest Rate (%)	Total amount saved (Zeds)
1	12	405	6	5000
2	48	92	6	5000
3	18	255	12	5000
4	48	82	12	5000
5				

Savings simulation

Introduction

Using the simulator involves two steps:

1. *Selecting the what you want to simulate; and*
2. *Entering the values of the relevant variables.*

The simulator allows you to save the details for up to five simulations at a time.

Explore the way that the simulator works then click on the NEXT arrow.

This screen does not appear in the unit. It is provided here to give the reader a sense of what the student will experience.

SAVINGS SIMULATOR

Step 1: Select what you want to simulate: How long it will take you to save an amount

Step 2: Complete the required information using the highlighted (red) sliders

Savings period: 49 Months

Monthly deposit: 80 Zeds

Annual interest rate: 12 % per year

Total saving: 5000 Zeds

Save the data | Clear the saved data

Simulation #	Savings Period (Months)	Monthly deposit (Zeds)	Annual Interest Rate (%)	Total amount saved (Zeds)
1	97	40	6	5000
2	55	80	6	5000
3	81	40	12	5000
4	49	80	12	5000
5				

Savings simulation

Question 1/3

Use the simulator to calculate the unknown amount in each situation.

1. How many Zeds will Sizwe save altogether if she:
   - Deposits 60 Zeds per month,
   - For a period of 48 months,
   - At an annual interest rate of 4%.

   Enter your answer here

2. How many Zeds must Sizwe deposit every month if she:
   - Wants to save 4,000 Zeds,
   - Over a period of 36 months,
   - At an annual interest rate of 8%.

   Enter your answer here

3. How long (in months) will it take Sizwe to:
   - Save 6000 Zeds,
   - If she deposits 100 Zeds per month,
   - At an annual interest rate of 10%.

   Enter your answer here

SAVINGS SIMULATOR

Step 1: Select what you want to simulate: Select what you want to simulate:

Step 2: Complete the required information using the highlighted (red) sliders

Savings period: 0 Months

Monthly deposit: 0 Zeds

Annual interest rate: 0 % per year

Total saving: 0 Zeds

Save the data

Clear the saved data

Simulation #	Savings Period (Months)	Monthly deposit (Zeds)	Annual Interest Rate (%)	Total amount saved (Zeds)
1				
2				
3				
4				
5				

Savings simulation

Question 2/3

For each simulation select **TWO STATEMENTS** to justify the use of the given simulator.

	Statement		
Simulation	You know how much money you will need	You know how much money you can save each month	You know when you will need the money
Savings period simulation	○	○	○
Monthly deposit simulation.	○	○	○
Total savings simulation	○	○	○

SAVINGS SIMULATOR

Step 1: Select what you want to simulate: Select what you want to simulate:

Step 2: Complete the required information using the highlighted (red) sliders

Savings period:	0	Months
Monthly deposit:	0	Zeds
Annual interest rate:	0	% per year
Total saving:	0	Zeds

Save the data

Clear the saved data

Simulation #	Savings Period (Months)	Monthly deposit (Zeds)	Annual Interest Rate (%)	Total amount saved (Zeds)
1				
2				
3				
4				
5				

Savings simulation

Question 3/3

Sizwe has done some simulations, She says: "***I notice than when I earn no interest and double the monthly deposit, the length of the savings period is halved. But, when I earn interest and double the monthly deposit the savings period is not halved***."

Select the appropriate tabs to study the records in Sizwe's simulation and to do your own simulations to answer the questions.

1. Complete the statement:

   Sizwe's observation is:

   - ○ always true
   - ○ sometimes true, it depends on the interest rate

2. Complete the statement:

   For a fixed total savings and a set monthly deposit, an increase in the interest rate reduces the length of the savings period more when:

   - ○ the monthly payment is smaller.
   - ○ the monthly payment is larger.

3. Provide a justification for the statement you completed in question 2.

Provide a justification

Sizwe's simulator | Blank simulator

SAVINGS SIMULATOR

Step 1: Select what you want to simulate: How long it will take you to save an amount

Step 2: Complete the required information using the highlighted (red) sliders

Savings period: 112 Months

Monthly deposit: 40 Zeds

Annual interest rate: 6 % per year

Total saving: 6000 Zeds

Save the data | Clear the saved data

Simulation #	Savings Period (Months)	Monthly deposit (Zeds)	Annual Interest Rate (%)	Total amount saved (Zeds)
1	300	20	0	6000
2	150	40	0	6000
3	184	20	6	6000
4	112	40	**6**	6000
5				

Savings simulation

Question 3/3

Sizwe has done some simulations, She says: "***I notice than when I earn no interest and double the monthly deposit, the length of the savings period is halved. But, when I earn interest and double the monthly deposit the savings period is not halved***."

Select the appropriate tabs to study the records in Sizwe's simulation and to do your own simulations to answer the questions.

1. Complete the statement:

   Sizwe's observation is:

   ○ always true
   ○ sometimes true, it depends on the interest rate

2. Complete the statement:

   For a fixed total savings and a set monthly deposit, an increase in the interest rate reduces the length of the savings period more when:

   ○ the monthly payment is smaller.
   ○ the monthly payment is larger.

3. Provide a justification for the statement you completed in question 2.

Provide a justification

Sizwe's simulator | Blank simulator

SAVINGS SIMULATOR

Step 1: Select what you want to simulate: Select what you want to simulate:

Step 2: Complete the required information using the highlighted (red) sliders

Savings period: 0 Months

Monthly deposit: 0 Zeds

Annual interest rate: 0 % per year

Total saving: 0 Zeds

Save the data | Clear the saved data

Simulation #	Savings Period (Months)	Monthly deposit (Zeds)	Annual Interest Rate (%)	Total amount saved (Zeds)
1				
2				
3				
4				
5				