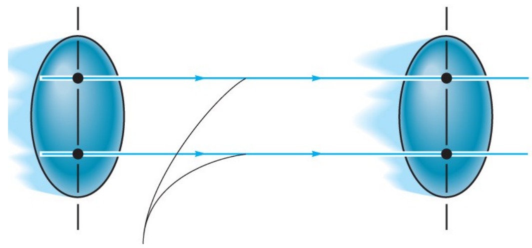
PLANAR KINEMATICS OF A RIGID BODY

I. Rigid body motion:

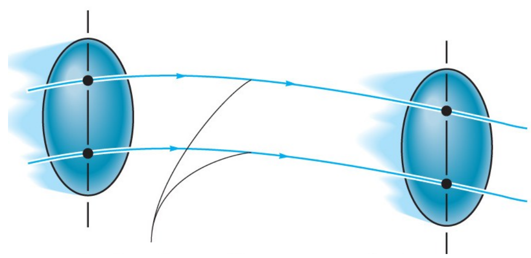
When all the particles of a rigid body move along paths which are equidistant from a fixed plane, the body is said to undergo *planar motion*. There are three types of rigid body planar motion

1. **Translation:** Every line segment in the body remains parallel to its original direction during the motion. Specifically, a body can undergo two types of translation:

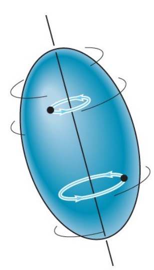
   (a) **Rectilinear translation:** All points follow parallel straight-line paths



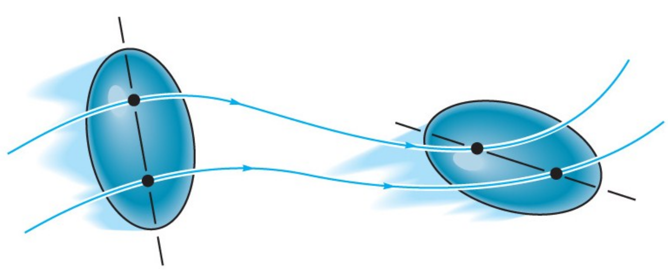
   (b) **Curvilinear translation:** All points follow curved paths that are of the same shape and are equidistant from one another.



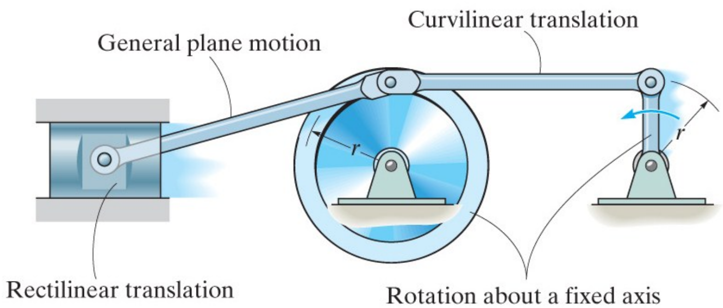
2. **Rotation about a fixed axis:** All the particles of the body, except those which lie on the axis of rotation move along circular paths



3. **General plane motion:** The body undergoes a combination of translation and rotation

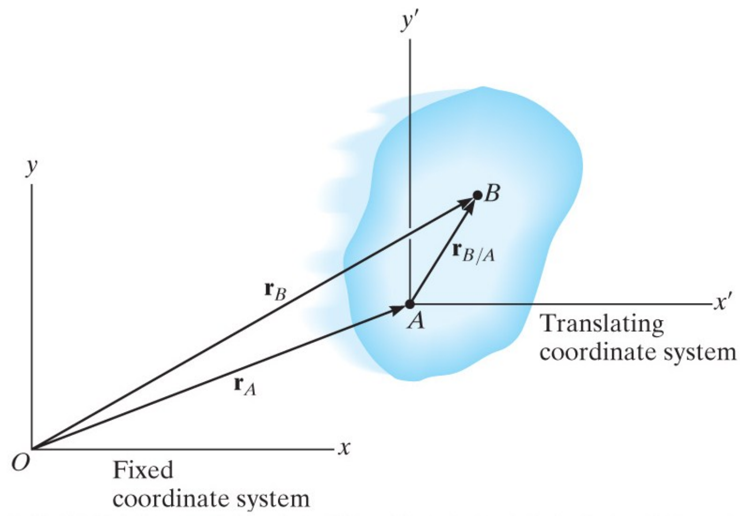


All the above types of rigid body planar motion are exemplified by the moving parts of the crank mechanism shown in the figure below



II. Translation:

Consider a rigid body which is subjected to either rectilinear or curvilinear translation in the x-y plane. x'-y' is a frame associated with the translating object.



A and B are two points in the rigid body. Because our object is a rigid body, all its points are at fixed distances from each other, moreover, for planar rigid body translation the direction $\vec{r}_{A/B}$ is constant then $\left(d\vec{r}_{B/A}/dt = 0 \ and \ d^2\vec{r}_{B/A}/dt^2 = 0\right)$. We have the following relations:

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

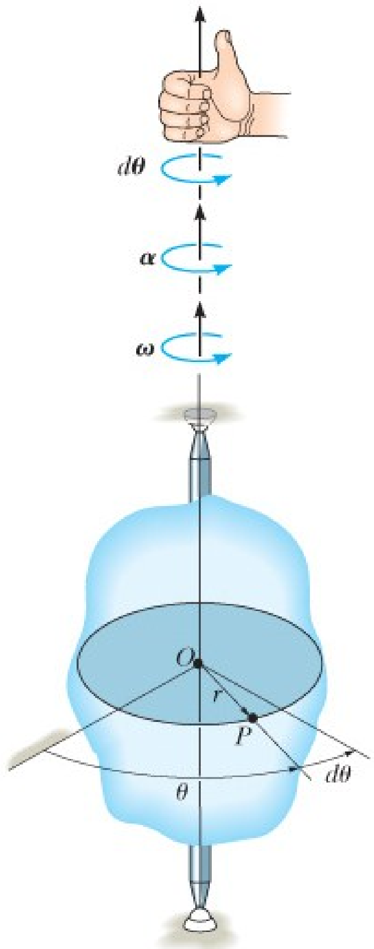
$$\vec{V}_B = \vec{V}_A$$

$$\vec{a}_B = \vec{a}_A$$

In other words, all the points on a translating rigid body move with the same velocity and acceleration. As a result, the kinematics of particle motion discussed in chapter 2, may also be used to specify the kinematics of points located in a translating rigid body.

II. Rotation about a fixed axis:

When a body is rotating about a fixed axis, any point P located in the body travels along a circular path. The motion of the body is described by its angular motion which involves three basic quantities: angular position (θ), angular velocity (ω), and angular acceleration (α) described as follow:



- **Angular velocity:** Measures the time rate of change of the angular position. If θ is the angular position of a radial line located in some representative plane of the body, the angular velocity ω is along the axis of rotation and its direction can be determined using the right hand rule; that is, the fingers of the right hand are curled with the sense of rotation, the thumb indicate the direction of the angular velocity.

$$\omega = \frac{d\theta}{dt}$$

  This vector has a magnitude which is often measured in rad/s. It is expressed here in scalar form since its direction is always along the axis of rotation. When indicating the angular motion, we can refer to the sense of rotation as clockwise or counterclockwise. counterclockwise rotations are usually chosen as positive

- **Angular acceleration:** Measures the time rate of change of the angular velocity. The angular acceleration α is along the axis of rotation, and its sense of direction depends on whether ω is increasing or decreasing. (if $\mid \omega \mid$ is decreasing α and ω have opposite direction and vise-versa

$$\alpha = \frac{d\omega}{dt}$$

- **useful relation between α, ω, and θ** By eliminating t from $\alpha = dw/dt$ and $w = d\theta/dt$ we obtain

$$\alpha \; d\theta = \omega \; d\omega$$

The similarity between the differential relations for angular motion and those developed for rectilinear motion of a particle ($v = ds/dt$, $a = dv/dt$, $and\ a\ ds = v\ dv$) should be apparent.

- **Constant angular acceleration:** If the angular acceleration of the body is constant $\alpha = \alpha_c$ then

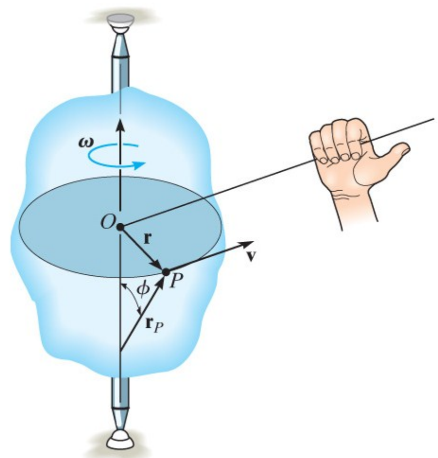
$$\omega = \omega_o + \alpha_c\ t$$

$$\theta = \theta_o + \omega_o\ t + \frac{1}{2}\ \alpha\ t^2$$

$$\omega^2 = \omega_o^2 + 2\ \alpha_c\ (\theta - \theta_o)$$

where θ_o and ω_o are the initial values of the body's angular position and angular velocity, respectively, and we have chosen counterclockwise rotation as positive.

- **Motion of point P:** As the rigid body rotates, point P travels along a circular path of radius r and center at O. This path is contained within the shaded plane shown in the figure below.



  - **Position:** The position of P is defined by the position vector $\vec{r}$ which extends from O to P
  - **Velocity:** can be found from its polar coordinate components $v_r = \dot{r}$, and $v_\theta = r\ \dot{\theta} = r\ \omega$. Since r is constant $\dot{r} = 0$ and

$$\vec{v} = r\ \omega\ \hat{u_\theta} = \vec{\omega} \times \vec{r}_p$$

    where $\vec{r}_p$, is directed from any point on the axis of rotation to point P. As a special case, the position vector $\vec{r}$ can be chosen for $\vec{r}_p$. Here $\vec{r}$ lies in the plane of motion and again the velocity of point P is

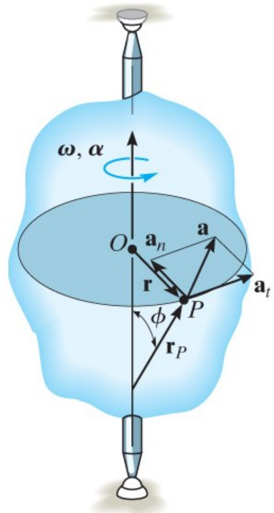
$$\vec{v} = \vec{\omega} \times \vec{r}$$

  - **acceleration:** The acceleration has two components. The tangential component of acceleration measures the rate of change of the magnitude of the velocity and can be determined using $a_t = dv/dt = r\ \alpha$. The normal component of acceleration measures the rate of change in direction of the velocity and can be determined from $a_n = v^2/r = r\ \omega^2$. In terms of vectors

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\vec{\omega}}{dt} \times \vec{r}_p + \vec{w} \times \frac{d\vec{r}_p}{dt} = \vec{\alpha} \times \vec{r}_p + \vec{\omega} \times (\vec{\omega} \times \vec{r}_p)$$

From the definition of the cross product, the first term $\vec{\alpha} \times \vec{r}_p$ has a a magnitude $a_t = \alpha\ r_p\ sin(\phi)$ and by the right hand rule, $\vec{\alpha} \times \vec{r}_p$ is in the direction of $\vec{a}_t$. Likewise, the second term has a magnitude of $a_n = \omega^2\ r_p\ sin(\phi)$, and applying the right hand rule twice, first to determine the result $\vec{v}_p = \vec{\omega} \times \vec{r}_p$ then $\vec{\omega} \times \vec{v}_p$, it can be seen that this result is in the same direction as $\vec{a}_n$. Noting that this is also the same direction as $-\vec{r}$, which lies in the plane of motion, we can express $\vec{a}_n$ in a much simpler form as $\vec{a}_n = -\omega^2\ \vec{r}$. Hence

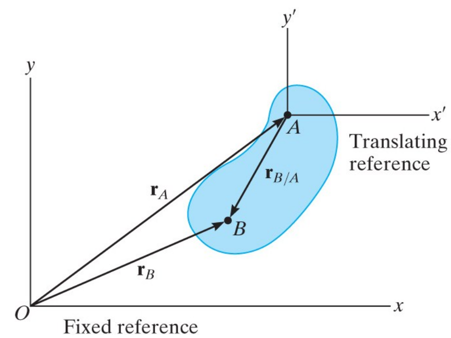
$$\vec{a} = \vec{a}_t + \vec{a}_n = \vec{\alpha} \times \vec{r} - \omega^2\ \vec{r}$$



- **Procedure for solving problems:** The velocity and acceleration of a point located on a rigid body that is rotating about a fixed axis can be determined using the following procedure
  - Angular motion
    - Establish the positive sense of rotation along the axis of rotation
    - If a relationship is known between any two of the four variables α, ω, θ, t, then a third variable can be obtained by using one of the following kinematic equations which relates all three variables
$$\omega = \frac{d\theta}{dt},\ \alpha = \frac{d\omega}{dt}, \alpha\ d\theta = \omega\ d\omega$$
    - If the body's angular acceleration is constant then the following equations can be used
$$\omega = \omega_o + \alpha_c\ t$$
$$\theta = \theta_o + \omega_o\ t + \frac{1}{2}\ \alpha\ t^2$$
$$\omega^2 = \omega_o^2 + 2\ \alpha_c\ (\theta - \theta_o)$$
    - Once the solution is obtained, the sense of α, ω, and θ is determined from the algebraic sings of their numerical quantities
  - Motion of P
    - In most cases, the velocity of P and its two components of acceleration can be determined from the scalar equations
$$v = r\ \omega,\ a_t = r\ \alpha,\ a_n = r\ \omega^2$$
    - If the geometry of the problem is difficult to visualize, the following vector equations should be used
$$\vec{v} = \vec{\omega} \times \vec{r}$$
$$\vec{a} = \vec{a}_t + \vec{a}_n = \vec{\alpha} \times \vec{r} - \omega^2\ \vec{r}$$

III. Relative-Motion analysis: Velocity:

A general plane motion of a rigid body can be described as a combination of translation and rotation. To view these "component" motions separately we will use a relative-motion analysis involving two sets of coordinate axes. The x, y coordinate system is fixed and measures the absolute position of two points A and B on the body. The origin of the x', y' coordinate system will be attached to the selected "base point", A, which generally has a known motion. The axes of this coordinate system translate with respect to the fixed frame but do not rotate with the body.

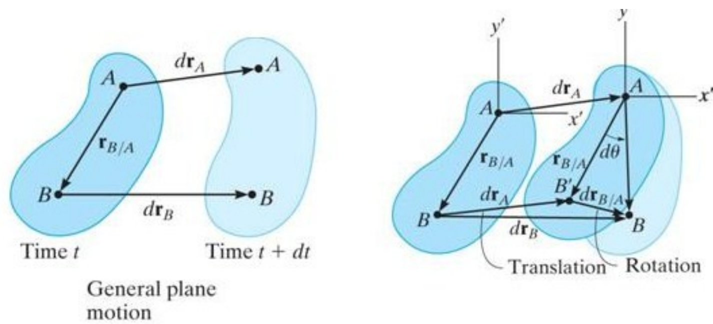


$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

During time dt, points A and B undergo displacements $d\vec{r}_A$ and $d\vec{r}_B$ such that

$$d\vec{r}_B = d\vec{r}_A + d\vec{r}_{B/A}$$

If we consider the general plane motion by its components parts then the entire body first translates by an amount $d\vec{r}_A$ so that the base point moves to its final position and point B moves to B'. The body is then rotated about A by an amount $d\theta$ so that B' undergoes a relative displacement $d\vec{r}_{B/A}$ and thus moves to its final position B (this is true because the body is rigid and the distance $\mid \vec{r}_{B/A} \mid$ is fixed). Due to the rotation about A $\mid d\vec{r}_{B/A} \mid = r_{B/A}\ d\theta$.

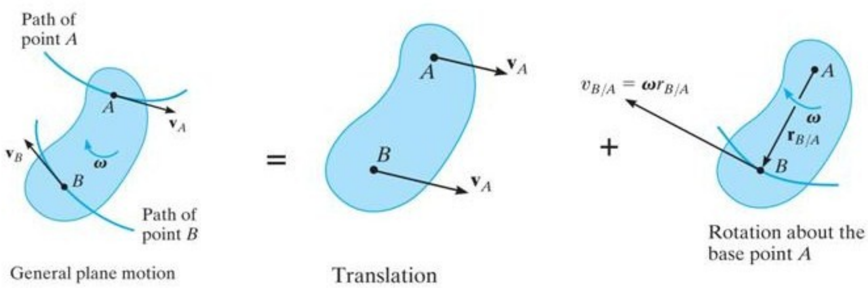


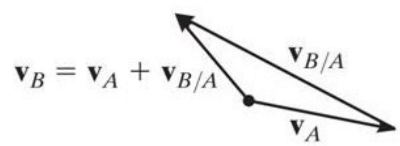
$$\frac{d\vec{r}_B}{dt} = \frac{d\vec{r}_A}{dt} + \frac{d\vec{r}_{B/A}}{dt} \ \ or\ \ \vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

The magnitude of $\vec{v}_{B/A}$ is $r_{B/A}\ d\theta/dt$ and its direction is along the z' axis.

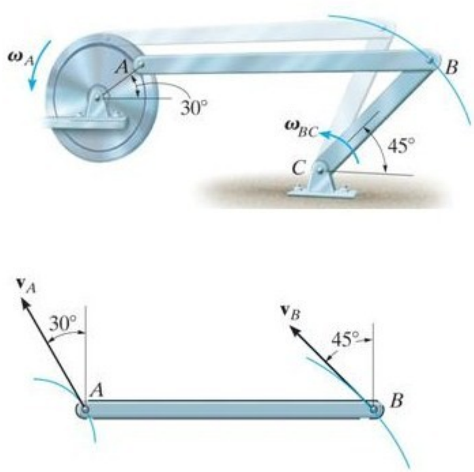
Each of the three terms in the above equation is represented graphically on the kinematic diagram in the figure below. Here it is seen that the velocity of B is determined by considering the entire body to translate with a velocity $\vec{v}_A$ and rotate about A with an angular velocity $\vec{\omega}$. Vector addition of these two effects is also shown. Since the relative velocity $\vec{v}_{B/A}$ represents the effect of circular motion about A, this term can be expressed by the cross product $\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}$. Hence

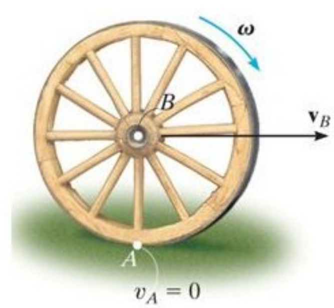
$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$





The velocity equation may be used in practical manner to study the general plane motion of a rigid body which is either pin connected to or in contact with other moving bodies. When applying this equation, points A and B should generally be selected as points on the body which are pin-connected to other bodies, or as points in contact with other adjacent bodies which have a known motion. For example both points A and B on the link AB have circular paths of motion since the wheel and link CB move in circular paths. The directions of $\vec{v}_A$ and $\vec{v}_B$ can therefore be established since they are always tangent to their paths of motion. In the case of the wheel, which rolls without slipping, point A can be selected at the ground. Here A (momentarily) has zero velocity since the ground does not move. Furthermore, the center of the wheel, B, moves along a horizontal path, so that $\vec{v}_B$ is horizontal.



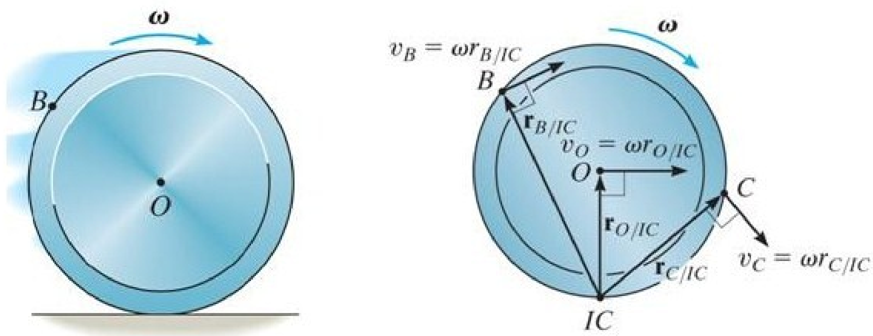


IV. Instantaneous center of zero velocity:

When using the equation $\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$, the velocity of any point B located on a rigid body can be obtained in a very direct way if one chooses the base point A to be a point that has a zero velocity at the instant considered. This point is called the *instantaneous center of zero velocity (IC)*, and it lies on the instantaneous axis of zero velocity which is always perpendicular to the plane of motion. Consequently since, if A is chosen as the IC, $\vec{v}_A = \vec{v}_{IC} = 0$ and

$$\vec{v}_B = \vec{\omega} \times \vec{r}_{B/IC}$$

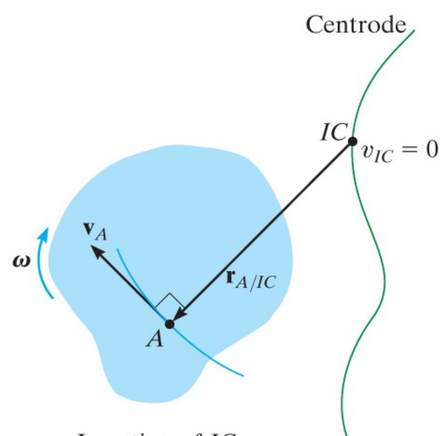
Hence, point B moves momentarily about the IC in a circular path i.e., the body appears to rotate about the instantaneous axis. For example, for a wheel which rolls without slipping, the point of contact with the ground is an IC. If it is imagined that the wheel is momentarily pinned at this point, the velocities of points, B, C, O, and so on can be found using $v = w\ r$. Here the radial distances $r_{B/IC}$, $r_{C/IC}$, $r_{O/IC}$, shown in the figure below, must be determined from the geometry of the wheel.



IV-1. Location of the IC:

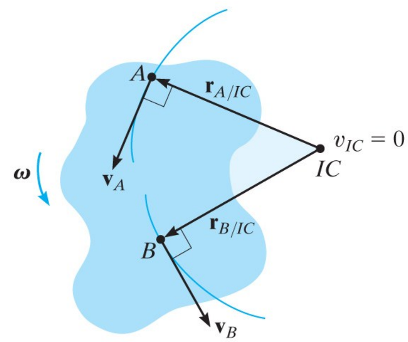
To locate the IC we can use the fact that the velocity of a point on the body is always perpendicular to the relative-position vector extending from the IC to the point. Several possibilities exist:

1. **Given the velocity of a point A on the body, and the angular velocity $\vec{\omega}$ of the body**. In this case, The IC is located along the perpendicular to $\vec{v}_A$ at A, such that the distance from A to the IC is $r_{A/IC} = v_A/\omega$. Note that the IC in the figure below lies up and to the right of A, since v_A must cause a clockwise angular velocity $\vec{\omega}$ about the IC

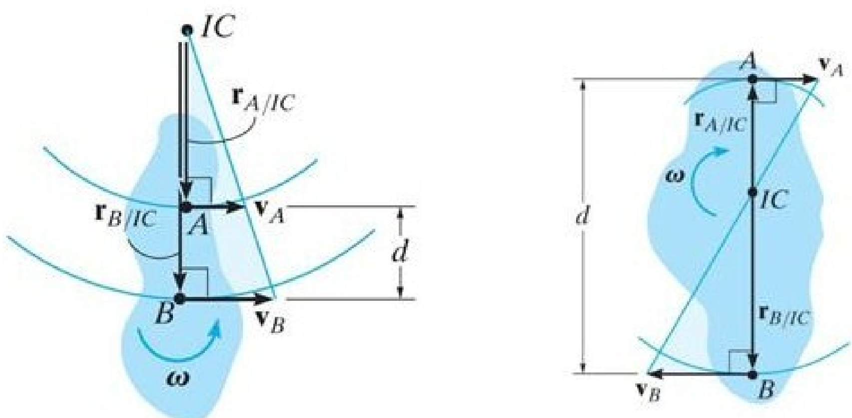


Location of IC
knowing $\mathbf{v}_A$ and $\boldsymbol{\omega}$

2. **Given the lines of action of two non-parallel velocities $\vec{v}_A$ and $\vec{v}_B$**. Construct at point A and B line segments that are perpendicular to $\vec{v}_A$ and $\vec{v}_B$. Extending these perpendiculars to their point of intersection as shown in the figure below, locates the IC at the instant considered



3. **Given the magnitude and direction of two parallel velocities $\vec{v}_A$ and $\vec{v}_B$.** Here the location of the IC is determined by proportional triangles. Examples are shown in the figure below. In both cases $r_{A/IC} = v_A/w$ and $r_{B/IC} = v_B/w$. If d is a known distance between points A and B then in the example to the left $d = r_{A/IC} + r_{B/IC}$ and in the example to the right $d = r_{B/IC} - r_{A/IC}$



4. Important notes
   - The point chosen as the IC for the body can be used only for an instant of time since the body changes its position from one instant to the next
   - The IC does not, in general, have zero acceleration and so should not be used for finding the acceleration of points in a body.

V. Relative motion analysis: Acceleration

An equation that relates the accelerations of two points on a rigid body subjected to general plane motion may be determined by differentiating the velocity equation $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$ with respect to time. This yields

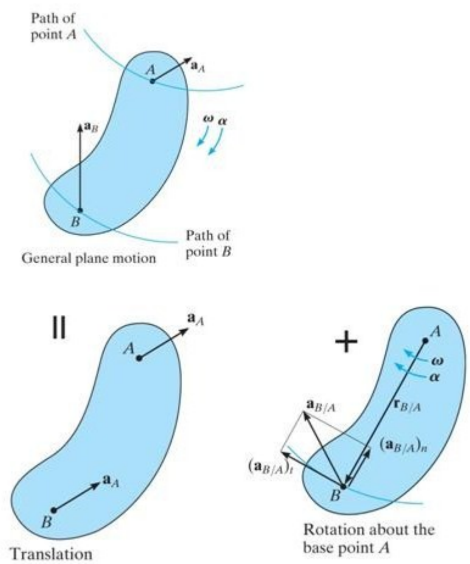
$$\frac{d\vec{v}_B}{dt} = \frac{d\vec{v}_A}{dt} + \frac{d\vec{v}_{B/A}}{dt}$$

$$\vec{a}_B = \vec{a}_A + \frac{d\vec{v}_{B/A}}{dt}$$

The last term represents the acceleration of B with respect to A as measured by an observer fixed to the translating x', y' axes which have their origin at the base point A. Since point B appears to moves along a circular arc that has a radius of curvature $r_{B/A}$, $\vec{a}_{B/A}$ can be expressed in terms of its tangential and normal components

$$\vec{a}_{B/A} = (\vec{a}_{B/A})_t + (\vec{a}_{B/A})_n$$
$$\vec{a}_B = \vec{a}_A + (\vec{a}_{B/A})_t + (\vec{a}_{B/A})_n$$

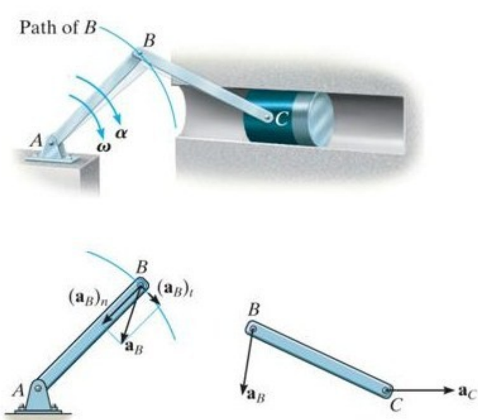
Each of the four terms in the equation above is represented graphically on the kinematic diagram shown in the figure below. here it is seen that at a given instant the acceleration of B is determined by considering the body to translate with an acceleration $\vec{a}_A$, and simultaneously rotate about the base point A with instantaneous angular velocity ω and angular acceleration α.



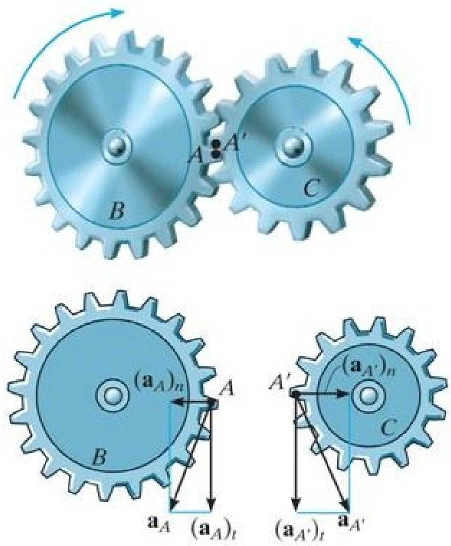
Since the relative acceleration components represent the effect of circular motion observed from translating axes having their origin at the base point A, these terms can be expressed as $(\vec{a}_{B/A})_t = \vec{\alpha} \times \vec{r}_{B/A}$ and $(\vec{a}_{B/A})_n = -\omega^2\ \vec{r}_{B/A}$

$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2\ \vec{r}_{B/A}$$

The above equation is applied in practical manner to study the accelerated motion of a rigid body which is pin connected to two other bodies, it should be realized that points which are coincident at the pin move with the same acceleration, since the path of motion over which they travel is the same. For example point B lying on either rod AB or BC of the crank mechanism shown in the figure below has the same acceleration, since the rods are pin connected at B. Here the motion of B is along a curved path, so that $\vec{a}_B$ can be expressed in terms of its tangential and normal components. At the other end of rod BC point C moves along a straight line path which is defined by the piston. Hence $\vec{a}_C$ is horizontal.



If two bodies contact one another without slipping, and the points in contact move along different paths, the tangential components of acceleration of the points will be the same; however the normal components will not be the same. For example, consider the two meshed gears in the figure below. Point A is located on gear B and a coincident point A' is located on gear C. Due to the rotational motion, $(a_A)_t = (a'_A)_t$; however, since both points follow different curved paths $(a_A)_n \neq (a_{A'})_n$ and therefore $\vec{a}_A \neq \vec{a}_{A'}$.

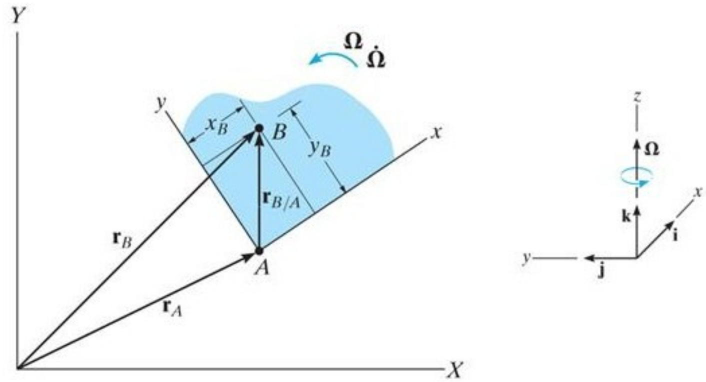


VI. Relative motion analysis using rotating axes

In the previous section the relative-motion analysis for velocities and acceleration was described using a translating coordinate system. This type of analysis is useful for determining the motion of points on the same rigid body, or the motion of points located on several pin-connected rigid bodies. In some problems however, rigid bodies (mechanisms) are constructed such that sliding will occur at their connections. The kinematic analysis for such cases is best performed if the motion is analyzed using a coordinate system which both translates and rotates. Furthermore, this frame of reference is useful for analyzing the motions of two points on a mechanism which are not located in the same rigid body and for specifying the kinematics of particle motion when the particle is moving along a rotating path.

In the following analysis two equations are developed which relate the velocity and acceleration of two points, one of which is the origin of a moving frame of reference subjected to both a translation and rotation in the plane. Due to the generality in the derivation which follows, these two points may represent either two particles moving independently of one another or two points located on the same (or different) rigid bodies.

Consider an x, y, z coordinate system (with origin at point A) which is assumed to be translating and rotating with respect to a fixed X, Y, Z coordinate system. We have the following equations which describe the position, velocity and acceleration of a point B. If Ω and $\dot{\Omega}$ are respectively the angular velocity and angular acceleration of the x, y axes. $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along the x, y, z axes respectively, and $\hat{I}$, $\hat{J}$ and $\hat{K}$ are the unit vectors along the X, Y, Z axes respectively.



- **Position:**

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

- **Velocity:**

$$\frac{d\vec{r}_B}{dt} = \frac{d\vec{r}_A}{dt} + \frac{d\vec{r}_{B/A}}{dt} \text{ or } \vec{v}_B = \vec{v}_A + \frac{d\vec{r}_{B/A}}{dt}$$

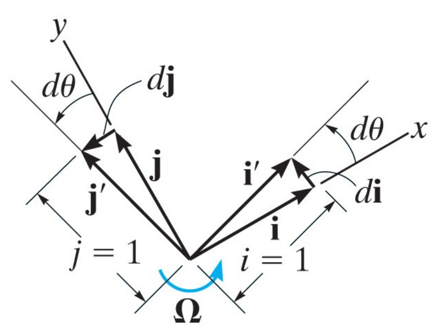
If x_B and y_B are the coordinates of point B along the x, y axes then

$$\vec{r}_{B/A} = x_B\ \hat{\imath} + y_B\ \hat{\jmath} \text{ and } \frac{d\vec{r}_{B/A}}{dt} = \frac{d}{dt}(x_B\ \hat{\imath} + y_B\ \hat{\jmath})$$

$$\frac{d\vec{r}_{B/A}}{dt} = \left(\frac{dx_B}{dt}\ \hat{\imath} + \frac{dy_B}{dt}\ \hat{\jmath}\right) + \left(x_B\ \frac{d\hat{\imath}}{dt} + y_B\ \frac{d\hat{\jmath}}{dt}\right)$$

One can show that

$$\frac{d\hat{\imath}}{dt} = \Omega\ \hat{\jmath} = \vec{\Omega} \times \hat{\imath} \text{ and } \frac{d\hat{\jmath}}{dt} = -\Omega\ \hat{\imath} = \vec{\Omega} \times \hat{\jmath}$$



$(\vec{v}_{B/A})_{xyz} = \left(\frac{dx_B}{dt}\ \hat{\imath} + \frac{dy_B}{dt}\ \hat{\jmath}\right)$, is the velocity of B with respect to A as measured by an observer attached to the rotating x, y, z reference.

Substituting these results in the previous equation leads to

$$\frac{d\vec{r}_{B/A}}{dt} = (\vec{v}_{B/A})_{xyz} + \Omega \times (x_B\ \hat{i} + y_B\ \hat{j}) = (\vec{v}_{B/A})_{xyz} + \Omega \times \vec{r}_{B/A}$$

$$\vec{v}_B = \vec{v}_A + (\vec{v}_{B/A})_{xyz} + \Omega \times \vec{r}_{B/A}$$

- **Acceleration:**

  In a similar manner the acceleration $\vec{a}_A$ and $veca_B$ of points A and B respectively with reference to the fixed frame X, Y, Z are related by

$$\vec{a}_B = \vec{a}_A + \dot{\vec{\Omega}} \times \vec{r}_{B/A} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}_{B/A}) + 2\vec{\Omega} \times (\vec{v}_{B/A})_{xyz} + (\vec{a}_{B/A})_{xyz}$$

  where $(\vec{a}_{B/A})_{xyz} = \left(\frac{d(v_{B/A})_x}{dt}\ \hat{i} + \frac{d(v_{B/A})_y}{dt}\ \hat{j} \right)$

  If this equation is compared to equation found using relative accelerations which valid for translating frame of reference, it can be seen that the difference between the equations is represented by the terms $2\Omega \times (\vec{v}_{B/A})_{xyz}$ and $(\vec{a}_{B/A})_{xyz}$. In particular $2\Omega \times (\vec{v}_{B/A})_{xyz}$ is called the Coriolis acceleration, named after the French engineer G.C. Coriolis who was the first to determine it. This term represents the difference in acceleration of B as measured from nonrotating and rotating x, y, z axes. The coriolis acceleration is always perpendicular to both $\vec{\Omega}$ and $(\vec{v}_{B/A})_{xyz}$. It is an important components of the acceleration which must be considered whenever rotating reference frames are used.