Planar Kinetics of a Rigid Body: Force and Acceleration

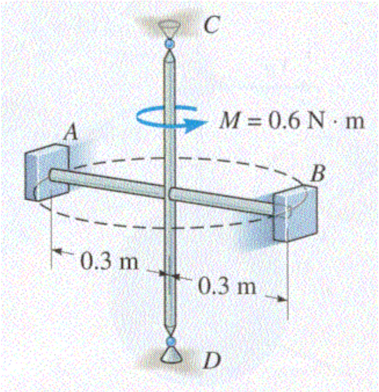
Prof. Nicholas Zabaras
Warwick Centre for Predictive Modelling
University of Warwick
Coventry CV4 7AL
United Kingdom

Email: nzabaras@gmail.com
URL: http://www.zabaras.com/

April 10, 2016

Moment and Angular Acceleration

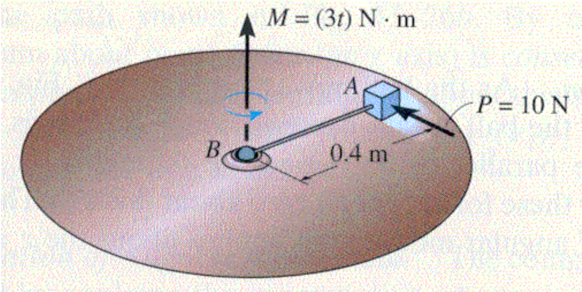
- When $M \neq 0$, rigid body experiences angular acceleration
- Relation between M and α is analogous to relation between F and a



$$F = ma, \qquad M = I\alpha$$

Mass = Resistance

Moment of Inertia

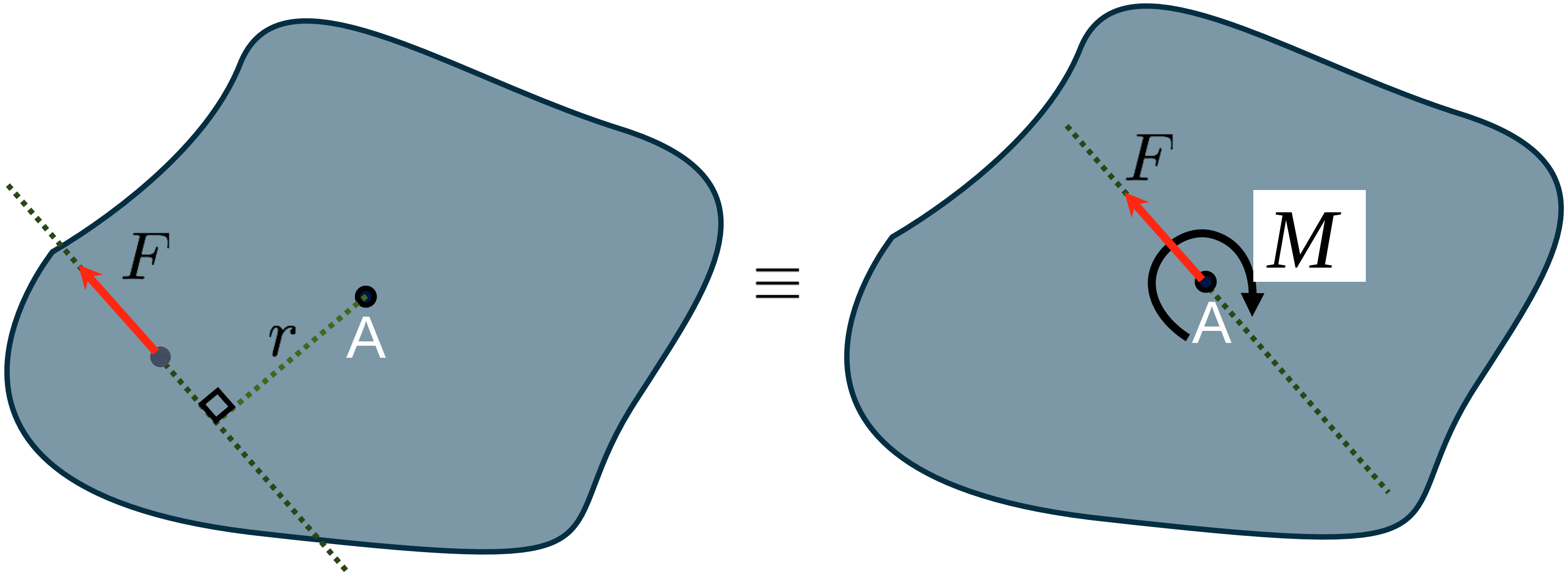


- Closely following the Vector Mechanics for Engineers, Beer and Johnston (Chapter 16), and Engineering Mechanics: Dynamics Hibbeler (Chapter 17)

Force and Torque

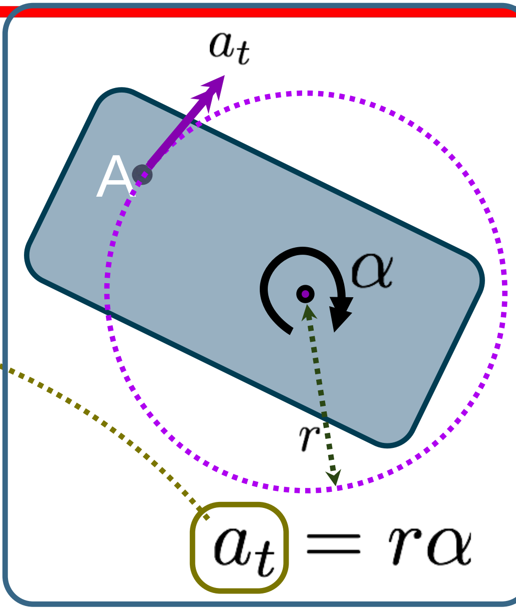
- A torque and a force provide the same angular acceleration when: $M = F \cdot r$

(r is the perpendicular distance between the force vector and the torque axis)

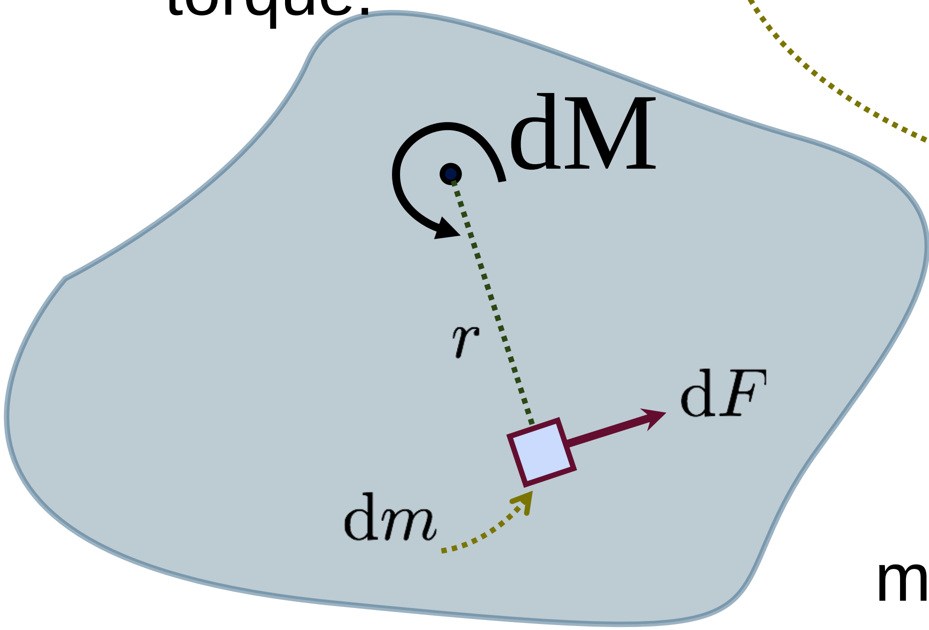


- A force on a rigid body is *dynamically equivalent* to the same force moved a perpendicular distance, r, plus an additional torque, $M = F \cdot r$

Moment of Inertia



- consider a small element, d*m*
- 2nd Law: $\mathrm{d}F = \mathrm{d}m \,.\, a$
- External torque: $\mathrm{dM} = dF \cdot r$



$$\mathrm{dM} = r^2 \alpha\, dm$$

$$\int \mathrm{dM} = \int r^2 \alpha\, dm$$

$$M = \left(\int r^2 dm\right) \cdot \alpha$$

moment of inertia: I

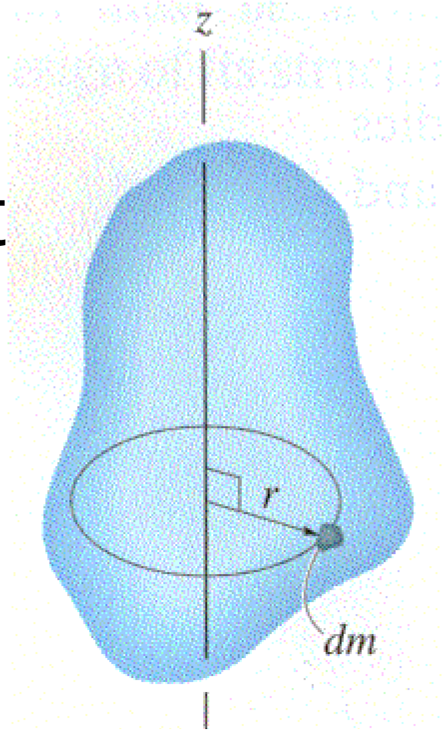
- Rotational 2nd Law: $\mathrm{M} = I\alpha$

Moment of Inertia

- This mass analog is called the *moment of inertia,* I, of the object

$$I = \int_m r^2 dm$$

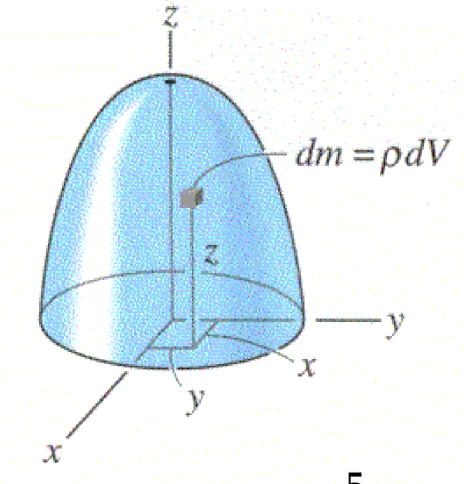
  - r = moment arm
  - SI units are kg m^2



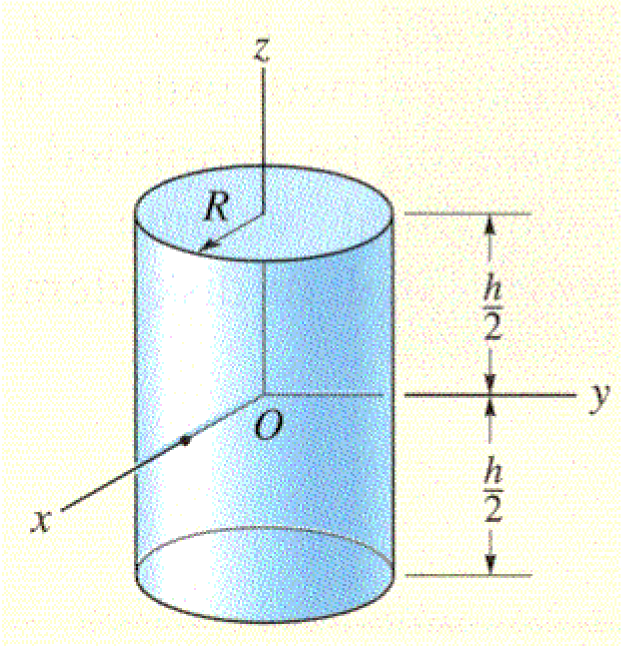
Using $dm = \rho\, dV$, where ρ is the volume density:

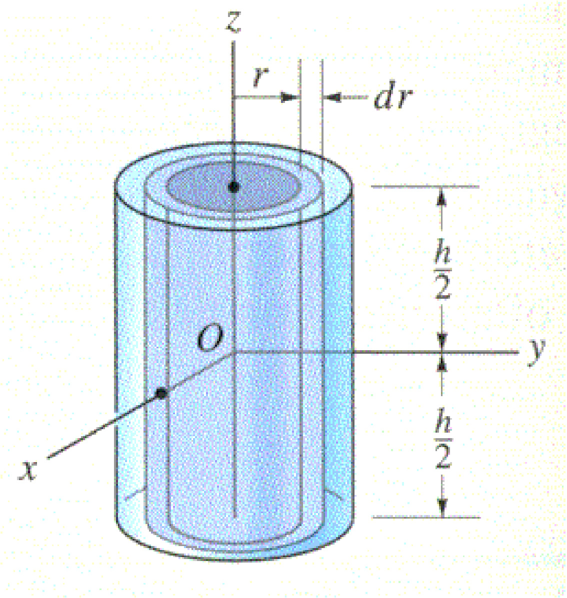
$$I = \int \rho\, r^2 dV$$

$$I = \rho \iiint r^2\, dx\, dy\, dz$$



Example





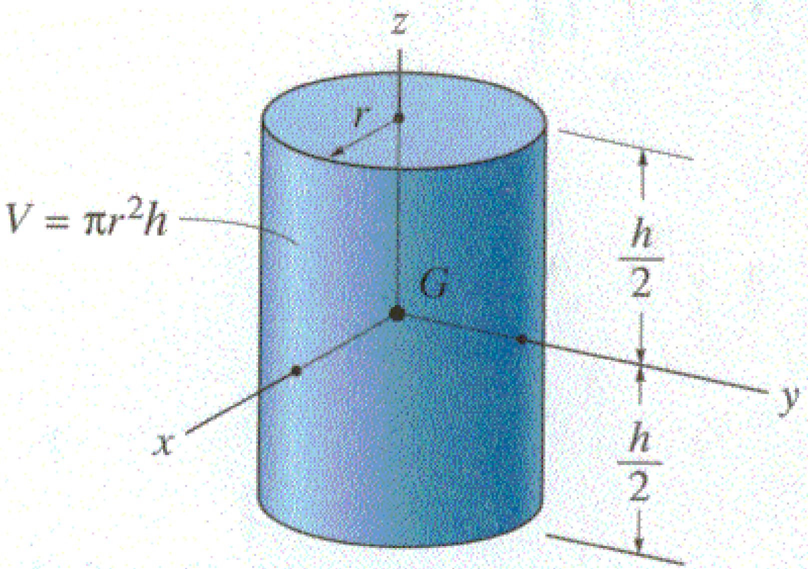
$$dm = \rho\, dV = \rho\,(2\pi\, r\, dr\ h)$$

$$I = \int_m r^2 dm = \rho\, 2\pi\, h \int_0^R r^3\, dr = \frac{\rho\pi}{2} R^4 h = \frac{1}{2} R^2 (\rho\pi R^2 h)$$

$$m = \rho\,\pi\, R^2 h$$

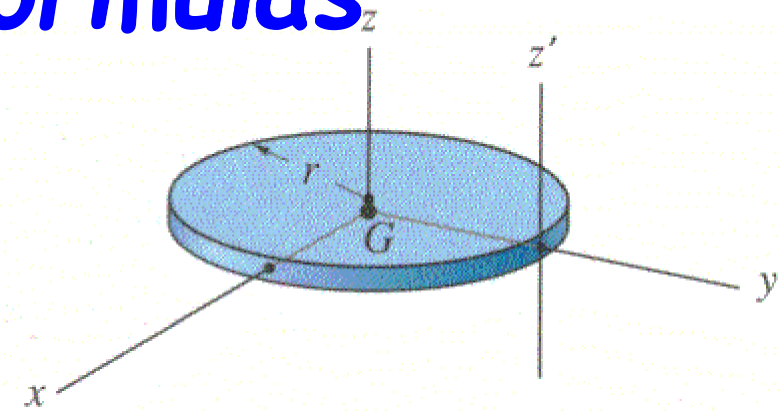
$$\boxed{I_z = \frac{1}{2} m R^2}$$

Useful Formulas



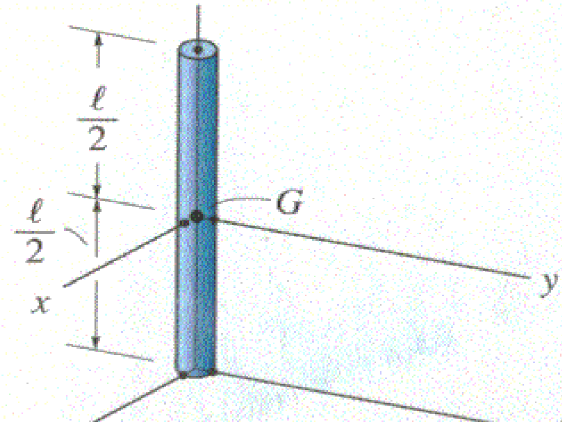
Cylinder

$$I_{xx} = I_{yy} = \frac{1}{12}m(3r^2 + h^2) \quad I_{zz} = \frac{1}{2}mr^2$$



Thin circular disk

$$I_{xx} = I_{yy} = \frac{1}{4}mr^2 \quad I_{zz} = \frac{1}{2}mr^2 \quad I_{z'z'} = \frac{3}{2}mr^2$$



Slender rod

$$I_{xx} = I_{yy} = \frac{1}{12}m\ell^2 \quad I_{x'x'} = I_{y'y'} = \frac{1}{3}m\ell^2 \quad I_{zz} = 0$$

$$I_{x'x'} = \frac{1}{3}ml^2$$

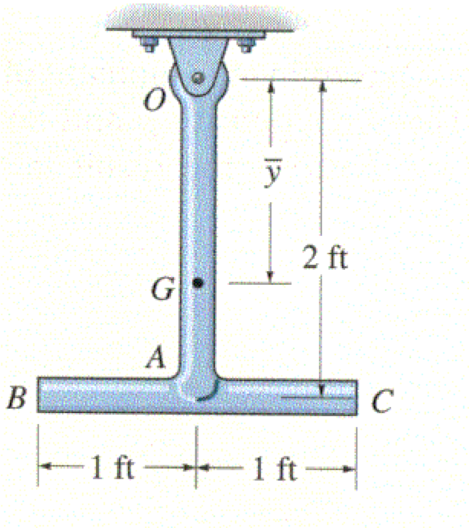
Radius of Gyration

Frequently tabulated data related to moments of inertia will be presented in terms of radius of gyration.

$$I = mk^2 \quad or \quad k = \sqrt{\frac{I}{m}}$$

Parallel Axis Theorem

- The moment of inertia about any axis parallel to and at distance d away from the axis that passes through the centre of mass is:

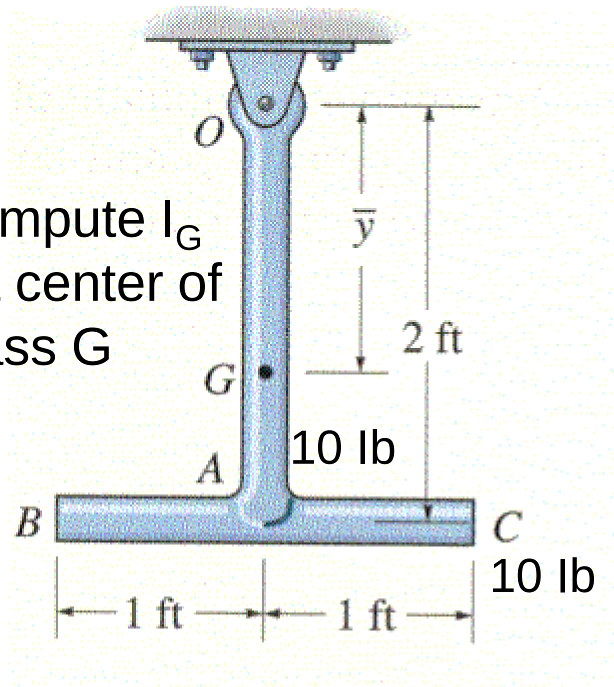


$$I_O = I_G + md^2$$

- Where
  - I_G= moment of inertia for mass centre G
  - m = mass of the body
  - d = perpendicular distance between the parallel axes.

Example: Parallel-axis theorem

Compute I_G wrt center of mass G



$$(I_{OA})_O = \frac{1}{3} m l^2 = \frac{1}{3} \left(\frac{10 lb}{32.2 \text{ft/s}^2}\right)(2\text{ft})^2 = 0.414 \text{ slug.ft}^2$$

$$(I_{BC})_O = \frac{1}{12} m l^2 + m d^2 = \frac{1}{12}\left(\frac{10}{32.2}\right)(2)^2 + \left(\frac{10}{32.2}\right)(2)^2$$

$$= 1.346 \quad \text{slug.ft}^2$$

$$I_O = 0.414 + 1.346 = 1.76 \text{ slug.ft}^2$$

$$I_O = I_G + m d^2$$

$$1.76 = I_G + \left(\frac{20}{32.2}\right)(1.5)^2$$

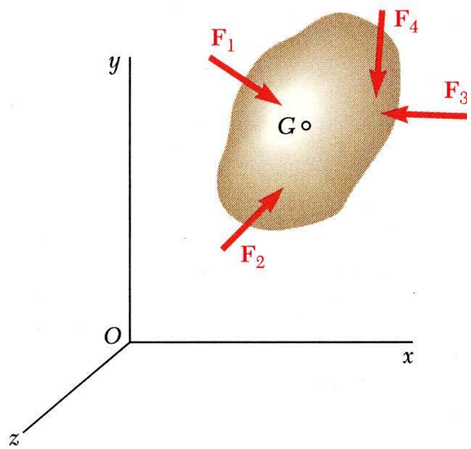
$$I_G = 0.362 \text{ slug.ft}^2$$

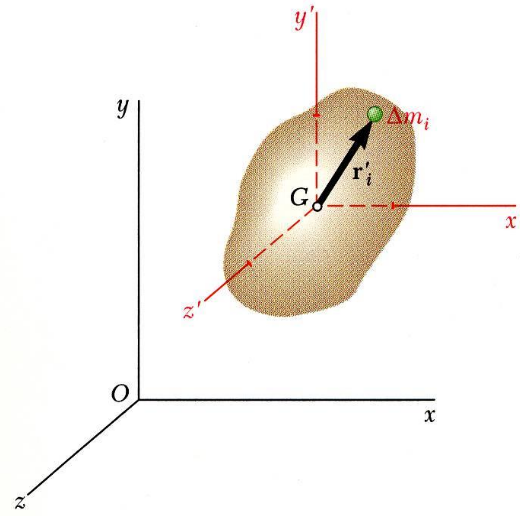
$$\bar{y} = \frac{}{\sum m}$$

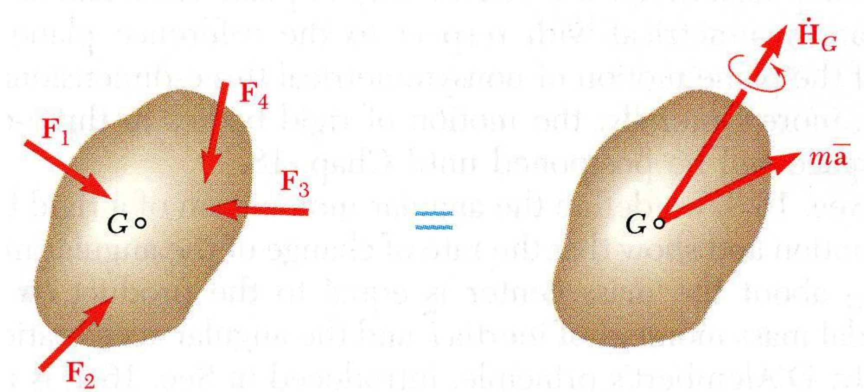
$$\bar{y} = \frac{\sum \tilde{y}_i m_i}{\sum m} = \frac{1(10/32.2) + 2(10/32.2)}{(10/32.2) + (10/32.2)} = 1.5 \text{ ft}$$

Note about units (not needed for any exam!): $1\, lb - m = \frac{1\, lb}{32.2\, ft/s^2} = \frac{1}{32.2} \text{slug}$

Equations of Motion for a Rigid Body





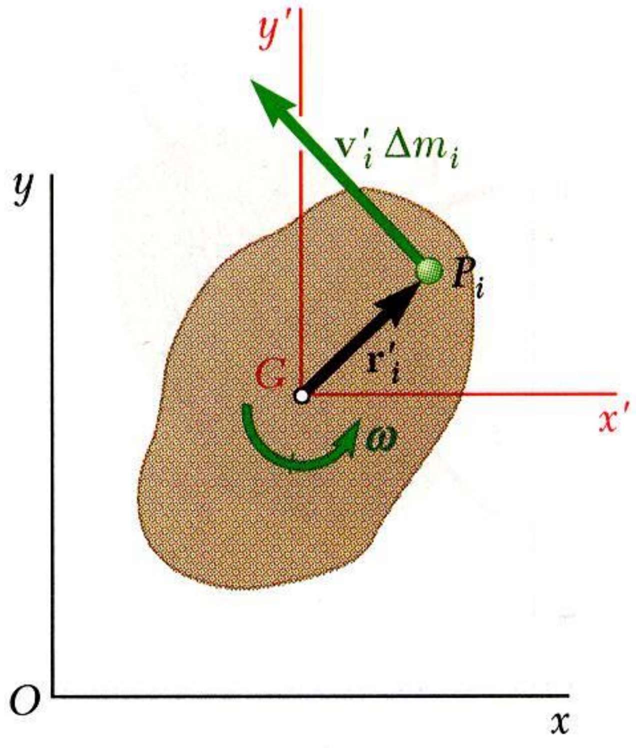


- Consider a rigid body acted upon by several external forces.
- Assume that the body is made of a large number of particles.
- For the motion of the mass center *G* of the body with respect to the Newtonian frame *Oxyz*, $\sum \vec{F} = m\vec{\bar{a}}$
- For the motion of the body with respect to the centroidal frame *Gx'y'z'*,

$$\sum \vec{M}_G = \dot{\vec{H}}_G$$

- System of external forces is equipollent to the system consisting of $m\vec{\bar{a}}$ and $\dot{\vec{H}}_G$.

Angular Momentum of a Rigid Body in Plane Motion



- Consider a rigid slab in plane motion.

- Angular momentum of the slab may be computed by

$$\vec{H}_G = \sum_{i=1}^{n}\left(\vec{r}_i{}' \times \vec{v}_i{}' \Delta m_i\right)$$
$$= \sum_{i=1}^{n}\left[\vec{r}_i{}' \times \left(\vec{\omega} \times \vec{r}_i{}'\right)\Delta m_i\right]$$
$$= \vec{\omega}\sum\left(r_i'^2 \Delta m_i\right)$$
$$= \bar{I}\vec{\omega}$$

- After differentiation,

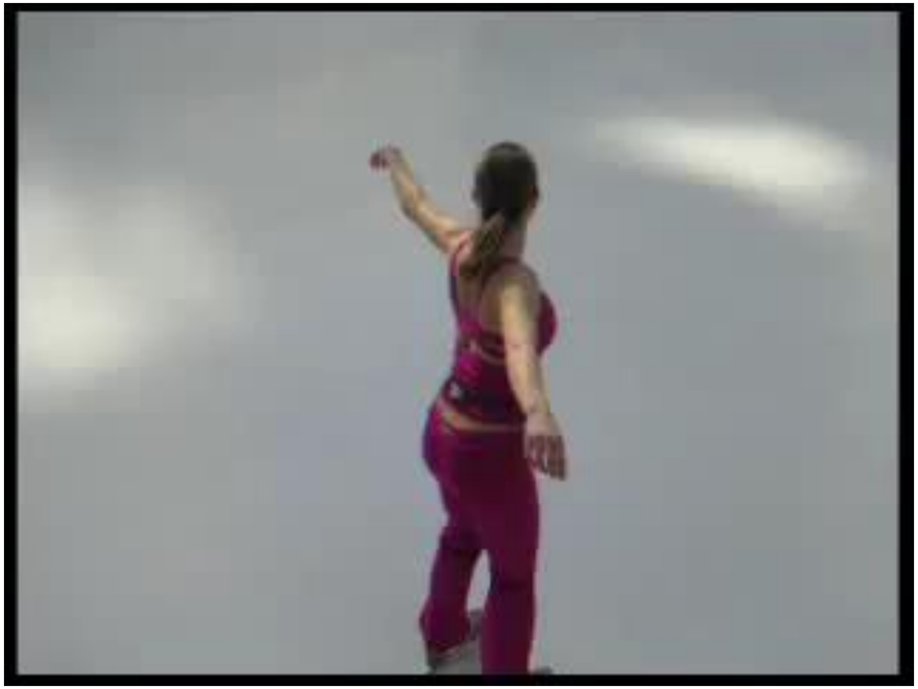
$$\dot{\vec{H}}_G = \bar{I}\dot{\vec{\omega}} = \bar{I}\vec{\alpha}$$

- Results are also valid for plane motion of bodies which are symmetrical with respect to the reference plane.

- Results are not valid for asymmetrical bodies or three-dimensional motion.

Angular Momentum $\dot{\vec{H}}_G = \bar{I}\dot{\vec{\omega}} = \bar{I}\vec{\alpha}$

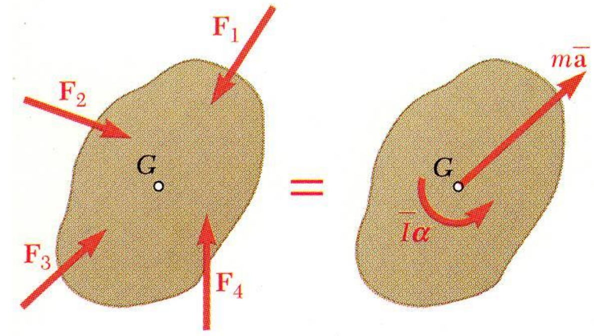
- Conservation of Angular Momentum: if there is no external torque, angular momentum stays the same.
- A spinning skater can *increase* their angular velocity by reducing their *I* value

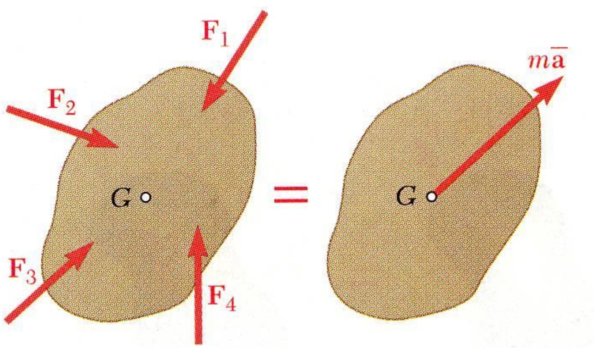


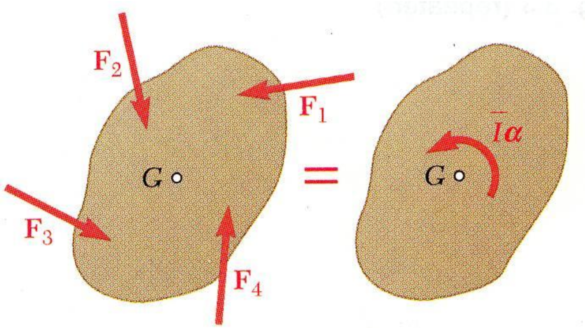
- Angular momentum has *direction* (along the axis of rotation)

  to change that direction is harder the more angular momentum the object has.

Plane Motion of a Rigid Body: D'Alembert's Principle







- Motion of a rigid body in plane motion is completely defined by the resultant and moment resultant about *G* of the external forces.

$$\sum F_x = m\bar{a}_x \quad \sum F_y = m\bar{a}_y \quad \sum M_G = \bar{I}\alpha$$

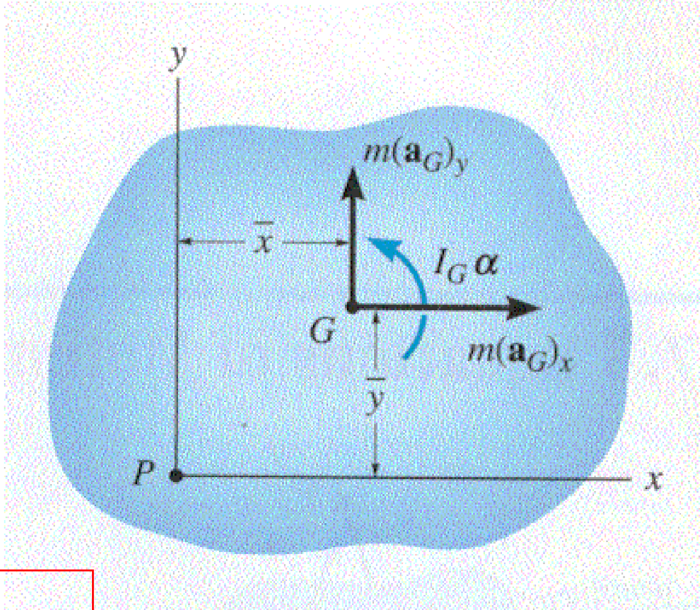
- The external forces and the collective effective forces of the slab particles are *equipollent* (reduce to the same resultant and moment resultant) and *equivalent* (have the same effect on the body).
- *d'Alembert's Principle*: The external forces acting on a rigid body are equivalent to the effective forces of the various particles forming the body.
- ***The most general motion of a rigid body that is symmetrical with respect to the reference plane can be replaced by the sum of a translation and a centroidal rotation.***

Kinetic Moment

$$\sum F_x = m(a_G)_x$$

$$\sum F_y = m(a_G)_y$$

$$\sum M_G = I_G \alpha$$



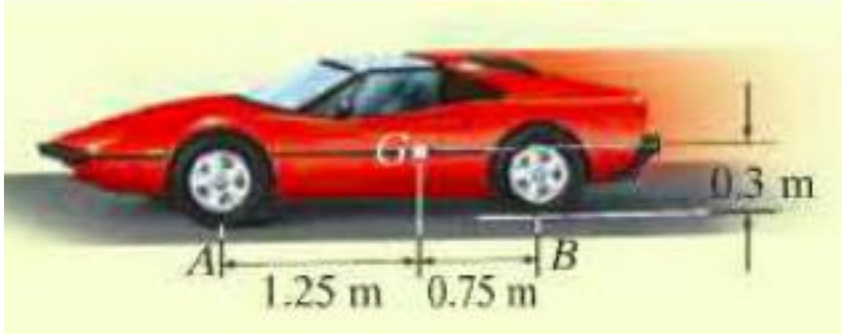
Kinetic diagram

$$\sum M_P = \sum (\mathcal{M}_k)_P$$

$$\sum M_P = \sum (\mathcal{M}_k)_P = \bar{x}\, m(a_G)_y - \bar{y}\, m(a_G)_x + I_G \alpha$$

Moments of the external forces about any point P in the body are equivalent to the sum of the kinetic moments of the components of $m\boldsymbol{a}_G$ plus the kinetic moment $I_G\alpha$

Example



$m = 2Mg$

$\mu_k = 0.25$

back wheels slipping

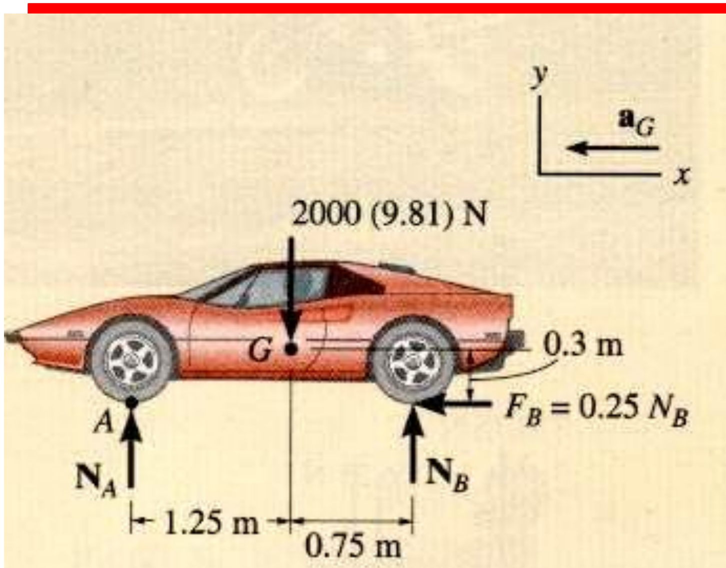
$a = ?$

The car shown has a mass of 2 Mg and a center of mass at *G.* Determine the acceleration if **the rear ·'driving" wheels are always slipping**, *whereas the front wheels are free to rotate*. Neglect the mass m of the wheels.

The coefficient of kinetic friction between the wheels and the road is μ_k = 0.25.

Hint: The rear-wheel frictional force pushes the car forward and since slipping occurs $F_B = 0.25 N_B$. With negligible wheel mass, Iα=0 and the required force to turn the front wheel is zero.

Example

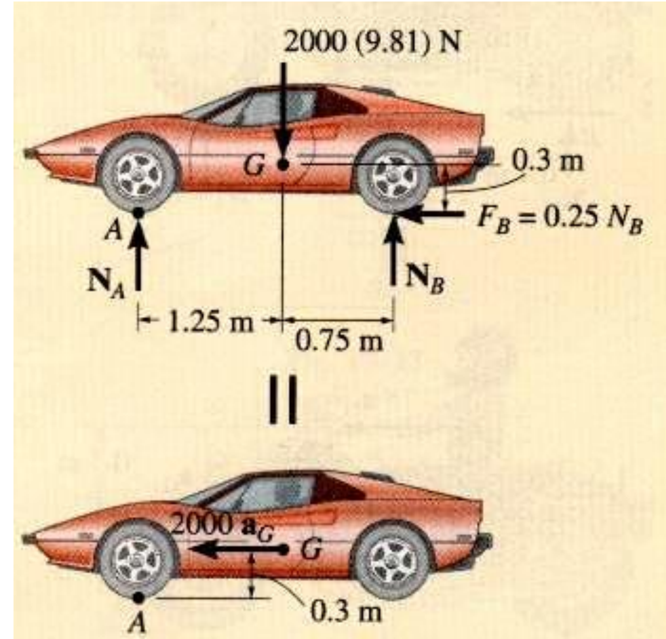


$m = 2Mg$

$\mu_k = 0.25$

back wheels slipping

$a = ?$



$$\rightarrow \sum F_x = m(a_G)_x; \quad -0.25N_B = -2000\,a_G$$

$$\uparrow \sum F_y = m(a_G)_y; \quad N_A + N_B - 2000(9.81) = 0$$

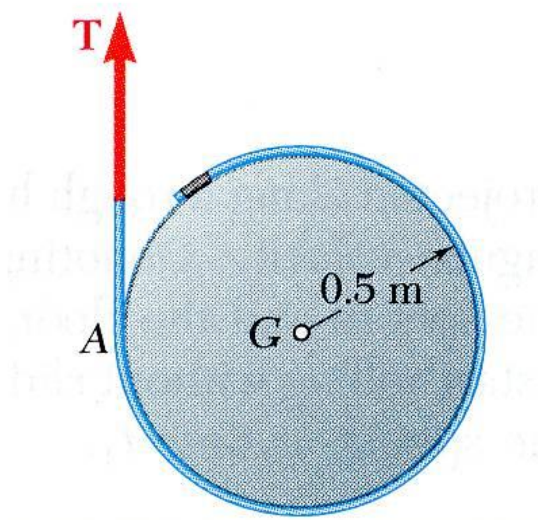
$$\circlearrowleft \sum M_G = 0 \qquad -N_A(1.25) + N_B(0.75) - 0.25N_B(0.3) = 0$$

$$N_A = 6.88\ kN,\ \ N_B = 12.7\ kN, \quad a_G = 1.59\ m/s^2$$

Alternatively: can use this with the 1st Eq. above (solve for a_G)

$$\circlearrowleft \sum M_A = \sum(\mathcal{M}_k)_A \qquad N_B(2) - 2000(9.81)(1.25) = 2000 \cdot a_G \cdot (0.3)$$

Sample Problem



A cord is wrapped around a disk of mass 15 kg. The cord is pulled upwards with a force T = 180 N.

Determine:

(a) the acceleration of the center of the disk,

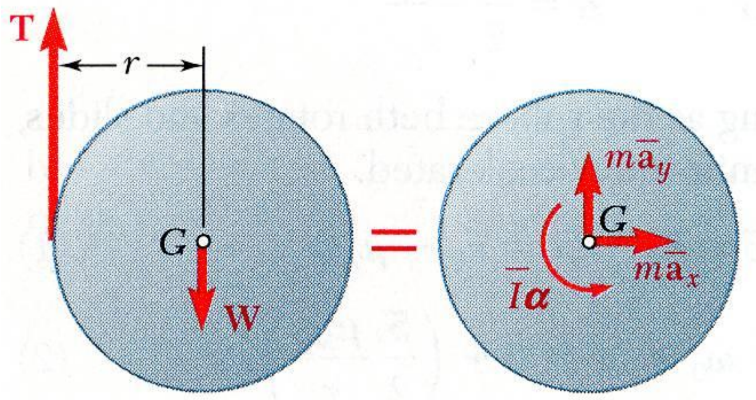
(b) the angular acceleration of the disk, and

(c) the acceleration of the cord.

SOLUTION:

- Draw the free-body-diagram equation expressing the equivalence of the external and effective forces on the disk.
- Solve the three corresponding scalar equilibrium equations for the horizontal, vertical, and angular accelerations of the disk.
- Determine the acceleration of the cord by evaluating the tangential acceleration of the point *A* on the disk.

Sample Problem



SOLUTION:

- Draw the free-body-diagram equation expressing the equivalence of the external and effective forces on the disk.
- Solve the three scalar equilibrium equations.

$$\xrightarrow{+} \sum F_x = \sum (F_x)_{eff}$$

$$0 = m\bar{a}_x$$

$$\boxed{\bar{a}_x = 0}$$

$$+\uparrow \sum F_y = \sum (F_y)_{eff}$$

$$T - W = m\bar{a}_y$$

$$\bar{a}_y = \frac{T - W}{m} = \frac{180\,\text{N} - (15\,\text{kg})(9.81\,\text{m/s}^2)}{15\,\text{kg}}$$

$$\boxed{\bar{a}_y = 2.19\,\text{m/s}^2 \uparrow}$$

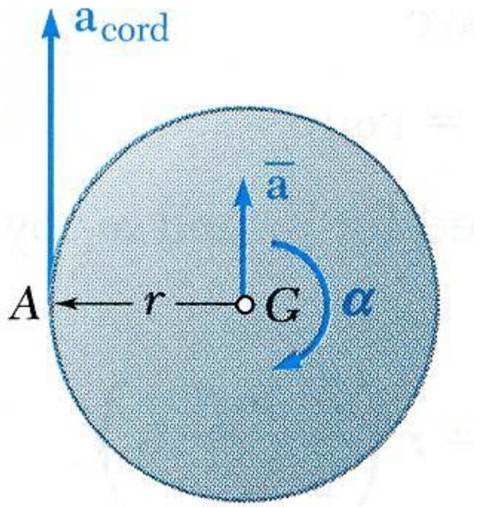
$$+\circlearrowleft \sum M_G = \sum (M_G)_{eff}$$

$$-Tr = \bar{I}\alpha = \left(\tfrac{1}{2}mr^2\right)\alpha$$

$$\alpha = -\frac{2T}{mr} = -\frac{2(180\,\text{N})}{(15\,\text{kg})(0.5\,\text{m})}$$

$$\boxed{\alpha = 48.0\,\text{rad/s}^2 \circlearrowright}$$

Sample Problem



- Determine the acceleration of the cord by evaluating the tangential acceleration of the point *A* on the disk.

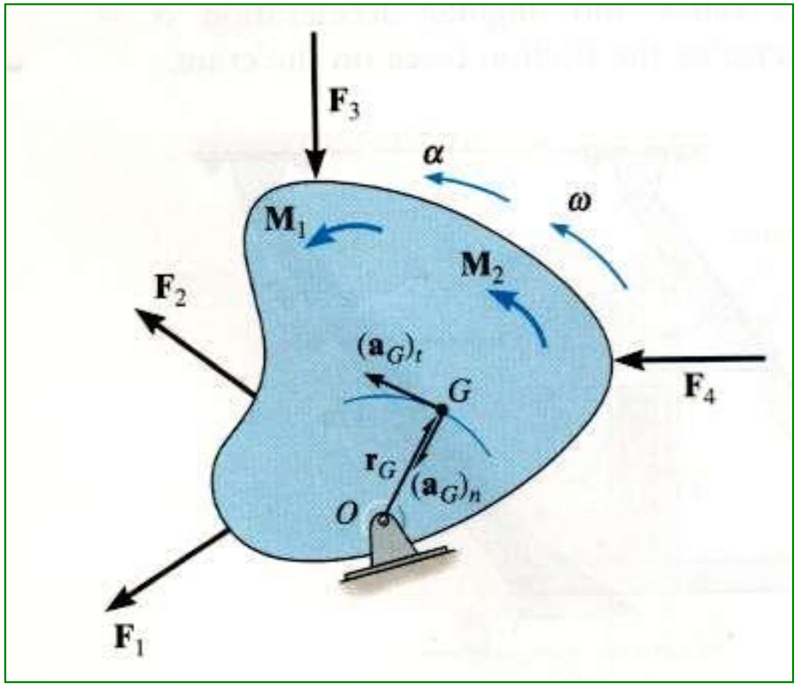
$$\vec{a}_{cord} = (a_A)_t = \bar{a} + (a_{A/G})_t$$
$$= 2.19\,\mathrm{m/s^2} + (0.5\,\mathrm{m})(48\,\mathrm{rad/s^2})$$

$$a_{cord} = 26.2\,\mathrm{m/s^2} \uparrow$$

$$\bar{a}_x = 0 \qquad \bar{a}_y = 2.19\,\mathrm{m/s^2} \uparrow$$

$$\alpha = 48.0\,\mathrm{rad/s^2} \circlearrowright$$

Rotation About a Fixed Axis



$$\sum F_n = m(a_G)_n = m\omega^2 r_G$$

$$\sum F_t = m(a_G)_t = m r_G \alpha$$

$$\sum M_G = I_G \alpha$$

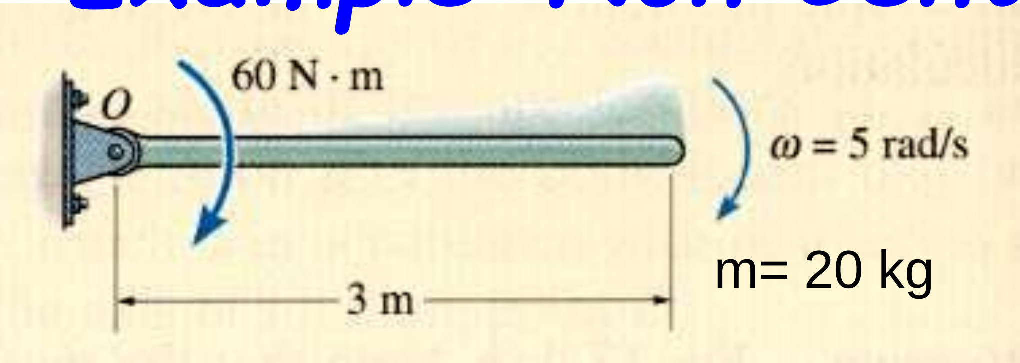
$$\sum M_O = \sum (\mathcal{M}_k)_O = I_G \alpha + r_G m(a_G)_t$$

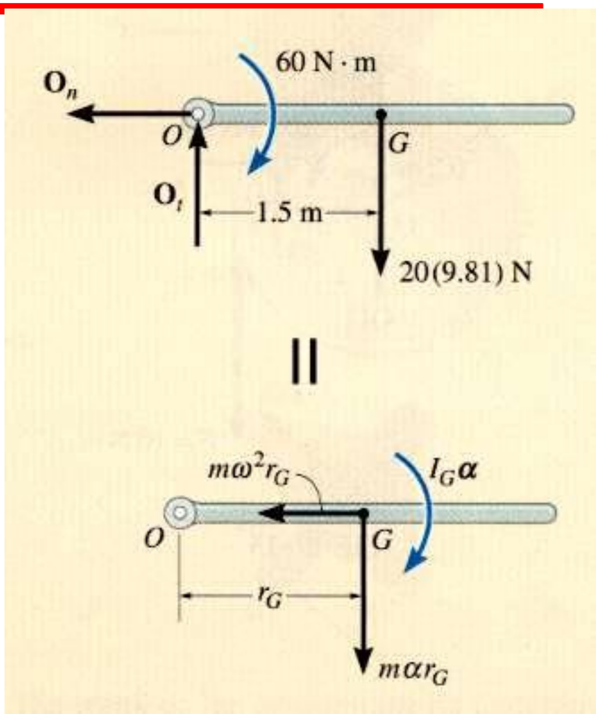
$$= I_G \alpha + r_G m \cdot r_G \alpha$$

$$= (I_G + m r_G^2)\alpha$$

$$= I_O \alpha$$

Example-Non Centroidal Rotation





$$\leftarrow \sum F_n = m\omega^2 r_G; \quad O_n = (20kg)(5rad/s)^2(1.5m)$$

$$\downarrow \sum F_t = m\alpha\, r_G; \quad -O_t + 20(9.81) = (20)(\alpha)(1.5m)$$

$$\sum M_G = I_G\alpha; \quad O_t(1.5) + 60N.m = [\frac{1}{12}(20)(3)^2]\alpha$$

$$O_n = 750N \quad O_t = 19.0N \quad \alpha = 5.90\ rad/s^2$$

Alternative moment Eqs:

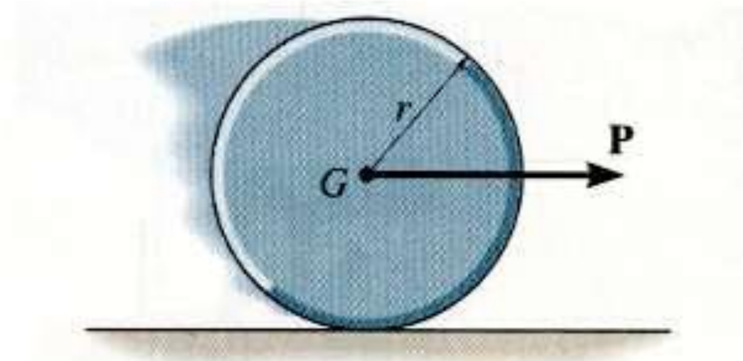
$$\sum M_o = \sum (M_k)_O; \quad 60N.m + 20(9.81)(1.5) = [\frac{1}{12}(20)(3)^2]\alpha + [20(\alpha)(1.5)(1.5)]$$

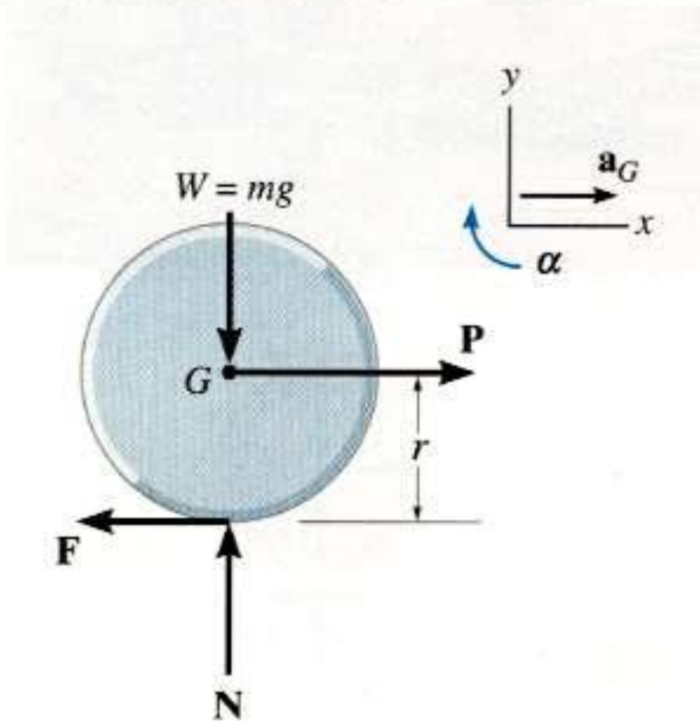
$$\alpha = 5.9\ rad/s^2$$

$$\sum M_o = I_o\alpha; \quad 60N.m + 20(9.81)(1.5) = [\frac{1}{3}(20)(3)^2]\alpha$$

$$\alpha = 5.9\ rad/s^2$$

Frictional Rolling Problems





- rolls without slipping
- slides as it rolls

$$\rightarrow \sum F_x = m(a_G)_x \qquad P - F = ma_G$$

$$\uparrow \sum F_y = m(a_G)_y \qquad N - mg = 0$$

$$\sum M_G = I_G \alpha \qquad Fr = I_G \alpha$$

(Rolling) No Slipping

$$a_G = \alpha r$$

valid if

$$F \leq \mu_s N$$

If

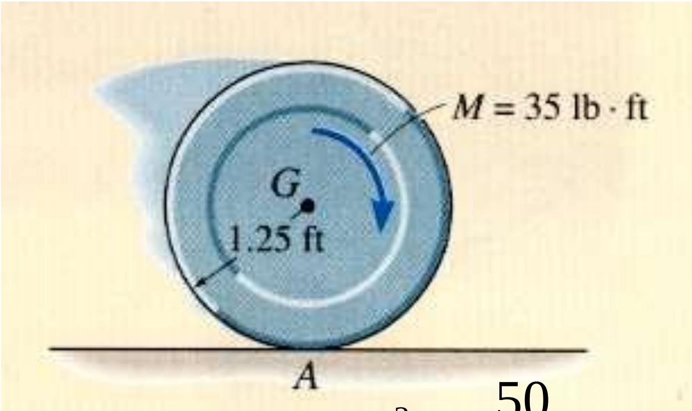
$$F > \mu_s N$$

→

Slipping

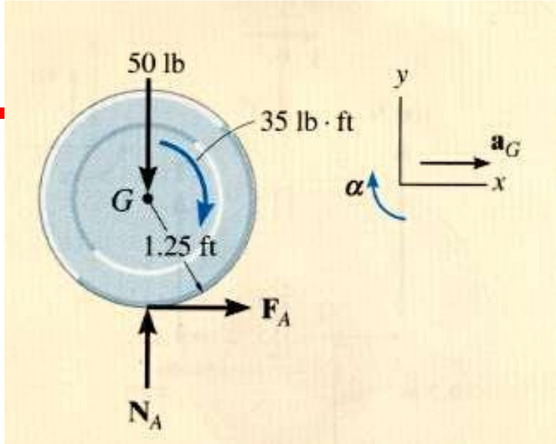
$$F = \mu_K N$$

Example



W= 50-lb
radius of gyration
k_G=0.70ft.
a_G=?.

$$\mu_s = 0.3 \text{ and } \mu_k = 0.25$$



The 50-lb wheel has a radius of gyration k_G=0.70 ft. If a 35-lb · ft couple moment is applied to the wheel, determine the acceleration of its mass center G. The coefficients of static and kinetic friction between the wheel and the plane at *A* are μ_s=0.3 and μ_k=0.25, respectively.

$$I_G = mk_G^{\,2} = \frac{50}{32.2}(0.7)^2 = 0.761\ slug \cdot ft^2$$

$$\rightarrow \sum F_x = m(a_G)_x \qquad F_A = \frac{50}{32.2} a_G$$

$$\uparrow \sum F_y = m(a_G)_y \qquad N_A - 50lb = 0$$

$$\circlearrowleft \sum M_G = I_G\,\alpha \qquad 35 - 1.25\,F_A = 0.761 \cdot \alpha$$

$$1)\ Rolling\ \ without\ slipping \qquad a_G = 1.25\,\alpha$$

$$N_A = 50\,lb \qquad F_A = 21.3lb$$

$$\alpha = 11.0\ rad/s^2 \qquad a_G = 13.7\,ft/s^2$$

$$\mu_s N(0.3*50=15)\ \le\ F_A = 21.3 \quad \text{slipping will take place}$$

$$2)\ F_A = \mu_k N = 0.25(50) = 12.5\,lb. \text{ Now substitute in } 35 - 1.25\,F_A = 0.761 \cdot \alpha \ \ \therefore\ \alpha = 25.5\ rad/s^2$$

$$Also\ \ from\ F_A = \frac{50}{32.2} a_G \ \ we\ obtain\ a_G = 8.05\,ft/s^2$$