

PLANE KINETICS OF RIGID BODIES

The kinetics of rigid bodies treats the relationships between the external forces acting on a body and the corresponding translational and rotational motions of the body.

In the kinetics of the particle, we found that two force equations of motion were required to define the plane motion of a particle whose motion has two linear components.

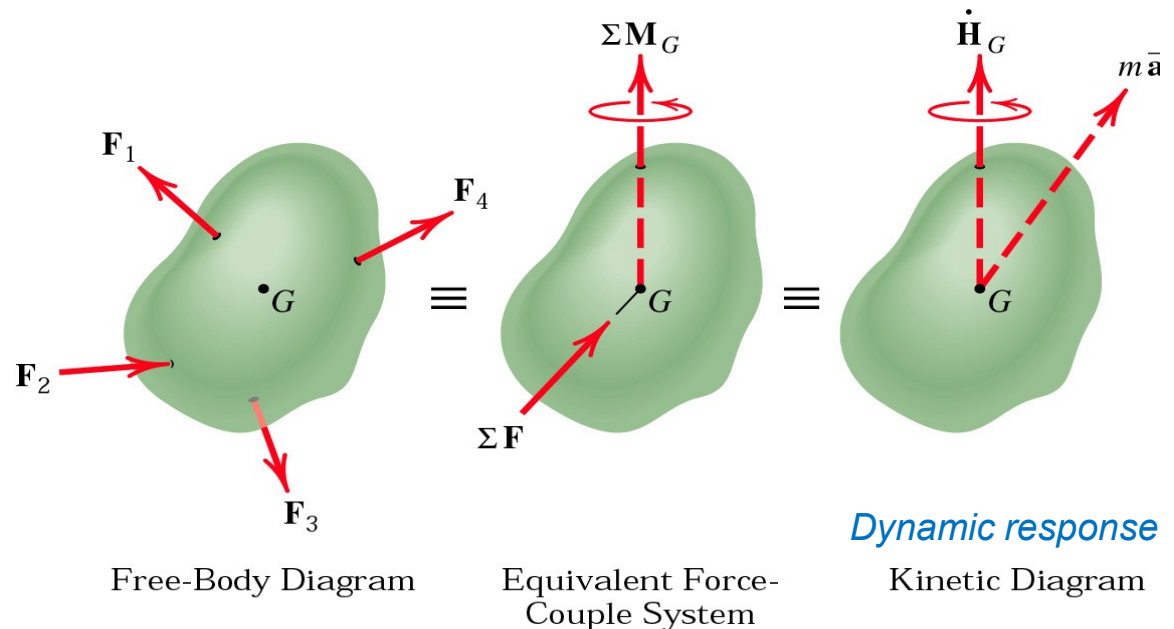
For the plane motion of a rigid body, an additional equation is needed to specify the state of rotation of the body.

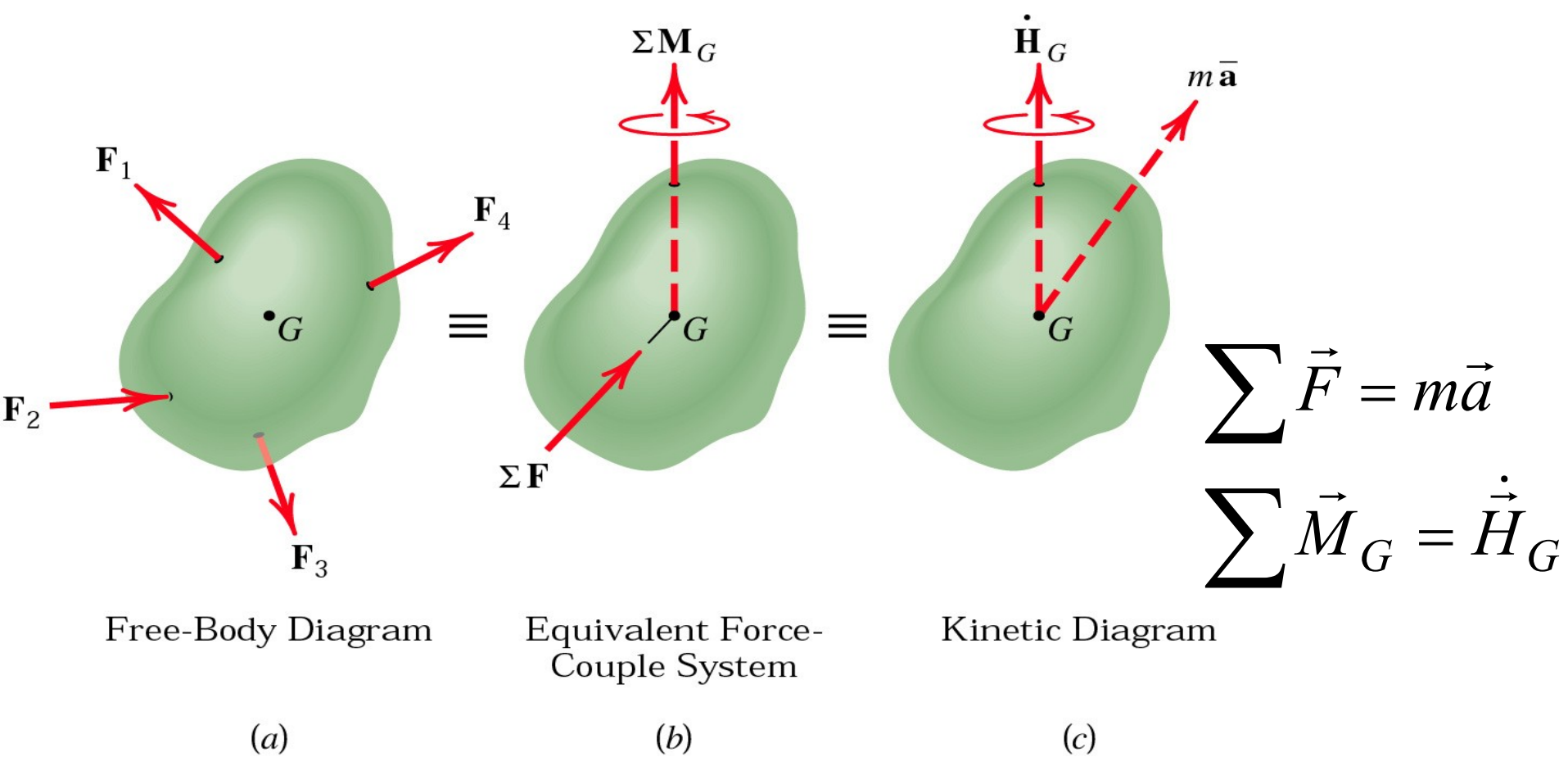
Thus, two force and one moment equations or their equivalent are required to determine the state of rigid-body plane motion.

GENERAL EQUATIONS OF MOTION

In our study of Statics, a general system of forces acting on a rigid body may be replaced by a resultant force applied at a chosen point and a corresponding couple.

By replacing the external forces by their equivalent force-couple system in which the resultant force acts through the mass center, we may visualize the action of the forces and the corresponding dynamic response.





a) Relevant free-body diagram (FBD)

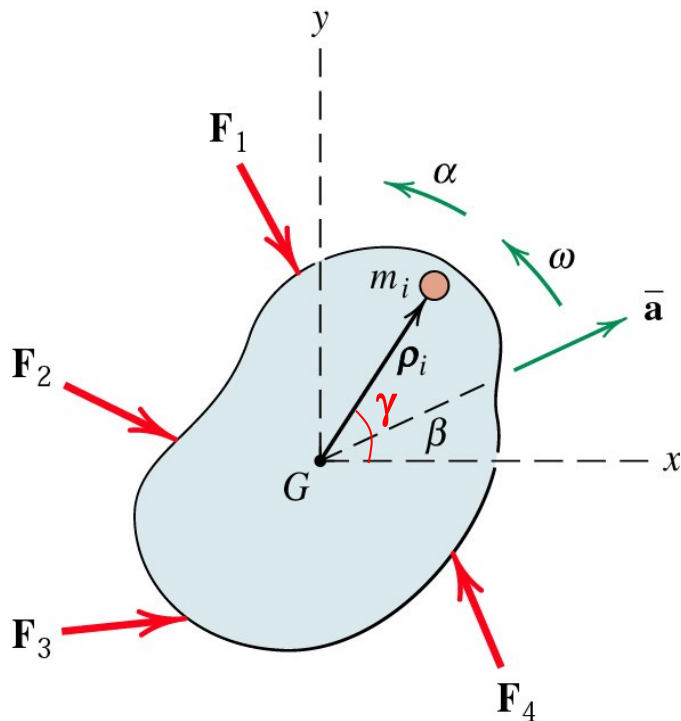
b) Equivalent force-couple system with resultant force applied through G

c) Kinetic diagram which represents the resulting dynamic effects

PLANE MOTION EQUATIONS

Figure shows a rigid body moving with plane motion in the x-y plane. The mass center **G** has an acceleration $\bar{\mathbf{a}}$ and the body has an angular velocity $\vec{\omega} = \omega \vec{k}$ and an angular acceleration $\vec{\alpha} = \alpha \vec{k}$.

The angular momentum about the mass center for the representative particle m_i :



$$\vec{H}_{Gi} = \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i$$

$\vec{\rho}_i$: position vector relative to **G** of particle m_i

$\dot{\vec{\rho}}_i = \vec{v}_i = \vec{\omega} \times \vec{\rho}_i$ Velocity of particle m_i

$\vec{\rho}_i = \rho_i (\cos \gamma \vec{i} + \sin \gamma \vec{j})$ $\vec{\omega} = \omega \vec{k}$

$$\vec{H}_{Gi} = \vec{\rho}_i \times m_i (\vec{\omega} \times \vec{\rho}_i)$$

The angular momentum about the mass center for the rigid body:

$$\vec{H}_G = \sum \vec{H}_{Gi} = \sum \vec{\rho}_i \times m_i (\vec{\omega} \times \vec{\rho}_i) = \sum \rho_i (\cos \gamma \vec{i} + \sin \gamma \vec{j}) \times m_i \underbrace{[\omega \vec{k} \times \rho_i (\cos \gamma \vec{i} + \sin \gamma \vec{j})]}_{m_i \omega \rho_i \cos \gamma \vec{j} - m_i \omega \rho_i \sin \gamma \vec{i}}$$

$$\vec{H}_G = \sum m_i \rho_i^2 \omega \underbrace{(\cos^2 \gamma + \sin^2 \gamma)}_1 \vec{k} = \sum \underbrace{(m_i \rho_i^2)}_{\bar{I}} \omega \vec{k}$$

$$\vec{H}_G = \bar{I} \omega \vec{k} \quad \Rightarrow \quad \vec{H}_G = \bar{I} \vec{\omega}$$

\bar{I} is a constant property of the body and is a measure of the rotational inertia or resistance to change in rotational velocity due to the radial distribution of mass around the z-axis through G.

(MASS MOMENT OF INERTIA of the body the about z-axis through G)

$$\bar{I} = \sum_{i=1}^{n \rightarrow \infty} m_i \rho_i^2 = \int_{m_i} \rho^2 dm \quad [\bar{I}] = \text{kgm}^2$$

$$\sum \vec{M}_G = \dot{\vec{H}}_G \quad \Rightarrow \quad \sum M_G = \frac{H_G}{dt} = \frac{d}{dt} (\bar{I} \omega) = \bar{I} \underbrace{\frac{d\omega}{dt}}_{\alpha} = \bar{I} \alpha$$

$$\boxed{\sum M_G = \bar{I} \alpha}$$

Analysis Procedure

In the solution of force-mass-acceleration problems for the plane motion of rigid bodies, the following steps should be taken after the conditions and requirements of the problem are clearly in mind.

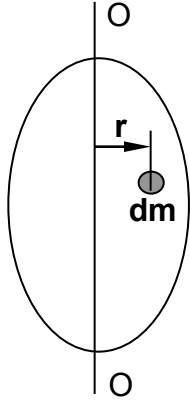
1) Kinematics : First, identify the class of motion and then solve any needed linear or angular accelerations which can be determined from given kinematic information.

2) Diagrams: Always draw the complete free-body diagram and kinetic diagram.

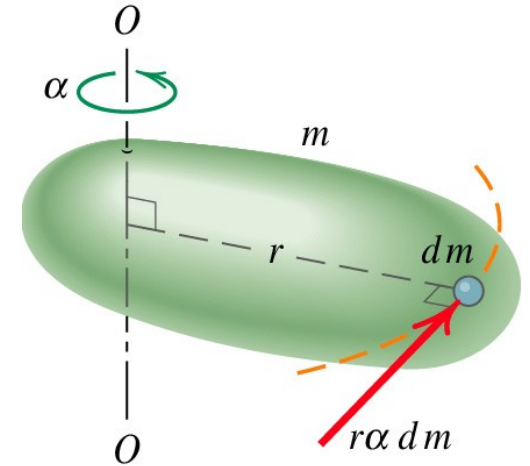
3) Apply the three equations of motion. ($\sum \vec{F} = m\bar{a}$, $\sum M_G = \bar{I}\alpha$)

Mass Moments of Inertia

Mass moment of inertia of dm about the axis OO , dI :



$$dI = r^2 dm$$



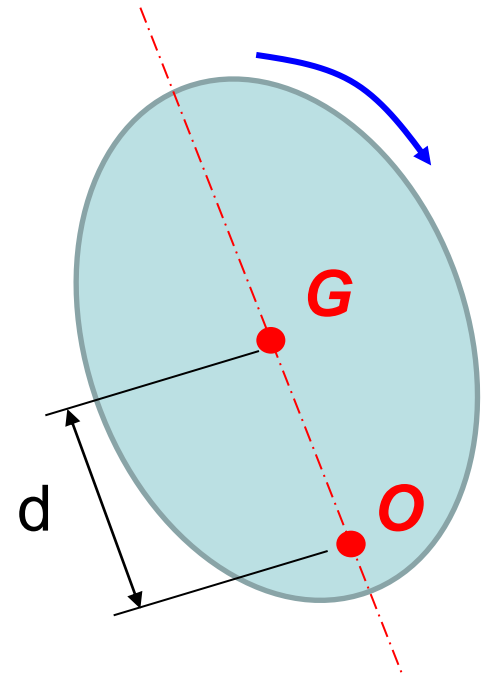
Total mass moment of inertia of mass m :

$$I = \int dI = \int r^2 dm \quad I \text{ is always positive and its units is } \text{kg}\cdot\text{m}^2.$$

Transfer of axes for mass moment of inertia:

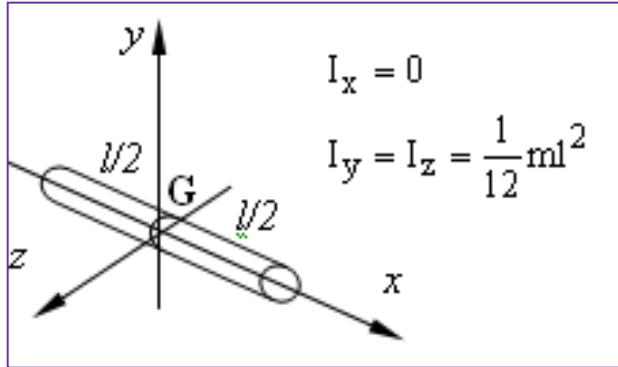
If the moment of inertia of a body is known about an axis passing through the mass center, it may be determined easily about any parallel axis.

$$I_O = \bar{I} + md^2$$

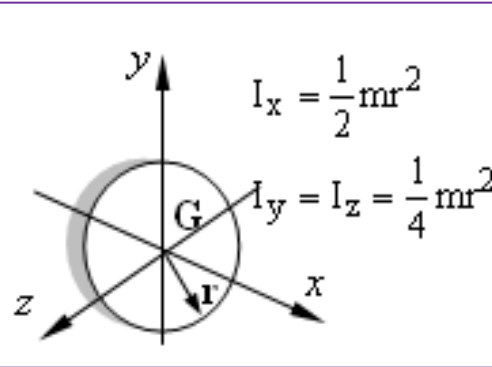


Mass Moments of Inertia for Some Common Geometric Shapes

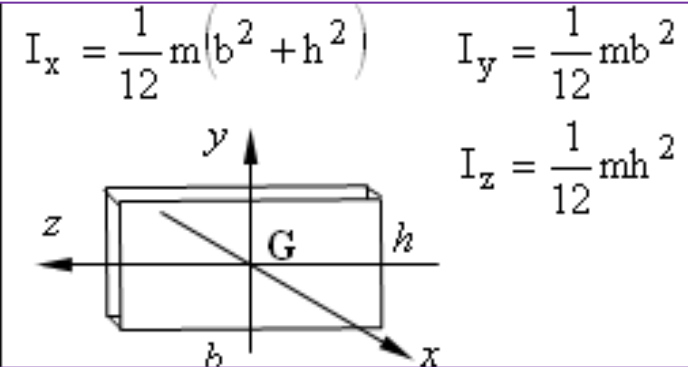
Thin bar



Thin circular plate



Thin rectangular plate



Radius of Gyration, k : The radius of gyration k of a mass m about an axis for which the moment of inertia is I is defined as

$$I = k^2 m \Rightarrow k = \sqrt{\frac{I}{m}}$$

Thus k is a measure of the distribution of mass of a given body about the axis in question, and its definition is analogous to the definition of the radius of gyration for area moments of inertia.

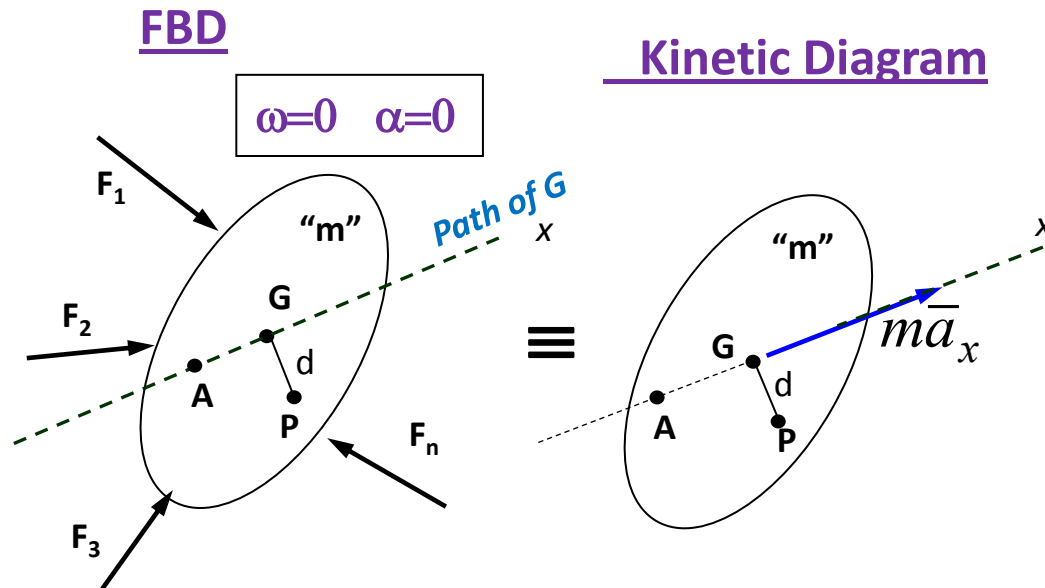
The moment of inertia of a body about a particular axis is frequently indicated by specifying the mass of the body and the radius of gyration of the body about the axis.

When the expressions for the radii of gyration are used, the equation becomes

$$k^2 = \bar{k}^2 + d^2$$

1) TRANSLATION

a) Rectilinear Translation:



$$\left(\sum F \right)_{\text{external force}} = m\bar{a}$$

$$\left(\sum F_x \right)_{e.f.} = m\bar{a}_x$$

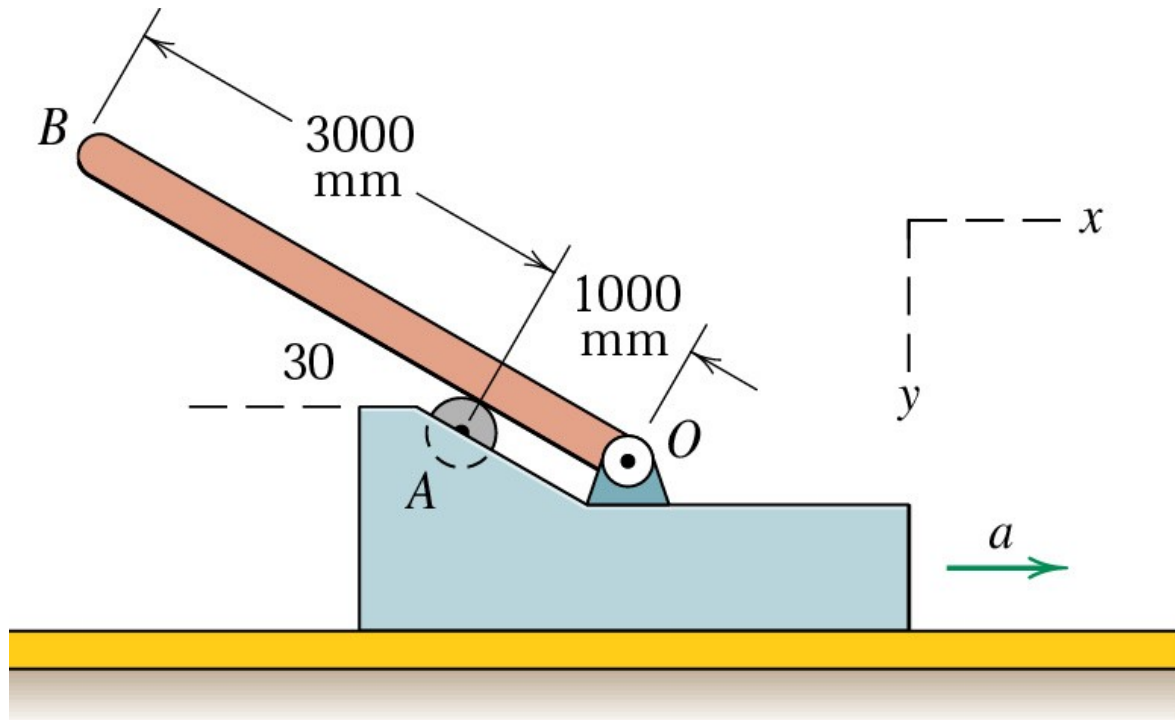
$$\left(\sum M_G \right)_{e.f.} = 0$$

$$\left(\sum M_A \right)_{e.f.} = 0$$

$$\left(\sum M_P \right)_{e.f.} = m\bar{a}_x d$$

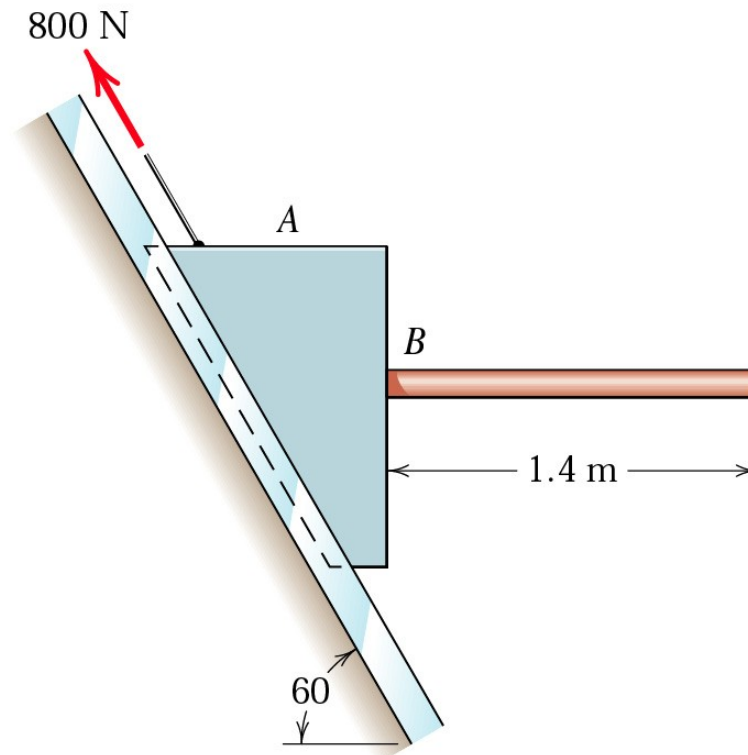
PROBLEMS

1. The uniform 30-kg bar OB is secured to the accelerating frame in the 30° position from the horizontal by the hinge at O and roller at A . If the horizontal acceleration of the frame is $a=20 \text{ m/s}^2$, compute the force F_A on the roller and the x - and y -components of the force supported by the pin at O .



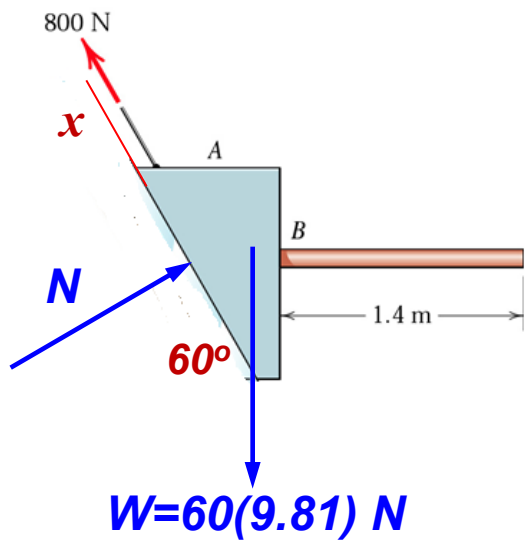
PROBLEMS

2. The block A and attached rod have a combined mass of 60 kg and are confined to move along the 60° guide under the action of the 800 N applied force. The uniform horizontal rod has a mass of 20 kg and is welded to the block at B . Friction in the guide is negligible. Compute the bending moment M exerted by the weld on the rod at B .

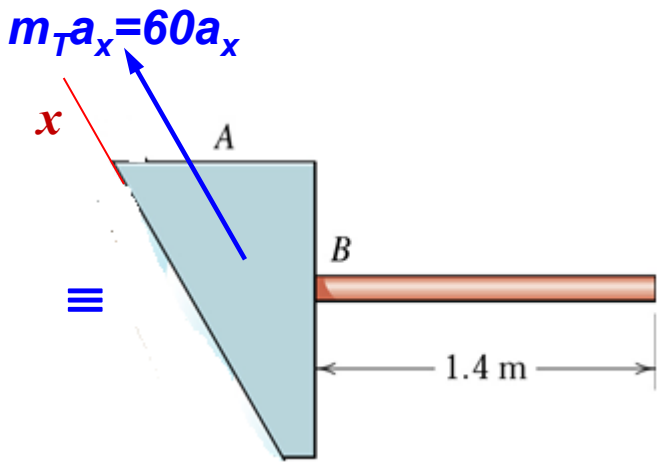


SOLUTION

FBD



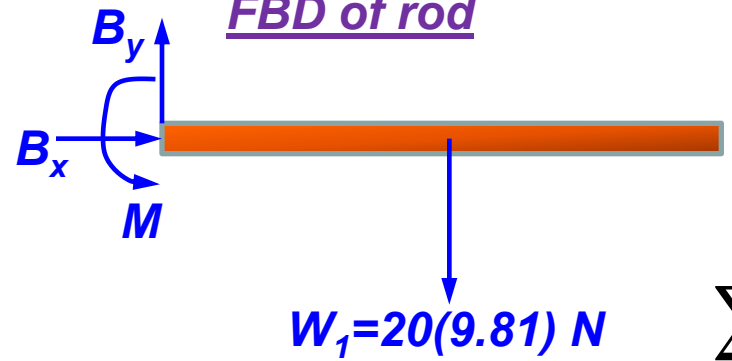
Kinetic Diagram



$$\sum F_x = m a_x \Rightarrow 800 - 60(9.81) \sin 60 = 60 a_x$$

$$a_x = 4.84\text{ m/s}^2$$

FBD of rod



KD of rod

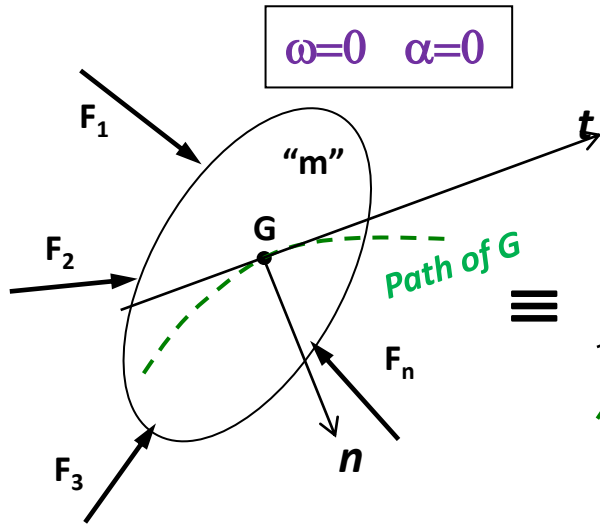


$$\sum M_B = m a_x d \Rightarrow M - 20(9.81)(0.7) = (20)(4.94)(0.7 \sin 60)$$

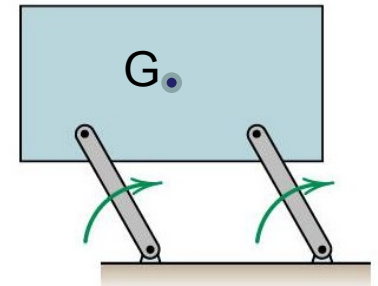
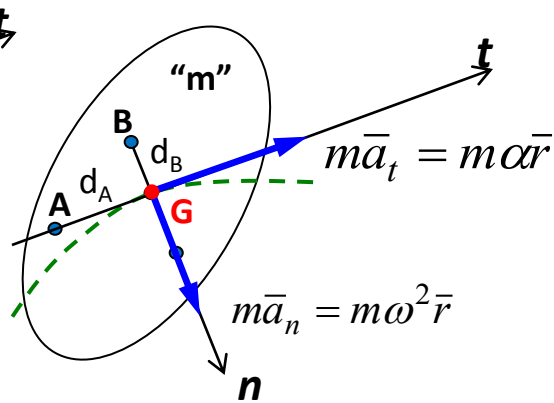
$$M = 196\text{ m/s}^2$$

b) Curvilinear Translation:

FBD



Kinetic Diagram



$$\left(\sum F_n\right)_{e.f.} = m\bar{a}_n$$

$$\left(\sum M_G\right)_{e.f.} = 0 \quad (\alpha = 0)$$

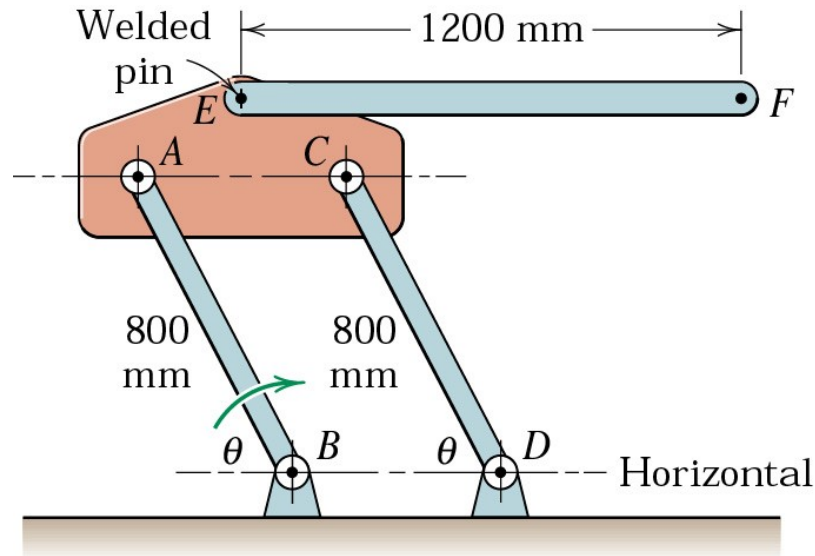
$$\left(\sum F_t\right)_{e.f.} = m\bar{a}_t$$

$$\left(\sum M_A\right)_{e.f.} = m\bar{a}_n d_A \quad \left. \begin{array}{l} + \\ - \end{array} \right\}$$

$$\left(\sum M_B\right)_{e.f.} = m\bar{a}_t d_B \quad \left. \begin{array}{l} + \\ - \end{array} \right\}$$

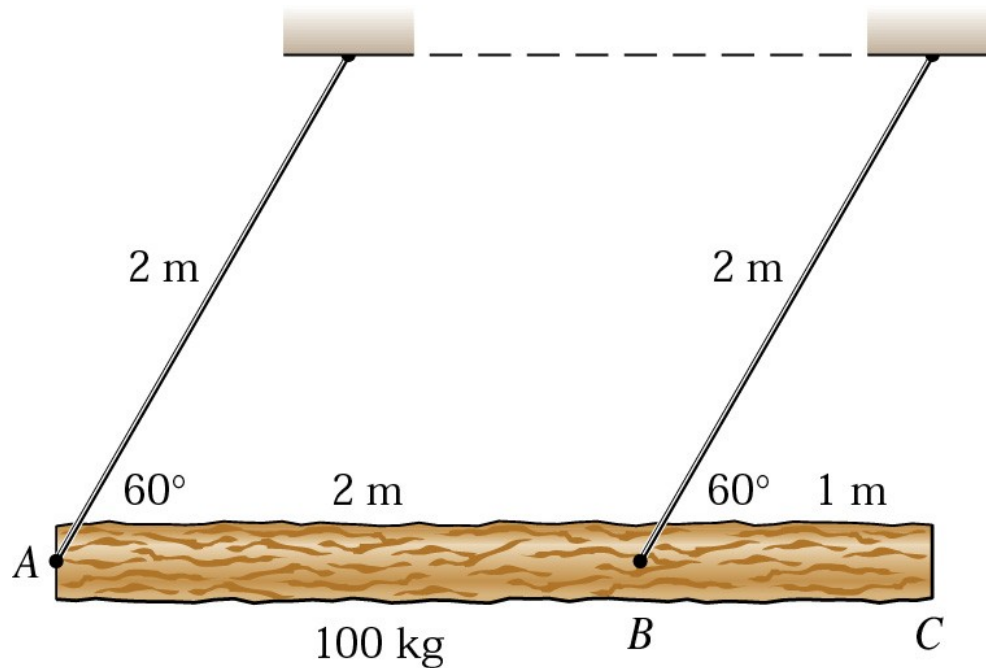
PROBLEMS

3. The parallelogram linkage shown moves in the vertical plane with the uniform 8 kg bar EF attached to the plate at E by a pin which is welded both to the plate and to the bar. A torque (not shown) is applied to link AB through its lower pin to drive the links in a clockwise direction. When θ reaches 60° , the links have an angular acceleration and an angular velocity of 6 rad/s^2 and 3 rad/s , respectively. For this instant calculate the magnitudes of the force F and torque M supported by the pin at E.

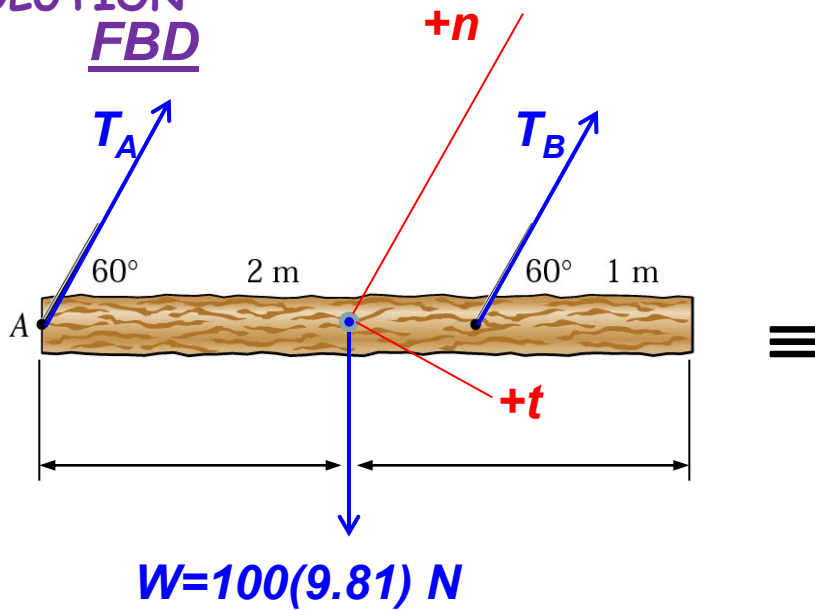


PROBLEMS

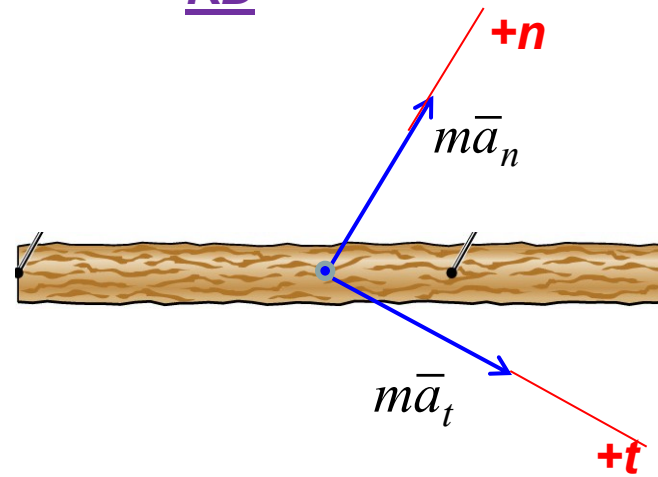
4. The uniform 100 kg log is supported by the two cables and used as a battering ram. If the log is released from rest in the position shown, calculate the initial tension induced in each cable immediately after release and the corresponding angular acceleration α of the cables.



SOLUTION
FBD



KD



When it starts to move, $v=0$, $\omega=0$ but $\alpha \neq 0$ $\bar{a}_n = \omega^2 r = 0$

$$\left(\sum F_n\right)_{d.k.} = 0 \Rightarrow T_A + T_B - mg \cos 30 = 0 \Rightarrow T_A + T_B = 849.57 *$$

$$\left(\sum F_t\right)_{d.k.} = m\bar{a}_t \Rightarrow mg \sin 30 = m\bar{a}_t \Rightarrow \bar{a}_t = 4.905 \text{ m/s}^2$$

$$\bar{a}_t = \alpha r \Rightarrow \alpha = \frac{4.905}{2} = 2.45 \text{ rad/s}^2$$

Length of the cables

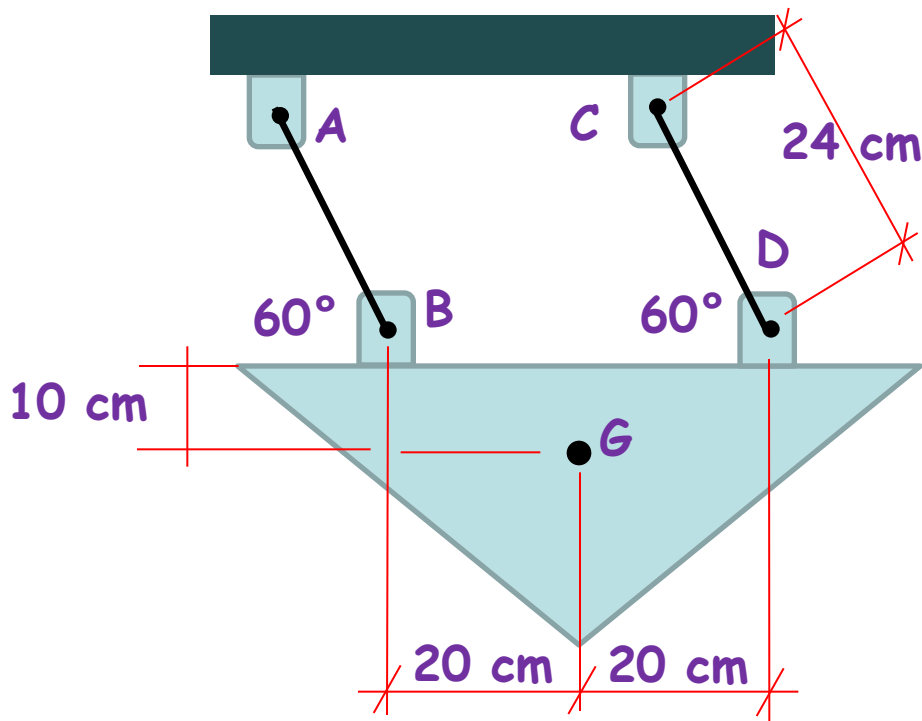
The motion of the log is curvilinear translation.

$$\left(\sum M_G\right)_{d.k.} = 0 \Rightarrow T_A \sin 60(1.5) - T_B \sin 60(0.5) = 0 \Rightarrow 3T_A = T_B *$$

$$T_A = 212.39 \text{ N} \quad T_B = 637.17 \text{ N}$$

PROBLEMS

5. An 18 kg triangular plate is supported by cables AB and CD . When the plate is in the position shown, the angular velocity of the cables is 4 rad/s ccw. At this instant, calculate the acceleration of the mass center of the plate and the tension in each of the cables.



Answer:

$$\bar{a} = 6.23 \text{ m/s}^2$$

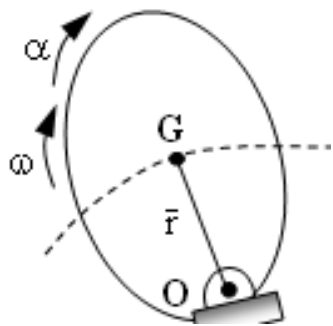
$$T_{AB} = 143.11 \text{ N} \quad T_{CD} = 78.93 \text{ N}$$

2) FIXED-AXIS ROTATION

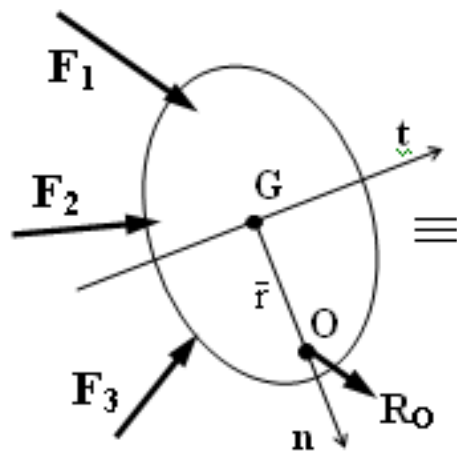
For this motion, all points in the body describe circles about the rotation axis, and all lines of the body have the same angular velocity ω and angular acceleration α .

The acceleration components of the mass center in **n-t** coordinates:

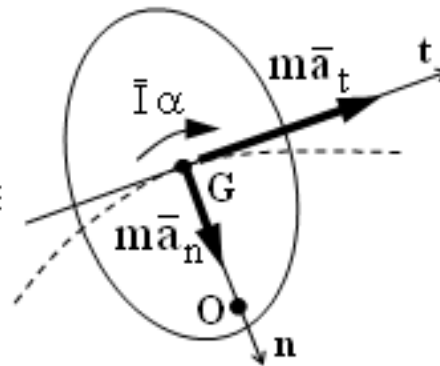
$$\bar{a}_t = \bar{r}\alpha \quad \bar{a}_n = \bar{r}\omega^2$$



FBD



Kinetic Diagram



Equations of Motion

$$\sum \vec{F} = m\bar{a} \quad *$$

$$\sum F_t = m\alpha\bar{r}$$

$$\sum F_n = m\omega^2\bar{r}$$

$$\sum M_G = \bar{I}\alpha \quad **$$

For fixed-axis rotation, it is generally useful to apply a moment equation directly about the rotation axis O .

$$\sum M_O = \bar{I}\alpha + m\bar{r}\bar{a}_t$$

Using transfer-of-axis relation for mass moments of inertia; $I_O = \bar{I} + m\bar{r}^2$ $\bar{I} = I_O - m\bar{r}^2$

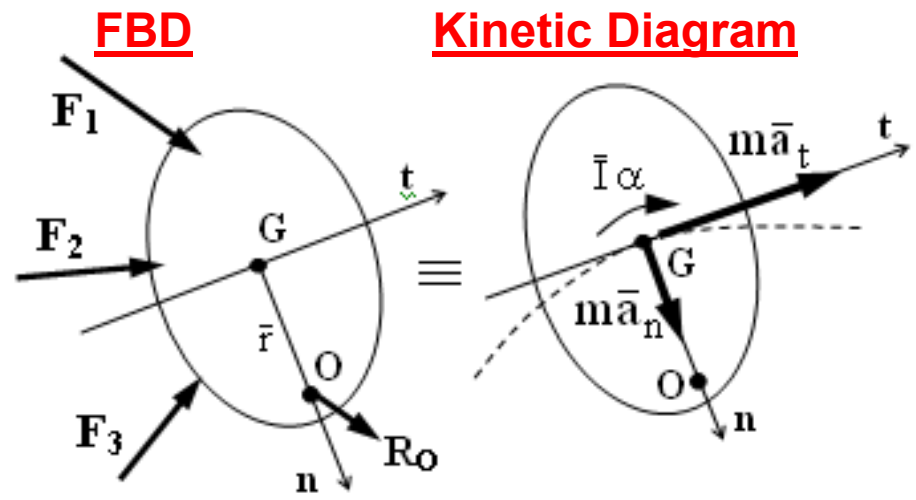
$$\sum M_O = (I_O - m\bar{r}^2)\alpha + m\bar{r}\bar{r}\alpha = I_O\alpha$$

$$\sum M_O = I_O\alpha$$

For the case of rotation axis through its mass center G :

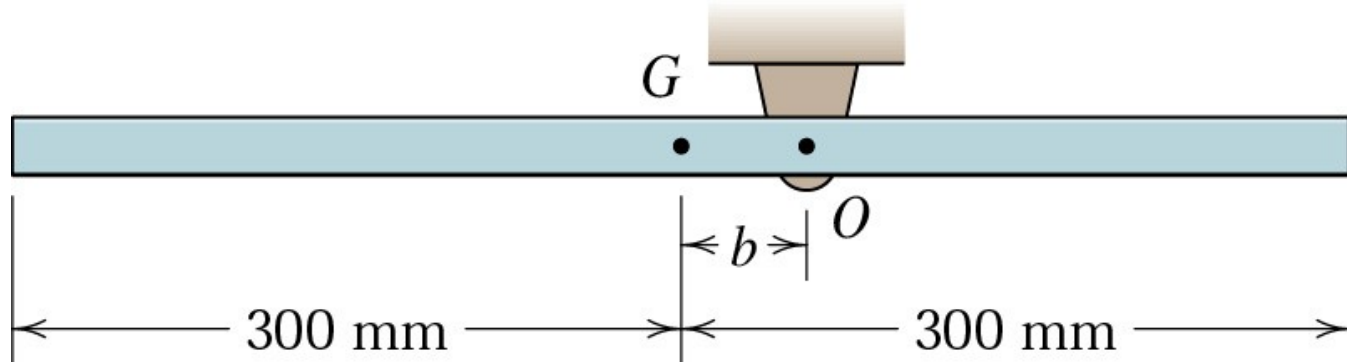
$$\bar{a} = 0 \text{ and } \sum \vec{F} = 0$$

$$\sum M_G = \bar{I}\alpha$$



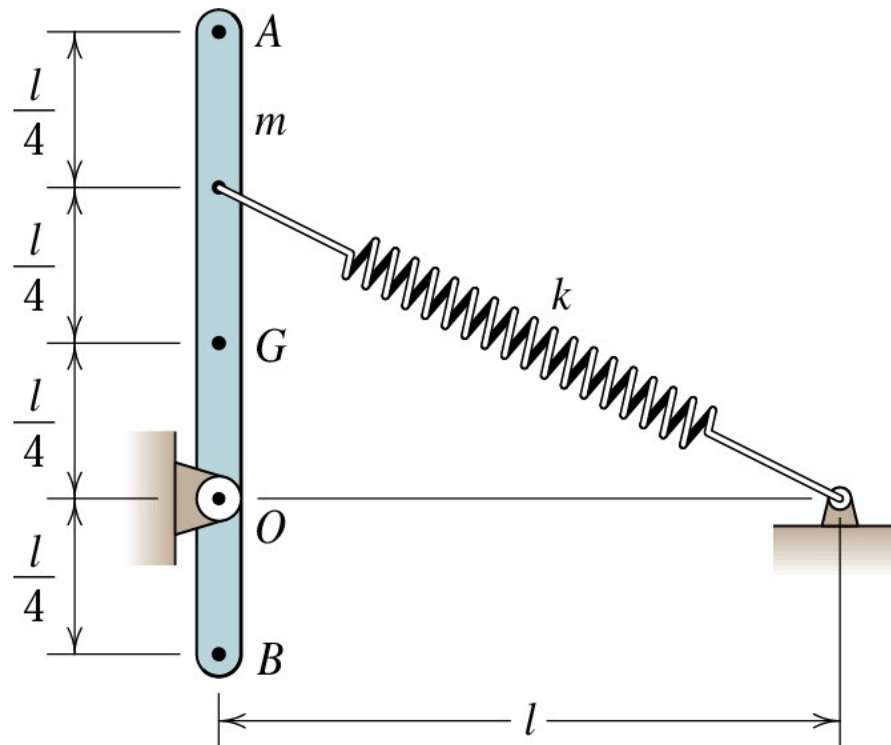
PROBLEMS

6. The uniform 8 kg slender bar is hinged about a horizontal axis through O and released from rest in the horizontal position. Determine the distance b from the mass center to O which will result in an initial angular acceleration of 16 rad/s^2 , and find the force R on the bar at O just after release.

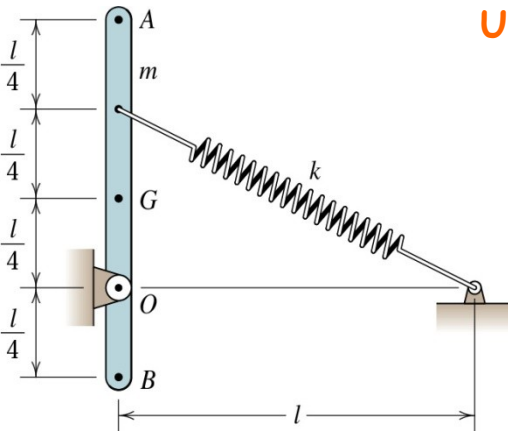


PROBLEMS

7. The spring is uncompressed when the uniform slender bar is in the vertical position shown. Determine the initial angular acceleration α of the bar when it is released from rest in a position where the bar has been rotated 30° clockwise from the position shown. Neglect any sag of the spring, whose mass is negligible.



SOLUTION



Unstretched length of the spring:

$$l_o = \sqrt{(2l/4)^2 + l^2} = \frac{\sqrt{5}}{2}l$$

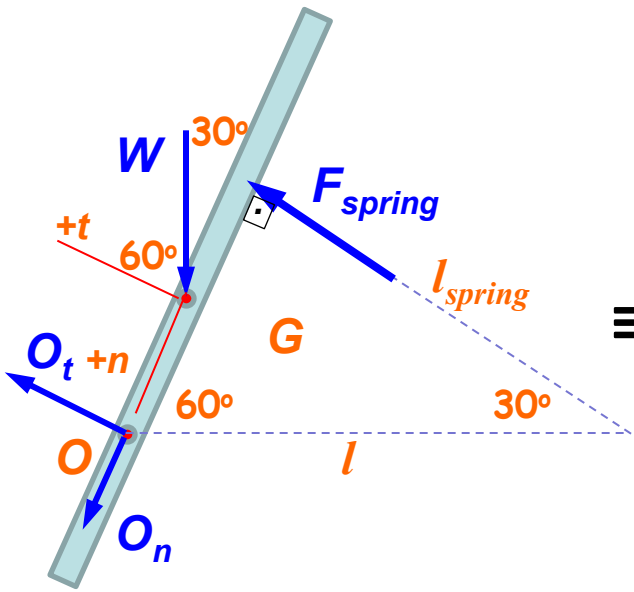
When $\theta=30^\circ$, length of the spring:

$$l_{spring} = \frac{\sqrt{3}}{2}l$$

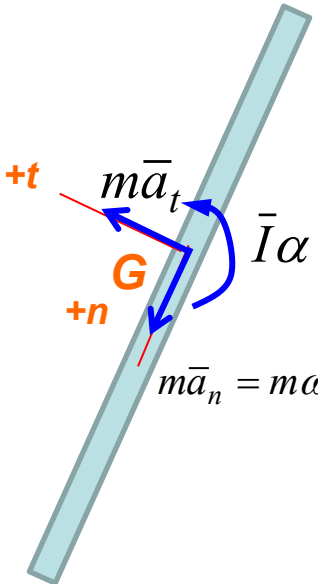
When $\theta=30^\circ$, spring force:

$$F_{spring} = k \left(\frac{\sqrt{5}}{2}l - \frac{\sqrt{3}}{2}l \right) = kl \left(\frac{\sqrt{5}}{2} - \frac{\sqrt{3}}{2} \right)$$

(in compression)



≡



$$\sum M_O = \bar{I}\alpha + m\bar{a}_t$$

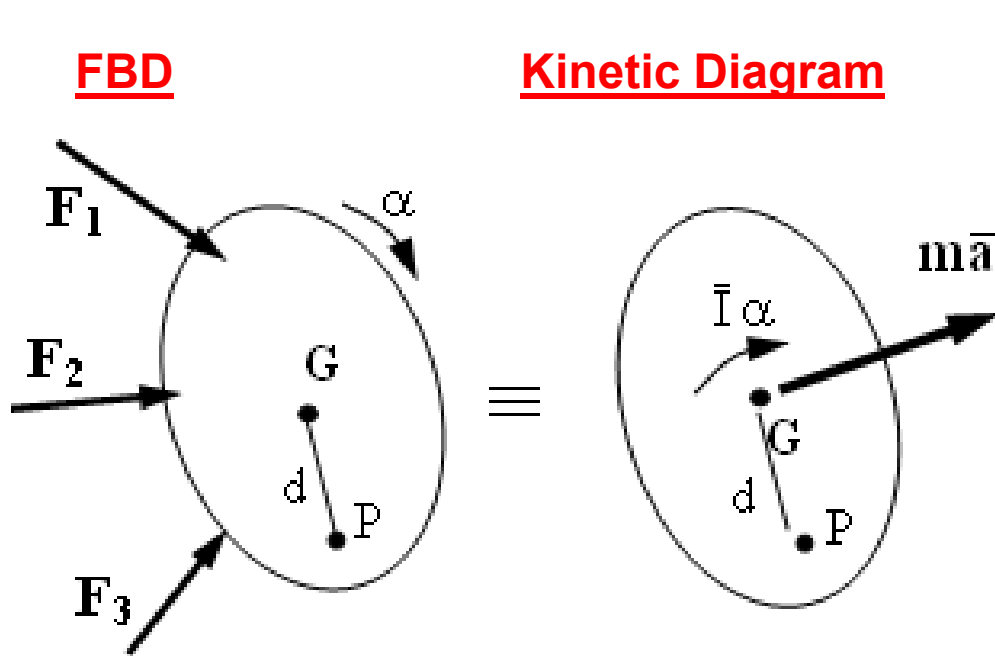
$$-mg \cos 60 \frac{l}{4} + F_{spring} \frac{l}{2} = \frac{1}{12}ml^2 + m\bar{a}_t \frac{l}{4}$$

$$\alpha = 0.864 \frac{k}{m} - 0.857 \frac{g}{l}$$

$$m\bar{a}_n = m\omega^2 \frac{l}{4} = 0$$

3) GENERAL PLANE MOTION

The dynamics of general plane motion of a rigid body combines translation and rotation.



Equations of motion:

$$\sum \vec{F} = m\bar{a}$$

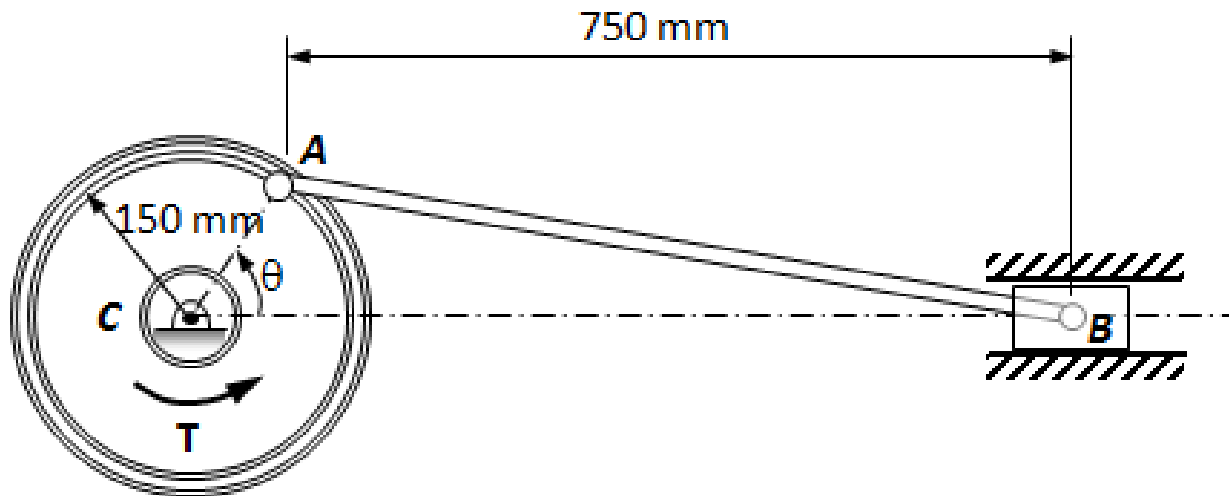
$$\sum M_G = \bar{I}\alpha$$

In some cases, it may be more convenient to use the alternative moment relation about any point P.

$$\sum M_P = \bar{I}\alpha + m\bar{a}d$$

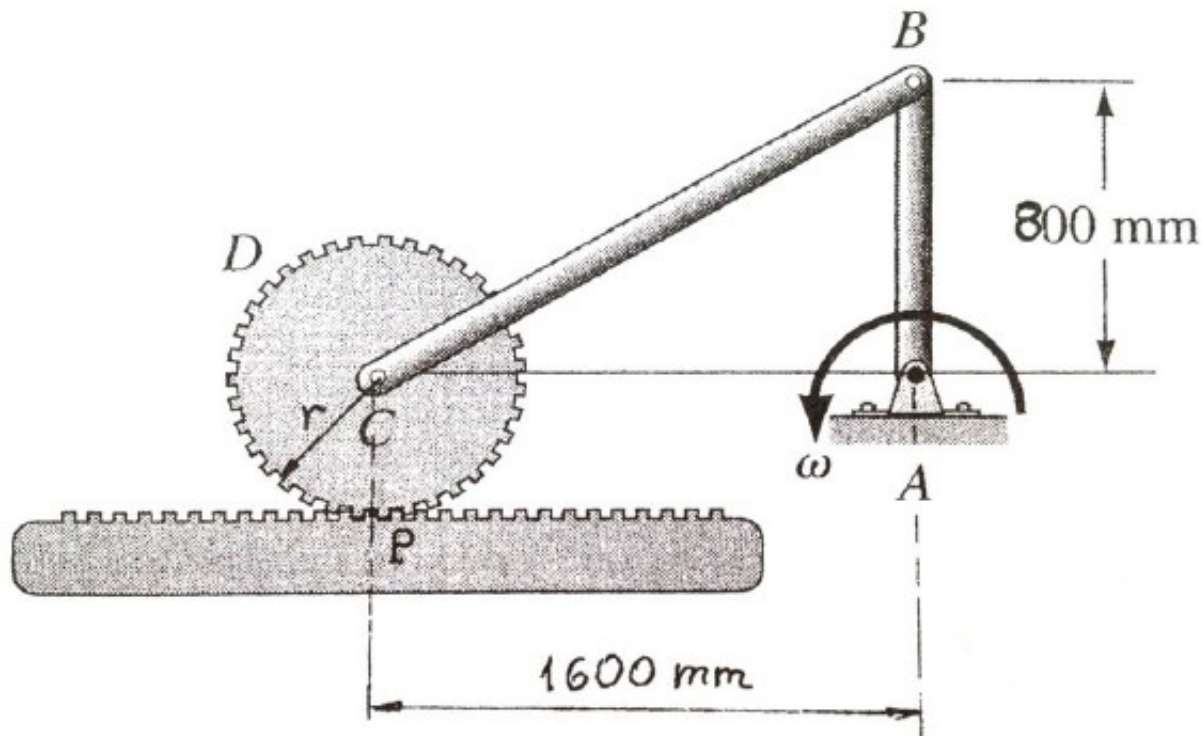
PROBLEMS

8. In the mechanism shown, the flywheel has a mass of 50 kg and radius of gyration about its center of 160 mm. Uniform connecting rod AB has a mass of 10 kg. Mass of the piston B is 15 kg. Flywheel is rotating by the couple T ccw at a constant rate 50 rad/s. When $\theta=53^\circ$ determine the angular velocity and angular acceleration of the connecting rod AB (ω_{AB} ve α_{AB}). What are the forces transmitted by the pins at A and B? Neglect the friction. Take $\sin 53=0.8$, $\cos 53=0.6$.



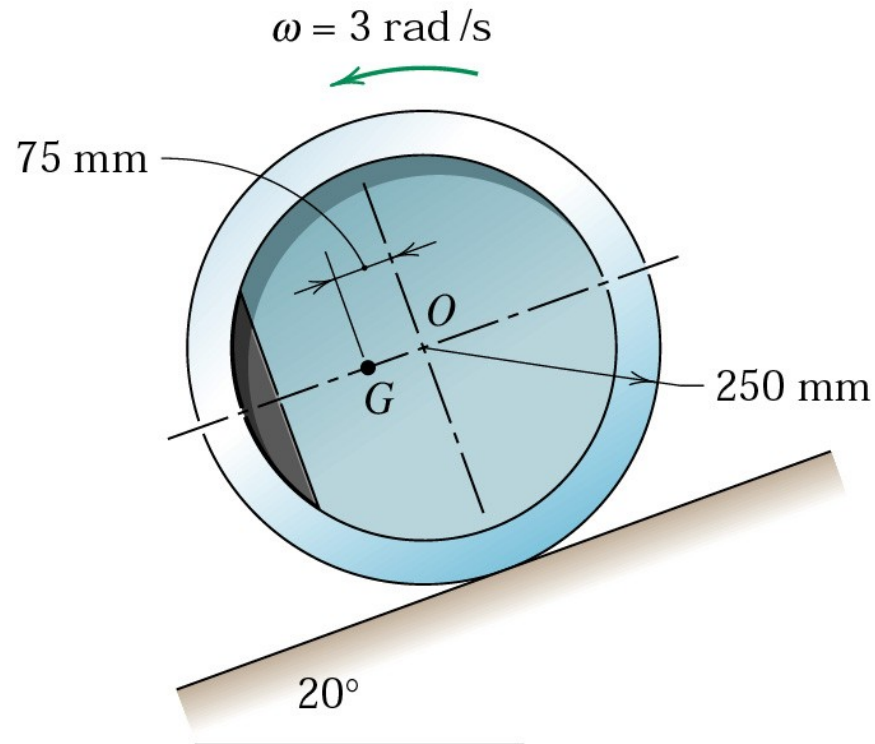
PROBLEMS

9. Member AB is being rotated at a constant angular velocity of $\omega = 10 \text{ rad/s}$ in ccw direction by a torque (not seen in the figure). Rotation of AB activates the 6 kg rod BC , which causes the 3 kg gear D to move. The radius of gyration of the gear about C is 200 mm. The radius of gear D is given as $r = 250 \text{ mm}$. For the instant represented determine the forces acting at pins B and C .



PROBLEMS

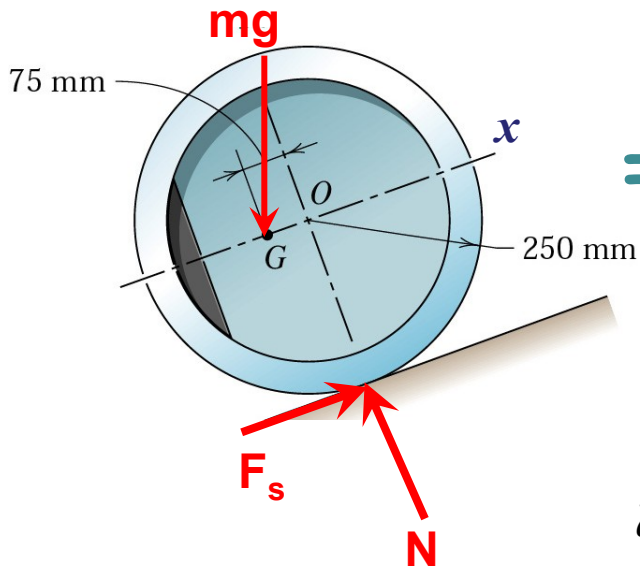
10. The unbalanced 20 kg wheel with the mass center at G has a radius of gyration about G of 202 mm. The wheel rolls down the 20° incline without slipping. In the position shown. The wheel has an angular velocity of 3 rad/s. Calculate the friction force F acting on the wheel at this position.



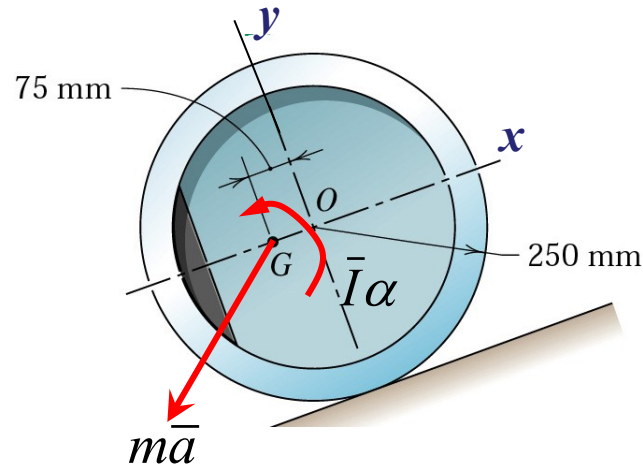
SOLUTION

"General Motion"

FBD



KD



$$\bar{I} = m\bar{k}^2 = 20(0.202)^2 = 0.816 \text{ kgm}^2$$

$$a_o = \alpha r = 0.25\alpha$$

$$a_o = \alpha r = 0.25\alpha$$

$$\begin{aligned} \vec{a}_G &= \vec{a}_O + \vec{a}_{G/O} = -0.25\alpha\vec{i} + \alpha\vec{k} \times (-0.075\vec{i}) + 3\vec{k} \times [3\vec{k} \times (-0.075\vec{i})] \\ \vec{a}_G &= \underbrace{(-0.25\alpha + 0.675)}_{a_x}\vec{i} - \underbrace{0.075}_{a_y}\vec{j} \end{aligned}$$

$$\left(\sum F_x\right)_{ef} = m\bar{a}_x \Rightarrow -mg \sin 20 + F = 20(-0.25\alpha + 0.675)$$

$$F + 5\alpha = 80.604$$

$$\left(\sum F_y\right)_{ef} = m\bar{a}_y \Rightarrow N - mg \cos 20 = 20(-0.075\alpha)$$

$$N = 184.367 - 1.5\alpha$$

$$\left(\sum M_G\right)_{ef} = \bar{I}\alpha \Rightarrow N(0.075) + F(0.25) = 0.816\alpha$$

$$\alpha = 15.597 \text{ rad/s}^2$$

$$F = 2.617 \text{ N}$$

$$N = 160.971 \text{ N}$$

PROBLEMS

11. The uniform 15 kg bar is supported on the horizontal surface at A by a small roller of negligible mass. If the coefficient of kinetic friction between end B and the vertical surface is 0.30, calculate the initial acceleration of end A as the bar is released from rest in the position shown.

