Polarization Imaging with Interferometry

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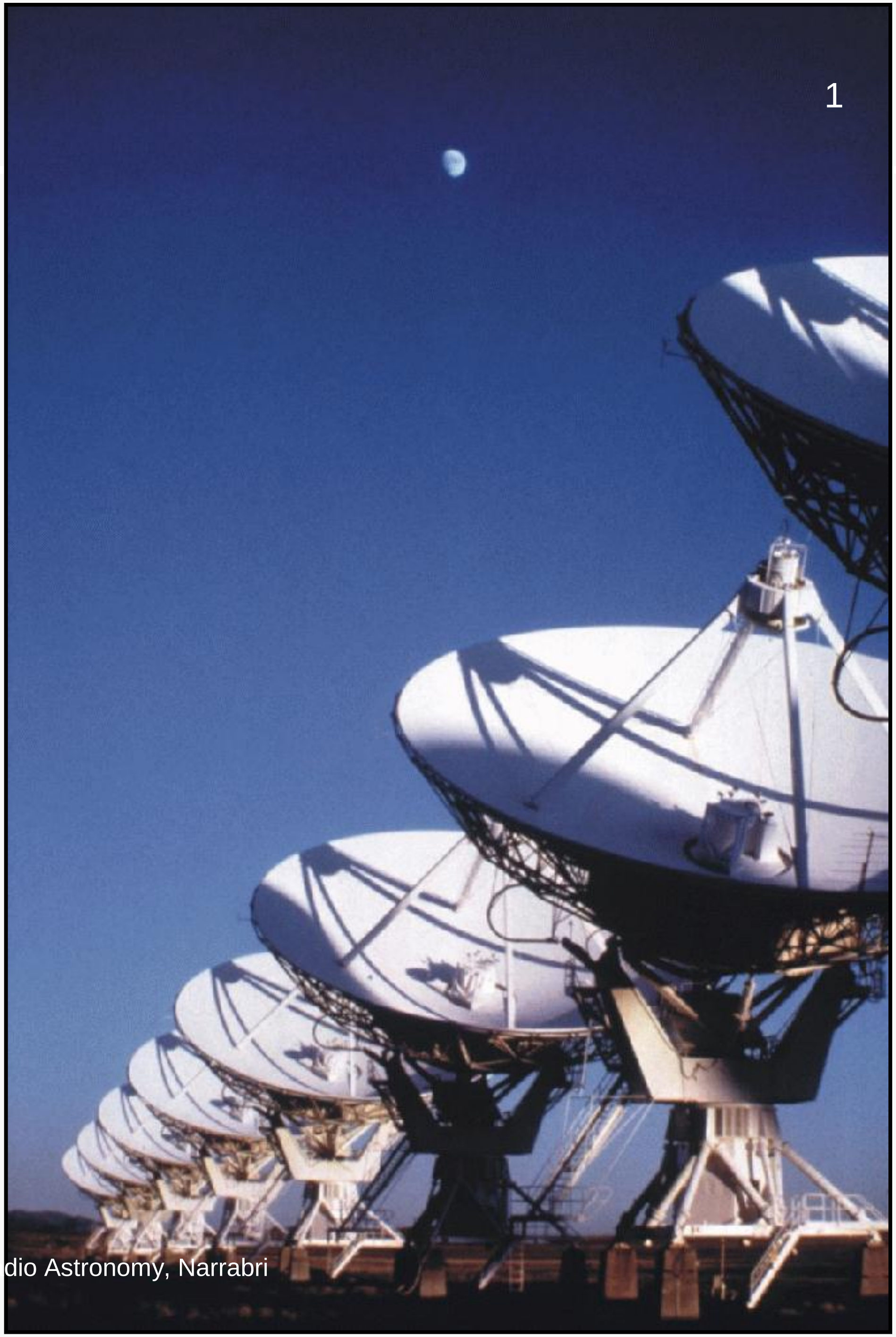

CASS School on Radio Astronomy

Narrabri, NSW

29 Sept – 03 Oct, 2014

List of Topics

- Review of Definitions – Stokes Parameters
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Stokes Parameters – Monochromatic Case

- Perfectly monochromatic EM waves have an E-vector which traces a perfect ellipse in a fixed plane.
- We utilize in radio astronomy the parameters defined by George Stokes (1852), and introduced to astronomy by Chandrasekhar (1946):

$$\left.\begin{aligned}
I &= A_X^2 + A_Y^2 && = A_R^2 + A_L^2 \\
Q &= A_X^2 - A_Y^2 && = 2A_R A_L \cos\delta_{RL} \\
U &= 2A_X A_Y \cos\delta_{XY} && = -2A_R A_L \sin\delta_{RL} \\
V &= -2A_X A_Y \sin\delta_{XY} && = A_R^2 - A_L^2
\end{aligned}\right\}$$

Units of power:
Jy, or Jy/beam

where A_X and A_Y are the cartesian amplitude components of the E-field, and δ_{XY} is the phase lag between them, and

A_R and A_L are the opposite circular amplitude components of the E-field, and δ_{RL} the phase lag between them.

- By (IAU) convention, the 'X' axis points to the NCP, the 'Y' axis to the east.
- Also by IAU convention, LCP has the E-vector rotating clockwise for approaching radiation.
- Monochromatic radiation is 100% polarized: $I^2 = Q^2 + U^2 + V^2$

Real Fields, Real Physics

- The monochromatic case is useful for visualizing the definitions, but is not realistic in astronomy.
- Real wideband signals comprise emission from an uncountable number of distant radiators, and are statistical in nature.
- 100% polarization is not possible with such systems.
- For analysis, we employ the ‘quasi-monochromatic’ representation:
  - Restrict attention to a very narrow slice of frequency of width $\Delta\nu$, for which the fields are described by a single amplitude and phase for a period $t \sim 1/\Delta\nu$.
  - Since the integration time $T >> t$, average the short-duration statistical measures to derive the Stokes parameters for timescales of interest.

Stokes Parameters – quasi-monochromatic

- For the 'quasi-monochromatic' approximation, we perform averages of the various quantities, over a period of time T>>1/Δν.
- This process provides a statistically robust set of values that can be related to the physics of the emitting processes.
- For the orthogonal linear, and opposite circular bases, we have:

$$
\begin{aligned}
I &= \left\langle A_X^2 \right\rangle + \left\langle A_Y^2 \right\rangle & &= \left\langle A_R^2 \right\rangle + \left\langle A_L^2 \right\rangle \\
Q &= \left\langle A_X^2 \right\rangle - \left\langle A_Y^2 \right\rangle & &= \left\langle 2 A_R A_L \cos \delta_{RL} \right\rangle \\
U &= \left\langle 2 A_X A_Y \cos \delta_{XY} \right\rangle & &= -\left\langle 2 A_R A_L \sin \delta_{RL} \right\rangle \\
V &= -\left\langle 2 A_X A_Y \sin \delta_{XY} \right\rangle & &= \left\langle A_R^2 \right\rangle - \left\langle A_L^2 \right\rangle
\end{aligned}
$$

- Note that in this case: $I^2 > Q^2 + U^2 + V^2$

- These four real numbers are a complete description of the polarization state of the incoming radiation.
- They are a function of frequency, source position, and time.

Complex Notation – Use of the Analytic Signal

- The ‘analytic signal’ is a construction which simplifies the mathematical analysis.
- Formally, it is a complex number whose:
  - Real part is the actual (real) signal
  - Imaginary part is the signal with all its spectral components shifted by 90 degrees.
  - This defines a Hilbert transform.
  - The analytic signal for $\cos(\omega t)$ is $e^{i\omega t}$.
- This is more than a mathematical trick – the analytic signal can be (and is) generated by analog or digital hardware.
- The justification for doing this will be seen in the following:

Stokes Parameters for Analytic Signal Representation

- If we denote the analytic Electric field for the various components by the script letter $\mathcal{E}$, we can show that the (real) Stokes parameters can be conveniently expressed as:

$$
\begin{aligned}
I &= \langle \mathcal{E}_X \mathcal{E}_X^* \rangle + \langle \mathcal{E}_Y \mathcal{E}_Y^* \rangle &&= \langle \mathcal{E}_R \mathcal{E}_R^* \rangle + \langle \mathcal{E}_L \mathcal{E}_L^* \rangle \\
Q &= \langle \mathcal{E}_X \mathcal{E}_X^* \rangle - \langle \mathcal{E}_Y \mathcal{E}_Y^* \rangle &&= \langle \mathcal{E}_R \mathcal{E}_L^* \rangle + \langle \mathcal{E}_L \mathcal{E}_R^* \rangle \\
U &= \langle \mathcal{E}_X \mathcal{E}_Y^* \rangle + \langle \mathcal{E}_Y \mathcal{E}_X^* \rangle &&= -i\left(\langle \mathcal{E}_R \mathcal{E}_L^* \rangle - \langle \mathcal{E}_L \mathcal{E}_R^* \rangle\right) \\
V &= -i\left(\langle \mathcal{E}_X \mathcal{E}_Y^* \rangle - \langle \mathcal{E}_Y \mathcal{E}_X^* \rangle\right) &&= \langle \mathcal{E}_R \mathcal{E}_R^* \rangle - \langle \mathcal{E}_L \mathcal{E}_L^* \rangle
\end{aligned}
$$

- These relations are valid for a single antenna. All derived values are real.
- What about interferometry?

Stokes Visibilities

- In terms of the (complex) fields at antennas1 and 2, the Stokes Visibilities can be written as

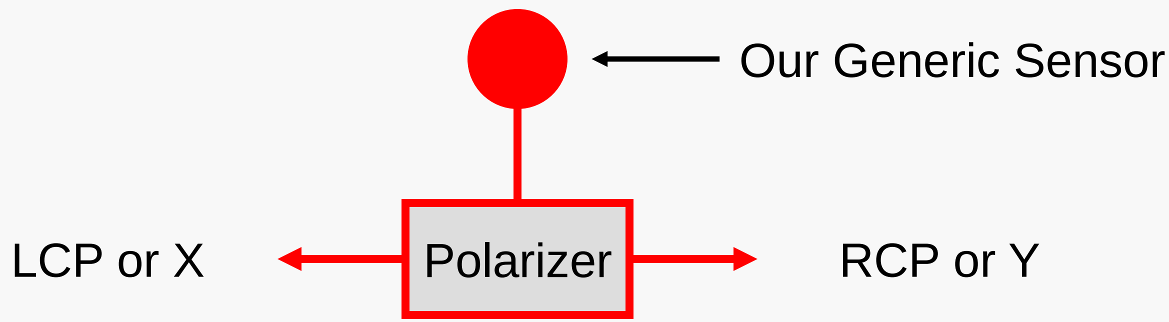
$$
\begin{aligned}
\mathcal{I} &= \left(\mathcal{E}_{X1}\mathcal{E}_{X2}^{*} + \mathcal{E}_{Y1}\mathcal{E}_{Y2}^{*}\right) &&= \left(\mathcal{E}_{R1}\mathcal{E}_{R2}^{*} + \mathcal{E}_{L1}\mathcal{E}_{L2}^{*}\right)\\
\mathcal{Q} &= \left(\mathcal{E}_{X1}\mathcal{E}_{X2}^{*} - \mathcal{E}_{Y1}\mathcal{E}_{Y2}^{*}\right) &&= \left(\mathcal{E}_{R1}\mathcal{E}_{L2}^{*} + \mathcal{E}_{L1}\mathcal{E}_{R2}^{*}\right)\\
\mathcal{U} &= \left(\mathcal{E}_{X1}\mathcal{E}_{Y2}^{*} + \mathcal{E}_{Y1}\mathcal{E}_{X2}^{*}\right) &&= -i\left(\mathcal{E}_{R1}\mathcal{E}_{L2}^{*} - \mathcal{E}_{L1}\mathcal{E}_{R2}^{*}\right)\\
\mathcal{V} &= -i\left(\mathcal{E}_{X1}\mathcal{E}_{Y2}^{*} - \mathcal{E}_{Y1}\mathcal{E}_{X2}^{*}\right) &&= \left(\mathcal{E}_{R1}\mathcal{E}_{R2}^{*} - \mathcal{E}_{L1}\mathcal{E}_{L2}^{*}\right)
\end{aligned}
$$

where I have dropped the angle brackets <> to reduce clutter.

- The script symbols $\mathcal{I}, \mathcal{Q}, \mathcal{U}, \mathcal{V}$ remind us that these Stokes Visibilities are complex numbers, related to the (real) source brightness through Fourier transform.
- $\mathcal{I} \Longleftrightarrow \mathrm{I}$, $\mathcal{Q} \Longleftrightarrow \mathrm{Q}$, $\mathcal{U} \Longleftrightarrow \mathrm{U}$, $\mathcal{V} \Longleftrightarrow \mathrm{V}$

Relation to Sensors

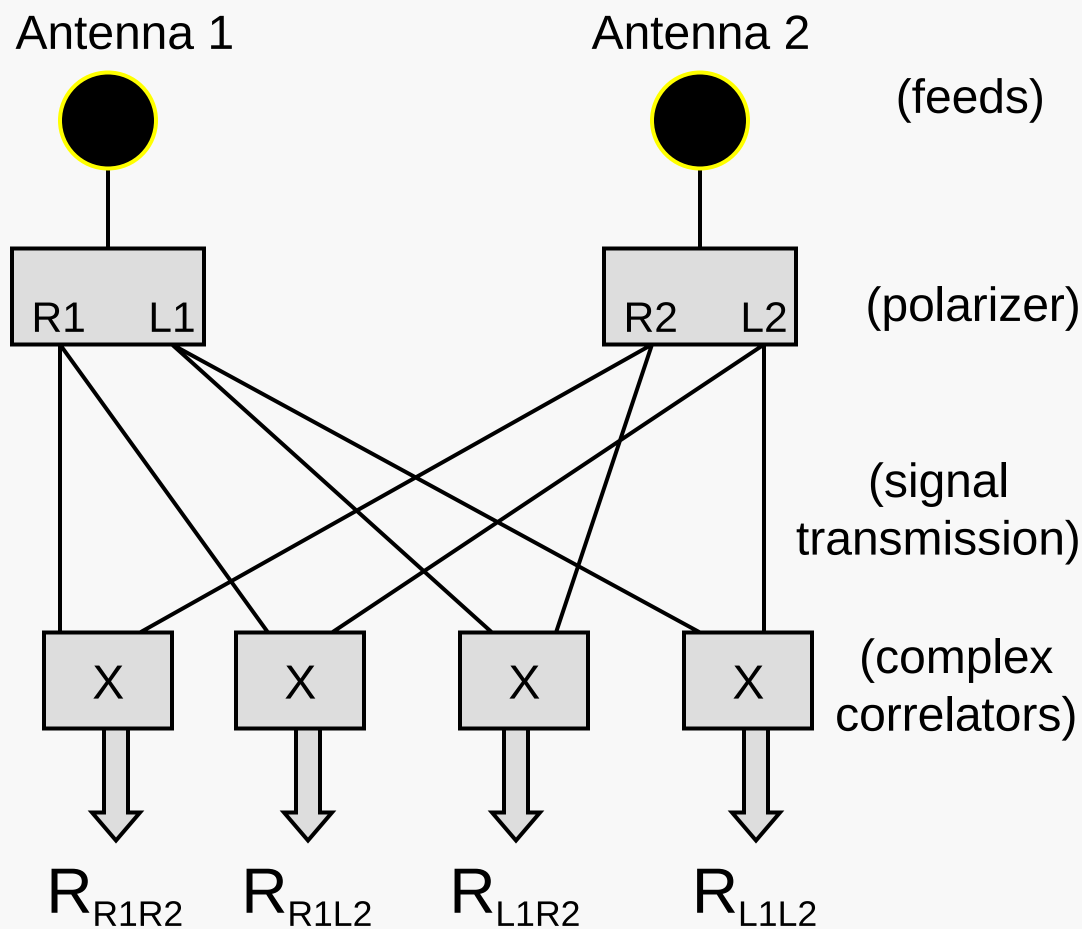
- The above is formulated in terms of electric fields measured at two locations.
- What is the relation to real sensors (antennas)?
- Antennas are polarized – they provide two simultaneous voltage signals whose values are (ideally) representations of the two electric field components – either in a circular or linear basis.



- We have two antennas, each with two differently polarized outputs.
- We can then form four complex correlations.

Four Complex Correlations per Pair of Antennas

- Two antennas, each with two differently polarized outputs, produce four complex correlations.
- From these four outputs, we want to generate the four complex Stokes' visibilities, $\mathcal{I}, \mathcal{Q}, \mathcal{U},$ and $\mathcal{V}$



Relating the Products to Stokes' Visibilities

- Let V_{R1}, V_{L1}, V_{R2} and V_{L2} be the complex representation (analytic signal) of the RCP and LCP voltages emerging from our (perfect) antennas.
- We can then utilize the definitions earlier given to show that the four complex correlations between these fields are related to the desired visibilities by (ignoring gain factors):

$$R_{R1R2} = \left\langle V_{R1} V_{R2}^{*} \right\rangle = (\mathcal{I} + \mathcal{V})/2$$

$$R_{L1L2} = \left\langle V_{L1} V_{L2}^{*} \right\rangle = (\mathcal{I} - \mathcal{V})/2$$

$$R_{R1L2} = \left\langle V_{L1} V_{R2}^{*} \right\rangle = (\mathcal{Q} + i\mathcal{U})/2$$

$$R_{L1R2} = \left\langle V_{R1} V_{L2}^{*} \right\rangle = (\mathcal{Q} - i\mathcal{U})/2$$

- So, if each antenna has two outputs whose voltages are faithful replicas of the EM wave's RCP and LCP components, then the simple equations shown are sufficient to obtain the Stokes Visibilities.

Solving for Stokes Visibilities -- Circular

- The solutions are straightforward:

$$\mathcal{I} = R_{R1R2} + R_{L1L2}$$
$$\mathcal{V} = R_{R1R2} - R_{L1L2}$$
$$\mathcal{Q} = R_{R1L2} + R_{L1R2}$$
$$\mathcal{U} = -i(R_{R1L2} - R_{L1R2})$$

- $\mathcal{V}$ is formed from the difference of two large quantities, while $\mathcal{Q}$ and $\mathcal{U}$ are formed from the sum and difference of small quantities.
- If calibration errors dominate (and they often do), the circular basis favors measurements of linear polarization.
- Although it is true that Q, U, and V are << I, it does not necessarily follow that $\mathcal{Q}$, $\mathcal{U}$, and $\mathcal{V}$ are much smaller than $\mathcal{I}$, (notable for extended objects).

For Linearly Polarized Antennas …

- We can go through the same exercise with perfectly linearly polarized feeds and obtain (presuming they are oriented such that the vertical feed lies along a line of constant HA, and again ignoring issues of gain):

$$R_{X1X2} = \left\langle V_{X1} V^*_{X2} \right\rangle = (\mathcal{I} + \mathcal{Q})/2$$

$$R_{Y1Y2} = \left\langle V_{Y1} V^*_{Y2} \right\rangle = (\mathcal{I} - \mathcal{Q})/2$$

$$R_{X1Y2} = \left\langle V_{X1} V^*_{Y2} \right\rangle = (\mathcal{U} + i\mathcal{V})/2$$

$$R_{Y1X2} = \left\langle V_{Y1} V^*_{X2} \right\rangle = (\mathcal{U} - i\mathcal{V})/2$$

- For each example, we have four measured quantities and four unknowns.
- The solution for the Stokes visibilities is easy.

Stokes' Visibilities for Pure Linear

- Again, the solution in straightforward:

$$\mathcal{I} = R_{X1X2} + R_{Y1Y2}$$
$$\mathcal{Q} = R_{X1X2} - R_{Y1Y2}$$
$$\mathcal{U} = R_{X1Y2} + R_{Y1X2}$$
$$\mathcal{V} = -i(R_{X1Y2} - R_{Y1X2})$$

- These equations hold provided that the outputs of the antennas match our model:
  - Perfectly polarized (linear or circular)
  - Exactly aligned (so X output always aligns with Right Ascension).
- But (sadly), neither of these is true.

Real Sensors

- Real Sensors are:
  - Imperfectly polarized.
    - Typically, the cross-polarization for circularly polarized systems is 5%. (Better with linear)
  - Misaligned with the sky frame.
    - Alt-Az antennas rotate w.r.t. the sky frame as they track a celestial source. The angle describing the misalignment is called the ‘parallactic angle’.
    - Equatorial antennas are fixed w.r.t. the sky, but there will be a (small) misalignment of the feed system with the sky.
- How do these imperfections affect the polarimetry?
- Start with Antenna rotation (it’s easier).

Antenna Rotation -- Circular

- I show (without derivation) how antenna rotation affects the results for the situation when both antennas are rotated by an angle Ψ_P.
- For perfectly circularly polarized antennas:

$$R_{R1R2} = (\mathcal{I} + \mathcal{V})/2 \qquad \mathcal{I} = R_{R1R2} + R_{L1L2}$$

$$R_{L1L2} = (\mathcal{I} - \mathcal{V})/2 \qquad \mathcal{V} = R_{R1R2} - R_{L1L2}$$

$$R_{R1L2} = (\mathcal{Q} + i\mathcal{U})e^{i2\psi_P}/2 \qquad \mathcal{Q} = R_{R1L2}e^{i2\Psi_P} + R_{L1R2}e^{-i2\Psi_P}$$

$$R_{L1R2} = (\mathcal{Q} - i\mathcal{U})e^{-i2\psi_P}/2 \qquad \mathcal{U} = -i(R_{R1L2}e^{i2\Psi_P} - R_{L1R2}e^{-i2\Psi_P})$$

- The effect of antenna rotation is to simply rotate the RL and LR visibilities. Parallel hand visibilities are unaffected.
- $\mathcal{Q}$ and $\mathcal{U}$ require only the cross-hand correlations. $\mathcal{I}$ and $\mathcal{V}$ require only the parallel hand correlations.

Antenna Rotation, Linear

- For perfect linearly polarized antennas, both rotated at an angle Ψ_P:

$$R_{V1V2} = (\mathcal{I} + \mathcal{Q}\cos 2\Psi_P + \mathcal{U}\sin 2\Psi_P)/2$$
$$R_{H1H2} = (\mathcal{I} - \mathcal{Q}\cos 2\Psi_P - \mathcal{U}\sin 2\Psi_P)/2$$
$$R_{V1H2} = (-\mathcal{Q}\sin 2\Psi_P + \mathcal{U}\cos 2\Psi_P + i\mathcal{V})/2$$
$$R_{H1V2} = (-\mathcal{Q}\sin 2\Psi_P + \mathcal{U}\cos 2\Psi_P - i\mathcal{V})/2$$

  - Note that I utilize ‘H’ and ‘V’ labels, since for an alt-az antenna, the orientation of the linear feeds will not align with the definitions for ‘X’ and ‘Y’

- With solution:

$$\mathcal{I} = R_{V1V2} + R_{H1H2})$$
$$\mathcal{Q} = \left(R_{V1V2} - R_{H1H2}\right)\cos 2\Psi_P - \left(R_{V1H2} + R_{H1V2}\right)\sin 2\Psi_P$$
$$\mathcal{U} = \left(R_{V1V2} - R_{H1H2}\right)\sin 2\Psi_P - \left(R_{V1H2} + R_{H1V2}\right)\cos 2\Psi_P$$
$$\mathcal{V} = i(R_{V1H2} - R_{H1V2})$$

- Note that $\mathcal{Q}$ and $\mathcal{U}$ require all four correlations. $\mathcal{I}$ and $\mathcal{V}$ require only two.

Circular vs. Linear

- One of the ongoing debates is the advantages and disadvantages of Linear and Circular systems.
- Point of principle: For full polarization imaging, both systems must provide the same results. Advantages/disadvantages of each are based on points of practicalities.

Circular System	Linear System
$\mathscr{I} = R_{R1R2} + R_{L1L2}$	$\mathscr{I} = R_{V1V2} + R_{H1H2}$
$\mathscr{V} = R_{R1R2} - R_{L1L2}$	$\mathscr{V} = i\left(R_{H1V2} - R_{V1H2}\right)$
$\mathscr{Q} = e^{i2\Psi_P} R_{R1L2} + e^{-i2\Psi_P} R_{L1R2}$	$\mathscr{Q} = \left(R_{V1V2} - R_{H1H2}\right)\cos 2\Psi_P - \left(R_{V1H2} + R_{H1V2}\right)\sin 2\Psi_P$
$\mathscr{U} = i\left(e^{-i2\Psi_P} R_{L1R2} - e^{i2\Psi_P} R_{R1L2}\right)$	$\mathscr{U} = \left(R_{V1V2} - R_{H1H2}\right)\sin 2\Psi_P + \left(R_{V1H2} + R_{H1V2}\right)\cos 2\Psi_P$

- For both systems, Stokes ‘I’ is the sum of the parallel-hands.
- Stokes ‘V’ is the difference of the crossed hand responses for linear, (good) and is the difference of the parallel-hand responses for circular (bad).
- Stokes ‘Q’ and ‘U’ are differences of cross-hand responses for circular (good), and differences of parallel hands for linear (bad).

Circular vs. Linear

- Both systems provide straightforward derivation of the Stokes' visibilities from the four correlations.
- Deriving useable information from differences of large values requires both good stability and good calibration. Hence
  - To do good circular polarization using circular system, or good linear polarization with linear system, we need special care and special methods to ensure good calibration.
- There are practical reasons to use linear:
  - Antenna polarizers are natively linear – extra components are needed to produce circular. This hurts performance.
  - These extra components are also generally of narrower bandwidth – it's harder to build circular systems with really wide bandwidth.
  - At mm wavelengths, the needed phase shifters are not available.
- One important practical reason favoring circular:
  - Calibrator sources are often significantly linearly polarized, but have imperceptible circular polarization.
  - Gain calibration is much simpler with circular feeds, especially for 'snapshot' style observations. (More on this, later).

Calibration Troubles …

- To understand this last point, note that for the linear system:

$$R_{V1V2} = G_{V1}G^*_{V2}(\mathcal{I} + \mathcal{Q}\cos 2\Psi_P + \mathcal{U}\sin 2\Psi_P)/2$$
$$R_{H1H2} = G_{H1}G^*_{H2}(\mathcal{I} - \mathcal{Q}\cos 2\Psi_P - \mathcal{U}\sin 2\Psi_P)/2$$

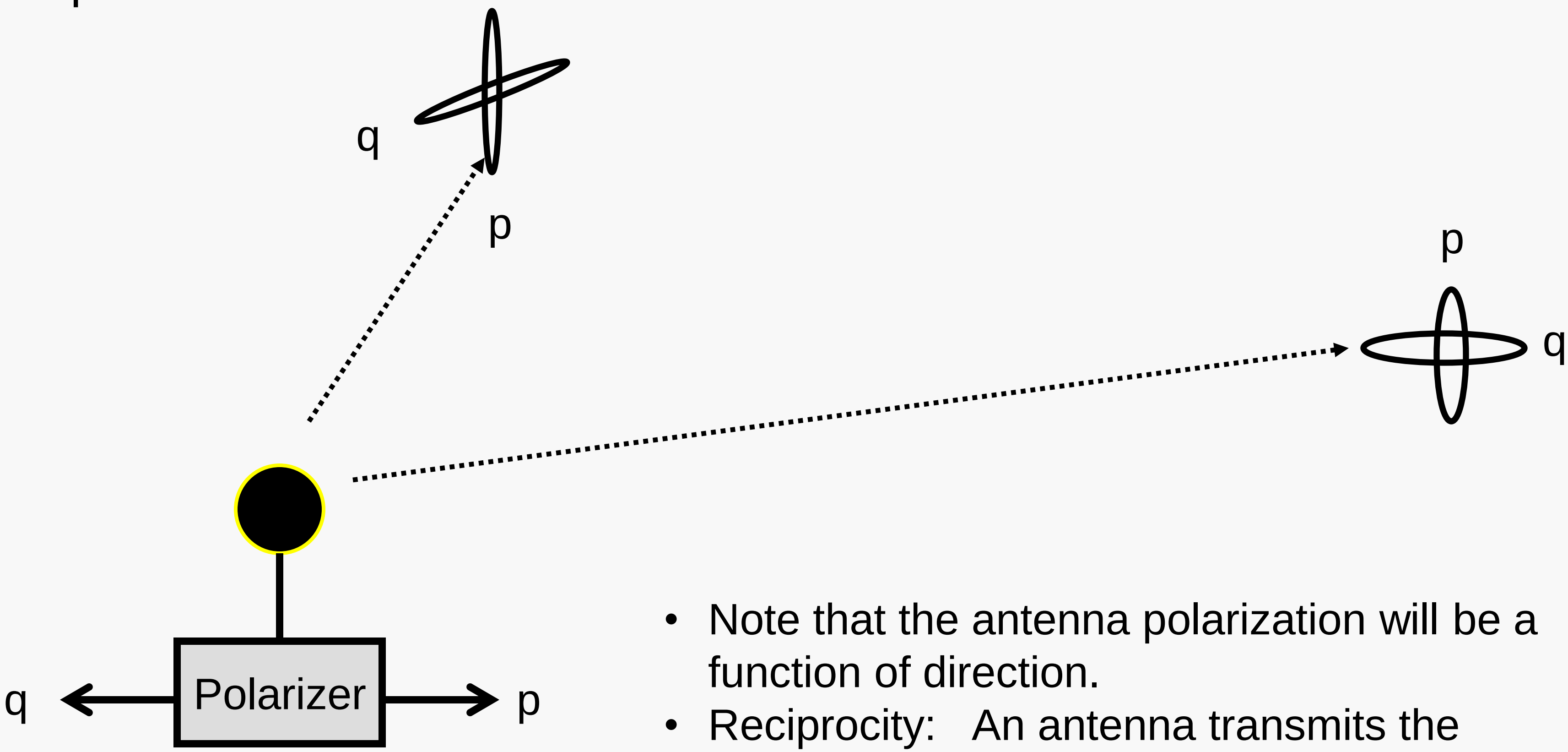
- To calibrate means to solve for the G_V and G_H terms.
- To do so requires knowledge of both Q and U.
- Virtually all calibrators have notable, and variable, linear pol.
- Meanwhile, for circular:

$$R_{R1R2} = G_{R1}G^*_{R2}(\mathcal{I} + \mathcal{V})/2$$
$$R_{L1L2} = G_{L1}G^*_{L2}(\mathcal{I} - \mathcal{V})/2$$

- Now we have *no* sensitivity to Q or U (good!). Instead, we have a sensitivity to V.
- But as it turns out – V is nearly always negligible for the 1000-odd sources that we use as standard calibrators.

Polarization of Real Antennas

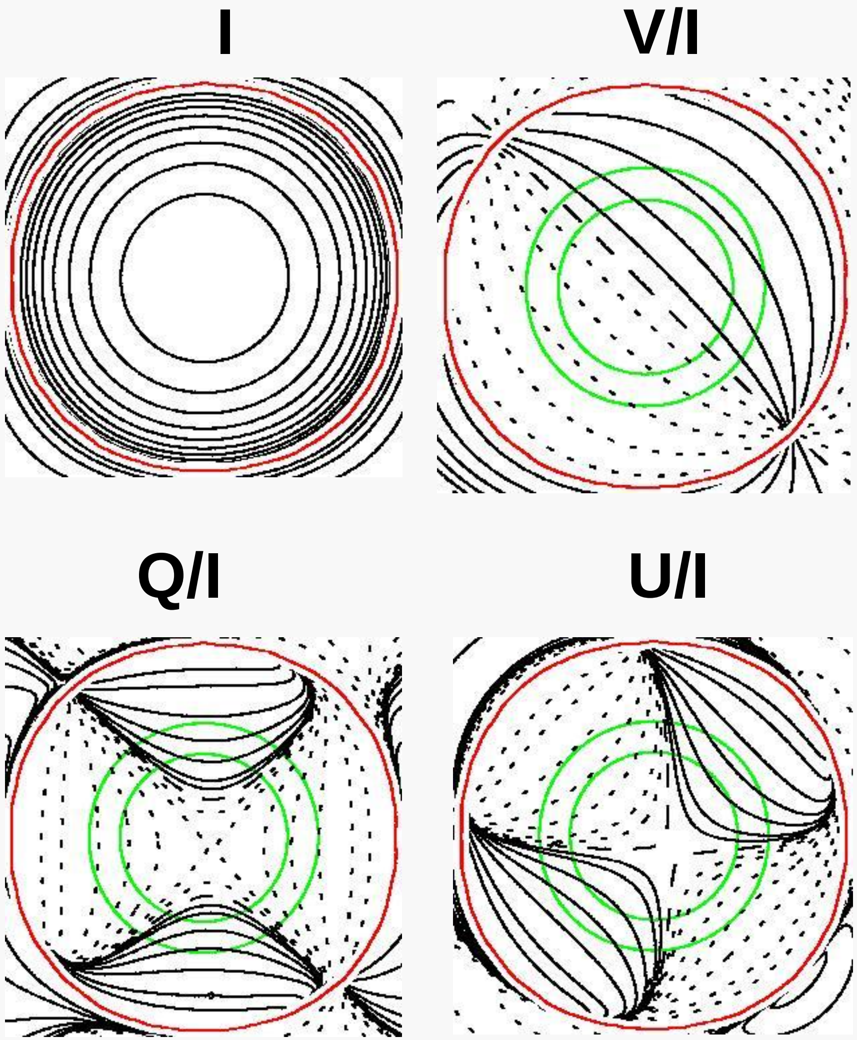
- Unfortunately, antennas never provide perfectly orthogonal outputs.
- In general, the two outputs from an antenna are elliptically polarized.



- Note that the antenna polarization will be a function of direction.
- Reciprocity: An antenna transmits the same polarization that it receives.

Beam Polarization Simulations for VLA

- The beam polarization is due to the antenna and feed geometry.
- Grasp8 calculation by Walter Brisken. (EVLA Memo # 58, 2003).
- Contour intervals: V/I = 4%, Q/I, U/I = 0.2%.
- Very large V/I polarization is due to the VLA's offset feeds.
- The more modest linear polarization is due to the parabolic antenna.
- The beam polarizations can be removed in software – if antenna patterns are known – at considerable computational cost.



Relating Output Voltages from Real Systems to Input Electric Fields

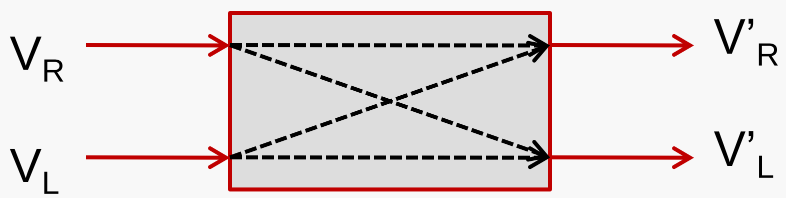
- The Stokes visibilities we want are defined in terms of the complex cross-correlations (coherencies) of electric fields: e.g. $<\mathcal{E}_{R1}\mathcal{E}^*_{R2}>$
- The quantities provided by the antenna are voltages, so what we get from our correlator are quantities like: $<V_{R1}V^*_{R2}>$
- In a real system, V_R isn't uniquely dependent upon E_R – it's a function of both polarizations and some gain factors:

$$V_R = G_R\left(C_{RR}E_R + S_{LR}E_L\right)$$

- We now develop a formalism to handle this general case.

Jones Matrix Algebra

- The analysis of how a real interferometer, comprising real antennas and real electronics, is greatly facilitated through use of Jones matrices.
- In this, we break up our general system into a series of 4-port components, each of which is presumed to be linear.
- We consider each component to have two inputs and two outputs:



- And write:

$$\begin{pmatrix} V^{'}_{R'} \\ V^{'}_{L} \end{pmatrix} = \begin{pmatrix} G_{RR} & G_{LR} \\ G_{RL} & G_{LL} \end{pmatrix} \begin{pmatrix} V_R \\ V_L \end{pmatrix}$$

- Or, in shorthand **V’ = JV**
- The four G components of the Jones matrix describe the linkages within the ‘grey box’.

Example Jones Matrices

- Each component of the overall system, including propagation effects, can be represented by a Jones matrix.
- These matrices can then be multiplied to obtain a ‘system Jones’ matrix.
- Examples (in a circular basis):
  - Faraday rotation by a magnetized plasma: $\begin{pmatrix} e^{-i\phi_R} & 0 \\ 0 & e^{-i\phi_L} \end{pmatrix}$
  - Atmospheric attenuation and phase retardation: $\begin{pmatrix} \alpha e^{i\phi} & 0 \\ 0 & \alpha e^{i\phi} \end{pmatrix}$
  - Antenna rotated by angle Ψ_P $\begin{pmatrix} e^{-i\Psi_P} & 0 \\ 0 & e^{i\Psi_P} \end{pmatrix}$
  - An imperfect polarizer (components are complex) $\begin{pmatrix} C_{RR} & S_{LR} \\ S_{RL} & C_{LL} \end{pmatrix}$
  - Post-polarizer electronic gains (complex): $\begin{pmatrix} G_R & 0 \\ 0 & G_L \end{pmatrix}$

The System Jones Matrix

- Now imagine a simple model, comprising of an antenna oriented at some angle Ψ_P to the sky, feeding an imperfect polarizer, followed by post-polarizer electronic gains.
- For this system, the output voltage (column vector) is related to the input electric fields by:

$$\mathbf{V} = \mathbf{J_G J_{pol} J_{rot} E} = \mathbf{J_{ant} E}$$

- Multiplying the various Jones matrices, we find

$$\begin{pmatrix} V_R \\ V_L \end{pmatrix} = \begin{pmatrix} G_R C_{RR} e^{-i\Psi_P} & G_R S_{LR} e^{i\Psi_P} \\ G_L S_{RL} e^{-i\Psi_P} & G_L C_{LL} e^{i\Psi_P} \end{pmatrix} \begin{pmatrix} E_R \\ E_L \end{pmatrix}$$

- This is normally factored into a product of two gain matrices:

$$\begin{pmatrix} V_R \\ V_L \end{pmatrix} = \underbrace{\begin{pmatrix} G_R C_{RR} e^{-i\Psi_P} & 0 \\ 0 & G_L C_{LL} e^{i\Psi_P} \end{pmatrix}}_{\text{Parallel Hand Gains}} \underbrace{\begin{pmatrix} 1 & D_{LR} e^{i2\Psi_P} \\ D_{RL} e^{-i2\Psi_P} & 1 \end{pmatrix}}_{\text{Cross-Polarization}} \begin{pmatrix} E_R \\ E_L \end{pmatrix}$$

The Mueller Matrix

- There are four cross products, and four complex Stokes visibilitities.
- The results (details at end of this talk) can be compactly written as a matrix equation:

$$\mathbf{R} = \mathbf{G} \cdot \mathbf{P} \cdot \mathbf{\Psi} \cdot \mathbf{S}$$

Where

- **R** = the response vector – the correlator output.
- **G** = the gain matrix – effect of post-polarizer amplifier
- **P** = the polarization mixing matrix (Mueller matrix)
- $\mathbf{\Psi}$ = the antenna rotation matrix (can include propagation)
- **S** = the Stokes vector – what we want.

When applied to our simple model:

Where **R** = the response vector – the correlator output.

G = the gain matrix – effect of post-polarizer amplifiers

P = the polarization mixing matrix (Mueller matrix)

Ψ = the antenna rotation matrix (can include propagation)

S = the Stokes vector – what we want.

$$\mathbf{R} = \mathbf{G} \cdot \mathbf{P} \cdot \mathbf{\Psi} \cdot \mathbf{S}$$

The various terms are:

- Response Vector, R:

$$\mathbf{R} = \begin{pmatrix} \langle V_{R1}V_{R2}^* \rangle \\ \langle V_{R1}V_{L2}^* \rangle \\ \langle V_{L1}V_{R2}^* \rangle \\ \langle V_{L1}V_{L2}^* \rangle \end{pmatrix}$$

- Gain Matrix, G:

$$\mathbf{G} = \begin{pmatrix} G_{R1}G_{R2}^* & 0 & 0 & 0 \\ 0 & G_{R1}G_{L2}^* & 0 & 0 \\ 0 & 0 & G_{L1}G_{R2}^* & 0 \\ 0 & 0 & 0 & G_{L1}G_{L2}^* \end{pmatrix}$$

- Polarization Matrix, P:

$$\mathbf{P} = \begin{pmatrix} C_{RR1}C_{RR2}^* & C_{RR1}S_{LR2}^* & S_{LR1}C_{RR2}^* & S_{LR1}S_{LR2}^* \\ C_{RR1}S_{RL2}^* & C_{RR1}C_{LL2}^* & S_{LR1}S_{RL2}^* & S_{LR1}C_{LL2}^* \\ S_{RR1}C_{RR2}^* & S_{RR1}S_{LR2}^* & C_{LL1}C_{RR2}^* & C_{LL1}S_{LR2}^* \\ S_{RR1}S_{RL2}^* & S_{RR1}C_{LL2}^* & C_{LL1}S_{RL2}^* & C_{LL1}C_{LL2}^* \end{pmatrix}$$

Terms, continued …

- Rotation Matrix, $\boldsymbol{\Psi}$:

$$\boldsymbol{\Psi} = \begin{pmatrix} e^{-i(\Psi_{R1}-\Psi_{R2})} & 0 & 0 & 0 \\ 0 & e^{-i(\Psi_{R1}+\Psi_{L2})} & 0 & 0 \\ 0 & 0 & e^{i(\Psi_{L1}+\Psi_{R2})} & 0 \\ 0 & 0 & 0 & e^{i(\Psi_{L1}-\Psi_{L2})} \end{pmatrix}$$

- Stokes Vector, **S**:

$$\mathbf{S} = \begin{pmatrix} (\mathcal{I}+\mathcal{V})/2 \\ (\mathcal{Q}+i\mathcal{U})/2 \\ (\mathcal{Q}-i\mathcal{U})/2 \\ (\mathcal{I}-\mathcal{V})/2 \end{pmatrix}$$

- <Whew!> Almost there.
- It gets easier from here …

Inverting the Polarization Equation

- We have, for the relation between the correlator output and the Stokes visibility:

$$\mathbf{R} = \mathbf{G} \cdot \mathbf{P} \cdot \mathbf{\Psi} \cdot \mathbf{S}$$

- The solution for S is trivial to write:

$$\mathbf{S} = \mathbf{\Psi}^{-1} \cdot \mathbf{P}^{-1} \cdot \mathbf{G}^{-1} \cdot \mathbf{R}$$

- The inverses for the rotation and gain matrices are trivial.
- More interesting is **P⁻¹:**

$$\mathbf{P}^{-1} = K \begin{pmatrix} C_{LL1}C^*_{LL2} & -C_{LL1}S^*_{LR2} & -S_{LR1}C^*_{LL2} & S_{LR1}S^*_{LR2} \\ -C_{LL1}S^*_{RL2} & C_{LL1}C^*_{RR2} & S_{LR1}S^*_{RL2} & -S_{LR1}C^*_{RR2} \\ -S_{RL1}C^*_{LL2} & S_{RL1}S^*_{LR2} & C_{RR1}C^*_{LL2} & -C_{RR1}S^*_{LR2} \\ S_{RL1}S^*_{RL2} & -S_{RL1}C^*_{RR2} & -C_{RR1}S^*_{RL2} & C_{RR1}C^*_{RR2} \end{pmatrix}$$

Where K is a normalizing factor:

$$K = \frac{1}{\left(C_{RR1}C_{LL1} - S_{LR1}S_{RL1}\right)\left(C^*_{RR2}C^*_{LL2} - S^*_{LR2}S^*_{RL2}\right)}$$

Obtaining the Stokes Visibilities

- All this shows that – in principle – the four complex outputs from an interferometer can be easily inverted to obtain the desired Stokes visibilities.
- Sadly, it's not quite that easy. To correctly invert, we need to know all the factors in the Jones matrices.
- In fact we do not, because …
  - Atmospheric gains are continually changing.
  - System gains change (but hopefully more slowly).
  - Antennas rotate on the sky (but we think we know this in advance …)
  - Antenna polarization may change (but probably very slowly)
  - Standard calibration techniques do not provide the correct values of C and S, but rather values relative to one antenna.

The Physical Meaning …

- To understand the meaning of the C and S terms, consider the antenna in 'transmission' mode.
- One can show (problem for the student!) that the elements in the polarization matrix are determined by the antenna's polarization, with:

$$C_R = \cos\beta_R e^{-i\varphi_R}$$
$$C_L = \cos\beta_L e^{i\varphi_L}$$
$$S_R = \sin\beta_R e^{i\varphi_R}$$
$$S_L = \sin\beta_L e^{-i\varphi_L}$$

$$\beta_R = \chi_R + \pi/4$$
$$\beta_L = \pi/4 - \chi_L$$

  - The β term is the deviation of the antenna polarization ellipse from perfectly circular.
  - The χ term is the antenna's ellipticity
  - The ϕ term is the position angle of the antenna's polarization ellipse, in the antenna frame.
- You can, by substituting the terms above into the polarization matrix, and including the antenna rotation terms, show that:

The response of one of the four correlations:

$$
\begin{aligned}
R_{pq} = G_{pq}\{ & [\cos(\Psi_p - \Psi_q)\cos(\chi_p - \chi_q) + i\sin(\Psi_p - \Psi_q)\sin(\chi_p + \chi_q)]\mathcal{I}/2 \\
& + [\cos(\Psi_p + \Psi_q)\cos(\chi_p + \chi_q) + i\sin(\Psi_p + \Psi_q)\sin(\chi_p - \chi_q)]\mathcal{Q}/2 \\
& - i[\cos(\Psi_p + \Psi_q)\sin(\chi_p - \chi_q) + i\sin(\Psi_p + \Psi_q)\cos(\chi_p + \chi_q)]\mathcal{U}/2 \\
& - [\cos(\Psi_p - \Psi_q)\sin(\chi_p + \chi_q) + i\sin(\Psi_p - \Psi_q)\cos(\chi_p - \chi_q)]\mathcal{V}/2\}
\end{aligned}
$$

This is the famous expression derived by Morris, Radhakrishnan and Seielstad (1964), relating the output of a single complex correlator to the complex Stokes visibilities, where the antenna effects are described in terms of the polarization ellipses of the two antennas.

R_{pq} is the complex output from the interferometer, for polarizations p and q from antennas 1 and 2, respectively.

Ψ and χ are the antenna polarization major axis and ellipticity for polarizations p and q.

$\mathcal{I}$, $\mathcal{Q}$, $\mathcal{U}$, and $\mathcal{V}$ are the Stokes Visibilities

G_{pq} is a complex gain, including the effects of transmission and electronics

The Generalized Formulation (circular basis)

- For an array with the same parallactic angle for each element, ignoring the gains, an alternate form can be written:

$$\begin{pmatrix} R_{R1R2} \\ R_{R1L2} \\ R_{L1R2} \\ R_{L1L2} \end{pmatrix} = \begin{pmatrix} 1 & D^*_{LR2} & D_{LR1} & D_{LR1}D^*_{LR2} \\ D^*_{RL2} & 1 & D_{LR1}D^*_{RL2} & D_{LR1} \\ D_{RL1} & D_{RL1}D^*_{LR2} & 1 & D^*_{LR2} \\ D_{RL1}D^*_{LR2} & D_{RL1} & D^*_{RL2} & 1 \end{pmatrix} \begin{pmatrix} (\mathcal{I}+\mathcal{V})/2 \\ e^{-2i\Psi_P}(\mathcal{Q}+i\mathcal{U})/2 \\ e^{2i\Psi_P}(\mathcal{Q}-i\mathcal{U})/2 \\ (\mathcal{I}-\mathcal{V})/2 \end{pmatrix}$$

Where:

$$\left.\begin{aligned} D_R &= \tan\beta_R e^{i2\varphi_R} \\ D_L &= \tan\beta_L e^{-i2\varphi_L} \end{aligned}\right\}$$

The β and φ parameters are related to the antenna polarization ellipse

- The D's are (unimaginatively) called the 'D-terms', and describe the amplitude and phase of the cross-over signals from R to L, and L to R.
- **Main Point:** The effect of an impure polarizer is to couple all four of the Stokes visibilities to all four cross-products.
- If the 'D' terms are known in advance, this matrix equation can be easily inverted, to solve for the Stokes visibilities in terms of the measured Rs, and the known Ds.

Approximations for Good Polarizers

- Considerable simplification occurs if the polarizers are good.
- Typically circular polarizers have |D| < 0.05.
- If |D|<<1, we can then ignore D*D products.
- Furthermore, if $|\mathcal{Q}|$ and $|\mathcal{U}| << |\mathcal{I}|$, we can ignore products between them and the Ds. (OK for point sources --- not always ok for extended sources).
- And $\mathcal{V}$ can be safely assumed to be zero.
- These approximations then allow us write:

$$R_{R1R2} = \mathcal{I}/2$$

$$R_{L1L2} = \mathcal{I}/2$$

$$R_{R1L2} = \left[\left(D_{R1} + D^*_{L2}\right)\mathcal{I} + e^{-2i\Psi_P}\left(\mathcal{Q} + i\mathcal{U}\right)\right]/2$$

$$R_{L1R2} = \left[\left(D_{L1} + D^*_{R2}\right)\mathcal{I} + e^{2i\Psi_P}\left(\mathcal{Q} - i\mathcal{U}\right)\right]/2$$

‘Nearly’ Circular Feeds (small D approximation)

- We get:

$$R_{R1R2} = \mathcal{I}/2$$

$$R_{L1L2} = \mathcal{I}/2$$

$$R_{R1L2} = \left[\left(D_{R1} + D^*_{L2}\right)\mathcal{I} + e^{-2i\Psi_P}\left(\mathcal{Q} + i\mathcal{U}\right)\right]/2$$

$$R_{L1R2} = \left[\underbrace{\left(D_{L1} + D^*_{R2}\right)\mathcal{I}}_{\text{Contamination}} + \underbrace{e^{2i\Psi_P}\left(\mathcal{Q} - i\mathcal{U}\right)}_{\text{Desired}}\right]/2$$

- The cross-hand responses are contaminated by a term proportional to ‘$\mathcal{I}$’.
- |D| ~ 0.05 ~ $|\mathcal{Q}|/|\mathcal{I}|$ => the two terms are of comparable magnitude.
- To recover the linear polarization, we must determine these D-terms, and remove their contribution.

Nearly Perfectly Linear Feeds

- In this case, assume that the ellipticity is very small ($\chi << 1$), and that the two feeds ('dipoles') are nearly perfectly orthogonal.
- We then define a *different* set of D-terms:

$$D_V = \varphi_V - i\chi_V$$
$$D_H = -\varphi_H + i\chi_H$$

- The angles φ_V and φ_H are the angular offsets from the exact horizontal and vertical orientations, w.r.t. the antenna.

$$R_{V1V2} = (\mathcal{I} + \mathcal{Q}\cos 2\Psi_P + \mathcal{U}\sin 2\Psi_P)/2$$
$$R_{H1H2} = (\mathcal{I} - \mathcal{Q}\cos 2\Psi_P - \mathcal{U}\sin 2\Psi_P)/2$$
$$R_{V1H2} = [\mathcal{I}\left(D_{V1} + D^*_{H2}\right) - \mathcal{Q}\sin 2\Psi_P + \mathcal{U}\cos 2\Psi_P + i\mathcal{V}]/2$$
$$R_{H1V2} = [\mathcal{I}\left(D_{H1} + D^*_{V2}\right) - \mathcal{Q}\sin 2\Psi_P + \mathcal{U}\cos 2\Psi_P - i\mathcal{V}]/2$$

- The situation is the same as for the circular system.

Measuring Cross-Polarization Terms

- Correction of the X-hand response for the ‘leakage’ is important, since the D-term amplitude is comparable to the fractional polarization.
- There are two standard ways to proceed:
  1. Observe a calibrator source of known polarization (preferably zero!)
  2. Observe a calibrator of unknown polarization over an extended period.
- **Case 1: Calibrator source known to have zero polarization**.

$$R_{V1V2} = \mathcal{I}/2$$

$$R_{H1H2} = \mathcal{I}/2$$

$$R_{V1H2} = \mathcal{I}\left(D_{V1} + D^{*}_{H2}\right)/2$$

$$R_{H1V2} = \mathcal{I}\left(D_{H1} + D^{*}_{V2}\right)/2$$

- Then a single observation should suffice to measure the leakage terms.
  - In fact, in this approximation, only $2N_{ant}$-1 terms can be determined. One must be assumed (usually = 0). All the others are referred to this. These are called the ‘relative’ D terms.

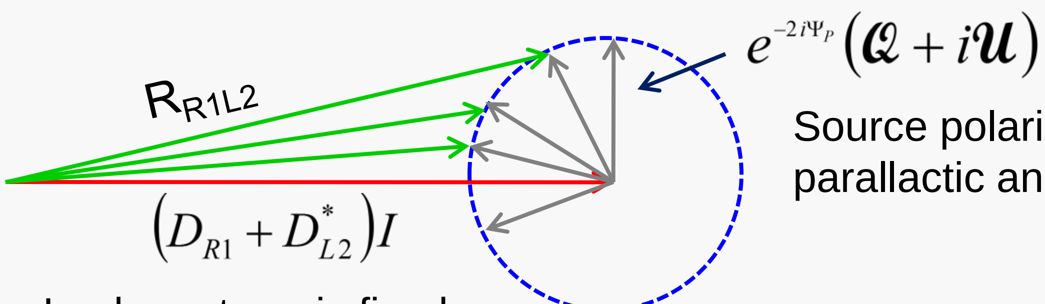
Determining Source and Antenna Polarizations

Case 2: Calibrator with significant (or unknown) polarization.

- You can determine both the (relative) D terms and the calibrator polarizations for an alt-az antenna by observing over a wide range of parallactic angle. (Conway and Kronberg first used this method.)

$$R_{L1R2} = \left[\left(D_{L1} + D^*_{R2}\right)\mathcal{I} + e^{2i\Psi_P}\left(\mathcal{Q} - i\mathcal{U}\right)\right]/2$$
$$R_{R1L2} = \left[\left(D_{R1} + D^*_{L2}\right)\mathcal{I} + e^{-2i\Psi_P}\left(\mathcal{Q} + i\mathcal{U}\right)\right]/2$$

- As time passes, Ψ_P changes in a known way.
- The source polarization term then rotates w.r.t. the antenna leakage term, allowing a separation.



Source polarization rotates with parallactic angle

Leakage term is fixed

Done with the Maths!

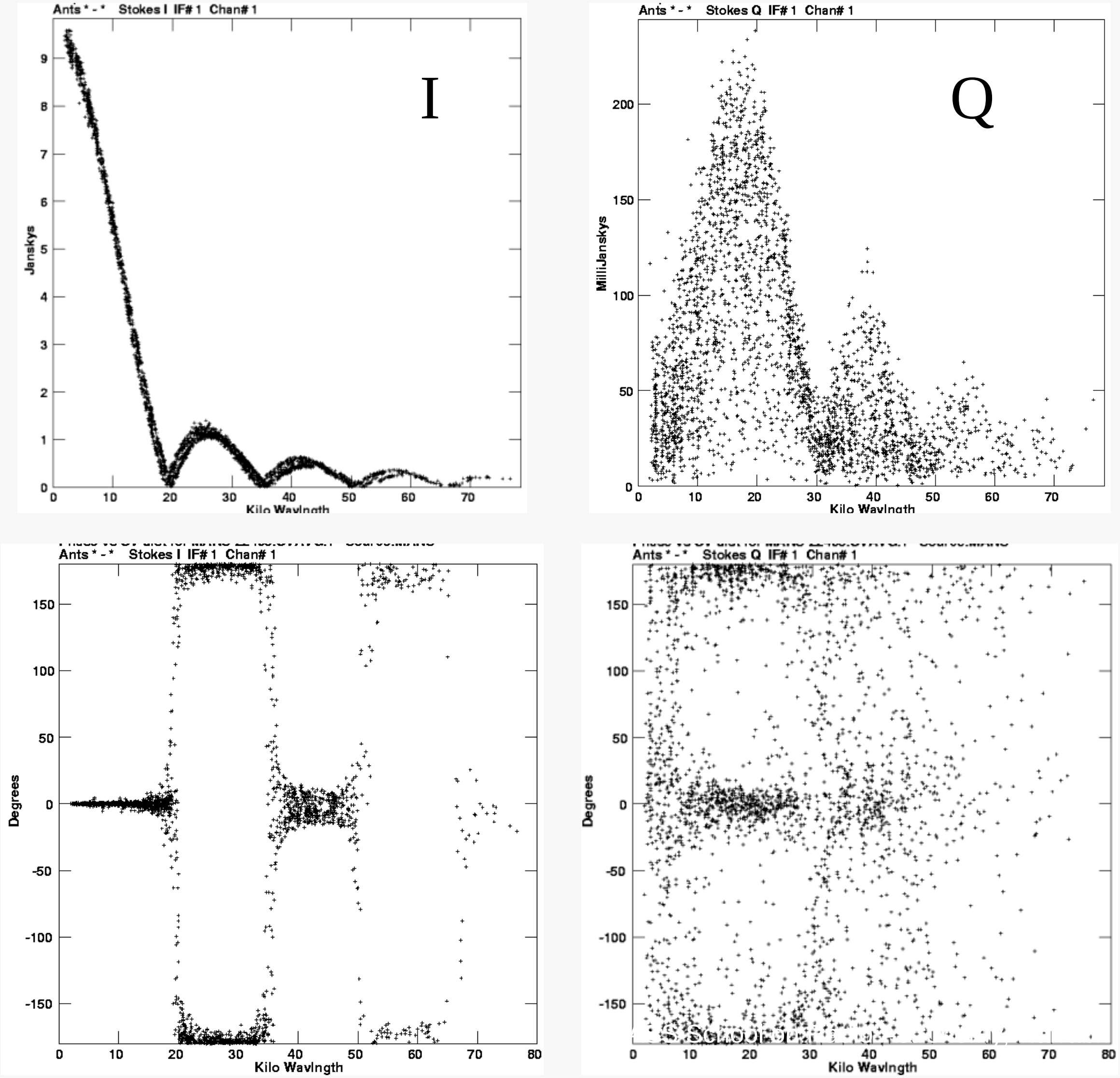
- You’ll (likely) be relieved to know we’re done with the mathematics.
  - But I hope you’ll realize this subject is mathematically very rich, to the delight of those of you who enjoy mathematics.
  - If you want to know more, refer to the papers by Sault and Bregman, and to the EVLA memos by me and Sault.
- The rest of this talk displays real data and images, to illustrate the various points.

$\mathcal{I}, \mathcal{Q}, \mathcal{U}, \mathcal{V}$ Visibilities

- Each of the Stokes parameters images we are interested in has an independent set of visibilities.
- I demonstrate a simple (but quite pretty) case for 23 GHz data from the planet Mars.
- VLA data, D-configuration (3 arcseconds resolution).

$\mathcal{I}$ and $\mathcal{Q}$ Visibilities for Mars at 23 GHz

VLA, 23 GHz, ‘D’ Configuration, January 2006



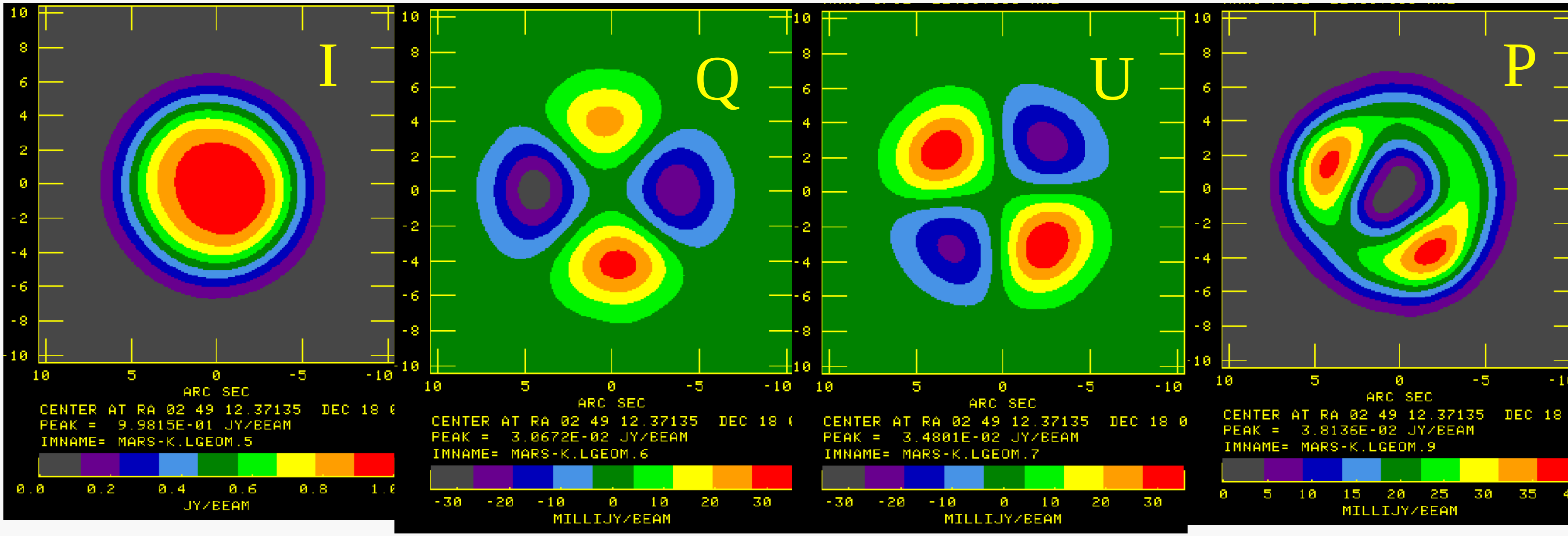
Amplitude

- $|\mathcal{I}|$ is close to a J_0 Bessel function.
- Zero crossing at 20kλ tells us Mars diameter ~ 10 arcsec.
- $|\mathcal{Q}|$ amplitude ~0 at zero baseline.
- $|\mathcal{Q}|$ zero at 30 kλ means polarization structures ~ 8 arcsec scale.

Phase

- $\mathcal{I}$ phase alternates between 0 and π.
- $\mathcal{Q}$ phase = both 0 and π in the ‘main lobe’ – this tells us there are both positive and negative structures, at different PA.

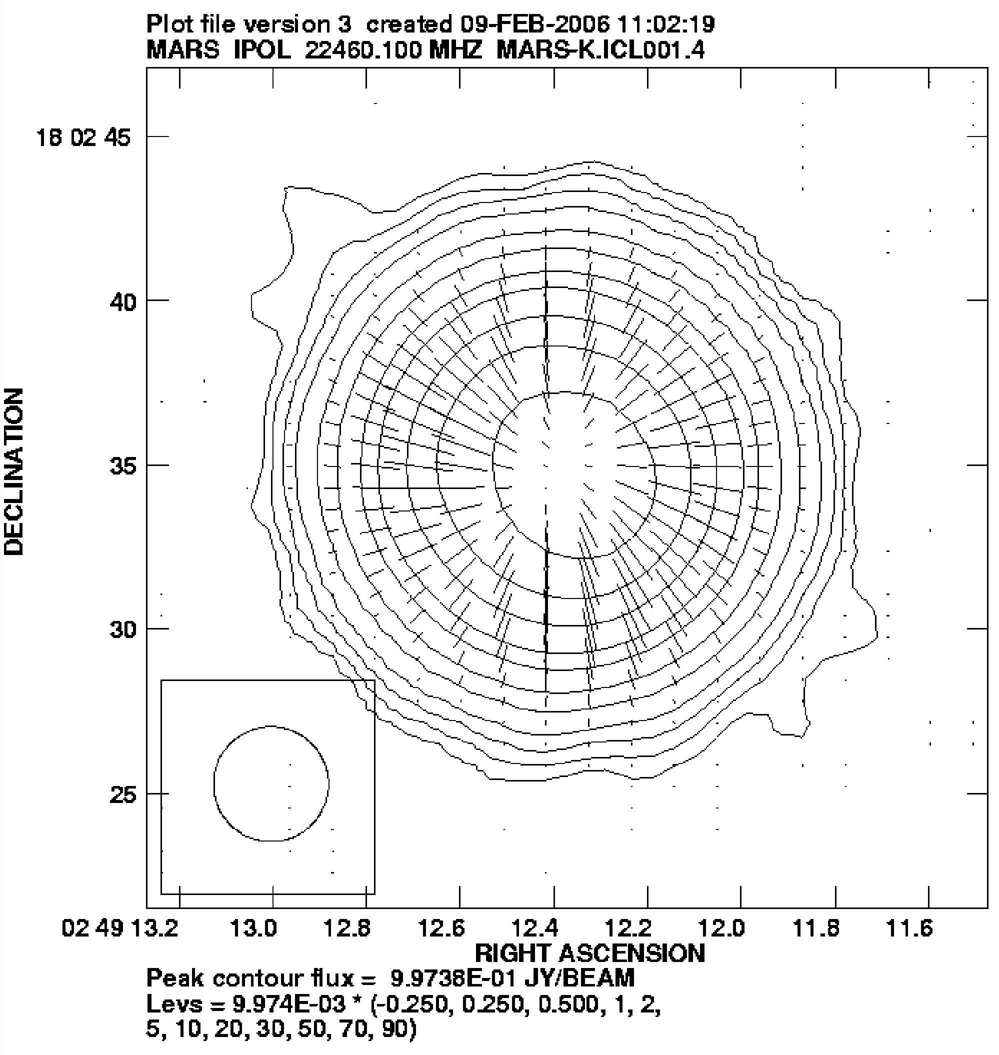
The Images -- Thermal Emission from Mars



- Mars emits in the radio as a black body.
- Shown are false-color coded I,Q,U,P images from Jan 2006 data at 23.4 GHz.
- V is not shown – all noise – no circular polarization.
- Resolution is 3.5”, Mars’ diameter is ~6”.
- From the Q and U images alone, we can deduce the polarization is radial, around the limb.
- Position Angle image not usefully viewed in color.

Mars – A Traditional Representation

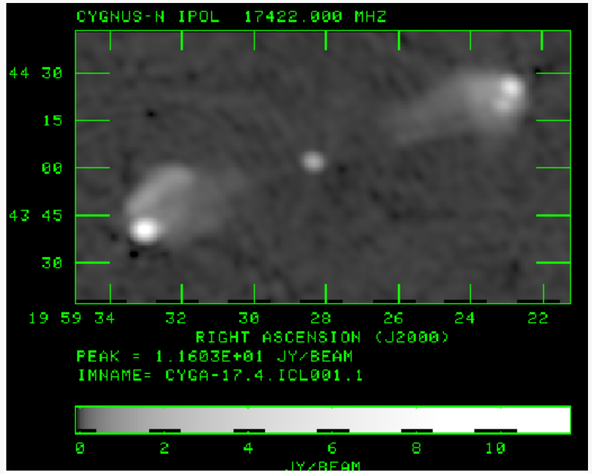
- Here, I, Q, and U are combined to make a more realizable map of the total and linearly polarized emission from Mars.
- The dashes show the direction of the E-field.
- The dash length is proportional to the polarized intensity.
- One could add the V components, to show little ellipses to represent the polarization at every point.



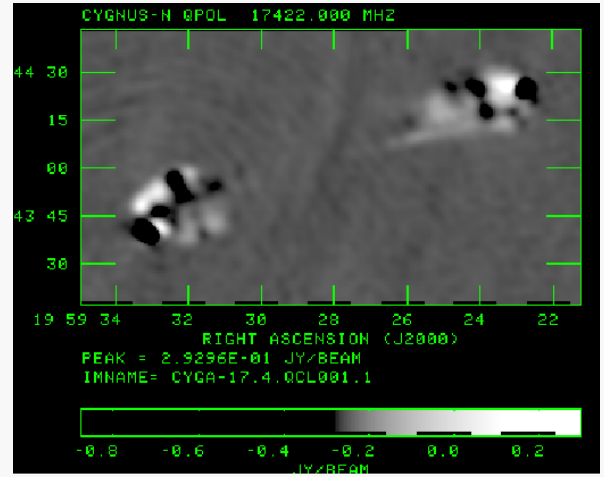
Cygnus A at 17.2 GHz

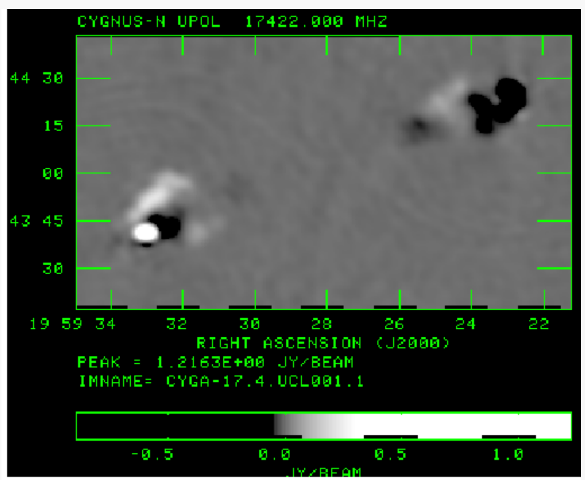
- Cygnus A is a luminous radio galaxy, one of the strongest sources in the sky.
- It is highly polarized at high (> 5 GHz) frequencies.
- Shown here are some D-configuration data, at 17.2 GHz.

I

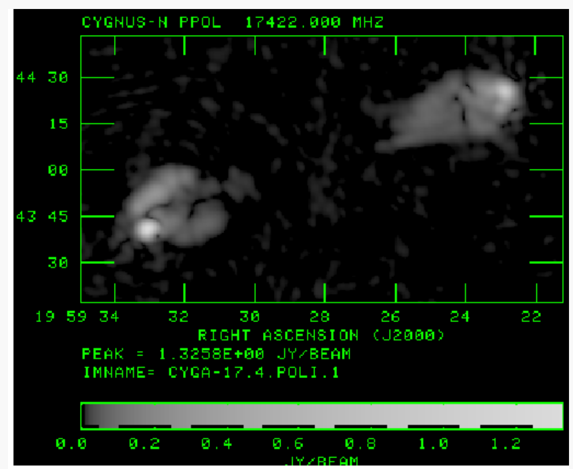


Q



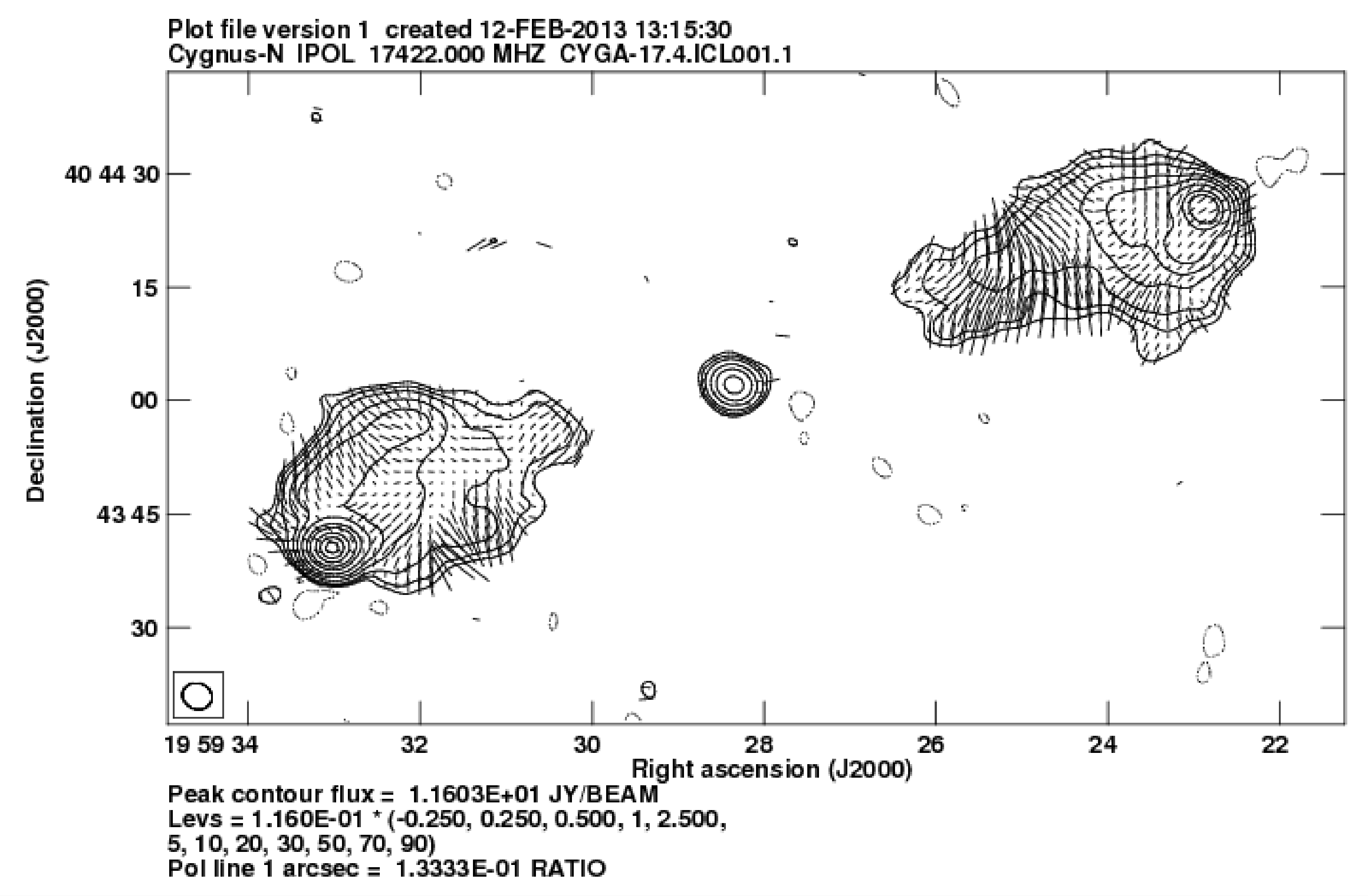


U



Pol I

A more traditional representation.



A Summary

- Polarimetry is a little complicated.
- But, the polarized state of the radiation gives valuable information into the physics of the emission.
- Well designed systems are stable, and have low cross-polarization, making correction relatively straightforward.
- Such systems easily allow estimation of polarization to an accuracy of 1 part in 10000.
- Beam-induced polarization can be corrected in software – development is under way.