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SCHUBERT CALCULUS ON A GRASSMANN ALGEBRA

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Aos meus pais Irandi e Vasconcelos

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Contents

Introduction

The Subject

This thesis is about *Schubert Calculus on Grassmann Algebras* (SCGA). Its main goal consists in proposing a (new) *axiomatic approach* able to describe, within a unified framework, different kind of *intersection theories* living on *grassmannians*, such as the *classical*, the *(small) quantum* and the *equivariant* one. The latter offers, according to the author's opinion, the best application of the "SCGA philosophy" among those discussed in this thesis and listed in the second part of the present introduction.

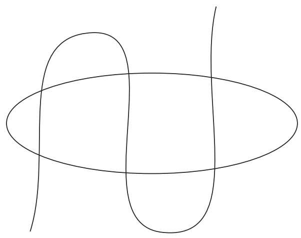
Grassmann Varieties. The *complex Grassmann scheme* (or grassmannian variety, or grassmannian *tout court*) $G_k(\mathbb{P}^n)$ (see Chapter 1) is a complete smooth projective scheme parametrizing k-dimensional linear subspaces of $\mathbb{P}^n$: it can be also thought as the parameter space $G(1+k, \mathbb{C}^{1+n})$ of all $(1+k)$-dimensional vector subspaces of $\mathbb{C}^{1+n}$. The most popular example is, perhaps, that of the grassmannian $G_1(\mathbb{P}^3)$ of all the lines of the complex projective 3-space, which can be realized as a smooth quadric hypersurface in the 5-dimensional projective space (the *Klein's quadric*).

Why Grassmann Algebras? To each module M on a commutative ring A, one can attach its *tensor algebra* $(T(M), \otimes)$. The *Grassmann algebra* $\bigwedge M$ is a suitable quotient of $T(M)$. It is more commonly known as *exterior algebra* but, in spite of daily habits, we prefer the former terminology to emphasize the fact that there is a *Schubert Calculus* (the subject of this thesis) living on two *"Grassmann things"*: a variety and an algebra. The whole point, which does not seem to have attracted any previous special attention in the literature, is that Grassmann algebras carry natural structures able to cope with the *intersection theory* of grassmannians, whose knowledge (amounting to know their *Schubert calculus*), allows, for example, to solve a relevant class of enumerative problems in projective spaces like:

How many lines intersect 4 *others in general position in* $\mathbb{P}^3$*?*

which is perhaps the most famous among them.

Intersection theory on Grassmannians. The *intersection theory* of a Grassmann variety tells us, roughly speaking, how general subvarieties of $G_k(\mathbb{P}^n)$ do intersect. Let us recall a basic and well known situation, that of the grassmannian $G(1, \mathbb{C}^3)$, a fancy way to refer to the projective plane $\mathbb{P}^2$, whose intersection theory is governed by *Bézout's theorem*: two general plane curves of degree d and d' intersect



A cubic and a conic curve in the plane intersect at 6 points by Bézout's theorem

at dd' points (counting multiplicities if the curves do not intersect *transversally*). The way one phrases this fact is that the *Chow intersection ring* $A^*(\mathbb{P}^2)$ (which can be identified with the *integral cohomology* $H^*(\mathbb{P}^2, \mathbb{Z})$) is isomorphic to the ring $\mathbb{Z}[\ell]/(\ell^3)$, where ℓ is the *class of a line*. The relation $\ell^3 = 0$, holding in such a ring, does not mean that three lines in the plane cannot intersect along a common subvariety, but that the general ones do not!

The intersection (or cohomology) ring of a general grassmannian $G_k(\mathbb{P}^n)$ is slightly more complicated: if $1 < k < n - k$, one needs at least $1 + k$ elements to generate the ring as a $\mathbb{Z}$-algebra and the relations can be expressed in terms of symmetric functions in the Chern roots of either the *universal quotient bundle* or the *tautological bundle* living on it.

SCGA. A *SCGA* (Section 2.2) is a pair $(\bigwedge M, D_t)$ where M is a module over an integral $\mathbb{Z}$-algebra A and $D_t := \sum_{i\geq 0} D_i t^i : \bigwedge M \longrightarrow \bigwedge M[[t]]$ is an algebra homomorphism whose *coefficients* $D_i \in End_A(\bigwedge M)$ are pairwise commuting, $D_i(\bigwedge^{1+k} M) \subseteq \bigwedge^{1+k} M$ and D_0 is an automorphism of $\bigwedge M$. The equation

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta, \qquad \forall \alpha, \beta \in \bigwedge M, \qquad (*)$$

holding by definition, is said to be the *fundamental equation of Schubert calculus* D_t on $\bigwedge M$. Notice that equality $(*)$ implies that the h^{th} coefficient D_h of D_t enjoys the h^{th}-Leibniz's rule with respect to the $\wedge$-product:

$$D_h(\alpha \wedge \beta) = \sum_{i=0}^{h} D_i\alpha \wedge D_{h-i}\beta, \qquad \forall \alpha, \beta \in \bigwedge M.$$

In particular D_0 is an algebra homomorphism ($D_0(\alpha \wedge \beta) = D_0\alpha \wedge D_0\beta$) and D_1 is a usual derivation ($D_1(\alpha \wedge \beta) = D_1\alpha \wedge \beta + \alpha \wedge D_1\beta$). The properties enjoyed by the endomorphisms D_h are also properties of any SCGA. For instance, *Newton's binomial formula* for the p^{th} iterated of D_1:

$$D_1^p(\alpha \wedge \beta) = \sum_{h=0}^{n} \binom{p}{h} D_1^h\alpha \wedge D_1^{p-h}\beta,$$

is itself a formula of (any) SCGA.

SCGP. Recall that $\bigwedge M$ is a graded algebra, being the direct sum of all the *exterior powers* of $\bigwedge^{1+k} M$ of M:

$$\bigwedge M = A \oplus \bigoplus_{k\geq 0} \bigwedge^{k+1} M, \qquad (\bigwedge^{0} M := A, \ \bigwedge^{1} M := M).$$

The pair $(\bigwedge^{1+k} M, D_t^{(k)})$, where $D_t^{(k)} := D_{t|_{\bigwedge^{1+k} M}}$, will be said to be the *Schubert Calculus on the k^{th} Grassmann Power* (Definition 2.4.2) associated to the SCGA

$(\bigwedge M, D_t)$ or, briefly, a k-SCGP. The pair $(M, D_t^{(0)})$ will be said to be the *root* of $(\bigwedge M, D_t)$ and it is the $(1+k)^{th}$-root of $(\bigwedge^{1+k} M, D_t^{(k)})$, for each $k \geq 0$. To keep track of the root of an SCGA we will also use the notation $\bigwedge(M, D_t^{(0)})$. Given any pair $(M, D_t : M \longrightarrow M[[t]])$, such that the coefficients of D_t are pairwise commuting, it is easy to see that there is a unique SCGA having (M, D_1) as a root. An SCGA is said to be *simple* if $D_i^{(0)} = (D_1^{(0)})^i$; it is said to be *regular* if there exists $m_0 \in M$ such that the set $\{D_i m_0 \mid i \geq 0\}$ generates M as A-module. A simple SCGA will be also written $\bigwedge(M, D_1^{(0)})$ or, since $D_1^{(0)} m = D_1 m$ for all $m \in M$, also simply as $\bigwedge(M, D_1)$, abusing notation.

In [26] (see also [27] for more examples) a suitable SCGA on a free $\mathbb{Z}$-module of rank $1+n$ is constructed in such a way that the corresponding k-SCGP describes, in a sense that will be explained in this thesis, *Schubert Calculus on the Grassmann Variety* $G_k(\mathbb{P}^n)$ (k-SCGV). Then, while a k-SCGA is taking account of a fixed k-SCGV, the SCGA deals with the intersection theory of $G_k(\mathbb{P}^n)$, for all $0 \leq k \leq n$ at once! In particular, one learns that the intersection theory of $G_k(\mathbb{P}^n)$ is induced by that of $\mathbb{P}^n$ by suitably extending the Chow ring $A^*(\mathbb{P}^n)$ to a ring of operators on the Chow group $A_*(G_k(\mathbb{P}^n))$ of $G_k(\mathbb{P}^n)$ – see the example below.

An example. Let M be a free $\mathbb{Z}$-module spanned by $\boldsymbol{\epsilon} := (\epsilon^0, \epsilon^1, \epsilon^2, \epsilon^3)$. Let $D_1 : M \longrightarrow M$ be the endomorphism defined by:

$$D_1 \epsilon^i = (1 - \delta_{i3}) \epsilon^{i+1-\delta_{i3}},$$

where δ_{i3} is the Kronecker's delta. The matrix of D_1 with respect to the basis $\boldsymbol{\epsilon}$ is a nilpotent Jordan block of maximal rank. Extend D_1 to an endomorphism of $\bigwedge^2 M$ by setting:

$$D_1(\epsilon^i \wedge \epsilon^j) = D_1 \epsilon^i \wedge \epsilon^j + \epsilon^i \wedge D_1 \epsilon^j.$$

Then one easily gets:

$$D_1^4(\epsilon^0 \wedge \epsilon^1) = 2 \cdot \epsilon^2 \wedge \epsilon^3,$$

and Theorem 2.9 of [26] implies that D_1 may be seen as the hyperplane class of the Plücker embedding of $G_1(\mathbb{P}^3)$, represented by the variety of all the lines of $\mathbb{P}^3$

intersecting another fixed line while the coefficient 2 multiplying $\epsilon^2 \wedge \epsilon^3$ ("the class of a point" in $A_*(G_1(\mathbb{P}^3))$ is precisely the number of lines intersecting 4 others in general position in $\mathbb{P}^3$.

Applications to (Equivariant) Cohomology of Grassmannians

The most important achievement of this thesis, according to the author's opinion, is the axiomatic characterization of a SCGA. It comes together with two computational tools, *Leibniz's rule* and *integration by parts*, which are the very abstract counterparts of the classical *Pieri's* and *Giambelli's* formulas holding in the intersection ring of the complex grassmannian $G_k(\mathbb{P}^n)$. In other words, if some kind of intersection theory on a Grassmann variety (or bundle) fits into any SCGA, we automatically know what Giambelli's and Pieri's type formulas may expect to compute, at least in principle, the structural constants of the involved intersection algebra. When applying this general philosophy to equivariant cohomology, we so get answers to questions raised in [43] about the construction of equivariant Pieri's formulas.

Regular Simple SCGAs. Our main applications come from a closer study of regular simple SCGA defined over a free module (of at most countable rank) over an integral $\mathbb{Z}$-algebra A of characteristic zero. In other words we shall deal with k-SCGP of the form $\bigwedge^{1+k}(M, D_1)$. For each $k \geq 0$, $\bigwedge^{1+k} M$ is obviously a module over the ring $A[\mathbf{T}] := A[T_1, T_2, \ldots]$ of the polynomials in infinitely many indeterminates. In fact any such polynomial can be evaluated at $D := (D_1, D_2, \ldots)$ yielding an operator on $\bigwedge^{1+k} M$. A key point of the theory is that if $\bigwedge(M, D_1)$ is regular, then for each $k \geq 0$ there exists an element generating $\bigwedge^{1+k} M$ over $A[\mathbf{T}]$. To any regular k-SCGA over a free module we then associate a ring

$$\mathcal{A}^*(\bigwedge^{1+k}(M, D_1))$$

which isomorphic, as an A-module, to $\bigwedge^{1+k} M$ itself, and will be said to be the *Poincaré dual* of the k-SCGP $\bigwedge^{1+k}(M, D_1)$.

Intersection Theory of Grassmann Bundles. The most important application of the SCGA idea, is the generalization of the main result of [26] to *Schubert Calculus on Grassmann bundles* (as in [23]). Let $p : E \longrightarrow X$ be a vector bundle of rank $1 + n$ over a reasonably well behaved base variety X. Then we show (Theorem 3.4.2) that the *intersection theory* of $G_k(\mathbb{P}(E))$, where $G_k(\mathbb{P}(E))$ is the Grassmann bundle associated to E, is described by $\bigwedge^{1+k}(M, D_1)$, where M can be identified with the Chow group $A_*(\mathbb{P}(E))$ and D_1 with the first Chern class of the tautological subbundle $O_{\mathbb{P}(E)}(-1)$ [1]. "Describes" here means that $A_*(G_k(\mathbb{P}(E))$, thought of as a module over $A^*(G_k(\mathbb{P}(E))$, is isomorphic to $\bigwedge^{1+k} M$, thought of as a module over $\mathcal{A}^*(\bigwedge^{1+k}(M, D_1))$. When X is a point we get the result proven in [26], and although such a generalization may taste as the n^{th} new way to see old results[2], it says that the intersection theory of $G_k(\mathbb{P}(E))$ ($k > 0$) can be achieved by that of the projective bundle $\mathbb{P}(E)$, by *taking exterior powers* of $A_*(\mathbb{P}(E))$, in a suitable sense. This remark has a quite relevant consequence, implying the strongest and most important application of the SCGA we have found up to now, and described below.

Equivariant Cohomology of Grassmannians Acted on by a Torus. Suppose that T is a $(1 + p)$-dimensional algebraic (i.e. $T = (\mathbb{C}^*)^{1+p}$) or compact (i.e. $T = (S^1)^{1+p}$) torus acting on $\mathbb{P}^n$ via some diagonal action with only isolated fixed points. For such a type of action the equivariant cohomology of the projective space is well understood and we wonder how that of the grassmannian $G_k(\mathbb{P}^n)$, under the induced action, looks like. Using Theorem (3.4.2) we can prove Theorem 4.1.10:

[1]The same general result has been achieved by Laksov and Thorup in [46], within the theory of splitting algebras. Their proof of the determinantal formula in a polynomial ring was crucial for understanding our situation.

[2]The structure of the Chow ring $A^*(G_k(\mathbb{P}(E)))$ was already well known!

if $A_T^*(\mathbb{P}^n) := A^*(\mathbb{P}^n \times_T ET)$[3] *is described by some* 0*-SCGP* (M, D_1)*, then* $A_T^*(G_k(\mathbb{P}^n))$ *is described by* $\bigwedge^{1+k}(M, D_1)$. We apply this theorem to study equivariant cohomology of grassmannians to the situation studied by Knutson and Tao in [39], who gave a combinatorial description based on results by Goresky-Kottwitz and MacPherson ([30]). Knutson-Tao's approach is based on the theory of *puzzles*. Ours, instead, is closer to classical Schubert calculus, based on Pieri's and Giambelli's formulas, which, by construction, are already at our disposal. In principle *Pieri's type* formulas can be deduced by the combinatorial description of the equivariant cohomology of a full flag variety offered in [65]. However, with the exceptions of a few, though relevant, cases (see [39], Proposition 2), this may be quite tricky in general.

Our very explicit description, by constrast, does not invoque neither Robinson's framework nor puzzles. The strategy is very simple: first we find a 0-SCGP which is equivalent to the equivariant cohomology of $\mathbb{P}^n$. This is done directly by hand via the explicit construction of an isomorphism. Then we construct the k-SCGP $\bigwedge^{1+k}(M, D_1)$ and applies Theorem 4.1.10 to guarantee that, indeed, we have achieved *equivariant cohomology*.

There are several problems studied by several authors in the equivariant cohomology of grassmannians, such as the determination of a corresponding Littlewood-Richardson rule for the structure constants of the cohomology ring, as well as their non negativity. We have not (yet?) a definitive answer to those type of questions. By the way we are able to construct some D_1-canonical bases for the equivariant cohomology: in such bases, Giambelli's formula coincides with the classical Giambelli's determinantal one. The same holds for Pieri's formula, provided that the codimension of the two multiplied cycles does not exceed the dimension of $G_k(\mathbb{P}^n)$. In this range, and for such a basis, the structure constants are the same as those of the classical cohomology of grassmannians: hence Littlewood-Richardson rule holds in this case. Since canonical bases always exist for any *simple regular SCGA*, we may say, in a sense, that all the $\bigwedge^{1+k}(M, D_1)$ look quite the same and all look

[3] Here $ET \longrightarrow BT$ is the universal T-principal bundle, see Chapter 4.

like classical Schubert calculus: this is true in particular for quantum Schubert calculus according to [5] and, as said, for the equivariant one. This important observation, however, as already remarked, does not prevent us to get general formulas in arbitrary bases. The best example is without doubt our Theorem 4.4.1, where *Pieri's formula for T-equivariant cohomology of grassmannians* is computed also for the cases not deduced in the paper [39][4], so realizing the hopes of the paper [43], in spite our Giambelli's formula being of a rather different nature.

Other Applications

The other applications of SCGA shown in the thesis have been divided into two parts. The former is more historical and pedagogical in its character and regards enumerative applications, classical and new. The second is a miscellanea of results regarding the Grassmannian of lines, obtained especially because of the simpler combinatorics involved in a 1-SCGP (only second exterior powers occur!)

Enumerative Applications. Recall that the intersection ring of the grassmannian $G_k(\mathbb{P}^n)$ is generated by some *Schubert cycles* classically denoted as $(\sigma_1, \ldots, \sigma_{k+1})$. The degree $d_{k,n}$ of the product $\sigma_1^{(1+k)(n-k)}$ is precisely the *Plücker degree* of $G_k(\mathbb{P}^n)$. Such a number had already been computed by Schubert in [70], using a combinatorial remark (see [23], p. 274) whose explanation is natural in our framework (Sect. 5.1). We can find a new easier proof computing the degree of any Schubert variety which has the advantage to generalize to other situations. For instance, we will produce a combinatorial formula for $\sigma_2^{2(d-3)}$, computing the number of projectively non equivalent rational curves of degree d in $\mathbb{P}^3$ having $2d-6$ flexes at specified positions[5] (see Section 5.2). Although the formula does not look very nice (many summations over many indices) one can easily plug it in **R**, a program for statistical computing, getting a table of the values up to $d = 10$.

[4]Even if in principle are deducible by Robinson's description; see [65].

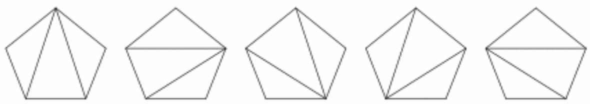
[5]This problem has been suggested to us by Prof. K. Ranestad

Playing with Grassmannian of Lines. The intersection theory of a grassmannian of lines is described by a 1-SCGP (as in [26]), where computations are very easy. Within this context

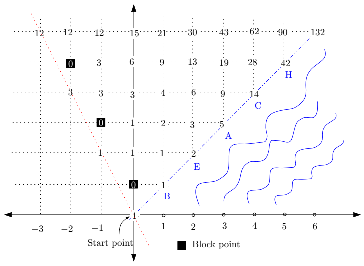
1. in Section 6.1 we (re)prove using SCGA framework that *in the Grassmannian* $G_1(\mathbb{P}^n)$, 0 *or* 1 *are the only possible Littlewood-Richardson coefficients.*

2. in Section 6.2 we deduce *a formula computing the degree of all the intersection products* $\kappa_{a,b} := \sigma_1^a \sigma_2^b$ in $A^*(G_1(\mathbb{P}^{n+1}))$ with $a + 2b = 2n$. Plugging this formula in `Matematica`© or **R**, one easily gets a table for $\kappa_{a,b}$, varying a and b;

3. in Section 6.4 some relationships with combinatorics are explored. In fact it is known (see e.g. [53]), that the Plücker degree of $G_1(\mathbb{P}^{n+1})$ coincide with the n^{th} Catalan number C_n (the number of different ways a convex polygon with $n+2$ sides can be decomposed into triangles, by drawing straight lines connecting vertices (see the figure below for C_3).

   Catalan's numbers occur also in several problems of lattice path enumeration. Niederhausen ([54], see also [76]) computed the number of different paths from the origin to the points of a *city map*, bounded by some *traffic blocks* and by a *beach* (whence the title of the paper "Catalan Traffic at the Beach"). Then he show that such numbers, for points on the beach line, are precisely the Catalans. We show that this fact is more than accidental: in a region of the city map bounded by the y-axis and the *beach* (Fig. below), the *solutions to Niederhausen's problem are precisely the numbers* $\kappa_{2m,n-m}$ *of lines of* $\mathbb{P}^{1+n}$ *intersecting a general configuration (in* $\mathbb{P}^{1+n}$*) of* $2m$ *subspaces of codimension* 2 *and* $n-m$ *subspaces of codimension* 3.



The easy proof relies on the differentiation formalism peculiar of a SCGA.

4. thinking of an exterior power of a module inside its exterior algebra, it is natural to expect that the enumerative geometry of grassmannians $G_k(\mathbb{P}^n)$ can be studied in terms of that of grassmannian $G_{k'}(\mathbb{P}^{n'})$ with $k' < k$ and $n' \leq n$. For instance we easily get formulas relating the degree of grassmannians $G_2(\mathbb{P}^n)$ with degrees of grassmannians $G_1(\mathbb{P}^n)$ (See Section 5.3). It is reasonable to believe that there may exist some generating function encoding all possible degrees of all possible grassmannians. To support this belief, *a generating function encoding the degrees of* $G_1(\mathbb{P}^n)$ *is computed*. It is expressed in terms of *modified Bessel functions* and as a byproduct we also got, a *new generating function for the Catalan's numbers* (See Section 6.3);

About the References

Schubert Calculus is related with many branches of mathematics, such as combinatorics, topology, analysis and representation theory. Since the pioneering papers by Schubert [70], Pieri [58] and Giambelli ([32], [33]; see also [44]) a rich literature has flourished on the subject. Besides the milestone papers such as [38] and [45], in recent years Schubert Calculus has been extended to more general (e.g. complete) flag varieties (see the beautiful lecture notes [25]) and to other types of Grassmannians (e.g. [60] and [61]). As a matter of combinatorial aspects, such as multiplication of Schubert classes ([63]) also related with small quantum cohomology of flag varieties, there is a wide production of many important authors (e.g. [3], [5], [6], [13] and references therein; see also [57] and, of course [80]). For combinatorial problems of recognizing Schubert cells there is the important [22]).

The bibliography at the end of this thesis is far from being complete. Moreover not all the papers and/or books included in it have been explicitly quoted in the text, but all have influenced and inspired, proportionally to my (not always) complete understanding, the way I tried to expose the subject of this thesis.

Chapter 1

Grassmann Varieties

In this chapter we review basics on grassmannian varieties and their intersection theory, rephrasing some known facts borrowed from the standard references on the subject like [23], [31], [50], to whom the reader who wants to get into more details is really referred. The reader who want to read the subject in a historical perspective should read [42]. An alternative exposition can be found in [27].

1.0.1 Notation. In this and subsequent chapters, we shall denote by $\mathcal{I}^k$ the subset of $\mathbb{N}^{1+k}$ of all increasing multi-indices of size $1+k$:

$$\mathcal{I}^k := \{(i_0, i_1, \ldots, i_k) \in \mathbb{N}^{1+k} \,|\, 0 \leq i_0 < i_1 < \ldots < i_k\}. \tag{1.1}$$

One also denotes by $\mathcal{I}^k_n$ the subset of $\mathcal{I}^k$ such that $i_n \leq n$:

$$\mathcal{I}^k_n := \{(i_0, i_1, \ldots, i_k) \in \mathbb{N}^{1+k} \,|\, 0 \leq i_0 < i_1 < \ldots < i_k \leq n\} \subset \mathcal{I}^k \tag{1.2}$$

1.1 Generalities

1.1.1 Let V be a finite dimensional complex vector space. For each integer $l \geq 0$, the *grassmannian variety* $G_l(V)$ parametrizes l-dimensional vector subspaces of V. If $l = 0$, then $G_l(V)$ is just a point (the null vectorspace [0]) while it is empty if l-exceeds the dimension of V. Let

$$U := \{\Lambda \in Hom_{\mathbb{C}}(\mathbb{C}^l, V) \mid \wedge^l \Lambda \neq 0\}$$

be the Zariski open subset in $Hom_{\mathbb{C}}(\mathbb{C}^l, V)$ parametrizing linear map $\Lambda : \mathbb{C}^l \longrightarrow V$ of maximal rank. Any $\Lambda \in U$ determines $[\Lambda] := Im(\Lambda) \subseteq G_l(V)$. Two linear maps $\Lambda, \Lambda' \in U$ determine the same subspace if and only if there exists $\phi \in Gl_k(\mathbb{C})$ such that $\Lambda' = \Lambda \circ \phi$. Hence, at least as a set, the grassmannian $Gl_l(V)$ can be seen as the orbit space $U/Gl_l(\mathbb{C})$. As a matter of fact, $Gl_l(\mathbb{C})$ acts on U algebraically and the quotient turns out to be a smooth projective variety or, in the analytic category, a smooth compact connected holomorphic manifold. The smoothness comes from the fact that $Gl(V)$ acts transitively on $G_l(V)$. Any $\Lambda \in U$ shall be identified with the l-tuple $(\mathbf{v}_i) := (\Lambda(e_i))$, where (e_i) is the canonical basis of $\mathbb{C}^l$. The corresponding l-plane shall be denoted also as $[\mathbf{v}_i]_{1 \leq i \leq l}$ – see below.

1.1.2 Definition. *The projective space associated to V is*

$$\mathbb{P}(V) := G_1(V).$$

In other words the space $\mathbb{P}(V)$ parametrizes the 1-dimensional subspaces of V. In this thesis we are more interested in working with the projective grassmannian. Let $k \geq 0$ be an integer.

1.1.3 Definition. *The grassmannian variety of the projective k-planes of $\mathbb{P}(V)$ is:*

$$G_k(\mathbb{P}(V)) := G_{1+k}(V).$$

Evidently $G_0(\mathbb{P}(V)) = \mathbb{P}(V)$, the set of points of $\mathbb{P}(V)$ itself.

From now on, one will assume that $\dim(V) = 1 + n$ and fix, once and for all, a basis $E := (e_0, e_1, \ldots, e_n)$. Let $\boldsymbol{\epsilon} := (\epsilon^0, \epsilon^1, \ldots, \epsilon^n)$ be the dual basis (i.e., $\epsilon^j(e_i) = \delta_i^j$). A (projective) k-plane in $\mathbb{P}(V)$ will be denoted as:

$$[\Lambda] = [\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k],$$

represented by the $k+1$-frame[1] $\Lambda = (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k)$. Let

$$\{e_{i_0} \wedge e_{i_1} \wedge \ldots \wedge e_{i_k} \,|\, 0 \leq i_0 < i_1 < \ldots < i_k \leq n\}$$

[1]A m-frame is an ordered set of $m \geq 1$ linearly independent vectors.

and

$$\{\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \,|\, 0 \leq i_0 < i_1 < \ldots < i_k \leq n\}$$

be the bases of $\bigwedge^k V$ and $\bigwedge^k V^\vee \cong (\bigwedge^k V)^\vee$, induced by E and $\boldsymbol{\epsilon}$ respectively. Then $G_k(\mathbb{P}(V))$ is covered by the following affine charts ([31]):

$$U_{i_0 i_1 \ldots i_k} := \{[\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k] \mid \epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\mathbf{v}_0 \wedge \mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k) \neq 0 \leq n\}. \quad (1.3)$$

Indeed there is a bijection between $U_I \longrightarrow \mathbb{C}^{(n-k)(1+k)}$ ($I \in \mathcal{I}_n^k$). Since U_I is a dense open set, then $\dim G_k(\mathbb{P}(V)) = (n-k)(1+k)$.

1.1.4 An element of $\bigwedge^{1+k} V$ is said to be *decomposable* if it is of the form $\mathbf{v}_0 \wedge \mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k$, for some $\mathbf{v}_i \in V$. The grassmannian $G_k(\mathbb{P}(V))$ can be then identified with the subvariety in $\mathbb{P}(\bigwedge^{1+k} V)$ of all non-zero decomposable elements. The map

$$\begin{array}{rccc} Pl_{\mathcal{E}} & : & G_k(\mathbb{P}(V)) & \longrightarrow & \mathbb{P}(\bigwedge^{1+k} V) \\ & & [\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k] & \longmapsto & \mathbf{v}_0 \wedge \mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k \end{array}$$

is said to be the Plücker map.

1.1.5 Proposition. *The Plücker map $Pl_{\mathcal{E}}$ is an embedding.*

Proof. It is not hard to see that the Plücker map is injective. The key point is to show that its tangent map is injective, too.

Let $B_\varepsilon := \{z \in \mathbb{C} \mid |z| < \varepsilon\}$, be a disc in the complex plane, $[\Lambda] \in G_k(\mathbb{P}(V))$ and $\gamma : B_\epsilon \longrightarrow G_k(\mathbb{P}(V))$ is a holomorphic curve such that $\gamma(0) := [\Lambda]$. Then, any tangent vector to $[\Lambda] \in G_k(\mathbb{P}(V))$ is of the form $(d\gamma/dz)_{z=0}$. Therefore,

$$T_{[\Lambda]} Pl_{\mathcal{E}} \left(\left. \frac{d\gamma}{dz} \right|_{z=0} \right) = \left(\left. \frac{d}{dz} (\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(z)) \right|_{z=0} \right)_{(i_0, i_1, \ldots, i_k) \in \mathcal{I}_n^k}.$$

is the tangent of the Plücker map at $[\Lambda]$. To prove the injectivity is then sufficient to show that

$$\left. \frac{d\gamma}{dz} \right|_{z=0} \neq 0 \Rightarrow T_{[\Lambda]} Pl_{\mathcal{E}} \left(\left. \frac{d}{dz} \right|_{z=0} \gamma(z) \right) \neq 0.$$

Since this is a local property, it suffices to check it on an affine open set of the Grassmannian of the form U_I ($I \in \mathcal{I}_n^k$) containing $[\Lambda]$. Up to a linear transformation permuting the elements of the basis E_n, one may assume that $I = (01 \ldots k)$. Any k-plane in U_I can be represented by a maximal rank matrix of the form:

$$\begin{pmatrix} & I_{(k+1)\times(k+1)} & & \\ & & & \\ x_{1,0} & \dots & x_{1,k} & \\ \vdots & \ddots & \vdots & \vdots \\ x_{n-k,0} & \dots & x_{n-k,k} & \end{pmatrix},$$

where a tangent vector can be written in the form $(dx_{ij}/dz)_{|z=0}$. For each pair (i,j), such that $0 \leq i \leq k$ and $k+1 \leq j \leq n$, one has

$$(\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \widehat{\epsilon^j} \wedge \ldots \wedge \epsilon^k \wedge \epsilon^{k+i})(\Lambda) = (-1)^{k-j+i-1} x_{ij},$$

so that the tangent map can be written as

$$\left((-1)^{k-j+i-1} \frac{dx_{ij}}{dz}\bigg|_{z=0}, \frac{d}{dz}\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\gamma(z))\bigg|_{z=0} \right)_{(i_0,i_1,\ldots,i_k)\in B},$$

where B is the set of all elements of $\mathcal{I}_n^k$ such that $\sharp(B \cap \{0,1\ldots,k\}) \leq k-2$. Then, the null tangent vector is the unique pre-image of null vector through the tangent of $\mathcal{P}l_{\mathcal{E}}$, i.e. this is injective and the Plücker map is an embedding. ∎

1.1.6 Another way to phrase the Plücker map is as follows,

$$[\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k] \mapsto (\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\mathbf{v}_0 \wedge \mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k)),$$

where one sets:

$$\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}(\mathbf{v}_0 \wedge \mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_k) = \begin{vmatrix} \epsilon^{i_0}(\mathbf{v}_0) & \epsilon^{i_0}(\mathbf{v}_1) & \ldots & \epsilon^{i_0}(\mathbf{v}_k) \\ \epsilon^{i_1}(\mathbf{v}_0) & \epsilon^{i_1}(\mathbf{v}_0) & \ldots & \epsilon^{i_1}(\mathbf{v}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon^{i_k}(\mathbf{v}_0) & \epsilon^{i_k}(\mathbf{v}_1) & \ldots & \epsilon^{i_k}(\mathbf{v}_k) \end{vmatrix}.$$

1.1.7 Example. Let $V := \mathbb{C} \oplus \mathbb{C}^3$. The Plücker map sends

$$\Lambda := \begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \in \mathbb{C}^{4\times 2} \mapsto Pl(\Lambda) := (\epsilon^i \wedge \epsilon^j(\Lambda))_{0\leq i<j\leq 3} =$$

$$= (\epsilon^0 \wedge \epsilon^1(\Lambda) : \epsilon^0 \wedge \epsilon^2(\Lambda) : \epsilon^0 \wedge \epsilon^3(\Lambda) : \epsilon^1 \wedge \epsilon^2(\Lambda) : \epsilon^1 \wedge \epsilon^3(\Lambda) : \epsilon^2 \wedge \epsilon^3(\Lambda)) =$$

$$= \left(\begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix}, \begin{vmatrix} a_0 & b_0 \\ a_2 & b_2 \end{vmatrix}, \begin{vmatrix} a_0 & b_0 \\ a_3 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}, \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \right) =$$

$$= (a_0b_1 - a_1b_0 : a_0b_2 - a_2b_0 : a_0b_3 - a_3b_0 : a_1b_2 - a_2b_1 : a_1b_3 - a_3b_1 : a_2b_3 - a_3b_2) \in \mathbb{P}^5.$$

Since, as it is apparent, the image of Λ depends only on its class modulo $Gl_2(\mathbb{C}^4)$, one has really got a map from $G_1(\mathbb{P}^3)$ to $\mathbb{P}^5$. Notice that

$$(\epsilon^0 \wedge \epsilon^1)(\Lambda)(\epsilon^2 \wedge \epsilon^3)(\Lambda) - (\epsilon^0 \wedge \epsilon^2)(\Lambda)(\epsilon^1 \wedge \epsilon^3)(\Lambda) + (\epsilon^0 \wedge \epsilon^3)(\Lambda)(\epsilon^1 \wedge \epsilon^2)(\Lambda) = 0,$$

for each $[\Lambda] \in G_1(\mathbb{P}^3)$, which is the equation of the Klein quadric in $\mathbb{P}^5 \cong \mathbb{P}(\bigwedge^2(\mathbb{C}^4))$.

The conclusion is that the elements $\{\epsilon^{i_0} \wedge \epsilon^{i_1} \in \bigwedge^2 V\}$ can be indeed thought of as Plücker coordinates of $\mathbb{P}(\bigwedge^2 V)$. More in general, $\{\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} \in \bigwedge^{1+k} V\}$ can be indeed thought of as Plücker coordinates of $\mathbb{P}(\bigwedge^{1+k} V)$.

1.2 Intersection Theory on $G_k(\mathbb{P}(V))$

This section aims to quickly review some intersection theory on Grassmann varieties to allow the suitable comparisons with the k-SCGPs studied later on.

1.2.1 Let $A_d(X)$ be the group of d-dimensional cycles classes modulo rational equivalence of a smooth projective complex variety. Let:

$$A^k(X) := A_{n-k}(X),$$

be the *Chow group of cycles of* X *of codimension* k. There is an obvious $\mathbb{Z}$-module isomorphism between $A^*(X) := \oplus A^i(X)$ and $A_*(X)$ (the Chow group). If X is smooth (as in our hypothesis), one can put on $A^*(X) \cong A_*(X)$ an *intersection product*

$$\begin{cases} \cdot \quad : \quad A^i(X) \times A^j(X) & \longrightarrow \quad A^{i+j}(X) \\ \qquad\qquad (\alpha, \beta) & \longmapsto \quad \alpha \cdot \beta \end{cases},$$

making it into a ring.

1.2.2 Geometrical interpretation of the intersection product.

Let V_1 and V_2 be two subvarieties of X. Recall that each irreducible component W of $V_1 \cap V_2$ satisfies $codim(W) \leq codim(V_1) + codim(V_2)$. We say that V_1 and V_2 intersect *properly* in X, if

$$codim(W) = codim(V_1) + codim(V_2), \text{ for each } W.$$

Thus, we have in $A^*(X)$:

$$[V_1] \cdot [V_2] = \sum_W m_W [W], \tag{1.4}$$

where the sum is over all irreducible components of the scheme theoretical intersection $V_1 \cap V_2$, and m_W is the *intersection multiplicity* of V_1 and V_2 along W. Moreover they intersect *transversally* along W, if and only if there exists a point $w \in W$ such that w is a non singular point of V_1 and V_2, and the tangent space at X satisfy

$$T_w V_1 + T_w V_2 = T_w X.$$

The point $w \in W$ is then non singular, and $T_w W = T_w V_1 \cap T_w V_2$. If V_1 and V_2 intersect transversally along W, the definition of intersection multiplicity is in such a way that $m_W = 1$. For this reason the product above is said to be *intersection product.*

1.2.3 Intersection product on homogeneous varieties.

Let X is a G-homogeneous variety and $[Y_1]$ and $[Y_2]$ are any two cycles of X. Kleiman's transversality Theorem (see [41]) ensures that there exists a dense Zariski open set $U \subset G$ such that for each $g \in U$, Y_1 and $L_g(Y_2)$ meet properly, where $L_g : X \longrightarrow X$ denotes the left translation. In this case, according to (1.4), $[Y_1] \cdot [gY_2] = [Y_1 \cap L_g(Y_2)]$. Moreover, in the case of the group $G = Gl_n(\mathbb{C})$, which is that acting on grassmannians, the map (L_g) is proper and

$$(L_g)_* : A_*(X) \longrightarrow A_*(X)$$

is the identity (Cf. [23], p. 207). In particular, the *self intersection* $[Y]^2$ is represented by the intersection of Y with a general translate of it.

1.2.4 Remark. The *cap product*:

$$\begin{cases} \cap: & A^*(X) \times A_*(X) & \longrightarrow & A_*(X) \\ & (\alpha, [V]) & \longmapsto & \alpha \cap [V] \end{cases},$$

gives the Chow Group a module structure over $A^*(X)$. The map $\alpha \mapsto \alpha \cap [X]$ induces an isomorphism (*Poincaré duality*)

$$A^j(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong A_{n-j}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

1.2.5 To do intersection theory on grassmannians, one first notice that the basis E of V induces a filtration $E^\bullet$:

$$E^\bullet : \quad E^0 := V \supset E^1 \supset \ldots \supset E^n \supset [\mathbf{0}],$$

where $E^i := [e_i, e_{i+1}, \ldots, e_n]$. If $[\Lambda]$ is any $(1+k)$-plane, one obviously gets the chain of inclusions:

$$[\Lambda] \supseteq [\Lambda] \cap E^1 \supseteq \ldots \supseteq [\Lambda] \cap E^n \supseteq [\mathbf{0}]$$

inducing the chain of inequalities

$$1 + k := \dim([\Lambda]) \geq \dim([\Lambda] \cap E^1) \geq \ldots \geq \dim([\Lambda] \cap E^n) \geq 0. \qquad (1.5)$$

Let

$$\begin{pmatrix} \epsilon^0 \\ \epsilon^1 \\ \vdots \\ \epsilon^n \end{pmatrix} ([\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k]) := \begin{pmatrix} \epsilon^0(\mathbf{v}_0) & \epsilon^0(\mathbf{v}_1) & \ldots & \epsilon^0(\mathbf{v}_k) \\ \epsilon^1(\mathbf{v}_0) & \epsilon^1(\mathbf{v}_1) & \ldots & \epsilon^1(\mathbf{v}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon^n(\mathbf{v}_0) & \epsilon^n(\mathbf{v}_1) & \ldots & \epsilon^n(\mathbf{v}_k) \end{pmatrix}$$

be the matrix whose entries are the components of the $(1+k)$-frame $[\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k]$ in the E-basis and denote by $\rho_i(E, \Lambda)$ the rank of the submatrix formed by the first i rows $(0 \leq i \leq k)$.

1.2.6 Proposition. *The following equality holds:*

$$\dim(E^i \cap [\Lambda]) := 1 + k - \rho_i(E, \Lambda).$$

Proof. In fact, the vectors $\mathbf{v} \in [\Lambda]$ belonging to E^i must satisfy the linear system of equations:

$$\left\{\epsilon^j(\mathbf{v}) = 0\ , \quad 0 \leq j \leq i.\right.$$

The dimension of the space of solutions is precisely $1 + k$ minus the rank of the system, which is exactly $\rho_i(E, \Lambda)$. ∎

Because of the obvious inequalities:

$$\rho_i(E, \Lambda) \leq \rho_{i+1}(E, \Lambda) \leq \rho_i(E, \Lambda) + 1$$

(adding a row to a matrix, the rank increases at most of 1), one deduces that

$$\dim(E^{i+1} \cap [\Lambda]) \leq \dim(E^i \cap [\Lambda]) \leq \dim(E^{i+1} \cap [\Lambda]) + 1,$$

i.e., any possible dimension "jump" is not bigger than 1. The upshot is that in the sequence:

$$k := \dim([\Lambda] \cap E^0) \geq \dim([\Lambda] \cap E^1) \geq \ldots \geq \dim([\Lambda] \cap E^{n+1}) = 0,$$

there are exactly $1 + k$ *dimension jumps.* Clearly, if $[\Lambda]$ is in general position with respect to the flag $E^\bullet$, the *jump sequence* is $1, 2, \ldots, k + 1$. Such a jump sequence shall be also called the $E^\bullet$-*Schubert index* associated to the $(1 + k)$-plane $[\Lambda]$. It is convenient to denote it as follows:

$$Sch_{E^\bullet}([\Lambda]) = (i_0 + 1, i_1 + 1, \ldots, i_k + 1),$$

where $0 \leq i_0 < i_1 < \ldots < i_k \leq n$.

1.2.7 Definition. *The* $E^\bullet$-*Schubert variety associated to the Schubert index* $(i_0 + 1, i_1 + 1, \ldots, i_k + 1)\}$ *is:*

$$\Omega_{i_0 i_1 \ldots i_k}(E^\bullet) = \{[\Lambda] \mid \dim([\Lambda] \cap E^{i_j}) \geq k - j$$

The Schubert variety $\Omega_{i_0 i_1 \ldots i_k}(E^\bullet)$ is the closure of

$$\mathring{\Omega}_{i_0 i_1 \ldots i_k}(E^\bullet) = \{[\Lambda] \mid \dim([\Lambda] \cap E^{i_j}) = k - j\ \}.$$

1.2.8 Proposition. *The set* $\overset{\circ}{\Omega}_{i_0 i_1 \ldots i_k}(E^\bullet)$ *is an affine cell of codimension* $(i_0 - 0) + (i_1 - 1) + \ldots + (i_k - k)$.

Proof. The proof is easy and standard. It is basically omitted for typographical reasons: it consists in finding suitable representatives of the k-planes lying in the set $\Omega_{i_0 i_1 \ldots i_k}$. See e.g. [31] and [42]. ∎

To any (projective) k-plane one can obviously attach one and only one Schubert index and therefore the Schubert cells form a partition of the grassmannian. A well known result of algebraic topology guarantees that the homology classes of their closures (the Schubert varieties themselves) generate the homology (or the Chow group). Furthermore, Chow's basis theorem says ([23], p.268) that $\Omega_{i_0 i_1 \ldots i_k} = [\Omega_{i_0 i_1 \ldots i_k}(E^\bullet)]$ form a $\mathbb{Z}$-basis of $A_*(G_k(\mathbb{P}(V))$. The notation reflects the fact that the Schubert class $\Omega_{i_0 i_1 \ldots i_k}$ does not depend on the chosen flag. Denote by $\sigma_{i_0 i_1 \ldots i_k} \in A^*(G_k(\mathbb{P}(V))$ its *Poincaré dual*, i.e.

$$\sigma_{i_0 i_1 \ldots i_k} \cap [G_k(\mathbb{P}(V)] = \Omega_{i_0 i_1 \ldots i_k},$$

where $[G_k(\mathbb{P}(V))]$ denotes the *fundamental class* of $G_k(\mathbb{P}(V))$.

1.2.9 Remark. In the current literature it is customary to index a Schubert variety using partitions: if $I = (i_0, i_1, \ldots, i_k) \in \mathcal{I}_n^k$, one writes $\boldsymbol{\Omega}_{\underline{\lambda}}$ instead of Ω_I (as it is done, e.g., in [55]), where $\underline{\lambda} = \underline{\lambda}(I) := (i_k - k, \ldots, i_1 - 1, i_0 - 0)$. Similarly one writes $\boldsymbol{\sigma}_{\underline{\lambda}}$ instead of $\sigma_{i_0 i_1 \ldots i_k}$. The $\boldsymbol{\sigma}_{\underline{\lambda}}$ are said to be *Schubert cycles* and they freely generate $A^*(G_k(\mathbb{P}(V))$ as a module over the integers.

1.2.10 The general (projective)k-plane has Schubert index $(1, 2, \ldots, k+1)$. In fact, a projective k-plane is general with respect to the flag $E^\bullet$ if its intersection with E^{1+k} (the subspace of codimension $1+k$) is the null vector[2]. Indeed, the $E^\bullet$-general k-plane lives in the complement of a Zariski closed set. Let us see that. If $[\Lambda]$ is not general, then $\dim([\Lambda] \cap E^{1+k}) > 0$. Hence there exists $0 \neq \mathbf{v} \in [\Lambda] \cap E^{1+k}$,

[2] the general homogeneous linear system of $1+k$ equations in $1+k$ unknowns, has no solution but the trivial one

i.e. there exists $\mathbf{u} \in \mathbb{C}^{1+k} \setminus \{\mathbf{0}\}$, such that $\Lambda \cdot \mathbf{u}$ satisfies the linear system:

$$\epsilon^i(\Lambda) \cdot \mathbf{u} = 0, \qquad 0 \le i \le k.$$

This is possible if $\det(\epsilon^i(\Lambda)) = 0$, i.e if $\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k(\Lambda) = 0$. If $\Lambda = (\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k)$, the condition can be recast as:

$$\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k([\Lambda]) = \begin{vmatrix} \epsilon^0(\mathbf{v}_0) & \epsilon^0(\mathbf{v}_1) & \ldots & \epsilon^0(\mathbf{v}_k) \\ \epsilon^1(\mathbf{v}_0) & \epsilon^1(\mathbf{v}_1) & \ldots & \epsilon^1(\mathbf{v}_k) \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon^k(\mathbf{v}_0) & \epsilon^k(\mathbf{v}_1) & \ldots & \epsilon^k(\mathbf{v}_k) \end{vmatrix} = 0.$$

1.2.11 Definition. (See [27]) *The expression* $\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k([\Lambda])$ *is said to be the* $E^\bullet$-Schubert Wronskian *at* $[\Lambda]$.

Hence the (projective) k-planes in $E^\bullet$-special position live in the zero-scheme $Z(\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k)$ of $\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k$. It is worth to remark that the latter depends only on the flag $E^\bullet$ and not on the adapted basis to the flag itself. For, were $(\phi^0, \phi^1, \ldots, \phi^n)$ another basis such that $E^i = [\phi^{i+1}, \ldots, \phi^n]$, then the transformation matrix T from (ϵ^j) to the (ϕ^j) would be triangular, so that

$$\phi^0 \wedge \phi^1 \wedge \phi^2 \wedge \ldots \wedge \phi^k = \det(T) \cdot \epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k.$$

Let

$$0 \longrightarrow \mathcal{T}_{1+k} \longrightarrow G_k(\mathbb{P}(V)) \times V \longrightarrow \mathcal{Q}_{1+k} \longrightarrow 0, \tag{1.6}$$

be the *tautological exact sequence*: $\mathcal{T}_{1+k}$ is the *universal tautological subbundle* of the trivial bundle $G_k(\mathbb{P}(V)) \times V$ and $\mathcal{Q}_{1+k}$ is the *universal quotient bundle*. Since $(\epsilon^0, \epsilon^1, \ldots, \epsilon^n)$ are sections of $\mathcal{T}_{1+k}^\vee$, it follows that $\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k$ is a section of the line bundle $\wedge^k \mathcal{T}_{1+k}^\vee$ and thus that the class of the $E^\bullet$-special k-planes in the Chow group $A_*(G_k(\mathbb{P}(V))$ is precisely:

$$[Z(\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k)] = c_1(\wedge^{1+k}\mathcal{T}_{1+k}^\vee) \cap [G_k(\mathbb{P}(V)].$$

One easily sees that this class does not depend on the flag chosen: any two Schubert-wronskians are sections of the same line bundle. Set $\boldsymbol{\sigma}_1 = c_1(\bigwedge^k \mathcal{T}_{1+k}^\vee)$, the first *special Schubert cycle*. Because of the exact sequence (1.6) and the fact that $c_t(\mathcal{T}_{1+k})c_t(\mathcal{Q}_{1+k}) = 1$ (c_t is the *Chern polynomial*, see e.g. [23]), it also follows that:

$$\boldsymbol{\sigma}_1 = c_1(\bigwedge^k \mathcal{T}_{1+k}^\vee) = -c_1(\bigwedge^k \mathcal{T}_{1+k}) = -c_1(\mathcal{T}_{1+k}) = c_1(\mathcal{Q}_{1+k}).$$

1.2.12 Suppose $F^\bullet$ is another flag of V. Because of the transitive action of $Gl(V)$ on $G_k(\mathbb{P}(V))$, there exists an automorphism g of V sending the flag $F^\bullet$ onto the flag $E^\bullet$ and, consequently, the Schubert variety $\Omega_I(F^\bullet)$ isomorphically onto $\Omega_I(E^\bullet)$. Since $Gl(V)$ is rational (and connected), their classes modulo rational equivalence in $A_*(G_k(\mathbb{P}(V)))$ are equal (Cf. Section 1.2.3). Then one lets:

$$\Omega_I = [\Omega_I(E^\bullet)] \in A_*(G_k(\mathbb{P}(V))),$$

for some complete flag $E^\bullet$ of V. One may also denote the same Schubert cycle as $\boldsymbol{\Omega}_{\underline{\lambda}}$, where $\underline{\lambda} = \underline{\lambda}(I)$ (Cf. Remark. 1.2.9). Clearly $\Omega_{01\ldots k} = \boldsymbol{\Omega}_{(0\ldots0)} = [G_k(\mathbb{P}(V))]$, the fundamental class of $G_k(\mathbb{P}(V))$. The class of $\Omega_I(E^\bullet)$ corresponds to a class in $A^*(G_k(\mathbb{P}(V)))$ classically denoted by $\boldsymbol{\sigma}_{\underline{\lambda}}$, related to it via the equality:

$$\boldsymbol{\sigma}_{\underline{\lambda}} \cap [G_k(\mathbb{P}(V))] = \boldsymbol{\Omega}_{\underline{\lambda}},$$

expressing *Poincaré duality* for grassmannians. The equality

$$\boldsymbol{\sigma}_{\underline{\lambda}} \cap \boldsymbol{\Omega}_{\underline{\mu}} = \boldsymbol{\sigma}_{\underline{\lambda}} \cap (\boldsymbol{\sigma}_{\underline{\mu}} \cap [G_k(\mathbb{P}(V))]) = (\boldsymbol{\sigma}_{\underline{\lambda}} \cdot \boldsymbol{\sigma}_{\underline{\mu}}) \cap [G_k(\mathbb{P}(V))].$$

expresses instead the fact that $A_*(G_k(V))$ is a module over $A^*(G_k(V))$. One also has:

1.2.13 Proposition (Chow basis theorem). *The classes $\boldsymbol{\Omega}_{\underline{\lambda}} := \boldsymbol{\sigma}_{\underline{\lambda}} \cap [G_k(V)]$, of Schubert varieties modulo rational equivalence, freely generate the Chow group $A_*(G_k(\mathbb{P}(V)))$.*

Proof [23], p. 268 or [31]. ∎

The following example serves as illustration of how Schubert Calculus should work.

1.2.14 Example. Let us look for the class in $A_*(G_2(\mathbb{P}^5))$ of all the planes intersecting a 3-dimensional projective linear subspace $H \subseteq \mathbb{P}^5$ along a line passing through a point $P \in H$ and incident to five 2-codimensional projective linear subspaces $(\Pi_1, \ldots, \Pi_5)$ in general position in $\mathbb{P}^5$. The first step to solve such an example is to identify the involved Schubert varieties. To do this is convenient to see this problem in the affine grassmannian $A_*(G(3, \mathbb{C}^6))$.

Let $E^\bullet$ be the complete flag:

$$\mathbb{C}^6 = E^0 \supset E^1 \supset E^2 \supset E^3 \supset E^4 \supset E^5 \supset E^6 = (0)$$

If $[\Lambda] \in G(3, \mathbb{C}^6)$ intersect a 4-dimensional subspace $\overline{H}$ of $\mathbb{C}^6$ along a 2-plane containing a given 1-plane, one has that $E^5 \subset [\Lambda] \cap E^2$, and that $\dim([\Lambda] \cap E^3) \geq 2$ thus, $dim(E^2 \cap [\Lambda]) \geq 2$ and $\dim(E^5 \cap [\Lambda]) = 1$. The most general such plane is when the equality holds, which corresponds indeed to the k-planes having $1, 3, 5$ as a Schubert index. If a 3-plane $[\Lambda]$, instead, meet E^3 along a positive dimensional vector subspace, then it belongs to the Schubert cycle $\Omega_{124}(E^\bullet)$, since for the most general such 3-plane one has $\dim([\Lambda] \cap E^4) = 0$. By Kleiman's theorem, one knows that it is possible to choose sufficiently general flags $F_0^\bullet, F_1^\bullet \ldots, F_5^\bullet$ such that the intersection:

$$X := \Omega_{013}(F_0^\bullet) \cap \ldots \cap \Omega_{013}(F_4^\bullet) \cap \Omega_{024}(F_5^\bullet)$$

is proper, i.e. such that the codimension of the intersection scheme coincides with the sum of the codimensions of those one is intersecting. The class of X in $A_*(G(3, \mathbb{C}^3))$ is then:

$$[X] = \boldsymbol{\sigma}_1^5 \cdot \boldsymbol{\sigma}_{31} \cap [A_*(G(3, \mathbb{C}^3))]$$

and the problem now amounts to compute explicitly the product $\boldsymbol{\sigma}_1^5 \boldsymbol{\sigma}_{31} \in A_*(G_2(\mathbb{P}^5))$.

1.2.15 Via the Poincaré isomorphism:

$$A^*(G_k(\mathbb{P}V)) \longrightarrow A_*(G_k(\mathbb{P}V)),$$

sending $\boldsymbol{\sigma}_{\underline{\lambda}} \mapsto \boldsymbol{\sigma}_{\underline{\lambda}} \cap [G_k(\mathbb{P}V)]$, and by Proposition 1.2.13, it follows that $A^*(G_k(\mathbb{P}V))$ is generated as a $\mathbb{Z}$-module by the Schubert cycles $\boldsymbol{\sigma}_{\underline{\lambda}}$. It turns out that $\boldsymbol{\sigma}_i = c_i(\mathcal{Q}_{1+k})$ (see [23], p. 271). Doing intersection theory on the grassmannian amounts to knowing how to multiply any two Schubert classes $\boldsymbol{\sigma}_{\underline{\lambda}}$ and $\boldsymbol{\sigma}_{\underline{\mu}}$, i.e. to know $\boldsymbol{\sigma}_{\underline{\lambda}} \cdot \boldsymbol{\sigma}_{\underline{\mu}}$ in $A^*(G_k(\mathbb{P}V))$ or, equivalently, $\boldsymbol{\sigma}_{\underline{\lambda}} \cap \boldsymbol{\Omega}_{\underline{\mu}} \in A_*(G_k(\mathbb{P}(V)))$. Using the combinatorial language of Young diagrams (see [23]), one may also say that the Chow ring $A^*(G_k(\mathbb{P}(V)))$ is freely generated, as a module over the integers, by the Schubert (co)cycles:

$$\{\boldsymbol{\sigma}_{\underline{\lambda}} \mid \underline{\lambda} \text{ is a partition contained in a } (k+1)(n-k) \text{ rectangle}\},$$

where $\boldsymbol{\sigma}_{\underline{\lambda}} \cap [G_k(\mathbb{P}(V))]$ is the class of a Schubert variety $\boldsymbol{\Omega}_{\underline{\lambda}}(E^\bullet)$ associated to any flag $E^\bullet$ of V. Schubert Calculus allows to write the product $\boldsymbol{\sigma}_{\underline{\lambda}} \cdot \boldsymbol{\sigma}_{\underline{\mu}}$ as an explicit linear combination of elements of the given basis of $A^*(G_k(\mathbb{P}V))$. It consists, indeed, in an explicit algorithm to determine the structure constants $\{C^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}}\}$ defined by:

$$\boldsymbol{\sigma}_{\underline{\lambda}} \cdot \boldsymbol{\sigma}_{\underline{\mu}} = \sum_{|\underline{\nu}|=|\underline{\lambda}|+|\underline{\mu}|} C^{\underline{\nu}}_{\underline{\lambda}\underline{\mu}} \boldsymbol{\sigma}_{\underline{\nu}}.$$

The coefficients $C^{\underline{\nu}}_{\underline{\lambda}\underline{\nu}}$ can be determined combinatorially via the *Littlewood-Richardson rule* ([49], p. 68).

The other recipe consists in determining any product via reduction to known cases. To this purpose, one first establishes a rule to multiply any Schubert cycle with a *special one*. A *special Schubert cycle* is a cycle indexed by a partition of length 1. Such a product is ruled by

1.2.16 Theorem (Pieri's Formula). *The following multiplication rule holds:*

$$\boldsymbol{\sigma}_h \cdot \boldsymbol{\sigma}_{\underline{\lambda}} = \sum_{\underline{\mu}} \boldsymbol{\sigma}_{\underline{\mu}} \tag{1.7}$$

[2] Pieri's formula can be also phrased by saying that sum (1.7) is over all the partitions $\underline{\mu}$ whose Young diagram $Y(\underline{\mu})$ is gotten by adding h boxes to $Y(\underline{\lambda})$, in all possible ways, not two on the same column. For instance, in $G_2(\mathbb{P}^{n-1})$, with $n \geq 9$, one has:

$$\boldsymbol{\sigma}_2 \cdot \boldsymbol{\sigma}_{331} = \boldsymbol{\sigma}_{531} + \boldsymbol{\sigma}_{432} + \boldsymbol{\sigma}_{333}.$$

the sum over all partitions such that $|\mu| = |\underline{\lambda}| + h$ *and*

$$n - k \geq \mu_1 \geq \lambda_1 \geq \ldots \geq \mu_k \geq \lambda_k.$$

where $n = \dim(V)$.

Proof. (see e.g. [31], p. 203). ∎

It is not difficult to prove that *Pieri's* formula determines, indeed, the ring structure of $A^*(G)$. In particular, one can see that $A^*(G)$ is generated, as a ring, by the first k special Schubert cycles $\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_k$. This is a consequence of another explicit consequence of Pieri's formula, i.e. the determinantal *Giambelli's formula*, expressing the Schubert cycle $\boldsymbol{\sigma}_{\underline{\lambda}}$ as a polynomial in the special ones:

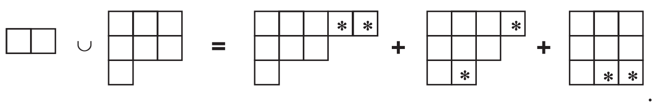
1.2.17 Proposition (Giambelli's Formula). *The Schubert cycle associated to a partition* $\underline{\lambda} = (r_k, \ldots, r_1)$ *is a (determinantal) polynomial expression in the special Schubert cycle* $\boldsymbol{\sigma}_i$*'s*:

$$\boldsymbol{\sigma}_{\underline{\lambda}} = \Delta_{\underline{\lambda}}(\boldsymbol{\sigma}) = \begin{vmatrix} \boldsymbol{\sigma}_{r_1} & \boldsymbol{\sigma}_{r_2+1} & \cdots & \boldsymbol{\sigma}_{r_k+k-1} \\ \boldsymbol{\sigma}_{r_1-1} & \boldsymbol{\sigma}_{r_2} & \cdots & \boldsymbol{\sigma}_{r_k+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{\sigma}_{r_1-k+1} & \boldsymbol{\sigma}_{r_2-k+2} & \cdots & \boldsymbol{\sigma}_{r_k} \end{vmatrix} = \det(\boldsymbol{\sigma}_{r_j+j-i}).$$

Proof. It will be given in Section 3.3.5, within the formalism of $\mathcal{S}$-derivations as a consequence of a suitable *"integration by parts"*. ∎

Therefore, the computation of an arbitrary product $\boldsymbol{\sigma}_{\underline{\lambda}} \cdot \boldsymbol{\sigma}_{\underline{\mu}}$ is reduced to a sequence of applications of Giambelli's and Pieri's formula: one first writes $\boldsymbol{\sigma}_{\underline{\lambda}}$ as a polynomial in the $\boldsymbol{\sigma}_i$'s, and then applies Pieri's formula as many times as necessary and then again Giambelli's and so on. Computations become intricate in big grassmannians and for lengthy partitions, but products are computable in principle.

The "graphical Pieri's formula", in terms of Young diagrams is depicted below:



1.2.18 Remarks. Giambelli's formula is a formal consequence of Pieri's formula, in spite trying to prove it may be rather tricky. See e.g. [31], p. 204–206, based on a case by case combinatorial analysis. In [50], p. 13, the Author proves Giambelli's formula in the realm of symmetric functions, as a consequence of the so-called *Jacobi-Trudy formula* (see [49], pp. 23 ff. and also [23], p. 422). In a subsequent chapter we will report Giambelli's formula on Grassmann algebras as in [27], which it is based on the definition of determinant of a square matrix.

However, that proposed by Laksov and Thorup, in a recent preprint ([46]), besides its elegance, seems to be the shortest and the most general.

1.2.19 Example (see [27]). Let $V := \mathbb{C}[X]/(X^{1+n})$ be the $\mathbb{C}$-vector space of polynomials of degree at most n. It is a $n+1$-dimensional $\mathbb{C}$-vector space spanned by the classes of $1, X, X^2, \ldots, X^n \in \mathbb{C}[X]$ modulo (X^{1+n}). Let $z_0 \in \mathbb{C}$ and consider the flag:

$$V \supset V(-z_0) \supset V(-2z_0) \supset \ldots \supset V(-(n+1)z_0) = 0$$

where

$$V(-iz_0) := \frac{(X - z_0)^i + (X^{1+n})}{X^{1+n}},$$

is the vector subspace of polynomials of degree less than or equal to n contained in the i^{th} power of the maximal ideal $(X - z_0)$ of $\mathbb{C}[X]$.

Let $[\Lambda]$ be a subspace of dimension $1+k$ of V and let $(p_0(X), p_1(X), \ldots, p_k(X))$ be a basis of it. Then $[\Lambda]$ is "special" with respect to the given flag, if there exists $0 \neq p(X) \in [\Lambda]$ vanishing at z_0 with multiplicity at least $1 + k$. Write:

$$p(X) = a^0 p_0(X) + a^1 p_1(X) + a^2 p_2(X) + \ldots + a^k p_k(X).$$

Then $p^{(i)}(z_0) = 0$ for all $0 \leq i \leq k$, where $p^{(i)}(X)$ is the i^{th} derivative of the polynomial $p(X)$. Hence $[\Lambda]$ is "special" if the following determinant:

$$W(p_0, p_1, \ldots, p_k)(z_0) := \begin{vmatrix} p_0(z_0) & p_1(z_0) & \ldots & p_k(z_0) \\ p_0'(z_0) & p_1'(z_0) & \ldots & p_k'(z_0) \\ \vdots & \ddots & \vdots & \\ p_0^{(k)}(z_0) & p_1^{(k)}(z_0) & \ldots & p_k^{(k)}(z_0) \end{vmatrix} = 0.$$

This is a true wronskian and hence motivates the terminology of Definition 1.2.11.

Chapter 2

Schubert Calculus on a Grassmann Algebra

The aim of this Chapter is to develop an easy flexible algebraic formalism which the paper [26] suggests to name *Schubert Calculus on Grassmann Algebras* (SCGA), by contrast with Schubert Calculus on Grassmann Varieties (SCGV) (see also [28]). The latter however motivated this work. To get into the matter of the subject, one should first begin with some review on exterior algebras.

2.1 Exterior Algebras.

2.1.1 Tensor Algebra of a module. Let A be a commutative ring with unit and let M and N be any two A-modules. A *tensor product* of M and N over A is a pair (T, ψ) where T is an A-module and ψ is a bilinear map $M \times N \longrightarrow T$ such that any bilinear map $\phi : M \times N \longrightarrow P$ factors through ψ via a unique module homomorphism $T \longrightarrow P$ (see [2], p. 24). The tensor product is unique up to a canonical isomorphism and it will be denoted by $M \otimes_A N$. Accordingly, the universal bilinear map will be denoted as $\psi(m, n) := m \otimes n$.

One may form the tensor algebra $(T(M), \otimes)$ of any A-module M. It is the direct sum:

$$T(M) = \bigoplus_{p \geq 0} M^{\otimes p}$$

where

$$M^0 := A, \quad M^{\otimes 1} := M, \quad M^{\otimes p} := M^{\otimes p-1} \otimes_A M.$$

The tensor product $\otimes$ is defined as follows: if $m \in M^{\otimes i}$ and $n \in M^{\otimes j}$, then $m \otimes n \in M^{\otimes i+j}$ is the universal image of (m, n) into $M^{\otimes i+j} = M^i \otimes_A M^j$. Such a product is then extended to all pairs of elements of $T(M)$ via A-bilinearity. As a classical example of tensor algebra one may recall the ring of polynomials $A[X]$ in one indeterminate X: it is the tensor algebra associated to A thought of as an A-module over itself.

2.1.2 Exterior (or Grassmann) Algebra. The *exterior algebra* $\bigwedge M$ of an A-module M is the quotient of the tensor algebra $T(M)$ modulo the bilateral ideal $I_A := \{m \otimes m \,|\, m \in M\}$. Let $a : T(M) \longrightarrow \bigwedge M$ be the canonical epimorphism and set:

$$m \wedge n := a(m \otimes n) = m \otimes n + I_A.$$

Clearly $m \wedge m = 0$, $\forall m \in \bigwedge M$ and $m \wedge n = -n \wedge m$ as a consequence of the fact that $(m + n) \wedge (m + n) = 0$ and then $m \otimes n + n \otimes m \in I_A$. The image of the submodule $M^{\otimes i}$ through a is denoted by $\bigwedge^i M$. Hence one can decompose the module $\bigwedge M$ into the direct sum

$$\bigwedge M := \bigoplus_{i \geq 0} \bigwedge^i M,$$

where $\bigwedge^0 M := A$ and $\bigwedge^1 M := M$. The submodule $\bigwedge^i M \subseteq \bigwedge M$ is said to be the i^{th}-*exterior power* of M. For each $i \geq 2$ (the cases $i = 0$ and $i = 1$ being trivial), $\bigwedge^i M$ enjoys a universal property, too. Recall that a multilinear map

$$\phi : \underbrace{M \times \ldots \times M}_{i \; times}$$

is said to be alternating if and only if

$$\begin{cases} \phi(x_{\tau(1)}, \ldots, x_{\tau(i)}) & = & (-1)^{|\tau|})\phi(x_1, \ldots, x_i) \\ \phi(x, x, x_3, \ldots, x_i) & = & 0, \qquad \forall x \in M \end{cases}$$

where τ is a permutation on the set $\{1, 2, \ldots, i\}$. There is a *universal alternating map*:

$$\underbrace{M \times \ldots \times M}_{i\ times} \longrightarrow \bigwedge^{i} M.$$

such that, for each alternating map $\tilde{a} : \underbrace{M \times \ldots \times M}_{i\ times} \longrightarrow P$, there exists a unique $\psi : \bigwedge^{i} M \longrightarrow P$ such that $\tilde{a}(m_1, \ldots, m_i) = \psi(m_1 \wedge \ldots \wedge m_i)$.

2.1.3 Exterior algebra of a free module. The exterior algebra of a free A-module M can be explicitly described as follows. Let $(\mu^0, \mu^1, \ldots)$ be an A-basis of M. Then $\bigwedge^{1+k} M$ is the A-module generated by all the expressions $\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k}$ subject to the relations

$$\mu^{i_{\tau(0)}} \wedge \mu^{i_{\tau(1)}} \wedge \ldots \wedge \mu^{i_{\tau(k)}} = (-1)^{|\tau|} \mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k}, \quad \forall \tau \in S_{1+k}.$$

Therefore $\bigwedge^{1+k} M$ is a free module spanned by the basis $\{\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k} : 0 \leq i_0 < i_1 < \ldots < i_k\}$. The following is a very important:

2.1.4 Definition. *The* weight *of the basis element* $\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k} \in \bigwedge^{1+k} M$ *is:*

$$\begin{aligned} wt(\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k}) &= (i_0 - 0) + (i_1 - 1) + \ldots + (i_k - k) = \\ &= \sum_{j=0}^{k} (i_j - j) = \sum_{j=0}^{k} i_j - \frac{k(k+1)}{2}, \end{aligned} \tag{2.1}$$

2.2 Derivations on Exterior Algebras

2.2.1 From now on, with an eye to the geometrical applications to be discussed later on, the ring A will be assumed to be an integral $\mathbb{Z}$-algebra of characteristic zero. Let t be an indeterminate over A and let $A[[t]]$ be the algebra of formal power series in t. If M is an A-module, let $M[[t]]$ be the formal power series with coefficients in M, (i.e. sequences of elements of M written as series). Similarly, $\bigwedge M[[t]]$ is the ring of formal power series with coefficients in the algebra $\bigwedge M$.

The main goal of this section is to begin a systematic study of the properties of algebra homomorphisms $D_t : \bigwedge M \longrightarrow \bigwedge M[[t]]$. They are in fact suited to model cohomologies theories on grassmannian varieties.

2.2.2 If $D_t : \bigwedge M \longrightarrow \bigwedge M[[t]]$ is an A-module homomorphism, denote by

$$D := (D_0, D_1, D_2, \dots)$$

its sequence of *coefficients*, defined through the equality:

$$D_t\alpha = \sum_{i\geq 0} D_i(\alpha)t^i, \quad \forall \alpha \in \bigwedge M.$$

The definition below is basically taken from [26].

2.2.3 Definition. *An A-module homomorphism $D_t : \bigwedge M \longrightarrow \bigwedge M[[t]]$ is a* derivation *on $\bigwedge M$ if it is also an A-algebra homomorphism, i.e. if the equality*

$$D_t(\alpha \wedge \beta) = D_t\alpha \wedge D_t\beta \tag{2.2}$$

holds for each $\alpha, \beta \in \bigwedge M$.

Since later on the coefficients $\{D_i\}$ will play the role of cohomology classes, we need indeed a further hypothesis.

2.2.4 Definition. *A derivation D_t on $\bigwedge M$ is said to be* homogeneous *if $D_i(M) \subseteq M$, for each $i \geq 0$, and* commutative *if $D_i \circ D_j = D_j \circ D_i$ for all $i, j \geq 0$. A homogeneous derivation is said to be* regular *if D_0 is an automorphism of $\bigwedge M$.*

2.2.5 Remark. Notice that equation (2.2) says that for a homogeneous derivation all the degrees of the exterior algebra $\bigwedge^{1+k} M$ are D_i invariants, i.e. $D_i(\bigwedge^{1+k} M) \subseteq \bigwedge^{1+k} M$: the restriction of D_i to $\bigwedge^{1+k} M$ defines an endomorphism of $\bigwedge^{1+k} M$ itself. One can easily check, by induction, that D_t is commutative if and only if ${D_i \circ D_j}_{|_M} = {D_j \circ D_i}_{|_M}$.

2.2.6 Definition. *A* Schubert derivation *($\mathcal{S}$-derivation) on $\bigwedge M$ is a regular commutative derivation on $\bigwedge M$.*

A $\mathcal{S}$-derivation will be denoted either by the symbol D_t (the formal power series) or D (the sequence), with no substantial distinction.

2.3 The Group $\mathcal{S}_t(\bigwedge M)$

In spite of the use of a quite different language, this section has been deeply inspired by [46] and [47].

2.3.1 Denote by $\mathcal{S}_t(\bigwedge M)$ the set of all $\mathcal{S}$-derivations of $\bigwedge M$. To each $D_t \in \mathcal{S}_t(\bigwedge M)$, one may associate an $A[[t]]$-algebra homomorphism

$$\widehat{D}_t : \bigwedge M[[t]] \longrightarrow \bigwedge M[[t]]$$

defined by:

$$\widehat{D}_t \sum_{i\geq 0} \alpha_i t^i = \sum_{i\geq 0} D_t\alpha_i \cdot t^i = \sum_{h\geq 0}(\sum_{i+j=h} D_i\alpha_j)t^h.$$

Confusing each $\alpha \in \bigwedge M$ with a constant formal power series with $\bigwedge M$-coefficients, one sees that $D_t\alpha = \widehat{D}_t\alpha$. Therefore:

$$(D_t * D'_t)(\alpha) = \sum_{h\geq 0}\sum_{i+j=h} D_i(D'_j\alpha)t^h = \widehat{D}_t(\sum_{j\geq 0} D'_j\alpha \cdot t^j) = \widehat{D}_t(D'_t\alpha) = (\widehat{D}_t \circ \widehat{D}'_t)\alpha. \quad (2.3)$$

Moreover $\widehat{D}_t : \bigwedge M[[t]] \longrightarrow \bigwedge M[[t]]$ is an algebra homomorphism. In fact:

$$\begin{aligned}\widehat{D}_t\Big(\sum_{i\geq 0}\alpha_i t^i \wedge \sum_{j\geq 0}\beta_j t^j\Big) &= \widehat{D}_t \sum_{h\geq 0}\Big(\sum_{i+j=h}\alpha_i\wedge\beta_j\Big)t^h = \sum_{h\geq 0}\Big(\sum_{i+j=h} D_t(\alpha_i\wedge\beta_j)\Big)t^h = \\ &= \sum_{h\geq 0}\Big(\sum_{i+j=h} D_t\alpha_i \wedge D_t\beta_j\Big)t^h = \sum_{i\geq 0} D_t\alpha_i t^i \wedge \sum_{j\geq 0} D_t\beta_j t^j = \\ &= \widehat{D}_t\sum_{i\geq 0}\alpha_i t^i \wedge \widehat{D}_t \sum_{j\geq 0}\beta_j t^j, \qquad (2.4)\end{aligned}$$

as desired. One can then show that:

2.3.2 Proposition. *The pair $(\mathcal{S}_t(\bigwedge M), *)$ is a group.*

Proof. Let us first show that if $D_t, D'_t \in \mathcal{S}_t(\bigwedge M)$ then $D_t * D'_t \in \mathcal{S}_t(\bigwedge M)$. In fact, using (2.3) and (2.4):

$$\begin{aligned}(D_t * D'_t)(\alpha\wedge\beta) &= \widehat{D_t}(D'_t(\alpha\wedge\beta)) = \widehat{D_t}(D'_t\alpha \wedge D'_t\beta) = \\ &= \widehat{D_t}(D'_t\alpha) \wedge \widehat{D_t}(D'_t\beta) = (D_t * D'_t)\alpha \wedge (D_t * D'_t)\beta,\end{aligned}$$

as desired. By its very definition, $*$ is obviously associative. The neutral element of $\mathcal{S}_t(\bigwedge M)$ is the map $\mathbf{1} : \bigwedge M \longrightarrow \bigwedge M[[t]]$ sending any $\alpha \in \bigwedge M$ to itself thought of as a constant formal power series. Moreover, let $\overline{D}_t$ be the formal inverse of D_t thought of as an invertible formal power series with coefficients in $End_A(\bigwedge M)$ (the existence of such an inverse is guaranteed by the invertibility of D_0). Then $\overline{D}_t \in \mathcal{S}_t(\bigwedge M)$. In fact:

$$\begin{aligned}\overline{D}_t(\alpha \wedge \beta) &= \widehat{\overline{D}}_t\big((D_t * \overline{D}_t)\alpha \wedge (D_t * \overline{D}_t)\beta)\big) = \\ &= \widehat{\overline{D}}_t(\widehat{D_t\overline{D}_t}\alpha \wedge \widehat{D_t\overline{D}_t}\beta) = \\ &= (\widehat{\overline{D}}_t \circ \widehat{D_t})(\overline{D}_t\alpha \wedge \overline{D}_t\beta) = \overline{D}_t\alpha \wedge \overline{D}_t\beta\end{aligned}$$

completing the proof of the proposition. ∎

2.3.3 Proposition. *If $D_t^{(0)} \in End_A(M)[[t]])$ such that $D_i^{(0)} \circ D_j^{(0)} = D_j^{(0)} \circ D_i^{(0)}$, there is a unique $D_t \in \mathcal{S}_t(\bigwedge M)$ extending $D_t^{(0)} : M \longrightarrow M[[t]]$, i.e, such that $D_{t|_M} = D_t^{(0)}$.*

Proof. It suffices to extend $D_t^{(0)} : M \longrightarrow M[[t]]$ to each degree $\bigwedge^{1+k} M$ $(k \geq 0)$ of the exterior algebra $\bigwedge M$. To this purpose, and for each $k \geq 0$, one first consider the map:

$$\phi_t : M^{\otimes(1+k)} \longrightarrow \bigwedge^{1+k} M[[t]],$$

defined by:

$$m_{i_0} \otimes m_{i_1} \otimes \ldots \otimes m_{i_k} \mapsto D_t^{(0)} m_{i_1} \wedge \ldots \wedge D_t^{(0)} m_{i_k}.$$

This map is clearly alternating and hence, by the universal property of exterior powers, it factors through a unique map $\bigwedge^{1+k} M \longrightarrow \bigwedge^{1+k} M[[t]]$, defined as:

$$D_t^{(1+k)}(m_{i_0} \wedge m_{i_1} \wedge \ldots \wedge m_{i_k}) = D_t^{(0)} m_{i_1} \wedge \ldots \wedge D_t^{(0)} m_{i_k},$$

on the basis elements, and extended by linearity. Then, for each $\alpha \in \bigwedge^{1+k} M$ and for all $k \geq 0$, one sets:

$$D_t\alpha = D_t^{(1+k)}\alpha.$$

It follows that if $\alpha \in \bigwedge^{1+k_1} M$ and $\beta \in \bigwedge^{k_2} M$, equation (2.2) holds by definition of D_t and the fact that $\alpha \wedge \beta$ is a finite A-linear combination of

$$\{m_{i_0} \wedge m_{i_1} \wedge \ldots \wedge m_{i_{k_1}} \wedge m_{i_{k_1+1}} \wedge \ldots \wedge m_{i_{k_1+k_2}}; \quad 0 \leq i_0 < i_1 < \ldots < i_{k_1+k_2}\}.$$

Since any element of $\bigwedge M$ is a finite sum of homogeneous ones, equation (2.2) holds for any arbitrary pair as well. The unicity part is straightforward: were D'_t another extension of $D_t^{(0)}$, one would have:

$$D'_t(m_{i_0}\wedge m_{i_1}\wedge\ldots\wedge m_{i_k}) = D_t^{(0)}m_{i_0}\wedge D_t^{(0)}m_{i_1}\wedge\ldots\wedge D_t^{(0)}m_{i_k} = D_t(m_{i_0}\wedge m_{i_1}\wedge\ldots\wedge m_{i_k}),$$

for each $m_{i_0} \wedge m_{i_1} \wedge \ldots \wedge m_{i_k}$ and each $k \geq 1$. Hence $D'_t = D_t$.
Remark 2.2.5 says that D_t is commutative and homogeneous. Moreover, $D_0 : \bigwedge M \longrightarrow \bigwedge M$ is an A-automorphism of $\bigwedge M$: in fact

$$D_0^{-1}(m_{i_0} \wedge m_{i_1} \wedge \ldots \wedge m_{i_k}) = D_0^{-1}(m_{i_0}) \wedge D_0^{-1}(m_{i_1}) \wedge \ldots \wedge D_0^{-1}(m_{i_k}).$$

is the unique D_0-preimage of a homogeneous element $m_{i_0} \wedge m_{i_1} \wedge \ldots \wedge m_{i_k}$. Hence $D_t \in \mathcal{S}_t(\bigwedge M)$. ∎

2.4 Schubert Calculus on $\bigwedge M$

2.4.1 Definition. *A* Schubert Calculus on a Grassmann Algebra (SCGA) *is a pair* $(\bigwedge M, D_t)$ *where* $D_t \in \mathcal{S}_t(\bigwedge M)$*. Equation (2.2) will be said to be the* fundamental equation of D_t-Schubert Calculus *on* $\bigwedge M$*. The exterior algebra* $\bigwedge M$ *is said to be the* support*, while* D_t *is the defining* $\mathcal{S}$*-derivation. Choosing* $D_t \in \mathcal{S}_t(\bigwedge M)$ *is the same as choosing a SCGA, the support being understood.*

By Proposition 2.3.3, each $(\bigwedge M, D_t)$ determines and is uniquely determined by the pair $(M, (D_t)_{|M})$, which shall be called the *root* of $(\bigwedge M, D_t)$. The notation $\bigwedge(M, D_t^{(0)})$ shall be used to denote the unique SCGA extending the homomorphism $D_t^{(0)} : M \longrightarrow M[[t]]$ to a $\mathcal{S}$-derivation of $\bigwedge M$. Clearly one has:

$$(\bigwedge^{1+k} M, D_{t|_{\bigwedge^{1+k} M}}) = \bigwedge^{1+k}(M, D_{t|_M}),$$

so that $(M, D_{t|_M})$ is the $(1+k)^{th}$-root of $\bigwedge^{1+k}(M, D_{t|_M})$.

2.4.2 Definition. *The pair* $(\bigwedge^{1+k} M, D_{t|_{\bigwedge^{1+k} M}}) = \bigwedge^{1+k}(M, D_{t|_M})$ *will be said to be the k-SCGP (Schubert Calculus on a Grassmann Power) associated to the SCGA* $(\bigwedge M, D_t)$.

Hence any $(M, D_t^{(0)})$ such that the coefficients of $D_t^{(0)}$ are pairwise commuting is the 0-SCGP of the corresponding SCGA $\bigwedge(M, D_t^{(0)})$ and $\bigwedge^{1+k}(M, D_t^{(0)})$ will be the corresponding k-SCGP. Notice that if $D_i^{(0)} = (D_1^{(0)})^i$, then the commutativity condition is automatically satisfied.

2.4.3 Definition. *Let* $(M, D_t^{(0)})$ *be the root of a SCGA, where* $D_t^{(0)} := \sum_{i\geq 0} D_i^{(0)} t^i$. *The SCGA* $\bigwedge(M, D_t^{(0)})$ *is said to be* regular *if and only if there exists* $m \in M$ *such that the sequence*

$$\bigoplus_{i\geq 0} A \cdot D_i m \longrightarrow M \longrightarrow 0$$

is exact. In this case m is said to be a fundamental element.

2.4.4 Definition. *If a* 0*-SCGP* $(M, D_t^{(0)})$ *is such that* $D_t^{(0)} = \sum_{i\geq 0} D_1^i t^i$, *then the SCGA* $\bigwedge(M, D_t)$ *(resp.* $\bigwedge^{1+k}(M, D_t)$*) is said to be* simple.

If $\bigwedge(M, D_t)$ is simple and regular, then, $(m, D_1 m, D_1^2 m, \ldots)$ generates M. In spite of having defined some distinguished classes of SCGAs (the regular and/or simple) the rules proven in next section holds *for all* SCGAs. Indeed they will be used also for making computation in SCGAs of the special kind defined above.

2.5 Some computations in a general SCGA

Once a $D_t \in \mathcal{S}_t(\bigwedge M)$ has been chosen, the corresponding Schubert calculus on $\bigwedge M$ is based on two important computational tools: *Leibniz's rule* and *integration by parts*. They are the abstract algebraic counterparts of Pieri's and Giambelli's formulas of *classical Schubert Calculus for Grassmann varieties or bundles* (as e.g. in [23], p. 266) as well as straightforward consequences of the *fundamental equation* (2.2).

2.5.1 Proposition (Leibniz Rule for D_h). *For each $h \geq 0$, the equality*

$$D_h(\alpha \wedge \beta) = \sum_{\{h_i \geq 0 \mid h_1+h_2=h\}} D_{h_1}\alpha \wedge D_{h_2}\beta, \tag{2.5}$$

holds.

Proof. It is a consequence of equation (2.2). In the left hand side one has

$$D_t(\alpha \wedge \beta) = \alpha \wedge \beta + D_1(\alpha \wedge \beta)t + D_2(\alpha \wedge \beta)t^2 + D_3(\alpha \wedge \beta)t^3 + \ldots$$

the right hand side instead

$$\begin{aligned} D_t(\alpha) \wedge D_t(\beta) &= (D_0\alpha + D_1\alpha t + D_2\alpha t^2 + \ldots) \wedge (D_0\beta + D_1\beta t + D_2\beta t^2 + \ldots) \\ &= D_0\alpha \wedge D_0\beta + (D_1\alpha \wedge D_0\beta + D_0\alpha \wedge D_1\beta)t + \\ &\quad +(D_2\alpha \wedge \beta + D_1\alpha \wedge D_1\beta + \alpha \wedge D_2\beta)t^2 + \ldots \end{aligned}$$

Then, comparing both equations one sees that $D_h(\alpha \wedge \beta)$ is the coefficient of t^h in the expansion of $D_t(\alpha \wedge \beta)$ and is also the coefficient of t^h in the expansion of the wedge product $D_t(\alpha) \wedge D_t(\beta)$, that is exactly the right hand side of Equation (2.5).

∎

In particular, D_0 is a module isomorphism

$$D_0(\alpha \wedge \beta) = D_0\alpha \wedge D_0\beta,$$

and D_1 satisfies the usual Leibniz's rule

$$D_1(\alpha \wedge \beta) = D_1\alpha \wedge \beta + \alpha \wedge D_1\beta.$$

2.5.2 Given a non-negative integer n and an ordered partition $(n_0, n_1, \ldots, n_h) \in \mathbb{Z}^{1+h}$ of it (i.e. $\sum n_i = n$), define the *multinomial coefficient*

$$\binom{n}{n_0, n_1, \ldots, n_h}, \tag{2.6}$$

through the equality

$$(x_0 + x_1 + \ldots + x_h)^n = \sum_{n_0+n_1+\ldots+n_h=n} \binom{n}{n_0, n_1, \ldots, n_h} x_0^{n_0} x_1^{n_1} \ldots x_h^{n_h}, \tag{2.7}$$

holding in the ring $\mathbb{Z}[x_0, x_1, \ldots, x_h]$. Notice that $\binom{n}{n_0,n_1,\ldots,n_h} = 0$, whenever any of the n_i's is negative instead, equals to

$$\binom{n}{n_0, n_1, \ldots, n_h} = \frac{n!}{n_0! n_1! \ldots n_h!}, \tag{2.8}$$

when all the n_i's are non-negative (with the convention $0! = 1$).

2.5.3 Notation. For notational uniformity, also the usual binomial coefficient

$$\binom{n}{n_0} = \frac{n!}{n_0!(n - n_0)!}$$

will be written as $\binom{n}{n_0, n_1}$.

2.5.4 Lemma. *The multinomial coefficients (2.6) satisfy the following identity:*

$$\binom{n}{n_0, n_1, \ldots, n_h} = \sum_{i=0}^{h} \binom{n-1}{n_0, n_1, \ldots, n_i - 1, \ldots, n_h}$$

Proof. To see this, one compares the coefficients of equality:

$$(x_0 + x_1 + \ldots + x_h)^n = (x_0 + x_1 + \ldots + x_h) \cdot (x_0 + x_1 + \ldots + x_h)^{n-1}.$$

On one hand:

$$\begin{aligned}
(x_0 + \cdots + x_h)^n &= (x_0 + x_1 + \ldots + x_h) \cdot (x_0 + x_1 + \ldots + x_h)^{n-1} = \\
&= (x_0 + x_1 + \ldots + x_h) \cdot \sum_{\sum n'_i = n-1} \binom{n-1}{n'_0, n'_1, \ldots, n'_h} x_0^{n'_0} x_1^{n'_1} \ldots x_h^{n'_h} \\
&= \sum_{i=0}^{h} \sum_{\sum n'_i = n-1} \binom{n-1}{n'_0, n'_1, \ldots, n'_h} x_0^{n'_0} x_1^{n'_1} \ldots x_i^{n'_i+1} x_h^{n'_h}.
\end{aligned}$$

Then, by setting $\begin{cases} n'_j + 1 & = & n_j & \text{if } j = i \\ n'_j & = & n_j & \text{otherwise} \end{cases}$, one sees that the equality:

$$(x_0 + \cdots + x_h)^n = \sum_{\sum n_i = n} \sum_{i=0}^{h} \binom{n-1}{n_0, \ldots, n_i - 1, \ldots, n'_h} x_0^{n_0} x_1^{n_1} \ldots x_i^{n_i} x_h^{n_h}, \qquad (2.9)$$

holds. On the other hand,

$$(x_0 + \cdots + x_h)^n = \sum_{\sum n_i = n} \binom{n}{n_0, n_1, \ldots, n_h} x_0^{n_0} x_1^{n_1} \ldots x_n^{n_h} \qquad (2.10)$$

Comparing the coefficients of the last sides of (2.9) and (2.10) respectively, one gets the claim. ∎

One needs to consider iterations of the operators D_i's.

2.5.5 Proposition. The n^{th} iterated of D_h satisfies the equality:

$$D_h^n(\alpha \wedge \beta) = \sum_{n_0 + n_1 + \ldots + n_h = n} \binom{n}{n_0, n_1, \ldots, n_h} \left(\prod_{i=0}^{h} D_i^{n_{h-i}} \right) \alpha \wedge \left(\prod_{i=0}^{h} D_i^{n_i} \right) \beta. \quad (2.11)$$

Proof. The proof is by induction on the integer n. For $n = 1$ formula (2.7) is nothing else than (2.5). Suppose it holds true for all $n - 1 \geq 1$. Then,

$$D_h^n(\alpha\wedge\beta) = D_h^{n-1}\Big(D_h(\alpha\wedge\beta)\Big) =$$

$$= D_h^{n-1}\Big(\sum_{\substack{l+m=h\\ l,m\geq 0}} D_l\alpha\wedge D_m\beta\Big) = \sum_{\substack{l+m=h\\ l,m\geq 0}} D_h^{n-1}\Big(D_l\alpha\wedge D_m\beta\Big) =$$

$$= \sum_{\substack{l+m=h\\ l,m\geq 0\\ n'_0+\ldots+n'_h=n-1\\ n_i\geq 0}} \binom{n-1}{n'_0,n'_1,\ldots,n'_h}\prod_{i=0}^{h} D_i^{n'_{h-i}}\cdot D_l\alpha\wedge\prod_{i=0}^{h} D_i^{n'_i}\cdot D_m\beta =$$

$$= \sum_{\substack{l=0\\ n'_0+\ldots+n'_h=n-1\\ n_i\geq 0}}^{h} \binom{n-1}{n'_0,n'_1,\ldots,n'_h} D_h^{n'_0}\ldots D_l^{n'_{h-l}+1}\ldots D_0^{n'_h}\alpha\wedge D_0^{n'_0}\ldots D_{h-l}^{n'_{h-l}+1}\ldots D_h^{n'_h}\beta.$$

Setting $n'_{h-l}=n_{h-l}-1$ and $n'_j=n_j$, for each $j\neq h-l$, one has:

$$D_h^n(\alpha\wedge\beta) = \sum_{\substack{l=0\\ n_0+\ldots+n_h=n\\ n_i\geq 0}}^{h} \binom{n-1}{n_0,n_1,\ldots,n_{h-l}-1,\ldots,n_h}\prod_{i=0}^{h} D_i^{n_{h-i}}\alpha\wedge\prod_{i=0}^{h} D_i^{n_i}\beta,$$

and by Lemma (2.5.4):

$$D_h^n(\alpha\wedge\beta) = \sum_{\substack{n_0+\ldots+n_h=n\\ n_i\geq 0}} \binom{n}{n_0,n_1,\ldots,n_h}\prod_{i=0}^{h} D_i^{n_{h-i}}\alpha\wedge\prod_{i=0}^{h} D_i^{n_i}\beta\quad.$$

∎

2.5.6 Corollary. *The Newton binomial formula holds for D_1:*

$$D_1^n(\alpha\wedge\beta) = \sum_{h+k=n}\binom{n}{h,k} D_1^h\alpha\wedge D_1^k\beta. \tag{2.12}$$

Proof. It's a consequence of Proposition 2.5.5, putting $h = 1$, $k = n_0$ and $m = n_1$ in equation (2.11). ∎

2.5.7 Proposition. *The following equality holds for each* $\alpha_0, \alpha_1, \dots, \alpha_k \in \bigwedge M$*:*

$$D_1^n(\alpha_0 \wedge \alpha_1 \wedge \dots \wedge \alpha_k) = \sum_{\substack{\sum n_i = n \\ n_i \geq 0}} \binom{n}{n_0, n_1, \dots, n_k} D_1^{n_0}\alpha_0 \wedge D_1^{n_1}\alpha_1 \wedge \dots \wedge D_1^{n_k}\alpha_k \quad (2.13)$$

Proof. By induction on the integer $k \geq 1$. For $k = 1$ the claim reduces to Corollary 2.5.6. Suppose that the formula holds for $k - 1$. Then:

$$D_1^n(\alpha_0 \wedge \alpha_1 \wedge \dots \wedge \alpha_k) = \sum_{n_0 + n'_1 = n} \binom{n}{n_0, n'_1} D_1^{n_0}\alpha_0 \wedge D_1^{n'_1}(\alpha_1 \wedge \dots \wedge \alpha_k),$$

still by Corollary 2.5.6. Now, by induction, last side can be written as:

$$\sum_{n_0 + n'_1 = n} \binom{n}{n_0, n'_1} \binom{n'_1}{n_1, n_2, \dots, n_k} D_1^{n_0}\alpha_0 \wedge D_1^{n_1}\alpha_1 \wedge \dots \wedge D_1^{n_k}\alpha_k =$$

$$= \sum_{n_0 + n_1 + \dots + n_k = n} \binom{n}{n_0, n_1, \dots, n_k} D_1^{n_0}\alpha_0 \wedge D_1^{n_1}\alpha_1 \wedge \dots \wedge D_1^{n_k}\alpha_k.$$

∎

A similar expression for $D_h^n(\alpha_0 \wedge \alpha_1 \wedge \dots \wedge \alpha_k)$ can be computed once one knows the corresponding one for $D_h^n(\alpha_0 \wedge \alpha_1 \wedge \dots \wedge \alpha_{k-1})$. To get it, let us consider the set:

$$\mathcal{R}_k^h = \left\{ \underline{\lambda} = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k \mid |\underline{\lambda}| := \sum_{j=0}^{k} \lambda_j \leq h \right\}.$$

2.5.8 Proposition. *The following equality holds for each* $\alpha_0, \alpha_1, \dots, \alpha_k \in \bigwedge M$

$$\begin{aligned} D_h^n(\alpha_0 \wedge \dots \wedge \alpha_k) &= \sum \binom{n}{N} \left(\prod_{i=0}^{h} D_i^{\sum_{|\underline{\delta}| = h-i} n_{\underline{\delta}}} \right) \alpha_0 \wedge \left(\prod_{i=0}^{h} D_i^{\sum_{\delta_k = i} n_{\underline{\delta}}} \right) \alpha_1 \wedge \dots \\ &\dots \wedge \left(\prod_{i=0}^{h} D_i^{\sum_{\delta_{k-j+1} = i} n_{\underline{\delta}}} \right) \alpha_j \wedge \dots \wedge \left(\prod_{i=0}^{h} D_i^{\sum_{\delta_1 = i} n_{\underline{\delta}}} \right) \alpha_k \quad (2.14) \end{aligned}$$

where $\underline{\delta} \in \mathcal{R}_k^h$, $N = (n_{(0,\ldots,0)}, \ldots, n_{\underline{\delta}}, \ldots, n_{(h,\ldots,0)})$[1] *and the sum is over all*

$$n_{(0,\ldots,0)} + \ldots + n_{\underline{\delta}} + \ldots, n_{(h,\ldots,0)} = n.$$

Proof. The proof is by induction on the integer k. For $k = 1$ the formula is true by Proposition (2.11). Suppose then that the formula is true for $k-1 \geq 1$. Then, we have :

$$D_h^n(\alpha_0 \wedge \ldots \wedge \alpha_k) = D_h^n\big((\alpha_0 \wedge \ldots \wedge \alpha_{k-1}) \wedge \alpha_k\big) =$$

$$= \sum \binom{n}{n_0, \ldots, n_h} \prod_{i=0}^{h} D_i^{n_{h-i}}(\alpha_0 \wedge \ldots \wedge \alpha_{k-1}) \wedge \prod_{i=0}^{h} D_i^{n_i}\alpha_k =$$

$$= \sum \binom{n}{n_0, \ldots, n_h} \binom{n_0}{N_0} \cdot \ldots \cdot \binom{n_h}{N_h} \prod_{i=0}^{h} D_i^{\sum_{|\underline{\delta}|=h-i} n_{\underline{\delta}}}\alpha_0 \wedge \prod_{i=0}^{h} D_i^{\sum_{\delta_k=i} n_{\underline{\delta}}}\alpha_1 \wedge \ldots$$

$$\ldots \wedge \prod_{i=0}^{h} D_i^{\sum_{\delta_{k-j+1}=i} n_{\underline{\delta}}}\alpha_j \wedge \ldots \wedge \prod_{i=0}^{h} D_i^{\sum_{\delta_1=i} n_{\underline{\delta}}}\alpha_{k-1} \wedge \prod_{i=0}^{h} D_i^{n_i}\alpha_k$$

where $\delta \in \mathcal{R}_k^h$, and:

$$N_j = (\ldots, n_{(j,\mu_2,\ldots,\mu_k)}, \ldots, n_{(j,\lambda_2,\ldots,\lambda_k)}, \ldots), \ \textit{with} \ \ (j, \mu_2, \ldots, \mu_k) <_{lex} (j, \lambda_2, \ldots, \lambda_k)$$

since $n_i = \displaystyle\sum_{\mu_2+\ldots+\mu_k \leq h-i} n_{(i,\mu_2,\ldots,\mu_k)}$, one may write as well:

$$\begin{aligned} D_h^n(\alpha_0 \wedge \ldots \wedge \alpha_k) \ &= \ \sum \binom{n}{N} \prod_{i=0}^{h} D_i^{\sum_{|\underline{\delta}|=h-i} n_{\underline{\delta}}}\alpha_0 \wedge \prod_{i=0}^{h} D_i^{\sum_{\delta_k=i} n_{\underline{\delta}}}\alpha_1 \wedge \ldots \\ &\ldots \wedge \prod_{i=0}^{h} D_i^{\sum_{\delta_{k-j+1}=i} n_{\underline{\delta}}}\alpha_j \wedge \ldots \wedge \prod_{i=0}^{h} D_i^{\sum_{\delta_1=i} n_{\underline{\delta}}}\alpha_k. \end{aligned}$$

The proposition follows by induction. ■

2.5.9 Example. As an application of Proposition 2.5.8, an expression for

$$D_2^n(\alpha_0 \wedge \alpha_1 \wedge \alpha_2).$$

will be computed. By definition, one has:

$$\mathcal{R}_2^2 = \{00, 01, 02, 10, 11, 20\}.$$

[1] ordered by (index)-lexicographical order, i.e.: $n_{\underline{\lambda}} < n_{\underline{\mu}} \Leftrightarrow \underline{\lambda} <_{lex} \underline{\mu}$ with $\underline{\lambda}$, $\underline{\mu} \in \mathcal{R}_k^h$

Therefore, using Proposition 2.5.8:

$$D_2^n(\alpha_0 \wedge \alpha_1 \wedge \alpha_3) = \sum \binom{n}{n_{00}, n_{01}, n_{02}, n_{11}, n_{20}} D_1^{n_{01}+n_{10}} D_2^{n_{00}} \alpha_0 \wedge D_1^{n_{01}+n_{11}} D_2^{n_{02}} \alpha_1 D_1^{n_{10}+n_{11}} D_2^{n_{20}} \alpha_2$$

where the sum is over $n_{00} + n_{01} + n_{02} + n_{11} + n_{20} = n$.

2.5.10 Proposition.

$$D_h D_l(\alpha_0 \wedge \ldots \wedge \alpha_k) = \sum_{\substack{\{h_i \geq 0 \mid h_0 + \ldots + h_k = h\} \\ \{l_i \geq 0 \mid l_0 + \ldots + l_k = l\}}} D_{h_0} D_{l_0} \alpha_0 \wedge \ldots \wedge D_{h_k} D_{l_k} \alpha_k$$

Proof. The proof is a straightforward consequence of Leibniz Rule for D_h (Proposition 2.5.5).

■

2.5.11 Proposition. *The formula for the composition of iterated holds:*

$$D_h^n D_l^m(\alpha_0 \wedge \ldots \wedge \alpha_k) =$$

$$= \sum_{N,M} \binom{n}{N}\binom{m}{M} \prod_{i=0}^{h} D_i^{\sum_{|\underline{\delta}|=h-i} n_{\underline{\delta}}} \prod_{j=0}^{l} D_j^{\sum_{|\underline{\lambda}|=l-j} m_{\underline{\lambda}}} \alpha_0 \wedge \prod_{i=0}^{h} D_i^{\sum_{\delta_k=i} n_{\underline{\delta}}} \prod_{j=0}^{l} D_j^{\sum_{\lambda_k=j} m_{\underline{\lambda}}} \alpha_1 \wedge \ldots$$
$$\ldots \wedge \prod_{i=0}^{h} D_i^{\sum_{\delta_{k-r+1}=i} n_{\underline{\delta}}} \prod_{j=0}^{l} D_r^{\sum_{\lambda_{k-r+1}=j} m_{\underline{\lambda}}} \alpha_r \wedge \ldots \wedge \prod_{i=0}^{h} D_i^{\sum_{\delta_1=i} n_{\underline{\delta}}} \prod_{j=0}^{l} D_j^{\sum_{\lambda_1=j} m_{\underline{\lambda}}} \alpha_k \quad (2.15)$$

where:

$$N = (n_{(0,\ldots,0)}, \ldots, n_{\underline{\delta}}, \ldots, n_{(h,\ldots,0)}), \text{ with } \underline{\delta} \in \mathcal{P}_k^h$$

and

$$M = (m_{(0,\ldots,0)}, \ldots, m_{\underline{\delta}}, \ldots, m_{(l,\ldots,0)}), \text{ with } \underline{\lambda} \in \mathcal{P}_k^l$$

Proof.

By applying twice Theorem 2.5.8 to $\alpha_0 \wedge \alpha_1 \wedge \ldots \wedge \alpha_k$. ■

The other important tool of a SCGA is described in the following:

2.5.12 Proposition. *Let* $D_t, E_t, G_t \in \mathcal{S}_t(\bigwedge M)$ *such that* $D_t * E_t = G_t$. *Then the* integration by parts formula *holds:*

$$\sum_{i+j=h} D_i\alpha \wedge G_j\beta = D_h(\alpha\wedge\beta) + D_{h-1}(\alpha\wedge E_1\beta) + \ldots + \alpha\wedge E_h\beta = \sum_{j=0}^{h} D_{h-j}(\alpha\wedge E_j\beta). \tag{2.16}$$

Proof. It is a consequence of the formula

$$\widehat{D_t}(\alpha \wedge E_t\beta) = D_t\alpha \wedge G_t\beta. \tag{2.17}$$

In fact, on the left hand side, one have

$$\begin{aligned}
\widehat{D_t}(\alpha \wedge E_t\beta) &= \widehat{D_t}\left(\alpha \wedge \left(\sum_{j\geq 0} E_j(\beta)t^j\right)\right) = \\
&= \widehat{D_t}\left(\sum_{j\geq 0}(\alpha \wedge E_j(\beta)) \cdot t^j\right) = \\
&= \sum_{j\geq 0} D_t(\alpha \wedge E_j(\beta)) \cdot t^j = \\
&= \sum_{j\geq 0}\left(\sum_{i\geq 0} D_i(\alpha \wedge E_j\beta) \cdot t^i\right) t^j = \\
&= \sum_{j\geq 0}\left(\sum_{i\geq 0} D_i(\alpha \wedge E_j\beta)\right) t^{i+j} = \sum_{j\geq 0}^{h} D_{h-j}(\alpha \wedge E_j\beta)t^h
\end{aligned} \tag{2.18}$$

On the right hand side, instead,

$$\begin{aligned}
D_t\alpha \wedge G_t\beta &= \left(\sum_{i\geq 0} D_i(\alpha)t^i\right) \wedge \left(\sum_{j\geq 0} G_j(\beta)t^j\right) = \\
&= \sum_{i\geq 0}\sum_{j\geq 0} D_i(\alpha) \wedge G_j(\beta)t^{i+j} = \sum_{i+j=h} D_i(\alpha) \wedge G_j(\beta)t^h.
\end{aligned} \tag{2.19}$$

Comparing the coefficients of t^h of the right sides of (2.18) and (2.19), the claim is proven. ∎

2.5.13 In particular, if $E_t = \overline{D}_t = \sum_{i\geq 0}(-1)^i\overline{D}_i t^i$, then $G_t = id_{\bigwedge M}$ (i.e. $G_0 = \mathbf{1}$ and, for $i > 0$, $G_i = 0$) and then:

$$D_h\alpha\wedge\beta = \sum_{i\geq 0}(-1)^i D_{h-i}(\alpha\wedge\overline{D}_i\beta) = D_h\alpha\wedge\beta - D_{h-1}\alpha\wedge\overline{D}_1\beta + \ldots + (-1)^i\alpha\wedge\overline{D}_h\beta. \quad (2.20)$$

So, e.g.:

$$D_1\alpha\wedge\beta = D_1(\alpha\wedge\beta) - \alpha\wedge\overline{D}_1\beta \quad \textit{and} \quad D_2\alpha\wedge\beta = D_2(\alpha\wedge\beta) - D_1(\alpha\wedge\overline{D}_1\beta) + \alpha\wedge\overline{D}_2\beta.$$

Chapter 3

Simple and Regular SCGAs on Free Modules

Classical Schubert Calculus on Grassmannian Varieties (SCGV) can be dealt with via a particular kind of SCGA on a free $\mathbb{Z}$-module ([26]). The natural hope is that allowing modules over an integral $\mathbb{Z}$-algebra of characteristic zero, one might describe more general situations of some geometrical interest, such as intersection theory on a Grassmann bundle. This hope can be indeed realized. In order to generalize the SCGA studied in [26] to other more general situations one must restricts the study to what one calls *simple* and *regular* SCGAs. A peculiarity of such SCGAs is that they induce on M certain *canonical bases* suited to perform computations almost like in the classical case.

3.1 A Convention

3.1.1 Let A be a graded $\mathbb{Z}$-algebra, $\mathbf{x} = (x_1, \ldots, x_{1+k})$ a set of indeterminates and M be a free A-module spanned by $\mathbf{m} := \{m_i\}_{0 \leq i \leq n}$ for some $n \in \mathbb{N} \cup \{\infty\}$. Let $\Phi := (\overline{\phi}_0 := id_M, \overline{\phi}_1, \ldots)$ be a sequence of pairwise commuting endomorphisms of M. Via Φ one may equip M with a structure of module over $A[\mathbf{x}] := A[x_1, x_2, \ldots, x_{1+k}]$, by defining

$$P \cdot m = \mathrm{ev}_\Phi(P)(m) = P(\Phi)(m),$$

where $P(\Phi)$ is the A-endomorphism given by $P \in A[\mathbf{x}]$ after "substituting" $x_i \mapsto \phi_i$. Suppose, furthermore, that there is an A-epimorphism

$$A[x_1, x_2, \ldots, x_{1+k}] \cdot m_0 \longrightarrow M \longrightarrow 0 \tag{3.1}$$

Denote by $\mathcal{A}^*(M, \Phi)$ the quotient $A[\mathbf{x}]/(\ker(\mathrm{ev}_{\Phi,m_0}))$ (which as an A-module is isomorphic to M) where

$$\mathrm{ev}_{\Phi,m_0} : A[\mathbf{x}] \longrightarrow M \tag{3.2}$$

sends $A[\mathbf{x}] \ni P \mapsto P(\Phi)(m_0)$.

3.1.2 Definition. *The ring $\mathcal{A}^*(M, \Phi)$ is said to be the Φ-Poincaré dual of M or also the intersection ring of the pair (M, Φ).*

By construction, such a ring is generated by $(\phi_1, \phi_2, \ldots)$ the classes of $\overline{\phi}_i$ modulo $\ker(\mathrm{ev}_{\Phi,m_0})$. Then M gets a free $\mathcal{A}^*(M, \Phi)$-module of rank 1 generated by m_0. The module structure is completely determined by the *Pieri's* products:

$$x_i \cdot m_j = \phi_i(m_j); \tag{3.3}$$

For each $m \in M$ there is $G_m \in A[\mathbf{x}]$ such that $G_m(\Phi) \cdot m_0 = m$. The polynomial G_m is unique up to an element of $\ker(\mathrm{ev}_{\Phi,m_0})$: it will be said *a Giambelli's polynomial for $m \in M$*. The element $G_m(\Phi)$, instead, is uniquely defined in $\mathcal{A}^*(M, \Phi)$: abusing notation will be denoted by the same symbol and abusing terminology will be said *the Giambelli's polynomial* of m. Since the m_is are an A-basis for M, clearly $G_{m_i}(\Phi)$ form a basis of $\mathcal{A}^*(M, \Phi)$ as an A-module.

Denote by $G_i^{\mathbf{m}}(\Phi) := G_{m_i}(\Phi)$ be the Giambelli's polynomial corresponding to m_i. To know the value of $x_i \cdot m_j$ is the same as knowing the expression $\phi_i \circ G_j^{\mathbf{m}}(\Phi)$ as linear combination of $\{G_i^{\mathbf{m}}(\Phi)\}$, i.e. the constant structures of the A-algebra $\mathcal{A}^*(M, \Phi)$.

3.1.3 Example. Let $M = A_*(\mathbb{P}^n)$ be the Chow group of $\mathbb{P}^n$. As a $\mathbb{Z}$-module it is freely generated by H_i, the class of a hyperplane of dimension i. It is a free $A^*(\mathbb{P}^n)$-module of rank 1 generated by $H_n := [\mathbb{P}^n]$. In fact $h^i \cap H_n = H_{n-i}$, where $h = c_1(O_{\mathbb{P}^n}(1))$. Hence the Giambelli's polynomial of H_i is h^{n-i}. Moreover, recalling that $A^*(\mathbb{P}^n) = \mathbb{Z}[h]/(h^{1+n})$ the Pieri's products are

$$h \cap H_i = H_{i-1}.$$

3.1.4 Example. Let $I \in \mathcal{I}_n^k$. Then $A^*(G_k(\mathbb{P}^n))$ is a free $\mathbb{Z}$-module generated by σ_I, where $\sigma_I \cap [G_k(\mathbb{P}^n)] = \Omega_I$. Classical Giambelli's formula says that $\sigma_I = \Delta_I(\sigma)$. Therefore

$$\Omega_I = \Delta_I(\sigma) \cap [G_k(\mathbb{P}^n)],$$

i.e. $\Delta_I(\sigma) \in A^*(G_k(\mathbb{P}^n))$ is the Giambelli's polynomial for Ω_I. Pieri's product are

$$\sigma_i \cap \Omega_I$$

and can be computed via Pieri's formula holding in $A^*(G_k(\mathbb{P}^n))$. In fact:

$$\sigma_i \cap \Omega_I = \sigma_i \cap (\sigma_I \cap [G_k(\mathbb{P}^n)]) = (\sigma_i \cdot \sigma_I) \cap [G_k(\mathbb{P}^n)]$$

and $\sigma_i \cap \sigma_I$ can be computed via Pieri's formula.

3.2 Simple and Regular SCGAs on a Free-Module over a Graded Algebra.

3.2.1 This section is strongly inspired by previous work already done by Laksov and Thorup ([46], where the author identify a free A-module of rank n with the quotient $A[X]/(p)$, where p is a monic irreducible poolynomial of degree n. We translate dear theory in a more elementary language which is suitable in view of the subsequent applications. The Reader is advised to look at that paper.

3.2.2 Let A be a graded $\mathbb{Z}$-algebra of characteristic zero:

$$A := A_0 \oplus A_1 \oplus A_2 \oplus \ldots, \qquad (A_0 = \mathbb{Z})$$

and let M be a free A-module of rank $1+n$, for some $n \in \mathbb{N} \cup \{\infty\}$, spanned by the basis $\boldsymbol{\mu} := (\mu^0, \mu^1, \ldots, \mu^n)$. The module M can be given the structure of a graded A-module:

$$M := M_0 \oplus M_1 \oplus M_2 \oplus \ldots \tag{3.4}$$

by setting

$$M_h := A_h \cdot \mu^0 \oplus A_{h-1} \cdot \mu^1 \oplus \ldots \oplus A_0 \cdot \mu^h.$$

If $m \in M_h$, one says that m has *weight* h ($\mathrm{wt}(m) = h$). In particular $\mathrm{wt}(\mu^h) = h$. For each $k \geq 0$, let $\bigwedge^{1+k} M$ be the $(1+k)^{th}$ exterior power of M. Let

$$\bigwedge^{1+k} \boldsymbol{\mu} = \{\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k} \,|\, 0 \leq i_0 \leq i_1 \leq \ldots \leq i_k\}$$

be the basis of $\bigwedge^{1+k} M$ induced by $\boldsymbol{\mu}$. Then $\bigwedge^{1+k} M$ is itself a graded A-module: if $a \in A_j$, define the *weight* of $a \cdot \mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k}$ as:

$$\mathrm{wt}(a\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k}) \quad = \quad j + \mathrm{wt}(\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k}),$$

where

$$\mathrm{wt}(\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k}) = i_0 + (i_1 - 1) + \ldots + (i_k - k) = \sum_{j=0}^{k} (i_j - j) = \sum_{j=0}^{k} i_j - \frac{k(k+1)}{2}.$$

Then one has $\bigwedge^{1+k} M = \bigoplus_{w \geq 0} (\bigwedge^{1+k} M)_w$ and $(\bigwedge^{1+k} M)_w$ is the $\mathbb{Z}$-submodule of $\bigwedge^{1+k} M$ of the elements of *weight* h

3.2.3 Let $D_1 : M \longrightarrow M$ be the unique endomorphism of M such that:

$$D_1 \mu^i = (1 - \delta_{in}) \mu^{i+1-\delta_{in}} + \sum_{j=1}^{i+1} a_j^{i+1} \mu^{i+1-j}, \quad 0 \leq i \leq n \tag{3.5}$$

and where:

- $n \in \mathbb{N} \cup \{\infty\}$ and is equal to $rk(M) - 1$;
- δ_{in} is the Kronecker's delta;
- $a_h^{j+1} \in A_h$, $\forall j \geq 0$ and all $0 \leq h \leq j+1$.

It follows that, with respect to the graduation (3.4), D_1 is an endomorphism of M homogeneous of degree 1. The definition of D_1 depends of course on the choice of the coefficients a^i_j.

3.2.4 Another way to write formula (3.5).

It is worth to explain a little bit more the notation used in formula (3.5), because it will be used again later on.

Case 1: $n = \infty$, i.e. when M has infinite countable rank over A (M is spanned by $(\mu^0, \mu^1, \ldots)$).

In this case, $\delta_{in} = 0$ for each $i \geq 0$ and then formula (3.5) simply says that

$$D_1\mu^i = \mu^{i+1} + \sum_{j=1}^{i+1} a^{i+1}_j \mu^{i+1-j}, \tag{3.6}$$

for all $i \geq 0$;

Case 2: n is finite, i.e. when M has finite countable rank over A (M is spanned by $(\mu^0, \mu^1, \ldots, \mu^n)$).

In this case expression (3.6) holds for all $0 \leq i \leq n-1$ and $D_1\mu^n$ is a (homogeneous of weight $n+1$) A-linear combination of $(\mu^0, \mu^1, \ldots, \mu^n)$, ie,

$$D_1\mu^n = \sum_{j=1}^{n} a^{i+1}_j \mu^{i+1-j}. \tag{3.7}$$

Let $\bigwedge(M, D_1)$ be the simple SCGA associated to the pair $(M, \sum D^i_1 t^i)$.

3.2.5 Proposition. *The SCGA $\bigwedge(M, D_1)$, where D_1 is defined by formula (3.5) is regular with fundamental element μ^0.*

Proof. For each $i \geq 0$, let $\epsilon^i = D_i\mu^0 = D^i_1\mu^0$ (agreeing that $D^0_1 = id_M$). Notice that $wt(\epsilon^i) = wt(\mu^i)$. The proof consists in showing that $\boldsymbol{\epsilon} = (\epsilon^i)_{0\leq i\leq n}$ is an A-basis of M, by checking that the matrix relating $\boldsymbol{\epsilon}$ to $\boldsymbol{\mu}$ is invertible. Indeed, we claim that

$$\epsilon^i = \mu^i + \sum_{j=1}^{i} b^i_j \mu^{i-j},$$

for some $b_j^i \in A_j$ and for each $0 \leq i \leq n$, showing that indeed $\boldsymbol{\epsilon}$ freely generates M as A-module. For $i = 0$, one has $\epsilon^0 = \mu^0$. Assume that the property holds for all $0 \leq h \leq i-1 \leq n-1$. Then:

$$\epsilon^i = D_1^i \mu^0 = D_1(D_1^{i-1}\mu^0) = D_1(\epsilon^{i-1}).$$

By the inductive hypothesis:

$$\epsilon^i = D_1\epsilon^{i-1} = D_1(\mu^{i-1} + \sum_{j=1}^{i-1} b_j^{i-1}\mu^{i-1-j}) = \mu^i + \sum_{j=1}^{i-1} b_j^{i-1} D_1\mu^{i-1-j}. \tag{3.8}$$

But $\sum_{j=1}^{i-1} b_j^{i-1} D_1 \mu^{i-1-j}$ is a homogeneous linear combination of weight i of $\mu^0, \ldots, \mu^{i-1}$, and hence equation (3.8) proves that $\boldsymbol{\epsilon}$ is a basis and that the SCGA $\bigwedge(M, D_1)$ is regular with fundamental element μ^0. ■

3.2.6 If $A = \mathbb{Z}$ one falls in the same situation studied in [26], while if

$$D_1\mu^j = (1 - \delta_{jn})\mu^{j+1-\delta_{jn}} + \delta_{jn}(a_1^{n+1}\mu^n + \ldots + a_{n+1}^{n+1}\mu^0),$$ [1]

for all $0 \leq j \leq n$, one says that $\boldsymbol{\mu}$ is a D_1-*canonical basis* for M. All the endomorphisms of the form (3.5) admit a D_1-canonical basis, which will be denoted by $\boldsymbol{\epsilon} := (\epsilon^0, \epsilon^1, \ldots)$: it is in fact sufficient to set

$$\epsilon^0 = \mu^0 \quad \text{and} \quad \epsilon^i = D_1^i\mu^0. \tag{3.9}$$

Notice that all the elements ϵ^i are homogeneous of weight i ($\text{wt}(\epsilon^i) = \text{wt}(\mu^i) = i$). The ordered set $\boldsymbol{\epsilon}$ is clearly a basis since it is related to $\boldsymbol{\mu}$ via an invertible triangular matrix B:

$$\epsilon^j = \sum_i B_i^j \cdot \mu^i,$$

where $B_0^j = 1$, $B_h^j = 0$ for $h > j$ and $B_i^j = a_i^j$ for $1 \leq i \leq j$. Similarly:

$$\bigwedge^{1+k} \boldsymbol{\epsilon} := \{\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} : 0 \leq i_0 < i_1 < \ldots < i_k\},$$

is a basis of $\bigwedge^{1+k} M$, again said to be D_1-canonical (notice, once more, that

$$\text{wt}(\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k}) = \text{wt}(\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}).$$

[1] One can write in another way as in 3.2.4.

3.2.7 Lemma. *Let* $(\bigwedge M, D_t)$ *be a simple SCGA and let* $\overline{D}_t := \sum_{j\geq 0}(-1)^j\overline{D}_j t^j$ *be the inverse of* D_t*. Then* $\overline{D}_{h|_{\bigwedge^{1+k} M}} = 0$*, for each* $h > 1+k$*.*

Proof. By induction on k. If $k = 0$ one has $\overline{D}_h(m) = 0$, for each $h \geq 2$ and each $m \in M$. In fact $D_{t_{|M}} = \sum_{i\geq 0} D_1^i t^i$. Therefore $\overline{D}_{t_{|M}} = 1 - D_1 t$, i.e. $\overline{D}_{h|_M} = 0$ for each $h \geq 2$. Suppose now the property true for k and let $h > 1+k$. Any $m_{1+k} \in \bigwedge^{1+k} M$ is a finite A-linear combination of elements of the form $m \wedge m_k$, for suitables $m \in M$ and $m_k \in \bigwedge^k M$. It suffices then to check the property for elements of this form. One has:

$$\overline{D}_h(m \wedge m_k) = \sum_{j=0}^{h} \overline{D}_j m \wedge \overline{D}_{h-j}(\alpha_k).$$

As $\overline{D}_j m = 0$ for $j \geq 2$, one has:

$$\sum_{j=0}^{h} \overline{D}_j m \wedge \overline{D}_{h-j} m_k = \overline{D}_1 m \wedge \overline{D}_{h-1} m_k$$

and, by the inductive hypothesis, this last term vanishes too, because $h - 1 > k$. ∎

As a consequence of the above property, one has a useful particular case of integration by parts:

$$D_h(\alpha \wedge m) = D_h\alpha \wedge m + D_{h-1}(\alpha \wedge D_1 m), \tag{3.10}$$

for each $\alpha \in \bigwedge M$ and each $m \in M$. This can be seen directly either by applying Leibniz's rule (2.5) for D_h or by applying formula (2.20) and observing that $\overline{D}_1 = D_1$ and $\overline{D}_j(\epsilon^i) = 0$, whenever $j \geq 2$.

3.2.8 Lemma. *For each* $h, m \geq 0$ *and* $k \geq 1$*, the following formula holds:*

$$D_h(\epsilon^m \wedge \epsilon^{m+1} \wedge \ldots \wedge \epsilon^{m+k-1} \wedge \epsilon^{m+k}) = \epsilon^m \wedge \epsilon^{m+1} \wedge \ldots \wedge \epsilon^{m+k-1} \wedge \epsilon^{m+k+h}. \tag{3.11}$$

Proof. By induction on k. Suppose the formula holds for all $k - 1 \geq 0$ and all $m \geq 0$. Then one has:

$$D_h(\epsilon^m \wedge \epsilon^{m+1} \wedge \ldots \wedge \epsilon^{m+k-1} \wedge \epsilon^{m+k}) =$$
$$= \epsilon^m \wedge D_h(\epsilon^{m+1} \wedge \ldots \wedge \epsilon^{m+k-1} \wedge \epsilon^{m+k}) + D_{h-1}(\epsilon^{m+1} \wedge \epsilon^{m+1} \wedge \ldots \wedge \epsilon^{m+k-1} \wedge \epsilon^{m+k}).$$

The second summand vanishes because $\epsilon^m \wedge \epsilon^m = 0$ and the first, by induction, is precisely

$$\epsilon^m \wedge \epsilon^{m+1} \wedge \ldots \wedge \epsilon^{m+k-1} \wedge \epsilon^{m+k+h}. \qquad \blacksquare$$

3.2.9 Let $\mathbb{Z}[\mathbf{T}] := \mathbb{Z}[T_1, T_2, \ldots]$ be the ring of polynomials in infinitely many indeterminates and let $A[\mathbf{T}] := \mathbb{Z}[\mathbf{T}] \otimes_{\mathbb{Z}} A$. For each $k \geq 0$, consider the map

$$\mathrm{ev}_{D,\mu^0\wedge\mu^1\wedge\ldots\wedge\mu^k} : A[\mathbf{T}] \longrightarrow \bigwedge^{1+k} M,$$

sending each $P \in A[\mathbf{T}]$ onto $\mathrm{ev}_{D,\mu^0\wedge\mu^1\wedge\ldots\wedge\mu^k}(P) = P(D) \cdot \mu^0 \wedge \mu^1 \wedge \ldots \wedge \mu^k$. Here, $P(D)$ is the endomorphism of $\bigwedge^{1+k} M$ gotten by "substituting" $T_i = D_i$ into the polynomial P.

3.2.10 Theorem. *The map* $\mathrm{ev}_{D,\mu^0\wedge\mu^1\wedge\ldots\wedge\mu^k}$ *is surjective.*

Proof. It is sufficient to prove that for each element $\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k} \in \bigwedge^{1+k} \boldsymbol{\mu}$, there exists a polynomial $G^{\boldsymbol{\mu}}_{i_0 i_1 \ldots i_k} \in A[\mathbf{T}]$ such that

$$\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k} = G^{\boldsymbol{\mu}}_{i_0 i_1 \ldots i_k}(D) \cdot \mu^0 \wedge \mu^1 \wedge \ldots \wedge \mu^k.$$

Since $\bigwedge^{1+k} \mathcal{E}$ is a basis of $\bigwedge^{1+k} M$ too and, moreover,

$$\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k = \mu^0 \wedge \mu^1 \wedge \ldots \wedge \mu^k,$$

it is sufficient to prove that for each $\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$, there exists $G^{\boldsymbol{\epsilon}}_{i_0 i_1 \ldots i_k} \in A[\mathbf{T}]$ such that

$$\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = G^{\boldsymbol{\epsilon}}_{i_0 i_1 \ldots i_k}(D) \cdot \epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k,$$

since any $\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k}$ is a unique A-linear combinations of elements of $\bigwedge^{1+k} \boldsymbol{\epsilon}$.

Let $0 \leq j \leq k$. Declare that $\bigwedge^{1+k} M$ enjoys the property $\mathbf{G}^{\boldsymbol{\epsilon}}_j$ if, for each

$$j < i_{j+1} < \ldots < i_k$$

there exists a polynomial $G^{\boldsymbol{\epsilon}}_{j,i_{j+1},\ldots,i_k} \in A[\mathbf{T}]$ such that

$$\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^j \wedge \epsilon^{i_{j+1}} \wedge \ldots \wedge \epsilon^{i_k} = G_{j,i_{j+1},\ldots,i_k}(D)\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k.$$

We shall show, by descending induction, that $\bigwedge^{1+k} M$ enjoys $\mathbf{G}^{\boldsymbol{\epsilon}}_j$ for each $0 \leq j \leq k$. In fact $\mathbf{G}^{\boldsymbol{\epsilon}}_k$ is trivially true, while $\mathbf{G}^{\boldsymbol{\epsilon}}_{k-1}$ is true by Lemma 3.2.8. Let us suppose that $\mathbf{G}^{\boldsymbol{\epsilon}}_j$ holds for some $1 \leq j \leq k-1$. Then $\mathbf{G}^{\boldsymbol{\epsilon}}_{j-1}$ holds. In fact, for each $j-1 < i_j < \ldots < i_k$,

$$\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^{j-1} \wedge \epsilon^{i_j} \wedge \ldots \wedge \epsilon^{i_k} = D_{i_j-j}(\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^{j-1} \wedge \epsilon^j) \wedge \epsilon^{i_{j+1}} \wedge \ldots \wedge \epsilon^{i_k},$$

by Lemma 3.2.8. Applying integration by parts (2.20) one has, therefore:

$$\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^{j-1} \wedge \epsilon^{i_j} \wedge \ldots \wedge \epsilon^{i_k} = \sum_{h=0}^{i_j-j} D_{i_j-j-h}(\epsilon^0 \wedge \epsilon^1 \wedge \ldots \epsilon^j \wedge \overline{D}_h(\epsilon^{i_{j+1}} \wedge \ldots \wedge \epsilon^{i_k})).$$

Since $\overline{D}_h(\epsilon^{i_{j+1}} \wedge \ldots \wedge \epsilon^{i_k})$ is a sum of elements of the form

$$\epsilon^{h_{j+1}} \wedge \ldots \wedge \epsilon^{h_k},$$

with $j < h_{j+1} < \ldots < h_j$, applying the inductive hypothesis, one concludes that $\mathbf{G}^{\boldsymbol{\epsilon}}_{j-1}$ holds, too. In particular $\mathbf{G}^{\boldsymbol{\epsilon}}_0$ holds and the claim is proven. ■

3.2.11 Example. Let us find a polynomial $G^{\boldsymbol{\epsilon}}_{i_0 i_1}(\mathbf{T})$ such that

$$\epsilon^{i_0} \wedge \epsilon^{i_1} = G^{\boldsymbol{\epsilon}}_{i_0 i_1}(D) \cdot \epsilon^0 \wedge \epsilon^1.$$

Applying twice formula (2.20), one has:

$$\begin{aligned} \epsilon^{i_0} \wedge \epsilon^{i_1} &= D_{i_0}\epsilon^0 \wedge \epsilon^{i_1} = D_{i_0}(\epsilon^0 \wedge \epsilon^{i_1}) - D_{i_0-1}(\epsilon^0 \wedge \epsilon^{i_1+1}) = \\ &= D_{i_0}D_{i_1-1}(\epsilon^0 \wedge \epsilon^1) - D_{i_0-1}D_{i_1}(\epsilon^0 \wedge \epsilon^1) = \begin{vmatrix} D_{i_0} & D_{i_1} \\ D_{i_0-1} & D_{i_1-1} \end{vmatrix} \cdot \epsilon^0 \wedge \epsilon^1. \end{aligned}$$

Thus, $G^{\boldsymbol{\epsilon}}_{i_0 i_1}(\mathbf{T}) = T_{i_0}T_{i_1-1} - T_{i_0-1}T_{i_1} = \begin{vmatrix} T_{i_0} & T_{i_1} \\ T_{i_0-1} & T_{i_1-1} \end{vmatrix}$

3.2.12 Definition. *The intersection ring of the k-SCGP $\bigwedge^{1+k}(M, D_1)$ (see Definition 2.4.2) is, by definition*

$$\mathcal{A}^*(\bigwedge^{1+k}(M, D_1)) = \frac{A[\mathbf{T}]}{(\ker(\mathrm{ev}_{D,\mu^0 \wedge \mu^1 \wedge \ldots \wedge \mu^k})}.$$

3.2.13 Definitions. *According to 3.1.1, $G^{\boldsymbol{\mu}}_{i_0i_1\ldots i_k} \in A[\mathbf{T}]$ such that*

$$\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k} = G^{\boldsymbol{\mu}}_{i_0i_1\ldots i_k}(D)\mu^0 \wedge \mu^1 \wedge \ldots \wedge \mu^k.$$

will be said to be a Giambelli polynomial *of $\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k}$. It is unique modulo* $\ker(\mathrm{ev}_{D,\mu^0\wedge\mu^1\wedge\ldots\wedge\mu^k})$*. On the other hand we shall say, abusing terminology, that $G^{\boldsymbol{\mu}}_{i_0i_1\ldots i_k}(D)$ is* the Giambelli's polynomial *of $\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k}$, when seen inside $\mathcal{A}^*(\bigwedge^{1+k}(M, D_1))$.*

Similarly, for each $\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$ we shall denote by $G^{\boldsymbol{\epsilon}}_{i_0i_1\ldots i_k} \in A[\mathbf{T}]$ a Giambelli polynomial *for $\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$, i.e. such that:*

$$\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = G^{\boldsymbol{\epsilon}}_{i_0i_1\ldots i_k}(D)\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k.$$

3.2.14 Proposition. *The ring $\mathcal{A}^*(\bigwedge^{1+k}(M, D_1))$ is generated by $(D_1, D_2, \ldots, D_{1+k})$.*

Proof. In fact, by Lemma 3.2.7, we know that $\overline{D}_h(\epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) = 0$, for each $h > 1 + k$. Then $\overline{D}_h = 0$ as element of $\mathcal{A}^*(\bigwedge^{1+k}(M, D_1))$. But one also has:

$$\overline{D}_h - \overline{D}_{h-1}D_1 + \ldots + (-1)^h D_h = 0$$

which, for $h > 1+k$ says that D_h is a polynomial expression in $D_{h-1}, D_{h-2}, \ldots, D_1$. In particular one knows that D_{k+2} is a polynomial expression in $D_1, D_2, \ldots, D_{1+k}$.

■

3.3 Pieri and Giambelli's Formulas in SCGA

3.3.1 Pieri's Formula in Canonical form. In the D_1-canonical basis $\bigwedge^{1+k} \boldsymbol{\epsilon}$ of $\bigwedge^{1+k} M$, the operators D_j have a particular simple expression:

$$D_h(\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) = \sum \epsilon^{i_0+h_0} \wedge \epsilon^{i_1+h_1} \wedge \ldots \wedge \epsilon^{i_k+h_k},$$

the sum being over all $(1 + k)$-tuples of non negative integers $(h_0, h_1, \ldots, h_k)$ such that $\sum h_i = h$. Because the alternating feature of the $\wedge$-product, some terms may cancel. The remaining terms are predicted by Pieri's formula for D_h.

3.3.2 Proposition (Pieri's formula in canonical form). *Pieri's formula for D_h in the canonical basis $\boldsymbol{\epsilon}$ holds:*

$$D_h(\epsilon^{i_0}\wedge\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k})=\sum_{(h_i)\in P(I,h)}\epsilon^{i_0+h_0}\wedge\epsilon^{i_1+h_1}\wedge\ldots\wedge\epsilon^{i_k+h_k}, \tag{3.12}$$

where $P(I,h)$ is the set of all $(1+k)$-tuples $(h_i)\in\mathbb{N}^{1+k}$ such that:

$$0\le i_0\le i_0+h_0<i_1\le i_1+h_1<i_2\le i_2+h_2<\ldots\le i_{k-1}+h_{k-1}<i_k$$

and $h_0+h_1+h_2+\ldots+h_k=h$.

Proof. Equation (3.12) is defined over the integers and then the same proof is as in [26], Theorem 2.4, where $A=\mathbb{Z}$, works in this case. We repeat it below, up to minor changes, for sake of completeness.

For $k=1$, formula (3.12) is trivially true. Let us prove it directly for $k=2$. For each $h\ge 0$, let us split sum (3.12) as:

$$D_h(\epsilon^{i_0}\wedge\epsilon^{i_1})=\sum_{h_0+h_1=h}\epsilon^{i_0+h_0}\wedge\epsilon^{i_1+h_1}=\mathcal{P}+\overline{\mathcal{P}}. \tag{3.13}$$

where

$$\mathcal{P}=\sum_{\substack{i_0+h_0<i_1\\ h_0+h_1=h}}\epsilon^{i_0+h_0}\wedge\epsilon^{i_1+h_1}\quad\text{and}\quad\overline{\mathcal{P}}=\sum_{\substack{i_0+h_0\ge i_1\\ h_0+h_1=h}}\epsilon^{i_0+h_0}\wedge\epsilon^{i_1+h_1}.$$

One contends that $\overline{\mathcal{P}}$ vanishes. In fact, on the finite set of all integers $i_1-i_0\le a\le i_1-i_0+h$, define the bijection $\rho(a)=i_1-i_0+h-a$. Then:

$$2\overline{\mathcal{P}}=\sum_{h_0=i_1-i_0}^{h}\epsilon^{i_0+h_0}\wedge\epsilon^{i_1+h-h_0}+\sum_{h_0=i_1-i_0}^{h}\epsilon^{i_0+\rho(h_0)}\wedge\epsilon^{i_1+h-\rho(h_0)}=$$

$$=\sum_{h_0=i_1-i_0}^{h}\epsilon^{i_1+h-h_0}\wedge\epsilon^{i_0+h_0}-\sum_{h_0=i_1-i_0}^{h}\epsilon^{i_0+h_0}\wedge\epsilon^{i_1+h_1}=0,$$

hence $\overline{\mathcal{P}}=0$ and (3.12) holds for $k=2$. Suppose now that (3.12) holds for all $1\le k'\le k-1$. Then, for each $h\ge 0$:

$$D_h(\epsilon^{i_0}\wedge\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k})=\sum_{h'_k+h_k=h}D_{h'_k}(\epsilon^{i_0}\wedge\ldots\wedge\epsilon^{i_{k-1}})\wedge D_{h_k}\epsilon^{i_k},$$

and, by the inductive hypothesis:

$$\sum_{(h_i)}(\epsilon^{i_0+h_0}\wedge\ldots\wedge\epsilon^{i_{k-2}+h_{k-2}}\wedge\epsilon^{i_{k-1}+h_{k-1}})\wedge\epsilon^{i_k+h_k}, \tag{3.14}$$

summed over all (h_i) such that $h_0+\ldots+h_k=h$ and

$$1\leq i_0+h_0<i_1\leq\ldots\ldots\leq i_{k-2}+h_{k-2}<i_{k-1}. \tag{3.15}$$

But now (3.14) can be equivalently written as:

$$\sum_{(h_i,h'')}\epsilon^{i_0+h_0}\wedge\ldots\wedge\epsilon^{i_{k-2}+h_{k-2}}\wedge D_{h''}(\epsilon^{i_{k-1}}\wedge\epsilon^{i_k}), \tag{3.16}$$

where the sum is over all $(h_0,\ldots,h_{k-2},h")$ such that $h_0+\ldots+h_{k-2}+h''=h$ and satisfying (3.15). Since

$$D_{h"}(\epsilon^{i_{k-1}}\wedge\epsilon^{i_k})=\sum_{\substack{i_{k-1}+h_{k-1}<i_k\\ h_{k-1}+h_k=h"}}\epsilon^{i_{k-1}+h_{k-1}}\wedge\epsilon^{i_k+h_k},$$

by the inductive hypothesis, substituting into (3.16) one gets exactly sum (3.12).■

3.3.3 Remark. The reason why one calls equality (3.12) Pieri's formula, is due to the fact that it coincides with the *combinatorial Pieri's formula*. In fact, to each finite increasing sequence

$$0\leq i_0<i_1<\ldots<i_k,$$

one may associate the partition of length $\leq 1+k$ (see e.g. [49]):

$$\underline{\lambda}=(\lambda_k,\lambda_{k-1},\ldots,\lambda_1,\lambda_0)=(i_k-k,i_{k-1}-(k-1),\ldots,i_1-1,i_0).$$

Writing for a moment $\epsilon^{\underline{\lambda}}$ for $\epsilon^{i_0}\wedge\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}$, then $D_h\epsilon^{\underline{\lambda}}=\sum_{\underline{\rho}}\epsilon^{\underline{\rho}}$, summed over all partitions $\underline{\rho}$ of length $\leq 1+k$ such that the Young diagram $Y(\underline{\rho})$ of $\underline{\rho}$ is gotten by the Young diagram $Y(\underline{\lambda})$ by adding h boxes in all possible ways, no two on the same column ([24])

3.3.4 As is well known, Pieri's formula implies in a purely formal way (using some Jacobi-Trudy identities, see [50]) *Giambelli's determinantal formula*, i.e. the Giambelli's polynomial of $\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$, as in Definition 3.2.13, can be chosen into the (Giambelli's) determinantal form:

$$G_{i_0 i_1 \ldots i_k}(D) = \Delta_{i_0 i_1 \ldots i_k}(D) = \begin{vmatrix} D_{i_0} & D_{i_1} & \ldots & D_{i_k} \\ D_{i_0-1} & D_{i_1-1} & \ldots & D_{i_k-1} \\ \vdots & \vdots & \ddots & \vdots \\ D_{i_0-k} & D_{i_1-k} & \ldots & D_{i_k-k} \end{vmatrix}. \tag{3.17}$$

In other words:

$$\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = \Delta_{i_0 i_1 \ldots i_k}(D) \cdot \epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k. \tag{3.18}$$

Below, a proof within SCGA formalism is offered.

3.3.5 Giambelli's Formula in Canonical Form. Let $I \in \mathcal{I}^k$ be a Schubert index. The Giambelli's determinant $\Delta_I(\mathbf{T}) \in \mathbb{Z}[\mathbf{T}]$ associated to $I = (i_0, i_1, \ldots, i_k) \in \mathcal{I}^{1+k}$ is:

$$\Delta_I(\mathbf{T}) = \sum_{\tau \in S_{1+k}} (-1)^{|\tau|} T_{i_{\tau(0)}} T_{i_{\tau(1)}-1} \cdot \ldots \cdot T_{i_{\tau(k)}-k} = \begin{vmatrix} T_{i_0} & T_{i_1} & \ldots & T_{i_k} \\ T_{i_0-1} & T_{i_1-1} & \ldots & T_{i_k-1} \\ \vdots & \vdots & \ddots & \vdots \\ T_{i_0-k} & T_{i_1-k} & \ldots & T_{i_k-k} \end{vmatrix} \tag{3.19}$$

If $\underline{\lambda} = (r_k, r_{k-1}, \ldots, r_0)$ is the partition associated to $\underline{\lambda}$, i.e., $r_j = i_j - j$, formula 3.19 can also be written as:

$$\Delta_{\underline{\lambda}}(\mathbf{T}) = \begin{vmatrix} T_{r_0} & T_{r_1+1} & \ldots & T_{r_k+k} \\ T_{r_0-1} & T_{r_1} & \ldots & T_{r_k+k-1} \\ \vdots & \vdots & \ddots & \vdots \\ T_{r_0-k} & T_{r_1-k-1} & \ldots & T_{r_k} \end{vmatrix} \tag{3.20}$$

Let us denote $\Delta_I(D)$ the elements of $End_A(\bigwedge M)$ defined by $\mathrm{ev}_D(\Delta_I(\mathbf{T}))$. Our target is to show that Giambelli's determinant $\Delta_I(D)$ is an explicit Giambelli's polynomial for $\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$. To this purpose, for any pair of positive integers (l, n), let $L_{l,n}$ be the set of all $(1+n)$-tuples $(l_0, l_1, \ldots, l_n)$ such that $0 \leq l_i \leq 1$ and $l_0 + l_1 + \ldots + l_n = l$. If $n < l-1$ the set $L_{l,n}$ is clearly empty.

3.3.6 Lemma. *The following identity holds in $\bigwedge M$:*

$$\begin{aligned}&(D_{h_0} D_{h_1} \ldots D_{h_{p-1}} D_{h_p} \alpha) \wedge \epsilon^i = \\ &= \sum_{l=0}^{p} (-1)^l \sum_{(l_j) \in L_{l,p}} D_{h_0 - l_0} D_{h_1 - l_1} \ldots D_{h_{p-1} - l_{p-1}} D_{h_p - l_p} (\alpha \wedge \epsilon^{i+l}).\end{aligned} \tag{3.21}$$

Proof. The proof is by induction on the integer p. For $p = 1$, formula (3.21) is nothing else than formula (3.10). Suppose that (3.21) holds for the integer $p-1 > 1$ and any $\alpha \in \bigwedge M$. One may then write

$$\begin{aligned}&(D_{h_0} D_{h_1} \ldots D_{h_{p-1}} D_{h_p} \alpha) \wedge \epsilon^i = (D_{h_0} D_{h_1} \ldots D_{h_{p-1}} (D_{h_p} \alpha)) \wedge \epsilon^i = \\ &= \sum_{l=0}^{p-1} (-1)^l \sum_{(l_j) \in L_{l,p-1}} D_{h_0 - l_0} D_{h_1 - l_1} \ldots D_{h_{p-1} - l_{p-1}} (D_{h_p} \alpha \wedge \epsilon^{i+l})\end{aligned} \tag{3.22}$$

Using (3.10), the last side of formula (3.22) becomes:

$$\begin{aligned} = & \sum_{l=0}^{p-1} (-1)^l \sum_{(l_j) \in L_{l,p-1}} D_{h_0 - l_0} D_{h_1 - l_1} \ldots D_{h_{p-1} - l_{p-1}} D_{h_p} (\alpha \wedge \epsilon^{i+l}) + \\ & + \sum_{l=0}^{p-1} (-1)^{l+1} \sum_{(l_j) \in L_{l,p-1}} D_{h_0 - l_0} D_{h_1 - l_1} \ldots D_{h_{p-1} - l_{p-1}} D_{h_p - 1} (\alpha \wedge \epsilon^{i+l+1}) = \\ = & \sum_{l=0}^{p} (-1)^l \sum_{(l_j) \in L_{l,p}} D_{h_0 - l_0} D_{h_1 - l_1} \ldots D_{h_{p-1} - l_{p-1}} D_{h_p - l_p} (\alpha \wedge \epsilon^{i+l}).\end{aligned}$$

■

Let $\underline{\lambda} = (r_k, \ldots, r_1, r_0)$, $I = (r_0, 1 + r_1, \ldots, k + r_k)$ and denote by $\Delta_I^{ij}(D)$ the determinant of the matrix one gets by erasing the i^{th} row and the j^{th} column.

3.3.7 Theorem. *Giambelli's formula on $\bigwedge M$ holds:*

$$\Delta_I^{k,k}(D)(\alpha) \wedge \epsilon^{k+r_k} = \sum_{l=0}^{k-1} (-1)^l \Delta_I^{k-l,k}(D)(\alpha \wedge \epsilon^{k+r_k+l}). \tag{3.23}$$

Proof. Since $\Delta_I^{k,k}(D) = \Delta_{(r_{k-1}\ldots r_1, r_0)}(D)$, one has:

$$\Delta_I^{k,k}(D)(\alpha) \wedge \epsilon^{k+r_k} = \sum_{\sigma\in S_{k-1}} (-1)^{|\sigma|} D_{i_{\sigma(1)}-1} \circ \ldots \circ D_{i_{\sigma(k-1)}-(k-1)}(\alpha) \wedge \epsilon^{k+r_k}$$

Now one applies formula (3.21) to the r.h.s. of the above equation, getting:

$$\sum_{\tau\in S_{k-1}} (-1)^{|\tau|} \sum_{l=0}^{k-1}(-1) \sum_{(l_\alpha)\in L_{l,k-1}} D_{i_\tau(0)-l_0} \circ D_{i_\tau(1)-1-l_1} \circ \ldots \circ D_{i_{\tau(k-1)}-(k-1)-l_{k-1}}(\alpha \wedge \epsilon^{k+r_k+l}) =$$

$$\sum_{l=0}^{k-1}(-1)^l \sum_{(l_\alpha)\in L_{l,k-1}} \sum_{\tau\in S_{k-1}} (-1)^{|\tau|} D_{i_\tau(0)-l_0} \circ D_{i_\tau(1)-1-l_1} \circ \ldots \circ D_{i_{\tau(k-1)}-(k-1)-l_{k-1}}(\alpha \wedge \epsilon^{k+r_k+l}) =$$

$$= \sum_{l=0}^{k-1}(-1)^l \sum_{(l_\alpha)\in L_{l,k-1}} \begin{vmatrix} D_{i_0-l_0} & D_{i_1-l_1} & \ldots & D_{i_k-l_k} \\ D_{i_0-1-l_0} & D_{i_1-1-l_1} & \ldots & D_{i_k-1-l_1} \\ \vdots & \vdots & \ddots & \vdots \\ D_{i_0-k-l_0} & D_{i_1-k-l_1} & \ldots & D_{i_k-k-l_k} \end{vmatrix} (\alpha \wedge \epsilon^{k+r_k+l}).$$

But

$$\sum_{(l_\alpha)\in L_{l,k-1}} \begin{vmatrix} D_{i_0-l_0} & D_{i_1-l_1} & \ldots & D_{i_k-l_k} \\ D_{i_0-1-l_0} & D_{i_1-1-l_1} & \ldots & D_{i_k-1-l_1} \\ \vdots & \vdots & \ddots & \vdots \\ D_{i_0-k-l_0} & D_{i_1-k-l_1} & \ldots & D_{i_k-k-l_k} \end{vmatrix} (\alpha \wedge \epsilon^{k+r_k+l}) = \Delta_I^{k-l,k}(D)(\alpha \wedge \epsilon^{k+r_k+l}),$$

proving Giambelli's formula (3.23). ∎

3.3.8 Corollary. *Giambelli's formula on $\bigwedge^{1+k} M$ holds:*

$$\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = \Delta_I(D) \cdot \epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k. \tag{3.24}$$

Proof. The proof is by induction on the integer k. For $k = 0$ one has $\epsilon^{i_0} = D_{i_0}\epsilon^0$ and the property holds. Suppose it holds for $k-1$. Then one has, using induction:

$$\begin{aligned} & \epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_{k-1}} \wedge \epsilon^{i_k} = \\ = \ & \Delta_{i_0 i_1 \ldots i_{k-1}}(D)(\epsilon^0 \wedge \ldots \wedge \epsilon^{k-1}) \wedge \epsilon^{i_k} = \\ = \ & \Delta_I^{k,k}(D)(\alpha) \wedge \epsilon^{i_k}, \end{aligned}$$

where one set $\alpha = \epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^{k-1}$. Since for such an α one has

$$\alpha \wedge \epsilon^{i_k+l} = D_{i_k-k+l}(\alpha \wedge \epsilon^k),$$

by applying Lemma 3.2.8, formula (3.23) can be written as:

$$\begin{aligned}\Delta_I^{k,k}(D)(\alpha)\wedge\epsilon^{i_k} &= \sum_{l=0}^{k-1}(-1)^l D_{i_k-k+l}\Delta_I^{k-l,k}(D)(\alpha\wedge\epsilon^k) = \\ &= \Delta_I(D)\epsilon^0\wedge\epsilon^1\wedge\ldots\wedge\epsilon^k,\end{aligned}$$

proving the claim. ∎

3.3.9 In general. Working with the basis $\bigwedge^{1+k}\boldsymbol{\mu}$, one can get some corresponding Pieri's and Giambelli's formulas. Since $\bigwedge\boldsymbol{\epsilon}$ and $\bigwedge\boldsymbol{\mu}$ are both A-bases of $\bigwedge^{1+k}M$, there exists a matrix:

$$\begin{array}{rcccc} B & : & \mathcal{I}_n^k\times\mathcal{I}_n^k & \longrightarrow & A \\ & & (I,J) & \longmapsto & B_J^I := B_{j_0j_1\ldots j_n}^{i_0i_1\ldots i_n}\end{array}$$

such that

$$\mu^{i_0}\wedge\mu^{i_1}\wedge\ldots\wedge\mu^{i_k} = \sum B_{j_0j_1\ldots j_k}^{i_0i_1\ldots i_k}\epsilon^{j_0}\wedge\epsilon^{j_1}\wedge\ldots\wedge\epsilon^{j_k}.$$

Let $\overline{B}$ the inverse matrix, i.e.:

$$\sum_{J\in\mathcal{I}_n^k} B_{j_0j_1\ldots j_k}^{i_0i_1\ldots i_k}\cdot\overline{B}_{l_0l_1\ldots l_k}^{j_0j_1\ldots j_k} = \delta_{l_0}^{i_0}\delta_{l_1}^{i_1}\ldots\delta_{l_k}^{i_k}$$

One has then

$$\begin{aligned}D_h(\mu^{i_0}\wedge\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}) &= \sum_{J\in\mathcal{I}_n^k} B_{j_0j_1\ldots j_k}^{i_0i_1\ldots i_k} D_h\epsilon^{j_0}\wedge\epsilon^{j_1}\wedge\ldots\wedge\epsilon^{j_k} = \\ &= \sum_{J\in\mathcal{I}_n^k} B_{j_0j_1\ldots j_k}^{i_0i_1\ldots i_k}\sum_{(h_i)\in P(J,h)}\epsilon^{j_0+h_0}\wedge\epsilon^{j_1+h_1}\wedge\ldots\wedge\epsilon^{j_k+h_k} = \\ &= \sum_{J\in\mathcal{I}_n^k}\sum_{(h_i)\in P(J,h)} B_{j_0j_1\ldots j_k}^{i_0i_1\ldots i_k}\overline{B}_{l_0l_1\ldots l_k}^{j_0+h_0,j_1+h_1,\ldots,j_k+h_k}\mu^{l_0}\wedge\mu^{l_1}\wedge\ldots\wedge\mu^{l_k},\end{aligned}$$

and then apply Pieri's for canonical bases. Moreover, using Giambelli's formula (3.18), one can write:

$$\mu^{i_0}\wedge\mu^{i_1}\wedge\ldots\wedge\mu^{i_k} = \sum B_{j_0j_1\ldots j_k}^{i_0i_1\ldots i_k}\Delta_{i_0i_1\ldots i_k}\epsilon^0\wedge\epsilon^1\wedge\ldots\wedge\epsilon^k,$$

which can be thought of as the corresponding Giambelli's for non canonical bases. However, from our point of view, Giambelli's polynomials can be found just by integration by parts, as in the following:

3.3.10 Example. The Giambelli's polynomials $G^{\boldsymbol{\mu}}_{02}(D)$, $G^{\boldsymbol{\mu}}_{03}(D)$ and $G^{\boldsymbol{\mu}}_{12}(D)$ are given by

$$G^{\boldsymbol{\mu}}_{02}(D) = D_1 - a_1^1 - a_1^2, \tag{3.25}$$

$$G^{\boldsymbol{\mu}}_{03}(D) = D_2 - (a_1^1 + a_1^2 + a_1^3)D_1 + a_1^1a_1^3 + a_1^1a_1^2 + a_1^2a_1^3 + a_1^2a_1^2 - a_2^3 - (a_1^2)^2 - a_2^2, \tag{3.26}$$

$$G^{\boldsymbol{\mu}}_{12}(D) = (D_1 - a_1^1 - a_1^3)G^{\boldsymbol{\mu}}_{02}(D) - G^{\boldsymbol{\mu}}_{03}(D) - a_2^3 \tag{3.27}$$

Let us check that via two different methods.

• **First method. Directly.** To check the expression for $G^{\boldsymbol{\mu}}_{02}(D)$, one notices that the expression for $D_1\mu^1$ gives $\mu^2 = D_1\mu^1 - (y_1 - y_0)\mu^0$. Hence:

$$\mu^0 \wedge \mu^2 = \mu^0 \wedge D_1\mu^1 - a_1^2\mu^0 \wedge \mu^1 = (D_1 - a_1^1 - a_1^2)(\mu^0 \wedge \mu^1) \tag{3.28}$$

proving the (3.25). Similarly, isolating μ^3 into the expression

$$\begin{aligned} D_2\mu^1 &= D_1^2\mu^1 = D_1(\mu^2 + a_1^2\mu^1 + a_2^2\mu^0) = \\ &= \mu^3 + (a_1^3 + a_1^2)\mu^2 + (a_2^3 + (a_1^2)^2 + a_2^2)\mu^1 + (a_3^3 + a_1^2a_2^2 + a_2^2a_1^1)\mu^0, \end{aligned}$$

one can write

$$\begin{aligned} \mu^0 \wedge \mu^3 &= \mu^0 \wedge D_2\mu^1 - (a_1^3 + a_1^2)\mu^0 \wedge \mu^2 - (a_2^3 + (a_1^2)^2 + a_2^2)\mu^0 \wedge \mu^1 = \\ &= \mu^0 \wedge D_2\mu^1 - (a_1^3 + a_1^2)\mu^0 \wedge D_1\mu^1 + a_1^2(a_1^3 + a_1^2)\mu^0 \wedge \mu^1 + \\ &\quad -(a_2^3 + (a_1^2)^2 + a_2^2)\mu^0 \wedge \mu^1 \end{aligned}$$

Now *integrating by parts*:

$$\mu^0 \wedge D_2\mu^1 = D_2(\mu^0 \wedge \mu^1) - D_1(D_1\mu^0 \wedge \mu^1) = (D_2 - a_1^1D_1)\mu^0 \wedge \mu^1$$

Similarly:

$$\mu^0 \wedge D_1\mu^1 = (D_1 - a_1^1)(\mu^0 \wedge \mu^1)$$

Hence:

$$\mu^0 \wedge \mu^3 = (D_2 - (a_1^1 + a_1^2 + a_1^3)D_1 + a_1^1a_1^3 + a_1^1a_1^2 + a_1^2a_1^3 + a_1^2a_1^2 - a_2^3 - (a_1^2)^2 - a_2^2)\mu^0 \wedge \mu^1 \tag{3.29}$$

proving the claimed expression for $G^{\boldsymbol{\mu}}_{03}(D)$. Finally one has:

$$\begin{aligned}
\mu^1 \wedge \mu^2 &= D_1\mu^0 \wedge \mu^2 - a^1_1\mu^0 \wedge \mu^2 = \\
&= D_1(\mu^0 \wedge \mu^2) - \mu^0 \wedge D_1\mu^2 - a^1_1\mu^0 \wedge \mu^2 = \\
&= (D_1 - a^1_1)G_{02}(D)\mu^0 \wedge \mu^1 - \mu^0 \wedge \mu^3 - a^3_1\mu^0 \wedge \mu^2 - a^3_2\mu^0 \wedge \mu^1 = \\
&= (D_1 - a^1_1 - a^3_1)G_{02}(D)\mu^0 \wedge \mu^1 - G^{\boldsymbol{\mu}}_{03}(D)\mu^0 \wedge \mu^1 - a^3_2\mu^0 \wedge \mu^1 = \\
&= ((D_1 - a^1_1 - a^3_1)G^{\boldsymbol{\mu}}_{02}(D) - G^{\boldsymbol{\mu}}_{03}(D) - a^3_2)\mu^0 \wedge \mu^1
\end{aligned}$$

proving also formula (3.27).

• **Second method. Via canonical bases.** One first write the first 4 elements of the D_1-canonical basis of the module M one is working on.

$$\begin{aligned}
\epsilon^0 &= \mu^0; \\
\epsilon^1 &= \mu^1 + a^1_1\mu^0; \\
\epsilon^2 &= \mu^2 + (a^2_1 + a^1_1)\mu^1 + (a^2_2 + (a^1_1)^2)\mu^0; \\
\epsilon^3 &= \mu^3 + (a^1_1 + a^2_1 + a^3_1)\mu^2 + (a^3_2 + a^2_2 + (a^2_1)^2 + a^2_1a^1_1 + (a^1_1)^2)\mu^1 + \\
&\quad +(a^3_3 + a^2_2a^2_1 + (a^1_1)^3 + 2a^1_1a^2_2)\mu^0.
\end{aligned}$$

Inverting the relations above:

$$\begin{aligned}
\mu^0 &= \epsilon^0; \\
\mu^1 &= \epsilon^1 - a^1_1\epsilon^0; \\
\mu^2 &= \epsilon^2 - (a^2_1 + a^1_1)\epsilon^1 + (a^2_1a^1_1 - a^2_2)\epsilon^0; \\
\mu^3 &= \epsilon^3 - (a^1_1 + a^2_1 + a^3_1)\epsilon^2 - (a^3_2 + a^2_2 - a^1_1a^2_1 - a^3_1a^2_1 - a^3_1a^1_1)\epsilon^1 + \\
&\quad +(2a^1_1a^2_2 + a^2_1a^2_2 + a^3_1a^2_2 + a^3_2a^1_1 + (a^1_1)^3 - a^3_1a^2_1a^1_1)\epsilon^0.
\end{aligned}$$

Therefore (recall that $\epsilon^0 \wedge \epsilon^1 = \mu^0 \wedge \mu^1$),

$$\begin{aligned}
\mu^0 \wedge \mu^2 &= \epsilon^0 \wedge [\epsilon^2 - (a^2_1 + a^1_1)\epsilon^1 + ((a^2_1 + a^1_1)a^1_1 - (a^2_2 + (a^1_1)^2))\epsilon^0] = \\
&= \epsilon^0 \wedge \epsilon^2 - (a^2_1 + a^1_1)\epsilon^0 \wedge \epsilon^1 =
\end{aligned}$$

At this point one uses the Giambelli's determinantal formulas for the canonical bases, getting:

$$= D_1\epsilon^0 \wedge \epsilon^1 - (a_1^2 + a_1^1)\epsilon^0 \wedge \epsilon^1 = (D_1 - (a_1^2 + a_1^1))\epsilon^0 \wedge \epsilon^1.$$

so proving formula (3.25). As for $G_{03}^{\boldsymbol{\mu}}(D)$ one has:

$$\begin{aligned}
\mu^0 \wedge \mu^3 &= \epsilon^0 \wedge (\epsilon^3 - (a_1^1 + a_1^2 + a_1^3)\epsilon^2 - (a_2^3 + a_2^2 - a_1^1 a_1^2 - a_1^3 a_1^2 - a_1^3 a_1^1)\epsilon^1 + \\
&\quad +(2a_1^1 a_2^2 + a_1^2 a_2^2 + a_1^3 a_2^2 + a_2^3 a_1^1 + (a_1^1)^3 - a_1^3 a_1^2 a_1^1)\epsilon^0) \\
&= \epsilon^0 \wedge \epsilon^3 - (a_1^1 + a_1^2 + a_1^3)\epsilon^0 \wedge \epsilon^2 - (a_2^3 + a_2^2 - a_1^1 a_1^2 - a_1^3 a_1^2 - a_1^3 a_1^1)\epsilon^0 \wedge \epsilon^1
\end{aligned}$$

and again, using the fact that $\epsilon^0 \wedge \epsilon^3 = D_2(\epsilon^0 \wedge \epsilon^1)$ one gets:

$$\Big(D_2 - (a_1^1 + a_1^2 + a_1^3)D_1 - (a_2^3 + a_2^2 - a_1^1 a_1^2 - a_1^3 a_1^2 - a_1^3 a_1^1)\Big)\epsilon^0 \wedge \epsilon^1$$

proving formula (3.26). The proof of (3.27) works as before:

$$\begin{aligned}
\mu^1 \wedge \mu^2 &= (\epsilon^1 - a_1^1\epsilon^0) \wedge (\epsilon^2 - (a_1^2 + a_1^1)\epsilon^1 + ((a_1^2 + a_1^1)a_1^1 - (a_2^2 + (a_1^1)^2))\epsilon^0) \\
&= \epsilon^1 \wedge \epsilon^2 - ((a_1^2 + a_1^1)a_1^1 - (a_2^2 + (a_1^1)^2))\epsilon^1 \wedge \epsilon^0 - a_1^1\epsilon^0 \wedge \epsilon^2 + a_1^1(a_1^2 + a_1^1)\epsilon^0 \wedge \epsilon^1 \\
&= \epsilon^1 \wedge \epsilon^2 + ((a_1^2 + a_1^1)a_1^1 - (a_2^2 + (a_1^1)^2) + a_1^1(a_1^2 + a_1^1))\epsilon^0 \wedge \epsilon^1 - a_1^1\epsilon^0 \wedge \epsilon^2. \\
&= \epsilon^1 \wedge \epsilon^2 + (a_1^2 a_1^1 - a_2^2 + a_1^1(a_1^2 + a_1^1))\epsilon^0 \wedge \epsilon^1 - a_1^1\epsilon^0 \wedge \epsilon^2
\end{aligned}$$

At this point one uses Giambelli's determinantal formula (3.17) for $\epsilon^1 \wedge \epsilon^2$:

$$\epsilon^1 \wedge \epsilon^2 = (D_1^2 - D_2)\epsilon^0 \wedge \epsilon^1,$$

easily deducible via integration by parts which is easier in this case because the basis $\boldsymbol{\epsilon}$ is canonical. As a consequence

$$\mu^1 \wedge \mu^2 = \Big(D_1^2 - D_2 - a_1^1 D_1 + (2a_1^1 a_1^2 + (a_1^1)^2 - a_2^2)\Big)\epsilon^0 \wedge \epsilon^1,$$

which coincides with formula (3.27) up substituting the expression of $G_{02}^{\boldsymbol{\mu}}(D)$ and $G_{03}^{\boldsymbol{\mu}}(D)$ in that same formula.

3.3.11 Remark. The moral of the second part of Example 3.3.10 is that whatever is the regular simple SCGA on a free A-module one is working with, Giambelli's formula for the basis element $\mu^{i_0}\wedge\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}$ is known once one knows Giambelli's formulas for the canonical bases, i.e. Giambelli's formulas holding in classical Schubert Calculus!

3.4 Intersection Theory on Projective Bundles.

3.4.1 Let $p : E \longrightarrow X$ be a vector bundle of rank $1+n$ over a smooth connected complex algebraic variety of dimension $m \geq 0$. For each $0 \leq k \leq n$, let $p_k : G_k(\mathbb{P}(E)) \longrightarrow X$ be the induced Grassmann bundle. Let

$$0 \longrightarrow \mathcal{T}_k \longrightarrow p_k^*E \longrightarrow \mathcal{Q}_k \longrightarrow 0,$$

be the k-tautological sequence over $G_k(\mathbb{P}(E))$: $\mathcal{T}_k$ is the (rank $1+k$) *universal tautological sub-bundle* of p_k^*E, while $\mathcal{Q}_k$ is the (rank $n-k$) *universal quotient bundle*. If $k=0$, $p_0 : \mathbb{P}(E) \longrightarrow X$ is the usual projective bundle and $\mathcal{T}_0 = O_{\mathbb{P}(E)}(-1)$. Recall that $A^*(\mathbb{P}(E))$ is an $A^*(X)$ algebra generated by $\zeta := c_1(\mathcal{T}_0)$ with a relation defining the Chern classes of E. More precisely (see [23], p. 141):

$$A^*(\mathbb{P}(E)) \cong \frac{A^*(X)[\zeta]}{(\zeta^{n+1}+p^*c_1(E)\zeta^n+\ldots+p^*c_n(E))}. \tag{3.30}$$

By Poincaré duality $A_*(\mathbb{P}(E)) \cong A^*(\mathbb{P}(E))$ is freely generated by $\boldsymbol{\epsilon} = (\epsilon^0, \epsilon^1, \ldots, \epsilon^n)$, where ϵ^i is gotten by capping with the fundamental class:

$$\epsilon^i := \zeta_1^i \cap [\mathbb{P}(E)].$$

Define $D_1 : M \longrightarrow M$ by setting

$$D_1\epsilon^i = \zeta\cap\epsilon^i = \zeta\cap(\zeta^i\cap[G_k(\mathbb{P}(E))] = \zeta^{i+1}\cap[G_k(\mathbb{P}(E)], \quad 0\leq i\leq n.$$

Clearly one has

$$D_1\epsilon^i = (1-\delta_{in})\epsilon^{i+1-\delta_{in}} - \delta_{in}(p^*c_1(E)D_1^n+\ldots+p^*c_n(E)) \ \ 0\leq i\leq n,$$

so that $\boldsymbol{\epsilon}$ is a D_1-canonical basis for M. By construction, the pair (M, D_1) is a 0-SCGP and there is a natural identification between $A^*(\mathbb{P}(E))$ and $\mathcal{A}^*(M, D_1)$ via the map $\zeta \mapsto D_1$. We contend that the k-SCGP $\bigwedge^{1+k}(M, D_1)$ describes the intersection theory of $G_k(\mathbb{P}(E))$. Let us recall some basic facts quoted from [23], p. 266ff: for our own commodity, some notation will be adapted in a obviously equivalent way. Let $C_t := c_t(\mathcal{Q}_k - p_k^* E)$ be the Chern polynomial of $\mathcal{Q}_k - p_k^* E \in K^0(G_k(\mathbb{P}(E)))$, the Grothendieck group of locally free sheaves on $G_k(\mathbb{P}(E))$, and let $\Delta_I(C) := \Delta_I(C_t)$ be the Giambelli's determinant associated to C_t and to $I \in \mathcal{I}_n^k$. Then, the basis theorem ([23], Proposition 14.6.5) implies that $\Delta_I(C)$ and $\Delta_I(C) \cap [G_k(\mathbb{P}(E)]$ freely generate $A^*(G_k(\mathbb{P}(E))$ and $A_*(G_k(\mathbb{P}(E))$, respectively, as modules over $A^*(X)$. In particular, $A^*(G_k(\mathbb{P}(E))$ is generated by $C_i := c_i(\mathcal{Q}_k - p_k^* E)$ as $A^*(X)$-algebra. Let

$$\begin{cases} \boldsymbol{\Delta} \quad : \quad \bigwedge^{1+k} M & \longrightarrow \quad A_*(G_k(\mathbb{P}(E)) \\ \\ \epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} & \longmapsto \quad \Delta_{i_0 i_1 \ldots i_k}(C) \cap [G_k(\mathbb{P}(E))] \end{cases}.$$

Clearly $\boldsymbol{\Delta}$ is an $A^*(X)$-module isomorphism. Moreover:

3.4.2 Theorem. *The map* $\mathbf{C} : \mathcal{A}^*(\bigwedge^{1+k}(M, D_1)) \longrightarrow A^*(G_k(\mathbb{P}(E))$, $D_i \mapsto C_i$ *is an* $A^*(X)$*-algebra isomorphism and the following diagram*

$$\begin{array}{ccc} \mathcal{A}^*(\bigwedge^{1+k}(M, D_1)) \otimes_{A^*(X)} \bigwedge^{1+k} M & \longrightarrow & \bigwedge^{1+k} M \\ \Big\downarrow \mathbf{C} \otimes \boldsymbol{\Delta} & & \Big\downarrow \boldsymbol{\Delta} \\ A^*(G_k(\mathbb{P}(E)) \otimes_{A^*(X)} A_*(G_k(\mathbb{P}(E)) & \stackrel{\cap}{\longrightarrow} & A_*(G_k(\mathbb{P}(E)) \end{array} \tag{3.31}$$

is commutative, where $\cap$ *is the* capping *bilinear map, the upper horizontal map sends* $(D_h, \epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) \mapsto D_h(\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k})$ *and the vertical maps are isomorphisms.*

Proof. To begin with, the map $\mathbf{C}$ is an $A^*(X)$-module isomorphism. In fact the $A^*(X)$-module $\mathcal{A}^*(\bigwedge^{1+k}(M, D_1))$ is freely generated by $\{\Delta_I(D) \,|\, I \in \mathcal{I}_n^k\}$, as well as $A^*(G_k(\mathbb{P}(E))$ is freely generated, by the basis theorem, by $\{\Delta_I(C) \,|\, I \in \mathcal{I}_n^k\}$, hence the map $D_i \mapsto C_i$ sends $\Delta_I(D)$ onto $\Delta_I(C)$. This map is indeed an $A^*(X)$-algebra isomorphism. To show this, since $\mathcal{A}^*(\bigwedge^{1+k}(M, D_1))$ and $A^*(G_k(\mathbb{P}(E))$ are generated, as $A^*(X)$-algebras, by D_i and C_i respectively, it suffices to show that:

$$\mathbf{C}(D_h\Delta_I(D)) = \mathbf{C}(D_h)\mathbf{C}(\Delta_I(D)) = C_h\Delta_I(C), \tag{3.32}$$

for every $0 \leq h \leq k$ and every $I = (i_0, i_1, \ldots, i_k) \in \mathcal{I}_n^k$.

Now,

$$D_h\Delta_I(D)\epsilon^0\wedge\epsilon^1 \wedge \ldots \wedge \epsilon^k = D_h\Delta_{i_0 i_1 \ldots i_k}(D)\epsilon^0\wedge\epsilon^1 \wedge \ldots \wedge \epsilon^k = D_h(\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}),$$

by Giambelli's formula (3.18). On the other hand, applying Pieri's formula (3.12), one has

$$\begin{aligned} D_h(\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}) &= D_h\left(\sum_{(h_i)\in P(I,h)} \epsilon^{i_0+h_0} \wedge \epsilon^{i_1+h_1} \wedge \ldots \wedge \epsilon^{i_k+h_k}\right) = \\ &= \sum_{(h_i)\in P(I,h)} \Delta_{I+H}(D)\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k, \end{aligned}$$

where one applied again Giambelli's formula (3.18), by setting $H = (h_0, h_1, \ldots, h_k) \in P(I, h)$. Therefore

$$D_h\Delta_{i_0 i_1 \ldots i_k}(D) = \sum_{(h_i)\in P(I,h)} \Delta_{I+H}(D)$$

in the ring $\mathcal{A}^*(\bigwedge^{1+k}(M, D_1))$ and, therefore:

$$\begin{aligned} \mathbf{C}(D_h\Delta_{i_0 i_1 \ldots i_k}(D)) &= \mathbf{C}(\sum_{(h_i)\in P(I,h)} \Delta_{I+H}(D)) = \sum_{(h_i)\in P(I,h)} \mathbf{C}(\Delta_{I+H}(D)) = \\ &= \sum_{(h_i)\in P(I,h)} \Delta_{I+H}(C) = C_h\Delta_I(C) \end{aligned}$$

where last equality holds by Pieri's formula in [23], p. 264, Lemma 14.5.2.

Hence equality (3.32) holds, implying that $\mathbf{C}$ is an $A^*(X)$-algebra isomorphism. This implies that $\mathbf{C} \otimes \mathbf{\Delta}$ is an isomorphism too and that diagram (3.31) is commutative, as a standard check easily shows. ∎

Notice that when X is a point one gets, as particular cases, the results of ([26]).

3.4.3 Remark. Notice that in (3.31) the pair (M, D_1) is indeed $(A_*(\mathbb{P}(E)), \zeta)$. The diagram can be re-written as:

$$\begin{array}{ccc}
\mathcal{A}^*(\bigwedge^{1+k}(A_*(\mathbb{P}(E)),\zeta)) \otimes_{A^*(X)} \bigwedge^{1+k} A_*(\mathbb{P}(E)) & \longrightarrow & \bigwedge^{1+k} A_*(\mathbb{P}(E)) \\
\Big\downarrow \mathbf{C}\otimes\mathbf{\Delta} & & \Big\downarrow \mathbf{\Delta} \\
A^*(G_k(\mathbb{P}(E)) \otimes_{A^*(X)} A_*(G_k(\mathbb{P}(E)) & \stackrel{\cap}{\longrightarrow} & A_*(G_k(\mathbb{P}(E))
\end{array}$$

To say that there is such a commutative diagram expressing the intersection theory of $G_k(\mathbb{P}(E))$ via that of $\mathbb{P}(E)$, by "taking exterior powers", one shall briefly write, as a reasonable notation:

$$A^*(G_k(\mathbb{P}(E)) = \bigwedge^{1+k}(A_*(\mathbb{P}(E))). \tag{3.33}$$

Equality (3.33) certainly holds in the category of $A^*(X)$-modules and will be understood at level of $A^*(X)$-algebras in the sense explained above (i.e. as algebra of operators over $\bigwedge^{1+k} A_*(\mathbb{P}(E))$.

Chapter 4

Equivariant Cohomology of Grassmannians

For space reasons we shall not recall in this chapter all preliminaries regarding the general definitions of equivariant cohomology and/or intersection ring. However there are very well established references on the subject, such as [11], [15] which combine the difficulty of the subject with an advanced expository skill. To these references we want also add [52], our first happy experience on the subject and the beautiful exposition in [68]. For the reader not yet aware of basics definitions of the theory, we want to address a quick introduction. The results of this chapter will be also collected in [29].

4.1 T-Equivariant Intersection Theory of Grassmannians

4.1.1 Grassmann bundle. Let $p : \mathcal{E} \longrightarrow X$ be a holomorphic vector bundle of rank $1 + n$ and let $m_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow Gl_{1+n}(\mathbb{C})$ be a cocycle defining it, where $\mathcal{U} := \{U_\alpha : \alpha \in \mathcal{A}\}$ is an open covering of X trivializing $\mathcal{E}$. The transition functions of the corresponding projective bundle $p_0 : \mathbb{P}(\mathcal{E}) \longrightarrow X$ are

$$[m_{\alpha\beta}] : U_\alpha \cap U_\beta \longrightarrow Sl_{1+n}(\mathbb{C}),$$

where one sets $[m_{\alpha\beta}] = m_{\alpha\beta}/\det(m_{\alpha\beta})$. The Grassmann bundle

$$p_k : G_k(\mathbb{P}(\mathcal{E})) \longrightarrow X,$$

can be realized as follows. First recall that $G_k(\mathbb{P}(\mathcal{E})) = G_{1+k}(\mathcal{E})$. Let

$$[\wedge^{1+k} p] : \mathbb{P}(\bigwedge^{1+k} \mathcal{E}) \longrightarrow X$$

be the projective bundle corresponding to the vector bundle

$$\wedge^{1+k} p : \bigwedge^{1+k} \mathcal{E} \longrightarrow X.$$

Then $G_k(\mathbb{P}(\mathcal{E})) \subseteq \mathbb{P}(\bigwedge^{1+k} \mathcal{E})$ is the closed subvariety of all points of $\mathbb{P}(\bigwedge^{1+k} \mathcal{E})$ which over a point $x \in X$ corresponds to the variety of $\mathbb{C}$-lines spanned by a totally decomposable vector of $\bigwedge^{1+k} \mathcal{E}_x$. The induced fibration $p_k : G_k(\mathbb{P}(\mathcal{E})) \longrightarrow X$ is precisely the sought for Grassmann bundle. Its transition functions are the same as those of the bundle $\mathbb{P}(\bigwedge^{1+k} \mathcal{E})$ which are themselves determined by those of $\mathcal{E}$.

4.1.2 Bundles associated to a principal bundle. Let G be an algebraic group and let $P \longrightarrow X$ be a holomorphic principal G-bundle: P is a smooth complex scheme acted on freely and algebraically on the right by G, in such a way that the orbit space $G \backslash P$ is isomorphic to X, and the transition functions of $P \longrightarrow X$ are holomorphic. For example, the scheme $\mathbb{C}^{1+m} \setminus \{0\}$ is acted on the right by $\mathbb{C}^*$ via componentwise multiplication and the orbit variety is precisely $\mathbb{P}^m$. Hence $\mathbb{C}^{1+m} \setminus \{0\} \longrightarrow \mathbb{P}^m$ is a holomorphic principal $\mathbb{C}^*$-bundle over $\mathbb{P}^m$ meeting our hypotheses.

Suppose now that F is a scheme equipped with any algebraic left G-action. The following theorem is copied by [14], p. 91, (16.14.7), there stated in the category of differentiable manifolds. However the same proof holds in the holomorphic category.

4.1.3 Theorem. *The group G acts holomorphically on the right on $P \times F$ via $(p, f)g = (pg, g^{-1}f)$. Suppose that the orbit scheme $P \times_G F : P \times F/G$ exists in the holomorphic category (it always exists in the differentiable category). Then $\pi_F : P \times_G F \longrightarrow X$ is a bundle* (the "associated bundle to F") *such that the fibers are holomorphically equivalent to F.*

Let us see a few applications of the above theorem. Suppose that $\rho : G \longrightarrow Gl(V)$ is a holomorphic representation, where V is a complex vector space. Then V becomes a left G-module via the action $g * \mathbf{v} = \rho(g)\mathbf{v}$. Let $P \times_\rho V \longrightarrow X$ be the associated bundle to V (we write $P \times_\rho V$ instead of $P \times_G V$ to emphasize the representation ρ). It is a vector bundle $\mathcal{E} \longrightarrow X$ whose transition functions are precisely

$$\rho(g_{\alpha\beta}) : U_\alpha \cap U_\beta \longrightarrow Gl(V),$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$ are the holomorphic transition functions of $P \longrightarrow X$ with respect to some open covering $\mathcal{U} = (U_\alpha : \alpha \in \mathcal{A})$ trivializing X. Similarly, one concludes that $P \times_\rho G_{1+k}(V) \longrightarrow X$ is a holomorphic grassmann bundle over X. One contends that indeed $P \times_\rho G_{1+k}(V) \longrightarrow X$ is isomorphic to $G_{1+k}(\mathcal{E}) \longrightarrow X$. In fact the transition function of the first bundle are $[\wedge^k \rho(g_{\alpha\beta})]$ where

$$\wedge^{1+k}\rho(g_{\alpha\beta}(x))(\mathbf{v}_0 \wedge \ldots \wedge \mathbf{v}_k) = \rho(g_{\alpha\beta}(x))(\mathbf{v}_0) \wedge \ldots \wedge \rho(g_{\alpha\beta}(x))(\mathbf{v}_k)$$

while those of the second bundle are $\wedge^{1+k}(\rho(g_{\alpha\beta}))$. But

$$\wedge^{1+k}(\rho(g_{\alpha\beta}))(x)(\mathbf{v}_0 \wedge \ldots \wedge \mathbf{v}_k) = \rho(g_{\alpha\beta}(x))(\mathbf{v}_0) \wedge \ldots \wedge \rho(g_{\alpha\beta}(x))(\mathbf{v}_k)$$

so that $\wedge^{1+k}\rho(g_{\alpha\beta}) = \wedge^{1+k}(\rho(g_{\alpha\beta}))$.

4.1.4 Let G be a complex Lie group. Then there exists a universal G-bundle $EG \longrightarrow BG$ satisfying the following property: for each G-bundle $P \longrightarrow X$, there exists a unique map $\phi : X \longrightarrow BG$, up to homotopy equivalence, such that $P = \phi^* BG$. It turns out that EG is contractible, i.e. all the homotopy groups are zero. Let now G being the compact torus $T' := (S^1)^{1+p}$, for some $p \geq 0$, and let $ET' \longrightarrow BT'$ be the corresponding *universal principal T'-bundle* : T' acts freely on ET', ET' is contractible and BT' is the product of $(1+p)$-copies of the complex infinite projective space $\mathbb{P}^\infty$. The latter is thought of as the inductive limit $\varinjlim \mathbb{P}^m$, with respect with the chain of natural inclusions

$$\ldots \hookrightarrow \mathbb{P}^m \hookrightarrow \mathbb{P}^{1+m} \hookrightarrow \ldots .$$

and equipped with the inductive topology. The principal T'-bundle ET' itself can be seen as the inductive limit of the bundles $E_mT' \longrightarrow B_mT'$, where $E_mT' = (S^{2m+1})^{1+p}$, $B_mT' = (S^{2m+1})^{1+p}/(S^1)^{1+p} \cong \mathbb{P}^m$. In particular E_mT' is $2m$-connected (i.e. $\pi_i(S^{2m+1}) = 0$ for all $1 \leq i \leq 2m$).

If X is a paracompact topological space equipped with a continuous right T'-action, the T'-equivariant cohomology ring of X is:

$$H^*_{T'}(X) := H^*(ET' \times_{T'} X)$$

The key result we shall need in the sequel is that for each $m > r \geq 0$, there is a map

$$\phi_{m,r,j} : H^j(E_mT' \times_{T'} X) \longrightarrow H^j(E_rT' \times_{T'} X)$$

which is an isomorphism for all $0 \leq j \leq 2r$ (see [11], [37], [16]). In particular $H^*_{T'}(X) = \bigoplus_{i\geq 0} H^i_{T'}(X)$, where each $H^i_{T'}(X) = \lim_{\leftarrow} H^i(E_mT' \times_{T'} X)$, where the inverse limit is taken with respect to the the system of (iso)morphisms $\phi_{m,r,j}$. It follows that for each $r \geq 0$, there is a morphism

$$\phi_{r,i} : H^i_{T'}(X) \longrightarrow H^i(E_rT' \times_{T'} X)$$

which is an isomorphism for each $i \leq 2m$ and such that, for each $m \geq r$, the natural inclusion $E_rT' \hookrightarrow E_mT'$ gives a map

$$E_rT' \times_{T'} X \hookrightarrow E_mT' \times_{T'} X$$

inducing a ring homomorphism

$$\phi_{m,r} : H^*(E_mT' \times_{T'} X) \longrightarrow H^*(E_rT' \times_{T'} X)$$

such that $\phi_{m,r,i} : H^i(E_mT' \times_{T'} X) \longrightarrow H^i(E_rT' \times_{T'} X)$ is an isomorphism for each $j \leq 2r$. Moreover

$$\phi_{m,r,j} \circ \phi_{m,j} = \phi_{r,j},$$

(by the universal property of the inverse limit).

It turns out that $H^*_{T'}(X)$ is a ring with respect to the *equivariant cup product*. If $\xi_1 \in H^{q_1}_{T'}(X)$ and $\xi_2 \in H^{q_2}_{T'}(X)$, then for each $r > q_1+q_2$, $H^i_{T'}(X) \cong H^i(E_rT' \times_{T'} X)$, for each $1 \leq i \leq r$. Then one set:

$$\xi_1 \cup \xi_2 = \phi^{-1}_{r,q_1+q_2}(\phi_{r,q_1}(\xi_1) \cup \phi_{r,q_2}(\xi_2)).$$

Notice that if $m \geq r$, one has:

$$\begin{aligned}
&\phi^{-1}_{r,q_1+q_2}(\phi_{r,q_1}(\xi_1) \cup \phi_{r,q_2}(\xi_2)) = \\
&= (\phi_{m,r,q_1+q_2} \circ \phi_{m,q_1+q_2})^{-1}(\phi_{m,r,q_1}(\phi_{m,q_1}(\xi_1)) \cup \phi_{m,r,q_2}(\phi_{m,q_2}(\xi_1))) = \\
&= \phi^{-1}_{m,q_1+q_2} \circ \phi^{-1}_{m,r,q_1+q_2} \circ \phi_{m,r,q_1+q_2}((\phi_{m,q_1}(\xi_1)) \cup \phi_{m,q_2}(\xi_1)) = \\
&= \phi^{-1}_{m,q_1+q_2}(\phi_{m,q_1}(\xi_1) \cup \phi_{m,q_2}(\xi_2)).
\end{aligned}$$

4.1.5 T-equivariant Chow groups. Let now X be a complex smooth projective variety acted on by $T := (\mathbb{C}^*)^{1+p}$, a $(1+p)$-dimensional algebraic torus. Then ET is the product of $(1+p)$ copies of $\mathbb{C}^\infty \setminus \{0\}$ and BT is again $(\mathbb{P}^\infty)^{1+p}$. If T acts on X, there is an obvious induced action of T' on X, and since S^1 is a deformation retract of $\mathbb{C}^*$, it turns out that

$$H^*_T(X) = H^*_{T'}(X).$$

From now on, then, we shall only deal with T-equivariant cohomology. It is possible to define a Chow equivariant intersection theory (see e.g. [15], [11], [52] for details). For the purposes of this exposition, we prefer to avoid the foundational details involved in the definition of such a ring by invoquing a result of Bialynicki-Birula ([7]), saying that there is a natural cycle map, doubling degrees,

$$cl_T : A^*_T(G_k(\mathbb{P}^n)) \longrightarrow H^*_T(G_k(\mathbb{P}^n)),$$

which is an isomorphism in this case (see also [11], p. 25). Hence, in the case that T is a $1+p$ dimensional torus, we define:

$$A^*_T(X) := H^*_T(X). \tag{4.1}$$

Up to now, (4.1) is just a different notation for denoting the T-equivariant cohomology. We shall relate it to the ordinary Chow intersection theory as follows. According to definition (4.1), for each $m \geq 0$ and for each $0 \leq i \leq 2m$, $A^i_T(X) = H^i(E_mT \times_T X)$. Now, if X is a grassmannian $G_k(\mathbb{P}^n)$ (or the projective space $\mathbb{P}^n$ itself, when $k = 0$), the bundle

$$E_mT \times_T G_k(\mathbb{P}^n) \longrightarrow B_mT,$$

is the grassmann bundle $G_k(\mathbb{P}(\mathcal{E}_m)) \longrightarrow (\mathbb{P}^m)^{1+p}$, where $\mathcal{E}_m \longrightarrow X$ is the holomorphic vector bundle $E_mT \times_T \mathbb{C}^{1+n} \longrightarrow B_mT$ associated to $E_mT \longrightarrow B_mT$. Therefore $G_k(\mathbb{P}(\mathcal{E}_m)) \longrightarrow B_mT$ is a flag bundle over a CW-complex (the product of projective spaces) and by [23], p. , it follows that $H^*(G_k(\mathbb{P}(\mathcal{E}_m))) = A^*(G_k(\mathbb{P}(\mathcal{E}_m))$. The latter, in particular, is a module over the ring $A^*(B_mT) = H^*(B_mT)$.

Therefore we can link the equivariant Chow ring $A^*_T(X)$ to Chow intersection theory by saying that there is a map

$$\phi_{m,i} : A^i_T(X) \longrightarrow A^i(E_mT \times_T X) = A^i(\mathbb{P}(\mathcal{E}_m))$$

which is an isomorphism for all $m \geq 0$ and all $0 \leq i \leq 2m$.

4.1.6 Recall that the equivariant Chow ring $A^*_T(\mathbb{P}^n)$ $(= H^*_T(X))$ of $\mathbb{P}^n$ is a free module of rank $n+1$ over $A = A^*_T(pt)$, generated by T-invariant cycles classes represented by intersections of the (T-invariant) coordinates hyperplanes. Suppose there is a regular 0-SCGP (M, D_1), where

$$M = A\mu^0 \oplus A\mu^1 \oplus \ldots \oplus A\mu^n,$$

such that:

i) M is a free $A^*_T(\mathbb{P}^n)$-module of rank 1 generated by μ^0. If $\xi \in A^*_T(\mathbb{P}^n)$ and $\eta \in M$, one writes $\xi \cap' \eta$ for the module multiplication;

ii) The rings $A_T^*(\mathbb{P}^n)$ and $\mathcal{A}^*(M, D_1) \cong A[\mathbf{T}]/(\ker(\mathrm{ev}_{D,\mu^0})$ are isomorphic, and the following diagram

$$\begin{array}{ccc} A_T^*(\mathbb{P}^n) \otimes_A M & \stackrel{\cap'}{\longrightarrow} & M \\ {\scriptstyle \iota_0}\Big\downarrow & & \Big\downarrow \mathbf{1}\,, \\ \mathcal{A}^*(M, D_1) \otimes_A M & \longrightarrow & M \end{array} \qquad (4.2)$$

is commutative, the vertical arrows being isomorphisms (the right one is just the identity).

4.1.7 Let $M(m) := A^*(B_mT)\mu^0 \oplus A^*(B_mT)\mu^1 \oplus \ldots \oplus A^*(B_mT)\mu^n$. Then one has

$$M(m)_j = A^j(B_mT)\mu^0 \oplus A^{j-1}(B_mT)\mu^1 \oplus \ldots A^{j-n}(B_mT)\mu^n$$

where one sets $A^i(B_mT) = 0$, if $i < 0$.

Now, for each $j \geq 0$ there exists m such that $A_T^i(pt) := A^i(BT) = A^i(B_mT)$, for all $0 \leq i \leq j$. Hence

$$M_j = A^j(BT)\mu^0 \oplus A^{j-1}(BT)\mu^1 \oplus \ldots A^{j-n}(BT)\mu^n = M(m)_j.$$

Hence $M(m)$ can be seen as an approximation of M. Since $A_T^i(BT) = \lim_{\longleftarrow} A^i(B_mT)$, it follows that $M_i = \lim_{\longleftarrow} M(m)_i$. Let $\psi_{m,i} : M_i \longrightarrow M(m)_i$ be the approximation map. Define $D_{1,m} : M(m) \longrightarrow M(m)$ as follows:

$$D_{1,m}\mu^j = \psi_{m,j+1}(D_1\mu^j)$$

Then $(M(m), D_{1,m})$ is a 0-SCGP, and one can consider the corresponding k-SCGP $(\bigwedge^{1+k}(M(m), D_{1,m})$. We have an isomorphism

$$\iota_{k,m} : A^*(G_k(\mathbb{P}(\mathcal{E}_m)) \longrightarrow \mathcal{A}^*(\bigwedge^{1+k}(M(m), D_{1,m})$$

which is that prescribed by Theorem 3.4.2.

Furthermore $M(m)$ is isomorphic to $A_*(\mathbb{P}(\mathcal{E}_m))$ through the $A^*(B_mT)$-isomorphism $\chi_{0,m} : M(m) \longrightarrow A_*(\mathbb{P}(\mathcal{E}_m))$ defined by

$$\mu^j \longmapsto c_1(\mathcal{T}_{0,m})^j \cap [\mathbb{P}(\mathcal{E}_m]$$

where $\mathcal{T}_{k,m}$ is the tautological bundle over $G_k(\mathbb{P}(\mathcal{E}_m))$. Hence the following diagram

$$\begin{array}{ccc} A^*(\mathbb{P}(\mathcal{E}_m)) \otimes M(m) & \xrightarrow{\cap'} & M(m) \\ \downarrow {\scriptstyle \mathbf{1}\otimes\chi_{0,m}} & & \downarrow \\ A^*(\mathbb{P}(\mathcal{E}_m)) \otimes A_*(\mathbb{P}(\mathcal{E}_m)) & \xrightarrow{\cap} & A_*(\mathbb{P}(\mathcal{E}_m)) \\ \downarrow {\scriptstyle \iota_{0,m}\otimes\chi_{0,m}{}^{-1}} & & \downarrow \\ \mathcal{A}^*(M(m), D_{1,m}) \otimes M(m) & \xrightarrow{d_m} & M(m) \end{array}$$

has the top commutative square if $\cap'$ is defined as $\xi \cap' \eta = \xi \cap \chi_{0,m}\eta$. As for the bottom one, is just a special case of Theorem 3.4.2 (for $k = 0$). Therefore the diagram

$$\begin{array}{ccc} A^*(\mathbb{P}(\mathcal{E}_m)) \otimes M(m) & \xrightarrow{\cap'} & M(m) \\ \downarrow {\scriptstyle \iota_{0,m}\otimes\mathbf{1}} & & \downarrow \\ \mathcal{A}^*(M(m), D_{1,m}) \otimes M(m) & \xrightarrow{d_m} & M(m) \end{array}$$

is commutative and is an approximation of the diagram (4.2).

4.1.8 Notice now that $\bigwedge^{1+k} M(m)$ is an approximation of $(\bigwedge^{1+k} M)$ in the following sense. For each $w \geq 0$, there exists $m > 0$ and a natural approximation isomorphism $(\bigwedge^{1+k} M)_i \longrightarrow (\bigwedge^{1+k} M(m))_i$. In fact

$$(\bigwedge^{1+k} M)_i = \bigoplus_{0\leq wt(I)\leq i} A^{w-wt(I)}(BT) \cdot \bigwedge^{I} \boldsymbol{\mu},$$

where if $I = (i_0, i_1, \ldots, i_k) \in \mathcal{I}_n^k$, then $\bigwedge^I \boldsymbol{\mu} = \mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k}$. Let $m > \max(w, i_k)$. Then $\phi_{m,j} : A^j(BT) \longrightarrow A^j(B_mT)$ is an isomorphism for all $0 \leq j \leq m$. One can then write

$$(\bigwedge^{1+k} M(m))_i = \bigoplus_{0\leq wt(I)\leq i} A^{w-wt(I)}(B_mT) \bigwedge^{I} \boldsymbol{\mu} = \bigoplus_{0\leq wt(I)\leq i} \phi_{m,w-wt(I)}(A^{i-wt(I)}(BT)) \bigwedge^{I} \boldsymbol{\mu}$$

4.1.9 By abuse, denote again by $\psi_{m,i} : \bigwedge^{1+k} M(m)_i \longrightarrow (\bigwedge^{1+k} M(m))_i$ the approximation homomorphism and define $\psi'_{m,i} : \mathcal{A}^i(\bigwedge^{1+k}(M, D_1) \longrightarrow \mathcal{A}^i(\bigwedge^{1+k}(M(m), D_{1,m})$

and $\psi'_{m,r} : \mathcal{A}^i(\bigwedge^{1+k}(M(m), D_{1,m}) \longrightarrow \mathcal{A}^i(\bigwedge^{1+k}(M(r), D_{1,r})$ imposing the commutativity of the following two diagrams:

$$\begin{array}{ccc} \mathcal{A}^i(\bigwedge^{1+k}(M, D_1)) & \overset{\mathrm{ev}_{\mu^{i_0}\wedge\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}}}{\longrightarrow} & \bigwedge^{1+k} M_i \\ \psi'_{m,i}\downarrow & & \downarrow \psi_{m,i} \\ \mathcal{A}^i(\bigwedge^{1+k}(M(m), D_{1,m})) & \overset{\mathrm{ev}_{\mu^{i_0}\wedge\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}}}{\longrightarrow} & \bigwedge^{1+k} M(m)_i \end{array}$$

and

$$\begin{array}{ccc} \mathcal{A}^i(\bigwedge^{1+k}(M(m), D_{1,m})) & \overset{\mathrm{ev}_{\mu^{i_0}\wedge\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}}}{\longrightarrow} & \bigwedge^{1+k} M(m)_i \\ \psi'_{m,r,i}\downarrow & & \downarrow \psi_{m,r,i} \\ \mathcal{A}^i(\bigwedge^{1+k}(M(r), D_{1,r})) & \overset{\mathrm{ev}_{\mu^{i_0}\wedge\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}}}{\longrightarrow} & \bigwedge^{1+k} M(r)_i \end{array}$$

Moreover $\psi'_{r,i} \circ \psi'_{m,r,i} = \psi'_{m,i}$ as well as $\psi_{r,i} \circ \psi_{m,r,i} = \psi_{m,i}$, by definition of inverse limits. Let

$$\iota_{k,m} : A^*(G_k(\mathbb{P}(\mathcal{E}_m)) \longrightarrow \mathcal{A}^*(\bigwedge^{1+k}(M, D_{1,m})$$

be the isomorphism defined in Theorem 3.4.2 (corresponding to the inverse of **C** in diagram (4.3). Then, The main result of this section (and of the thesis) allows a sharp description of the equivariant Schubert Calculus, alternative to the puzzle's techniques described in [39].

4.1.10 Theorem. *There is an A-algebra isomorphism*

$$\iota_k : A_T^*(G_k(\mathbb{P}^n)) \longrightarrow \mathcal{A}^*(\bigwedge^{1+k}(M, D_1)),$$

making $\bigwedge^{1+k} M$ into a free $A_T^*(G_k(\mathbb{P}^n))$-module of rank 1 spanned by $\mu^0\wedge\mu^1\wedge\ldots\wedge\mu^n$ isomorphic to $\bigwedge^{1+k} M$ thought of as a module over $\mathcal{A}^*(\bigwedge^{1+k}(M, D_1))$.

Proof. Let $\xi \in A_T^*(G_k(\mathbb{P}^n))$. It is a finite sum of homogeneous elements. So, we may assume, without loss of generality, that $\xi \in A_T^i(G_k(\mathbb{P}^n))$. Define ι_k : $A_T^i(G_k(\mathbb{P}^n)) \longrightarrow \mathcal{A}^*(\bigwedge^{1+k}(M, D_1))$ as follows. Let m such that $\phi_{m,i} : A_T^i(G_k(\mathbb{P}^n)) \longrightarrow A^i(G_k(\mathbb{P}(\mathcal{E}_m))$ is an isomorphism, and define:

$$\iota_k(\xi) = \psi'^{-1}_{m,i}(\iota_{k,m}(\phi_{m,i}(\xi)))$$

One has, for each $q > m$:

$$\psi'^{-1}_{m,i}(\iota_{k,m}(\phi_{m,i}(\xi))) = \psi'^{-1}_{q,i}(\iota_{k,q}(\phi_{q,i}(\xi)))$$

and hence $\iota_k(\xi)$ is well defined. We claim that ι_k is an A-module homomorphism. In fact

$$\begin{aligned}\iota_k(\xi_1\cup\xi_2) &= {\psi'}^{-1}_{m,i+j}(\iota_{k,m}(\phi_{m,i+j}(\xi_1\cup\xi_2)))=\\ &= {\psi'}^{-1}_{m,i+j}\iota_{k,m}(\phi_{m,i}(\xi_1)\cup\phi_{m,j}(\xi_2))=\\ &= {\psi'}^{-1}_{m,i+j}(\iota_{k,m}(\phi_{m,i}(\xi_1)\iota_{k,m}(\phi_{m,j}(\xi_2))=\\ &= {\psi'}^{-1}_{m,i}(\iota_{k,m}(\phi_{m,i}(\xi_1)\cup{\psi'}^{-1}_{m,i}(\iota_{k,m}(\phi_{m,j}(\xi_2))=\iota_k(\xi_1)\cup\iota_k(\xi_2).\end{aligned}$$

Moreover, if $\iota_k(\xi)=0$, then ${\psi'}^{-1}_{m,i}\iota_{k,m}(\phi_{m,i}(\xi))=0$, then $\phi_{m,i}(\xi)=0$, because ${\psi'}^{-1}_{m,i}$ and $\iota_{k,m}$ are isomorphisms, too. Then $\xi=0$, because $\phi_{m,i}$ is an isomorphism. Moreover, each $\eta\in\bigwedge^{1+k}M_i$ is sent onto $\psi_{m,i}(\eta)$ and $\iota^{-1}_{k,m}\psi_{m,i}(\eta)$ is a pre-image of η in $A^i_T(G_k(\mathbb{P}^n))$. Hence ι_k is an isomorphism. ∎

Define $\xi\cap'\eta=\iota_k(\xi)\eta$. Then the following diagram is commutative.

$$\begin{array}{ccc} A^*_T(G_k(\mathbb{P}^n))\otimes_A\bigwedge^{1+k}M & \stackrel{\cap'}{\longrightarrow} & \bigwedge^{1+k}M\\ \iota_k\otimes\mathbf{1}\Big\downarrow & & \Big\downarrow\mathbf{1}\\ \mathcal{A}^*(\bigwedge^{1+k}(M,D_1))\otimes_A\bigwedge^{1+k}M & \stackrel{d}{\longrightarrow} & \bigwedge^{1+k}M\end{array}\quad . \tag{4.3}$$

and exhibits $\bigwedge^{1+k}M$ as a free $A^*_T(G_k(\mathbb{P}^n))$-module of rank 1 isomorphic to $\bigwedge^{1+k}M$ thought of as a free $\mathcal{A}^*(\bigwedge^{1+k}(M,D_1))$ of rank 1, generated by $\mu^{i_0}\wedge\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}$.

4.2 Tao-Knutson T-Equivariant Schubert Calculus.

4.2.1 In this section we shall apply Theorem 4.1.10 to offer the promised alternative description of the equivariant Schubert Calculus investigated by Knutson and Tao in [39]. There the authors use the combinatorial tool of puzzles, which is interesting in its own. Our approach, however, will consist in identifying a 0-SCGP (M,D_1) describing the equivariant intersection ring of the projective space, and looking at the corresponding $\bigwedge^{1+k}(M,D_1)$. The situation is as follows: a compact $n+1$-dimensional torus $T:=(S^1)^{n+1}$ acts diagonally on $\mathbb{P}^n$, with isolated fixed points and one aims to compute the T-equivariant Schubert Calculus, i.e. the structure constant of the algebra. Our way to *compute Schubert calculus* is, instead using puzzles, via Pieri's and Giambelli's formulas for equivariant cohomology.

4.2.2 The model. Let M be a free A-module of rank $1+n$ spanned by $(\mu^0, \mu^1, \ldots, \mu^n)$, where $A = \mathbb{Z}[y_0, y_1, \ldots, y_n]$ (playing the role of the equivariant cohomology of a point). Consider the 0-SCGP (M, D_1), where $D_1 : M \longrightarrow M$ is the unique A-endomorphism such that:

$$D_1\mu^j = (1-\delta_{jn})\mu^{j+1-\delta_{jn}} + (y_j - y_0)\mu^j, \quad 0 \le j \le n, \tag{4.4}$$

where δ_{jn} is the Kronecker's delta. Denote by $D_0 = D_1^0$ the identity of M and let $G_i^{\boldsymbol{\mu}} \in \mathbb{Z}[T_1] \subset \mathbb{Z}[\mathbf{T}]$ such that $G_0 = 1$ and, for each $1 \le i \le n$, $G_i^{\boldsymbol{\mu}} \in A[T_1] \subset A[\mathbf{T}]$ is defined by

$$G_i^{\boldsymbol{\mu}} = \prod_{j=0}^{i-1}(T_1 - (y_j - y_0)),$$

so that

$$\mathrm{ev}_D(G_i^{\boldsymbol{\mu}}) := G_i^{\boldsymbol{\mu}}(D) = \prod_{j=0}^{i-1}(D_1 - (y_j - y_0)D_0) \in End_A(M).$$

In particular $G_1^{\boldsymbol{\mu}}(D) = D_1$. An easy check shows that, for $1 \le i \le n$,

$$\mu^i = G_i^{\boldsymbol{\mu}}(D) \cdot \mu^0. \tag{4.5}$$

In other words, $G_i^{\boldsymbol{\mu}}$ is a Giambelli's polynomial for μ^i, in the sense of for each $0 \le i \le n$. Because of the surjection $\mathrm{ev}_{D,\mu^0} : A[\mathbf{T}] \longrightarrow M$, one has:

$$\mathcal{A}^*(M, D_1) = \frac{A[\mathbf{T}]}{(\ker(ev_{D,\mu^0}))} \cong M. \tag{4.6}$$

Hence, $\mathcal{A}^*(M, D_1)$, is freely generated as a module over A, by

$$G_0^{\boldsymbol{\mu}}(D) := 1, \quad G_1^{\boldsymbol{\mu}}(D), \quad \ldots, \quad G_n^{\boldsymbol{\mu}}(D).$$

Since each $G_i^{\boldsymbol{\mu}}(D)$ is an (explicit) A-polynomial expression in $G_1^{\boldsymbol{\mu}}(D) = D_1$ with A-coefficients, it follows that D_1 generates $\mathcal{A}^*(M, D_1)$ as an A-algebra. Moreover,

D_1^{n+1} is a (unique) A-linear combination of $1, D_1, \ldots, D_1^n$ since the last are A-linearly independent in $\mathcal{A}^*(M, D_1)$. The corresponding relation can be given by noticing that $(D_1 - (y_n - y_0))\mu^n = 0$ i.e. $(D_1 - (y_n - y_0))G_n^{\boldsymbol{\mu}}(D)\mu^0 = 0$, easily implying that

$$\mathcal{A}^*(M, D_1) = \frac{A[D_1]}{(\prod_{i=0}^n (D_1 - (y_i - y_0)))} \tag{4.7}$$

4.2.3 We contend that the ring $\mathcal{A}^*(M, D_1)$ is canonically isomorphic to the T-equivariant intersection (or cohomology) ring of $\mathbb{P}^n$, as implicitly described in [39], to whom the reader is referred for additional details. Consider the module $A^{1+n} := \bigoplus_{i=0}^n H_T^*(pt)$ together with the A-algebra structure defined by componentwise multiplication of polynomials. Then, in [39], one identifies the equivariant cohomology $H_T^*(\mathbb{P}^n)$ of $\mathbb{P}^n$ with the A-subalgebra of A^{1+n} which is freely generated, as A-module, by the *classes*

$$S_0 := \tilde{S}_{011\ldots1}, \quad S_1 := \quad \tilde{S}_{101\ldots1}, \quad S_2 := \tilde{S}_{110\ldots1}, \quad S_n := \tilde{S}_{111\ldots0}$$

where the subscript of S denotes the position of the "0" in the indexing string of $\tilde{S}$ (notice that our indeterminates y are indexed by the set $\{0, 1, \ldots, n\}$, so all our notation are 1-shifted with respect to those of [39]). The first generator S_0 is the $(1+n)$-tuple whose components are all equal to 1 (the identity of $H_T^*(\mathbb{P}^n)$), while the components of S_i ($i > 0$) must satisfy the Goresky–Kottwitz–MacPherson (GKM) conditions ([30]): applying the recipe of [39], p. 230, one sees that S_i can be chosen in such a way that its j^{th} component is:

$$S_i^j = (y_j - y_0) \cdot (y_j - y_1) \cdot \ldots \cdot (y_j - y_{i-1}) = \prod_{h=0}^{i-1} (y_j - y_h).$$

From these data it is obvious that $H_T^*(\mathbb{P}^n)$ is indeed generated as a ring by S_1, because each S_i is an A-polynomial expression of S_1. In fact:

$$S_i = G_i^{\boldsymbol{\mu}}(S_1). \tag{4.8}$$

The proof of (4.8) is straightforward: it suffices to check equality for each component of S_i. Then:

$$(G_i^{\boldsymbol{\mu}}(S_1))^j = \prod_{h=0}^{i-1}(S_1^j - (y_h - y_0)) = \prod_{h=0}^{i-1}(y_j - y_0 - y_h + y_0) = \prod_{h=0}^{i-1}(y_j - y_h) = S_i^j.$$

4.2.4 Proposition. *There is a ring isomorphism* $\iota : \mathcal{A}^*(M, D_1) \longrightarrow H_T^*(\mathbb{P}^n)$ *given by* $D_1 \mapsto S_1$*.*

Proof. Since any generator $G_i^{\boldsymbol{\mu}}(D)$ of $\mathcal{A}^*(M, D_1)$ (resp. $H_T^*(\mathbb{P}^n)$) is a polynomial expression in D_1 (resp. in S_1), it is sufficient to show that $\iota(G_i^{\boldsymbol{\mu}}(D)) = S_i$, and this is true because of formula (4.8). ∎

4.2.5 Notation. From now on, for notational simplicity, we shall use variables $Y_1, \ldots, Y_n$, defined by

$$Y_i = y_i - y_0.$$

4.2.6 The k-SCGP $\bigwedge^{1+k}(M, D_1)$. By Theorem 4.1.10, $\bigwedge^{1+k} M$ is a free module of rank 1 over $A_T^*(G_k(\mathbb{P}^n))$ which is itself isomorphic to $\mathcal{A}^*(\bigwedge^{1+k}(M, D_1))$ and hence with it identified. One then knows that $A_T^*(G_k(\mathbb{P}^n))$ is indeed generated by

$$(D_1, \ldots, D_k, D_{1+k})$$

as an A-algebra, by Proposition 3.2.14. As an A-module, instead, $A_T^*(G_k(\mathbb{P}^n))$ is freely generated by Giambelli's polynomials $G_{i_0 i_1 \ldots i_k}^{\boldsymbol{\mu}}(D)$ (recall our various abuse of notation and terminology in Section 3.1.1), one for each $\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k}$. Moreover we have Pieri's type formulas at our disposal: they are just Leibniz's rule. As a matter of fact, Giambelli's polynomial of $\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k}$ correspond to the A-module basis indicated by Knutson and Tao. However we can use several different bases, including the canonical ones, as we shall see in a moment.

4.2.7 Example. This example is a revisitation of [39], p. 231. Let $A := \mathbb{Z}[y_0, y_1, y_2, y_3]$, $M := A\mu^0 \oplus \ldots \oplus A\mu^3$ and $D_1 : M \longrightarrow M$ defined by

$$\begin{cases} D_1\mu^i & = & \mu^{i+1} + Y_i\mu^i \quad \text{if} \quad 0 \le i < n \\ D_1\mu^n & = & Y_n\mu^n \end{cases}$$

(recall Notation 4.2.5).

Let us lexicographically order the basis $\mu^i \wedge \mu^j$ of $\bigwedge^2 M$. Then one easily finds, either directly or via canonical bases as in Example 3.3.10,

$$\begin{aligned}
G^{\boldsymbol{\mu}}_{01}(D) &= 1;\\
G^{\boldsymbol{\mu}}_{02}(D) &= D_1 - Y_1 D_0;\\
G^{\boldsymbol{\mu}}_{03}(D) &= D_2 - e_1(Y_1, Y_2)D_1 + e_2(Y_1, Y_2)D_0;\\
G^{\boldsymbol{\mu}}_{12}(D) &= D_1^2 - D_2;\\
G^{\boldsymbol{\mu}}_{13}(D) &= \Big(D_1 - Y_3\Big)G^{\boldsymbol{\mu}}_{0,3}(D);\\
G^{\boldsymbol{\mu}}_{23}(D) &= (D_2 - Y_3^2)G^{\boldsymbol{\mu}}_{03}(D) - (Y_1 + Y_3)G^{\boldsymbol{\mu}}_{13}(D).
\end{aligned}$$

Notice that for each $(i, j) \in \mathcal{I}^1_3$, $G^{\boldsymbol{\mu}}_{03}(D) \in \mathcal{A}^*(\bigwedge^2(M, D_1))$ is an A-endomorphisms of M. Let us write the matrices associated to $G_{ij}(D)$ in the basis $\wedge^2\boldsymbol{\mu}$. $G^{\boldsymbol{\mu}}_{01}(D)$ is just the identity matrix (the 6 diagonal entries are all 1);

$$(G^{\boldsymbol{\mu}}_{01}) = \begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0\\ 0 & \mathbf{1} & 0 & 0 & 0 & 0\\ 0 & 0 & \mathbf{1} & 0 & 0 & 0\\ 0 & 0 & 0 & \mathbf{1} & 0 & 0\\ 0 & 0 & 0 & 0 & \mathbf{1} & 0\\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{pmatrix}$$

As for $G^{\boldsymbol{\mu}}_{02}(D)$ one has:

$$\begin{aligned}
\Big(D_1 - Y_1 D_0\Big)\mu^0 \wedge \mu^1 &= \mu^1 \wedge \mu^1 + \mu^0 \wedge (\mu^2 + Y_1\mu^1) - (y_1 - y_0)\mu^0 \wedge \mu^1 =\\
&= \mu^0 \wedge \mu^2\\
\Big(D_1 - Y_1 D_0\Big)\mu^0 \wedge \mu^2 &= \mu^1 \wedge \mu^2 + \mu^0 \wedge (\mu^3 + Y_2\mu^1) - (y_1 - y_0)\mu^0 \wedge \mu^2 =\\
&= \mu^1 \wedge \mu^2 + \mu^0 \wedge \mu^3 + (\mathbf{Y_2} - \mathbf{Y_1})\mu^0 \wedge \mu^2\\
\Big(D_1 - Y_1 D_0\Big)\mu^0 \wedge \mu^3 &= \mu^1 \wedge \mu^3 + \mu^0 \wedge ((Y_1 - Y_0)\mu^3) - (Y_1 - Y_0)\mu^0 \wedge \mu^3 =\\
&= \mu^1 \wedge \mu^3 + (\mathbf{Y_3} - \mathbf{Y_1})\mu^0 \wedge \mu^3
\end{aligned}$$

$$\begin{aligned}
\Big(D_1 - Y_1 D_0\Big)\mu^1\wedge\mu^2 &= (\mu^2 + Y_1\mu^1)\wedge\mu^2 + \mu^1\wedge(\mu^3 + Y_2\mu^2) - Y_1\mu^1\wedge\mu^2 = \\
&= Y_1\mu^1\wedge\mu^2 + \mu^1\wedge\mu^3 + Y_2\mu^1\wedge\mu^2 - Y_1\mu^1\wedge\mu^2 = \\
&= \mu^1\wedge\mu^3 + \mathbf{Y_2}\mu^1\wedge\mu^2 \\
\Big(D_1 - Y_1 D_0\Big)\mu^1\wedge\mu^3 &= (\mu^2 + Y_1\mu^1)\wedge\mu^3 + \mu^1\wedge(Y_3\mu^3) - Y_1\mu^1\wedge\mu^3 \\
&= \mu^2\wedge\mu^3 + \mathbf{Y_3}\mu^1\wedge\mu^3 \\
\Big(D_1 - Y_1 D_0\Big)\mu^2\wedge\mu^3 &= (\mu^3 + Y_2\mu^2)\wedge\mu^3 + \mu^2\wedge(Y_3\mu^3) - Y_1\mu^2\wedge\mu^3 \\
&= (\mathbf{Y_3} + \mathbf{Y_2} - \mathbf{Y_1})\mu^2\wedge\mu^3
\end{aligned}$$

Then, matrix of $G^{\boldsymbol{\mu}}_{02}$ in the basis $\{\mu^i\wedge\mu^j\}$ is:

$$(G^{\boldsymbol{\mu}}_{02}(D)) = \begin{pmatrix}
\mathbf{0} & 0 & 0 & 0 & 0 & 0 \\
1 & \mathbf{Y_2}-\mathbf{Y_1} & 0 & 1 & 0 & 0 \\
0 & 1 & \mathbf{Y_3}-\mathbf{Y_1} & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{Y_2} & 1 & 0 \\
0 & 0 & 1 & 0 & \mathbf{Y_3} & 0 \\
0 & 0 & 0 & 0 & 1 & \mathbf{Y_3}+\mathbf{Y_2}-\mathbf{Y_1}
\end{pmatrix}$$

In the same way, one can compute the matrices of the remaining $G^{\boldsymbol{\mu}}_{ij}(D)$, getting

$$(G^{\boldsymbol{\mu}}_{0,3}(D)) = \begin{pmatrix}
\mathbf{0} & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{0} & 0 & 0 & 0 & 0 \\
1 & Y_3-Y_1 & (\mathbf{Y_3}-\mathbf{Y_1})(\mathbf{Y_3}-\mathbf{Y_2}) & 0 & 0 & 0 \\
0 & 0 & 0 & \mathbf{0} & 0 & 0 \\
0 & 1 & Y_3-Y_2 & Y_3 & \mathbf{Y_3}(\mathbf{Y_3}-\mathbf{Y_2}) & 0 \\
0 & 0 & 1 & 0 & Y_3 & \mathbf{Y_3}(\mathbf{Y_3}-\mathbf{Y_1})
\end{pmatrix}$$

$$(G^{\boldsymbol{\mu}}_{1,2}(D)) = \begin{pmatrix}
\mathbf{0} & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{0} & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{0} & 0 & 0 & 0 \\
1 & Y_2 & 0 & \mathbf{Y_1 Y_2} & 0 & 0 \\
0 & 1 & Y_3 & Y_1 & \mathbf{Y_1 Y_3} & 0 \\
0 & 0 & 0 & 1 & Y_3 & \mathbf{Y_2 Y_3}
\end{pmatrix}$$

$$(G^{\boldsymbol{\mu}}_{1,3}(D))=\begin{pmatrix}\mathbf{0}&0&0&0&0&0\\0&\mathbf{0}&0&0&0&0\\0&0&\mathbf{0}&0&0&0\\0&0&0&\mathbf{0}&0&0\\1&Y_3&Y_3(Y_3-Y_2)&Y_1Y_3&\mathbf{Y_1Y_3(Y_3-Y_2)}&0\\0&1&Y_3&Y_3&Y_3^2&\mathbf{Y_2(Y_3-Y_1)Y_3}\end{pmatrix}$$

$$(G^{\boldsymbol{\mu}}_{2,3}(D))=\begin{pmatrix}\mathbf{0}&0&0&0&0&0\\0&\mathbf{0}&0&0&0&0\\0&0&\mathbf{0}&0&0&0\\0&0&0&\mathbf{0}&0&0\\0&0&0&0&\mathbf{0}&0\\1&Y_3+Y_2-Y_1&Y_3(Y_3-Y_1)&Y_3Y_2&Y_3Y_2(Y_3-Y_1)&\mathbf{Y_2(Y_3-Y_1)Y_3(Y_2-Y_1)}\end{pmatrix}$$

to each (i,j) such that $0\leq i<j\leq 3$, associates the sequence

$$\mathbf{a}_{ij}:\{0,1,\dots,3\}\longrightarrow\{0,1\}$$

such that $\mathbf{a}_{ij}(i)=0$, $\mathbf{a}_{ij}(j)=0$ and $\mathbf{a}_{ij}(m)=1$ if $m\neq i,j$, one sees that the polynomials occurring in the diagonal of $G^{\boldsymbol{\mu}}_{ij}(D)$ satisfy GKM conditions prescribed to the restriction $\alpha\in A^*_T(G_1(\mathbb{P}^3))$ to the equivariant cohomology of a point. In other words we got the basis described by Knutson and Tao: the map sends (remind that we are using a left 1-shifted notation for the indices)

$$G^{\boldsymbol{\mu}}_{i,j}(D)\mapsto\tilde{S}_{\mathbf{a}_{ij}}.$$

The polynomials occurring in the picture at p. 231 of [39] are precisely the diagonals elements of the triangular matrices associated $G^{\boldsymbol{\mu}}_{i,j}(D)$: notice that our description of the basis of equivariant cohomology is even more explicit of that occurring in [39], since one “physically” sees, here, that setting all the y to zero one gets the basis of the classical intersection ring of the Grassmannian (as a consequence of the fact that our Giambelli's are the classical ones setting all y to zero).

In particular one gets the identification

$$\begin{array}{rcccc} \gamma & : & \bigwedge^2 M & \longrightarrow & A_T^*(G_1(\mathbb{P}^3)) \\ & & \mu^i \wedge \mu^j & \stackrel{\gamma}{\longmapsto} & \tilde{S}_{\mathbf{a}_{ij}} \end{array} \tag{4.9}$$

and one can easily check by hands that

$$\tilde{S}_{\mathbf{a}_{ij}} \tilde{S}_{\mathbf{a}_{hr}} = \gamma(G_{ij}^{\boldsymbol{\mu}}(D)\mu^h \wedge \mu^r)$$

and this means that one may identify the A-basis described in [39] with our $\mu^i \wedge \mu^j$.

4.2.8 Consider the following diagram, depicting the Chevalley-Bruhat order inside $\mathcal{I}_3^1$:

$$\begin{array}{ccccc} & & (01) & & \\ & & \downarrow & & \\ & & (02) & & \\ & \swarrow & & \searrow & \\ (12) & & & & (03) \\ & \searrow & & \swarrow & \\ & & (13) & & \\ & & \downarrow & & \\ & & (23) & & \end{array}$$

Let $\alpha \in A_T^*(G_1(\mathbb{P}^3))$ and let $\alpha_I = (\alpha_{ij})$ be the restriction of α to each fixed point of the T-action: in particular each $\alpha_{ij} \in A_T^*(pt) = \mathbb{Z}[y_0, y_1, y_2, y_3]$. Then the GKM conditions spelled in the article [39], can be translated into the following: $\alpha_{ij} - \alpha_{ij'}$ is divisible by $y_{j'} - y_i$ and $\alpha_{ij} - \alpha_{i'j}$ is divisible by $y_i - y_{i'}$.

4.3 Equivariant Schubert Calculus in Canonical bases.

Before continuing the analysis of equivariant Schubert calculus in the bases proposed by Knutson and Tao, let us see what it looks like when canonical bases are used. First of all we have a combinatorial proposition:

4.3.1 Proposition. *The following formula holds for all $i > 0$:*

$$D_i(\mu^j) = D_1^i(\mu^j) = \sum_{l=0}^{i} h_{i-l}(Y_j, \dots, Y_{j+l})\mu^{j+l}, 0 < j \leq n \tag{4.10}$$

where

$$h_m(X_1, \dots, X_k) = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq k} X_{i_1} \cdot X_{i_2} \cdot \dots \cdot X_{i_m}, \quad h_0 = 1$$

are the complete homogeneous symmetric functions.

Proof.

The proof is by induction on the integer i. If $i = 1$, by definition

$$\begin{aligned} D_1\mu^j &= \mu^{j+1} + Y_j\mu^j \\ &= \mu^{j+1} + h_1(Y_j)\mu^j. \end{aligned}$$

Suppose that the formula is true for $i \geq 1$. Thus, since $D_i(\mu^j) = D_1 D_1^{i-1}(\mu^j)$, the induction's hypothesis says that:

$$\begin{aligned} D_i(\mu^j) &= D_1\Big(\sum_{l=0}^{i-1} h_{i-l-1}(Y_j, \dots, Y_{j+l})\mu^{j+l}\Big) \\ &= \sum_{l=0}^{i-1} h_{i-l-1}(Y_j, \dots, Y_{j+l})D_1(\mu^{j+l}) = \end{aligned}$$

Using the first statement, one has:

$$\begin{aligned} &= \sum_{l=0}^{i-1} h_{i-l-1}(Y_j, \dots, Y_{j+l})(\mu^{j+1} + h_1(Y_j)\mu^j) = \\ &= \sum_{l=0}^{i-1} h_{i-l-1}(Y_j, \dots, Y_{j+l})\mu^{j+l+1} + \sum_{l=0}^{i-1} h_{i-l-1}(Y_j, \dots, Y_{j+l})h_1(Y_{j+l})\mu^{j+l} = \end{aligned}$$

Putting $l' = l + 1$ in the first summation , one has:

$$= \sum_{l'=1}^{i} h_{i-l'}(Y_j, \dots, Y_{j+l'-1})\mu^{j+l'} + \sum_{l=0}^{i-1} h_{i-l-1}(Y_j, \dots, Y_{j+l})h_1(Y_{j+l})\mu^{j+l} =$$

$$= \quad h_0(Y_j,\ldots,Y_{j+i-1})\mu^{j+i} + \sum_{l'=1}^{i-1} h_{i-l'}(Y_j,\ldots,Y_{j+l'-1})\mu^{j+l'} +$$

$$+h_{i-1}(Y_j)h_1(Y_j)\mu^j + \sum_{l=1}^{i-1} h_{i-l-1}(Y_j,\ldots,Y_{j+l})h_1(Y_{j+l})\mu^{j+l} =$$

Simplifying the summations, one sees that:

$$\begin{aligned} &= \quad h_0(Y_j,\ldots,Y_{j+i-1})\mu^{j+i} + h_i(Y_j)\mu^j + \\ &\quad +\sum_{l'=1}^{i-1}\Big(h_{i-l'}(Y_j,\ldots,Y_{j+l'-1}) + h_{i-l-1}(Y_j,\ldots,Y_{j+l})h_1(Y_{j+l})\Big)\mu^{j+l'} = \\ &= \quad h_0(Y_j,\ldots,Y_{j+i-1})\mu^{j+i} + h_i(Y_j)\mu^j + \sum_{l'=1}^{i-2} h_{i-l'}(Y_j,\ldots,Y_{j+l'})\mu^{j+l'} \end{aligned}$$

Therefore

$$D_i(\mu^j) \quad = \quad \sum_{l=0}^{i} h_{i-l}(Y_j,\ldots,Y_{j+l})\mu^{j+l}, \quad 0 < j \leq n.$$

Using the inductive hypothesis the Proposition is proven.

■

4.3.2 In particular:

$$D_{i+1}\mu^0 = D_i\mu^1 = \sum_{l=0}^{i} h_{i-l}(Y_1,\ldots,Y_{1+l})\mu^{1+l} = \mu^i + \sum_{l=0}^{i-1} h_{i-l}(Y_1,\ldots,Y_{1+l})\mu^{1+l}.$$

It follows that the explicit expression of the D_1-canonical basis $\boldsymbol{\epsilon}$ is:

$$\epsilon^0 = \mu^0 \qquad \text{and} \qquad \epsilon^{j+1} = \sum_{m=0}^{j} h_{j-m}(Y_1,\ldots,Y_{1+m})\mu^{1+m}, \tag{4.11}$$

Canonical bases solve the problem of computing equivariant Schubert calculus in the most classical possible combinatorial way. In fact:

4.3.3 Proposition. *Let $h \geq 0$. If $i_k + h \leq n$*

$$D_h\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$$

can be computed via Pieri's formula of classical Schubert calculus.

Proof. In fact, if $i+h\leq n$ one has that

$$D_h\epsilon^i=\epsilon^{i+h},$$

and hence the same proof as Proposition 3.3.2 works in this case. ∎

More than that one has:

4.3.4 Proposition. *Giambelli's determinantal formula holds:*

$$\epsilon^{i_0}\wedge\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}=\Delta_{i_0i_1\ldots i_k}(D)\epsilon^0\wedge\epsilon^1\wedge\ldots\wedge\epsilon^k.$$

Proof. It is Giambelli's determinatal formula for canonical bases already proven in Corollary 3.3.8. ∎

It follows that $\{\Delta_I(D)\,|\,I\in\mathcal{I}_n^k\}$ is an A-basis for the equivariant cohomology of the grassmannian $G_k(\mathbb{P}^n)$. The structure constants $\{C_{IJ}^K\}$ defined by

$$\Delta_I(D)\cdot\Delta_J(D)=\sum_{K\in\mathcal{I}_n^k}C_{IJ}^K\Delta_K(D),$$

are the same as Littlewood-Richardson coefficients as soon as $wt(I)+wt(J)\leq(1+k)(n-k)$. When $wt(I)+wt(J)>(1+k)(n-k)$, one uses the relation

$$D_1\epsilon^n=e_1(Y_1,\ldots,Y_n)\epsilon^n-e_2(Y_1,\ldots,Y_n)\epsilon^{n-1}+\ldots+(-1)^ne_n(Y_1,\ldots,Y_n)\epsilon^1$$

to get the desired expression.

4.3.1 The presentation of $A_T^*(G_1(\mathbb{P}^3))$.

As a further illustration of our methods, let us deduce the presentation of the T-equivariant intersection ring of $A_T^*(G_1(\mathbb{P}^3))$. This is generated, as a $\mathbb{Z}[y_0,y_1,y_2,y_3]$-algebra, by D_1 and D_2. Let us explicitly write the canonical bases in this case:

$$\begin{aligned}\epsilon^0&=\mu^0\\ \epsilon^1&=\mu^1\\ \epsilon^2&=\mu^2+Y_1\mu^1\\ \epsilon^3&=\mu^3+(Y_1+Y_2)\mu^2+Y_1^2\mu^1\end{aligned}\qquad(4.12)$$

from which

$$\mu^0 = \epsilon^0$$

$$\mu^1 = \epsilon^1 \qquad (4.13)$$

$$\mu^2 = \epsilon^2 - Y_1\epsilon^1 \qquad (4.14)$$

$$\mu^3 = \epsilon^3 - (Y_1 + Y_2)\epsilon^2 + Y_1Y_2\epsilon^1 \qquad (4.15)$$

Using expresion (4.12), one has:

$$D_1\epsilon^3 = (Y_1 + Y_2 + Y_3)\mu^3 + (Y_1^2 + Y_1Y_2 + Y_2^2)\mu^2 + Y_1^3\mu^1. \qquad (4.16)$$

$$(4.17)$$

Thus substituting expressions (4.13), (4.14), (4.15) into (4.16) and taking $e_i := e_i(\mathbf{Y}) = e_i(Y_1, Y_2.Y_3)$, one has:

$$D_1\epsilon^3 = e_1\epsilon^3 - e_2\epsilon^2 + e_3\epsilon^1 \qquad (4.18)$$

Also,

$$\begin{aligned} D_2\epsilon^3 &= D_1^2\epsilon^3 = e_1(e_1\epsilon^3 - e_2\epsilon^2 + e_3\epsilon^1) - e_2\epsilon^3 + e_3\epsilon^2 = \\ &= e_1^2\epsilon^3 - (e_1e_2 - e_3)\epsilon^2 + e_1e_3\epsilon^1 \qquad (4.19) \end{aligned}$$

We claim that there is no relation in degree 2. In fact $D_1^2\epsilon^0\wedge\epsilon^1 = \epsilon^0\wedge\epsilon^3+\epsilon^1\wedge\epsilon^2$ and $D_2\epsilon^0\wedge\epsilon^1 = \epsilon^0\wedge\epsilon^3$. Any non trivial A-relation between D_1^2 and D_2 would hence imply an A-relation between $\epsilon^0\wedge\epsilon^3$ and $\epsilon^1\wedge\epsilon^2$, and this is impossible because they are A-linearly independent. One can find a relation between D_1^3 and D_1D_2. In fact

$$D_1^3(\epsilon^0 \wedge \epsilon^1) = D_1(\epsilon^0 \wedge \epsilon^3 + \epsilon^1 \wedge \epsilon^2) = 2\epsilon^1 \wedge \epsilon^3 + \epsilon^0 \wedge D_1\epsilon^3$$

and

$$D_1D_2(\epsilon^0 \wedge \epsilon^1) = D_1(\epsilon^0 \wedge \epsilon^3) = \epsilon^1 \wedge \epsilon^3 + \epsilon^0 \wedge D_1\epsilon^3.$$

Hence:

$$(D_1^3 - 2D_1D_2)\epsilon^0 \wedge \epsilon^1 + \epsilon^0 \wedge D_1\epsilon^3 = 0 \tag{4.20}$$

Substituting (4.18) into (4.20) one gets

$$\begin{aligned} 0 &= (D_1^3 - 2D_1D_2)\epsilon^0 \wedge \epsilon^1 + e_1\epsilon^0 \wedge \epsilon^3 - e_2\epsilon^0 \wedge \epsilon^2 + e_3\epsilon^0 \wedge \epsilon^1 = \\ &= (D_1^3 - 2D_1D_2 + e_1D_2 - e_2D_1 + e_3D_0)\epsilon^0 \wedge \epsilon^1 \end{aligned}$$

Therefore

$$R_3(D_1, D_2) := D_1^3 - 2D_1D_2 + e_1D_2 - e_2D_1 + e_3D_0 = 0$$

in $\mathcal{A}^*(\bigwedge^2(M, D_1)) = A_T^*(G_1(\mathbb{P}^3))$.

Similarly one has

$$D_1^4\epsilon^0 \wedge \epsilon^1 = D_1(2\epsilon^1 \wedge \epsilon^3 + \epsilon^0 \wedge D_1\epsilon^3) = 2\cdot\epsilon^2 \wedge \epsilon^3 + 2\cdot\epsilon^1 \wedge D_1\epsilon^3 + \epsilon^1 \wedge D_1\epsilon^3 + \epsilon^0 \wedge D_1^2\epsilon^1$$

On the other hand

$$D_2^2(\epsilon^0 \wedge \epsilon^1) = D_2(\epsilon^0 \wedge \epsilon^3) = \epsilon^2 \wedge \epsilon^3 + \epsilon^1 \wedge D_1\epsilon^3 + \epsilon^0 \wedge D_1^2\epsilon^3$$

and then

$$(D_1^4 - 2D_2^2)\epsilon^0 \wedge \epsilon^1 - \epsilon^0 \wedge D_1^2\epsilon^3 + \epsilon^1 \wedge D_1\epsilon^3 = 0 \tag{4.21}$$

Plugging into (4.21) the expressions (4.18) and (4.19) and arguing as above, one finally gets the relation:

$$D_1^4 - 2D_2^2 - (e_1^2 - e_2)D_2 + (e_1e_2 - e_3)D_1 + e_1e_3 + e_1(D_1D_2 - D_3) - e_2(D_1^2 - D_2) = 0$$

Still, in the above formula, one must substitute D_3 in terms of D_1 and D_2: we know a priori that this relation exists, in fact:

$$0 = \overline{D}_3 = \overline{D}_2D_1 - \overline{D}_1D_2 + D_3,$$

from where $D_3 = D_1^3 - 2D_1D_2$. The final expression is hence:

$$R_4(D_1, D_2) := D_1^4 - 2D_2^2 - e_1^2D_2 + (e_1e_2 - e_3)D_1 + e_1e_3 + e_1(3D_1D_2 - D_1^3) - e_2D_1^2 = 0$$

Hence one can conclude that:

$$A_T^*(G_1(\mathbb{P}^3)) = \frac{A[D_1, D_2]}{(R_3(D_1, D_2), R_4(D_1, D_2))} \tag{4.22}$$

Notice that if one sets all y-indeterminates to be zero one gets the classical presentation of the intersection ring of the grassmannian $G_1(\mathbb{P}^3)$:

$$A^*(G_1(\mathbb{P}^3)) = \frac{A[D_1, D_2]}{(D_1^3 - 2D_1D_2, D_1^4 - 2D_2^2)}.$$

4.3.5 The equivariant intersection ring of $G_1(\mathbb{P}^3)$ in the bases $\tilde{S}_{\mathbf{a}_{ij}}$.

In the basis $\tilde{S}_{\mathbf{a}_{ij}}$ used in [39], one has that $A_T^*(G_1(\mathbb{P}^3))$ is generated by $\tilde{S}_{\mathbf{a}_{02}}$ and $\tilde{S}_{\mathbf{a}_{03}}$. In fact

$$D_1 - Y_1 \mapsto \tilde{S}_{\mathbf{a}_{02}} \qquad \textit{and} \qquad D_2 - (Y_1 + Y_2)D_1 + Y_1Y_2 \mapsto \tilde{S}_{\mathbf{a}_{03}}$$

Hence one gets

$$D_1 = \tilde{S}_{\mathbf{a}_{02}} + Y_1\tilde{S}_{\mathbf{a}_{01}} \tag{4.23}$$

$$D_2 = \tilde{S}_{\mathbf{a}_{03}} + (Y_1 + Y_2)\tilde{S}_{\mathbf{a}_{02}} - Y_1^2\tilde{S}_{\mathbf{a}_{01}} \tag{4.24}$$

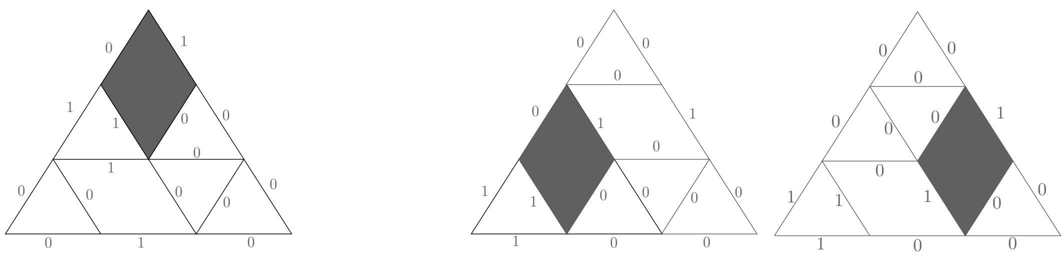
expressions that, substituted into (4.22) give the desired presentation.

4.3.6 Some Examples of Computations.

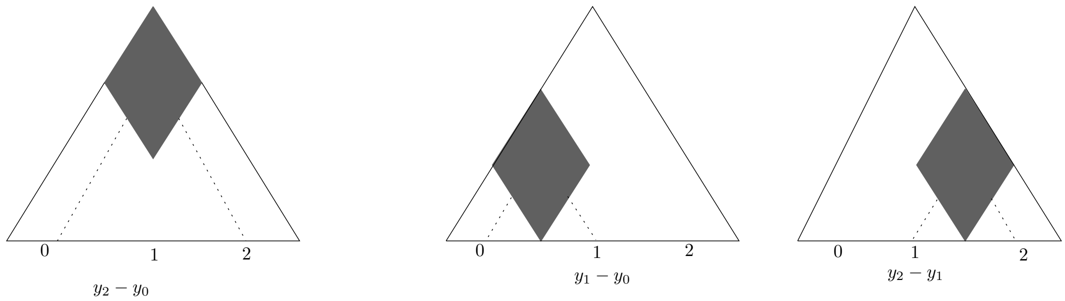
1) In the first part of this example the reader is assumed to have some knowledge of the *puzzle technique*, especially that involving the equivariant piece (see [39]). In $G_1(\mathbb{P}^2) \cong \mathbb{P}^2$ one wants to compute

$$\tilde{S}_{\mathbf{a}_{02}} \cdot \tilde{S}_{\mathbf{a}_{12}} \qquad (\mathbf{a}_{02} = (010), \quad \mathbf{a}_{12} = (100)).$$

• **Via puzzles.** To make computations, one has to construct all the *equivariant puzzles* with borders labelled with 010 and 100 (or 100 and 010) i.e.,



in order to compute the weight of the *equivariant piece*, i.e.,



so that, finally, one has:

$$\tilde{S}_{\mathbf{a}_{02}}\cdot\tilde{S}_{\mathbf{a}_{12}}=(y_2-y_0)\tilde{S}_{\mathbf{a}_{12}}=(y_2-y_1+y_1-y_0)\tilde{S}_{\mathbf{a}_{12}}=\tilde{S}_{\mathbf{a}_{02}}\cdot\tilde{S}_{\mathbf{a}_{12}}$$

• **Via SCGA.** Recall that $\tilde{S}_{\mathbf{a}_{02}}\equiv D_1-Y_1D_0$ seen as element of $A_T^*(\mathbb{P}^2)$ while $\tilde{S}_{\mathbf{a}_{12}}\equiv\mu^1\wedge\mu^2$ seen as element of $\bigwedge^2(A\epsilon^0\oplus A\epsilon^1\oplus A\epsilon^2)$. Therefore (see formula (4.9)):

$$\begin{aligned}\tilde{S}_{\mathbf{a}_{02}}\cdot\tilde{S}_{\mathbf{a}_{12}} &= \gamma((D_1-Y_1)\mu^1\wedge\mu^2)=\gamma(D_1(\mu^1\wedge\mu^2)-Y_1\mu^1\wedge\mu^2)=\\ &= \gamma(D_1\mu^1\wedge\mu^2+\mu^1\wedge D_1\mu^2-Y_1\mu^1\wedge\mu^2)=\\ &= \gamma(Y_1\mu^1\wedge\mu^2+Y_2\mu^1\wedge\mu^2-Y_1\mu^1\wedge\mu^2)=\\ &= \gamma(Y_2\mu^1\wedge\mu^2)=Y_2\tilde{S}_{\mathbf{a}_{12}}=\tilde{S}_{\mathbf{a}_{02}}\cdot\tilde{S}_{\mathbf{a}_{12}}\end{aligned}$$

2) In this example we shall compute the product

$$G^{\boldsymbol{\mu}}_{123}(D)\cdot G^{\boldsymbol{\mu}}_{014}(D)$$

in $A_T^*(G_2(\mathbb{P}^n)$ with n very large. This means that we shall work with a free A-module M of sufficiently high rank, spanned by $\boldsymbol{\mu}=(\mu^0,\mu^1,\ldots)$ studying the 2-SCGP $\bigwedge^3(M,D_1)$, D_1 as in (4.4). This example is combinatorially tricky with puzzles. First we observe that:

$$G^{\boldsymbol{\mu}}_{123}(D)\cdot G^{\boldsymbol{\mu}}_{014}(D)\mu^0\wedge\mu^1\wedge\mu^2=G^{\boldsymbol{\mu}}_{123}(D)\mu^0\wedge\mu^1\wedge\mu^4$$

Although it is not strictly necessary, we shall use canonical bases to speed up computations. To this purpose, in order to make notation easier, we shall write $e_j(Y_{1\leq i\leq n})$ (resp. $h_j(Y_{1\leq i\leq n})$) to denote the j^{th} elementary symmetric polynomial (resp. the j^{th} complete symmetric polynomial in $(Y_1,\ldots,Y_n)$. Of course $e_1(Y_{1\leq i\leq n})=h_1(Y_{1\leq i\leq n})$. First

$$\mu^1\wedge\mu^2\wedge\mu^3=\epsilon^1\wedge\epsilon^2\wedge\epsilon^3.$$

On the other hand

$$\mu^0\wedge\mu^1\wedge\mu^4=\epsilon^0\wedge\epsilon^1\wedge\epsilon^4-e_1(Y_{1\leq i\leq 3})\epsilon^0\wedge\epsilon^1\wedge\epsilon^3+e_2(Y_{1\leq i\leq 3})\epsilon^0\wedge\epsilon^1\wedge\epsilon^2.$$

We shall compute

$$G^{\boldsymbol{\epsilon}}_{123}(D)\cdot(\epsilon^0\wedge\epsilon^1\wedge\epsilon^4-e_1(Y_{1\leq i\leq 3})\epsilon^0\wedge\epsilon^1\wedge\epsilon^3+e_2(Y_{1\leq i\leq 3})\epsilon^0\wedge\epsilon^1\wedge\epsilon^2)$$

But $G^{\boldsymbol{\epsilon}}_{123}(D)=\overline{D}_3:=\Delta_{123}(D)$. Therefore

$$\begin{aligned}
& G^{\boldsymbol{\mu}}_{123}(D)\cdot G^{\boldsymbol{\mu}}_{014}(D)\mu^0\wedge\mu^1\wedge\mu^2=\\
= \;& \overline{D}_3(\epsilon^0\wedge\epsilon^1\wedge\epsilon^4-e_1(Y_{1\leq i\leq 3})\epsilon^0\wedge\epsilon^1\wedge\epsilon^3+e_2(Y_{1\leq i\leq 3})\epsilon^0\wedge\epsilon^1\wedge\epsilon^2)=\\
= \;& \epsilon^1\wedge\epsilon^2\wedge\epsilon^5-e_1(Y_{1\leq i\leq 3})\epsilon^1\wedge\epsilon^2\wedge\epsilon^4+e_2(Y_{1\leq i\leq 3})\epsilon^1\wedge\epsilon^2\wedge\epsilon^3=
\end{aligned}$$

Returning to the original basis $\bigwedge^3\boldsymbol{\mu}$, one easily gets:

$$\begin{aligned}
& G^{\boldsymbol{\mu}}_{123}(D)\cdot G^{\boldsymbol{\mu}}_{014}(D)\mu^0\wedge\mu^1\wedge\mu^2=\\
= \;& \mu^1\wedge\mu^2\wedge\mu^5+[h_1(Y_{1\leq i\leq 4})-e_1(Y_{1\leq i\leq 3})]\mu^1\wedge\mu^2\wedge\mu^4+\\
+ \;& [h_2(Y_{1\leq i\leq 4})-e_1(Y_{1\leq i\leq 3})h_1(Y_{1\leq i\leq 3})+e_2(Y_{1\leq i\leq 3})]\mu^1\wedge\mu^2\wedge\mu^3=\\
= \;& \mu^1\wedge\mu^2\wedge\mu^5+Y_4\mu^1\wedge\mu^2\wedge\mu^4+(Y_1+Y_2+Y_3+Y_4)Y_4\mu^1\wedge\mu^2\wedge\mu^3=\\
= \;& \Big(G^{\boldsymbol{\mu}}_{125}(D)+Y_4G^{\boldsymbol{\mu}}_{124}(D)+(Y_1+Y_2+Y_3+Y_4)Y_4G^{\boldsymbol{\mu}}_{123}(D)\Big)\mu^0\wedge\mu^1\wedge\mu^2
\end{aligned}$$

showing the power of our methods.

4.4 Equivariant Pieri's Formula

Let $l\geq 0$ and $\mu^{i_0}\wedge\mu^{i_1}\wedge\ldots\wedge\mu^{i_k}\in\bigwedge^{1+k}M$. Leibniz's rule of SCGA gives:

$$D_l(\mu^{i_0} \wedge \ldots \wedge \mu^{i_k}) = \sum_{l_0+l_1+\ldots+l_k=l} D_{l_0}\mu^{l_0} \wedge \ldots \wedge D_{l_k}\mu^{l_k}$$

Thus, using Equation (4.10), and defining (for computational reasons) $Y_j := y_j - y_0$, one has:

$$= \sum_{l_0+l_1+\ldots+l_k=l} \left[\left(\sum_{m_0=0}^{l_0} h_{l_0-m_0}(Y_{i_0}, Y_{i_0+1}, \ldots, Y_{i_0+m_0})\mu^{i_0+m_0} \right) \wedge \ldots \right.$$
$$\left. \ldots \wedge \left(\sum_{m_k=0}^{l_k} h_{l_k-m_k}(Y_{i_k}, Y_{i_k+1}, \ldots, Y_{i_k+m_k})\mu^{i_k+m_k} \right) \right] =$$

expanding the wedge products:

$$= \sum_{l_0+\ldots+l_k=l} \left[\sum_{m_0+\ldots+m_k=0}^{l_0+l_1+\ldots+l_k} \left(\prod_{j=0}^{k} h_{l_j-m_j}(Y_{i_j}, Y_{i_j+1}, \ldots, Y_{i_j+m_j})\mu^{i_0+m_0} \wedge \ldots \wedge \mu^{i_k+m_k} \right) \right] =$$
$$= \sum_{m_0+\ldots+m_k=0}^{l} \left[\left(\sum_{l_0+\ldots+l_k=l} \prod_{j=0}^{k} h_{l_j-m_j}(Y_{i_j}, Y_{i_j+1}, \ldots, Y_{i_j+m_j}) \right) \mu^{i_0+m_0} \wedge \ldots \wedge \mu^{i_k+m_k} \right] =$$

Therefore, by property of symmetric polynomials,

$$= \sum_{m_0+\ldots+m_k=0}^{l} \left[h_{l-\sum_{j=0}^{k} m_j}(Y_{i_0}, \ldots, Y_{i_0+m_0}, \ldots, Y_{i_k}, \ldots, Y_{i_k+m_k})\mu^{i_0+m_0} \wedge \ldots \wedge \mu^{i_k+m_k} \right] =$$

Thus, putting $u = l - \sum_{j=0}^{k} m_j$, one finally gets:

$$D_l(\mu^{i_0} \wedge \ldots \wedge \mu^{i_k}) =$$
$$= \sum_{u=0}^{l} \sum_{m_0+m_1+\ldots+m_k+u=l} h_u(Y_{i_0}, \ldots, Y_{i_0+m_0}, \ldots, Y_{i_k}, \ldots, Y_{i_k+m_k})\mu^{i_0+m_0} \wedge \ldots \wedge \mu^{i_k+m_k}$$

Relying on what has been said, and keeping into account the alternating feature of the $\wedge$-product, causing cancellations of terms, we can finally state (solving a problem proposed in [43]):

4.4.1 Theorem. *Pieri's formula for T-equivariant cohomology of grassmannians holds:*

$$D_l(\mu^{i_0} \wedge \ldots \wedge \mu^{i_k}) =$$
$$= \sum_{u=0}^{l} \sum_{(m_i)\in P(I,l-u)} h_u(Y_{i_0}, \ldots, Y_{i_0+m_0}, \ldots, Y_{i_k}, \ldots, Y_{i_k+m_k})\mu^{i_0+m_0} \wedge \ldots \wedge \mu^{i_k+m_k} \tag{4.25}$$

where $P(I, l-u)$ is the set of all $(1+k)$-tuples $(m_i) \in \mathbb{N}^{1+k}$ such that:

$$0 \le i_0 \le i_0 + m_0 < i_1 \le i_1 + m_1 < i_2 \le i_2 + m_2 < \ldots \le i_{k-1} + m_{k-1} < i_k$$

and $m_0 + m_1 + m_2 + \ldots + m_k = l - u$.

Proof.
The proof is inspired by [26], and works by induction on the integer k. For $k = 1$, formula (4.25) is trivially true. Let us prove it directly for $k = 2$. For each $h \ge 0$, let us split sum (4.25) as:

$$\begin{aligned} D_l(\mu^{i_0} \wedge \mu^{i_1}) &= \sum_{u=0}^{l} \sum_{m_0+m_1+u=l} h_u(Y_{i_0}, \ldots, Y_{i_0+m_0}, Y_{i_1}, \ldots, Y_{i_1+m_1})\mu^{i_0+m_0} \wedge \mu^{i_1+m_1} = \\ &= \mathcal{P} + \overline{\mathcal{P}} \end{aligned}$$

where

$$\mathcal{P} = \sum_{u=0}^{l} \sum_{\substack{m_0+m_1=l-u \\ i_0+m_0<i_1}} h_u(Y_{i_0}, \ldots, Y_{i_0+m_0}, Y_{i_1}, \ldots, Y_{i_1+m_1})\mu^{i_0+m_0} \wedge \mu^{i_1+m_1}$$

and

$$\overline{\mathcal{P}} = \sum_{u=0}^{l} \sum_{\substack{m_0+m_1=l-u \\ i_0+m_0\ge i_1}} h_u(Y_{i_0}, \ldots, Y_{i_0+m_0}, Y_{i_1}, \ldots, Y_{i_1+m_1})\mu^{i_0+m_0} \wedge \mu^{i_1+m_1}$$

One contends that $\overline{\mathcal{P}}$ vanishes. In fact, on the finite set of all integers $i_1 - i_0 \leq a \leq i_1 - i_0 + l - u$, define the bijection $\rho(a) = i_1 - i_0 + l - u - a$. Then:

$$\begin{aligned}
2\overline{\mathcal{P}} &= \sum_{u=0}^{l} \sum_{m_0=i_1-i_0}^{l-u} h_u(Y_{i_0}, \ldots, Y_{i_0+m_0}, Y_{i_1}, \ldots, Y_{i_1+l-u-m_0})\mu^{i_0+m_0} \wedge \mu^{i_1+l-u-m_0} + \\
&+ \sum_{u=0}^{l} \sum_{m_0=i_1-i_0}^{l-u} h_u(Y_{i_0}, \ldots, Y_{i_0+\rho(m_0)}, Y_{i_1}, \ldots, Y_{i_1+l-u-\rho(m_0)})\mu^{i_0+\rho(m_0)} \wedge \mu^{i_1+l-u-\rho(m_0)} = \\
&= -\sum_{u=0}^{l} \sum_{m_0=i_1-i_0}^{l-u} h_u(Y_{i_0}, \ldots, Y_{i_0+m_0}, Y_{i_1}, \ldots, Y_{i_1+l-u-m_0})\mu^{i_1+l-u-m_0} \wedge \mu^{i_0+m_0} + \\
&+ \sum_{u=0}^{l} \sum_{m_0=i_1-i_0}^{l-u} h_u(Y_{i_0}, \ldots, Y_{i_0+m_0}, Y_{i_1}, \ldots, Y_{i_1+l-u-m_0})\mu^{i_1+l-u-m_0} \wedge \mu^{i_0+m_0} = 0
\end{aligned}$$

hence $\overline{\mathcal{P}} = 0$ and (4.25) holds for $k = 2$. Suppose now that (4.25) holds for all $1 \leq k' \leq k - 1$. Then, for each $h \geq 0$:

$$D_l(\mu^{i_0} \wedge \mu^{i_1} \wedge \ldots \wedge \mu^{i_k}) = \sum_{l'_k + l_k = l} D_{l'_k}(\mu^{i_0} \wedge \ldots \wedge \mu^{i_{k-1}}) \wedge D_{l_k}\mu^{i_k},$$

and, by the inductive hypothesis:

$$= \left(\sum_{u=0}^{l'_k} \sum_{(m_i)} h_u(Y_{i_0}, \ldots, Y_{i_0+m_0}, \ldots, Y_{i_{k-1}}, \ldots, Y_{i_{k-1}+m_{k-1}})\mu^{i_0+m_0} \wedge \ldots \wedge \mu^{i_{k-1}+m_{k-1}} \right) \wedge$$
$$\wedge \left(\sum_{m_k=0}^{l_k} h_{l_k-m_k}(Y_{i_k}, Y_{i_k+1}, \ldots, Y_{i_k+m_k})\mu^{i_k+m_k} \right) \tag{4.26}$$

summed over all (m_i) such that $m_0 + \ldots + m_k = l - u$ and

$$1 \leq i_0 + m_0 < i_1 \leq \ldots\ldots \leq i_{k-2} + m_{k-2} < i_{k-1}. \tag{4.27}$$

But now (4.26) can be equivalently written as:

$$= \left(\sum_{u=0}^{l-l''} \sum_{(m_i)} h_u(Y_{i_0}, \ldots, Y_{i_0+m_0}, \ldots, Y_{i_{k-2}}, \ldots, Y_{i_{k-2}+m_{k-2}})\mu^{i_0+m_0} \wedge \ldots \wedge \mu^{i_{k-2}+m_{k-2}} \right) \wedge$$
$$\wedge D_{l''}\left(\mu^{i_{k-1}} \wedge \mu^{i_k}\right) \tag{4.28}$$

where the sum is over all $(m_0, \ldots, m_{k-2}, l'')$ such that $m_0 + \ldots + m_{k-2} + l'' = l$ and satisfying (4.27). Since

$$D_{l''}(\mu^{i_{k-1}} \wedge \mu^{i_k}) =$$
$$= \sum_{u=0}^{l''} \sum_{\substack{m_{k-1} + m_k = l'' - u \\ i_{k-1} + m_{k-1} < i_k}} h_u(Y_{i_{k-1}}, \ldots, Y_{i_{k-1}+m_{k-1}}, Y_{i_k}, \ldots, Y_{i_k+m_k}) \mu^{i_k+m_k} \wedge \mu^{i_k+m_k}$$

by the inductive hypothesis, substituting into (4.28) one gets exactly sum (4.25).■

4.4.2 Example. The coefficient of $\mu^2 \wedge \mu^3 \wedge \mu^7$ in the computation of $D_3(\mu^2 \wedge \mu^3 \wedge \mu^5)$, is:

$$h_1(Y_2, Y_3, Y_5, Y_6, Y_7) = Y_2 + Y_3 + Y_5 + Y_6 + Y_7 = y_2 + y_3 + y_5 + y_6 + y_7 - 5y_0.$$

4.4.3 In particular Pieri's rule for codimension 1 subvarieties, is given by

$$D_1(\mu^{i_0} \wedge \ldots \wedge \mu^{i_k}) = \sum_{n_0+n_1+\ldots+n_k=1} D_{m_0}\mu^{i_0} \wedge \ldots \wedge D_{m_k}\mu^{i_k} =$$
$$= \sum_{n_0+n_1+\ldots+n_k=1} \left(\sum_{m_0=0}^{n_0} h_{n_0-m_0}(Y_{i_0}, Y_{i_0+1}, \ldots, Y_{i_0+m_0})\mu^{i_0+m_0} \wedge \ldots \right.$$
$$\left. \ldots \wedge \sum_{m_k=0}^{n_k} h_{n_k-m_k}(Y_{i_k}, Y_{i_k+1}, \ldots, Y_{i_k+m_k})\mu^{i_k+m_k} \right)$$
$$= \sum_{j=0}^{k} \mu^{i_0} \wedge \ldots \wedge \left(\sum_{m_j=0}^{1} h_{1-m_j}(Y_{i_j}, Y_{i_j+1}, \ldots, Y_{i_j+m_j})\mu^{i_j+m_j} \right) \wedge \ldots \mu^{i_k} =$$
$$= \sum_{j=0}^{k} \mu^{i_0} \wedge \ldots \wedge \left(h_0(Y_{i_j}, Y_{i_j+1})\mu^{i_j+1} + h_1(Y_{i_j})\mu^{i_j} \right) \wedge \ldots \mu^{i_k} =$$
$$= \sum_{j=0}^{k} \mu^{i_0} \wedge \ldots \wedge \left(\mu^{i_j+1} + Y_{i_j}\mu^{i_j} \right) \wedge \ldots \mu^{i_k} =$$

$$
\begin{aligned}
&= \sum_{j=0}^{k} \mu^{i_0} \wedge \ldots \wedge \mu^{i_j+1} \wedge \ldots \mu^{i_k} + \sum_{j=0}^{k} Y_{i_j} \mu^{i_0} \wedge \ldots \wedge \mu^{i_j} \wedge \ldots \mu^{i_k} = \\
&= \sum_{j=0}^{k} \mu^{i_0} \wedge \ldots \wedge \mu^{i_j+1} \wedge \ldots \mu^{i_k} + (Y_{i_0} + \ldots + Y_{i_k}) \mu^{i_0} \wedge \ldots \wedge \mu^{i_k} \\
&= \sum_{j=0}^{k} \mu^{i_0} \wedge \ldots \wedge \mu^{i_j+1} \wedge \ldots \mu^{i_k} + \sum_{r=0}^{k} Y_{i_k} \mu^{i_0} \wedge \ldots \wedge \mu^{i_k}
\end{aligned}
$$

4.4.4 Remark. Knutson and Tao, in [39], computes a Pieri's formula for codimension 1 subvarieties. It can be gotten within our formalism as follows (see 4.4.5 for the detailed comparison).

$$
\begin{aligned}
& G^{\mu}_{0,1,\ldots,k-1,k+1}(D) \mu^{i_0} \wedge \ldots \wedge \mu^{i_k} = \\
&= \left(D_1 - \sum_{r=1}^{k} Y_r \right) \mu^{i_0} \wedge \ldots \wedge \mu^{i_k} = \\
&= \sum_{j=0}^{k} \mu^{i_0} \wedge \ldots \wedge \mu^{i_j+1} \wedge \ldots \mu^{i_k} + \left(\sum_{r=0}^{k} Y_{i_k} - \sum_{r=1}^{k} Y_r \right) \mu^{i_0} \wedge \ldots \wedge \mu^{i_k} = \\
&= \sum_{j=0}^{k} \mu^{i_0} \wedge \ldots \wedge \mu^{i_j+1} \wedge \ldots \mu^{i_k} + \left(\sum_{r=0}^{k} (y_{i_k} - y_r) \right) \mu^{i_0} \wedge \ldots \wedge \mu^{i_k} \qquad (4.29)
\end{aligned}
$$

In fact,

$$
\begin{aligned}
D_1(\mu^0 \wedge \mu^1 \wedge \ldots \wedge \mu^k) &= \sum_{j=0}^{k} \mu^0 \wedge \mu^1 \wedge \ldots \mu^{j-1} \wedge D_1(\mu^j) \wedge \mu^{j+1} \wedge \ldots \wedge \mu^k \\
&= \sum_{j=0}^{k} \mu^0 \wedge \mu^1 \wedge \ldots \mu^{j-1} \wedge (\mu^{j+1} + Y_j \mu^j) \wedge \mu^{j+1} \wedge \ldots \wedge \mu^k \\
&= \sum_{j=0}^{k} Y_j \mu^0 \wedge \mu^1 \wedge \ldots \mu^{j-1} \wedge \mu^j \wedge \mu^{j+1} \wedge \ldots \wedge \mu^k + \\
&\quad + \mu^0 \wedge \mu^1 \wedge \ldots \wedge \mu^{k-1} \wedge \mu^{k+1} \\
&= (Y_1 + \ldots + Y_k) \mu^0 \wedge \mu^1 \wedge \ldots \wedge \mu^k + \mu^0 \wedge \mu^1 \wedge \ldots \wedge \mu^{k-1} \wedge \mu^{k+1} \\
&= \sum_{r=1}^{k} Y_r \mu^0 \wedge \mu^1 \wedge \ldots \wedge \mu^k + \mu^0 \wedge \mu^1 \wedge \ldots \wedge \mu^{k-1} \wedge \mu^{k+1}.
\end{aligned}
$$

4.4.5 Comparison with Pieri's Formula as in [39].

Let $0 \leq k \leq n$ be a fixed integer. Denote, as in [39], by $\left\{ {n \atop k} \right\}$ the set of all the sequences $\mathbf{a}_{i_0 i_1 \ldots i_{k-1}} : \{0, 1, \ldots, n-1\} \longrightarrow \{0, 1\}$ such that $\mathbf{a}_{i_0 i_1 \ldots i_{k-1}}(j) = 1$ for all j but $\{i_0, i_1, \ldots, i_{k-1}\}$ where it takes the value 0. For example

$$\left\{ {4 \atop 2} \right\} = \{1100, 1010, 1001, 0110, 0101, 0011\}.$$

The elements of $\left\{ {n \atop k} \right\}$ may be clearly employed to index Schubert classes and are used by Knutson and Tao to spell the *puzzle rule* as in [40]. The cardinality of this set$\left\{ {n \atop k} \right\}$ is $\binom{n}{k}$.

Let $\widetilde{S}_\lambda$ be the equivariant Schubert variety corresponding to some $\lambda \in \left\{ {n \atop k} \right\}$.

Then Pieri's rule for a codimension 1 equivariant Schubert variety, as is spelled in [39], is given by :

$$\widetilde{S}_{div} \widetilde{S}_\lambda = \left(\widetilde{S}_{div} | \lambda \right) \widetilde{S}_\lambda + \sum_{\lambda' : \lambda' \to \lambda} \widetilde{S}_{\lambda'}$$

where

i) $div := \mathbf{a}_{0,1,\ldots,k-1,k+1}$ denotes the unique element of $\left\{ {n+1 \atop k+1} \right\}$ with one inversion;

ii) the expression $\lambda' : \lambda' \to \lambda$ means that λ' differs by λ in only two spots $i, i+1$, where λ has 01 and λ' has 10;

iii) the coefficient $\left(\widetilde{S}_{div} | \lambda \right)$ is given by ([39], p.236, Lemma 3):

$$\left(\widetilde{S}_{div} | \lambda \right) = \sum_{j=0}^{n} (1 - \lambda(j)) y_j - \sum_{i=0}^{k} y_i$$

Thus, if $\lambda = \mathbf{a}_{i_0, i_1, \ldots, i_k}$ one has:

$$\left(\widetilde{S}_{div} | \lambda \right) = (y_{i_0} + y_{i_1} + \ldots + y_{i_k}) - (y_0 + y_1 + \ldots + y_k) = \sum_{r=0}^{k} (y_{i_r} - y_r).$$

Therefore Knutson and Tao's formula reads as:

$$\widetilde{S}_{div}\widetilde{S}_{\lambda} = \sum_{r=0}^{k}(y_{i_r} - y_r)\widetilde{S}_{\lambda} + \sum_{\lambda':\lambda'\to\lambda}\widetilde{S}_{\lambda'}$$

which is precisely formula 4.29.

Chapter 5

A few Enumerative Examples

In this section we go back to classical Schubert calculus to see how much SCGA point of view may help in solving enumerative problems. Our main examples will be i) offering a new way to prove the formula for computing the Plücker degree $d_{k,n}$ of Schubert varieties in $G_k(\mathbb{P}^n)$ and ii) a combinatorial formula expressing the number of projectively non equivalent rational curves of degree d in $\mathbb{P}^3$ having flexes at $2d-6$ prescribed points.

5.0.6 Let M_{1+l} be a free $\mathbb{Z}$-module spanned by $(\epsilon^0, \epsilon^1, \ldots, \epsilon^l)$ and let $D_t \in \mathcal{S}_t(\bigwedge M)$ induced by the shift polynomial $D_t : M_l \longrightarrow M_l[[t]]$ defined by:

$$D_t(\epsilon^j) = \sum_{i\geq 0}(D_i\epsilon^j)t^i \qquad \text{where} \qquad D_i\epsilon^j = \begin{cases} \epsilon^{i+j} & \text{if} \quad i+j \leq l \\ 0 & \text{if} \quad i+j > l \end{cases}$$

Because of Pieri's formula (3.12), the main result of [26], or its generalization (Theorem 3.4.2), there is the following dictionary between the k-SCGP $\bigwedge^{1+k}(M, D_1)$ and the SCGV on $G_k(\mathbb{P}^l)$:

5.0.7 Dictionary.

$\sigma_h \in H^*(G_k(\mathbb{P}^n))$	$\leftrightarrow$	$D_h \in \mathcal{A}^*(\bigwedge^{k+1} M_{1+n}, D)$
$[G_k(\mathbb{P}^n)]$	$\leftrightarrow$	$\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k$
$\sigma_{\underline{\rho}} \cap [G_k(\mathbb{P}^n)]$	$\leftrightarrow$	$\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k}$
$(\sigma_h \cup \sigma_{\underline{\rho}}) \cap [G_k(\mathbb{P}^n)]$	$\leftrightarrow$	$D_h(\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k})$
$(\sigma_{\underline{\lambda}} \cup \sigma_{\underline{\rho}}) \cap [G_k(\mathbb{P}^n)]$	$\leftrightarrow$	$\Delta_{\underline{\lambda}}(D)(\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k})$

where $\rho = (i_0, \ldots, i_k)$. In particular the degree $d_{k,n}$ of the Plücker image of $G_k(\mathbb{P}^n)$ is given by

$$d_{k,n} = \int \sigma_1^{(k+1)(n-k)},$$

and therefore $d_{k,n}$ fits into the equality:

$$D_1^{(k+1)(n-k)}\epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k = d_{k,n} \cdot \epsilon^{n-k} \wedge \ldots \wedge \epsilon^n.$$

5.1 Degree of Schubert Varieties

Let $E = (e_0, e_1, \ldots, e_n)$ be a $E^\bullet$-adapted basis of $\mathbb{C}^{1+n}$, $\boldsymbol{\epsilon} = (\epsilon^0, \epsilon^1, \ldots, \epsilon^n)$ its dual and let

$$E^\bullet : \mathbb{P}^n := V^0 \supset V^1 \supset \ldots \supset V^n \supset V^{1+n} = \emptyset$$

be a complete flag of projective subspaces of $\mathbb{P}^n$.

5.1.1 Proposition. Let $d_{i_0,i_1,\ldots,i_k}$ be the Plücker degree of the Schubert variety $\Omega_{i_0,i_1,\ldots,i_k}(F^\bullet)$. Then:

$$d_{i_0,i_1,\ldots,i_k} = \sum_{\tau \in S_{1+k}} (-1)^{|\tau|} \binom{(n-k)(1+k)-w}{n-k+\tau(0)-i_0, n-k+\tau(1)-i_1, \ldots, n-k+\tau(k)-i_k} \quad (5.1)$$

$$= \sum_{\tau \in S_{1+k}} (-1)^{|\tau|} \binom{(n-k)(1+k)-w}{n-i_{\tau(0)}-k, n-i_{\tau(1)}-(k-1), \ldots, n-i_{\tau(k)}-(k-k)} \quad (5.2)$$

where w is the codimension of $\Omega_{i_0,i_1,\ldots,i_k}(F^\bullet)$.

Proof.

The degree of the Schubert variety $(\Omega_{i_0,i_1,\ldots,i_k}(F^\bullet))$ satisfies the following equality:

$$D_1^{(1+k)(n-k)-w}\epsilon^{i_0} \wedge \epsilon^{i_1} \wedge \ldots \wedge \epsilon^{i_k} = d_{i_0,i_1,\ldots,i_k} \cdot \epsilon^0 \wedge \epsilon^1 \wedge \ldots \wedge \epsilon^k$$

Hence all the matter consists in evaluating the left hand side of the equality above. This will be done by using the general formula (2.13).

$$D_1^{N(n,k,w)}\epsilon^{i_0}\wedge\epsilon^{i_1}\wedge\ldots\wedge\epsilon^{i_k}=\sum\binom{N(n,k,w)}{h_0,h_1,\ldots,h_k}\epsilon^{i_0+h_0}\wedge\epsilon^{i_1+h_1}\wedge\ldots\wedge\epsilon^{i_k+h_k}$$

the sum over all distinct $(1+k)$-tuple of non negative integers $(h_0,h_1,\ldots,h_k)$ such that $h_0+h_1+\ldots+h_k=N(n,k,w)$, where for sake of notational brevity we set $N(n,k,w):=(1+k)(n-k)-w$. The only surviving terms are those for which $(i_0+h_0,i_1+h_1,\ldots,i_k+h_k)$ is a permutation of $(n-k,n-k+1,\ldots,n)$, i.e.:

$$\{(i_0+h_0=n-k+\tau(0),i_1+h_1=n-k+\tau(1),\ldots,i_k+h_k=n-k+\tau(k):\tau\in S_{1+n}\}.$$

This gives $h_j=n-k+\tau(j)-i_j$. Therefore one concludes that:

$$d_{i_0,i_1,\ldots,i_k}=\sum_{\tau\in S_{1+n}}(-1)^{|\tau|}\binom{N(n,k,w)}{n-k+\tau(0)-i_0,n-k+\tau(1)-i_1,\ldots,n-k+\tau(k)-i_k},$$

which is expression (5.1). Equation (5.2) is simply the "transpose" of (5.1).

∎

In particular one has:

$$d_{k,n}:=d_{01\ldots k}=\sum_{\tau\in S_{1+n}}(-1)^{|\tau|}\binom{(1+k)(n-k)}{n-k+\tau(0),n-k+\tau(1)-1,\ldots,n-k+\tau(k)-k},\quad(5.3)$$

5.1.2 Formula (5.2) is suited to be put in the classical form the degree of Schubert varieties is known with. In fact:

$$\sum_{\tau\in S_{1+k}}(-1)^{|\tau|}\binom{(n-k)(1+k)-w}{n-i_{\tau(0)}-k,n-i_{\tau(1)}-(k-1),\ldots,n-i_{\tau(k)}-(k-k)}=$$

$$\sum_{\tau\in S_{1+k}}(-1)^{|\tau|}\frac{N(n,k,w)!}{(n-i_{\tau(0)}-k)!(n-i_{\tau(1)}-(k-1))!\ldots(n-i_{\tau(k)})!}\quad(5.4)$$

The common denominator of the alternating sum (5.4) is precisely

$$(n-i_0)!(n-i_1)!\cdot(n-i_k)!=(n-i_\tau(0))!(n-i_\tau(1))!\cdot(n-i_\tau(k))!$$

for each $\tau\in S_{1+k}$. Therefore sum (5.4) can be written as:

$$\sum_{\tau\in S_{1+k}} \frac{(-1)^{|\tau|}N(n,k,w)!}{(n-i_0)!(n-i_1)!\dots(n-i_k)!}\cdot\frac{(n-i_\tau(0))!(n-i_\tau(1))!\cdot\ldots\cdot(n-i_\tau(k))!}{(n-i_\tau(0)-k)!(n-i_\tau(1)-(k-1))!\dots(n-i_\tau(k))!} =$$

$$= N(n,k,w)!\frac{\sum_{\tau\in S_{1+k}}(-1)^{|\tau|}\prod_{j_0=0}^{k-1}(n-i_{\tau(0)}-j_0)\cdot\prod_{j_1=0}^{k-2}(n-i_{\tau(1)}-j_1)\cdot\ldots\cdot(n-i_{\tau(k)})}{(n-i_0)!(n-i_1)!\dots(n-i_k)!} \quad (5.5)$$

Since the numerator of (5.5) is a homogeneous polynomial of degree $k(k+1)/2$ in $n, n-1, \dots, n-k+1$ and $i_0, i_1, \dots i_k$ and it changes sign if one permutes $j \leftrightarrow k$, it must be an integral multiple of $\prod_{k<j}(i_j - i_k)$. To determine this multiple it is sufficient to compare the coefficients of a same monomial occurring on both sides. In the expansion of the left hand side, the monomial

$$(-1)^{\frac{k(k-1)}{2}} i_0^{k-1} i_1^{k-2} i_2^{k-3}\cdot\ldots\cdot i_{k-1}$$

occurs, corresponding to the identical substitution $\tau = (012\dots k)$; the same monomial, with the same sign, occurs on the right hand side, proving that the numerator of formula (5.5) is indeed equal to $\prod_{k<j}(i_j - i_k)$. Then the final formula is:

$$d_{i_0,i_1,\dots,i_k} = \frac{((n-k)(k+1)-w)!\prod_{j<k}(i_k-i_j)}{(n-i_0)!(n-i_1)!\dots(n-i_k)!}. \quad (5.6)$$

In particular:

$$d_{k,n} := d_{012\dots k} = \frac{1!2!\cdot\ldots\cdot k!((n-k)(k+1))!}{n!(n-1)!\dots(n-k)!}. \quad (5.7)$$

To generalize formulas (5.6) or (5.7) to other kind of integrals (=top intersection products in the cohomology ring) seems to be a hard task. However, the generalization (5.3) is straightforward.

5.1.3 Remark. Proposition (5.1.1) can be seen as a particular case of the following Theorem (one takes $I = (0, \dots, k)$ and $J = (n-k, \dots, n)$).

More in general, it is possible to compute the number

$$\int((\sigma_1)^{|\underline{\rho}|-|\underline{\lambda}|}\cup\sigma_{\underline{\rho}})\cap[G_k(\mathbb{P}^n)],$$

where $\underline{\lambda}$, $\underline{\rho}$, are any partitions such that $\underline{\lambda} \preceq \underline{\rho}$ in the Bruhat-Chevalley order.

Let $I = (i_0, i_1, \ldots, i_k)$ and $J = (j_0, j_1, \ldots, j_k) \in \mathcal{I}_n^k$ be the index associated respectively to the partitions $\underline{\lambda}$ and $\underline{\rho}$. By our dictionary,

$$\int ((\sigma_1)^{|\underline{\rho}|-|\underline{\lambda}|} \cup \sigma_{\underline{\rho}}) \cap [G_k(\mathbb{P}^n)] = \mathbf{K_{I,J}}$$

where $\mathbf{K_{I,J}}$ is given by the equality:

$$D_1^{|J|-|I|}(\varepsilon^{i_0} \wedge \ldots \wedge \varepsilon^{i_k}) \quad = \quad \mathbf{K_{I,J}} \cdot \varepsilon^{j_0} \wedge \ldots \wedge \varepsilon^{j_k}.$$

Since $\underline{\lambda} \preceq \underline{\rho}$, also $I \preceq J$. Thus, this number is given by:

5.1.4 Proposition.

$$\begin{aligned} \mathbf{K_{I,J}} &= \sum_{\sigma \in S_k} (-1)^{|\sigma|} \binom{|J|-|I|}{\sigma(j_0)-i_0, \ldots, \sigma(j_k)-i_k} \\ &= (|J|-|I|)! \cdot |(a_{lm})| \end{aligned}$$

where $|(a_{lm})|$ is the determinant of the matrix (a_{lm}), where:

$$a_{lm} = \frac{1}{(j_m - i_l)!} \ , \ 0 \le l, m \le k$$

Proof.

By Corollary 2.5.6, Newton's binomial formula (2.12) holds. The proof is by induction on the integer k. If $k = 0$, the formula is easily seen to be true, by definition of the D_i's. In fact

$$D_1^{j_0-i_0}\varepsilon^{i_0} = \binom{j_0-i_0}{j_0-i_0}\varepsilon^{j_0} \Rightarrow D_1^{j_0-i_0}\varepsilon^{i_0} = \varepsilon^{j_0}.$$

If $k = 1$ we have, by (2.12):

$$\begin{aligned} D_1^{|J|-|I|}(\varepsilon^{i_0} \wedge \varepsilon^{i_1}) &= \sum_{l=0}^{|j|-|I|} \binom{|J|-|I|}{l, |J|-|I|-l} D_1^l \varepsilon^{i_0} \wedge D_1^{|J|-|I|-l}\varepsilon^{i_1} \\ &= \sum_{l=0}^{|J|-|I|} \binom{|J|-|I|}{l, |J|-|I|-l} \varepsilon^{i_0+l} \wedge \varepsilon^{i_1+|J|-|I|-l} \end{aligned}$$

In the above sum one needs to check only the coefficient of the terms $\varepsilon^{j_0} \wedge \varepsilon^{j_1}$ and $\varepsilon^{j_1} \wedge \varepsilon^{j_0}$. Then

$$\begin{cases} i_0 + l = j_0 & \Rightarrow \quad l = j_0 - i_0 \\ i_0 + l = j_1 & \Rightarrow \quad l = j_1 - i_0 \end{cases}$$

For this

$$\begin{aligned} D_1^{|J|-|I|}(\varepsilon^{i_0} \wedge \varepsilon^{i_1}) &= \binom{|J|-|I|}{j_0 - i_0, j_0 + j_1 - i_0 - i_1 - j_0 + i_0} - \binom{|J|-|I|}{j_1 - i_0, j_0 + j_1 - i_0 - i_1 - j_1 + i_0} \\ &= \binom{|J|-|I|}{j_0 - i_0, j_1 - i_1} - \binom{|J|-|I|}{j_1 - i_0, j_0 - i_1}. \end{aligned}$$

Now, suppose that the formula is true for each $k \geq 0$. We have:

$$\begin{aligned} D_1^{|J|-|I|}(\varepsilon^{i_0} \wedge \ldots \wedge \varepsilon^{i_{k-1}} \wedge \varepsilon^{i_k}) &= \sum_{l=0}^{|J|-|I|} \tbinom{|J|-|I|}{l} D_1^l(\varepsilon^{i_0} \wedge \ldots \wedge \varepsilon^{i_{k-1}}) \wedge D_1^{|J|-|I|-l} \varepsilon^{i_k} \\ &= \sum_{l=0}^{|J|-|I|} \tbinom{|J|-|I|}{l} D_1^l(\varepsilon^{i_0} \wedge \ldots \wedge \varepsilon^{i_{k-1}}) \wedge \varepsilon^{i_k + |J|-|I|-l} \end{aligned}$$

In the above sum one needs to check the coefficient of the terms $\varepsilon^{\sigma(j_1)} \wedge \ldots \wedge \varepsilon^{\sigma(j_k)}$. Therefore,

$$i_k + |J| - |I| - l = \sigma(j_k) \quad \Rightarrow \quad l = |J| - |I| + i_k - \sigma(j_k)$$

Then, by induction's hypothesis, $D_1^{|J|-|I|}(\varepsilon^{i_0} \wedge \ldots \wedge \varepsilon^{i_k})$ is equal to

$$= \sum_{h=0}^{k} \tbinom{|J|-|I|}{l} \sum_{\overline{\sigma} \in S_{k-1}} (-1)^{|\overline{\sigma}|} \binom{|J|-|I|}{\overline{\sigma}(j_0) - i_0, \ldots, \overline{\sigma}(j_{h-1}) - i_{h-1}, \overline{\sigma}(j_{h+1}) - i_h, \ldots, \overline{\sigma}(j_k) - i_{k-1}} \cdot$$
$$\cdot (\varepsilon^{j_0} \wedge \ldots \wedge \widehat{\varepsilon^{j_h}} \wedge \ldots \wedge \varepsilon^{j_k}) \wedge \varepsilon^{j_h}$$

with $j_h = \sigma(j_k)$ and $l = |J| - |I| + i_k - \sigma(j_k)$

$$= \sum_{h=0}^{k} \binom{|J|-|I|}{l} (-1)^{k-h} \sum_{\overline{\sigma} \in S_{k-1}} (-1)^{|\overline{\sigma}|} \binom{|J|-|I|}{\overline{\sigma}(j_0)-i_0, \ldots, \overline{\sigma}(j_{h-1})-i_{h-1}, \overline{\sigma}(j_{h+1})-i_h, \ldots, \overline{\sigma}(j_k)-i_{k-1}} \cdot$$

$$\cdot (\varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_k})$$

$$= \sum_{h=0}^{k} \sum_{\overline{\sigma} \in S_{k-1}} (-1)^{k-h+|\overline{\sigma}|} \binom{|J|-|I|}{\overline{\sigma}(j_0)-i_0, \ldots, \overline{\sigma}(j_{h-1})-i_{h-1}, \overline{\sigma}(j_{h+1})-i_h, \ldots, \overline{\sigma}(j_k)-i_{k-1}, j_h - i_k} \cdot$$

$$\cdot (\varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_k})$$

$$= \sum_{\sigma \in S_k} (-1)^{|\sigma|} \binom{|J|-|I|}{\sigma(j_0)-i_0, \ldots, \sigma(j_k)-i_{k-1}, \sigma(j_h)-i_k} \varepsilon^{j_1} \wedge \ldots \wedge \varepsilon^{j_k}$$

Then, by induction, the formula is true $\forall\ k \geq 0$

■

5.2 Flexes of Rational Curves

Our last example has been suggested us by K. Ranestad. One may like to compute the number N_d of all projectively non equivalent rational curves of degree d in $\mathbb{P}^3$ which have flexes at $2d-6$ prescribed points. These can be constructed by projection of a rational normal curve in $\mathbb{P}^d$ with $2d-6$ marked points from a $\mathbb{P}^{d-4}$ intersecting the osculating plane of the curves at the marked points. Such a number is counted precisely by

$$N_d = \int \sigma_2^{2(d-3)}$$

in the grassmannian $G_{d-3}(\mathbb{P}^d) \cong G_3(\mathbb{P}^d)$. Basing on our dictionary, one has:

$$D_2^{2(d-3)} \epsilon^0 \wedge \epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 = N_d \cdot \epsilon^{d-3} \wedge \epsilon^{d-2} \wedge \epsilon^{d-1} \wedge \epsilon^d.$$

An expression for N_d can be figured out by applying Proposition 2.5.8. One has:

$$= \sum \binom{n}{n_{000}, n_{001}, n_{002}, n_{010}, n_{011}, n_{020}, n_{100}, n_{101}, n_{110}, n_{200}} D_2^{n_{000}} D_1^{n_{001}+n_{010}+n_{100}} \alpha_0 \wedge$$

$$\wedge D_1^{n_{001}+n_{011}+n_{101}} D_2^{n_{002}} \alpha_1 \wedge D_1^{n_{010}+n_{011}+n_{110}} D_2^{n_{020}} \alpha_2 \wedge D_1^{n_{100}+n_{101}+n_{011}} D_2^{n_{200}} \alpha_3$$

Renaming the n_δ for commodity, one has:

$$\begin{aligned} & D_2^{2(d-3)} \epsilon^0 \wedge \epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3 = \\ = \ & \sum \binom{2(d-3)}{n_0, \ldots, n_9} \epsilon^{2n_0+n_1+n_3+n_6} \wedge \epsilon^{1+2n_2+n_1+n_4+n_7} \wedge \epsilon^{2+2n_5+n_3+n_4+n_8} \wedge \epsilon^{3+2n_9+n_4+n_6+n_7}, \end{aligned}$$

the sum being over all non negative integers $n_0, \ldots, n_9$ such that $\sum_{i=0}^{9} n_i = 2(d-3)$.

We are interested in the coefficient of $\epsilon^{d-3} \wedge \epsilon^{d-2} \wedge \epsilon^{d-1} \wedge \epsilon^d$, the class of a point. Such a coefficient gets the contributions of all 10-uples of non-negative integers such that:

$$\begin{cases} 2n_0 + n_1 + n_3 + n_6 & = & d - 3 + \tau(0) \\ 1 + 2n_2 + n_1 + n_4 + n_7 & = & d - 3 + \tau(1) \\ 2 + 2n_5 + n_3 + n_4 + n_8 & = & d - 3 + \tau(2) \\ 3 + 2n_9 + n_4 + n_6 + n_7 & = & d - 3 + \tau(3) \end{cases} \tag{5.8}$$

where $\tau \in S_4$, the group of permutation on $\{0, 1, 2, 3\}$, or:

$$\begin{cases} 2n_0 + n_1 + n_3 + n_6 & = & d - 3 + \tau(0) \\ 2n_2 + n_1 + n_4 + n_7 & = & d - 3 + \tau(1) - 1 \\ 2n_5 + n_3 + n_4 + n_8 & = & d - 3 + \tau(2) - 2 \\ 2n_9 + n_4 + n_6 + n_7 & = & d - 3 + \tau(3) - 3 \end{cases} \tag{5.9}$$

Then

$$N_d = \sum_{\tau} (-1)^{|\tau|} \binom{2(d-3)}{n_0, n_1, \ldots, n_9} \tag{5.10}$$

the sum being over all $\tau \in S_4$ and over all $(n_0, n_1, \ldots, n_9)$ satisfying (5.8) or (5.9).

Using the final formula in **R** (a program for statistical computing) (or *Mathematica* slower than **R** for these computations) one gets the following table:

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
N_d	1	0	1	5	126	3396	114675	4430712	190720530	8942188632	449551230102	$\cdots$

which can also be achieved via the package Schubert developed by Katz and Strømme. However we have a way to see (namely formula (5.10) together with conditions (5.9) and (5.8)) how complicated a general formula expressing such number may be.

5.3 Playing with $\mathcal{S}$-Derivations.

Working in $\bigwedge^k M$ suggests that Schubert calculus for grassmannians $G_k(\mathbb{P}^n)$ must be linked recursively with Schubert calculus of $G_{k'}(\mathbb{P}^{n'})$ with $k' < k$ and $n' \leq n$. This is also observed in [31]. To this purpose, and as a matter of example, we will determine the way of linking these Schubert Calculus for the grassmannians $G_1(\mathbb{P}^{1+n})$, $G_2(\mathbb{P}^{2+n})$ and $G_3(\mathbb{P}^{3+n})$.

5.3.1 Example. Working on $\bigwedge^2 M_{2+n}$, one has:

$$D_1^{2n}(\epsilon^0 \wedge \epsilon^1) = \sum_{h=0}^{2n} \binom{2n}{h} D_1^h \epsilon^0 \wedge D_1^{2n-h}\epsilon^1 = \sum_{h=0}^{2n} \binom{2n}{h} \epsilon^h \wedge \epsilon^{2n+1-h} \tag{5.11}$$

Since D_1^{2n} sends $(\bigwedge^2 M)_0$ onto $(\bigwedge^2 M)_{2n}$ (D_1 is homogeneous of degree 1 with respect to the weight graduation of $\bigwedge^2 M$) and $D_n\epsilon^0 = d_{0,n}\epsilon^n$ and $D_n\epsilon^0 = d_{0,1+n}\epsilon^{1+n}$ it follows that only the sum

$$\binom{2n}{n} d_{0,n}\epsilon^n \wedge \epsilon^n + \binom{2n}{n} d_{0,1+n}\epsilon^{1+n} \wedge \epsilon^n$$

can survive in expression (5.11). Therefore:

$$D_1^{2n}\epsilon^0 \wedge \epsilon^1 = \left[\binom{2n}{n} d_{0,n} - \binom{2n}{1+n} d_{0,1+n}\right] \epsilon^n \wedge \epsilon^{1+n} \tag{5.12}$$

so that

$$d_{1,1+n} = \binom{2n}{n} d_{0,n} - \binom{2n}{1+n} d_{0,1+n} = \frac{(2n)!}{(n+1)!n!} \quad (\text{since } \ d_{0,1+m} = 1, \ \forall m).$$

5.3.2 Example. Working on $\bigwedge^3 M_{3+n}$, one has:

$$D_1^{3n}(\epsilon^0 \wedge \epsilon^1 \wedge \epsilon^3) = \sum_{h=0}^{3n} \binom{3n}{h} D_1^h(\epsilon^0 \wedge \epsilon^1) \wedge \epsilon^{2+3n-h} \tag{5.13}$$

In the above sum only will survive multiples of the terms:

$$\epsilon^n \wedge \epsilon^{n+1} \wedge \epsilon^{n+2}, \quad \epsilon^n \wedge \epsilon^{n+2} \wedge \epsilon^{n+1} \quad \epsilon^{n+1} \wedge \epsilon^{n+2} \wedge \epsilon^n$$

Then, the possible values of h are $2n$, $2n+1$ and $2n+2$. Thus,

$$\begin{aligned} D_1^{3n}(\epsilon^0 \wedge \epsilon^1 \wedge \epsilon^3) &= \binom{3n}{2n} D_1^{2n}(\epsilon^0 \wedge \epsilon^1) \wedge \epsilon^{2+n} + \binom{3n}{2n+1} D_1^{2n+1}(\epsilon^0 \wedge \epsilon^1) \wedge \epsilon^{1+n} + \\ &+ \binom{3n}{2n+2} D_1^{2n+2}(\epsilon^0 \wedge \epsilon^1) \wedge \epsilon^n \end{aligned}$$

Using Example (5.3.1) one has the coefficient of the first addendum. For the second one only observe that the coefficient of $\epsilon^n \wedge \epsilon^{n+2}$ in the expansion of $D_1^{2n+1}(\epsilon^0 \wedge \epsilon^1)$ is equal to the degree $d_{1,2+n}$. In the last addendum, instead one observes that the coefficient of $\epsilon^{n+1} \wedge \epsilon^{n+2}$ in the expansion of $D_1^{2n+2}(\epsilon^0 \wedge \epsilon^1)$ is the degree $d_{1,1+n}$. Thus,

$$\begin{aligned} D_1^{3n}(\epsilon^0 \wedge \epsilon^1 \wedge \epsilon^3) &= \binom{3n}{2n} d_{1,2+n} \epsilon^n \wedge \epsilon^{n+1} \wedge \epsilon^{n+2} + \binom{3n}{2n+1} d_{1,2+n} \epsilon^n \wedge \epsilon^{n+2} \wedge \epsilon^{n+1} + \\ &+ \binom{3n}{2n+2} d_{1,1+n} \epsilon^{n+1} \wedge \epsilon^{n+2} \wedge \epsilon^n \end{aligned}$$

$$\begin{aligned} d_{2,2+n} &= \left[\binom{3n}{2n} - \binom{3n}{2n+1}\right] d_{1,2+n} + \binom{3n}{2n+2} d_{1,1+n} \\ &= \left[\binom{3n}{n} - \binom{3n}{n-1}\right] d_{1,2+n} + \binom{3n}{n-2} d_{1,1+n} \end{aligned}$$

5.3.3 Example. Working on $\bigwedge^4 M_{4+n}$, one has:

$$D_1^{4n}(\epsilon^0 \wedge \epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3) = \sum_{h=0}^{4n} \binom{4n}{h} D_1^h(\epsilon^0 \wedge \epsilon^1 \wedge \epsilon^2) \wedge \epsilon^{3+4n-h} \tag{5.14}$$

In the above sum will only survive multiples of the terms:

$$\epsilon^n\wedge\epsilon^{n+1}\wedge\epsilon^{n+2}\wedge\epsilon^{n+3},\quad \epsilon^n\wedge\epsilon^{n+1}\wedge\epsilon^{n+3}\wedge\epsilon^{n+2},\quad \epsilon^n\wedge\epsilon^{n+2}\wedge\epsilon^{n+3}\wedge\epsilon^{n+1}\quad \epsilon^{n+1}\wedge\epsilon^{n+2}\wedge\epsilon^{n+3}\wedge\epsilon^{n}$$

Then, the possible values of h are $3n$, $3n+1$, $3n+2$ and $3n+3$. Thus,

$$\begin{aligned} &D_1^{4n}(\epsilon^0\wedge\epsilon^1\wedge\epsilon^2\wedge\epsilon^3)= \\ &\quad=\binom{4n}{3n}D_1^{3n}(\epsilon^0\wedge\epsilon^1\wedge\epsilon^2)\wedge\epsilon^{n+3}+\binom{4n}{3n+1}D_1^{3n+1}(\epsilon^0\wedge\epsilon^1\wedge\epsilon^2)\wedge\epsilon^{n+2}+ \\ &\quad+\binom{4n}{3n+2}D_1^{3n+2}(\epsilon^0\wedge\epsilon^1\wedge\epsilon^2)\wedge\epsilon^{n+1}+\binom{4n}{3n+3}D_1^{3n+3}(\epsilon^0\wedge\epsilon^1\wedge\epsilon^2)\wedge\epsilon^{n} \end{aligned}$$

Using Example (5.3.3) one easily sees that the first addendum is $d_{2,2+n}$. In the same way, one see that the third and fourth addendum are $d_{2,3+n}$. To end computations one only needs to know the coefficient of the second addendum:

$$D_1^{3n+1}(\epsilon^0\wedge\epsilon^1\wedge\epsilon^2)=\sum_{h=0}^{3n+1}\binom{3n+1}{h}D_1^h(\epsilon^0\wedge\epsilon^1)\wedge\epsilon^{3+3n-h}$$

In the above sum will only survive multiples of the terms:

$$\epsilon^n\wedge\epsilon^{n+1}\wedge\epsilon^{n+3},\quad \epsilon^n\wedge\epsilon^{n+3}\wedge\epsilon^{n+1}\quad \epsilon^{n+1}\wedge\epsilon^{n+3}\wedge\epsilon^{n}$$

Then, the possible values of h are $2n$, $2n+2$ and $2n+3$.

Thus,

$$\begin{aligned} D_1^{3n+1}(\epsilon^0\wedge\epsilon^1\wedge\epsilon^2) &= \binom{3n+1}{2n}D_1^{2n}(\epsilon^0\wedge\epsilon^1)\wedge\epsilon^{n+3}+ \\ &+\binom{3n+1}{2n+2}D_1^{2n+2}(\epsilon^0\wedge\epsilon^1)\wedge\epsilon^{n+1}+\binom{3n+1}{2n+3}D_1^{2n+3}(\epsilon^0\wedge\epsilon^1)\wedge\epsilon^{n} \end{aligned}$$

Using the same idea of previous examples one sees that the coefficients of the first, second and third addenda are, respectively $d_{1,1+n}$, $d_{1,3+n}-d_{1,2+n}$ and $d_{1,3+n}$.

Thus,

$$\begin{aligned} D_1^{3n+1}(\epsilon^0\wedge\epsilon^1\wedge\epsilon^2) &= \binom{3n+1}{2n}d_{1,1+n}\epsilon^n\wedge\epsilon^{n+1}\wedge\epsilon^{n+3}+ \\ &+\binom{3n+1}{2n+2}(d_{1,3+n}-d_{1,2+n})\epsilon^n\wedge\epsilon^{n+3}\wedge\epsilon^{n+1}+ \\ &+\binom{3n+1}{2n+3}d_{1,3+n}\epsilon^{n+1}\wedge\epsilon^{n+3}\wedge\epsilon^{n}= \end{aligned}$$

$$= \left[\binom{3n+1}{2n} d_{1,1+n} - \binom{3n+1}{2n+2} (d_{1,3+n} - d_{1,2+n}) + \binom{3n+1}{2n+3} d_{1,3+n} \right] \epsilon^n \wedge \epsilon^{n+1} \wedge \epsilon^{n+3} =$$

$$= \left\{ \binom{3n+1}{2n} d_{1,1+n} + \binom{3n+1}{2n+2} d_{1,2+n} + \left[\binom{3n+1}{2n+3} - \binom{3n+1}{2n+2} \right] d_{1,3+n} \right\} \epsilon^n \wedge \epsilon^{n+1} \wedge \epsilon^{n+3}.$$

Reorganizing the results, one has:

$$\begin{aligned} D_1^{4n}(\epsilon^0 \wedge \epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3) \quad = \quad & \binom{4n}{3n} d_{2,2+n} \epsilon^n \wedge \epsilon^{n+1} \wedge \epsilon^{n+2} \wedge \epsilon^{n+3} + \\ & + \binom{4n}{3n+1} \left\{ \binom{3n+1}{2n} d_{1,1+n} + \binom{3n+1}{2n+2} d_{1,2+n} + \right. \\ & \left. + \left[\binom{3n+1}{2n+3} - \binom{3n+1}{2n+2} \right] d_{1,3+n} \right\} \epsilon^n \wedge \epsilon^{n+1} \wedge \epsilon^{n+3} \wedge \epsilon^{n+2} + \\ & + \binom{4n}{3n+2} d_{2,3+n} \epsilon^n \wedge \epsilon^{n+2} \wedge \epsilon^{n+3} \wedge \epsilon^{n+1} + \\ & + \binom{4n}{3n+3} d_{2,3+n} \epsilon^{n+1} \wedge \epsilon^{n+2} \wedge \epsilon^{n+3} \wedge \epsilon^n \end{aligned}$$

Thus,

$$\begin{aligned} d_{3,3+n} \quad = \quad & \binom{4n}{n} d_{2,2+n} + \left[\binom{4n}{n-2} - \binom{4n}{n-3} \right] d_{2,3+n} - \binom{4n}{n-1} \left\{ \binom{3n+1}{n+1} d_{1,1+n} + \right. \\ & \left. + \binom{3n+1}{n-1} d_{1,2+n} + \left[\binom{3n+1}{n-2} - \binom{3n+1}{n-1} \right] d_{1,3+n} \right\}. \end{aligned}$$

Chapter 6

The Grassmannian of Lines

The intersection theory of the Grassmannians of lines is described by a 1-SCGP (as in [26]), where computations get very easy. The model for its intersection theory is the same of Chapter 5 when $k = 1$.

This Chapter shall be devote to show some applications of our theory in the grassmannian of Lines.

6.1 Littlewood-Richardson Coefficients

Littlewood-Richardson Coefficients have a variety of interpretations, often in terms of symmetric functions, representation theory and geometry. In each case they appear as structure coefficients of rings. In geometry, Littlewood-Richardson Coefficients are structure coefficients of the cohomology ring of the Grassmannian with respect to the basis of Schubert cycles. In this section we prove, using our framework, a well know result about Littlewood-Richardson coefficients: *In the Grassmannian $G_1(\mathbb{P}^n)$ Littlewood-Richardson coefficients are always* 0 *or* 1. First recall:

6.1.1 Definition. *Let $\Delta_I(D)$ and $\Delta_J(D)$ be the Giambelli's polynomial for $\epsilon^I = \epsilon^{i_0} \wedge \ldots \wedge \epsilon^{i_k}$ and $\epsilon^J = \epsilon^{j_0} \wedge \ldots \wedge \epsilon^{j_k}$ respectively. Then:*

$$[\epsilon^I] * [\epsilon^J] \quad = \quad \Delta_I(D)(\epsilon^J)$$

$$= \sum_{|M|=|I|+|J|} C_{I,J}^{M} \epsilon^{M} \qquad (6.1)$$

with $\epsilon^M = \epsilon^{m_0} \wedge \ldots \wedge \epsilon^{m_k}$ *and* $m_0 < m_1 \ldots < m_k$.

The $C_{I,J}^M$ *are the Littlewood-Richardson Coefficients.*

6.1.2 Examples. Computing some LR coefficients in the Grassmannian $G(1, \mathbb{P}^n)$.

$$\begin{aligned} 1) \quad [\epsilon^1 \wedge \epsilon^2] * [\epsilon^2 \wedge \epsilon^3] &= \Delta_{12}(D)(\epsilon^0 \wedge \epsilon^1) * [\epsilon^2 \wedge \epsilon^3] = \\ &= \Delta_{12}(D)(\epsilon^2 \wedge \epsilon^3) = \\ &= \begin{vmatrix} D_1 & D_2 \\ D_0 & D_1 \end{vmatrix} (\epsilon^2 \wedge \epsilon^3) = \\ &= (D_1^2 - D_2)(\epsilon^2 \wedge \epsilon^3) = \epsilon^3 \wedge \epsilon^4. \end{aligned}$$

Then,

$$C_{(1,2),(2,3)}^{(0,7)} = 0, \quad C_{(1,2),(2,3)}^{(1,6)} = 0, \quad C_{(1,2),(2,3)}^{(2,5)} = 0, \quad C_{(1,2),(2,3)}^{(3,4)} = 1.$$

$$\begin{aligned} 2) \quad [\epsilon^1 \wedge \epsilon^3] * [\epsilon^2 \wedge \epsilon^4] &= \Delta_{13}(D)(\epsilon^0 \wedge \epsilon^1) * [\epsilon^2 \wedge \epsilon^4] = \\ &= \Delta_{13}(D)(\epsilon^2 \wedge \epsilon^4) = \\ &= \begin{vmatrix} D_1 & D_3 \\ D_0 & D_2 \end{vmatrix} (\epsilon^2 \wedge \epsilon^4) = \\ &= (D_1 D_2 - D_3)(\epsilon^2 \wedge \epsilon^4) = \\ &= \epsilon^2 \wedge \epsilon^7 + \epsilon^3 \wedge \epsilon^6 + \epsilon^3 \wedge \epsilon^6 + \epsilon^4 \wedge \epsilon^5 + \\ &\quad -\epsilon^2 \wedge \epsilon^7 - \epsilon^3 \wedge \epsilon^6 - \epsilon^4 \wedge \epsilon^5 - \epsilon^5 \wedge \epsilon^4 \\ &= \epsilon^3 \wedge \epsilon^6 + \epsilon^4 \wedge \epsilon^5 \end{aligned}$$

Then,

$$C_{(1,3),(2,4)}^{(2,7)} = 0 \quad C_{(1,3),(2,4)}^{(3,6)} = 1 \quad C_{(1,3),(2,4)}^{(4,5)} = 1$$

6.1.3 Proposition. (LR coefficients on $\mathbf{G}(\mathbf{1}, \mathbb{P}^{\mathbf{n}})$) *Let* I, J *and* M *be Schubert indices of length* 2, *such that* $|I| + |J| = |M|$. *Then,*

$$0 \leq C_{I,J}^M \leq 1.$$

Proof.

$$\Delta_{(i_0,i_1)}(D)(\epsilon^{j_0}\wedge\epsilon^{j_1}) = \begin{vmatrix} D_{i_0} & D_{i_1} \\ D_{i_0-1} & D_{i_1-1} \end{vmatrix}(\epsilon^{j_0}\wedge\epsilon^{j_1}) = (D_{i_0}D_{i_1-1} - D_{i_1}D_{i_0-1})(\epsilon^{j_0}\wedge\epsilon^{j_1}) =$$

$$= \sum_{\substack{a_0+a_1 = i_0 \\ c_0+c_1 = i_1-1}} \epsilon^{j_0+a_0+c_0}\wedge\epsilon^{j_1+a_1+c_1} - \sum_{\substack{b_0+b_1 = i_0-1 \\ d_0+d_1 = i_1}} \epsilon^{j_0+b_0+d_0}\wedge\epsilon^{j_1+b_1+d_1} =$$

$$= \sum_{a_0=0}^{i_0}\sum_{c_0=0}^{i_1-1}\epsilon^{j_0+a_0+c_0}\wedge\epsilon^{j_1+i_0+i_1-(a_0+c_0+1)} - \sum_{b_0=0}^{i_0-1}\sum_{d_0=0}^{i_1}\epsilon^{j_0+b_0+d_0}\wedge\epsilon^{j_1+i_0+i_1-(b_0+d_0+1)} =$$

$$= \sum_{a_0=0}^{i_0-1}\sum_{c_0=0}^{i_1-1}\epsilon^{j_0+a_0+c_0}\wedge\epsilon^{j_1+i_0+i_1-(a_0+c_0+1)} + \sum_{c_0=0}^{i_1-1}\epsilon^{j_0+i_0+c_0}\wedge\epsilon^{j_1+i_1-(c_0+1)} -$$

$$-\sum_{b_0=0}^{i_0-1}\sum_{d_0=0}^{i_1-1}\epsilon^{j_0+b_0+d_0}\wedge\epsilon^{j_1+i_0+i_1-(b_0+d_0+1)} - \sum_{b_0=0}^{i_0-1}\epsilon^{j_0+i_1+b_0}\wedge\epsilon^{j_1+i_0-(b_0+1)} =$$

$$= \sum_{c_0=0}^{i_1-1}\epsilon^{j_0+i_0+c_0}\wedge\epsilon^{j_1+i_1-(c_0+1)} - \sum_{b_0=0}^{i_0-1}\epsilon^{j_0+i_1+b_0}\wedge\epsilon^{j_1+i_0-(b_0+1)}$$

Since $i_1 - i_0 > 0$, then,

$$\begin{aligned}
\Delta_{(i_0,i_1)}(D)(\epsilon^{j_0}\wedge\epsilon^{j_1}) &= \sum_{c_0=0}^{i_1-1}\epsilon^{j_0+i_0+c_0}\wedge\epsilon^{j_1+i_1-(c_0+1)} - \sum_{b_0=0}^{i_0-1}\epsilon^{j_0+i_1+b_0}\wedge\epsilon^{j_1+i_0-(b_0+1)} \\
&= \sum_{c_0=0}^{i_1-i_0-1}\epsilon^{j_0+i_0+c_0}\wedge\epsilon^{j_1+i_1-(c_0+1)} + \sum_{c_0=i_1-i_0}^{i_1-1}\epsilon^{j_0+i_0+c_0}\wedge\epsilon^{j_1+i_1-(c_0+1)} - \\
&\quad -\sum_{b_0=0}^{i_0-1}\epsilon^{j_0+i_1+b_0}\wedge\epsilon^{j_1+i_0-(b_0+1)} \\
&= \sum_{c_0=0}^{i_1-i_0-1}\epsilon^{j_0+i_0+c_0}\wedge\epsilon^{j_1+i_1-(c_0+1)}
\end{aligned}$$

To prove that all coefficients are bigger or equal to zero, it is sufficient proves that:

$$j_0 + i_0 + c_0 < j_1 + i_1 - (c_0+1), \quad \forall\ c_0 \text{ s.t. } 0 \le c_0 \le i_1 - i_0 - 1 \ \ (j_0 < j_1,\ i_0 < i_1).$$

Let us suppose that $j_1 - j_0 \ge i_1 - i_0$.

$$\Big(j_1 + i_1 - (c_0+1)\Big) - \Big(j_0 + i_0 + c_0\Big) = (j_1 - j_0) + (i_1 - i_0) - (2c_0 + 1)$$

$$
\begin{aligned}
&> (j_1 - j_0) + (i_1 - i_0) - (2(i_1 - i_0 - 1) + 1) \\
&> (j_1 - j_0) - (i_1 - i_0) + 1 > 0
\end{aligned}
$$

Then,

$$j_1 + i_1 - (c_0 + 1) > j_0 + i_0 + c_0, \quad \forall\ c_0 \text{ s.t. } 0 \le c_0 \le i_1 - i_0 - 1 \quad (j_0 < j_1,\ i_0 < i_1).$$

For this, one concludes that $0 \le C^M_{I,J} \le 1$ ∎

6.1.4 Example A. Computing some LR coefficients in the Grassmannian $G(1, \mathbb{P}^n)$.

$$
\begin{aligned}
1A) \quad [\epsilon^1 \wedge \epsilon^2] * [\epsilon^2 \wedge \epsilon^3] &= \Delta_{12}(D)(\epsilon^2 \wedge \epsilon^3) = \\
&= \sum_{c_0=0}^{0} \epsilon^{3+c_0} \wedge \epsilon^{4-c_0} = \epsilon^3 \wedge \epsilon^4;
\end{aligned}
$$

$$
\begin{aligned}
2A) \quad [\epsilon^1 \wedge \epsilon^3] * [\epsilon^2 \wedge \epsilon^4] &= \Delta_{13}(D)(\epsilon^2 \wedge \epsilon^4) = \\
&= \sum_{c_0=0}^{1} \epsilon^{3+c_0} \wedge \epsilon^{6-c_0} = \epsilon^3 \wedge \epsilon^6 + \epsilon^4 \wedge \epsilon^5.
\end{aligned}
$$

6.2 Top Intersection Numbers in $G_1(\mathbb{P}^{1+n})$

Let M_{2+n} the free $\mathbb{Z}$-module spanned by $(\epsilon^0, \epsilon^1, \ldots, \epsilon^{n+1})$ and $\bigwedge^2(M_{2+n}, D_1)$ the 2-SCGP as in Chapter 5.

Let $a, b \ge 0$ such $a + 2b = 2n$. Then, in $\bigwedge^2 M_{2+n}$, the following equality holds:

$$D_1^a D_2^b(\epsilon^0 \wedge \epsilon^1) = \kappa_{a,b} \cdot \epsilon^n \wedge \epsilon^{n+1}$$

By virtue of our dictionary, $\kappa_{a,b}$ is nothing else than

$$\int \sigma_1^a \sigma_2^b \cap [G_1(\mathbb{P}^{1+n})].$$

One has:

$$\sum \binom{a}{a_0, a_1} \binom{b}{b_0, b_1, b_2} \prod_{i=0}^{1} D_i^{a_{1-i}} \prod_{i=0}^{2} D_i^{b_{2-i}} \epsilon^0 \wedge \prod_{i=0}^{1} D_i^{a_i} \prod_{i=0}^{2} D_i^{b_i} \epsilon^1 =$$

$$= \sum \binom{a}{a_0, a_1} \binom{b}{b_0, b_1, b_2} D_1^{a_0} D_1^{b_1} D_2^{b_0} \epsilon^0 \wedge D_1^{a_1} D_1^{b_1} D_2^{b_2} \epsilon^1 =$$

$$= \sum \binom{b}{b_0, b_1, b_2} \binom{a}{a_0, a_1} \epsilon^{a_0+b_1+2b_0} \wedge \epsilon^{1+a_1+b_1+2b_2} =$$

$$= \sum_{b_0=0}^{b} \sum_{a_0=0}^{2n-2b} \binom{2n-2b}{a_0, 2n-2b-a_0} \cdot C_{a_0,b_0} \tag{6.2}$$

where

$$C_{a_0,b_0} = \binom{b}{b_0, n-a_0-2b_0, b+b_0+a_0-n} - \binom{b}{b_0, n-a_0-2b_0+1, b+b_0+a_0-n-1}$$

Formula (6.2) can be considered a new formula in Schubert calculus. Here is a table computing $\kappa_{a,b}$ asking *Mathematica* 5.1 to use formula (6.2).

n	a	b	$\kappa_{a,b}$
1	**0**	**0**	**1**
2	0	1	0
	2	**0**	**1**
3	0	2	1
	2	1	1
	4	**0**	**2**
4	0	3	1
	2	2	2
	4	1	3
	6	**0**	**5**
5	0	4	3
	2	3	4
	4	2	6
	6	1	9
	8	**0**	**14**

n	a	b	$\kappa_{a,b}$
6	0	5	6
	2	4	9
	4	3	13
	6	2	19
	8	1	28
	10	**0**	**42**
7	0	6	15
	2	5	21
	4	4	30
	6	3	43
	8	2	62
	10	1	90
	12	**0**	**132**

n	a	b	$\kappa_{a,b}$
8	0	7	36
	2	6	51
	4	5	72
	6	4	102
	8	3	145
	10	2	207
	12	1	297
	14	**0**	**429**
9	0	8	91
	2	7	127
	4	6	178
	6	5	250
	8	4	352
	10	3	497
	12	2	704
	14	1	1001
	16	**0**	**1430**

n	a	b	$\kappa_{a,b}$
10	0	9	232
	2	8	323
	4	7	450
	6	6	628
	8	5	878
	10	4	1230
	12	3	1727
	14	2	2431
	16	1	3432
	18	**0**	**4862**

6.2.1 Remark. The term C_{a_0,b_0} in expression (6.2) can be simplified, one has

$$\begin{aligned} C_{a_0,b_0} &= \frac{b!}{b_0!(n-a_0-2b_0)!(b+b_0+a_0-n)!} - \frac{b!}{b_0!(n-a_0-2b_0+1)!(b+b_0+a_0-n-1)!} \\ &= \frac{b!\left[(n-a_0-2b_0+1)-(b+b_0+a_0-n)\right]}{b_0!(n-a_0-2b_0+1)!(b+b_0+a_0-n)!} \\ &= \frac{b!(2n-b-2a_0-3b_0+1)}{b_0!(n-a_0-2b_0+1)!(b+b_0+a_0-n)!} \\ &= \binom{b+1}{b_0,\ n-a_0-2b_0+1,\ b+b_0+a_0-n}\frac{2n-b-2a_0-3b_0+1}{b+1} \end{aligned}$$

and then,

$$\kappa_{a,b} = \sum_{b_0=0}^{b}\sum_{a_0=0}^{a}\binom{a}{a_0}\binom{b+1}{b_0,\ n-a_0-2b_0+1,\ b+b_0+a_0-n}\frac{2n-b-2a_0-3b_0+1}{b+1},$$

6.2.2 In particular, if $b=0$, we know that $\kappa_{a,0}$ is the Plücker degree of the grassmannian $G(1,\mathbb{P}^{n+1})$. Computing this number, we have:

$$\begin{aligned} \kappa_{a,0}=\kappa_{2n,0} &= \sum_{a_0=0}^{2n}\binom{2n}{a_0}\binom{1}{0,n-a_0+1,a_0-n}\frac{2n-2a_0+1}{1} \\ &= -\binom{2n}{n+1}\binom{1}{0,0,1}+\binom{2n}{n}\binom{1}{0,1,0} \\ &= \binom{2n}{n}-\binom{2n}{n+1} \\ &= \frac{1}{n+1}\cdot\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n} \end{aligned}$$

6.3 Bessel Function and Degree of the Grassmannian of Lines

In Example 5.3.1(also in (6.2.2)) we have seen that the degree of the grassmannian $G_1(\mathbb{P}^{1+n})$ is the coefficient $\kappa_{2n,0} := d_{1,1+n}$ of the expansion

$$D_1^{2n}(\epsilon^0 \wedge \epsilon^1) = d_{1,1+n}\epsilon^n \wedge \epsilon^{1+n},$$

in $\bigwedge^2 M$ where M is a free $\mathbb{Z}$-module of rank $2+n$ and D_1 is the extension of the shift endomorphism to $\bigwedge^2 M$. This coefficient, as remarked, is equal to the Catalan number:

$$d_{1,1+n} = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}. \tag{6.3}$$

In this chapter we will find an expression involving modified Bessel's functions as generating function for the Plücker degrees of the grassmannians of lines in a projective space.

The problem is motivated because using our formalism (see Section 5.3, and also [27]) we can easily prove that the degree of the grassmaniann $G_k(\mathbb{P}^{k+n})$ can be written as an explicit linear combination of degrees $d_{k',k'+n'}$, with $k' < k$. This fact seems to suggest that it should be possible to collect all the degrees of the grassmannians (for all k and n) in some general generating function. Below we analyze a very special case, inspired by this philosophy.

6.3.1 A Few Words about Bessel Functions. As well known, for each $z \in \mathbb{C}$, the function $f_z(t) = e^{\frac{1}{2}z(t-\frac{1}{t})}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and admits therefore a Laurent series expansion converging at the function itself at all points of its domain:

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n \mathbf{J_n}(z). \tag{6.4}$$

The coefficient of t^n ($n \in \mathbb{Z}$) of such an expansion will be said *Bessel function*. An explicit expression for $\mathbf{J_n}(z)$ is given by the equality:

$$e^{\frac{1}{2}z(t-\frac{1}{t})} = \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}z\right)^r t^r}{r!} \sum_{s=0}^{\infty} \frac{\left(-\frac{1}{2}z\right)^s t^{-s}}{s!}.$$

Hence, for $n \geq 0$ one has:

$$\mathbf{J_n}(z) = \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{1}{2}z\right)^{n+2s}}{s!(n+s)!} \quad \text{and} \quad \mathbf{J_{-n}}(z) = (-1)^n \mathbf{J_n}(z). \tag{6.5}$$

Differentiating expression (6.4) with respect to t, one gets the recurrence formulae:

$$\mathbf{J_{n-1}}(z) + \mathbf{J_{n+1}}(z) = \frac{2n}{z}\mathbf{J_n}(z) \quad \text{and} \quad \mathbf{J_{n-1}}(z) - \mathbf{J_{n+1}}(z) = 2\mathbf{J_n'}(z),$$

from which the *Bessel differential equation*:

$$z^2 \frac{d^2\mathbf{J_n}(z)}{dz^2} + z\frac{d\mathbf{J_n}(z)}{dz} + (z^2 - n^2)\mathbf{J_n}(z) = 0.$$

For each $n \in \mathbb{Z}$, then, $\mathbf{J_n}(z)$ is a solution of the (Bessel) differential equation:

$$z^2 y'' + zy' + (z^2 - n^2)y = 0. \tag{6.6}$$

Another linearly independent solution is given by the Weber's function:

$$\mathbf{Y_n}(z) = \frac{\mathbf{J_n}(z)\cos(n\pi) - \mathbf{J_{-n}}(z)}{\sin n\pi}. \tag{6.7}$$

The function $\mathbf{J_n}(z)$ is also said to be the *n-Bessel function of first kind* while $\mathbf{Y_n}(z)$ the *n-Bessel function of second kind.* For each $\nu \in \mathbb{C}$ one defines the Bessel function of order ν, $\mathbf{J}_\nu(z)$ and $\mathbf{Y}_\nu(z)$, to be solutions of the equation:

$$z^2 y'' + zy' + (z^2 - \nu^2)y = 0 \tag{6.8}$$

which will be called *Bessel differential equation for functions of order* ν.

The Bessel functions are related to the Hankel's functions, also called Bessel functions of the third kind,

$$\mathbf{H}_\nu^{(1)}(z) = \mathbf{J}_\nu(z) + i\mathbf{Y}_\nu(z) \tag{6.9}$$

$$\mathbf{H}_\nu^{(2)}(z) = \mathbf{J}_\nu(z) - i\mathbf{Y}_\nu(z) \tag{6.10}$$

This functions have many interesting properties.

6.3.2 Modified Bessel functions.

Now we consider the differential equation

$$z^2y'' + zy' - (z^2 + n^2)y = 0 \tag{6.11}$$

called *modified Bessel differential equation* and occurring in many problems of Mathematical Physics.

One of the solutions of this equation is called *modified Bessel function of first kind* $\mathbf{I_n}(z)$. This solution is related with $\mathbf{J_n}(z)$ in this way :

$$\mathbf{I_n}(z) = i^{-n}\mathbf{J_n}(iz)$$

Then,

$$\begin{aligned}
\mathbf{I_n}(z) &= i^{-n}\sum_{s=0}^{\infty}\frac{(-1)^s\left(\frac{1}{2}iz\right)^{n+2s}}{s!(n+s)!} \\
&= i^{-n}i^{n}\sum_{s=0}^{\infty}\frac{(-1)^s i^{2s}\left(\frac{1}{2}z\right)^{n+2s}}{s!(n+s)!} \\
&= \sum_{s=0}^{\infty}\frac{\left(\frac{1}{2}z\right)^{n+2s}}{s!(n+s)!}
\end{aligned} \tag{6.12}$$

Then we can see that:

$$\mathbf{I_{-n}}(z) = \mathbf{I_n}(z) \tag{6.13}$$

In fact,

$$\begin{aligned}
\mathbf{I_{-n}}(z) &= i^{n}\mathbf{J_{-n}}(iz) = \\
&= i^{n}(-1)^n\mathbf{J_n}(iz) = \\
&= i^{2n}i^{-n}(-1)^n\mathbf{J_n}(iz) = \\
&= (-1)^n i^{-n}(-1)^n\mathbf{J_n}(iz) = \\
&= i^{-n}\mathbf{J_n}(iz) = \mathbf{I_n}(z)
\end{aligned}$$

The second solution of this equation is called *modified Bessel function of second kind*, $\mathbf{K_n}(z)$ with

$$\mathbf{K_n}(z) = \left(\frac{\pi}{2}\right)\frac{\mathbf{I_n}(z) - \mathbf{I_{-n}}(z)}{\sin n\pi} \tag{6.14}$$

6.3.3 A Generating Function for the $d_{1,n+1}$'s. Easy computations show that the numbers $d_{1,1+n}$ of formula (6.3) satisfy the recurrence below:

$$(n+2)d_{1,n+2} - 2(2n+1)d_{1,n+1} = 0, \tag{6.15}$$

holding for each $n \geq 1$. Let us organize them into a formal power series:

$$F(z) = \sum_{n=0}^{\infty} \frac{d_{1,n+1}}{n!} z^n. \tag{6.16}$$

The power series $F(z)$ is indeed an entire holomorphic function. In fact:

6.3.4 Proposition. *The series (6.16) converges for all $z \in \mathbb{C}$.*

Proof. One simply applies the ratio test using the recursive relation (6.15). One has:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{\frac{d_{1,n+2}}{(n+1)!}}{\frac{d_{1,n+1}}{n!}} \right| &= \lim_{n\to\infty} \left| \frac{n! d_{1,n+2}}{(n+1)! d_{1,n+1}} \right| = \\
&= \lim_{n\to\infty} \left| \frac{\frac{2(2n+1)}{n+2} d_{1,n+1}}{(n+1) d_{1,n+1}} \right| = \\
&= \lim_{n\to\infty} \left| \frac{2(2n+1)}{(n+1)(n+2)} \right| = 0. \qquad \blacksquare
\end{aligned}$$

6.3.5 Proposition. *The function $w = F(z)$ (formula (6.16)) is solution of the Cauchy problem:*

$$\begin{cases} zw'' + 2(1-2z)w' - 2w = 0 \\ w(0) = 1 \\ w'(0) = 1 \end{cases} . \tag{6.17}$$

Proof. First of all one can immediately check that $F(0) = F'(0) = 1$. Secondly, using (6.15), one has that:

$$zF'' + 2(1-2z)F' - 2F = (zF')' + F' - 4(zF)' + 2F.$$

Substituting expression (6.16) in the above equality, using (6.15), one then gets, for each $n \geq 0$:

$$[(zF)' + F' - 4(zF)' + 2F](z) =$$

$$= \sum_{n\geq 0}\left(\frac{(n+1)d_{1,n+2}}{n!}+\frac{d_{1,n+2}}{n!}-\frac{4(n+1)d_{1,n+1}}{n!}+\frac{2d_{1,n+1}}{n!}\right)z^n =$$

$$= \sum_{n\geq 0}\left(\frac{(n+2)d_{1,n+2}-2(2n+1)d_{1,n+1}}{n!}\right)z^n = 0. \qquad \blacksquare$$

6.3.6 Theorem. *The solution of the Cauchy problem (6.17) is :*

$$w(z) = e^{2z}(\mathbf{I_0}(2z) - \mathbf{I_1}(2z)), \tag{6.18}$$

where

$$\mathbf{I_n}(z) = \sum_{s=0}^{\infty}\frac{\left(\frac{1}{2}z\right)^{n+2s}}{s!(n+s)!} \tag{6.19}$$

is the modified Bessel function of first kind, *satisfying* Bessel's differential equation

$$z^2w'' + zw' - (z^2+n^2)w = 0.$$

Proof. One first multiplies by z the differential equation occurring in (6.17), getting

$$z^2w'' + 2(1-2z)zw' - 2zw = 0. \tag{6.20}$$

Then one looks for a solution in the form of a convergent power series convergent in a neighborhood of 0. This will be achieved using Frobenius method ([4], p. 1), looking for a solution of the form:

$$w(z,m) = \sum_{n=0}^{\infty} A_n x^{n+m}, \qquad \textit{with} \qquad A_0 \neq 0. \tag{6.21}$$

This is possible, in this case, because $2(1-2z)z$ and $2z$ are (entire) holomorphic function. From equation (6.21) one has :

$$\frac{dw}{dz} = mA_0z^{m-1} + \sum_{k=1}^{\infty}(m+k)A_kz^{m+k-1},$$

.

$$\frac{d^2w}{dz^2} = m(m-1)A_0z^{m-2} + \sum_{k=1}^{\infty}(m+k)(m+k-1)A_kz^{m+k-2}$$

Then, in order to satisfy equation (6.17), for $w = w(z)$, one must have:

$$z^2 \frac{d^2 w}{dz^2} + z(2-4z)\frac{dw}{dz} - 2zw = 0,$$

which, using the power series expansion, will be written as:

$$z^2 \left[m(m-1)A_0 z^{m-2} + \sum_{k=1}^{\infty} (m+k)(m+k+1)A_k z^{m+k-2} \right] +$$
$$+2(1-2z)z \left[mA_0 z^{m-1} + \sum_{k=1}^{\infty} (m+k)A_k z^{m+k-1} \right] - 2z \left[A_0 z^m + \sum_{k=1}^{\infty} A_k z^{m+k} \right] = 0.$$

Dividing by z^m, this gives:

$$[m(m-1)A_0 + 2mA_0]\, z^m + [(m+1)mA_1 - 4mA_0 - 2A_0]\, z^{m+1} + \ldots$$
$$\ldots + [(m+n)(m+n-1)A_n + 2(m+n)A_n +$$
$$-4(m+n-1)A_{n-1} - 2A_{n-1}]\, z^{m+n} + \ldots = 0,$$

and then,

$$(m^2+m)A_0 z^m + \big[(m+1)mA_1 - (m+2)A_0\big] z^{m+1} + \ldots +$$
$$+ \{(m+n)(m+n+1)A_n - [4(m+n-1)+2]A_{n-1}\}\, z^{m+n} = 0$$

The left-hand side of equation (6.22) identically vanishes in its convergence domain if and only if all the coefficients occurring in the z-power series vanish. Imposing such a vanishing one deduces that

$$m(m+1)A_0 = 0, \tag{6.22}$$

together with the recursive relation:

$$A_n = \frac{4(m+n-1)+2}{(m+n)(m+n+1)} A_{n-1}, \tag{6.23}$$

holding for each $n \geq 1$. Therefore, using (6.23) one has:

$$w(z,m) = A_0 z^m \left[1 + \frac{4m+2}{(m+1)(m+2)} z + \frac{(4m+6)(4m+2)}{(m+1)(m+2)^2(m+3)} z^2 + \right.$$
$$\left. + \frac{(4m+10)(4m+6)(4m+2)}{(m+1)(m+2)^2(m+3)^2(m+4)} z^3 + \ldots \right] \tag{6.24}$$

We know, by the way $A_1, A_2, \ldots$ have been constructed, that :

$$z^2\frac{d^2w}{dx^2} + 2(1-2z)z\frac{dw}{dz} - 2zw = m(m+1)A_0 z^{m-2}.$$

That last equation can be rewritten in the form :

$$\left(z^2\frac{d^2}{dz^2} + 2(1-2z)z\frac{d}{dz} - 2z\right) w = m(m+1)A_0 z^m. \tag{6.25}$$

Since $A_0 \neq 0$, the right hand side of (6.25) is zero if and only if $m = 0$ or $m = -1$. The case $m = -1$ must be excluded, since contrarily one would get solutions not converging in any neighborhood of the origin of the complex plane. On the other hand, setting $m = 0$, one has:

$$\left(z^2\frac{d^2}{dz^2} + 2(1-2z)z\frac{d}{dz} - 2z\right) w(z,0) = 0,$$

which shows that $w(z,0)$ is a solution of equation (6.20). Using (6.24) one has found, then:

$$\begin{aligned} w(z,0) &= 1 + z + z^2 + \frac{5}{6}z^3 + \frac{7}{12}z^4 + \ldots = \\ &= 1 + z + \frac{2}{2!}z^2 + \frac{5}{3!}z^3 + \frac{14}{4!}z^4 + \ldots \end{aligned}$$

Using (6.19), the fact that $e^{2z} = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!}$ and an easy computation, one finally gets :

$$w(z,0) = e^{2z}(I_0(2z) - I_1(2z)),$$

a function solving the given Cauchy problem, as an immediate check shows. ∎

We have hence proven that

6.3.7 Corollary.

$$F(z) = e^{2z}(I_0(2z) - I_1(2z))$$

is a (exponential) generating function for the degrees of the grassmannians $G_1(\mathbb{P}^{n+1})$ $(n \geq 0)$. ∎

6.4 Catalan Traffic and Top Intersection Numbers

6.4.1 Catalan Numbers. As remarked e.g. in [53] the n^{th} *Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

is equal to the Plücker degree of the grassmannian of lines $G_1(\mathbb{P}^{n+1})$. Catalan numbers occur in several combinatorial situations[1], especially in problems of *lattice paths enumeration*.

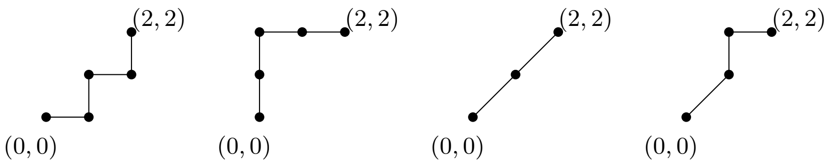


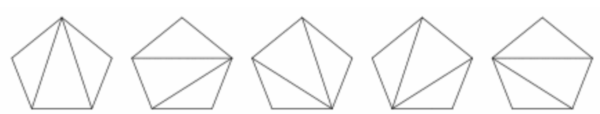
Fig. 1. Paths (1), (2), (3), (4) are given respectively by the sequence of points

$\varsigma_1 = (0,0),(1,0),(1,1),(2,1),(2,2)$, $\varsigma_2 = (0,0),(0,1),(0,2),(1,2),(2,2)$, $\varsigma_2(0,0),(1,1),(2,2)$,

$\varsigma_4 = (0,0),(1,1),(1,2),(2,2)$.

Recall that a lattice path ς of length ℓ is a finite sequence $(a_0, b_0), \ldots, (a_\ell, b_\ell)$ of points of $\mathbb{Z}^2$, where (a_0, b_0) is the *starting point*, (a_ℓ, b_ℓ) is the *end point* and a *path* reaches (a_ℓ, b_ℓ) starting from (a_0, b_0) through a sequence of *steps* obeying certain rules. Figure 1 depicts some examples of lattice paths starting from $(0, 0)$ to $(2, 2)$, where only unitary steps are allowed. Within this context, it is well

[1] Just to give an example of the many combinatorial occurrences of Catalan's, one can think of C_n as the number of different ways a convex polygon with $n+2$ sides can be decomposed into triangles, by drawing straight lines connecting vertices (the figure below depicts the case $n = 3$),



or as the number of finite sequences of $2n$ terms, such that n elements are equal to 1 and n elements are equal to -1, and the sum of the first i elements is ≥ 0, for every $1 \leq i \leq 2n$.

known (see e.g. [75]), that C_n is the number of lattice paths joining $(0,0)$ to $(x,x) \in S := \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \,|\, 0 \le x \le y\}$, allowing unitary steps only, along the x or y direction.

6.4.2 The Catalan Traffic Game. In the paper [54], Niederhausen constructed and studied the following *traffic game*. One is given of a *city map* $\mathcal{C} \subseteq \mathbb{Z}^2$ where one aims to enumerate lattice paths joining the origin to a point $(m,n) \in \mathcal{C}$. The city map is bounded by the line $m+n=0$ (the *beach*) and the traffic rules (the steps conditions) are subject to constraints, such as *gates* and *block points* (see Fig. below).

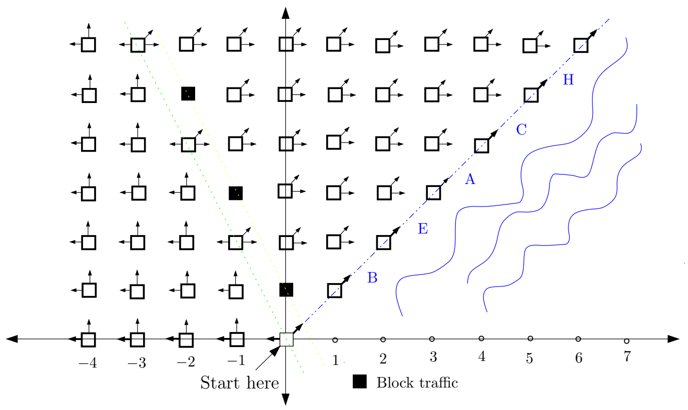


Fig. 2. The city map with the traffic rule. The beach separates the town from the sea.

A first rule of the game is that one cannot walk beyond the beach line[2]. Furthermore:

i) At lattice points strictly below the line $2m+n=0$, only North ($\uparrow$) or West ($\leftarrow$) directions are allowed;

ii) At lattice points strictly above the line $2m+n=0$, only East ($\rightarrow$) or NE ($\nearrow$) directions are allowed;

iii) Block out all traffic at the points $(m,n) \in \mathbb{Z}^2$ lying on the line $2m+n=1$ ($\blacksquare$);

iv) On the line $2m+n=0$ (gates), allow $W(\leftarrow)$, E ($\rightarrow$), and NE ($\nearrow$) (because of the road blocks at $2m+n=1$).

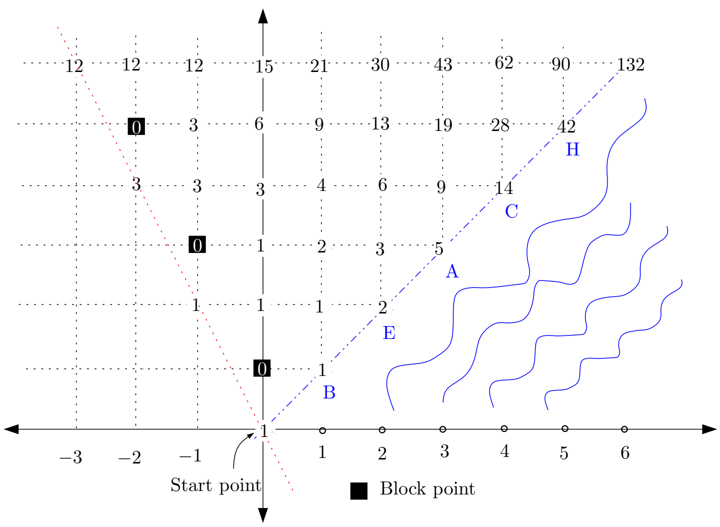
[2] Whence the title of the paper [54]: *Catalan traffic at the beach.*

The city map of Fig. 2 is precisely that occurring in [54] after a harmless counterclockwise rotation of 90 degrees.

Let $\Upsilon(m,n)$ be the number of distinct paths joining $(0,0)$ to $(m,n) \in \mathcal{C}$. The main result of [54], there proven in three different ways, is that $\Upsilon(n,n) = C_n$. One way this is proven is to show that the following recursive formula:

$$\begin{cases} \Upsilon(m+1,n) &= \Upsilon(m,n) + \Upsilon(m,n-1) \\ \Upsilon(0,0) &= 1 \\ \Upsilon(0,1) &= 0 \end{cases} \tag{6.26}$$

holds for all $(m,n) \in \mathbb{Z}^2$ such that $-n \leq 2m \leq 2n$. Using recursive formula (6.26), the city map can be completed by attaching the number $\Upsilon(m,n)$ to each $(m,n) \in \mathcal{C}$:



Since $\Upsilon(n,n)$ turns out to be the n^{th} Catalan number, it is also the degree of the grassmannian $G_1(\mathbb{P}^{n+1})$ in its Plücker embedding. This fact can be generalized. In fact the main result of this section is to prove the following:

6.4.3 Theorem. *For all* $0 \leq m \leq n$*, the number* $\Upsilon(m,n)$ *is the number* $\kappa_{2m,n-m}$ *of lines in* $\mathbb{P}^{n+1}$ *incident* $2m$ *linear subspaces of codimension* 2 *and* $n-m$ *subspaces of codimension* 3 *in general position in* $\mathbb{P}^{n+1}$.

The proof will consist in showing that the numbers $\kappa_{2m,n-m}$ enjoy the same recursion enjoyed by Υ.

6.4.4 Setup for the proof of Theorem 6.4.3. Let M be a free $\mathbb{Z}$-module rank $n+2$ spanned by $\mathcal{E} := (\epsilon^0, \epsilon^1, \ldots, \epsilon^{n+1})$. Let $D_j : M \longrightarrow M$ be D_1^j, where D_1 is the unique $\mathbb{Z}$-endomorphism of M whose matrix with respect to the basis $\mathcal{E}$ is a Jordan block of maximal rank (See Section (3.2). Let $D_t : \bigwedge M \longrightarrow \bigwedge M[[t]]$ be the $\mathcal{S}$-derivation gotten by extending $D_t := \sum_{j\geq 0} D_j t^j : M \longrightarrow M[[t]]$.

Moreover let $\overline{D}_t := \sum_{j\geq 0} \overline{D}_j t^j : \bigwedge M \longrightarrow \bigwedge M[[t]]$ the inverse of $D_t \in \mathcal{S}_t(\bigwedge M)$. By intersection theory on Grassmann varieties, the number of lines in $\mathbb{P}^{n+1}$ incident $2m$ linear subspaces of codimension 2 and $n-m$ subspaces of codimension 3 in general position in $\mathbb{P}^{n+1}$ is computed by the degree

$$\int_{G_1(\mathbb{P}^{n+1})} \sigma_1^{2m} \sigma_2^{n-m},$$

which, by virtue of the dictionary 5.0.7, is equal to the coefficient $\kappa_{2m,n-m}$ occurring in the equality:

$$D_1^{2m} D_2^{n-m}(\epsilon^0 \wedge \epsilon^1) = \kappa_{2m,n-m} \cdot \epsilon^n \wedge \epsilon^{n+1} \tag{6.27}$$

6.4.5 Proof of Theorem 6.4.3. For commodity, we set $K(m,n) := \kappa_{2m,n-m}$. Clearly $K(0,0) = 1 = \Upsilon(0,0)$ and $K(0,1) = 0 = \Upsilon(0,1)$. In fact

$$D_1^0(\epsilon^0 \wedge \epsilon^1) = 1 \cdot \epsilon^0 \wedge \epsilon^1 \qquad \textit{and} \qquad D_2(\epsilon^0 \wedge \epsilon^1) = 0.$$

Recall that $\overline{D}_2 = D_1^2 - D_2$. Now, using 6.27 and by definition of $K(m,n)$, one has

$$\begin{aligned} D_1^{2m} D_2^{n-m}(\epsilon^0 \wedge \epsilon^1) &= D_1^{2m} D_2^{n-m-1}(D_1^2 - \overline{D}_2)(\epsilon^0 \wedge \epsilon^1) = \\ &= (D_1^{2m+2} D_2^{n-m-1} - D_1^{2m} D_2^{n-m-1} \overline{D}_2)(\epsilon^0 \wedge \epsilon^1) = \\ &= D_1^{2m+2} D_2^{n-m-1}(\epsilon^0 \wedge \epsilon^1) - D_1^{2m} D_2^{n-m-1} \overline{D}_2(\epsilon^0 \wedge \epsilon^1) \end{aligned} \tag{6.28}$$

Now, on the r.h.s of formula (6.28), the former summand is precisely $K(m+1,n)\epsilon^n \wedge \epsilon^{n+1}$ while the latter, is equal to $D_1^{2m} D_2^{n-m-1}(\epsilon^1 \wedge \epsilon^2)$. This last expression can be read in $\bigwedge^2 M'$, where M' is a free $\mathbb{Z}$-module of rank $n+1$, generated by

$(\epsilon^1, \ldots, \epsilon^{n+1})$, and, therefore, is equal to $\kappa_{2m,n-m-1}\epsilon^n \wedge \epsilon^{n+1}$. Hence, keeping in mind that $\kappa_{2m,n-m-1} = K(m, n-1)$, one has, using (6.28):

$$K(m,n) \cdot \epsilon^n \wedge \epsilon^{n+1} = (K(m+1,n) - K(m,n-1)) \cdot \epsilon^n \wedge \epsilon^{n+1}.$$

As a conclusion

$$K(m+1,n) = K(m,n) + K(m,n-1), \qquad (6.29)$$

which is the sought for recursive formula (compare with formula (6.26)), which together with the initial conditions proves that $\Upsilon(m,n) = K(m,n)$, as claimed. ■

6.4.6 Then,

$$\kappa_{2m,n-m} = \sum_{b=0}^{n-m}\sum_{a=0}^{2m}\binom{2m}{a}\binom{n-m+1}{b,\ n-a-2b+1,\ b+a-m}\frac{m+n-2a-3b+1}{n-m+1}, \forall\ n \geq m \geq 0,$$

gives the way to arrive in the point (m, n) in the described city map starting from $(0, 0)$.

6.5 Bessel Functions and Catalan Traffic

In Chapter 6.3 we proved that:

$$F(z) = e^{2z}(I_0(2z) - I_1(2z))$$

is a (exponential) generating function for the degrees of the grassmannians $G_1(\mathbb{P}^{n+1})$ $(n \geq 0)$.

Then, using recursive relations 6.29, we have that:

6.5.1 Theorem.

$$H_k(z) = \sum_{h=0}^{k}(-1)^h\binom{k}{h}F^{(k-h)}(z), \quad 0 \leq k \leq n. \qquad (6.30)$$

is a generating function for the integrals in the grassmannians $G_2(\mathbb{C}^{n+2})$, i.e.

$$\kappa_{2(n-k),k} = H_k^{(n-k)}(0), \;\; 0 \leq k \leq n.$$

Proof. The proof is by induction on the integer k. If $k = 0$, we have:

$$H_0(z) = F(z).$$

Let us suppose that the formula is true for all $k > 1$:

$$H_{k-1}(z) = \sum_{h=0}^{k-1} (-1)^h \binom{k-1}{h} F^{(k-1-h)}(z).$$

Then,

$$\begin{aligned}
H_k(z) &= H'_{k-1}(z) - H_{k-1}(z) \\
&= \left(\sum_{h=0}^{k-1} (-1)^h \binom{k-1}{h} F^{(k-1-h)}(z) \right)' - \sum_{h=0}^{k-1} (-1)^h \binom{k-1}{h} F^{(k-1-h)}(z) \\
&= \sum_{h=0}^{k-1} (-1)^h \binom{k-1}{h} F^{(k-h)}(z) - \sum_{h=0}^{k-1} (-1)^h \binom{k-1}{h} F^{(k-1-h)}(z) \\
&= F^{(k)} + \sum_{h=1}^{k-1} (-1)^h \binom{k-1}{h} F^{(k-h)}(z) - \sum_{h=0}^{k-2} (-1)^h \binom{k-1}{h} F^{(k-1-h)}(z) + (-1)^{k-1} F \\
&= F^{(k)} + \sum_{r=0}^{k-2} (-1)^{r+1} \binom{k-1}{r+1} F^{(k-r-1)}(z) + \sum_{h=0}^{k-2} (-1)^{h+1} \binom{k-1}{h} F^{(k-1-h)}(z) + (-1)^{k-1} F \\
&= F^{(k)} + \sum_{r=0}^{k-2} (-1)^{r+1} \left(\binom{k-1}{r+1} + \binom{k-1}{r} \right) F^{(k-r-1)}(z) + (-1)^{k-1} F \\
&= F^{(k)} + \sum_{r=0}^{k-2} (-1)^{r+1} \binom{k}{r} F^{(k-r-1)}(z) + (-1)^{k-1} F \\
&= \sum_{h=0}^{k} (-1)^h \binom{k}{h} F^{(k-h)}(z) \qquad \blacksquare
\end{aligned}$$

6.5.2 In particular, since

$$\frac{dI_\mu(z)}{dz} = I'_\mu(z) = \frac{1}{2}[I_{\mu-1}(z) + I_{\mu+1}(z)],$$

$H_1(z) = e^{2z}(I_1(2z) - I_2(2z))$ is a (exponential) generating function for $(\kappa_{2(n-1),1})$, the number of planes incident "$2n-2$" linear subspaces of codimension 2 and "1" linear subspaces of codimension 3 in general position in $\mathbb{P}^{n+1}$. In fact, by the theorem:

$$\begin{aligned} H_1(z) &= F'(z) - F(z), m \geq 1 \\ &= 2e^{2z}(I_0(2z) - I_1(2z)) + e^{2z}[2I_1(2z) - I_0(2z) - I_2(2z)] - \\ &\quad -e^{2z}(I_0(2z) - I_1(2z)) = e^{2z}(I_1(2z) - I_2(2z)). \end{aligned}$$

In the same way, one sees that the generating function for $(\kappa_{2(n-2),2})$ is $H_2(z) = e^{2z}(I_0(2z) - I_3(2z))$:

$$\begin{aligned} H_2(z) &= F''(z) - 2F'(z) + F(z), m \geq 2 \\ &= H_1'(z) - H_1(z) = \\ &= e^{2z}(I_0(2z) - I_3(2z)). \end{aligned}$$

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