Polyhedral Combinatorics (ADM III)

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Diese Vorlesung wird im Rahmen der Berlin Mathematical School angeboten und auf Englisch gehalten.

General description

Polyhedral combinatorics can be viewed as a technique that uses methods from polyhedral theory and linear algebra in order to solve combinatorial problems. The main idea is to transform a combinatorial problem into a polyhedral problem by, for instance, considering the convex hull of the incidence vectors of the feasible solutions of the combinatorial problem, and to employ techniques from linear and integer programming in order to solve the combinatorial problem. At the end of this class the students will be able to handle this methodology, apply it to practically relevant cases, and prove important results. The focus will be on the solution of NP-hard combinatorial optimization problems.

More specific description

Important combinatorial problems that will be addressed include the travelling salesman, the max-cut, the Steiner tree, the matroid optimization, the assignment, the linear ordering, the stable set (including perfect graphs and the theta-body), and the matching problem. We will, for example, study the facet structure of some of the polytopes associated with these problems and show how to employ cutting plane techniques for the solution of the corresponding optimization problems. Instances from the real-world will illustrate the range of problems that come up and can be solved in practice.

Literature:

The following list of books contains background material for the class. Specific literature will be referenced in the notes below.

- (**ADMII**) Martin Grötschel, *Lineare und Ganzzahlige Programmierung* (Algorithmische Diskrete Mathematik II), Skriptum zur Vorlesung im WS 2009/2010, see http://www.zib.de/groetschel/teaching/WS0910/skriptADMII-WS0910.pdf

These lecture notes provide the basic knowledge expected from students attending the course and can be downloaded from the Web site.

- (**GLS**) Martin Grötschel, László Lovász, Alexander Schrijver, *Geometric Algorithms and Combinatorial Optimization*, Springer, 1988, second edition 1993. The book can be downloaded from http://www.zib.de/groetschel/pubnew/paper/groetschellovaszschrijver1988.pdf

This book describes the ellipsoid method (and many other algorithms) and the consequences of these algorithms for combinatorial optimization. It has a strong polyhedral flavor. Various parts of this book will be presented in the class.

- (**ASCO**) Alexander Schrijver, *Combinatorial Optimization - Polyhedra and Efficiency*, Springer, 2002.

This is THE book on (actually I would say the bible of) polyhedral combinatorics. In three volumes everything known in this area is covered with extreme detail and concise proofs.

- (**ASLP**) Alexander Schrijver, *Theory of Linear and Integer Programming*, Wiley, 1986.

This book covers everything on linear and integer programming known up to 1986. It is another outstanding and valuable reference.

- (**GZ**) Günter M. Ziegler, *Lectures on Polytopes*, Springer-Verlag, Revised Edition, 1998.

This book is a general reference on polyhedral theory providing a lot of background material on polyhedra not covered in the class.

Content

Due to insufficient time, chapters 11 and 12 would not be presented in this class.

Chapter 1: Introduction and Examples

1.1 Based on the lecture notes (ADMII) a brief introduction into polyhedral theory (chapters 2, 6-8) and on linear programming is given. Relevant results on polyhedra, polytopes, faces, vertices, extreme rays, facets, recession cones, integral polyhedral, total unimodularity, etc. will be repeated throughout the course whenever necessary.

1.2 An overview about questions asked (and answered) in *extremal combinatorics* is given. One of the earliest results in this area is Sperner's Lemma (sometimes also called Sperner's Theorem). It states that the maximum cardinality of an antichain (which is a family $\mathcal{A}$ of subsets of a set so that no two elements of $\mathcal{A}$ contain each other) of a set *E* of *n* elements is $m := \binom{n}{k} = \frac{n!}{k!(n-k)!}$, where $k := \left\lfloor \frac{n}{2} \right\rfloor$ is the round-down of $\frac{n}{2}$. Lubell, see D. Lubell, A short proof of Sperner's Lemma, *Journal of Combinatorial Theory* 1 (1966) 299. has given a proof of this result that can be interpreted as showing the validity of the inequality $\sum_{S \subseteq E} x_S \le m$ for the polytope which is defined as the convex hull of all incidence vectors of antichains $\mathcal{A}$ of *E* $P_{AC}(n) := conv\{\chi^A \in \mathbb{R}^{P(E)} \mid \mathcal{A}$ antichain in *E*}, where P(*E*) denotes the set of all subsets of *E* (power set of *E*).

1.3 A brief survey of issues in *extremal graph theory* is provided, some simple results such as "What is the largest number of edges in a planar or a bipartite graph?" are proved. A generalization of the last observation is Turán's Theorem, one of the "model results" of extremal graph theory. It states that a graph *G = (V,E)* on *n* nodes without any *k*-clique has no more than $\frac{k-2}{2k-2} n^2$ edges.

First, an attempt is shown to prove Turán's theorem by copying the ideas of Lubell's proof of the Sperner Lemma. My attempt to make this work failed, and I show where things go wrong. Thereafter, I present an adaptation of the proof of Motzkin and Straus, see T.S. Motzkin and E. G. Straus, Maxima for graphs and a new proof of a theorem of Turan, *Canadian Journal of Mathematics*, 17(1965) 533-540, that employs the idea of maximizing the quadratic function $2\sum_{ij \in E} x_i x_j$ over a simplex.

1.4 The Galai identities $\alpha(G) + \tau(G) = |V| = \nu(G) + \rho(G)$

relating the *matching* $\nu(G)$ and the *edge covering number* $\rho(G)$ and the *stability* $\alpha(G)$ and the *node covering number* $\tau(G)$ to the number of nodes of a graph G, are introduced as well as *König's matching* and *edge covering theorems for bipartite graphs*, see (GLS), pages 229-232.

Then the question of finding a *node cover* (also called *blocking set*) of minimum cardinality in a hypergraph $\mathcal{H}$ is introduced. An analysis of the greedy algorithm for this problem is provided that is based on a linear programming relaxation of the *hypergraph blocking set problem*. The result, independently shown by Johnson (1974), Stein (1974) and Lovász (1975), states that the size of a greedy blocking set is not larger than $(1+\ln(\Delta))$ times the minimum size $\tau(\mathcal{H})$ of a blocking set, where $\triangle$ is the largest degree of a node in $\mathcal{H}$. The "linear programming proof" of this result provides an even better bound, namely, the minimum size $\tau(\mathcal{H})$ of a blocking set can be replaced by the optimum value $\tau^*(\mathcal{H})$ of its natural LP relaxation, see, e.g. (ASCO), p. 1380-1381 or Martin Grötschel, Lászlo Lovász, Combinatorial Optimization, in Ronald L. Graham, Martin Grötschel, Lászlo Lovász (eds.), *Handbook of Combinatorics*, Volume II, Elsevier (North-Holland), 1995, 1541-1597, which can be downloaded from:

http://www.zib.de/groetschel/pubnew/paper/groetschellovasz1995.pdf

1.5 Integer polyhedra
In this class we will mainly deal with polyhedra that arise as the convex hull of a finite number of integral vectors. These are typically the incidence vectors of certain combinatorial objects such as spanning trees, Hamiltonian cycles, or stable sets in a graph. Let us define such polyhedra in more general terms.

A polyhedron P is called an *integer polyhedron* if it is the convex hull of the integer vectors contained in P. This is equivalent to: P is rational (i.e., P can be described by an inequality system with rational coefficients only) and each face of P contains an integer vector. This immediately implies that a polytope P is integer if and only if each vertex of P is integer. If a polyhedron $P = \{x \mid Ax \le b\}$ is integer, then the linear programming problem

$$\max\{c^T x \mid Ax \le b\}$$

has an integer optimum solution, in case it has a finite optimum solution at all. Hence, in this case,

$$\max\{c^T x \mid Ax \le b;\ x \text{ integer}\} = \max\{c^T x \mid Ax \le b\}.$$

This, in fact, characterizes integer polyhedral.

Theorem. *Let P be a rational polyhedron. Then P is integer if and only if, for each rational vector c, the linear programming problem* $\max\{c^T x \mid Ax \le b\}$ *has an integer optimum solution if it is finite.*

A stronger characterization is due to Edmonds and Giles [1977]:

Theorem. *A rational polyhedron P is integer if and only if for each integral vector c the value of* $\max\{c^T x \mid x \in P\}$ *is an integer if it is finite.*

A *0/1-polytope* is a polytope with all vertices being 0/1-vectors.

Chapter 2: The Travelling Salesman Problem: An Introduction into Research in Polyhedral Combinatorics

The presentation follows closely the paper:

Martin Grötschel, Manfred W. Padberg, Polyhedral Theory, in Eugene L. Lawler, Jan Karel Lenstra, A. H. G. Rinnooy Kan, David B. Shmoys (eds.), *The Traveling Salesman Problem. A Guided Tour of Combinatorial Optimization*, Wiley, 1985, 251-306,
where various polytopes related to the symmetric and asymmetric travelling salesman problem are introduced. Several proofs are given (for instance two proofs that determine the dimension of the symmetric travelling salesman polytope) with the intention to illustrate various proof techniques in polyhedral combinatorics. The lectures on this topic also include a survey of the paper:

Manfred W. Padberg, Martin Grötschel, Polyhedral Computations, in Eugene L. Lawler, Jan Karel Lenstra, A. H. G. Rinnooy Kan, David B. Shmoys (eds.), *The Traveling Salesman Problem. A Guided Tour of Combinatorial Optimization*, Wiley, 1985, 307-360,

where the basics of separation and cutting plane algorithms are described. The two papers mentioned can be downloaded from:

http://www.zib.de/groetschel/pubnew/paper/groetschelpadberg1985.pdf

http://www.zib.de/groetschel/pubnew/paper/padberggroetschel1985.pdf

The lecture on polyhedral aspects of the TSP was finished with a Power Point presentation summarizing the results and surveying additional material, URL:

http://www.zib.de/groetschel/teaching/SS2010/100518ADMIII-Vorlesung-TSP-Survey.ppt

Chapter 3: The Steiner Tree Problem

This lecture introduces and compares several integer programming formulations for the *undirected Steiner tree problem* in graphs. The presentation starts with the classical *undirected cut formulation*, which can be strengthened by a facet-defining class of *Steiner partition inequalities*. These are *NP*-hard to separate. Surprisingly, an associated *directed cut formulation*, which is based on directed Steiner cuts w.r.t. some arbitrarily chosen root terminal, is stronger than the undirected formulation including all Steiner partition inequalities. Since the LP-relaxation of the directed formulation can be solved in polynomial time, we have the surprising result that a class of inequalities that subsumes the Steiner partition inequalities can be separated in polynomial time. The directed cut formulation is equivalent to a third formulation, the *flow formulation* that sends one unit of flow to each non-root terminal. This formulation can be strengthened by a simple class of flow balance constraints, that foreclose flow rejoins for every commodity. Extending this idea to prevent flow rejoins for sets of commodities leads to the *common flow formulation*, which is currently the strongest known formulation for the undirected Steiner tree problem. In fact, there is no instance known, for which this formulation has a duality gap. The lecture covers material from the following publications:

S. Chopra & M. R. Rao, The Steiner tree problem I: Formulations, compositions and extension of facets, *Mathematical Programming* 64(1-3) 209-229, 1994, URL:

http://www.springerlink.com/content/mt4703015h9j8334/?p=37d4917b629f45a3ac6ebd8c1a36a7f7&pi=9

S. Chopra & M. R. Rao, The Steiner tree problem II: Properties and classes of facets, Mathematical Programming 64(1-3), 231-246, 1994,
URL: http://www.springerlink.com/content/j812n3p7793x387x/

T. Polzin, *Algorithms for the Steiner problem in networks*, Ph.D. Thesis, U Saarbrücken, 2003, URL:

http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.16.1056&rep=rep1&type=pdf

Chapter 4: Polyhedra Related to Matroids and Independence Systems

Most of the results presented in this chapter can be found in Chapters 40 and 41 (pages 688 – 724) of (ASCO) and in a somewhat more condensed form in Section 7.5 (pages 210 – 218) of (GLS). The Power Point presentation: ”Independence Systems, Matroids, the Greedy Algorithm, and related Polyhedra" summarizes the material, URL:

http://www.zib.de/groetschel/teaching/SS2010/100601ADMIII-Vorlesung-MatroidPolyhedra.ppt

Chapter 5: Path and Flow Polyhedra, Total Unimodularity

This chapter introduces polyhedra associated with paths and flows in graphs and digraphs and characterizes total unimodularity. The content is a concise overview of some of the polyhedral results presented in Chapter 13 (pages 198-216) of (ASCO).

Chapter 6: Cardinality Homogeneous Set Systems

A subset $\mathcal{C}$ of the power set of a finite set *E* is called *cardinality homogeneous* if, whenever $\mathcal{C}$ contains some set *F*, $\mathcal{C}$ contains all subsets of *E* of cardinality |*F*|. Examples of such set systems $\mathcal{C}$ are the sets of all even or of all odd cardinality subsets of *E*, or, for each uniform matroid, its set of circuits and its set of cycles. With each cardinality homogeneous set system $\mathcal{C}$, we associate the polytope $P(\mathcal{C})$, the convex hull of the incidence vectors of all sets in $\mathcal{C}$. We provide a complete and non-redundant linear description of $P(\mathcal{C})$. We show that a greedy algorithm optimizes any linear function over $P(\mathcal{C})$, we construct, by a dual greedy procedure, an explicit optimum solution of the dual linear program; and we describe a polynomial time separation algorithm for the class of polytopes of type *P(C).*

This chapter is based on the paper: Martin Grötschel, Cardinality Homogeneous Set Systems, Cycles in Matroids, and Associated Polytopes, in Martin Grötschel (ed.), *The Sharpest Cut: The Impact of Manfred Padberg and His Work*, MPS-SIAM, 2004, 99-120, which can be downloaded from

http://www.zib.de/groetschel/pubnew/paper/groetschel2004b.pdf

The results can be generalized in various ways, some of the extensions contained in the PhD Thesis: Rüdiger Stephan, *Polyhedral Aspects of Cardinality Constrained Combinatorial Optimization Problems,* TU Berlin, 2009, downloadable from URL:

http://opus.kobv.de/tuberlin/volltexte/2009/2401/pdf/stephan_ruediger.pdf

will be surveyed. The Power Point presentation contains some of the material discussed, see URL:

http://www.zib.de/groetschel/teaching/SS2010/100608ADMIII-VorlesungCardHomSetSystems.ppt .

Chapter 7: Matching Polyhedra

The polyhedral theory of the various versions of the matching problem is extensively described in Parts II and III of Volume A of (ASCO). In this chapter we give a proof that, for a graph $G=(V,E)$, the system of equations and inequalities

$$x(\delta(v))=1 \qquad \forall\ v\in V$$

$$x(\delta(W))\geq 1 \qquad \forall\ W\subseteq V, |W|\ odd$$

$$x_e\geq 0 \qquad \forall\ e\in E$$

provides a complete description of the *perfect matching polytope*,

$$P(G):=conv\{\chi^M\in\mathbb{R}^E \mid M\subseteq E\ \ perfect\ matching\}$$

i.e., the convex hull of all incidence vectors of perfect matchings in G. We derive several consequences by “transformation tricks” providing complete descriptions of polyhedra related to *c*-capacitated b-matchings and other matching problems.

Chapter 8: Acyclic Subgraphs and Linear Ordering

This chapter introduces the linear ordering and the acyclic subgraph problem and some of the applications of these problems as well as the polyhedral theory developed for these problems. The polyhedral aspects can be found in the papers:

Martin Grötschel, Michael Jünger, Gerhard Reinelt, On the acyclic subgraph polytopes, *Mathematical Programming*, 33:1 (1985) 28-42 and Martin Grötschel, Michael Jünger, Gerhard Reinelt, Facets of the linear ordering polytope, *Mathematical Programming*, 33:1 (1985) 43-60, downloadable from

http://www.zib.de/groetschel/pubnew/paper/groetscheljuengerreinelt1985b.pdf

http://www.zib.de/groetschel/pubnew/paper/groetscheljuengerreinelt1985c.pdf

Chapter 9: The Max-Cut Problem

The max-cut problem is the task to find, in a graph with edge weights, a cut of maximum weight. This problem has interesting applications and generalizations. We describe the polyhedral approach to this problem following the papers:

F. Barahona and A. R. Mahjoub, On the cut polytope, *Mathematical Programming* 36 (1986) 157-173,

Francisco Barahona, Martin Grötschel, Ali Ridha Mahjoub, Facets of the Bipartite Subgraph Polytope, *Mathematics of Operations Research*, 10 (1985) 340-358

http://www.zib.de/groetschel/pubnew/paper/barahonagroetschelmahjoub1985.pdf

Francisco Barahona, Martin Grötschel, Michael Jünger, Gerhard Reinelt, An application of combinatorial optimization to statistical physics and circuit layout design, *Operations Research*, 36:3 (1988) 493-513

http://www.zib.de/groetschel/pubnew/paper/barahonagroetscheljuengeretal1988.pdf

Chapter 10: Stable Sets, Theta Bodies, and Perfect Graphs

This chapter is a condensed presentation of chapter 9 of (GLS).

For the "Strong Perfect Graph Theorem" see the Web page:

http://users.encs.concordia.ca/~chvatal/perfect/spgt.html

provided by V. Chvátal.

Chapter 11: Cycles in Binary Matroids

Cycles are disjoint unions of circuits of a matroid. The Chinese postman problem and the max-cut problem are special cases of the task to find a maximum weight cycle in a binary matroid. The presentation follows the papers:

Francisco Barahona, Martin Grötschel, On the Cycle Polytope of a Binary Matroid, *Journal of Combinatorial Theory, Series B*, 40 (1986) 40-62, see:

http://www.zib.de/groetschel/pubnew/paper/barahonagroetschel1986.pdf

Martin Grötschel, Klaus Truemper Decomposition and Optimization over Cycles in Binary Matroids, *Journal of Combinatorial Theory, Series B*, 46:3 (1989) 306-337

http://www.zib.de/groetschel/pubnew/paper/groetscheltruemper1989a.pdf

Martin Grötschel, Klaus Truemper, Master Polytopes for Cycles of Binary Matroids, *Linear Algebra and its Applications*, 114/115 (1989) 523-540

http://www.zib.de/groetschel/pubnew/paper/groetscheltruemper1989b.pdf

Chapter 12: Proof Techniques in Polyhedral Combinatorics

If there is time left we summarize all the proof techniques presented in this class.