



# PORTFOLIO OPTIMIZATION

## A GENERAL FRAMEWORK FOR PORTFOLIO CHOICE



It is widely accepted among investment professionals that, while portfolio optimization has compelling theoretical merit, it is not useful in practice. Practitioners are concerned that optimization is an “error maximizing”<sup>1</sup> process fraught with insurmountable estimation issues. The abstract of an early academic critique of mean-variance optimization, (Michaud 1989) states:

The indifference of many investment practitioners to mean-variance optimization technology, despite its theoretical appeal, is understandable in many cases. The major problem with mean-variance optimization is its tendency to maximize the effects of errors in the input assumptions. Unconstrained mean-variance optimization can yield results that are inferior to those of simple equal-weighting schemes.

Many nay-sayers selectively quote the above passage as reason to dismiss optimization altogether. However, this same abstract continues with the following thoughts:

Nevertheless, mean-variance optimization is superior to many ad hoc techniques in terms of integration of portfolio objectives with client constraints and efficient use of information. Its practical value may be enhanced by the sophisticated adjustment of inputs and the imposition of constraints based on fundamental investment considerations and the importance of priors. The operating principle should be that, to the extent that reliable information is available, it should be included as part of the definition of the optimization procedure.

(Chopra and Ziemba 1993) compared the performance of a portfolio formed on historical data using mean-variance optimization to a portfolio formed with perfect information of the future. They found that historical data was quite useful in estimating volatilities and covariances. The performance shortfall from using mean-variance optimization was almost entirely due to errors in estimates of the mean. They suggested that portfolio optimization with constrained means, such as Minimum Variance, would solve this issue quite effectively.

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<sup>1</sup> Michaud (1989)

Some prominent practitioners have offered compelling defenses of optimization. (Kritzman 2014) makes an especially strong empirical and analytical case, suggesting that:

Cynics often refer to mean-variance optimizers as error maximizers because they believe that small input errors lead to large output errors. This cynicism arises from a misunderstanding of sensitivity to inputs. Consider optimization among assets that have similar expected returns and risk. Errors in the estimates of these values may substantially misstate optimal allocations. Despite these misallocations, however, the return distributions of the correct and incorrect portfolios will likely be quite similar. Therefore, the errors do not matter because the resultant incorrect portfolio is nearly as good as the correct portfolio.

Now consider optimization among assets that have significantly dissimilar expected returns and risk. Errors in these estimates will have little impact on optimal allocations; hence again the return distributions of the correct and incorrect portfolios will not differ much. There may be some cases in which small input errors matter, but in most cases sensitivity to estimation error is more hype than reality.

It's clear that portfolio optimization is a powerful tool that must be used thoughtfully and responsibly. However, even the critics agree that optimization is the only mechanism to make best use of all the information available to investors.

Despite this consensus, many investors default to naive methods of portfolio construction, which typically take three forms:

- Conviction weight - investments are held in proportion to the portfolio manager's conviction in the investments' relative prospects
- Market cap weight - investments are held in proportion to their market capitalizations
- Equal weight - investments are held in equal weight

What is not well known - or at least rarely discussed - among practitioners is that each of these heuristic portfolio choices implies strong views about the nature of the investments under consideration.

If we acknowledge that most investors are seeking to achieve the highest returns subject to a maximum risk threshold, portfolios constructed using the above heuristics express strong views about market inefficiency. In fact, we are confident that many practitioners would be uncomfortable with the extreme implications expressed by these portfolios.

For example, managers who form portfolios on capitalization weights are expressing the strict view that the chosen investments should produce returns that are a linear function of their beta to the portfolio. In other words, that the investments all have equal Treynor ratios<sup>2</sup>. While this assumption is consistent with CAPM, it is grossly inconsistent with the empirical evidence, as we shall see below.

Equal weight portfolios express a view that market returns are entirely random, so that nothing can be known about the investments' relative risks, correlations, or expected returns. Additionally, an equal weight portfolio is only optimal when the expected returns on the underlying investments have no relationship with their respective risks whatsoever, or in the unlikely case that all assets have equal expected returns, volatilities, and homogeneous correlations.

For reasons to be discussed below, it may indeed be reasonable to assume that the returns to investments drawn from certain universes are equal, regardless of risk. However, for the vast majority of cases it is naive in the extreme to assert that there is no information or edge to be gleaned by attempts to forecast volatility and/or correlation. The literature on forecasting these variables is vast, and the benefits are widely acknowledged by practitioners and academics alike.

Conviction weighting expresses strong active views on relative returns. We will discuss robust ways to incorporate active views on returns in a future article.

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<sup>2</sup> The Treynor ratio is excess returns divided by beta or  $\frac{E(\mu_i - Rf)}{\beta_{i,P}}$ .



The most surprising revelation, which this paper will explore in excruciating detail, is that for each of the heuristic methods above there is an advanced portfolio optimization approach that is likely to be more efficient; more consistent with broadly held views about the relationships between risk and return; and more optimal out of sample than their naive counterparts.

## ARTICLE STRUCTURE

In this article, we will first build a theoretical framework that will enable us to determine the most appropriate method of portfolio construction for most situations. We'll introduce the Portfolio Optimization Machine™ and suggest how an investor might decide which type of optimization is most consistent with the qualities, beliefs, and assumptions he holds about the assets under consideration.

Our primary focus will be on ways investors can make best use of opportunities for diversification, without relying on active return estimates. We will discover that it is possible to form optimally diversified portfolios based exclusively on intuitive relationships between risk and return, which obviates the need for independent return estimates altogether.

While financial theory is replete with well-meaning theories describing various intuitive relationships between risk and return, we will show that many of these theories do not hold up to empirical scrutiny. In many cases, there is a tenuous or even inverse relationship between risk and return. We'll describe optimal ways to form portfolios that are consistent with the evidence.

A natural question in relation to optimization methods is how they might improve on the performance of other types of index strategies, such as “Fundamental Index” and “Smart Beta” strategies. We introduce practical solutions incorporating risk-based views that we will revisit in greater detail later.

Skepticism about the usefulness of optimization methods is largely fueled by a few seminal papers that seem to favor naive optimization. We will investigate and attempt to replicate the results

of a frequently cited paper, namely “Optimal Versus Naive Diversification: How Inefficient is the 1/N Portfolio Strategy?” (DeMiguel, Garlappi, and Uppal 2007). As the title suggests, the paper describes the results of empirical tests of various portfolio optimization methods on several datasets to compare the performance of optimal versus naive approaches. They run tests of different portfolio construction methods on U.S. industries, sectors, developed market country equity indices, and U.S. equity factor portfolios.

Unfortunately, the paper suffers in a few key areas. We’ll demonstrate why we might expect optimization methods to disappoint when applied to these universes, *ex ante*, and run some tests to confirm their results. Spoiler alert: despite low expectations for optimization methods due to theoretical reasons, the observed performance for many of these methods rival the results from naive approaches.

In later articles in this series, we’ll examine the major challenges that sometimes cause portfolio optimization methods to underperform naive methods. Using futures trend following as a case study, we’ll show how a thoughtful application of basic principles of finance, coupled with creative solutions to overcome certain technical hurdles, can unleash the true potential of factor strategies. We show that combining thoughtful optimization with reliable factor signals - which we call “Adaptive Asset Allocation” - can produce results that dominate more naive approaches, with fewer assumptions.

These results generalize to virtually any problem of portfolio choice.

## THE OPTIMIZATION MACHINE

Whether you are conscious of it or not, the choices you make about portfolio construction express views about the properties of the asset universe under consideration. This is true whether you choose to allocate to investments based on naive methods, such as capitalization weights or equal weights; apply heuristic methods like inverse volatility or variance; or deploy full-scale portfolio optimization.

**This fact is inescapable. The only question is whether you want to express these views consciously or unconsciously.**

To facilitate this discussion, we must first set some fundamental assumptions. First, we assume that investors have preferences that are well captured by mean-variance utility. That is, investors prefer to own the portfolio that maximizes their expected return subject to a maximum tolerable portfolio volatility. We acknowledge that many investors care less about volatility than risks like “permanent loss of capital”, “maximum drawdown”, or “expected shortfall”. However, these alternative definitions of risk are well captured by mean and variance. Consider that, when a systematic strategy is regularly rebalanced, a large expected mean relative to volatility strongly implies a smaller risk of permanent loss, a smaller expected maximum drawdown, and a smaller expected shortfall.

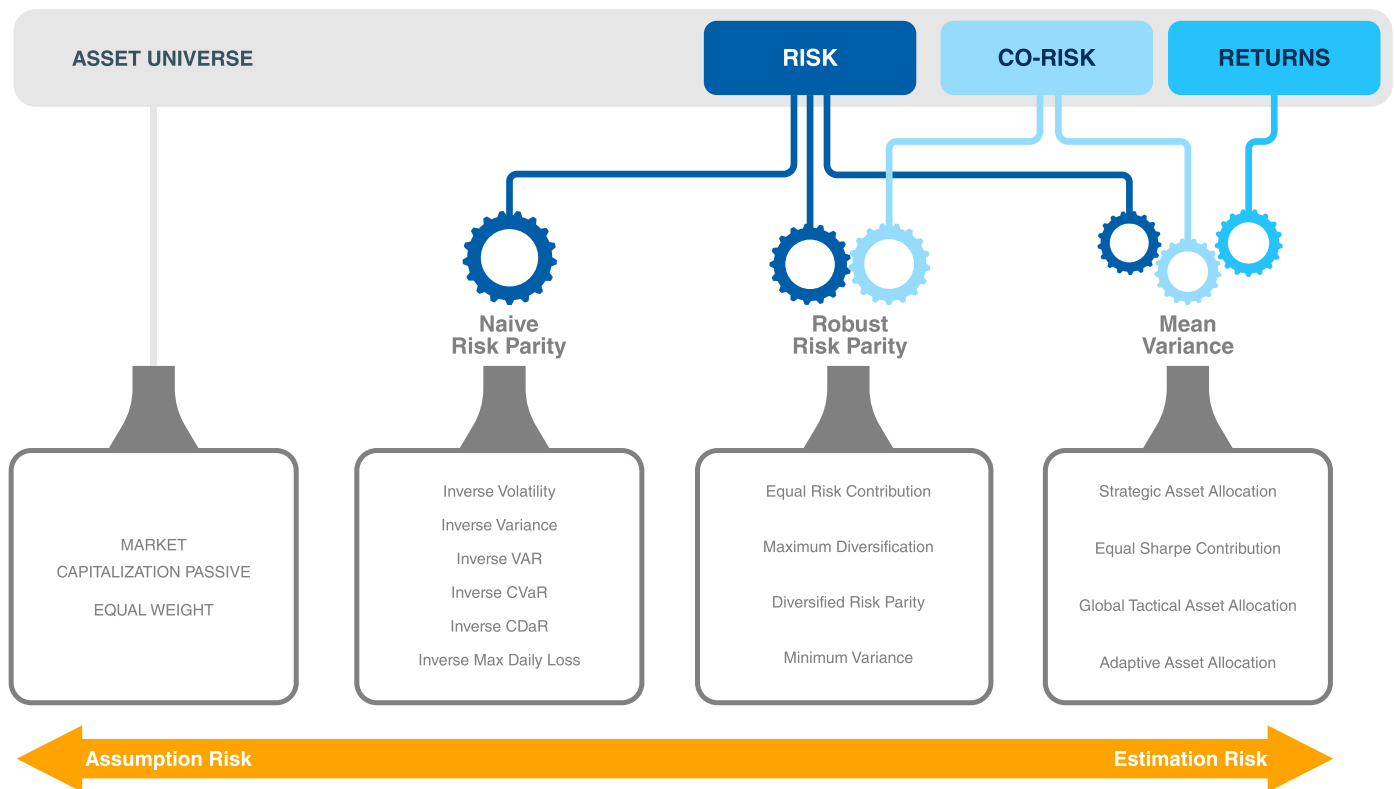
Some investors might raise the objection that certain strategies, especially those that are definitionally “short volatility” like high-yield credit, option selling, and merger arbitrage exhibit highly asymmetric return distributions with extreme negative skew. These are well-known exceptions to the rule. Remember that optimization under mean-variance utility does not require that the underlying investments be normally distributed. Rather, it requires that the returns to investments be reasonably well described by the first two moments (i.e. mean and variance) of their return distributions. Entire domains of research are devoted to the analysis of distributions of security prices, but from a practical perspective we can reasonably claim this is true for diversified portfolios of stocks, bonds, commodities, and currencies.

In addition to assuming mean-variance utility, we assume that investors are not bound by leverage constraints. This is not strictly necessary, but it simplifies the discussion, because the mean-variance optimal portfolio will always be the one that maximizes the portfolio’s Sharpe ratio. Under leverage constraints, investors will usually be forced to compromise on diversification in order to achieve higher returns.

Given that the Optimization Machine is designed to facilitate optimal portfolio choice for assets with positive expectancy, we assume all portfolios are formed with long-only constraints ( $w > 0$ ). In addition, we constrain all portfolios to weights that sum to 1 ( $\sum_i^N = 1$ ).

Subject to these conditions, Figure 1 illustrates how our beliefs about the nature of the investments under consideration inform which optimization is most likely to produce a mean-variance efficient portfolio.

Figure 1: The Portfolio Optimization Machine.



Source: ReSolve Asset Management. For illustrative purposes only.



It is clear from Figure 1 that portfolio choice is subject to three estimates about the investments being considered for allocation: returns, risk, and co-movement. For simplicity, we will define risk as volatility and co-movement as Pearson correlation of returns, but there are many valid ways for these concepts to be expressed. For example, investors might feel implied duration of cash-flows, or “margin of safety” better reflect their own perception of risk, and/or choose to focus on the correlation of cash flows rather than returns, etc.

The choice of which optimization is most likely to approach mean-variance efficiency is thus subject to each investor’s assumptions and active views about the investments. For example, when an investor has no active views about the relative volatility, correlations, or returns of a group of investments, their choice is limited to equal weighting, or weighting by market capitalization.

There are situations where investors may legitimately feel they have no information about their investments. However, many investors believe they can estimate, with better than random precision, a subset of the variables that would allow the construction of more optimal portfolios. When paired with estimates of volatilities and/or correlations, risk based optimizations may express reliable relationships between risk and return to inform portfolio choice. This obviates the need for investors to express active views on returns based on, for example, macroeconomic variables or the historical sample, which are known to be noisy and highly flawed. Instead of requiring an independent return estimate for each investment, risk based optimizations simply require a declaration of the dominant relationship between risk and return.

Of course, some investors do believe they can distinguish between securities on the basis of returns. Factor based investing approaches like fundamental indexing and smart beta take the view that certain investment characteristics imply relatively favorable or unfavorable expected returns. While most factor-oriented products avoid the use of optimization, this is an active decision based on strong assumptions. We explore the veracity of these assumptions below.

## CAPITALIZATION WEIGHTING: A SPECIAL CASE

The Capital Asset Pricing Model (CAPM) described by Sharpe and Treynor in the 1960s proposed a linear relationship between returns and an investment's  $\beta$  with the market. They suggested that the market rewards investors for the risk of being invested in the market itself, and not for any stock-specific risks that can be diversified away. Investors who believe that the CAPM expresses a reasonable relationship between risk and return would choose to hold the market portfolio.

(Haugen and Baker 1991) questioned the fundamental assumptions that are necessary for the market cap portfolio to be mean-variance efficient:

"A prediction that cap-weighted stock portfolios have the lowest possible volatility given their expected return must be based on a set of assumptions that includes the following:

1. All investors agree about the risk and expected return for all securities.
2. All investors can short-sell all securities without restriction.
3. No investor's return is exposed to federal or state income tax liability now in effect.
4. The investment opportunity set for all investors holding any security in the index is restricted to the securities in the cap-weighted index."

In addition to questions about the plausibility of its foundational assumptions, CAPM suffers from poor explanatory power in practice. That is, returns have historically exhibited only a weak relationship with market beta. For example, (Fama and French 1992) show that the positive relationship between stock beta and returns conflates beta with the size effect. More specifically, the fact that stock returns appear linearly proportional to beta is because small cap stocks have larger residual variance (i.e. risk not explained by the market portfolio).

“Our tests do not support the central production of the [CAPM], that average stock returns are positively related to beta.”

—(Fama and French 1992)

This mistake is obvious in retrospect. The market portfolio is disproportionately dominated by the largest capitalization stocks. Small stocks have almost no impact on the returns to the market portfolio. Therefore, small stocks will have larger residual variance. This residual variance is captured by beta. Thus, beta is really just measuring the small-cap effect.

When (Fama and French 1992) examined the relationship between returns and stock betas adjusted for size, they found no residual effect. In other words, after adjusting for size, the slope of the Security Market Line is flat, and the market does not reward investors for taking on more systematic risk.

We discuss this in the context of Figure 3 below.

## **NAIVE VERSUS OPTIMAL DIVERSIFICATION**

While CAPM is perhaps the most profound empirical failure in the history of finance, over the past few decades market cap based indexing has completely dominated all other forms of indexing for two reasons:

- it is grounded in a well formed, macro-consistent theory and
- investment capacity.

Also, until recently there was very little in the way of academically validated literature to inform alternative methods. However, over the past decade or so academics have proposed several alternatives for portfolio choice that express equally compelling relationships between risk and return.

Given the empirical failure of CAPM to explain investment returns, investors have shown increasing receptivity to these heuristic approaches. The most optimal method for portfolio construction is informed by the properties of the investment universe itself. Both naive- and optimization-based methods express strong assumptions about the properties of the investment universe under evaluation. Optimization will produce better results than naive methods under certain conditions, and vice versa.

For the purpose of this article, we label portfolio weighting methods that rely on estimates of volatilities, correlations, or returns optimal. On the other hand, naive methods do not require any information about these variables.

The most naive method, which requires no information whatsoever about the investments under consideration for investment, is equal weighting. Cap weighting, discussed at length above, also does not require estimates of portfolio variables, but does require information about relative market caps.

Figure 2 is a Portfolio Optimization Decision Tree to help investors understand which allocation method is most likely to produce mean-variance optimal portfolios under a variety of assumptions. For example, an investor who is seeking to form an optimal portfolio of investments where he has active views on volatility and correlation, and believes the returns to the assets are independent of risk (i.e. markets are NOT risk efficient) would follow the Decision Tree down and to the left to discover that the Minimum Variance portfolio is mean-variance optimal under this set of assumptions.

Table 1 supplements the Optimization Machine and Decision Tree with more detail on the estimates and optimality equivalence conditions for each approach. We describe the Objectives, Parameters, Optimality Equivalence Conditions, and typical Portfolio Characteristics for portfolios formed using several popular optimization methods.

The Objective column describes what the optimization is seeking to optimize. The Parameters for each optimization are the variables that are required to solve for the optimal weights.

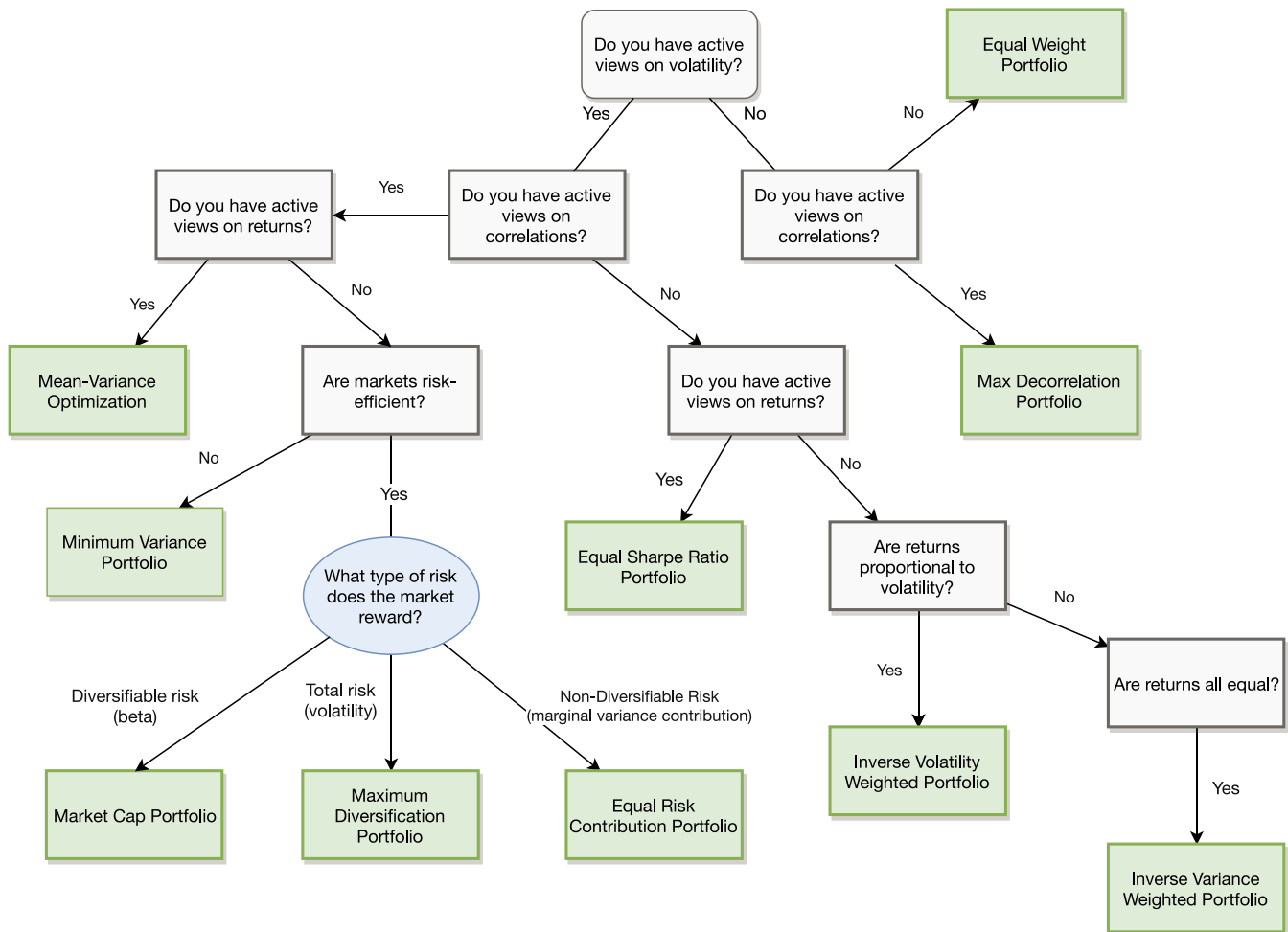
The Optimality Equivalence Conditions are the set of assumptions that are required to be true in order for the optimization to produce a mean-variance optimal portfolio. For example, for the Maximum Diversification optimization to produce a true mean-variance optimal portfolio, we must assume that assets have similar Sharpe ratios.

The final column, “Portfolio Characteristics”, describes the typical character of the portfolios that are formed using each optimization. Some optimizations produce portfolios that seem highly concentrated when viewed through the lens of portfolio weights. For example, Minimum Variance and Maximum Diversification optimizations will often produce optimal portfolios that hold a small fraction of all of the assets, and some assets may have very large weights.

However, when investments have materially different levels of volatility and/or low correlations, a portfolio that has large weights in non-volatile assets and smaller weights in volatile assets may actually be quite well balanced from a risk perspective. On the other hand, an equally weighted portfolio may be quite concentrated from the perspective of risk distribution. So it is important to distinguish between concentration in terms of capital weights and concentration in terms of risk weights. In the end, one should be most concerned with concentration of risk, and less concerned with the often illusory concentration of capital.

We also comment on how sensitive the optimization should be to the character of the investment universe. When investments are relatively homogeneous in terms of expected volatilities, correlations, and returns, then naive methods and optimal methods are likely to produce similar results. However, investment universes containing highly diverse assets are not well suited to naive methods. Thus, naive methods are more sensitive to the character of the investment universe than optimization-based methods, all things equal.

Figure 2: Portfolio Optimization Decision Tree



Source: ReSolve Asset Management. For illustrative purposes only.



Table 1: A taxonomy of portfolio optimizations.

	Objective	Parameters	Optimality Conditions	Portfolio Characteristics
Equal Weight	Equalize capital weights	Number of assets	Similar returns and covariances	Minimal concentration, highly sensitive to investment universe
Inverse Volatility	Weights proportional to inverse of volatility	Asset volatilities	Similar Sharpe ratios and homogeneous correlations	Low concentration, highly sensitive to investment universe
Capitalization Weight	Weights proportional to market cap	Covariance with market portfolio	Similar Treynor ratios	Low concentration, sensitive to investment universe
Equal Risk Contribution	Equalize risk contributions	Covariance matrix	Similar <i>marginal</i> Sharpe ratios	Low concentration, moderately sensitive to covariances, moderately sensitive to investment universe
Maximum Diversification	Maximize portfolio diversification	Covariance matrix	Similar Sharpe ratios	Sometimes concentrated, sensitive to correlations, moderately sensitive to volatilities, robust to investment universe
Minimum Variance	Minimize portfolio volatility	Covariance matrix	Similar returns	Sometimes concentrated, sensitive to covariances, robust to investment universe
Mean-Variance	Maximize return subject to risk tolerance	Expected returns + Covariance matrix	NA	Sometimes concentrated, sensitive to return and covariance estimates, robust to investment universe

Source: ReSolve Asset Management. For illustrative purposes only.

## RISK BASED OPTIMIZATION

An exploration of risk-based optimization methods must be linked directly to a discussion of what risks are compensated in markets - if risk is compensated at all! We discuss the most popular heuristic optimization methods below, and link them to appropriate theories of risk. We examine the empirical performance of these methods on a variety of data sets later in our series.

### MINIMUM VARIANCE

If all investments have the same expected return independent of risk, investors seeking maximum returns for minimum risk should concentrate exclusively on minimizing risk. This is the explicit objective of the Minimum Variance portfolio.

$$w^{MV} = \arg \min w^T \cdot \Sigma \cdot w$$

where  $\Sigma$  is the covariance matrix.

<sup>2</sup> Where daily index histories did not extend to 1991 we calculated from inception. Portfolio volatility was calculated from pairwise complete covariances.

(Haugen and Baker 1991) proposed dispensing with any relationship between risk and return, at least for equities. Their paper was one of the first to demonstrate that stock returns are not well explained by beta. In fact, they observed a negative relationship between returns and volatility.

In the face of a spurious link between risk and return, (Haugen and Baker 1991) suggested that a regularly reconstituted long-only Minimum Variance portfolio might dominate the capitalization weighted portfolio for stocks.

## MAXIMUM DIVERSIFICATION

(Choueifaty and Coignard 2008) proposed that markets are risk-efficient, such that investments will produce returns in proportion to their total risk, as measured by volatility. This differs from CAPM, which assumes returns are proportional to non-diversifiable (i.e. systematic) risk. Choueifaty et al. described their method as Maximum Diversification, for reasons that will become clear below.

Consistent with the view that returns are directly proportional to volatility, the Maximum Diversification optimization substitutes asset volatilities for returns in a maximum Sharpe ratio optimization, taking the following form.

$$w^{MD} = \arg \max \frac{w \times \sigma}{\sqrt{w^T \cdot \Sigma \cdot w}}$$

where  $\sigma$  and  $\Sigma$  reference a vector of volatilities, and the covariance matrix, respectively.

Note that the optimization seeks to maximize the ratio of the weighted average volatility of the portfolio's constituents to total portfolio volatility. This is analagous to maximizing the weighted average return, when return is directly proportional to volatility.

An interesting implication, explored at length in a follow-on paper by (Choueifaty, Froidure, and Reynier 2012) is that the ratio maximized in the optimization function quantifies the amount of diversification in the portfolio. This is quite intuitive.

The volatility of a portfolio of perfectly correlated investments would be equal to the weighted sum of the volatilities of its constituents, because there is no opportunity for diversification. When assets are imperfectly correlated, the weighted average volatility becomes larger than the portfolio volatility in proportion to the amount of diversification that is available.

## MAXIMUM DECORRELATION

Maximum Decorrelation described by (Christoffersen et al. 2010) is closely related to Minimum Variance and Maximum Diversification, but applies to the case where an investor believes all assets have similar returns and volatility, but heterogeneous correlations. It is a Minimum Variance optimization that is performed on the correlation matrix rather than the covariance matrix. And when the weights derived from the Maximum Decorrelation optimization are divided through by their respective volatilities and re-standardized so they sum to 1, we retrieve the Maximum Diversification weights.

The Maximum Decorrelation portfolio is found by solving for:

$$w^{MDec} = \arg \min w^T \cdot A \cdot w$$

where A is the correlation matrix.

## RISK PARITY

Both the Minimum Variance and Maximum Diversification portfolios are mean-variance efficient under intuitive assumptions. Minimum Variance is efficient if assets have similar returns while Maximum Diversification is efficient if assets have similar Sharpe ratios. However, both methods have the drawback that they can be quite concentrated in a small number of assets. For example, the Minimum Variance portfolio will place disproportionate weight in the lowest volatility asset

while the Maximum Diversification portfolio will concentrate in assets with high volatility and low covariance with the market. In fact, these optimizations may result in portfolios that hold just a small fraction of all available assets.

There are situations where this may not be preferable. Concentrated portfolio may not accommodate large amounts of capital without high market impact costs. In addition, concentrated portfolios are more susceptible to mis-estimation of volatilities or correlations.

These issues prompted a search for heuristic optimizations that meet similar optimization objectives, but with less concentration. The equal weight and capitalization weight portfolios are common examples of this, but there are other methods that are compelling under different assumptions.

## **INVERSE VOLATILITY AND INVERSE VARIANCE**

When investments have similar expected Sharpe ratios, and an investor cannot reliably estimate correlations (or we can assume correlations are homogeneous), the optimal portfolio would be weighted in proportion to the inverse of the assets' volatilities. When investments have similar expected returns (independent of volatility) and unknown correlations, the Inverse Variance portfolio is mean-variance optimal.

Note that the Inverse Volatility portfolio is consistent with the Maximum Diversification portfolio, and the Inverse Variance portfolio approximates a Minimum Variance portfolio, when all investments have identical pairwise correlations.

The weights for the Inverse Volatility and Inverse Variance portfolios are found by:

$$w^{IV} = \frac{1/\sigma}{\sum_{i=1}^n 1/\sigma}$$
$$w^{IVar} = \frac{1/\sigma^2}{\sum_{i=1}^n 1/\sigma^2}$$

where  $\sigma$  is the vector of asset volatilities and  $\sigma^2$  is the vector of asset variances.

## EQUAL RISK CONTRIBUTION

(Maillard, Roncalli and Teiletche 2008) described the Equal Risk Contribution portfolio, which is satisfied when all assets contribute the same volatility. It has been shown that the Equal Risk Contribution portfolio is a compelling balance between the objectives of the equal weight and Minimum Variance portfolios. It is also a close cousin to the Inverse Volatility portfolio, except that it is less vulnerable to the case where assets have vastly different correlations.

The weights for the Equal Risk Contribution Portfolio are found through the following convex optimization, as formulated by (Spinu 2013):

$$w^{ERC} = \arg \min \frac{1}{2} w^T \cdot \Sigma \cdot w - \frac{1}{n} \sum_{i=1}^n \ln(w_i)$$

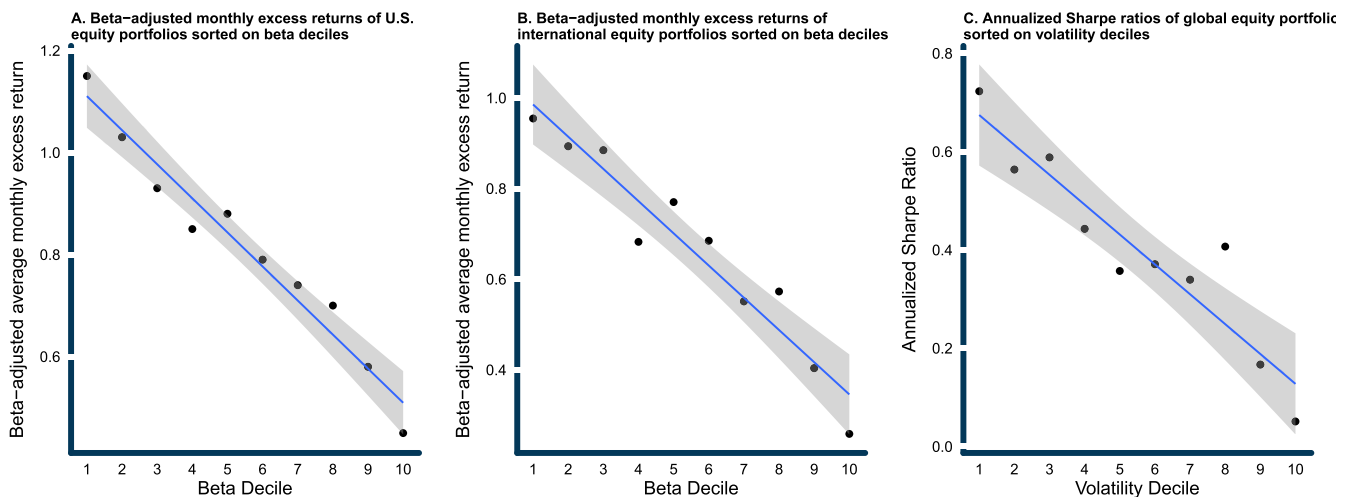
where  $\Sigma$  is the covariance matrix.

The Equal Risk Contribution portfolio will hold all assets in positive weight, and is mean-variance optimal when all assets are expected to contribute equal marginal Sharpe ratios (relative to the Equal Risk Contribution portfolio itself). Thus, optimality equivalence relies on the assumption that the Equal Risk Contribution portfolio is macro-efficient. It has been shown that the portfolio will have a volatility between that of the Minimum Variance Portfolio and the Equal Weight portfolio.

## THE HISTORICAL RELATIONSHIP BETWEEN RISK AND RETURNS IN STOCKS

Rational utility theory suggests there should be a positive linear relationship between expected risk and return. If an investment has twice the volatility of another investment, investors should require twice the excess return. Under the CAPM, stock returns should scale with a stock's  $\beta$  to the market portfolio. However, Figure 3 shows that the empirical relationship between risk and return for stocks has a complicated history.

Figure 3: Excess returns regressed on beta and volatility deciles.



Source: Calculations by ReSolve Asset Management. Analysis of U.S. decile portfolios sorted on beta\* from Junkai Qian and Marat Molyboga based on data from CRSP and Compustat, 1963 - 2017. Figures are average beta-adjusted monthly excess returns divided by ex-post average beta for portfolios sorted on beta decile, and controlled for market capitalization. Data for international decile portfolios sorted on beta from Frazzini & Pedersen "Betting Against Beta" (2013). Data for decile portfolios sorted on volatility from Blitz & van Vliet "The Volatility Effect: Lower Risk Without Lower Return" (2007).\*



Chart A of Figure 3 plots the long-term average beta-adjusted excess monthly returns for portfolios of U.S. stocks sorted on beta, while Chart B illustrates the same relationship for international stocks. Beta-adjusted returns are equivalent to the Treynor ratio :

$$\frac{R-rf}{\beta}$$

CAPM would predict that stock portfolios should produce similar beta-adjusted returns, regardless of the portfolio's  $\beta$  . Instead, we see that portfolios with higher  $\beta$ s to the market cap weighted portfolio are less efficient at producing returns. In theory, an investor can borrow money to lever a portfolio with a  $\beta$  of 0.5 to achieve a  $\beta$  of 1, and earn a higher return than an investor who invests in a portfolio with a natural  $\beta$  of 1.

Chart C examines a similar relationship between stocks' Sharpe ratio and volatility. Rational utility theory would predict that stock portfolios should produce similar Sharpe ratios, regardless of volatility. Instead we see that stocks with high historical volatility have produced substantially lower Sharpe ratios than stocks with low volatility. Which means that an investor can theoretically borrow money to lever a low volatility portfolio to the same risk as a high volatility portfolio, and earn substantially higher return than an investor who simply invests in high volatility stocks.

The effects described above have been well known by academics for almost three decades. (Haugen and Baker 1991) and (Fama and French 1992) documented what would later be called the “Low Volatility” and “Betting Against Beta” anomalies by (Blitz and Vliet 2007) and (Frazzini and Pedersen 2014), respectively. Moreover, academics have offered convincing arguments for why these effects may be rooted in certain types of non-classical sources of risk, such as leverage and tracking error aversion.

(diBartolomeo 2007) makes the case that the low volatility effect is fully explained by misspecification of the CAPM as a single period model. In a multi-period context investors are concerned

with geometric means rather than the arithmetic means most often cited in academic papers. After adjusting for the combined impact of algebraic differences between arithmetic and geometric measures of return, and real-world features of markets, such as skew, kurtosis, borrowing and transaction costs, stock returns are probably independent of risk.

If this is true, it follows from Figure 2 that the mean-variance optimal portfolio might be the Minimum Variance portfolio. A future article in this series we will test this theory on live data.

## **THE HISTORICAL RELATIONSHIP BETWEEN RISK AND RETURNS ACROSS ASSET CLASSES**

The paper “Betting Against Beta” by (Frazzini and Pedersen 2014) demonstrated a clear negative relationship between  $\beta$  and return for individual stocks in both U.S. and international equity markets; country equity indices; country bonds; U.S. Treasuries; foreign exchange; credit indices; corporate bonds; and commodities.

However, (Frazzini and Pedersen 2014) examined the relationship between  $\beta$  and returns at an intramarket level. In other words, they examined the performance of  $\beta$ -neutral self-financed portfolios that went long U.S. low-beta stocks and short high-beta U.S. stocks, or long low- $\beta$  commodities and short high-beta commodities, etc. They did not examine the relationship between the returns of high- $\beta$  assets and low- $\beta$  assets across markets, which might have formed portfolios that were long low-beta asset classes (i.e. bonds) and short high- $\beta$  asset classes (i.e. equities).

We would argue that analyzing the relationship between risk and return across diverse asset classes requires a deeper understanding of diversification than a similar analysis within a given asset category. At the asset class level, the role of diversification is to ensure the portfolio is positioned to deliver on its objectives in any macroeconomic environment.

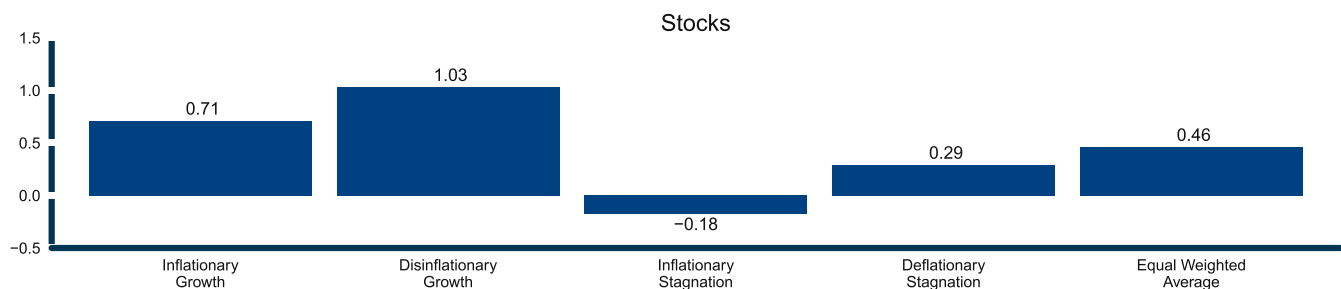
If we view diversification through a macroeconomic lens, we are interested in how asset classes perform during different market regimes. Specifically, we are interested in how assets respond to changes in expectations about inflation and growth, as these dynamics explain most of the movements of asset classes through time.

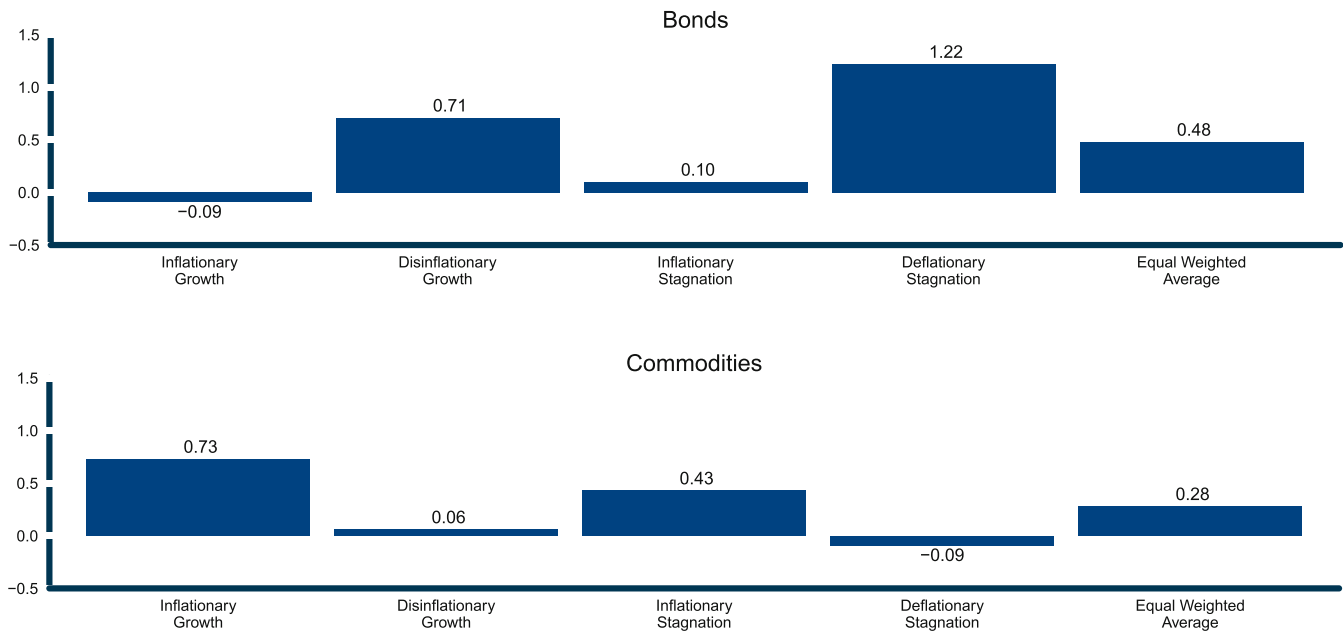
Macroeconomic regimes can last for many years at a time. For example, the world experienced a sustained period of negative growth and high inflation shocks from 1970-1982. The period from 1982-1999 represented eighteen almost uninterrupted years of disinflationary positive growth shocks. From 2003 - 2008 markets experienced an inflationary growth trend.

Given that macroeconomic regimes often last for a decade or more at a time, and we only have about a century's worth of granular economic and financial market data, history offers a very small sample size from which to determine expected performance of asset classes in different regimes. Practically speaking, we have less than fifteen non-overlapping ten-year periods to populate four regimes (i.e. inflationary growth / disinflationary growth / inflationary stagnation / deflationary stagnation), or less than four data points per regime.

Figure 4 describes the average Sharpe ratios of global stocks, bonds and commodities conditioned on the four regimes described above over the period 1970 - 2014. The far right bar in Figure 4 shows the average Sharpe ratio across the four different regimes. It reflects the expected long-term Sharpe ratio assuming each regime occurs with equal probability.

Figure 4: Sharpe ratio for asset classes in different macroeconomic regimes.





Source: Charts by ReSolve Asset Management. Data from (Ilmanen, Maloney, and Ross 2014)

With such a small sample of sustained macroeconomic regimes to study, we must not assume our sample is representative of the true population. For example, it would be perfectly normal, from a statistical perspective, to draw a sample of four or five decades where inflationary stagnation regimes are vastly underrepresented, and disinflationary growth periods overrepresented, relative to their true distribution in the long-term.

This means that, if we simply use the historical sample of asset class returns to inform expectations, we risk biasing our estimates toward those assets that happened to benefit from the mostly disinflationary growth-oriented environment over our historical record. For this reason, we believe it's better to estimate returns for each asset class conditioned on the different economic environments. Then we can weight returns in each environment against our estimates of the probability of experiencing those environments to derive final return forecasts.

Importantly, even if we use the regime based analysis described above, we must recognize that the mean returns are accompanied by large standard errors. This is unavoidable because of the incredibly small number of samples per regime. Thus, even though the average Sharpe ratio has been higher for stocks and bonds than commodities in the historical sample, it may be more likely that commodities simply drew a set of low Sharpes from a population of Sharpes that is on par with stocks and bonds. With such a small sample size, it is difficult to reject this rational expectation.

Since stocks, bonds, and commodities are fundamentally expected to respond in unique ways to certain economic regimes, it is natural to assume they are uncorrelated. In fact, the long-term average pairwise correlations between stocks, bonds, and commodities are almost exactly zero since 1970<sup>3</sup>.

Per Figure 2, when major asset classes have similar expected Sharpe ratios, but different expected volatilities, there are several risk-based optimizations that produce mean-variance optimal portfolios. In fact, if the assets have identical correlations, the Inverse Volatility portfolio = the Maximum Diversification portfolio = the Equal Risk Contribution portfolio, and all of these would fall under the heading “Risk Parity”.

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<sup>3</sup> Calculations by ReSolve Asset Management. Data from Global Financial Data. Stocks = MSCI World TR Index; Bonds = US 10-Year Continuous Contract Treasury Bond Total Return; Commodities = GSCI Commodity Index Total Return.

## SUMMARY

In this article, we introduced the Portfolio Optimization Machine and described how portfolio choice is informed by what an investor believes about certain qualities of the asset universe under consideration. We describe a continuum of approaches to portfolio formation ranging from those that assume extreme naïveté about any features of the underlying assets, to methods that make use of all available information about volatilities, correlations, and excess returns.

The core of our discussion focused on optimizations that relieve investors from the burden of forecasting returns. Instead, we described optimizations that express certain relationships between risk and expected return. While most investors will be familiar with CAPM and market cap weighting, we show that there are other equally compelling ways to form portfolios based on alternative definitions of risk.

If we are to attempt to form portfolios that are mean-variance optimal under specific assumptions about the relationships between risk and return, we must have a framework for evaluating the nature of this relationship. We provide case studies to exhibit this framework for both stocks and asset classes, and show that the relationships are complicated, and require substantial interpretation by investors.

In a follow-up report, we will put the concepts in this section to the test by running simulations on a variety of investment universes.



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