POTENTIAL ENERGY APPLIED IN FINITE ELEMENT METHOD

by

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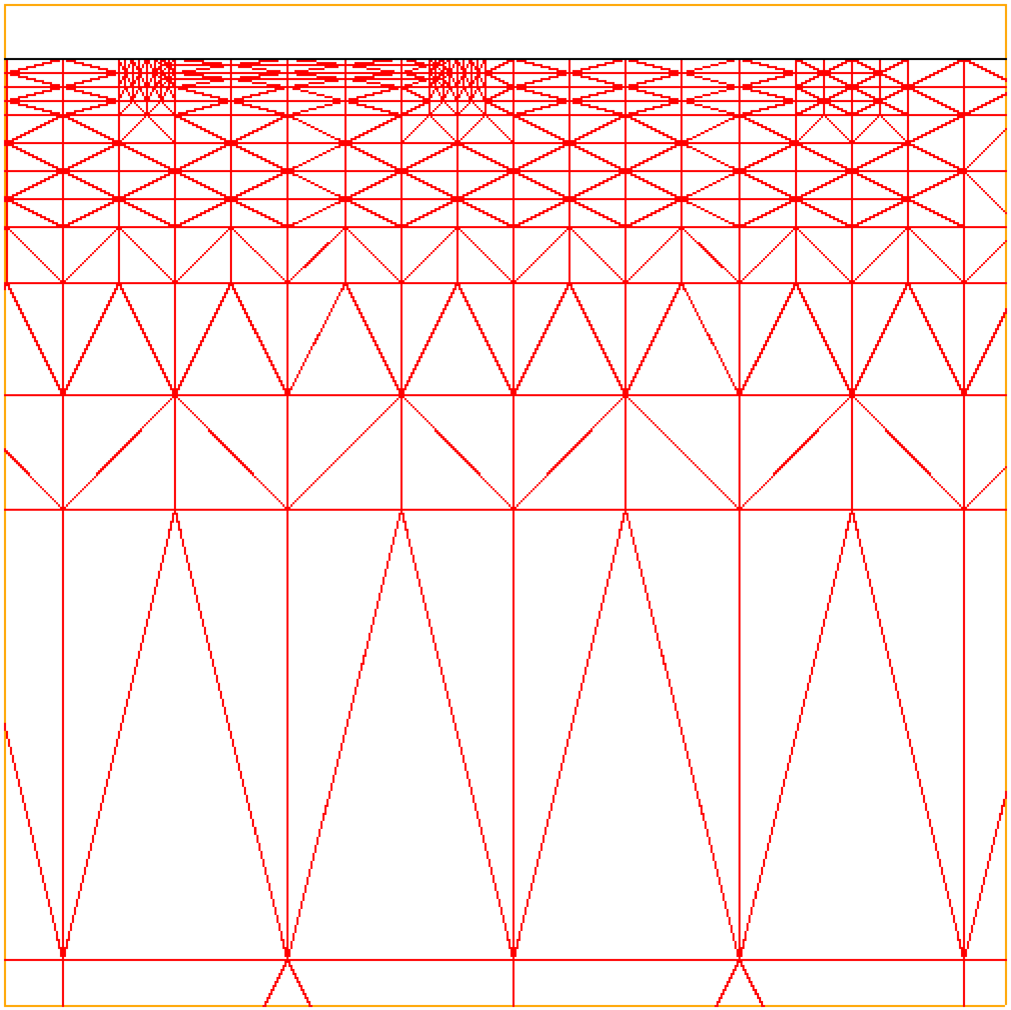
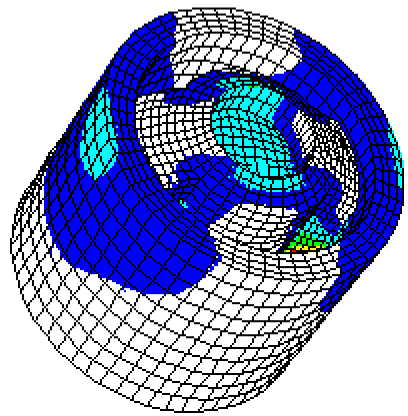
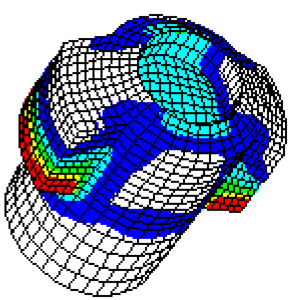






INTRODUCTION

Finite difference method is suitable to used if the structure to be analysed has the specific governing differential equation.

For structures which are so complex that is difficult or impossible to determine the governing differential equation, a powerful method for analysing such complex structures is the finite element method.

The way finite element analysis obtains the temperatures, stresses, flows, or other desired unknown parameters in the finite element model are by minimizing an energy functional. An energy functional consists of all the energies associated with the particular finite element model. Based on the law of conservation of energy, the finite element energy functional must equal zero.

The finite element method obtains the correct solution for any finite element model by minimizing the energy functional. The minimum of the functional is found by setting the derivative of the functional with respect to the unknown grid point potential for zero. Thus, the basic equation for finite element analysis is:

$$\frac{\partial F}{\partial p} = 0$$

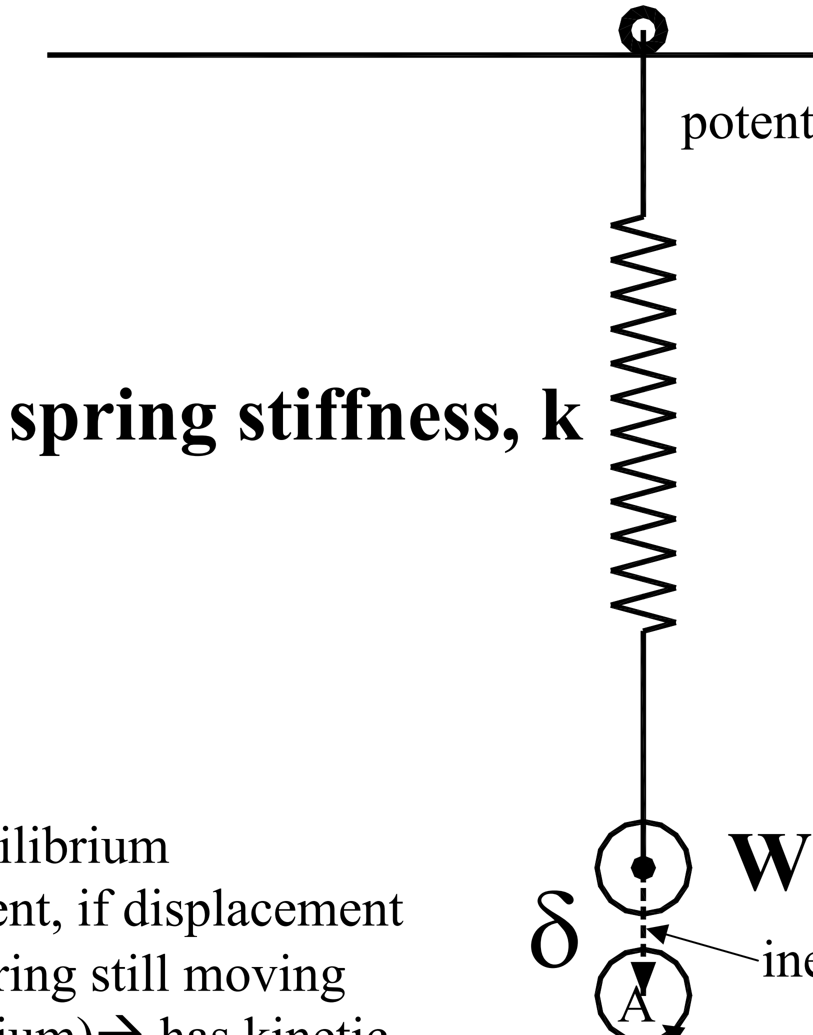
where F is the energy functional and p is the unknown grid point potential (In mechanics, the potential is displacement) to be calculated. This is based on the principle of virtual work, which states that if a particle is under equilibrium, under a set of a system of forces, then for any displacement, the virtual work is zero. Each finite element will have its own unique energy functional.

PRINCIPLE OF MINIMUM POTENTIAL ENERGY

For conservative systems, of all the kinematically admissible displacement fields, those corresponding to equilibrium extremize the total potential energy. If the extremum condition is a minimum, the equilibrium state is stable.

Satisfy the single-valued nature of displacements (compatibility) and the boundary conditions.

SPRING EXAMPLE



inequilibrium

equilibrium

potential energy, strain energy, work done

$$\Pi = \frac{1}{2}k\delta^2 - W\delta$$

minimum potential energy

$$\frac{d\Pi}{d\delta} = 0:$$

external force

$$\frac{d\Pi}{d\delta} = k\delta - W = 0$$

internal force in the spring

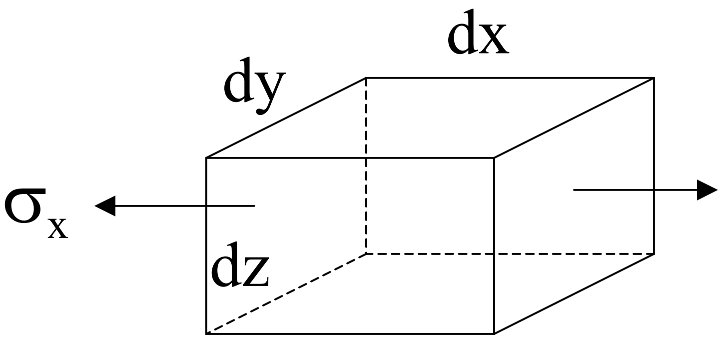
We get equation of equilibrium → satisfy equilibrium condition

δ is an equilibrium displacement, if displacement < δ, the spring still moving (inequilibrium)→ has kinetic energy

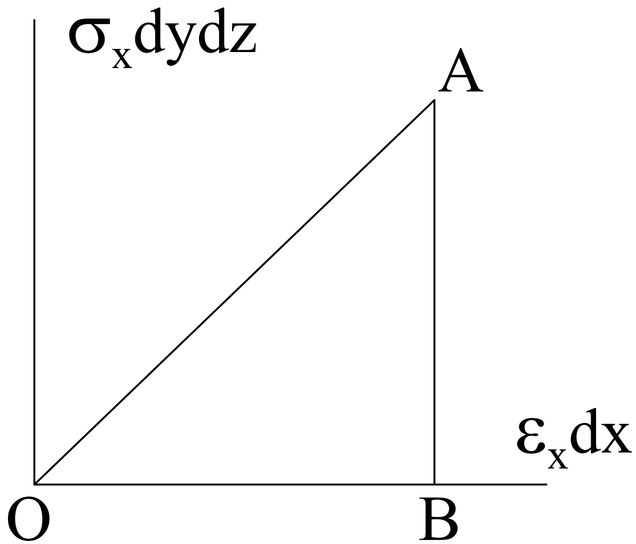
STRAIN ENERGY

When loads are applied to a body, they will deform the materials. Provided no energy is lost in the form of heat, the external work done by the loads will be *converted* into internal work called **strain energy**. This energy, which is always positive, is stored in the body and is caused by the action of either normal or shear stress.

STRAIN ENERGY DUE TO NORMAL STRESS



The force $\sigma_x dydz$ does work on an extension $\varepsilon_x dx$



Work done during deformation, dW= Area of triangle OAB

$$dW = \frac{1}{2}\sigma_x \varepsilon_x dxdydz$$

or

$$dW = \frac{1}{2}\sigma_x \varepsilon_x dV$$

Strain energy is an amount of energy stored in the material due to work done

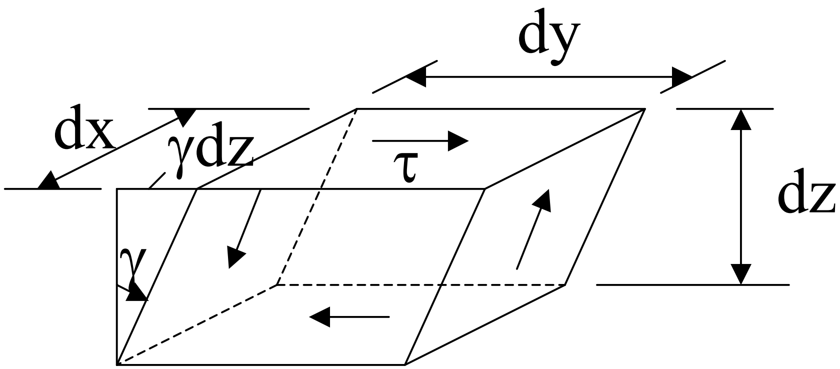
In general, if the body is subjected only to a uniaxial *normal stress* σ, acting in a specified direction, the strain energy in the body is then

$$W = \int_V \frac{\sigma \varepsilon}{2} dV$$

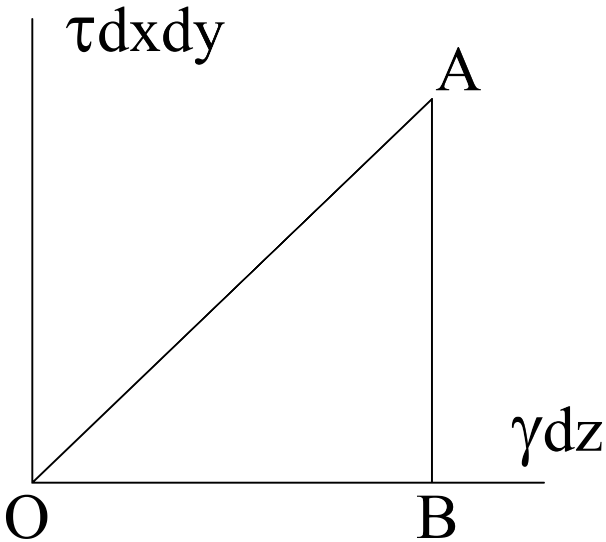
Also, if the material behaves in a linear-elastic manner, Hooke's law applies, $\sigma = E\varepsilon$, and therefore:

$$W = \int_V \frac{\sigma^2}{2E} dV$$

STRAIN ENERGY DUE TO SHEAR STRESS



The force τdxdy does work on a shear slip γdz



Work done during deformation, dW= Area of triangle OAB

$$dW = \frac{1}{2}\left(\tau dxdy\right)\gamma dz$$

or

$$dW = \frac{1}{2}\tau\gamma dV$$

Strain energy is an amount of energy stored in the material due to work done

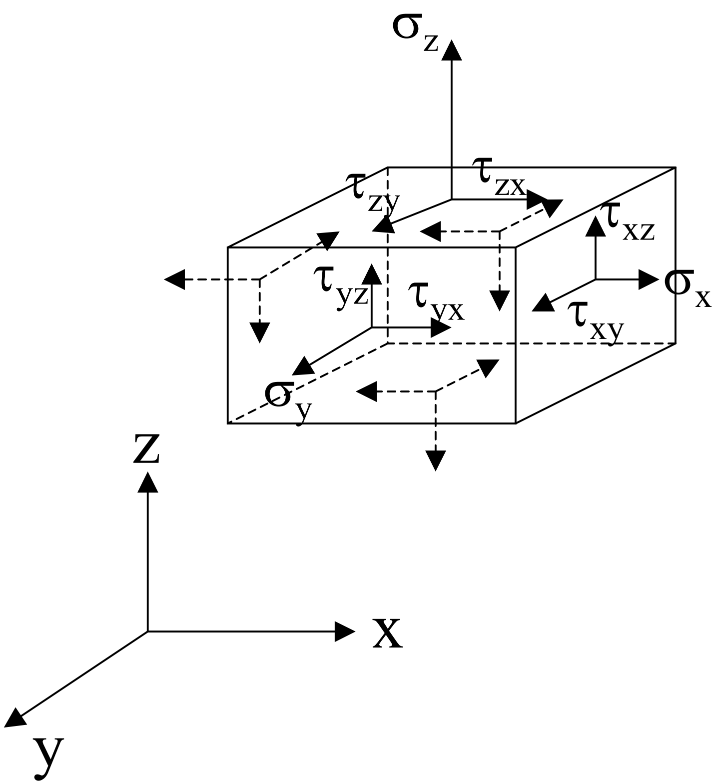
In general, if the body is subjected only to a *shear stress* τ, the strain energy due to shear stress in the body is then

$$W = \int_V \frac{\tau\gamma}{2} dV$$

Also, if the material behaves in a linear-elastic manner, Hooke's law applies, $\tau = G\gamma$, therefore:

$$W = \int_V \frac{\tau^2}{2G} dV$$

STRAIN ENERGY DUE TO MULTIPLE STRESSES



$$dW = \frac{1}{2}\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz}\right)$$

or

$$dW = \frac{1}{2}\int_{\Omega} \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}\, dV$$

where

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} \qquad \boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}$$

STRAIN ENERGY – PURE BENDING

$$dU = \int_V \frac{1}{2}\sigma_x \varepsilon_x dV$$

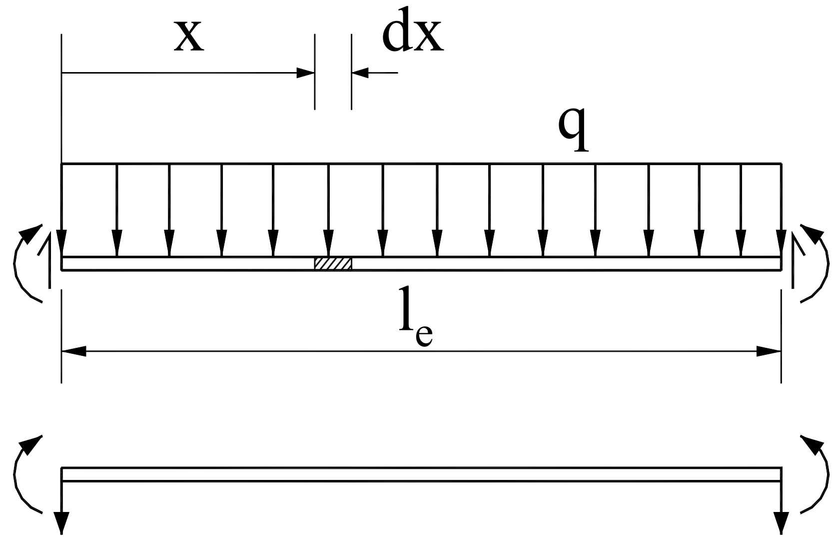
$$= \left(\int_A \frac{1}{2}\left(\frac{My}{I}\right)\left(\frac{\sigma_x}{E}\right) dA\right) dx$$

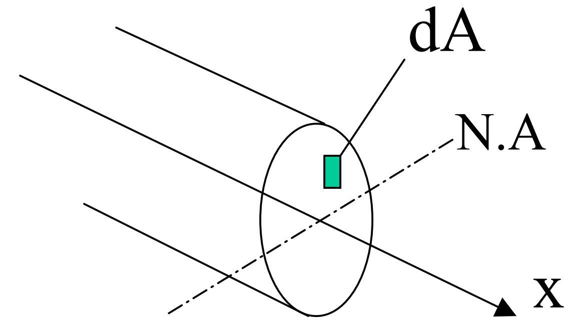
$$= \int_A \frac{1}{2}\left(\frac{My}{I}\right)\left(\frac{My}{EI}\right) dA dx$$

$$= \frac{1}{2}\frac{M^2}{EI^2}\int_A y^2 dA dx$$

$$= \frac{1}{2}\frac{M^2}{EI^2} I dx$$

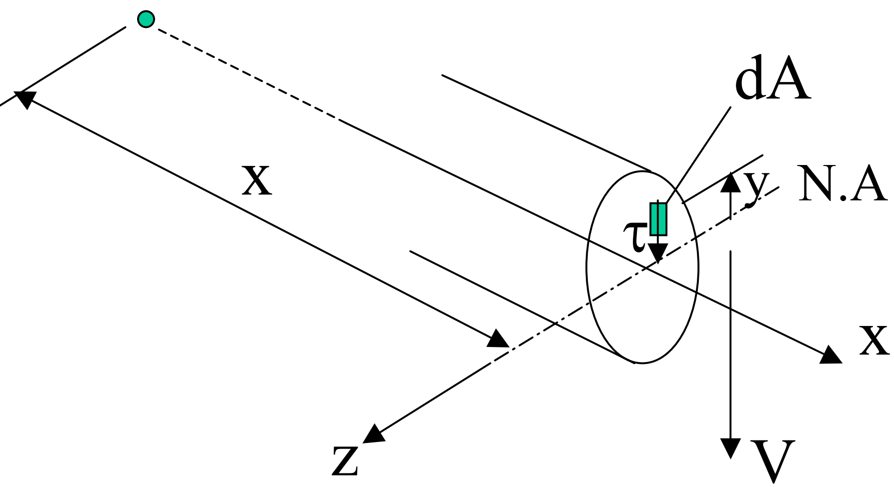
$$= \frac{1}{2}\frac{M^2}{EI} dx$$





Strain energy stored due to bending of a very small length of beam element, dx

STRAIN ENERGY IN BEAM DUE TO TRANSVERSE SHEAR



$$\tau = \frac{VQ}{Ib}$$

$$W = \int_V \frac{\tau^2}{2G} dV = \int_V \frac{1}{2G}\left(\frac{VQ}{Ib}\right)^2 dA dx$$

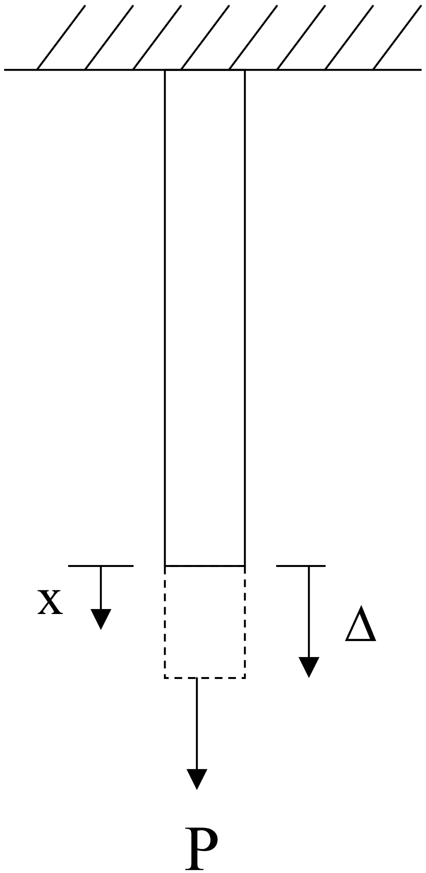
$$W = \int_0^L \frac{V^2}{2GI^2}\left(\int_A \frac{Q^2}{b^2} dA\right) dx$$

Define the *form factor* for shear as $f_s = \frac{A}{I^2}\int_A \frac{Q^2}{b^2} dA$

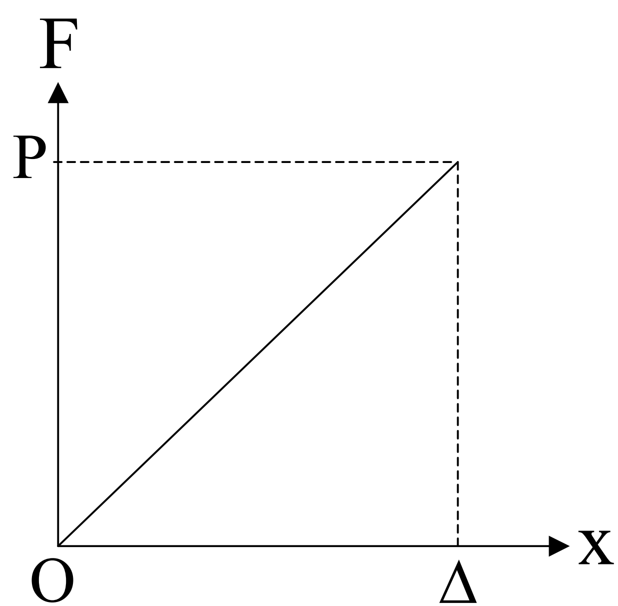
Substituting into the above equation, we get

$$W = \int_0^L \frac{f_s V^2}{2GA} dx$$

EXTERNAL WORK – BAR ELEMENT



Consider the work done by an axial force applied to the end of the bar. As the magnitude of force, F is *gradually* increased from zero to some limiting value F=P, the bar displaced from x=0 to x=Δ.



$$\text{External Work, } W = \int_0^\Delta F dx$$

$$= \int_0^\Delta (kx)\, dx$$

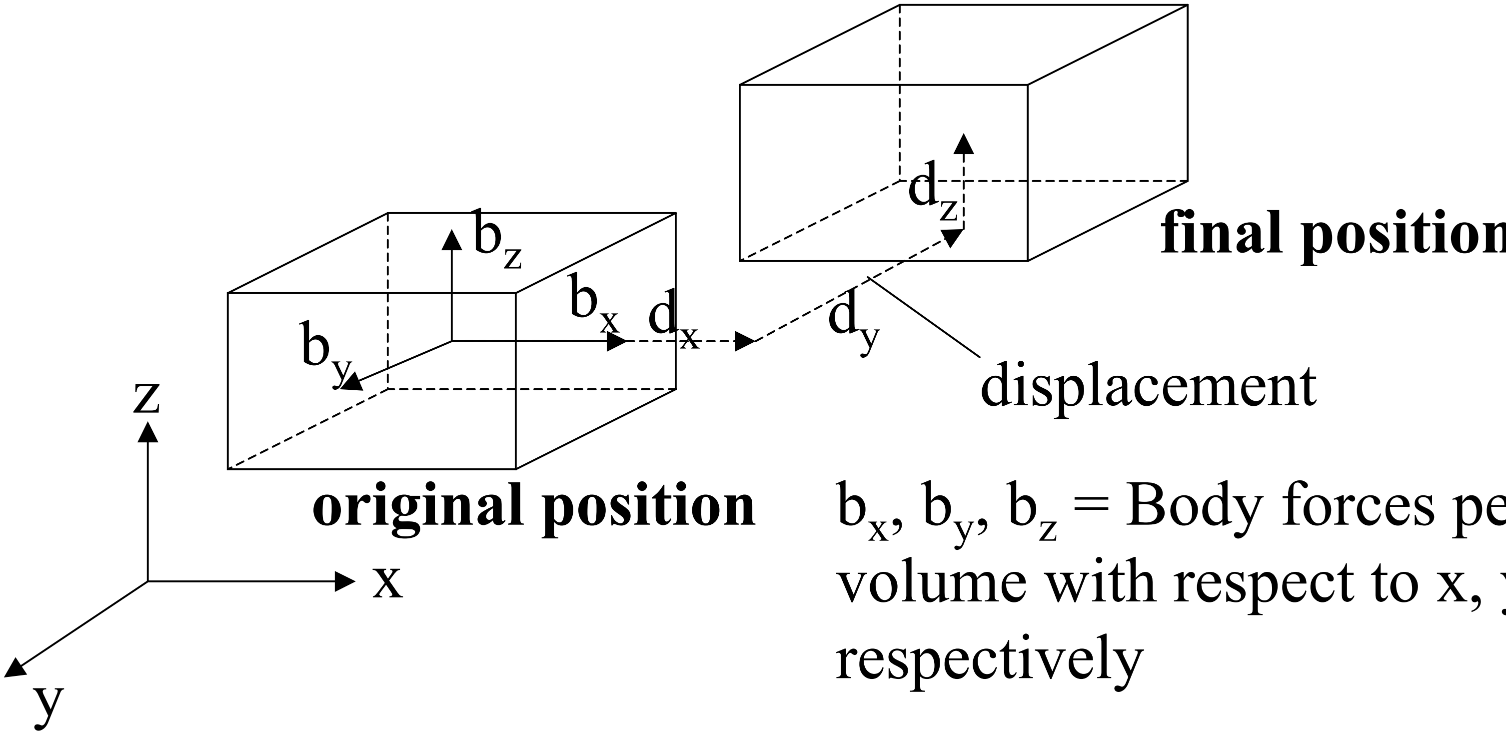
$$= k\int_0^\Delta x dx$$

$$= k\left(\Delta^2/2\right)$$

$$= \frac{1}{2}(k\Delta)\Delta$$

$$= \frac{1}{2}P\Delta$$

EXTERNAL WORK DUE TO BODY FORCES



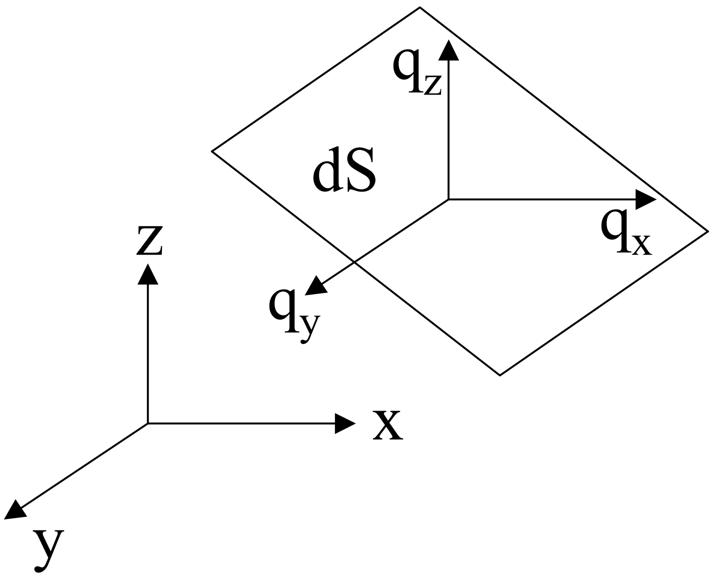
b_x, b_y, b_z = Body forces per unit volume with respect to x, y and z axes, respectively

$$\boldsymbol{d} = \begin{Bmatrix} d_x \\ d_y \\ d_z \end{Bmatrix} \qquad \boldsymbol{b} = \begin{Bmatrix} b_x \\ b_y \\ b_z \end{Bmatrix}$$

Work due to body forces =

$$\int_{\Omega} \boldsymbol{d}^T \boldsymbol{b} \, dV$$

EXTERNAL WORK DUE TO SURFACE TRACTIONS



q_x, q_y, q_z = Surface tractions with respect to x, y and z axes, respectively

$$\boldsymbol{q} = \begin{Bmatrix} q_x \\ q_y \\ q_z \end{Bmatrix}$$

Work due to surface tractions =

$$\int_{\Gamma} \boldsymbol{d}^T \boldsymbol{q}\, dS$$

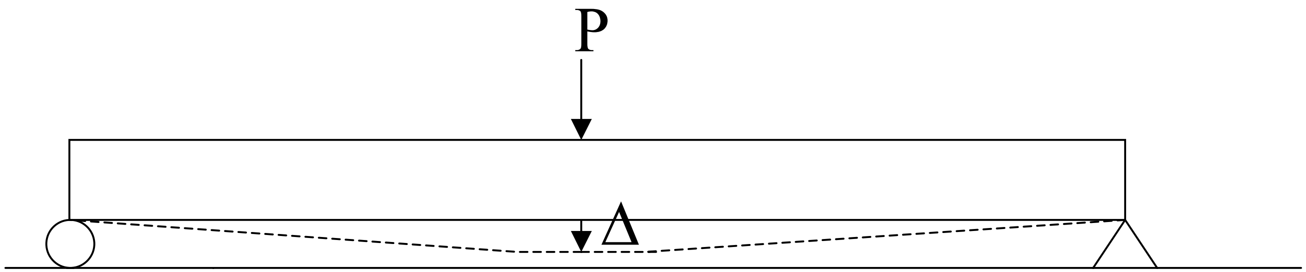
CONSERVATION OF ENERGY

Most energy methods used in mechanics are based on a balance of energy, often referred to as the conservation of energy. The energy developed by heat effects will be neglected. As a result, if a loading is applied *slowly* to a body, so that kinetic energy can also be neglected, then physically the external loads tend to deform the body so that the loads do *external work* W_e as they are displaced. This external work caused by the loads is transformed into *internal work* or strain energy W_i, which is stored in the body. Furthermore, when the loads are removed, the strain energy restores the body back to its original undeformed position, provided the material's elastic limit is not exceeded. The conservation of energy for the body can therefore be stated mathematically as

$$W_e = W_i$$

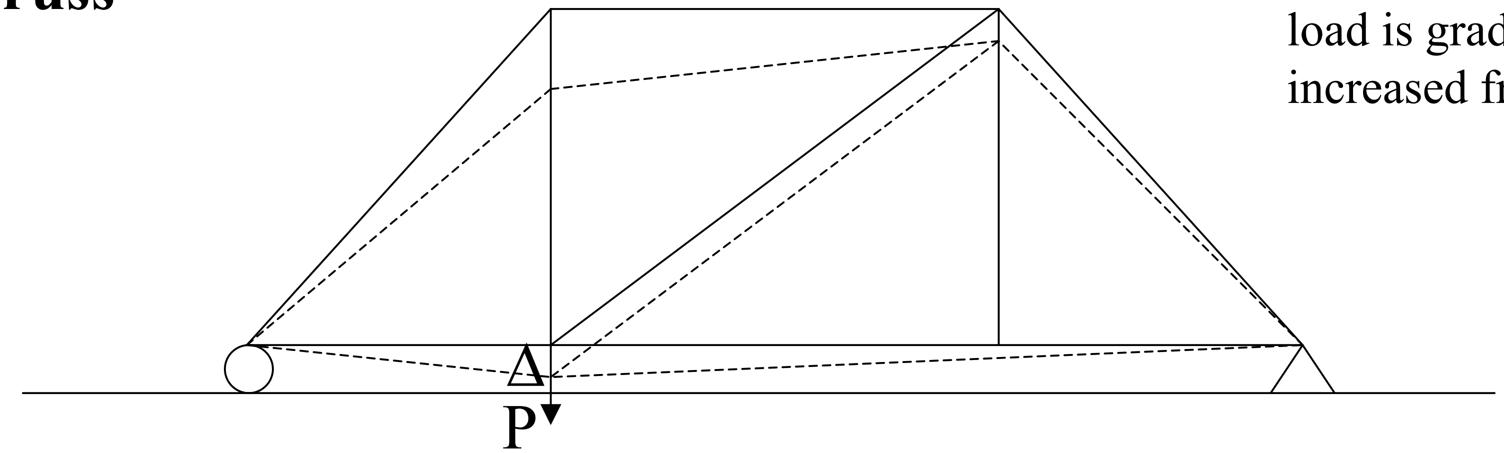
CONSERVATION OF ENERGY - EXAMPLE

Beam



$$\frac{1}{2}P\Delta = \int_0^L \frac{M^2}{2EI}dx$$

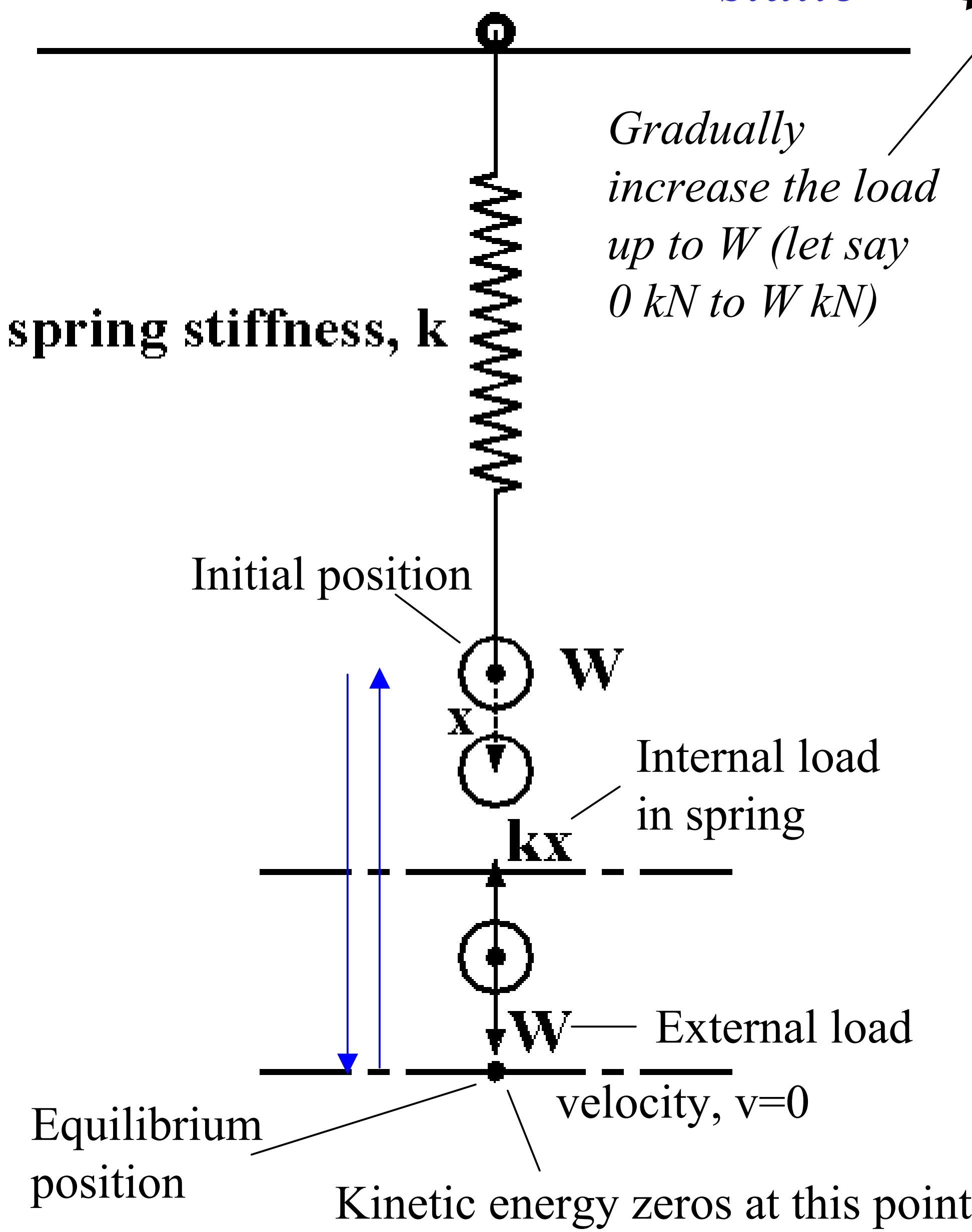
Truss



We assumed that the load is gradually increased from 0 to P

$$\frac{1}{2}P\Delta = \sum \frac{N^2 L}{2AE}$$

DIFFERENT CONCEPT IN ENERGY METHOD



static

External Work = Internal Work

strain energy

$$\frac{1}{2}Wx = \frac{1}{2}kx^2$$

gives $\longrightarrow$ $x = \dfrac{W}{k}$

Equilibrium Principle

$$W = kx$$

gives $\longrightarrow$ $x = \dfrac{W}{k}$

Same answer

Potential Energy Concept

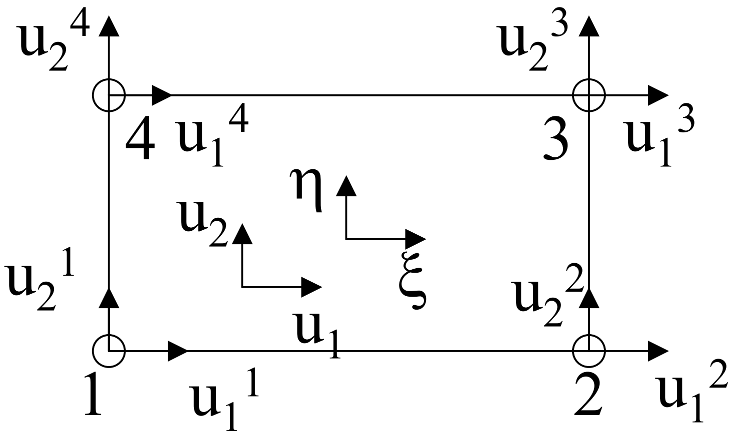
dynamic

$$\Pi = \frac{1}{2}kx^2 - Wx$$

$$\frac{d\Pi}{dx} = kx - W = 0$$

gives $\longrightarrow$ $x = \dfrac{W}{k}$

THE FOUR-NODE QUADRILATERAL (2D)



$$u_1 = N_1 u_1^1 + N_2 u_1^2 + N_3 u_1^3 + N_4 u_1^4$$

$$u_2 = N_1 u_2^1 + N_2 u_2^2 + N_3 u_2^3 + N_4 u_2^4$$

d=**Nu**

where

$$\boldsymbol{d} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\boldsymbol{u} = \begin{Bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \\ u_1^3 \\ u_2^3 \\ u_1^4 \\ u_2^4 \end{Bmatrix}$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

N=Shape function

$$\boldsymbol{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$\varepsilon = \boldsymbol{Ld}$ where $\longrightarrow$ $\boldsymbol{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$

$\varepsilon = \boldsymbol{LNu}$

$\varepsilon = \boldsymbol{Bu}$ where $\boldsymbol{B} = \boldsymbol{LN}$

STRAIN-STRESS RELATIONSHIP

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

STRESS-STRAIN RELATIONSHIP

For plane stress:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{E} & -\dfrac{\nu}{E} & 0 \\ -\dfrac{\nu}{E} & \dfrac{1}{E} & 0 \\ 0 & 0 & \dfrac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

POTENTIAL ENERGY

$$\Pi = \frac{1}{2}\int_{\Omega} \sigma^T \varepsilon dV - \int_{\Omega} \boldsymbol{d}^T \boldsymbol{b} dV - \int_{\Gamma} \boldsymbol{d}^T \boldsymbol{q} dS$$

$$= \frac{1}{2}\int_{\Omega} \left(\boldsymbol{D}\varepsilon\right)^T \varepsilon dV - \int_{\Omega} \boldsymbol{d}^T \boldsymbol{b} dV - \int_{\Gamma} \boldsymbol{d}^T \boldsymbol{q} dS$$

$$= \frac{1}{2}\int_{\Omega} \left(\boldsymbol{DBu}\right)^T \boldsymbol{Bu} dV - \int_{\Omega} \boldsymbol{d}^T \boldsymbol{b} dV - \int_{\Gamma} \boldsymbol{d}^T \boldsymbol{q} dS$$

$$= \frac{1}{2}\int_{\Omega} \left(\boldsymbol{DBu}\right)^T \boldsymbol{Bu} dV - \int_{\Omega} \left(N\boldsymbol{u}\right)^T \boldsymbol{b} dV - \int_{\Gamma} \left(N\boldsymbol{u}\right)^T \boldsymbol{q} dS$$

$$= \frac{1}{2}\int_{\Omega} \boldsymbol{u}^T \left(\boldsymbol{DB}\right)^T \boldsymbol{Bu} dV - \int_{\Omega} \boldsymbol{u}^T N^T \boldsymbol{b} dV - \int_{\Gamma} \boldsymbol{u}^T N^T \boldsymbol{q} dS \qquad Note: \left(PQ\right)^T = Q^T P^T$$

$$= \frac{1}{2}\int_{\Omega} \boldsymbol{u}^T \left(\boldsymbol{DB}\right)^T \left(\boldsymbol{B}^{\boldsymbol{T}}\right)^{\boldsymbol{T}} \boldsymbol{u} dV - \int_{\Omega} \boldsymbol{u}^T \boldsymbol{N}^T \boldsymbol{b} dV - \int_{\Gamma} \boldsymbol{u}^T \boldsymbol{N}^T \boldsymbol{q} dS$$

$$= \frac{1}{2}\int_{\Omega} \boldsymbol{u}^T \left(\mathrm{B}^{\mathrm{T}} \boldsymbol{DB}\right)^T \boldsymbol{u} dV - \int_{\Omega} \boldsymbol{u}^T \boldsymbol{N}^T \boldsymbol{b} dV - \int_{\Gamma} \boldsymbol{u}^T \boldsymbol{N}^T \boldsymbol{q} dS$$

$$\Pi = \sum_e \Pi_e$$

Thus,

$$\Pi_e = \frac{1}{2}\int_{\Omega_e} \boldsymbol{u}^T\left(\boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}\right)^T \boldsymbol{u}\,dV - \int_{\Omega_e} \boldsymbol{u}^T \boldsymbol{N}^T \boldsymbol{b}\,dV - \int_{\Gamma} \boldsymbol{u}^T \boldsymbol{N}^T \boldsymbol{q}\,dS$$

$$= \frac{1}{2}\boldsymbol{u}^T \int_{\Omega_e} \left(\mathrm{B}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{B}\right)^T dV\boldsymbol{u} - \boldsymbol{u}^T \int_{\Omega_e} \boldsymbol{N}^T \boldsymbol{b}\,dV - \boldsymbol{u}^T \int_{\Gamma} \boldsymbol{N}^T \boldsymbol{q}\,dS$$

$$= \frac{1}{2}\boldsymbol{u}^T \int_{\Omega_e} \left(\mathrm{B}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{B}\right)^T dV\boldsymbol{u} - \boldsymbol{u}^T \left(\int_{\Omega_e} \boldsymbol{N}^T \boldsymbol{b}\,dV + \int_{\Gamma} \boldsymbol{N}^T \boldsymbol{q}\,dS \right)$$

$$= \frac{1}{2}\boldsymbol{u}^T \boldsymbol{k}\boldsymbol{u} - \boldsymbol{u}^T \boldsymbol{f}$$

k=element stiffness

from $\Pi_e = \frac{1}{2}\boldsymbol{u}^T \boldsymbol{k}\boldsymbol{u} - \boldsymbol{u}^T \boldsymbol{f}$ **:**

Consider 4 nodes/element

$$\Pi_e = \frac{1}{2}\begin{pmatrix} u_1^1 k_{11} u_1^1 + u_1^1 k_{12} u_2^1 + \cdots + u_1^1 k_{1n} u_2^4 + \\ u_2^1 k_{21} u_1^1 + u_2^1 k_{22} u_2^1 + \cdots + u_2^1 k_{2n} u_2^4 + \\ \cdots + \\ u_2^4 k_{n1} u_1^1 + u_2^4 k_{n2} u_2^1 + \cdots + u_2^4 k_{nn} u_2^4 \end{pmatrix} - \left(u_1^1 f_1 + u_2^1 f_2 + \cdots + u_2^4 f_n \right)$$

Note: n=2x4=8

Taking the derivative $\frac{\partial \Pi_e}{\partial u_1^1} = 0$:

$$\frac{\partial \Pi_e}{\partial u_1^1} = \left(k_{11} u_1^1 + \frac{1}{2} k_{12} u_2^1 + \cdots + \frac{1}{2} k_{1n} u_2^4 + \frac{1}{2} u_2^1 k_{21} + \cdots + \frac{1}{2} u_2^4 k_{n1} \right) - \left(f_1 \right) = 0$$

or

$$\left(k_{11} u_1^1 + k_{12} u_2^1 + \cdots + k_{1n} u_2^4 \right) - \left(f_1 \right) = 0 \qquad (A1)$$

$$\Pi_e = \frac{1}{2}\begin{pmatrix} u_1^1 k_{11} u_1^1 + u_1^1 k_{12} u_2^1 + \cdots + u_1^1 k_{1n} u_2^4 + \\ u_2^1 k_{21} u_1^1 + u_2^1 k_{22} u_2^1 + \cdots + u_2^1 k_{2n} u_2^4 + \\ \cdots + \\ u_2^4 k_{n1} u_1^1 + u_2^4 k_{n2} u_2^1 + \cdots + u_2^4 k_{nn} u_2^4 \end{pmatrix} - \left(u_1^1 f_1 + u_2^1 f_2 + \cdots + u_2^4 f_n\right)$$

original equation

Taking the derivative $\frac{\partial \Pi_e}{\partial u_2^1} = 0:$

$$\frac{\partial \Pi_e}{\partial u_2^1} = \left(\frac{1}{2}u_1^1 k_{12} + \frac{1}{2}k_{21}u_1^1 + k_{22}u_2^1 + \cdots + \frac{1}{2}k_{2n}u_2^4 + \cdots + \frac{1}{2}u_2^4 k_{n2}\right) - \left(f_2\right) = 0$$

or $\left(k_{21}u_1^1 + k_{22}u_2^1 + \cdots + k_{2n}u_2^4\right) - \left(f_2\right) = 0 \qquad (A2)$

Similarly, taking the derivative $\frac{\partial \Pi_e}{\partial u_2^4} = 0:$

$$\left(k_{n1}u_1^1 + k_{n2}u_2^1 + \cdots + k_{nn}u_2^4\right) - \left(f_n\right) = 0 \qquad (A'n')$$

Collecting equations A1, A2, …, A'n' one gets the element equilibrium equation:

by minimizing the potential energy, $\frac{\partial \Pi_e}{\partial u} = 0$

ku-f = 0

vector

where

$$\boldsymbol{k} = \int_{\Omega_e} \left(\mathrm{B}^{\mathrm{T}}\boldsymbol{DB}\right)^T dV$$

$$\boldsymbol{f} = \int_{\Omega_e} \boldsymbol{N}^T \boldsymbol{b} dV + \int_{\Gamma} \boldsymbol{N}^T \boldsymbol{q} dS$$

Load vector

stiffness matrix

$$\boldsymbol{k} = \int_{\Omega_e} \left(\mathrm{B}^{\mathrm{T}}\boldsymbol{DB}\right)^T dV$$

$$= \int_{\Omega_e} \left(\boldsymbol{DB}\right)^T \left(\boldsymbol{B}^T\right)^T dV$$

$$= \int_{\Omega_e} \left(\boldsymbol{B}\right)^T \left(\boldsymbol{D}\right)^T \left(\boldsymbol{B}^T\right)^T dV$$

$$= \int_{\Omega_e} \left(\boldsymbol{B}\right)^T \left(\boldsymbol{D}\right)^T \boldsymbol{B} dV$$

$$= \int_{\Omega_e} \boldsymbol{B}^T \boldsymbol{DB} dV$$

$$\boldsymbol{D}^T = \boldsymbol{D}$$

PROCEDURE OF FINITE ELEMENT ANALYSIS

- Assemble the element equilibrium equation (**ku**=**f**) for all finite elements in the structure and form global equilibrium equation, **KU**=**F**
- Consider boundary condition and modify the **K** matrix, **U** and **F** vectors. Form the following equation: **AX=B** where **X** **=** unknown vectors, **B** = known vectors.
- Solve the equation **AX**=**B**
- Calculate other parameters – displacements, strains, stresses etc