Potential Flow of Viscous Fluids: Historical Notes

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Abstract

In this essay I will attempt to identify the main events in the history of thought about irrotational flow of viscous fluids. I am of the opinion that when considering irrotational solutions of the Navier-Stokes equations it is never necessary and typically not useful to put the viscosity to zero. This observation runs counter to the idea frequently expressed that potential flow is a topic which is useful only for inviscid fluids; many people think that the notion of a viscous potential flow is an oxymoron. Incorrect statements like "… irrotational flow implies inviscid flow but not the other way around" can be found in popular textbooks.

Though convenient, phrases like "inviscid potential flow" or "viscous potential flow" confuse properties of the flow (potential or irrotational) with properties of the material (inviscid, perfect, viscous or viscoelastic); it is better and more accurate to speak of the irrotational flow of an inviscid, perfect or viscous fluid.

EVERY THEOREM ABOUT POTENTIAL FLOW OF PERFECT FLUIDS WITH CONSERVATIVE BODY FORCES APPLIES EQUALLY TO VISCOUS FLUIDS IN REGIONS OF IRROTATIONAL FLOW

Key words: perfect fluid, inviscid fluid, viscous fluid, potential flow, irrotational flow, vorticity, viscous decay, viscous irrotational pressure, dissipation method, boundary layer, drag, lift

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I. Navier-Stokes equations

The history of Navier-Stokes equations begins with the 1822 memoir of Navier who derived equations for homogeneous incompressible fluids from a molecular argument. Using similar arguments, Poisson (1829) derived the equations for a compressible fluid. The continuum derivation of the Navier-Stokes equation is due to Saint Venant (1843) and Stokes (1845). In his 1847 paper, Stokes wrote that

> Let P_1, P_2, P_3 be the three normal, and T_1, T_2, T_3 the three tangential pressures in the direction of three rectangular planes parallel to the co-ordinate planes, and let D be the symbol of differentiation with respect to t when the particle and not the point of space remains the same. Then the general equations applicable to a heterogeneous fluid, (the equations (10) of my former (1845) paper,) are
>
> $$\rho\left(\frac{\mathrm{D}u}{\mathrm{D}t}-X\right)+\frac{\mathrm{d}P_1}{\mathrm{d}x}+\frac{\mathrm{d}T_3}{\mathrm{d}y}+\frac{\mathrm{d}T_2}{\mathrm{d}z}=0, \tag{132}$$
>
> with the two other equations which may be written down from symmetry. The pressures P_1, &c. are given by the equations
>
> $$P_1 = p-2\mu\left(\frac{\mathrm{d}u}{\mathrm{d}x}-\delta\right),\; T_1=-\mu\left(\frac{dv}{dz}+\frac{dw}{dy}\right), \tag{133}$$
>
> and four other similar equations. In these equations
>
> $$3\delta=\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}\cdot \tag{134}$$

The equations written by Stokes in his 1845 paper are the same ones we use today:

$$\rho\left(\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}-\mathbf{X}\right)=\mathrm{div}\mathbf{T}, \tag{I.1}$$

$$\mathbf{T}=\left(-p-\frac{2}{3}\mu\mathrm{div}\mathbf{u}\right)\mathbf{1}+2\mu\mathbf{D}[\mathbf{u}], \tag{I.2}$$

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}=\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}, \tag{I.3}$$

$$\mathbf{D}[\mathbf{u}]=\frac{1}{2}\left[\nabla\mathbf{u}+\nabla\mathbf{u}^T\right], \tag{I.4}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t}+\rho\mathrm{div}\mathbf{u}=0\,. \tag{I.5}$$

Inviscid fluids are fluids with zero viscosity. Viscous effects on the motion of fluids were not understood before the notion of viscosity was introduced by Navier in 1822. Perfect fluids, following the usage of Stokes and other 19th century English mathematicians, are inviscid fluids which are also incompressible. Statements like Truesdell's (1954), In 1781 Lagrange presented his celebrated velocity-potential theorem: if a velocity potential exists at one time in a motion of an inviscid incompressible fluid, subject to conservative extraneous force, it exists at all past and future times.

though perfectly correct, could not have been asserted by Lagrange, since the concept of an inviscid fluid was not available in 1781.

II. Stokes theory of potential flow of viscous fluid

The theory of potential flow of a viscous fluid was introduced by Stokes (1850). All of his work on this topic is framed in terms of the effects of viscosity on the attenuation of small amplitude waves on a liquid-gas surface. Everything he said about this problem is cited below. The problem treated by Stokes was solved exactly using the linearized Navier-Stokes equations, without assuming potential flow, was solved exactly by Lamb (1932).

Stokes discussion is divided into three parts discussed in §51, 52, 53.

(1) The dissipation method in which the decay of the energy of the wave is computed from the viscous dissipation integral where the dissipation is evaluated on potential flow (§51).

(2) The observation that potential flows satisfy the Navier-Stokes together with the notion that certain viscous stresses must be applied at the gas-liquid surface to maintain the wave in permanent form (§52).

(3) The observation that if the viscous stresses required to maintain the irrotational motion are relaxed, the work of those stresses is supplied at the expense of the energy of the irrotational flow (§53).

Lighthill (1998) discussed Stokes' ideas but he did not contribute more to the theory of irrotational motions of a viscous fluid. On page 234 he notes that

"Stokes ingenious idea was to recognize that the average value of the rate of working given by sinusoidal waves of wave number

$$2\mu\left[(\partial\phi/\partial x)\partial^2\phi/\partial x\partial z+(\partial\phi/\partial z)\partial^2\phi/\partial z^2\right]_{z=0}$$

which is required to maintain the unattenuated irrotational motions of sinusoidal waves must exactly balance the rate at which the same waves when propagating freely would lose energy by internal dissipation."

Lamb (1932) gave an exact solution of the problem considered by Stokes in which vorticity and boundary layers are not neglected. He showed that the value given for the decay constant computed by Stokes is twice the correct value. Joseph and Wang (2004) computed the decay constant for gravity waves directly as an ordinary stability problem in which the velocity is irrotational, the pressure is given by Bernoulli's equation and the viscous component of the normal stress is evaluated on the irrotational flow. This kind of analysis we call viscous potential flow or VPF. The decay constant computed by VPF is one half the correct values computed by the dissipation method when the waves are longer than critical value for which progressive waves give way to monotonic waves. For waves shorter than the critical value the decay constant is given by $g/2\upsilon k$; the decay constant from Lambs exact solution agrees with the dissipation value for long waves and with the VPF value for short waves. All these facts can be obtained from two quite distinct irrotational approximations (VPF and VCVPF) discussed by Wang & Joseph (2005) in section *VIII*.

II.1 The dissipation method 51. By means of the expression given in Art. 49, for the loss of *vis viva* due to internal friction, we may readily obtain a very approximate solution of the problem: To determine the rate at which the motion subsides, in consequence of internal friction, in the case of a series of oscillatory waves propagated along the surface of a liquid. Let the vertical plane of xy be parallel to the plane of motion, and let y be measured vertically downwards from the mean surface; and for simplicity's sake suppose the depth of the fluid very great compared with the length of a wave, and the motion so small that the square of the velocity may be neglected. In the case of motion which we are considering, $udx + vdy$ is an exact differential $d\phi$ when friction is neglected, and

$$\phi = c\varepsilon^{-my}\sin(mx - nt), \tag{140}$$

where c, m, n are three constants, of which the last two are connected by a relation which it is not necessary to write down. We may continue to employ this equation as a near approximation when friction is taken into account, provided we suppose c, instead of being constant, to be parameter which varies slowly with the time. Let V be the *vis viva* of a given portion of the fluid at the end of the time t, then

$$V = \rho c^2 m^2 \iiint \varepsilon^{-2my}\mathrm{d}x\mathrm{d}y\mathrm{d}z\cdot \tag{141}$$

But by means of the expression given in Art.49, we get for the loss of vis viva during the time dt, observing that in the present case μ is constant, $w = 0$, $\delta = 0$, and $u\mathrm{d}x + v\mathrm{d}y = \mathrm{d}\phi$, where ϕ is independent of z,

$$4\mu\mathrm{d}t\iiint\left\{\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}\right)^2 + \left(\frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}\right)^2 + 2\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}x\mathrm{d}y}\right)^2\right\}\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

which becomes, on substituting for ϕ its value,

$$8\mu c^2 m^4 \mathrm{d}t\iiint \varepsilon^{-2my}\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

But we get from (141) for the decrement of *vis viva* of the same mass arising from the variation of the parameter c

$$-2\rho m^2 c\frac{\mathrm{d}c}{\mathrm{d}t}\mathrm{d}t\iiint \varepsilon^{-2my}\mathrm{d}x\mathrm{d}y\mathrm{d}z\cdot$$

Equating the two expressions for the decrement of *vis viva*, putting for m its value $2\pi\lambda^{-1}$, where λ is the length of a wave, replacing μ by $\mu'\rho$, integrating, and supposing c_0 to be the initial value of c, we get

$$c = c_0 \varepsilon^{-\frac{16\pi^2\mu' t}{\lambda^2}}.$$

In a footnote on page 624, Lamb notes that "Through an oversight in the original calculation the value $\lambda^2/16\pi^2\nu$ was too small by one half". The value 16 should be 8.

It will presently appear that the value of $\sqrt{\mu'}$ for water is about 0.0564, an inch and a second being the units of space and time. Suppose first that λ is two inches, and t ten seconds. Then $16\pi^2\mu' t\lambda^{-2} = 1.256$, and $c : c_0 :: 1 : 0.2848$, so that the height of the waves, which varies as c, is only about a quarter of what it was. Accordingly, the ripples excited on a small pool by a puff of wind rapidly subside when the exciting cause ceases to act.

Now suppose that λ is to fathoms or 2880 inches, and that t is 86400 seconds or a whole day. In this case $16\pi^2\mu' t\lambda^{-2}$ is equal to only 0.005232, so that by the end of an entire day, in which time waves of this length would travel 574 English miles, the height would be diminished by little more than the one two hundredth part in consequence of friction. Accordingly, the long swells of the ocean are but little allayed by friction, and at last break on some shore situated at the distance of perhaps hundreds of miles from the region where they were first excited.

II.2 The distance a wave will travel before it decays by a certain amount. The observations made by Stokes about the distance a wave will travel before its amplitude decays by a given amount, point the way to a useful frame for the analysis of the effects of viscosity on wave propagation. Many studies of nonlinear irrotational waves can be found in the literature but the only study of the effects of viscosity on the decay of these waves known to me are due to M. Longuet-Higgins (1997) who used the dissipation method to determine the decay due to viscosity of irrotational steep capillary-gravity waves in deep water. He finds that that the limiting rate of decay for small amplitude solitary waves are twice those for linear periodic waves computed by the dissipation method. The dissipation of very steep waves can be more than ten times more than linear waves due to the sharply increased curvature in wave troughs. He assumes that that the nonlinear wave maintains its steady form while decaying under the action of viscosity. The wave shape could change radically from its steady shape in very steep waves. These changes could be calculated for irrotational flow using VPF as in the work of Miksis, et al. (1982) (see XI).

Stokes (1847) studied the motion of nonlinear irrotational gravity waves in two dimensions which are propagated with a constant velocity, and without change of form. This analysis led Stokes (1880) to the celebrated maximum wave whose asymptotic form gives rise to a pointed crest of angle 120°. The effects of viscosity on such extreme waves has not been studied but they may be studied by the dissipation method or same potential flow theory used by Stokes (1847) for inviscid fluids with the caveat that the normal stress condition that p vanish on the free surface be replaced by the condition that

$$p + \mu \partial u_n / \partial n = 0$$

on the free surface with normal **n** where the velocity component $u_n = \partial \phi / \partial n$ is given by the potential.

II.3 The stress of a viscous fluid in potential flow. 52. It is worthy of remark, that in the case of a homogeneous incompressible fluid, whenever $u\mathrm{d}x + v\mathrm{d}y + w\mathrm{d}z$ is an exact differential, not only are the ordinary equations of fluid motion satisfied*, but the equations obtained when friction is taken into account are satisfied likewise. It is only the equations of condition which belong to the boundaries of the fluid that are violated. Hence any kind of motion which is possible according to the ordinary equations, and which is such that $u\mathrm{d}x + v\mathrm{d}y + w\mathrm{d}z$ is an exact differential, is possible likewise when friction is taken into account, provided we suppose a certain system of normal and tangential pressures to act at the boundaries of the fluid, so as to satisfy the equations (133). Since μ disappears from the general equations (1), it follows that p is the same function as before. But in the first case the system of pressures at the surface was $P_1 = P_2 = P_3 = p$, $T_1 = T_2 = T_3 = 0$. Hence if ΔP_1 &c. be the additional pressures arising from friction, we get from (133), observing that $\delta = 0$, and that $u\mathrm{d}x + v\mathrm{d}y + w\mathrm{d}z$ is an exact differential $\mathrm{d}\phi$,

$$\Delta P_1 = -2\mu \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}, \quad \Delta P_2 = -2\mu \frac{\mathrm{d}^2\phi}{\mathrm{d}y^2}, \quad \Delta P_3 = -2\mu \frac{\mathrm{d}^2\phi}{\mathrm{d}z^2}, \tag{142}$$

$$\Delta T_1 = -2\mu \frac{\mathrm{d}^2\phi}{\mathrm{d}y\mathrm{d}z}, \quad \Delta T_2 = -2\mu \frac{\mathrm{d}^2\phi}{\mathrm{d}z\mathrm{d}x}, \quad \Delta T_3 = -2\mu \frac{\mathrm{d}^2\phi}{\mathrm{d}x\mathrm{d}y}, \tag{143}$$

Let dS be an element of the bounding surface, l', m', n' the direction-cosines of the normal drawn outwards, ΔP, ΔQ, ΔR the components in the direction of x, y, z of the additional pressure on a plane in the direction of dS. Then by the formula (9) of my former paper applied to the equations (142), (143) we get

$$\Delta P = -2\mu \left\{ l' \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + m' \frac{\mathrm{d}^2\phi}{\mathrm{d}x\mathrm{d}y} + n' \frac{\mathrm{d}^2\phi}{\mathrm{d}x\mathrm{d}z} \right\}, \tag{144}$$

with similar expressions for ΔQ and ΔR, and ΔP, ΔQ, ΔR are the components of the pressure which must be applied at the surface, in order to preserve the original motion unaltered by friction.

II.4 Viscous stresses needed to maintain an irrotational wave. Viscous decay of the free wave. 53. Let us apply this method to the case of oscillatory waves, considered in Art. 51. In this case the bounding surface is nearly horizontal, and its vertical ordinates are very small, and since the squares of small quantities are neglected, we may suppose the surface to coincide with the plane of xz in calculating the system of pressures which must be supplied, in order to keep up the motion. Moreover, since the motion is symmetrical with respect to the plane of xy, there will be no tangential pressure in the direction of z, so that the only pressures we have to calculate are ΔP_2 and ΔT_3. We get from (140), (142), and (143), putting $y = 0$ after differentiation,

$$\Delta P_2 = -2\mu m^2 c \sin(mx - nt), \ \Delta T_3 = 2\mu m^2 c \cos(mx - nt). \tag{145}$$

If u_1, v_1 be the velocities at the surface, we get from (140), putting $y = 0$ after differentiation,

$$u_1 = mc\cos(mn - nt), \ v_1 = -mc\sin(mx - nt). \tag{146}$$

It appears from (145) and (146) that the oblique pressure which must be supplied at the surface in order to keep up the motion is constant in magnitude, and always acts in the direction in which the particles are moving.

The work of this pressure during the time dt corresponding to the element of surface dxdz, is equal to $\mathrm{d}x\mathrm{d}z(\Delta T_3 \cdot u_1 \mathrm{d}t + \Delta P_1 \cdot v_1 \mathrm{d}t)$. Hence the work exerted over a given portion of the surface is equal to

$$2\mu m^3 c^2 \mathrm{d}t \iint \mathrm{d}x\mathrm{d}z \cdot$$

In the absence of pressures ΔP_2, ΔT_3 at the surface, this work must be supplied at the expense of *vis viva*. Hence $4\mu m^3 c^2 \mathrm{d}t \iint \mathrm{d}x\mathrm{d}z$ is the *vis viva* lost by friction, which agrees with the expression obtained in Art. 51, as will be seen on performing in the latter the integration with respect to y, the limits being $y = 0$ to $y = \infty$.

III. Irrotational solutions of the Navier-Stokes equations; irrotational viscous stresses.

Consider first the case of incompressible fluids $\operatorname{div}\mathbf{u} = 0$. If $\mathbf{X}$ has a potential ψ and the fluid is homogeneous (ρ and μ are constants independent of position at all times) then it is readily shown that

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla|\mathbf{u}|^2 - \mathbf{u}\wedge\boldsymbol{\omega}\right] = -\nabla(p+\psi) + \mu\nabla^2\mathbf{u}\,, \tag{III.1}$$

where $\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}$. It is evident that $\boldsymbol{\omega} = 0$ is a solution of the curl (III.1). In this case

$$\mathbf{u} = \nabla\phi\,,\ \nabla^2\phi = 0\,. \tag{III.2}$$

Since $\mu\nabla^2\mathbf{u} = \mu\nabla\nabla^2\phi = 0$ independent of μ, for large viscosities as well as small viscosities, (III.1) shows that

$$\nabla\left(\rho\frac{\partial\phi}{\partial t} + \frac{\rho}{2}|\nabla\phi|^2 + p + \psi\right) = 0\,, \tag{III.3}$$

and $p = p_I$ is determined by Bernoulli's

$$\rho\frac{\partial\phi}{\partial t} + \frac{\rho}{2}|\nabla\phi|^2 + p_I + \psi = F(t)\,. \tag{III.4}$$

Potential flow $\mathbf{u} = \nabla\phi$, $\nabla^2\phi$ is a solution of the homogeneous, incompressible Navier-Stokes with a pressure $p = p_I$ determined by Bernoulli's equation, independent of viscosity. All of this known, maybe even well known, but largely ignored by the fluid mechanics community from the days of Stokes up till now.

Much less well known, and totally ignored, is the formula (I.2) for the viscous stress evaluated on potential flow $\mathbf{u} = \nabla\phi$,

$$\mathbf{T} = p\mathbf{1} + 2\mu\nabla\otimes\nabla\phi\,. \tag{III.5}$$

The formula shows directly and with no ambiguity that viscous stresses are associated irrotational flow. This formula is one of the most important that could be written about potential flows. It is astonishing, that aside from Stokes (1850), this formula which should be in every book on fluid mechanics, can not be found in any.

The resultants of the irrotational viscous stresses (III.5) do not enter into the Navier-Stokes equations (III.1). Irrotational motions are determined by the condition that the solenoidal velocity is curl free and the evolution of the potential is associated with the irrotational pressure in the Bernoulli equation. However, the dissipation of the energy of potential flows and the power of viscous irrotational stresses do not vanish. Regions of curl free motions of the Navier-Stokes equations are guaranteed by various theorems concerning the persistence of irrotationality in the motions of parcels of fluid emanating from regions of irrotational flow (see section IX). All flows on unbounded domains which tend asymptotically to rest or uniform motion and all the irrotational flows outside of vorticity boundary layers give rise to an additional irrotational viscous dissipation which deserves consideration.

The effects of viscous irrotational stresses which are balanced internally enter into the dynamics of motion at places where they become unbalanced such as at free surfaces and at the boundary of regions in which vorticity is important such as boundary layers and even at internal points in the liquid at which stress induced tensions exceed the breaking strength of the liquid. Irrotational viscous stresses enter as an important element in a theory of stress induced cavitation in which the field of principal stresses which determine the places and times at which the tensile stress exceed the breaking strength or cavitation threshold of the liquid must be computed (see Funada, et al. 2005 and Padrino et al. 2005).

Irrotational flows cannot satisfy no-slip and associated conditions at boundaries when $\mu > 0$ (and also when $\mu = 0$). No real fluid has $\mu = 0$. It is an act of self deception to put away no-slip by positing a fictitious fluid which has no viscosity.

Irrotational flows of a viscous fluid scale with the Reynolds number as do rotational solutions of the Navier-Stokes equations generally. The solutions of the Navier-Stokes equations, rotational and irrotational, are thought to become independent of the Reynolds number at large Reynolds numbers. They can be said to converge to a common set of solutions corresponding to irrotational motion of an inviscid fluid. This limit should be thought to correspond a condition of flow, large Reynolds numbers, and not to a weird material without viscosity; the viscosity should not be put to zero.

Stokes thought that the motion of perfect fluids is an ideal abstraction from the motion of real fluids with small viscosity, like water. He did not mention irrotational flows of very viscous fluids which are associated with normal stresses

$$\tau_n = \mu \mathbf{n} \cdot (\nabla \otimes \nabla \phi) \cdot \mathbf{n}$$

in situations in which the dynamical effects of shear stresses in the direction $\mathbf{t}$

$$\tau_s = \mu\, \mathbf{t} \cdot (\nabla \otimes \nabla \phi) \cdot \mathbf{n}$$

are negligible. The irrotational purely radial motion of a gas bubble in a liquid (the Rayleigh-Poritsky bubble (Poritsky 1951), usually incorrectly attributed to Rayleigh-Plesset (Plesset 1949)) is a potential flow. The shear stresses are zero everywhere but the irrotational normal stresses scale with the viscosity for any viscosity, large or small.

Another exact irrotational solution of the Navier Stokes equations is the flow between rotating cylinders in which the angular velocities of the cylinders are adjusted to fit the potential solution in circles with

$$\mathbf{u} = \mathbf{e}_\theta u,$$
$$u = a^2\omega_a / r = b^2\omega_b / r.$$

The torques necessary to drive the cylinders are proportional to the viscosity of the liquid for any viscosity, large or small. This motion may be realized approximately in a cylinder of large height with a free surface on top anchored in a bath of mercury below.

A less special example is embedded in almost every complex flow of a viscous fluid at each and every stagnation point. The flow at a point of stagnation is a purely extensional flow, a potential flow with extensional stresses proportional to the product of viscosity times the rate extension there. The irrotational viscous extensional stresses at points of extension can be huge even when the viscosity is small.

A somewhat more complex set of flows of viscous fluids which are very nearly irrotational are generated by waves on free surfaces. The shear stresses on the free surfaces vanish but the normal stresses generated by the up and down motion of the waves do not vanish; gravity waves on highly viscous fluids are greatly retarded by viscosity. It is not immediately obvious that the effects of vorticity on such waves are so well approximated by purely irrotational motions (see Lamb 1932 and Wang and Joseph 2005). Very rich theories of common irrotational flows of a viscous fluid which update and greatly improve conventional studies of perfect fluids are assembled and can be downloaded from PDF files at

(http://www.aem.umn.edu/people/faculty/joseph/ViscousPotentialFlow/).

IV. Irrotational solutions of the compressible Navier-Stokes equations and the equations of motion for certain viscoelastic fluids.

The velocity may be obtained from a potential provided that the vorticity ζ=curl u=0 at all points in a simply connected region. This is a kinematic condition which may or may not be compatible with the equations of motion. For example, if the viscosity varies with position or the body forces are not potential, then extra terms, not containing the vorticity will appear in the vorticity equation and $\boldsymbol{\zeta=0}$ will not be a solution in general. Joseph and Liao (1994) formulated a compatibility condition for irrotational solutions $\mathbf{u} = \nabla\phi$ of (I.1) in the form

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \mathrm{grad}x = \frac{1}{\rho}\mathrm{div}\mathbf{T}[\mathbf{u}] \tag{IV.1}$$

If

$$\frac{1}{\rho}\mathrm{div}\mathbf{T}[\nabla\phi] = -\nabla\psi\ , \tag{IV.2}$$

the $\boldsymbol{\zeta} = 0$ is a solution of (IV.1) and

$$\rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \chi\right) + \psi = f(t) \tag{IV.3}$$

is the Bernoulli equation.

Consider first the case of viscous compressible flow (Joseph 2003) for which the stress is given by (I.2). The gradient of density and viscosity which are spoiler for the general Vorticity equation do not enter the equations which perturb uniform states of pressure p_0, density ρ_0 and velocity U_0.

To study acoustic propagation, the equations are linearized; putting

$$[\mathbf{u}, p, p] = [\mathbf{u}', p_0 + p', \rho_0 + \rho'], \tag{IV.4}$$

where $\mathbf{u}'$, p' and ρ' are small quantities, we obtain

$$T_{ij} = -\left(p_0 + p' + \frac{2}{3}\mu_0 \mathrm{div}\mathbf{u}' \right)\delta_{ij} + \mu_0 \left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right), \tag{IV.5}$$

$$\rho_0 \left(\frac{\partial \mathbf{u}'}{\partial t} + \nabla \chi \right) = -\nabla p' + \mu_0 \left(\nabla^2 \mathbf{u}' + \frac{1}{3}\nabla \mathrm{div}\mathbf{u}' \right), \tag{IV.6}$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \mathrm{div}\mathbf{u}' = 0\,, \tag{IV.7}$$

where p_0, ρ_0 and μ_0 are constants. For acoustic problems, we assume that a small change in ρ induces small changes in p by fast adiabatic processes; hence

$$p' = C_0^2 \rho'\,, \tag{IV.8}$$

where C_0 is the speed of sound.

Forming now the curl of (IV.6) we find that $\mathrm{curl}\mathbf{u}' = 0$ is a solution and $\mathbf{u}' = \nabla\phi$. This gives rise to a viscosity dependent Bernoulli equation

$$\rho_0 \left(\frac{\partial \phi}{\partial t} + \chi \right) + \psi = f(t),$$

where

$$\psi = -\frac{4}{3}\mu_0 \nabla^2 \phi.^{*}$$

When $\chi = 0$, the stress (IV.5) is given in terms of the potential ϕ by

$$T_{ij} = -\left(p_0 - \rho_0 \frac{\partial \phi}{\partial t} + 2\mu_0 \nabla^2 \phi \right)\delta_{ij} + 2\mu_0 \frac{\partial^2 \phi}{\partial x_i \partial x_j}\,. \tag{IV.9}$$

To obtain the equation satisfied by the potential ϕ, we eliminate ρ' in (IV.7) with p' using (IV.8), then eliminate $\mathbf{u}' = \nabla\phi$ and p' in terms of ϕ using $\rho_0 \frac{\partial \phi}{\partial t} + p' - \frac{4}{3}\mu_0 \nabla^2 \phi = 0$ to find

$$\frac{\partial^2 \phi}{\partial t^2} = \left(C_0^2 + \frac{4}{3}\nu_0 \frac{\partial}{\partial t} \right)\nabla^2 \phi\,, \tag{IV.10}$$

where the potential ϕ depends on the speed of sound and the kinematic viscosity $\nu_0 = \mu_0 / \rho_0$.

Joseph and Liao (1994) showed that most models of a viscoelastic fluid do not satisfy the compatibility condition (IV.2) in general but it may be satisfied for particular irrotational flows like stagnation point flow of any fluid. The equations of motion satisfy the compatibility equation (IV.2) in the case of inviscid, viscous and linear viscoelastic fluids for which $\psi = 0$ is the usual Bernoulli pressure and the second order fluid model (Joseph 1992 extending results of Pipkin 1970) for which

$$\psi = p - \hat{\beta}(\nabla \otimes \nabla \phi)^2$$

where $\beta = n_2 - n_1/2$ and n_1 and n_2 are the coefficients of the first and second normal stress difference.

*Truesdell (1950) discussed Bernoulli's theorem for viscous compressible fluids under some exotic hypothesis for which in general the vorticity is not zero. He notes "…Long ago Craig (1890) noticed that in the degenerate and physically improbable case of steady irrotational flow of a viscous incompressible fluid…the classical Bernoulli theorem of type (A) still holds…" Type (A) is a Bernoulli equation for a compressible fluid which holds throughout the fluid. Craig does not consider the linearized case for which the Bernoulli equation for compressible fluids has an explicit dependence on viscosity which is neither degenerate or improbable.

V. Irrotational solutions of the Navier-Stokes equations: viscous contributions to the pressure

A viscous contribution to the pressure in irrotational flow is a new idea which is required to resolve the discrepancy between the direct VPF calculation of the decay of an irrotational wave and the calculation based on the dissipation method. The calculation by VPF differs from the calculation based on potential flow of an inviscid fluid because the viscous component of the normal stress at the free surface is included in the normal stress balance. The viscous component of the normal stress is evaluated on potential flow. The dissipation calculation starts from the evolution of energy equation in which the dissipation integral is evaluated on the irrotational flow; the pressure does not enter into this evaluation. Why does the decay rate computed by these two methods give rise to different values? The answer to this question is associated with a viscous correction of the irrotational pressure which is induced by the uncompensated irrotational shear stress at the free surface; the shear stress should be zero there but the irrotational shear stress, proportional to viscosity, is not zero. The irrotational shear stress cannot be made to vanish in potential flow but the explicit appearance of this shear stress in the traction integral in the energy balance can be eliminated in the mean by the selection of an irrotational pressure which depends on viscosity.

The idea of a viscous contribution to the pressure seems to have been first suggested to Moore (1963) by Batchelor as a method of reconciling the discrepancy in the values of the drag on a spherical gas bubble calculated on irrotational flow by the dissipation method and directly by VPF (section XI). The first successful calculation of this extra pressure was carried out for the spherical bubble by Kang and Leal (1988a,b). Their work suggested that this extra viscous pressure could be calculated from irrotational flow without reference to boundary layers or vorticity. This idea was first implemented by Joseph and Wang (2004) using an energy argument in which the viscous pressure was selected to eliminate the uncompensated irrotational shear stress from the power of traction integral at the bubble surface. The idea was further developed by Wang and Joseph (2005) in their study of viscous decay of irrotational gravity waves that showed that this viscous contribution to the pressure could be calculated from a purely irrotational theory. Their study is valuable because it can be compared with the exact solution of Lamb (1932) in which boundary layers and vorticity are present but not important.

Purely irrotational theories of the effect of the viscosity on the decay of free gravity waves

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Abstract

A purely irrotational theory of the effect of viscosity on the decay of free gravity waves is derived and shown to be in excellent agreement with Lamb's (1932) exact solution. The agreement is achieved for all waves numbers k excluding a small interval around a critical $k=k_c$ where progressive waves change to monotonic decay. Very detailed comparisons are made between the purely irrotational and exact theory.

1. Introduction

Lamb (1932, §348, §349) performed an analysis of the effect of viscosity on free gravity waves. He computed the decay rate by a dissipation method using the irrotational flow only. He also constructed an exact solution for this problem, which satisfies both the normal and shear stress conditions at the interface.

Joseph and Wang (2004) studied Lamb's problem using the theory of viscous potential flow (VPF) and obtained a dispersion relation which gives rise to both the decay rate and wave-velocity. They also computed a viscous correction for the irrotational pressure and used this pressure correction in the normal stress balance to obtain another dispersion relation. This method is called a viscous correction of the viscous potential flow (VCVPF). Since VCVPF is an irrotational theory the shear stress cannot be made to vanish. However, the corrected pressure eliminates this uncompensated shear stress from the power of traction integral arising in an energy analysis of the irrotational flow.

Here we find that the viscous pressure correction of the irrotational motion gives rise to a higher order irrotational correction to the irrotational velocity which is proportional to the viscosity and does not have a boundary layer structure. The corrected velocity depends strongly on viscosity and is not related to vorticity; the whole package is purely irrotational. The corrected irrotational flow gives rise to a dispersion relation which is in splendid agreement with Lamb's exact solution, which has no explicit viscous pressure.

The agreement with the exact solution holds for fluids even 10^4 times more viscous than water and for small and large wave numbers where the cutoff wave number k_c marks the place where progressive waves give rise to monotonic decay. We find that VCVPF gives rise to the same decay rate as in Lamb's exact solution and in his dissipation calculation when $k < k_c$. The exact solution agrees with VPF when $k > k_c$. The effects of vorticity are sensible only in a small interval centered on the cutoff wave number. We will do a comprehensive comparison for the decay rate and wave-velocity given by Lamb's exact solution and Joseph and Wang's VPF and VCVPF theories.

2. Irrotational viscous corrections for the potential flow solution

The gravity wave problem is governed by the continuity equation

$$\nabla\cdot\mathbf{u}=0, \tag{1}$$

and the linearized Navier-Stokes equation

$$\frac{\partial\mathbf{u}}{\partial t}=-\frac{1}{\rho}\nabla p-g\mathbf{e}_y+\nu\nabla^2\mathbf{u}, \tag{2}$$

with the boundary conditions at the free surface ($y\approx 0$)

$$T_{xy}=0,\quad T_{yy}=0, \tag{3}$$

where T_{xy} and T_{yy} are components of the stress tensor and the surface tension is neglected. We divide the velocity and pressure field into two parts

$$\mathbf{u}=\mathbf{u}_{\mathrm{p}}+\mathbf{u}_{\mathrm{v}},\; p=p_{\mathrm{p}}+p_{\mathrm{v}}, \tag{4}$$

where the subscript p denotes potential solutions and v denotes viscous corrections. The potential solutions satisfy

$$\mathbf{u}_{\mathrm{p}}=\nabla\phi,\; \nabla^2\phi=0, \tag{5}$$

and

$$\frac{\partial\mathbf{u}_{\mathrm{p}}}{\partial t}=-\frac{1}{\rho}\nabla p_{\mathrm{p}}-g\mathbf{e}_y\cdot \tag{6}$$

The viscous corrections are governed by

$$\nabla\cdot\mathbf{u}_{\mathrm{v}}=0, \tag{7}$$

$$\frac{\partial\mathbf{u}_{\mathrm{v}}}{\partial t}=-\frac{1}{\rho}\nabla p_{\mathrm{v}}+\nu\nabla^2\mathbf{u}_{\mathrm{v}}\cdot \tag{8}$$

We take the divergence of (8) and obtain

$$\nabla^2 p_{\mathrm{v}}=0, \tag{9}$$

which shows that the pressure correction must be harmonic. Next we introduce a stream function ψ so that (7) is satisfied identically:

$$u_{\mathrm{v}}=-\frac{\partial\psi}{\partial y},\; \mathrm{v}_{\mathrm{v}}=\frac{\partial\psi}{\partial x}\cdot \tag{10}$$

We eliminate p_{v} from (8) by cross differentiation and obtain following equation for the stream function

$$\frac{\partial}{\partial t}\nabla^2\psi=\nu\nabla^4\psi\cdot \tag{11}$$

To determine the normal modes which are periodic in respect of x with a prescribed wave-length $\lambda=2\pi/k$, we assume a time-factor e^{nt}, and a space-fact e^{ikx}

$$\psi=B\mathrm{e}^{nt+\mathrm{i}kx}\mathrm{e}^{my}, \tag{12}$$

where m is to be determined from (11). Inserting (12) into (11), we obtain

$$(m^2-k^2)\left[n-\nu(m^2-k^2)\right]=0\,. \tag{13}$$

The root $m^2=k^2$ gives rise to irrotational flow; the root $m^2=k^2+n/\nu$ leads to the rotational component of the flow. The rotational component cannot give rise to a non-zero harmonic pressure because

$$\nabla^2\mathrm{e}^{nt+\mathrm{i}kx}\mathrm{e}^{my}=(m^2-k^2)\mathrm{e}^{nt+\mathrm{i}kx}\mathrm{e}^{my} \tag{14}$$

can not vanish if $m^2\neq k^2$. The only harmonic pressure for the rotational component is zero. Thus the governing equation for the rotational part of the flow can be written as

$$\frac{\partial\psi}{\partial t}=\nu\nabla^2\psi, \tag{15}$$

which is the equation used by Lamb (1932) in his exact solution.

The effect of viscosity on the decay of a free gravity wave can be approximated by a purely irrotational theory in which the explicit dependence of the power of traction of the irrotational shear stress is eliminated by a viscous contribution p_v to irrotational pressure. In this theory $\mathbf{u}=\nabla\phi$ and a stream function, which is associated with vorticity, is not introduced. The kinetic energy, potential energy and dissipation of the flow can be computed using the potential flow solution

$$\phi = A\mathrm{e}^{nt+ky+\mathrm{i}kx}. \tag{16}$$

We insert the potential flow solution into the mechanical energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_V \rho|\mathbf{u}|^2/2\,\mathrm{d}V + \int_0^\lambda \rho g\eta^2/2\,\mathrm{d}x\right) = \int_0^\lambda\left[\mathrm{v}(-p+\tau_{yy}) + u\tau_{xy}\right]\mathrm{d}x - \int_V 2\mu\mathbf{D}:\mathbf{D}\,\mathrm{d}V, \tag{17}$$

where η is the elevation of the surface and $\mathbf{D}$ is the rate of strain tensor. Motivated by previous authors (Moore 1963, Kang and Leal 1988), we add a pressure correction to the normal stress which satisfies

$$\int_0^\lambda \mathrm{v}(-p_v)\,\mathrm{d}x = \int_0^\lambda u\tau_{xy}\,\mathrm{d}x, \tag{18}$$

But in our problem here, there is no explicit viscous pressure function in the exact solution [see (24) and (25)]. It turns out that the pressure correction defined here in the purely irrotational flow is related to quantities in the exact solution in a complicated way which requires further analysis [see (31)].

Joseph and Wang (2004) solved for the harmonic pressure correction from (9), then determined the constant in the expression of p_v using (18), and obtained

$$p_v = -2\mu k^2 A\mathrm{e}^{nt+ky+\mathrm{i}kx}. \tag{19}$$

The velocity correction associated with this pressure correction can be solved from (8). We seek normal modes solution $\mathbf{u}_v \sim \mathrm{e}^{nt+ky+\mathrm{i}kx}$ and equation (8) becomes

$$\rho n\mathbf{u}_v = -\nabla p_v. \tag{20}$$

Hence, curl($\mathbf{u}_v$) = 0 and $\mathbf{u}_v$ is irrotational. After assuming $\mathbf{u}_v = \nabla\phi_1$ and $\phi_1 = A_1\mathrm{e}^{nt+ky+\mathrm{i}kx}$, we obtain

$$\rho n\phi_1 = -p_v \quad \Rightarrow \quad \phi_1 = \frac{2\mu k^2}{\rho n}A\mathrm{e}^{nt+ky+\mathrm{i}kx}. \tag{21}$$

We compute the viscous normal stress due to the velocity correction

$$2\mu\frac{\partial \mathrm{v}_v}{\partial y} = 2\mu\frac{\partial^2\phi_1}{\partial y^2} = \frac{4\mu^2k^4}{\rho n}A\mathrm{e}^{nt+ky+\mathrm{i}kx}. \tag{22}$$

Since for mobile fluids such as water or even glycerin, $\nu=\mu/\rho$ is small, this viscous normal stress is negligible compared to p_v when k is small. Therefore, the viscous normal stress induced by the velocity correction can be neglected in the normal stress balance in the VPVPF theory. The viscous normal stress (22) could be large when k is large, however, we will show in the following sections that the flow is nearly irrotational at large values of k and no correction is needed.

The calculation shows that the velocity correction $\mathbf{u}_v$ associated with the pressure correction is irrotational. Our pressure correction (19) is proportional to μ and on the same order of the viscous stresses evaluated using ϕ (16). This pressure correction is associated with a correction for the velocity potential ϕ_1 (21), which is also proportional to μ. The shear stress evaluated using ϕ_1 is proportional to μ^2 and non-zero. To balance the power of this non-physical shear stress, one can add a pressure correction proportional to μ^2, which will in turn induce a correction for the velocity potential proportional to μ^2. One can continue to build higher order corrections and they will all be irrotational. The final velocity potential has the following form

$$\phi = (A + A_1 + A_2 + \cdots)\mathrm{e}^{nt+ky+\mathrm{i}kx}, \tag{23}$$

where $A_1{\sim}\mu$, $A_2{\sim}\mu^2$... Thus the VCVPF theory is an approximation to the exact solution based on solely potential flow solutions. The higher order corrections are small for liquids with small viscosities; the most important correction is the first pressure correction proportional to μ. In our application of VCPVF to the gravity wave problem, only the first pressure correction (19) is added to the normal stress balance and higher order normal stress terms such as (22) are not added. We obtain a dispersion relation in excellent agreement with Lamb's exact solution (see the comparison in the next section); adding the higher order corrections to the normal stress balance does not improve the VCVPF approximation. It should be pointed out that no matter how many correction terms are added to the potential (23), the shear stress evaluated using (23) is still non-zero unless $(A + A_1 + A_2 + ...) = 0$. Therefore, VCVPF is only an approximation to the exact solution and cannot satisfy the shear stress condition at the free surface.

VI. Irrotational solutions of the Navier-Stokes equations: classical theorems.

An authorative and readable exposition of irrotational flow theory and its applications can be found in chapter 6 of the book on fluid dynamics by Batchelor (1967). He speaks of the role of the theory of flow of an inviscid fluid. He says

> In this and the following chapter, various aspects of the flow of a fluid regarded as entirely inviscid (and incompressible) will be considered. The results presented are significant only inasmuch as they represent an approximation to the flow of a real fluid at large Reynolds number, and the limitations of each result must be regarded as information as the result itself.

Most of the classical theorems reviewed in Chapter 6 do not require that the fluid be inviscid. These theorems are as true for viscous potential flow as they are for inviscid potential flow. Kelvin's minimum energy theorem holds for the irrotational flow of a viscous fluid. The results for the positions of the maximum speed the minimum of the pressure given by the Bernoulli equation follow from the assumption that the flow is irrotational independent of the viscosity of the fluid.

The theory of the acceleration reaction leads to the concept of added mass; it follows from the analysis of unsteady irrotational flow. The theory applies to viscous and inviscid fluids alike.

Harold Jeffrey (1928) derived an equation (his (20)) which replaces the circulation theorem of classical (inviscid) hydrodynamics. When the fluid is homogeneous, Jeffrey's equation may be written as

$$\frac{\mathrm{d}C}{\mathrm{d}t} = -\frac{\mu}{\rho}\oint \mathrm{curl}\,\boldsymbol{\omega} \cdot d\mathbf{l} \tag{VI.1}$$

where

$$C(t) = \oint \mathbf{u} \cdot d\mathbf{l}$$

is the circulation round a closed material curve drawn in the fluid. This equation shows that

> "… the initial value of $\mathrm{d}C/\mathrm{d}t$ around a contour in a fluid originally moving irrotationally is zero, whether or not there is a moving solid within the contour. This at once provides an explanation of the equality of the circulation about an aero plane and that about the vortex left behind when it starts; for the circulation about a large contour that has never been cut by the moving solid or its wake remains zero, and therefore the

circulations about contours obtained by subdividing it must also add up to zero. It also indicates why the motion is in general nearly irrotational except close to a solid or to fluid that has passed near one".

St. Venant (1869) interpreted the result of Lagrange about the invariance of circulation $\mathrm{d}C/\mathrm{d}t = 0$ to mean that

vorticity cannot be generated in the interior of a viscous incompressible fluid, subject to conservative extraneous force, but is necessarily diffused inward from the boundaries.

The circulation formula (VI.1) is an important result in the theory of irrotational flows of a viscous fluid. A particle which is initially irrotational will remain irrotational in motions which do not enter into the vortical layers at the boundary.

VII. Critical remarks about the "The impossibility of irrotational motions in general".

This topic is treated in § 37 of the monograph by Truesdell (1954). The basic idea is that, in general, irrotational motions of incompressible fluids satisfy Laplace's equation and the normal and tangential velocities at the bounding surfaces can not be simultaneously prescribed. The words "in general" allow for rather special cases in which the motion of the bounding surfaces just happens to coincide with the velocities given by the derivatives of the potential. Such special motions were studied for viscous incompressible fluids by Hamel (1941). A bounding surface must always contact the fluid so the normal component of the velocity of the fluid must be exactly the same as the normal component of the velocity of the boundary. The no-slip condition cannot then "in general" be prescribed. Truesdell uses "adherence condition" meaning "sticks fast" rather than the usual no-slip condition of Stokes. The no slip condition is even now a topic of discussion and the mechanisms by which fluids stick fast are not clear. Truesdell does not consider liquid-gas surfaces or, more exactly, liquid-vacuum surfaces on which slip is allowed.

Truesdell's conclusion

"…that the boundary condition customarily employed in the theory of viscous fluids makes irrotational motion is a virtual impossibility."

is hard to reconcile with the idea that flows outside boundary layers, are asymptotically irrotational. Ever so many examples of non-exotic calculations of irrotational motions of viscous fluids which approximate exact solutions of the Navier-Stokes equations and even agree with experiments at low Reynolds numbers are listed on Joseph's web based archive.

VIII. The drag on a spherical gas bubble

As in the case of irrotational waves, the problem of the drag on gas bubbles in a viscous liquid may be studied using viscous potential flow directly and by the dissipation method and the two calculations do not agree.

The idea that viscous forces in regions of potential flow may actually dominate the dissipation of energy was first expressed by Stokes (1950), and then, with more details, by Lamb 1932 who studied the viscous decay of free oscillatory waves on deep water § 348 and small oscillations of a mass of liquid about the spherical form § 355, using the dissipation method. Lamb showed that in these cases the rate of dissipation can be calculated with sufficient accuracy by regarding the motion as irrotational.

VIII.1 Dissipation calculation. The computation of the drag D on a sphere in potential flow using the dissipation method seems to have been given first by Bateman (1932) (see Dryden, et al. 1956) and repeated by Ackeret (1952). They found that $D = 12\pi a\mu U$ where μ is the viscosity, a radius of the sphere and U its velocity. This drag is twice the Stokes drag and is in better agreement with the measured drag for Reynolds numbers in excess of about 8.

The same calculation for a rising spherical gas bubble was given by Levich (1949). Measured values of the drag on spherical gas bubbles are close to $12\pi a\mu U$ for Reynolds numbers larger than about 20. The reasons for the success of the dissipation method in predicting the drag on gas bubbles have to do with the fact that vorticity is confined to thin layers and the contribution of this vorticity to the drag is smaller in the case of gas bubbles, where the shear traction rather than the relative velocity must vanish on the surface of the sphere. A good explanation was given by Levich (1962) and by Moore (1959, 1963); a convenient reference is Batchelor (1967). Brabston and Keller (1975) did a direct numerical simulation of the drag on a gas spherical bubble in steady ascent at terminal velocity U in a Newtonian fluid and found the same kind of agreement with experiments. In fact, the agreement between experiments and potential flow calculations using the dissipation method are fairly good for Reynolds numbers as small as 5 and improves (rather than deteriorates) as the Reynolds number increases.

The idea that viscosity may act strongly in the regions in which vorticity is effectively zero appears to contradict explanations of boundary layers which have appeared repeatedly since Prandtl. For example, Glauert (1943) says (p.142) that

> … Prandtl's conception of the problem is that the effect of the viscosity is important only in a narrow boundary layer surrounding the surface of the body and that the viscosity may be ignored in the free fluid outside this layer.

According to Harper (1972), this view of boundary layers is correct for solid spheres but not for spherical bubbles. He says that

> … for R>>1, the theories of motion past solid spheres and tangentially stress-free bubbles are quite different. It is easy to see why this must be so. In either case vorticity must be generated at the surface because irrotational flow does not satisfy all the boundary conditions. The vorticity remains within a boundary layer of thickness $\delta = O\left(aR^{-1/2}\right)$, for it is convected around the surface in a time t of order a/U, during which viscosity can diffuse it away to a distance δ if $\delta^2 = O(\nu t) = O\left(a^2/R\right)$. But for a solid sphere the fluid velocity must change by $O(U)$ across the layer, because it vanishes on the sphere, whereas for a gas bubble the normal derivative of velocity must change by $O(U/a)$ in order that the shear stress be zero. That implies that the velocity itself changes by $O(U\delta/a) = O\left(UR^{-1/2}\right) = o(U)$…
>
> In the boundary layer on the bubble, therefore, the fluid velocity is only slightly perturbed from that of the irrotational flow, and velocity derivatives are of the same order as in the irrotational flow. Then the viscous dissipation integral has the same value as in the irrotational flow, to the first order, because the total volume of the boundary layer, of order $a^2\delta$, is much less than the volume, of order a^3, of the region in which the velocity derivatives are of order U/a. The volume of the wake is not small, but the velocity derivatives in it are, and it contributes to the dissipation only in higher order terms….

The drag on a spherical gas bubble in steady flow at modestly high Reynolds numbers (say, $R_e > 50$) can be calculated using the dissipation method assuming irrotational flow without any reference to boundary layers or vorticity. The dissipation calculation gives $D = 12\pi a\mu U$ or $C_D = 48/R$ where $R = 2aU\rho/\mu$.

VIII.2 Direct calculation of the drag using viscous potential flow (VPF). Moore (1959) calculated the drag directly by integrating the pressure and viscous normal stress of the potential flow. The irrotational shear stress is not zero but is not used in the drag calculation. The shear stress which is zero in the real flow was put to zero in the direct calculation. The pressure is computed from Bernoulli's equation and it has no drag resultant. If the irrotational shear stress was not neglected the drag by direct calculation would vanish, even though the dissipation is not zero. Moore's direct calculation gave $D = 8\pi a\mu U$ or $C_D = 32/R$ instead of $C_D = 48/R$.

VIII.3 Pressure correction (VCVPF). The discrepancy between the dissipation calculation leading to $C_D = 48/R$ and the direct VPF calculation leading to $C_D = 32/R$ led Batchelor, as reported in Moore (1963), to suggest the idea of a pressure correction to the irrotational pressure. In that paper, Moore performed a boundary layer analysis and his pressure correction is readily obtained by setting $y = 0$ in his equation (2.37):

$$p_v = \frac{4}{R\sin^2\theta}(1-\cos\theta)^2(2+\cos\theta), \tag{VIII.1}$$

which is singular at the separation point where $\theta = \pi$. The presence of separation is a problem for the application of boundary layers to the calculation of drag on solid bodies. To find the drag coefficient Moore calculated the momentum defect, and obtained the Levich value 48/R plus contributions of order $R^{-3/2}$ or lower.

The first successful calculation of a viscous pressure correction was carried out by Kang and Leal (1988a). They calculated a viscous correction of the irrotational pressure by solving the Navier-Stokes equations under the condition that the shear stress on the bubble surface is zero. Their calculation could not be carried out to very high Reynolds numbers, and it was not verified that the dissipation in the liquid is close to the value given by potential flow. They find indications of a boundary layer structure but they do not establish the existence of properties of a layer in which the vorticity is important. They obtain the drag coefficient 48/R by direct integration of the normal stress and viscous pressure over the boundary. This shows that the force resultant of the pressure correction does indeed contribute exactly the 16/R which is needed to reconcile the difference between the dissipation calculation and the direct calculation of drag.

Kang and Leal (1988a) obtain their drag result by expanding the pressure correction as a spherical harmonic series and noting that only one term of this series contributes to the drag, no appeal to the boundary layer approximation being necessary. Kang and Leal (1988b) remark that

> "In the present analysis, we therefore use an alternative method which is equivalent to Lamb's dissipation method, in which we ignore the boundary layer and use the potential flow solution right up to the boundary, with the effect of viscosity included by adding a viscous pressure correction and the viscous stress term to the normal stress balance, using the inviscid flow solution to estimate their values."

The VCVPF approach to problems of gas-liquid flows taken by Joseph and Wang (2004) and by Wang and Joseph (2005), in which the viscous contribution to the pressure is selected to remove the uncompensated irrotational shear stress from the traction integral (see VIII.18), is different than that used by Kang and Leal (1988a, b).

For the case of a gas bubble rising with the velocity U in a viscous fluid, it is possible to prove that the drag D_1 computed indirectly by the dissipation method is equal to the drag D_2 computed directly by our formulation of VCVPF. Suppose that (VIII.18) holds and that the drag on the bubble is given as $D_1 = \mathcal{D}/U$, where $\mathcal{D}$ is the dissipation. Then

$$
\begin{aligned}
D_1 &= \mathcal{D}/U \\
&= \int_V 2\mu \mathbf{D} : \mathbf{D} \mathrm{d}V/U = \int_A n \cdot 2\mu \mathbf{D} \cdot u \mathrm{d}A/U \\
&= \int_A (\tau_n u_n + \tau_s u_s) \mathrm{d}A/U = \int_A \left(-p^v + \tau_n\right) u_n \mathrm{d}A/U \\
&= \int_A e_x \cdot e_n (-p^v - p^i + \tau_n) \mathrm{d}A = \int_A e_x \cdot \mathbf{T} \cdot e_n \mathrm{d}A = D_2
\end{aligned}
$$

where we have used the normal velocity continuity $u_n = Ue_x \cdot e_n$, the zero-shear-stress condition at the gas-liquid interface and the fact that the Bernoulli pressure does not contribute to the drag.

Dissipation calculations for the drag on a rising oblate ellipsoidal bubble was given by Moore (1965) and for the rise of a spherical liquid drop, approximated by Hill's spherical vortex in another liquid in irrotational motion. The drag results from these dissipation calculations were obtained by Joseph and Wang (2004), using the VCVPF pressure correction formula (VIII.18).

VIII.4 Acceleration to steady flow of a spherical gas bubble. A spherical gas bubble accelerates to steady motion in an irrotational flow of a viscous liquid induced by a balance of the acceleration of the added mass of the liquid with the Levich drag. The equation of rectilinear motion is linear and may be integrated giving rise to exponential decay with decay constant $18\nu t/a^2$ where ν is the kinematic viscosity of the liquid and a is the bubble radius. The problem of decay to rest of a bubble moving initially when the forces maintaining motion are inactivated and the acceleration of a bubble initially at rest to terminal velocity are considered. The equation of motion follows from the assumption that the motion of the viscous liquid is irrotational. It is an elementary example of how potential flows can be used to study the unsteady motions of a viscous liquid suitable for the instruction of undergraduate students.

Consider a body moving with the velocity U in an unbounded viscous potential flow. Let M be the mass of the body and M' be the added mass, then the total kinetic energy of the fluid and body is

$$T = \frac{1}{2}(M + M')U^2. \tag{VIII.2}$$

Let D be the drag and F be the external force in the direction of motion, then the power of D and F should be equal to the rate of the total kinetic energy,

$$(F + D)U = \frac{\mathrm{d}T}{\mathrm{d}t} = (M + M')U\frac{\mathrm{d}U}{\mathrm{d}t}. \tag{VIII.3}$$

We next consider a spherical gas bubble, for which $M = 0$ and $M' = \frac{2}{3}\pi a^3 \rho_f$. The drag can be obtained by direct integration using the irrotational viscous normal stress and a viscous pressure correction: $D = -12\pi\mu a U$. Suppose the external force just balances the drag, then the bubble moves with a constant velocity $U = U_0$. Imagine that the external force suddenly disappears, then (VIII.3) gives rise to

$$-12\pi\mu a U = \frac{2}{3}\pi a^3 \rho_f \frac{\mathrm{d}U}{\mathrm{d}t}. \tag{VIII.4}$$

The solution is

$$U = U_0 \mathrm{e}^{-\frac{18\nu}{a^2}t}, \tag{VIII.5}$$

which shows that the velocity of the bubble approaches zero exponentially.

If gravity is considered, then $F = \frac{4}{3}\pi a^3 \rho_f g$. Suppose the bubble is at rest at $t = 0$ and starts to move due to the buoyant force. Equation (VIII.3) can be written as

$$\frac{4}{3}\pi a^3 \rho_f g - 12\pi\mu a U = \frac{2}{3}\pi a^3 \rho_f \frac{\mathrm{d}U}{\mathrm{d}t}. \tag{VIII.6}$$

The solution is

$$U = \frac{a^2 g}{9\nu}\left(1 - \mathrm{e}^{-\frac{18\nu}{a^2}t}\right), \tag{VIII.7}$$

which indicates the bubble velocity approaches the steady state velocity

$$U = \frac{a^2 g}{9\nu}. \tag{VIII.8}$$

VIII.5 The rise velocity and deformation of a gas bubble computed using VPF. The shape of a rising bubble, or of a falling drop, in an incompressible viscous liquid was computed numerically by Miksis et al. (1982), omitting the condition on the tangential traction at the bubble or drop surface. The shape is found, together with the flow of the surrounding fluid, by assuming that both are steady and axially symmetric, with the Reynolds number being large. The flow is taken to be a potential flow and the viscous normal stress, evaluated on the irrotational flow, is included in the normal stress balance. This study is exactly what we have called VPF; it follows the earlier study of Moore (1957), but it differs markedly from Moore's study because the bubble shape is computed.

When the bubble is sufficiently distorted, its top is found to be spherical and its bottom is found to be rather flat. Then the radius of its upper surface is in fair agreement with the formula of Davies and Taylor (1950). This distortion occurs when the effect of gravity is large while that of surface tension is small. When the effect of surface tension is large, the bubble is nearly a sphere. The difference in these two cases is associated with large and small Morton numbers.

VIII.6 The rise velocity of a spherical cap bubble computed using VPF. Davies and Taylor (1950) studied the rise velocity of a lenticular or spherical cap bubble assuming that motion was irrotational and the liquid inviscid. The spherical cap as if some fraction of the sphere much less than 1/2, say 1/8, is cut off with the spherical cap in the front and a nearly flat trailing edge. These are the shapes of large volume bubbles of gas rising in the liquid. They measured the bubble shape and showed that it indeed had a spherical cap when rising in water. Brown (1965) did experiments which shows the cap is very nearly spherical even when the liquid in which the gas bubble rises is very viscous.

Joseph (2002) applied the theory of viscous potential flow VPF to the problem of finding the rise velocity U of a spherical cap bubble. The rise velocity is given by

$$\frac{U}{\sqrt{gD}} = -\frac{8}{3}\frac{\nu}{\sqrt{gD^3}} + \frac{\sqrt{2}}{3}\left[1 + \frac{32\nu^2}{gD^3}\right]^{1/2}, \tag{VIII.9}$$

where $R = D/2$ is the radius of the cap and ν is the kinematic viscosity of the liquid. Davies and Taylor's (1950) result follows from (VIII.9) when the viscosity is zero. Equation (VIII.9) may be expressed as a drag law

$$C_D = 6 + 32/R_e. \qquad \text{(VIII.10)}$$

This drag law is in excellent agreement with experiments at large Morton numbers reported by Bhaga and Weber (1981) after the drag law is scaled so that the effective diameter used in the experiments and the spherical cap radius of Davies and Taylor (1950) are the same (see Figure 1).