Power series and Taylor series

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First . . . a review of what we have done so far:

1. We examined series of constants and learned that we can say everything there is to say about geometric and telescoping series.
2. We developed tests for convergence of series of constants.
3. We considered power series, derived formulas and other tricks for finding them, and know them for a few functions.

1. Geometric and telescoping series

The geometric series is

$$\sum_{n=0}^{\infty} a_n r^n = a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$$

provided $|r| < 1$ (when $|r| \geq 1$ the series diverges).

We often use partial fractions to detect telescoping series, for which we can calculate explicitly the *partial sums* S_n.

2. Tests for convergence of series of constants

1. Fundamental divergence test (nth term must go to zero for convergence to be possible)
2. Integral test
3. Comparison and limit comparison tests
4. Ratio test
5. Root test
6. Alternating series test

3. Power series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \qquad \text{where } a_n = \frac{f^{(n)}(0)}{n!}$$

or

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

and we know the series for e^x, $\sin x$ and $\dfrac{1}{1-x}$.

Convergence of power series

Before we get too excited about finding series, let's make sure that, at the very least, the series converge.

Later, we'll deal with the question of whether they converge to the function we expect. But for now, we'll assume that if they converge, they converge to the function they "came from".

(Strictly speaking, this is not always true — but it is true for a large class of functions, which includes nearly all the ones encountered in basic science and mathematics. This fact was not fully appreciated until the early part of the twentieth century.)

Fortunately, most of the question of whether power series converge is answered fairly directly by the *ratio test*.

Ratio test review

Recall that for a series of constants $\sum_{n=0}^{\infty} b_n$, we have that the series converges (absolutely) if

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right|$$

is less than one, diverges if the limit is greater than one, and the test is indeterminate if the limit equals one.

To use the ratio test on power series, just leave the x there and calculate the limit for each value of x. This will give an inequality that x must satisfy in order for the series to converge.

For the exponential function

The power series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Therefore

$$\lim_{n\to\infty} \left| \frac{b_{n+1}}{b_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$

No matter what x is, the limit is 0, which is less than 1. So *the series for the exponential function converges for all values of x*.

Your turn!

For which values of x does the series for $f(x) = \sin x$ converge?

A more interesting example:

For the series $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$, This time the ratio test gives

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \frac{n}{x^n} \right| = \lim_{n\to\infty} |x| \frac{n}{n+1} = |x|.$$

So the series converges if $|x| < 1$ and diverges if $|x| > 1$ (reminiscent of the geometric series).

It remains to check the endpoints $x = 1$ and $x = -1$

For $x = 1$ the series is $\sum_{n=1}^{\infty} \frac{1}{n}$, the (divergent) harmonic series.

For $x = -1$ the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, the alternating harmonic series, which we know to be (conditionally) convergent.

So $\sum_{n=1}^{\infty} \frac{x^n}{n}$ **converges if $-1 \le x < 1$ and diverges otherwise.**

OK, your turn...

For which values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2x)^n}{n^2}$ converge?

A. $-1 < x < 1$

B. $-2 < x < 2$

C. $-\frac{1}{2} \le x < \frac{1}{2}$

D. $-2 \le x \le 2$

E. $-\frac{1}{2} \le x \le \frac{1}{2}$

For which values of x does the series $\displaystyle\sum_{n=1}^{\infty} nx^n$ converge?

A. $-1 < x < 1$

B. $-1 \le x < 1$

C. $-1 < x \le 1$

D. $-1 \le x \le 1$

E. $0 \le x < 1$

From these examples. . .

. . . it should be apparent that power series converge for values of x in an interval that is centered at zero, i.e., an interval of the form $[-a, a]$, $(-a, a]$, $[-a, a)$ or $(-a, a)$ (where a might be either zero or infinity).

The interval is called the *interval of convergence* and the number a is called the *radius of convergence*.

Let's go back to finding series for functions

There are two ways:

The standard way:

Use the formula $a_n = \dfrac{f^{(n)}(0)}{n!}$ to find the coefficients. We've found series for e^x and $\sin x$ this way:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

We could calculate other series this way (and sometimes we do have to resort to this), but the other way is more fun:

The other way

Magic tricks: Start from known series and use algebraic and/or analytic manipulation to get others:

Substitute x^2 for x everywhere in the series for e^x to get:

$$\begin{aligned} e^{x^2} &= 1 + [x^2] + \frac{[x^2]^2}{2!} + \frac{[x^2]^3}{3!} + \frac{[x^2]^4}{4!} + \cdots \\ &= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \end{aligned}$$

Another example

Take the derivative of the series for $\sin x$ to get:

$$\begin{aligned}
\cos x = \frac{d}{dx}\sin x &= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\
&= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}
\end{aligned}$$

Yet another example

Integrate both sides of the geometric series from 0 to x to get:

$$\int_0^x \frac{1}{1-t}\,dt = \int_0^x (1 + t + t^2 + t^3 + \cdots)\,dt$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

Negate both sides and replace x by $-x$ everywhere to get:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$$

Put $x = 1$ to learn that the fourth series from before sums to $\ln 2$.

Start from the geometric series again:

and substitute $-x^2$ for x everywhere it appears to get:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n\,2n}}{x}$$

Now integrate both sides from 0 to x to get:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}.$$

Put $x = 1$ to show that the first series from before sums to $\pi/4$.

We still have the challenge of showing that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Applications of series I: Limits at $x = 0$.

We know for other reasons that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

But we could prove this using series:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \lim_{x\to 0} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$
$$= \lim_{x\to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots \right) = 1 + 0 + 0 + \cdots = 1$$

You can do this for complicated limits at 0 — substitute the series for the functions and do algebra.

Calculate the limit: $\lim_{x\to 0} \frac{x - \sin x}{1 - e^{-x^3}}$

A. 0 B. $\frac{1}{6}$ C. 1 D. $\frac{1}{12}$ E. does not exist

Applications of series II: Approximate evaluation of integrals

Many integrals that cannot be evaluated in closed form (i.e., for which no elementary anti-derivative exists) can be approximated using series (and we can even estimate how far off the approximations are).

Calculate $\int_0^1 e^{-x^2}\,dx$ to the nearest 0.001.

We begin by substituting $-x^2$ for x in the known series for e^x, and then integrating it. This will give us a numerical series that converges to the answer:

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots\,dx = 1-\frac{1}{3}+\frac{1}{5\cdot 2!}-\frac{1}{7\cdot 3!}+\cdots$$

So far we have

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \cdots$$

The series on the right is alternating, so if we want the error to be less than $0.001 = \frac{1}{1000}$, we need to take all the terms before the first one that is less than that. In other words, which is the first term with denominator greater than 1000?

Well, $9\cdot 4! = 216$ and $11\cdot 5! = 1320$, so the term $\dfrac{1}{11\cdot 5!}$ should be the first omitted term. So we write

$$\int_0^1 e^{-x^2}\,dx \approx 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} = \frac{5651}{7560} \approx 0.747$$

(according to Maple, the answer to 5 decimal places is 0.74669).

Try this:

Which of the following is closest to

$$\int_0^1 \cos\sqrt{x}\,dx \ ?$$

A. 0.7635 B. 0.5637 C. 0.3567 D. 0.6357 E. 0.6735

How good are our approximations?

We've been associating series with functions and using them to evaluate limits, integrals and such.

In the integrals we've estimated, we've been fortunate that the resulting numerical series were alternating series so the error is easy to estimate. What happens when the series are not alternating?

We'll continue to concentrate on questions like:

1. If I use only the first three terms of the series, how big is the error?
2. How many terms do I need to get the error smaller than 0.0001?

To get error estimates:

Use a generalization of the Mean Value Theorem for derivatives.

The Mean Value Theorem approach:

Recall the mean-value theorem:

$$f'(\text{somewhere between } a \text{ and } b) = \frac{f(b) - f(a)}{b - a}$$

Set $a = 0$ and $b = x$ and get that

$$f'(\text{somewhere between } 0 \text{ and } x) = \frac{f(x) - f(0)}{x}$$

Rearrange this to get

$$f(x) = f(0) + f'(\text{somewhere})x.$$

Turn this into an error estimate

Start from

$$f(x) = f(0) + f'(\text{somewhere})x.$$

Conclude that if you know that the absolute value of the derivative of f is always less than M, then you know that

$$|f(x) - f(0)| < M|x|.$$

The derivative form of the error estimate for series is a generalization of this.

Lagrange's form of the remainder

Suppose you write the approximation obtained using the terms up to x^n of the series for $f(x)$ and let the "remainder" (the difference between the actual value of $f(x)$ and the part of the series you are using) be $R_n(x)$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x)$$

Lagrange's form of the remainder looks a lot like what would be the next term of the series, except the $n+1$st derivative is evaluated at an unknown point between 0 and x, rather than at 0:

$$R_n(x) = \frac{f^{(n+1)}(\text{somewhere between } 0 \text{ and } x)}{(n+1)!}x^{n+1}$$

So if we know bounds on the $n+1$st derivative of f, we can bound the error in the approximation.

Example: The series for $\sin x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

If we use the first two (nonzero) terms, we have

$$\sin x = x - \frac{x^3}{3!} + R_4(x)$$

because the x^4 term of the series is zero anyhow.

For $f(x) = \sin x$, the fifth derivative is $f'''''(x) = \cos x$. And we know that $|\cos t| < 1$ for all t between 0 and x. We can conclude from this that:

$$|R_4(x)| < \frac{|x|^5}{5!}.$$

So for instance, we can conclude that the approximation

$$\sin(1) = 1 - \frac{1}{6} = \frac{5}{6}$$

is accurate to within $1/5! = 1/120$ — i.e., to two decimal places.

Your turn

How accurate is the approximation

$$\sqrt{e} = e^{0.5} \approx 1 + 0.5 + \frac{0.5^2}{2!} + \frac{0.5^3}{3!} = 1.645833\ldots\ ?$$

Now turn the question around —

How many terms of the series do we need to add together to get $\sqrt{e}$ to 5 decimal places?

Convergence of series to their functions

Another application of Lagrange's form of the remainder is to prove that the series of a function actually converges to the function.

For example

For the series for $\sin x$, we have (since all the derivatives of $\sin x$ are less than or equal to 1 in absolute value for all x):

$$R_n(x) < \left| \frac{x^{n+1}}{(n+1)!} \right|$$

and for any fixed value of x, this quantity will approach zero as $n \to \infty$. Thus, the remainder becomes arbitrarily small – and zero in the limit.

So we are now justified in writing $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

Shifting the origin — Taylor vs Maclaurin

So far, we've been writing all of our series as infinite polynomials and using values of the function $f(x)$ and its derivatives evaluated at $x = 0$. It is possible to change one's point of view and use values of the function and derivatives at another point.

A first example — Start with the geometric series:

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

If we define a new function $g(x) = f(x+1) = \dfrac{1}{1-(x+1)} = -\dfrac{1}{x}$ then we could write

$$g(x) = -\frac{1}{x} = 1 + (x+1) + (x+1)^2 + (x+1)^3 + (x+1)^4 + \cdots$$

This expansion is valid for $-1 < x+1 < 1$, in other words for $-2 < x < 0$.

Taylor series for $g(x) = -1/x$ expanded at $x = -1$

By taking derivatives of the function $g(x) = -1/x$ and evaluating them at $x = -1$, we will discover that the expansion of $g(x)$ we have found is the Taylor series for $g(x)$ expanded around -1:

$$g(x) = g(-1) + \frac{g'(-1)}{1!}(x+1) + \frac{g''(-1)}{2!}(x+1)^2 + \frac{g'''(-1)}{3!}(x+1)^3 + \cdots$$

In general, we have the Taylor expansion of $f(x)$ around $x = a$:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Note that this specializes to our old friend (which we will now call the Maclaurin series) when $a = 0$.

Reducing Taylor to Maclaurin

Series expansions around points other than zero are useful when trying to approximate function values for x far from zero, but close to a different point where much is known about the function.

But note that by defining a new function $g(x) = f(x + a)$, you can use Maclaurin expansions for g instead of general Taylor expansions for f.

Binomial series

An important series that arises in many applications is a generalization of the binomial theorem:

The binomial theorem

If p is a positive integer, then

$$(1+x)^p = \sum_{k=0}^{p} \binom{p}{k} x^k$$

where

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}$$

are the binomial coefficients (the numbers in Pascal's triangle).

If p is not a positive integer

The same expansion works except it doesn't stop (i.e., it gives a series instead of a polynomial) and we need a new definition for $\binom{p}{k}$.

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

For instance, if $p = -1$,

this gives the alternating harmonic series:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

A more involved example: the Maclaurin series for $\arcsin x$

How could we find the series for $\arcsin x$ without resorting to the general formula?

Well,

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \quad \text{and} \quad \arcsin(0) = 0$$

so we have

$$\arcsin x = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt$$

and $\dfrac{1}{\sqrt{1-t^2}} = (1+u)^{-1/2}$, where $u = -t^2$.

So we start with the binomial series with $p = -\frac{1}{2}$:

$$\frac{1}{\sqrt{1+u}} = 1 - \frac{u}{2} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}u^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}u^3 + \cdots$$

$$= 1 - \frac{1}{2}u + \frac{1 \cdot 3}{2^2 \cdot 2!}u^2 - \frac{1 \cdot 3 \cdot 5}{2^3 \cdot 3!}u^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^4 \cdot 4!}u^4 + \cdots$$

$$= 1 - \frac{1}{2}u + \frac{1 \cdot 3}{2 \cdot 4}u^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}u^3 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}u^4 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} u^n$$

If we put $u = -t^2$, all the minus signs will cancel and we get:

$$\frac{1}{\sqrt{1-t^2}} = 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} t^{2n}.$$

So we can get the series for arcsin x by integrating:

$$\begin{aligned}
\arcsin x &= \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt \\
&= \int_0^x \left(1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\,t^{2n}\right)\,dt \\
&= x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\,\frac{x^{2n+1}}{(2n+1)}
\end{aligned}$$

Use that $0!=1$ and that $2\cdot 4\cdot 6\cdots(2n) = 2^n(n!)$ to rewrite this as

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2}\,\frac{x^{2n+1}}{(2n+1)}$$

Exercise: Determine the interval of convergence of this series.

A digression: The Fibonacci numbers

Everyone is probably familiar with the famous sequence of Fibonacci numbers. The idea is that you start with 1 (pair of) rabbit(s) the zeroth month. The first month you still have 1 pair. But then in the second month you have $1+1=2$ pairs, the third you have $1+2=3$ pairs, the fourth, $2+3=5$ pairs, etc.... The pattern is that if you have f_n pairs in the nth month, and f_{n+1} pairs in the $n+1$st month, then you will have $f_{n+2} = f_n + f_{n+1}$ pairs in the $n+2$nd month.

The first several terms of the sequence are thus:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Problem: Is there a general formula for f_n ?

Generating functions

Seeking a formula for the terms of a recursively-defined sequence is a common problem in many parts of mathematics and science. And a powerful method for solving such problems involves series — which in this case are called *generating functions* for their sequences.

For the Fibonacci numbers $\{f_0, f_1, \ldots\}$, we will simply define a function $F(x)$ via the series:

$$F(x) = f_0 + f_1x + f_2x^2 + f_3x^3 + \cdots = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$$

Now we have to get the recurrence relation $f_{n+2} = f_{n+1} + f_n$ into the mix.

Using the recurrence relation

To do this, we'll use the fact that multiplication by x "shifts" the series for $F(x)$ as follows:

$$
\begin{aligned}
F(x) &= f_0 + f_1x + f_2x^2 + f_3x^3 + f_4x^4 + \cdots \\
xF(x) &= f_0x + f_1x^2 + f_2x^3 + f_3x^4 + f_4x^5 + \cdots \\
x^2F(x) &= f_0x^2 + f_1x^3 + f_2x^4 + f_3x^5 + f_4x^5 + \cdots
\end{aligned}
$$

Now, subtract the second two from the first – almost everything will cancel because of the recurrence relation!
The result is $(1 - x - x^2)F(x) = f_0 + (f_1 - f_0)x$. But $f_0 = f_1 = 1$, so we have deduced that

$$F(x) = \frac{1}{1 - x - x^2} \ !$$

What good does this do?

Find the series. . .

Since $F(x) = \dfrac{1}{1-x-x^2}$ is the generating function for the Fibonacci numbers, if we can find a formula for the coefficients of the series of $F(x)$, we'll have a formula for the Fibonacci numbers.

Partial fractions to the rescue!

Factor the denominator of $F(x)$ as $1 - x - x^2 = (\alpha - x)(\beta + x)$, where $\alpha = \dfrac{\sqrt{5}-1}{2}$ and $\beta = \dfrac{\sqrt{5}+1}{2}$ (We write α and β to avoid having complicated expressions the whole way along).
Now use partial fractions to write:

$$F(x) = \frac{1}{(\alpha - x)(\beta + x)} = \frac{\frac{1}{\alpha+\beta}}{\alpha - x} + \frac{\frac{1}{\alpha+\beta}}{\beta + x}$$

So if we can get the series for $\dfrac{1}{\alpha - x}$ and $\dfrac{1}{\beta + x}$ we'll be (almost) done!

Two geometric series

First,

$$\frac{1}{\alpha - x} = \frac{1}{\alpha\left(1 - \frac{x}{\alpha}\right)} = \frac{1}{\alpha}\left(1 + \frac{x}{\alpha} + \frac{x^2}{\alpha^2} + \frac{x^3}{\alpha^3} + \cdots\right).$$

And

$$\frac{1}{\beta + x} = \frac{1}{\beta\left(1 + \frac{x}{\beta}\right)} = \frac{1}{\beta}\left(1 - \frac{x}{\beta} + \frac{x^2}{\beta^2} - \frac{x^3}{\beta^3} + \cdots\right).$$

Now recall that $\alpha = \dfrac{\sqrt{5}-1}{2}$ and $\beta = \dfrac{\sqrt{5}+1}{2}$ Two important facts about α and β are:

$$\alpha + \beta = \sqrt{5} \quad \text{and} \quad \alpha = \frac{1}{\beta}.$$

Therefore,

$$
\begin{aligned}
F(x) &= \frac{\frac{1}{\alpha+\beta}}{\alpha - x} + \frac{\frac{1}{\alpha+\beta}}{\beta + x} \\
&= \frac{1}{\sqrt{5}}\left[\frac{1}{\alpha}\left(1 + \frac{x}{\alpha} + \frac{x^2}{\alpha^2} + \cdots\right) + \frac{1}{\beta}\left(1 - \frac{x}{\beta} + \frac{x^2}{\beta^2} + \cdots\right)\right] \\
&= \frac{1}{\sqrt{5}}\left[(\beta + \beta^2 x + \beta^3 x^2 + \cdots) + (\alpha - \alpha^2 x + \alpha^3 x^2 + \cdots)\right] \\
&= \frac{1}{\sqrt{5}}\left[(\beta + \alpha) + (\beta^2 - \alpha^2)x + (\beta^3 + \alpha^3)x^2 + (\beta^4 - \beta^4)x^3 + \cdots\right]
\end{aligned}
$$

Therefore we have $f_0 = \frac{1}{\sqrt{5}}(\beta + \alpha)$, $f_1 = \frac{1}{\sqrt{5}}(\beta^2 - \alpha^2)$, $f_2 = \frac{1}{\sqrt{5}}(\beta^3 + \alpha^3)$, $f_3 = \frac{1}{\sqrt{5}}(\beta^4 - \alpha^4)$ and so forth.

And in general

$$f_n = \frac{1}{\sqrt{5}}\left(\beta^{n+1} + (-1)^n \alpha^{n+1}\right)$$
$$= \frac{1}{\sqrt{5}}\left(\left(\frac{\sqrt{5}+1}{2}\right)^{n+1} + (-1)^n\left(\frac{\sqrt{5}-1}{2}\right)^{n+1}\right),$$

which is the general solution for the Fibonacci numbers.

Since $\dfrac{\sqrt{5}-1}{2} < 1$, we have that

$$f_n = O\left(\left(\frac{\sqrt{5}+1}{2}\right)^n\right).$$