

# Power Spectral Density for Continuous-Phase Frequency Shift Keying (FSK)

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# Power Spectral Density for Continuous-Phase Frequency Shift Keying (FSK)

The power spectral density for continuous-phase frequency shift keying (FSK) is derived in this document. The words “power spectrum” will be used instead of “power spectral density” sometimes for brevity. Actually, the spectrum for the complex envelope of the FSK signal is derived. The first section shows how the actual FSK spectrum can be easily found by simply translating the spectrum of the complex envelope by the carrier frequency. As an example, the spectrum for M-ary FSK is presented.

## 1 Finding the Spectrum of a Frequency Modulated Signal from the Spectrum of its Complex Envelope

Suppose the input to a frequency modulator is the baseband message  $m(t)$  and the carrier frequency is  $\omega_c$ . The transmitted FM signal is

$$s(t) = \cos\left(\omega_c t + \int_0^t m(\tau) d\tau + \theta_0\right) \quad (1)$$

where  $\theta_0$  is a random initial phase uniformly distributed over  $[0, 2\pi)$ . In terms of complex notation, this signal is

$$s(t) = \Re\left\{e^{j[\omega_c t + \int_0^t m(\tau) d\tau + \theta_0]}\right\} = \Re\left\{e^{j(\omega_c t + \theta_0)} e^{j \int_0^t m(\tau) d\tau}\right\} \quad (2)$$

The complex envelope of the FM signal is

$$x(t) = e^{j \int_0^t m(\tau) d\tau} \quad (3)$$

Using the fact that the real part of a complex number  $z$  is  $(z + \bar{z})/2$  it follows that

$$s(t) = \frac{1}{2}x(t)e^{j(\omega_c t + \theta_0)} + \frac{1}{2}\overline{x(t)}e^{-j(\omega_c t + \theta_0)} \quad (4)$$

The power spectral density for  $s(t)$  is the Fourier transform of its autocorrelation function  $R_{ss}(\tau)$  which is defined as

$$\begin{aligned} R_{ss}(\tau) &= \mathcal{E}\{s(t+\tau)\overline{s(t)}\} \\ &= \frac{1}{4}\mathcal{E}\left\{\left[x(t+\tau)e^{j[\omega_c(t+\tau)+\theta_0]} + \overline{x(t+\tau)}e^{-j[\omega_c(t+\tau)+\theta_0]}\right]\right. \\ &\quad \left.\times \left[\overline{x(t)}e^{-j[\omega_c t + \theta_0]} + x(t)e^{j[\omega_c t + \theta_0]}\right]\right\} \end{aligned} \quad (5)$$

where  $\mathcal{E}$  signifies statistical expectation. Assuming that  $\theta_0$  and  $x(t)$  are statistically independent, (5) can be expressed as

$$\begin{aligned} 4R_{ss}(\tau) &= \mathcal{E}\left\{e^{j(2\omega_c t + \omega_c \tau + 2\theta_0)}\right\} \mathcal{E}\{x(t+\tau)x(t)\} \\ &\quad + e^{-j\omega_c t} \mathcal{E}\left\{\overline{x(t+\tau)}x(t)\right\} + e^{j\omega_c t} \mathcal{E}\left\{x(t+\tau)\overline{x(t)}\right\} \\ &\quad + \mathcal{E}\left\{e^{-j(2\omega_c t + \omega_c \tau + 2\theta_0)}\right\} \mathcal{E}\left\{\overline{x(t+\tau)}\overline{x(t)}\right\} \end{aligned} \quad (6)$$

The first and last lines on the right of (6) are zero because  $\theta_0$  is uniformly distributed over  $[0, 2\pi)$  and

$$\mathcal{E} \{ e^{j2\theta_0} \} = \frac{1}{2\pi} \int_0^{2\pi} e^{j2x} dx = 0 \quad (7)$$

The autocorrelation function for the complex envelope is  $R_{xx}(\tau) = \mathcal{E} \{ x(t + \tau) \overline{x(t)} \}$  and its power spectral density is the Fourier transform of  $R_{xx}(\tau)$  which is

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau \quad (8)$$

Therefore,

$$\begin{aligned} R_{ss}(\tau) &= \frac{1}{4} R_{xx}(\tau) e^{j\omega_c\tau} + \frac{1}{4} \overline{R_{xx}(\tau)} e^{-j\omega_c\tau} \\ &= \frac{1}{2} \Re \{ R_{xx}(\tau) e^{j\omega_c\tau} \} \end{aligned} \quad (9)$$

**Lemma 1**  $S_{xx}(\omega)$  is real.

**Proof:**

$$\overline{R_{xx}(\tau)} = \mathcal{E} \{ \overline{x(t)x(t+\tau)} \} = R_{xx}(-\tau) \quad (10)$$

Therefore

$$\overline{S_{xx}(\omega)} = \int_{-\infty}^{\infty} \overline{R_{xx}(\tau)} e^{j\omega\tau} d\tau = \int_{-\infty}^{\infty} R_{xx}(-\tau) e^{j\omega\tau} d\tau = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau = S_{xx}(\omega) \quad (11)$$

A complex number must be real if it is equal to its complex conjugate. ■

Taking the Fourier transform of (9) and using the frequency translation and complex conjugate theorems for Fourier transforms, the power spectral density for the FM signal is found to be

$$S_{ss}(\omega) = \frac{1}{4} S_{xx}(\omega - \omega_c) + \frac{1}{4} S_{xx}(-\omega - \omega_c) \quad (12)$$

Thus the power spectral density of the complex envelope completely determines the power spectral density of the FM signal. The first term on the right side of (12) is a translation of the scaled power spectrum of the complex envelope up to the carrier frequency  $\omega_c$  and the second term is the translation of the power spectrum of the complex envelope down to  $-\omega_c$ .

## 2 Definition of the Continuous-Phase Frequency Shift Keyed (FSK) Signal

A sequence of binary bits arriving at the rate of  $R$  bits per second is to be transmitted. The transmitter segments the received bits into successive blocks of  $K$  bits. Thus the blocks are

formed at the rate of  $f_b$  blocks per second and each block occupies  $T = 1/f_b$  seconds. The block frequency  $f_b$  is called the baud or symbol rate. A block of  $K$  bits can have  $M = 2^K$  possible values. In continuous-phase frequency shift keying each block selects one of  $M$  instantaneous frequency functions to be transmitted by a frequency modulator during that baud. The special case where the possible frequencies are equally spaced and remain constant during the baud is presented in Section 4. Lucky, Salz, and Weldon<sup>1</sup> present a more general situation where the data block for a baud selects the instantaneous frequency from a set of  $M$  signals  $\{a_1(t), a_2(t), \dots, a_M(t)\}$ . Each of these signals is zero except for  $0 \leq t < T$ . Let the block at time  $nT$  have the decimal value  $k_n$ . Then the instantaneous frequency function selected for the baud starting at time  $nT$  is  $a_{k_n}(t - nT)$  and the instantaneous frequency of the complete transmitted signal assuming transmission is started at  $t = 0$  is

$$m(t) = \sum_{n=0}^{\infty} a_{k_n}(t - nT) \quad (13)$$

The phase of the transmitted frequency modulated (FM) signal is the integral of the instantaneous frequency up to the current time, so the complex envelope of the transmitted signal normalized to amplitude 1 is

$$x(t) = e^{j \int_0^t m(\tau) d\tau} = e^{j \int_0^t \sum_{n=0}^{\infty} a_{k_n}(\tau - nT) d\tau} = e^{j \sum_{n=0}^{\infty} \int_0^t a_{k_n}(\tau - nT) d\tau} \quad (14)$$

Suppose that the current time  $t$  satisfies  $LT \leq t < (L + 1)T$  for some integer  $L$ . In words, the interval  $[0, t)$  includes  $L$  complete bauds for the interval  $[0, LT)$  and a fraction of a baud for  $LT \leq t < (L + 1)T$ . The frequency pulse  $a_{k_n}(\tau - nT)$  is nonzero only for  $nT \leq \tau < (n + 1)T$ . Then

$$\begin{aligned} \theta(t) &= \sum_{n=0}^{\infty} \int_0^t a_{k_n}(\tau - nT) d\tau = \sum_{n=0}^{L-1} \int_{nT}^{(n+1)T} a_{k_n}(\tau - nT) d\tau + \int_{LT}^t a_{k_L}(\tau - LT) d\tau \\ &= \sum_{n=0}^{L-1} \int_0^T a_{k_n}(\tau) d\tau + \int_0^{t-LT} a_{k_L}(\tau) d\tau \quad \text{for } LT \leq t < (L + 1)T \end{aligned} \quad (15)$$

Each integral in (15) is the phase change caused by the selected instantaneous frequency pulse for that baud. The phase change from the beginning of the baud starting at time  $nT$  is

$$b_n(t') = \int_0^{t'} a_{k_n}(\tau) d\tau \quad \text{for } t' = t - nT \quad (16)$$

The total phase change over baud  $n$  is  $b_n(T)$ . With this definition, (15) can be written as

$$\theta(t) = \sum_{n=0}^{\infty} \int_0^t a_{k_n}(\tau - nT) d\tau = \sum_{n=0}^{L-1} b_n(T) + b_L(t - LT) \quad \text{for } LT \leq t < (L + 1)T \quad (17)$$

The complex envelope can then be expressed as

$$x(t) = e^{j[\sum_{n=0}^{L-1} b_n(T) + b_L(t-LT)]} = e^{jb_L(t-LT)} e^{j\sum_{n=0}^{L-1} b_n(T)} \quad \text{for } LT \leq t < (L + 1)T \quad (18)$$

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<sup>1</sup>R.W. Lucky, J. Salz, and E.J. Weldon, *Principles of Data Communication*, McGraw-Hill, 1968, pp. 202-207 and 242-245.

### 3 Derivation of the Power Spectral Density for the FSK Signal

Lucky, Salz, and Weldon contains a derivation of the power spectral density of  $x(t)$  but many details are not included. This section includes all the details. The power spectral density can be defined as

$$S_{xx}(\omega) = \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} \mathcal{E} \{ |X_\lambda(\omega)|^2 \} \quad (19)$$

where

$$X_\lambda(\omega) = \int_0^\lambda x(t) e^{-j\omega t} dt \quad (20)$$

Thus

$$X_\lambda(\omega) = \int_0^\lambda e^{j \int_0^t \sum_{n=0}^{\infty} a_{kn}(\tau-nT) d\tau} e^{-j\omega t} dt \quad (21)$$

The derivation can be somewhat simplified by letting the interval  $[0, \lambda)$  cover a complete number of bauds. So, for an integer  $N$ , let  $\lambda = NT$ . The limit will now be taken for  $N \rightarrow \infty$  and

$$X_{NT}(\omega) = \int_0^{NT} e^{j \int_0^t \sum_{n=0}^{\infty} a_{kn}(\tau-nT) d\tau} e^{-j\omega t} dt \quad (22)$$

The integration interval  $[0, NT)$  can be partitioned into  $N$  intervals of length  $T$  and the integral can be computed as the sum of the integrals over the partitions. Thus

$$X_{NT}(\omega) = \sum_{k=0}^{N-1} \int_{kT}^{(k+1)T} e^{j \int_0^t \sum_{n=0}^{\infty} a_{kn}(\tau-nT) d\tau} e^{-j\omega t} dt \quad (23)$$

Replacing the complex envelope by (18) gives

$$\begin{aligned} X_{NT}(\omega) &= \sum_{k=0}^{N-1} e^{j \sum_{n=0}^{k-1} b_n(T)} \int_{kT}^{(k+1)T} e^{j b_k(t-kT)} e^{-j\omega t} dt \\ &= \sum_{k=0}^{N-1} e^{j \sum_{n=0}^{k-1} b_n(T)} \int_0^T e^{j b_k(t)} e^{-j\omega t} dt e^{-j\omega kT} \\ &= \sum_{k=0}^{N-1} e^{j \sum_{n=0}^{k-1} b_n(T)} F_k(\omega) e^{-j\omega kT} \end{aligned} \quad (24)$$

where

$$F_k(\omega) = \int_0^T e^{j b_k(t)} e^{-j\omega t} dt \quad (25)$$

The function  $F_k(\omega)$  is the Fourier transform of a typical transmitted signal during the baud starting at time 0.

Using these results, the squared magnitude of the Fourier transform of the signal segment over the time interval  $[0, NT)$  is

$$\begin{aligned}
|X_{NT}(\omega)|^2 &= X_{NT}(\omega)\overline{X_{NT}(\omega)} \\
&= \left\{ \sum_{k=0}^{N-1} e^{j\sum_{n=0}^{k-1} b_n(T)} F_k(\omega) e^{-j\omega kT} \right\} \left\{ \sum_{m=0}^{N-1} e^{-j\sum_{n=0}^{m-1} b_n(T)} \overline{F_m(\omega)} e^{j\omega mT} \right\} \\
&= \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} e^{j\left[\sum_{n=0}^{k-1} b_n(T) - \sum_{n=0}^{m-1} b_n(T)\right]} F_k(\omega) \overline{F_m(\omega)} e^{-j\omega(k-m)T} \tag{26}
\end{aligned}$$

Let the summand in (26) be  $D_{km}$ . The sum can be split into three parts as follows:

$$|X_{NT}(\omega)|^2 = \sum_{m=k=0}^{N-1} D_{kk} + \sum_{0 \leq m < k \leq N-1} D_{km} + \sum_{0 \leq m < k \leq N-1} D_{km} \tag{27}$$

Consider the square lattice of points  $(k, m)$  in a  $k, m$  plane for  $0 \leq k \leq N-1, 0 \leq m \leq N-1$ . The first sum is along the diagonal of the square from  $(0, 0)$  to  $(N-1, N-1)$ . The second sum is over the points in the square below that diagonal, and the third sum is over the points above that diagonal. Notice that  $D_{mk} = \overline{D_{km}}$  so the third sum is the complex conjugate of the second sum and thus

$$\begin{aligned}
|X_{NT}(\omega)|^2 &= \sum_{m=k=0}^{N-1} D_{kk} + 2\Re e \left\{ \sum_{0 \leq m < k \leq N-1} D_{km} \right\} \\
&= \sum_{k=0}^{N-1} |F_k(\omega)|^2 + 2\Re e \left\{ \sum_{0 \leq m < k \leq N-1} e^{-j\omega(k-m)T} F_k(\omega) \overline{F_m(\omega)} e^{j\sum_{n=m}^{k-1} b_n(t)} \right\} \tag{28}
\end{aligned}$$

Let

$$\begin{aligned}
S_{xx}^{(N)}(\omega) &= \frac{1}{NT} \mathcal{E} \left\{ |X_{NT}(\omega)|^2 \right\} \\
&= \frac{1}{NT} \sum_{k=0}^{N-1} \mathcal{E} \left\{ |F_k(\omega)|^2 \right\} + \frac{2}{NT} \Re e \left[ \sum_{0 \leq m < k \leq N-1} e^{-j\omega(k-m)T} \right. \\
&\quad \left. \times \mathcal{E} \left\{ F_k(\omega) \overline{F_m(\omega)} \prod_{n=m}^{k-1} e^{jb_n(T)} \right\} \right] \tag{29}
\end{aligned}$$

It will be assumed that the input bits are independent and equally likely to be 0 or 1. Then the blocks of  $K$  bits selected by the transmitter are independent and equally likely and the functions of these blocks in different bauds are independent. Therefore,

$$\mathcal{E} \left\{ F_k(\omega) \overline{F_m(\omega)} \prod_{n=m}^{k-1} e^{jb_n(T)} \right\} = \mathcal{E} \left\{ F_k(\omega) \right\} \mathcal{E} \left\{ \overline{F_m(\omega)} e^{jb_m(T)} \right\} \prod_{n=m+1}^{k-1} \mathcal{E} \left\{ e^{jb_n(T)} \right\} \tag{30}$$

for  $k \neq m$  and  $m < k$ .

At this point it is convenient to introduce the following functions:

1. Characteristic Function for  $b_n(t)$

$$C(\alpha; t) = \mathcal{E} \left\{ e^{j\alpha b_n(t)} \right\} \quad (31)$$

2. Average Fourier Transform of a Baud Signal

$$\begin{aligned} F(\omega) &= \mathcal{E} \{ F_n(\omega) \} = \mathcal{E} \left\{ \int_0^T e^{jb_n(t)} e^{-j\omega t} dt \right\} \\ &= \int_0^T \mathcal{E} \left\{ e^{jb_n(t)} \right\} e^{-j\omega t} dt = \int_0^T C(1; t) e^{-j\omega t} dt \end{aligned} \quad (32)$$

3.

$$\begin{aligned} G(\omega) &= \mathcal{E} \left\{ \overline{F_n(\omega)} e^{jb_n(T)} \right\} = \mathcal{E} \left\{ \int_0^T e^{-jb_n(t)} e^{j\omega t} dt e^{jb_n(T)} \right\} \\ &= \int_0^T \mathcal{E} \left\{ e^{j[b_n(T) - b_n(t)]} \right\} e^{j\omega t} dt \end{aligned} \quad (33)$$

4. Average Squared Amplitude Spectrum of a Baud Signal

$$P(\omega) = \mathcal{E} \left\{ |F_n(\omega)|^2 \right\} = \mathcal{E} \left\{ \left| \int_0^T e^{jb_n(t)} e^{-j\omega t} dt \right|^2 \right\} \quad (34)$$

Each of these four functions does not depend on the baud time index  $n$  because the signal probabilities have been assumed to be the same for each baud. With these definitions, (29) can be written as

$$S_{xx}^{(N)}(\omega) = \frac{1}{T} P(\omega) + \frac{2}{NT} \Re e \left\{ F(\omega) G(\omega) \sum_{0 \leq m < k \leq N-1} e^{-j\omega(k-m)T} C^{k-m-1}(1; T) \right\} \quad (35)$$

The sum in (35) can be evaluated by summing along the the diagonal lines  $k - m = n$  shown in Figure 1. Along each line the summand is

$$e^{-j\omega n T} C^{n-1}(1; T)$$

and there are  $N - n$  points on each line. Therefore,

$$\begin{aligned} \frac{1}{N} \sum_{0 \leq m < k \leq N-1} e^{-j\omega(k-m)T} C^{k-m-1}(1; T) &= \sum_{n=1}^{N-1} \frac{N-n}{N} e^{-j\omega n T} C^{n-1}(1; T) \\ &= e^{-j\omega T} \sum_{n=0}^{N-2} \frac{N-n-1}{N} e^{-j\omega n T} C^n(1; T) \end{aligned} \quad (36)$$

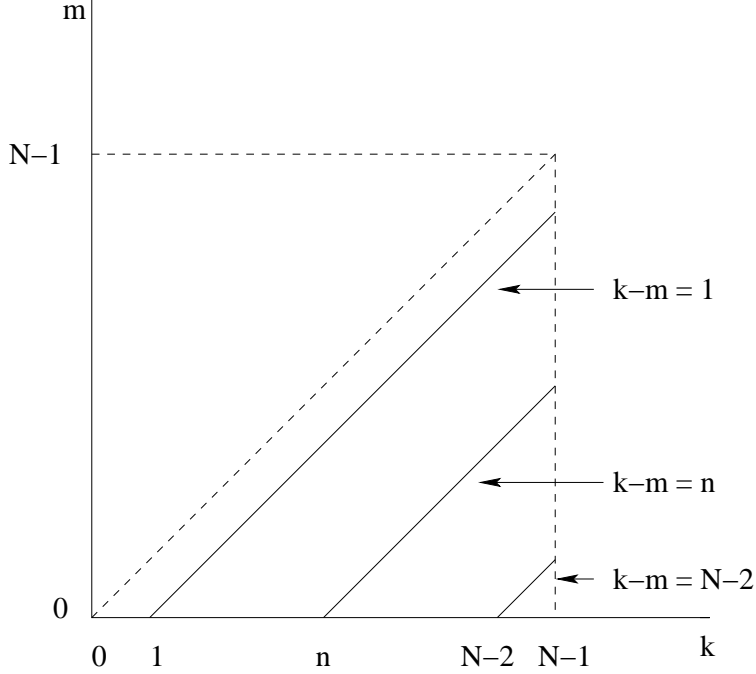


Figure 1: Paths for Evaluating the Sum in Equation (35)

Since

$$\lim_{N \rightarrow \infty} \frac{N - n - 1}{N} = 1$$

it follows that

$$\lim_{N \rightarrow \infty} \sum_{n=0}^{N-2} \frac{N - n - 1}{N} \left[ e^{-j\omega T} C(1; T) \right]^n = \lim_{N \rightarrow \infty} \sum_{n=0}^{N-2} \left[ e^{-j\omega T} C(1; T) \right]^n \quad (37)$$

For  $|C(1; T)| < 1$ , Equation (37) is the sum of a geometric series and

$$\lim_{N \rightarrow \infty} \sum_{n=0}^{N-2} \left[ e^{-j\omega T} C(1; T) \right]^n = \frac{1}{1 - e^{-j\omega T} C(1; T)} \quad (38)$$

Therefore,

$$S_{xx}(\omega) = \frac{1}{T} P(\omega) + \frac{2}{T} \Re \left\{ F(\omega) G(\omega) \frac{e^{-j\omega T}}{1 - C(1; T) e^{-j\omega T}} \right\} \quad \text{for } |C(1; T)| < 1 \quad (39)$$

Under some conditions  $|C(1; T)| = 1$  and (37) does not converge in the ordinary sense. However, it will be shown below to converge to a series of Dirac delta functions in the distribution sense. Since  $C(\alpha; t)$  is a characteristic function, its magnitude can never exceed 1. The following lemma shows when it has magnitude equal to 1.

**Lemma 2**  $|C(1; T)| = 1$  when  $b_n(T)$  is a discrete random variable with possible values  $2\pi k + d$  for  $k = 0, \pm 1, \pm 2, \dots$  with probabilities  $p_k = P\{b(nT) = 2\pi k + d\}$



**Proof:**

In this case

$$C(\alpha; T) = \sum_{k=-\infty}^{\infty} p_k e^{\alpha(2\pi k+d)} = e^{j\alpha d} \sum_{k=-\infty}^{\infty} p_k e^{\alpha 2\pi k} \quad (40)$$

Since  $\sum_{k=-\infty}^{\infty} p_k = 1$  and  $e^{2\pi k} = 1$ ,

$$C(1; T) = e^{jd} \sum_{k=-\infty}^{\infty} p_k = e^{jd} = e^{j\gamma T} \quad \text{where } \gamma = d/T = df_b \quad (41)$$

and  $|e^{jd}| = 1$ . ■

This lemma implies that when each elementary frequency pulse  $a_k(t)$  causes a total phase change  $b_n(T)$  over a baud of  $d$  radians modulo  $2\pi$ , spectral lines appear. When  $|C(1; T)| = 1$ , consider the finite sum

$$A_N = \sum_{n=0}^{N-2} \left[ e^{-j\omega T} C(1; T) \right] = \sum_{n=0}^{N-2} e^{-j(\omega-\gamma)Tn} = \frac{1 - e^{-j(\omega-\gamma)T(N-1)}}{1 - e^{-j(\omega-\gamma)T}} \quad (42)$$

The real part of  $A_N$  is

$$\Re\{A_N\} = \frac{1}{2}(A_N + \overline{A_N}) = \frac{1}{2} \sum_{n=0}^{N-2} \left[ e^{-j(\omega-\gamma)Tn} + e^{j(\omega-\gamma)Tn} \right] = \frac{1}{2} + \frac{1}{2} \sum_{n=-(N-2)}^{N-2} e^{-j(\omega-\gamma)Tn} \quad (43)$$

**Lemma 3** Let  $\omega_b = 2\pi/T$ . Then

$$\omega_b \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_b) = \sum_{n=-\infty}^{\infty} e^{-j\omega Tn} \quad (44)$$

**Proof:**

The function of  $\omega$  on the left side is periodic in  $\omega$  with period  $\omega_b$  and can be represented by a Fourier series  $\sum_{n=-\infty}^{\infty} c_n e^{-j\omega Tn}$ . The Fourier coefficients are

$$c_n = \frac{1}{\omega_b} \int_{-\omega_b/2}^{\omega_b/2} \omega_b \delta(\omega) d\omega = 1 \quad (45)$$

Using this lemma it follows that ■

$$\lim_{N \rightarrow \infty} \Re\{A_N\} = \frac{1}{2} + \frac{\omega_b}{2} \sum_{n=-\infty}^{\infty} \delta(\omega - \gamma - n\omega_b) \quad (46)$$

The imaginary part of  $A_N$  times  $2j$  is

$$\begin{aligned} 2j \Im\{A_N\} &= A_N - \overline{A_N} = \frac{1 - e^{-j(\omega-\gamma)T(N-1)}}{1 - e^{-j(\omega-\gamma)T}} - \frac{1 - e^{j(\omega-\gamma)T(N-1)}}{1 - e^{j(\omega-\gamma)T}} \\ &= \left[ \frac{1}{1 - e^{-j(\omega-\gamma)T}} - \frac{1}{1 - e^{j(\omega-\gamma)T}} \right] - \left[ \frac{e^{-j(\omega-\gamma)T(N-1)}}{1 - e^{-j(\omega-\gamma)T}} + \frac{e^{j(\omega-\gamma)T(N-1)}}{e^{j(\omega-\gamma)T}} \right] \end{aligned} \quad (47)$$

The second term on the right of (47) in rectangular brackets converges to zero with  $N$  in the distribution sense as a result of the Riemann-Lebesgue lemma. The first term can be put in a simpler form as follows:

$$\begin{aligned} \frac{1}{1 - e^{-j(\omega-\gamma)T}} - \frac{1}{1 - e^{j(\omega-\gamma)T}} &= \frac{e^{j(\omega-\gamma)T/2}}{e^{j(\omega-\gamma)T/2} - e^{-j(\omega-\gamma)T/2}} - \frac{e^{-j(\omega-\gamma)T/2}}{e^{-j(\omega-\gamma)T/2} - e^{j(\omega-\gamma)T/2}} \\ &= \frac{(e^{j(\omega-\gamma)T/2} + e^{-j(\omega-\gamma)T/2})/2}{j(e^{j(\omega-\gamma)T/2} - e^{-j(\omega-\gamma)T/2})/(2j)} \\ &= -j \cot[(\omega - \gamma)T/2] \end{aligned} \quad (48)$$

Therefore,

$$\lim_{N \rightarrow \infty} A_N = \frac{1}{2} + \frac{\omega_b}{2} \sum_{n=-\infty}^{\infty} \delta(\omega - \gamma - n\omega_b) - \frac{j}{2} \cot[(\omega - \gamma)T/2] \quad (49)$$

When  $|C(1; T)| = 1$ ,  $G(\omega)$  defined in (50) is

$$\begin{aligned} G(\omega) &= \mathcal{E} \{ \overline{F_n(\omega)} e^{j b_n(T)} \} = \mathcal{E} \{ \overline{F_n(\omega)} e^{j(2\pi k_n + \gamma T)} \} \\ &= e^{j\gamma T} \mathcal{E} \{ \overline{F_n(\omega)} \} = e^{\gamma T} \overline{F(\omega)} \end{aligned} \quad (50)$$

since  $e^{j2\pi k_n} = 1$ .

Combining (35 – 37) and (48 – 50) yields

$$\begin{aligned} TS_{xx}(\omega) &= P(\omega) + |F(\omega)|^2 \Re e \left\{ e^{-j(\omega-\gamma)T} [1 - j \cot((\omega - \gamma)T/2)] \right. \\ &\quad \left. + e^{-j(\omega-\gamma)T} \omega_b \sum_{n=-\infty}^{\infty} \delta(\omega - \gamma - n\omega_b) \right\} \end{aligned} \quad (51)$$

The real part of the first product in the braces can be reduced to just  $-1$ . To simplify the formulas, let  $x = (\omega - \gamma)T$ . Then

$$\begin{aligned} \Re e \{ e^{-jx} (1 - j \cot(x/2)) \} &= \Re e \{ (\cos x - j \sin x) (1 - j \cot(x/2)) \} \\ &= \cos x - \sin x \frac{\cos(x/2)}{\sin(x/2)} \\ &= (\cos^2(x/2) - \sin^2(x/2)) - (2 \sin(x/2) \cos(x/2)) \frac{\cos(x/2)}{\sin(x/2)} \\ &= \cos^2(x/2) - \sin^2(x/2) - 2 \cos^2(x/2) \\ &= -\cos^2(x/2) - \sin^2(x/2) = -1 \end{aligned} \quad (52)$$

The impulses in the second term in (51) pop up when  $\omega - \gamma = n\omega_b$  so that

$$e^{-j(\omega-\gamma)T} = e^{-jn\omega_b T} = e^{-jn2\pi} = 1$$

The (51) simplifies to

$$S_{xx}(\omega) = \frac{1}{T} P(\omega) + \frac{1}{T} |F(\omega)|^2 \left[ -1 + \omega_b \sum_{n=-\infty}^{\infty} \delta(\omega - \gamma - n\omega_b) \right] \quad (53)$$

### 3.1 Summary of the Formulas for the Power Spectral Density of Continuous-Phase FSK

The formulas for the power spectral density for the complex envelope of the general form of frequency shift keying (FSK) defined in this report are summarized here.

Definition of functions used in the power spectral density formulas:

1. Phase change caused by a frequency pulse over one baud

$$b_n(t) = \int_0^t a_n(\tau) d\tau \quad \text{for } 0 \leq t < T \quad (54)$$

2. Fourier transform of the FSK signal over baud 0

$$F_n(\omega) = \int_0^T e^{jb_n(t)} e^{-j\omega t} dt \quad (55)$$

3. A special characteristic function value

$$C(1; T) = \mathcal{E} \left\{ e^{jb_n(T)} \right\} \quad (56)$$

- 4.

$$\gamma = \frac{1}{T} \arg C(1; T) \quad (57)$$

5. Expected value of the transforms of the signals transmitted over baud 0

$$F(\omega) = \mathcal{E} \{ F_n(\omega) \} \quad (58)$$

- 6.

$$G(\omega) = \mathcal{E} \left\{ \overline{F_n(\omega)} e^{jb_n(T)} \right\} \quad (59)$$

7. Expected value of the squared magnitude of the signals transmitted in baud 0

$$P(\omega) = \mathcal{E} \left\{ |F_n(\omega)|^2 \right\} \quad (60)$$

**The power spectral density for the complex envelope**

$$x(t) = e^{j \int_0^t \sum_{n=0}^{\infty} a_{k_n}(\tau - nT) d\tau} \quad (61)$$

is when  $|C(1; T)| < 1$

$$S_{xx}(\omega) = \frac{1}{T} P(\omega) + \frac{2}{T} \Re \left\{ F(\omega) G(\omega) \frac{e^{-j\omega T}}{1 - C(1; T) e^{-j\omega T}} \right\} \quad (62)$$

$$(63)$$

and is when  $|C(1;T)| = 1$

$$S_{xx}(\omega) = \frac{1}{T}P(\omega) - \frac{1}{T}|F(\omega)|^2 + \frac{\omega_b}{T} \sum_{n=-\infty}^{\infty} |F(\gamma + n\omega_b)|^2 \delta(\omega - \gamma - n\omega_b) \quad (64)$$

The power spectral density for the transmitted signal

$$s(t) = \Re e \left\{ x(t) e^{j(\omega_c t + \theta_c)} \right\} \quad (65)$$

is

$$S_{ss}(\omega) = \frac{1}{4}S_{xx}(\omega - \omega_c) + \frac{1}{4}S_{xx}(-\omega - \omega_c) \quad (66)$$

## 4 Power Spectral Density for Constant Frequency During a Baud and Equally Spaced Frequencies

In many actual FSK transmitters the frequency during a baud remains constant and the possible transmitted frequencies are equally spaced. Each input bit block of  $K$  bits is used to select one of  $M = 2^K$  radian frequencies from the set

$$\begin{aligned} \Lambda_k &= \omega_c + \omega_d[2k - (M - 1)] \\ &= 2\pi\{f_c + f_d[2k - (M - 1)]\} \quad \text{for } k = 0, 1, \dots, M - 1 \end{aligned} \quad (67)$$

The frequency  $\omega_c = 2\pi f_c$  is called the carrier frequency. The radian frequencies

$$\Omega_k = \omega_d[2k - (M - 1)] = 2\pi f_d[2k - (M - 1)] \quad \text{for } k = 0, 1, \dots, M - 1 \quad (68)$$

are the possible frequency deviations from the carrier frequency during each symbol. The deviations range from  $-\omega_d(M - 1)$  to  $\omega_d(M - 1)$  in steps of  $\Delta\omega = 2\omega_d$ . Each selected frequency is sent for  $T_b = 1/f_b$  seconds. The modulation index  $h$  is defined to be the tone separation divided by the baud rate, that is,

$$h = \frac{2\omega_d}{\omega_b} = \frac{2f_d}{f_b} \quad (69)$$

The sinusoid transmitted during a baud is called the FSK symbol or tone specified by the bit block. Let  $k_n$  be the decimal value of an input bit block during the symbol period  $nT_b \leq t < (n + 1)T_b$ . The input block selects the frequency deviation

$$\Omega_{k_n} = \omega_d[2k_n - (M - 1)] \quad (70)$$

The frequency signal for baud  $n$  is  $\Omega_{k_n}p(t - nT_b)$  where  $p(t)$  is the unit height pulse of duration  $T_b$  defined as

$$p(t) = \begin{cases} 1 & \text{for } 0 \leq t < T_b \\ 0 & \text{elsewhere} \end{cases} \quad (71)$$

The elementary frequency deviation set of signals is

$$a_k(t) = \Omega_k p(t) = \omega_d [2k - (M - 1)] p(t) \quad \text{for } k = 0, 1, \dots, M - 1 \quad (72)$$

Assuming transmission starts at  $t = 0$ , the complete frequency signal is the staircase signal

$$m(t) = \sum_{n=0}^{\infty} \Omega_{k_n} p(t - nT_b) \quad (73)$$

This baseband signal is applied to an FM modulator with carrier frequency  $\omega_c$  and frequency sensitivity  $k_\omega = 1$  to generate the FSK signal. The complex envelope of the transmitted FM signal is

$$x(t) = A_c e^{j \int_0^t m(\tau) d\tau} e^{j\phi_0} \quad (74)$$

where  $\phi_0$  is an arbitrary initial phase.

The functions required to compute the power spectral density assuming the  $M$  frequencies are equally likely are:

1.

$$b_n(t) = \int_0^t \Omega_n p(t) dt = \Omega_n t \quad \text{for } 0 \leq t < T \quad (75)$$

2.

$$F_n(\omega) = \int_0^T e^{j b_n(t)} e^{-j\omega t} dt = T \frac{\sin \frac{(\omega - \Omega_n)T}{2}}{\frac{(\omega - \Omega_n)T}{2}} e^{-j(\omega - \Omega_n)T/2} \quad (76)$$

3.

$$C(1; T) = \mathcal{E} \{ e^{j b_n(T)} \} = \frac{1}{M} \sum_{n=0}^{M-1} e^{j\omega_d T [2n - (M-1)]} = \frac{2}{M} \sum_{n=1}^{M/2} \cos[\omega_d T (2n - 1)] \quad (77)$$

$$= \frac{\sin(M\pi h)}{M \sin(\pi h)} \quad (78)$$

4. When the modulation index  $h$  is an integer  $k$

$$\gamma = \begin{cases} 0 & \text{for } k \text{ even} \\ \omega_b/2 & \text{for } k \text{ odd} \end{cases} \quad (79)$$

5.

$$F(\omega) = \frac{T}{M} \sum_{n=0}^{M-1} \frac{\sin \frac{(\omega - \Omega_n)T}{2}}{\frac{(\omega - \Omega_n)T}{2}} e^{-j(\omega - \Omega_n)T/2} \quad (80)$$

6.

$$G(\omega) = \mathcal{E} \left\{ \overline{F_n(\omega)} e^{jb_n(T)} \right\} = \frac{1}{M} \sum_{n=0}^{M-1} T \frac{\sin \frac{(\omega - \Omega_n)T}{2}}{(\omega - \Omega_n)T} e^{j(\omega - \Omega_n)T/2} e^{j\Omega_n T} \quad (81)$$

Also, the power spectral density formula requires

$$\begin{aligned} e^{-j\omega T} G(\omega) &= e^{-j\omega T} \frac{1}{M} \sum_{n=0}^{M-1} T \frac{\sin \frac{(\omega - \Omega_n)T}{2}}{(\omega - \Omega_n)T} e^{j(\omega - \Omega_n)T/2} e^{j\Omega_n T} \\ &= \frac{1}{M} \sum_{n=0}^{M-1} T \frac{\sin \frac{(\omega - \Omega_n)T}{2}}{(\omega - \Omega_n)T} e^{j(\omega - \Omega_n)T/2} e^{-(\omega - j\Omega_n)T} \\ &= \frac{1}{M} \sum_{n=0}^{M-1} T \frac{\sin \frac{(\omega - \Omega_n)T}{2}}{(\omega - \Omega_n)T} e^{-j(\omega - \Omega_n)T/2} = F(\omega) \end{aligned} \quad (82)$$

7.

$$P(\omega) = \frac{T^2}{M} \sum_{n=0}^{M-1} \left[ \frac{\sin \frac{(\omega - \Omega_n)T}{2}}{(\omega - \Omega_n)T} \right]^2 \quad (83)$$

Putting these results together, the power spectral density is

$$TS_{xx}(\omega) = \begin{cases} P(\omega) + 2\Re e \left[ \frac{F^2(\omega)}{1 - C(1; T)e^{-j\omega T}} \right] & \text{for } h = \frac{2\omega_d}{\omega_b} \text{ not an integer} \\ P(\omega) - |F(\omega)|^2 + \omega_b \sum_{n=-\infty}^{\infty} |F(\gamma + n\omega_b)|^2 \delta(\omega - \gamma - n\omega_b) & \\ \text{for } h = \text{an integer } k & \end{cases} \quad (84)$$

where

$$\gamma = \begin{cases} 0 & \text{for } k \text{ even} \\ \omega_b/2 & \text{for } k \text{ odd} \end{cases} \quad (85)$$

### Comments on the Properties of this Power Spectral Density

$F_n(\omega)$  has its peak magnitude at the tone frequency  $\Omega_n = \omega_d[2n - (M - 1)]$  and zeros at multiples of the symbol rate,  $\omega_b$ , away from the tone frequency. This is exactly what would be expected for a burst of duration  $T$  of a sinusoid at the tone frequency.

Another implementation of an FSK transmitter could use a bank of oscillators at the tone frequencies and each baud the transmitter could switch to the oscillator specified by

the input bit block. This approach does not guarantee continuous phase. The term  $P(\omega)$  is what would result for this switched oscillator case when the phases of the oscillators are independent random variables uniformly distributed over  $[0, 2\pi)$ .

The remaining terms account for the continuous phase property and give a narrower spectrum than if the the phase were discontinuous.

The power spectrum has impulses at the  $M$  tone frequencies when  $h$  is an integer. However, the impulses at other frequencies disappear because they are multiplied by the nulls of  $F(\gamma + n\omega_b)$ .

#### 4.1 Power Spectral Density Examples for $M = 2$ FSK

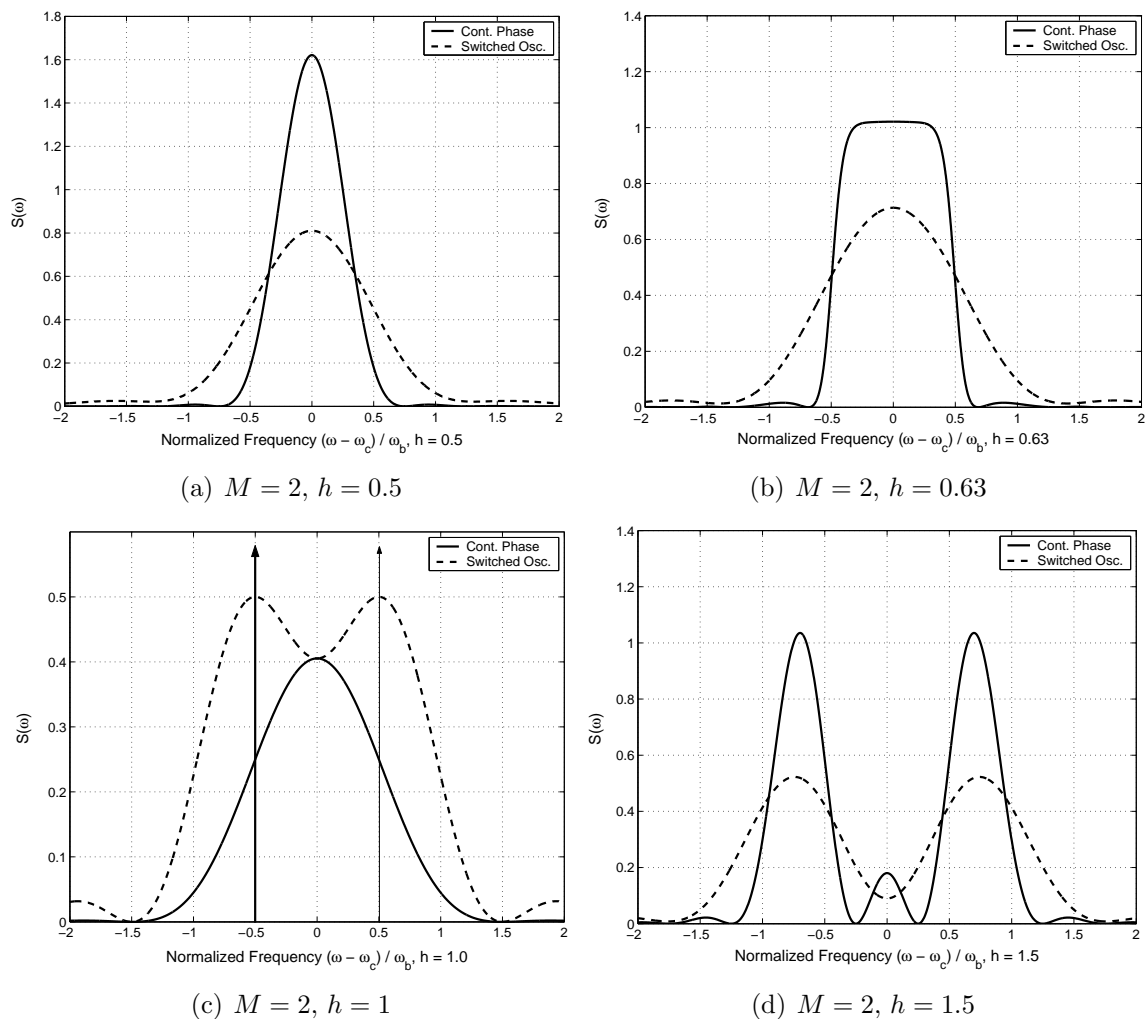


Figure 2: Normalized Power Spectral Densities  $TS_{xx}(\omega)$  for Continuous Phase and Switched Oscillator Binary FSK for Several Values of  $h$