PRACTICE MATH SUBJECT GRE PROBLEMS.

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All logarithms with unspecified base are natural logarithms with base e.

1. Calculus.

Problem 1. $\lim_{x\to 0} \dfrac{\sin(x^3)}{x\,(\cos(2x)-1)} =$

(A) 2 (B) -2 (C) $\dfrac{1}{2}$ (D) $-\dfrac{1}{2}$ (E) 1

Problem 2. $\lim_{x\to\infty} \dfrac{\log(x)}{x^{1/4}} =$

(A) 0 (B) 1 (C) $\dfrac{1}{4}$ (D) 4 (E) Does not exist

Problem 3. $\lim_{x\to\infty} \sqrt{x^2-x}-x =$

(A) 2 (B) -2 (C) $\dfrac{1}{2}$ (D) $-\dfrac{1}{2}$ (E) 1

Problem 4. $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} =$

(A) 0 (B) 1 (C) $\dfrac{1}{2}$ (D) 2 (E) ∞

Problem 5. $\lim_{x\to\infty} \left(1+\dfrac{2}{x}+\dfrac{3}{x^2}\right)^x =$

(A) 1 (B) $1/e$ (C) e (D) $1/e^2$ (E) e^2

Problem 6. $\displaystyle\int_1^4 \frac{\log x}{\sqrt{x}}dx =$

(A) $\log(64)-1$ (B) $\log(64)-4$ (C) $\log(128)-1$
(D) $\log(128)-4$ (E) $\log(256)-4$

Problem 7. $\displaystyle\int_1^2 x^3\sqrt{x^2+1}dx =$

(A) $\dfrac{2}{15}\left(\sqrt{5}-\sqrt{2}\right)$ (B) $\dfrac{2}{15}\left(25\sqrt{5}-\sqrt{2}\right)$ (C) $\dfrac{2}{5}\left(25\sqrt{5}-1\right)$
(D) $\dfrac{2}{5}\left(\sqrt{5}-\sqrt{2}\right)$ (E) $\dfrac{1}{15}\left(\sqrt{5}-\sqrt{2}\right)$

Problem 8. $\displaystyle\int_1^3 \frac{e^{1/x}}{x^2}dx =$

(A) e (B) $e-\sqrt{e}$ (C) $e-\sqrt[3]{e}$ (D) $\dfrac{1}{e^3}$ (E) $\dfrac{1}{e}-\dfrac{1}{e^3}$

Problem 9. $\displaystyle\int_0^1 \frac{e^x+1}{e^x+x}dx =$

(A) 1 (B) e (C) $1+e$ (D) $\log(1+e)$ (E) $\log(e+e^e)$

Problem 10. $\displaystyle\int_0^{\pi/2} \cos^2 x\,dx =$

(A) $\pi/3$ (B) $\pi/4$ (C) $\pi/8$ (D) $1/2$ (E) $1/4$

Problem 11. Which of the following shows the numbers $27^{30}, 5^{40}, 4^{60}$ in increasing order?
(A) $27^{30} < 5^{40} < 4^{60}$ (B) $27^{30} < 4^{60} < 5^{40}$
(C) $4^{60} < 27^{30} < 5^{40}$ (D) $4^{60} < 5^{40} < 27^{30}$
(E) $5^{40} < 4^{60} < 27^{30}$

Problem 12. Which of the following shows the numbers $10^{1/9}, 9^{1/10}, 8^{1/11}$ in increasing order?
(A) $10^{1/9} < 9^{1/10} < 8^{1/11}$ (B) $10^{1/9} < 8^{1/11} < 9^{1/10}$
(C) $8^{1/11} < 10^{1/9} < 9^{1/10}$ (D) $8^{1/11} < 9^{1/10} < 10^{1/9}$
(E) $9^{1/10} < 8^{1/11} < 10^{1/9}$

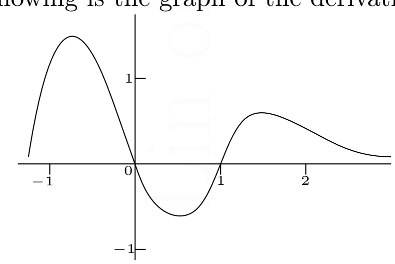
Problem 13. Which of the following shows the numbers e^π, e^e, π^e in increasing order?
(A) $e^\pi < e^e < \pi^e$ (B) $e^e < e^\pi < \pi^e$
(C) $e^e < \pi^e < e^\pi$ (D) $\pi^e < e^e < e^\pi$
(E) $\pi^e < e^\pi < e^e$

Problem 14. Which of the following shows the numbers $200!, 100^{200}, 200^{100}$ in increasing order?
(A) $200! < 100^{200} < 200^{100}$ (B) $200! < 200^{100} < 100^{200}$
(C) $200^{100} < 200! < 100^{200}$ (D) $200^{100} < 100^{200} < 200!$
(E) $100^{200} < 200^{100} < 200!$

Problem 15. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^4}{n^5} =$
(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$ (E) $\frac{1}{5}$

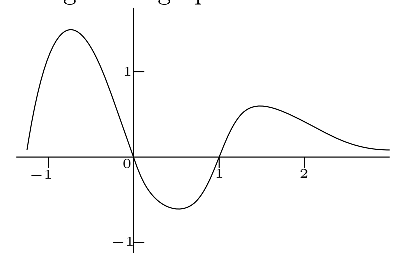
Problem 16. Let $p(x)$ be a polynomial such that $p(3) = p'(3) = 0$. What must be true about p?
I. The degree of p must be at least 2.
II. There is a local maximum at $x = 3$.
III. There is a local minimum at $x = 3$.
(A) I only (B) II only (C) III only
(D) I, II, and III (E) None of I, II, III

Problem 17. The following is the graph of the derivative f' of a function f:



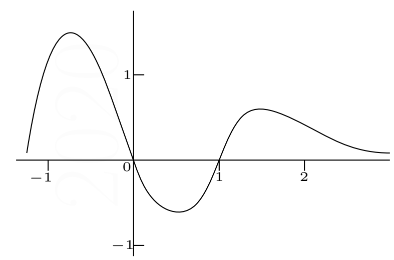
Which of the following must be true?
I. f is decreasing on the interval $(0, 1)$
II. $f(-1) > f(0)$
III. $f(-1) > f(1)$
(A) I only (B) II only (C) III only
(D) I and II only (E) I and III only

Problem 18. The following is the graph of the derivative f' of a function f:



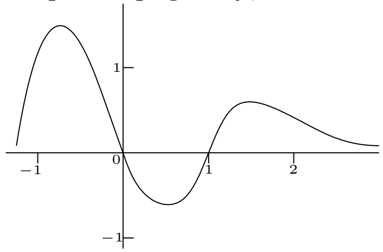
Which of the following must be true?
I. $f(0) > f(1)$
II. f has a maximum between the interval $(0, 2)$
III. f has a maximum between the interval $(-1, 1)$
(A) I only (B) II only (C) III only
(D) I and II only (E) I and III only

Problem 19. The following is the graph of $F(x) = \int_0^x f(t)dt$, where f is some continuous function:



Which of the following must be true?
I. $f(x) > 0$ for all $x \in (-1, 0)$
II. f has exactly one root in the interval $(0, 2)$
III. $f(0) < 0$
(A) I only (B) II only (C) III only
(D) I and II only (E) I and III only

Problem 20. The following is the graph of f, some twice differentiable function:



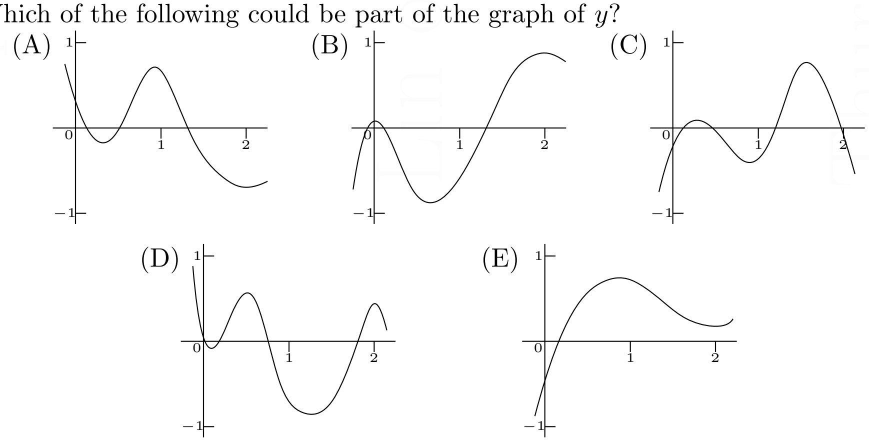
Which of the following CANNOT be true?
I. There exists at least two points p, q in $(-1, 2)$ such that $f''(p) = f''(q)$.
II. There exists a point $r \in (-1, 0)$ such that $f'(r) < -1$.
III. There exists a point $s \in (0, 1)$ such that $f''(s) < 0$.
(A) I only (B) II only (C) III only
(D) I and II only (E) I and III only

Problem 21. Let g be a twice differentiable function. Suppose $g'(x) < 0$ and $g''(x) > 0$ for all $x \in \mathbb{R}$. Which of the following must be true?
I. $\lim_{x\to\infty} g(x) = -\infty$ II. $\lim_{x\to-\infty} g(x) = +\infty$ III. g is injective
(A) I only (B) II only (C) III only
(D) I and III only (E) II and III only

Problem 22. Suppose y is a continuous real-valued function such that

$$y''(1) > 0, y'(1) < 0, \text{ and } y'(2) = 0.$$

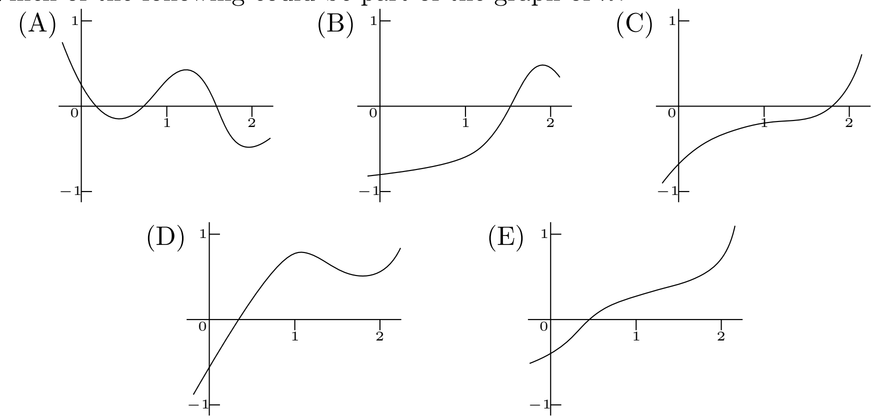
Which of the following could be part of the graph of y?



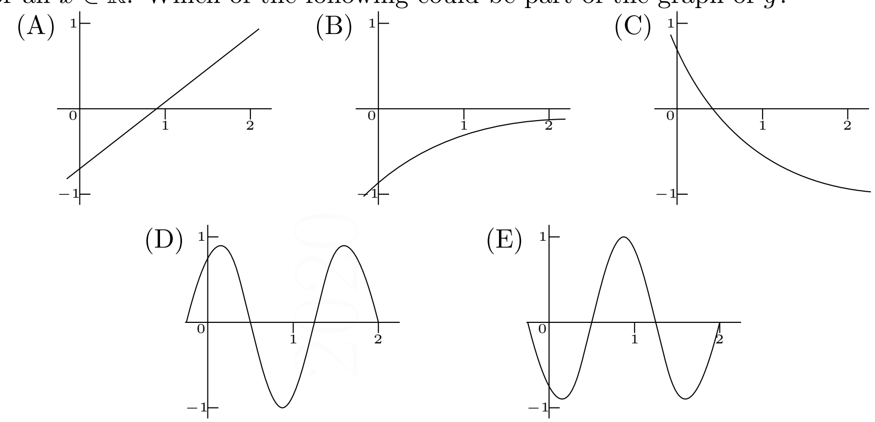
Problem 23. Suppose h is a continuous real-valued function such that

$$h'(x) > 0 \text{ on } (0, 1), \text{ and } h(1) + h''(2) = 0.$$

Which of the following could be part of the graph of h?



Problem 24. Suppose g is a differentiable function such that $g'(x) + g(x) = 0$ for all $x \in \mathbb{R}$. Which of the following could be part of the graph of g?



Problem 25. The region bounded by the curves $y = 1/x$, $x = 1$, $x = 2$, $y = 0$ is rotated about the x-axis. The volume of the resulting solid of revolution is
(A) π (B) $\pi/2$ (C) $\pi/3$ (D) $\pi \ln 2$ (E) $\pi \ln 3$

Problem 26. Let p be a nonconstant polynomial of degree 5. Which of the following must be true?
I. The graph of p has three inflection points, points where the second derivative changes signs.
II. Between any two distinct roots of p, there exists a local maximum or minimum of p.
III. The graph of p crosses the x-axis at least twice.
(A) I only (B) II only (C) III only
(D) I, II, and III (E) None of I, II, III

Problem 27. The region bounded by curves $y = x$ and $y = x^n$ for some integer $n \geqslant 2$ in the first quadrant of the xy-plane is rotated about the x-axis. If the volume of the resulting solid is $\frac{4\pi}{15}$, what is n?
(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Problem 28. The region bounded by curves $y = x$ and $y = x^n$ for some integer $n \geqslant 2$ in the first quadrant of the xy-plane is rotated about the y-axis. If the volume of the resulting solid is $\frac{4\pi}{15}$, what is n?
(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Problem 29. The region bounded by curves $y = x^2$ and $x = y^2$ is rotated about the line $x = -1$. The volume of the resulting solid of revolution is
(A) $\frac{1}{3}\pi$ (B) $\frac{3}{10}\pi$ (C) $\frac{14}{15}\pi$ (D) $\frac{29}{30}\pi$ (E) $\frac{16}{45}\pi$

Problem 30. The region bounded by curves $y = 1 + \sec x$ and $y = 3$ is rotated about the line $y = 1$. The volume of the resulting solid of revolution is
(A) $\pi\left(\frac{4}{3}\pi - \sqrt{3}\right)$ (B) $2\pi\left(\frac{4}{3}\pi - \sqrt{3}\right)$ (C) $\pi\left(\frac{2}{3}\pi - \frac{\sqrt{3}}{3}\right)$
(D) $2\pi\left(\frac{2}{3}\pi - \frac{\sqrt{3}}{3}\right)$ (E) $2\pi\left(\frac{2}{3}\pi - \sqrt{3}\right)$

Problem 31. If g is differentiable on the interval $(-2, 4)$ and that $g'(x) \geqslant -3$ for all x. If $g(3) = -9$, what is the largest possible value for $g(-1)$?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 32. If g is differentiable on the interval $(-3, 3)$ and that $g'(x) \leqslant 2$ for all x. If $g(-1) = 1$, what is the largest possible value of $g(2)$?
(A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Problem 33. If g is differentiable on the interval $(0, 5)$ and that $g'(x) \geqslant 2$ for all x. If $g(1) = 5$, what is the smallest possible value of $g(4)$?
(A) 5 (B) 7 (C) 9 (D) 11 (E) 13

Problem 34. Consider the function $f(x) = x - x^3 + 2^x$. Which of the following statements must be true?
I. f has at least one root on the interval $(-3, 0)$
II. f has at least one root on the interval $(1, 2)$
III. f has at least two roots on $\mathbb{R}$
(A) I only (B) II only (C) III only
(D) I and II only (E) II and III only

Problem 35. Consider the function $f(x) = 1 + 2x + x^3 + 4x^5$. How many real roots does f have?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 5

Problem 36. Let g be a continuous real-valued function such that $g(x) = \int_0^{\sin(x)} \cos(x)dx$. What is $g'(0)$?
(A) 0 (B) 1 (C) $\cos(1)$ (D) $\cos(\sin(1))$ (E) $\sin(\sin(1))$

Problem 37. Let g be a continuous real-valued function such that $g(x) = \int_{\sqrt{x}}^{x^2} e^t dt$. What is $g'(4)$?
(A) $e^8 - e$ (B) $e^{16} - e^2$ (C) $2e^{16} - e^2$
(D) $2e^{16} - \frac{1}{2}e^2$ (E) $8e^{16} - \frac{1}{4}e^2$

Problem 38. Let g be a continuous real-valued function such that $12e^{2x} - 1 = \int_x^c g(t)dt$. What is the value of c?
(A) 1 (B) -1 (C) $\log 12$ (D) $\frac{1}{2}\log 12$ (E) $-\frac{1}{2}\log 12$

Problem 39. Let g be a continuous real-valued function such that for all positive x we have $x^3 - x = \int_c^{\sqrt{x}} g(t)dt$, for some real number c. How many possible values of c are there?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 40. If $F(x) = \int_1^x f(t)dt$, where $f(t) = \int_1^{t^2} \frac{\sqrt{1+u^4}}{u} du$, then $F''(2) =$
(A) $2\sqrt{257}$ (B) $\sqrt{257}$ (C) $2\sqrt{17}$ (D) $\sqrt{17}$ (E) $\frac{\sqrt{17}}{4}$

Problem 41. How many real solutions does the equation $x^2 - 2^x = 0$ have?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 42. Let $f(x) = \sin(x)$ and $g(x) = \ln|x|$ both defined on $\mathbb{R}\backslash\{0\}$. For how many $x \in \mathbb{R}\backslash\{0\}$ do we have $f(x) = g(x)$?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 43. Let $f(x) = \sin(2x)$ and $g(x) = \ln|x|$ both defined on $\mathbb{R}\backslash\{0\}$. For how many $x \in \mathbb{R}\backslash\{0\}$ do we have $f(x) = g(x)$?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 44. Which of the following equations has the greatest number of real solutions?
(A) $x^5 + 2x - 3 = 0$
(B) $x^2 + x - 1 = 0$
(C) $x^4 + x^2 + 1 = e^x$
(D) $\sin x = x$
(E) $e^{-x^2} = x^2$

Problem 45. How many real solutions does the equation $\frac{x+1}{x^2+1} = x^2$ have?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 46. Let f be the function defined by $f(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} x^n$ on $-1 < x < 1$. Then $f(\frac{1}{2}) =$
(A) 1 (B) 2 (C) 4 (D) $\log 2$ (E) $\log 4$

Problem 47. Let f be the function defined by $f(x) = \sum_{n=0}^{\infty} n^2 x^n$ on $-1 < x < 1$. Then $f(\frac{1}{2}) =$
(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

Problem 48. Let f be the function defined by $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$ on $-1 < x < 1$. Then $\int_0^x f(t)dt =$
(A) $(1-x)\log(1-x)$ (B) $x + (1-x)\log(1-x)$ (C) $x - x\log\frac{1}{1-x}$
(D) $1 + x - x\log\frac{1}{1-x}$ (E) $1 - x - x\log\frac{1}{1-x}$

Problem 49. Let f be the function defined by $f(x) = \sum_{n=0}^{\infty} \frac{(\sin x)^n}{n!}$. Then $f'(\pi) =$
(A) -1 (B) 0 (C) 1 (D) 2 (E) 3

Problem 50. Let f be the function defined by $f(x) = \sum_{n=0}^{\infty} (x+1)^n$ on $-2 < x < 0$. Then $f'(x) =$
(A) $\frac{1}{x}$ (B) $\frac{1}{x^2}$ (C) $\frac{1}{1+x}$ (D) $\frac{1}{(1+x)^2}$ (E) $\frac{x}{1+x}$

Problem 51. Let g be a function that $g(x) = \log(x+1)$. Then $\lim_{x\to 0} \frac{g(x)}{g(x^2)} =$
(A) 0 (B) 1 (C) $\log 2$ (D) $\log 4$ (E) The limit does not exist.

Problem 52. Let g be a function that $g(x) = \log(x+1)$. Then $\lim_{x\to 0} \frac{g(x^2)}{g(x)^2} =$
(A) 0 (B) 1 (C) $\log 2$ (D) $\log 4$ (E) The limit does not exist.

Problem 53. Let g be a function that $g(x) = \arctan(x^2)$. Then $\lim_{x\to 0} \frac{x^4}{g(g(x))} =$
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 54. $\lim_{x\to 0}\left(\frac{1}{\sin^2 x}-\frac{1}{x^2}\right)=$

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$ (E) The limit does not exist.

Problem 55. $\lim_{x\to 0^+}\left(\frac{\sin x}{x}\right)^{1/x^2}=$

(A) $\cos e$ (B) $e^{-1/6}$ (C) $e^{1/3}$ (D) 0 (E) 1

Problem 56. What is the value of $\int_{-3}^{3}\left(xe^{x^2}\cos(x)-|x|\right)dx$?

(A) 9 (B) 6 (C) $\sqrt{2}e^9\sin(3)$ (D) $\frac{\sqrt{2}}{2}e^9\sin(3)$ (E) 0

Problem 57. What is the value of $\int_{0}^{2}\left(\sin^3(1-x)e^{1-2x+x^2}+\frac{1}{1+x}\right)dx$?

(A) 0 (B) $\frac{1}{4}e^{1/4}-\log 3$ (C) $\log 3$ (D) 1/3 (E) 1/5

Problem 58. What is the value of $\int_{-1}^{1}\frac{e^x-e^{-x}}{e^x+e^{-x}}dx$?

(A) 0 (B) e^2 (C) e^{-2} (D) $\frac{e-1/e}{e+1/e}$ (E) $\frac{1}{e+1/e}$

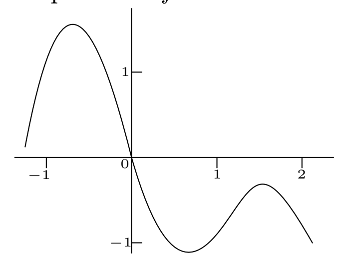
Problem 59. What is the value of $\int_{0}^{2}\left(xe^{-x^2}+x\right)dx$?

(A) 0 (B) $\frac{1}{e^4}$ (C) $\frac{5}{2}-\frac{1}{e^4}$ (D) $\frac{5}{4}-\frac{1}{e^4}$ (E) $\frac{5}{2}-\frac{1}{2e^4}$

Problem 60. What is the value of $\int_{-2}^{2}\frac{1}{1+e^x}dx$

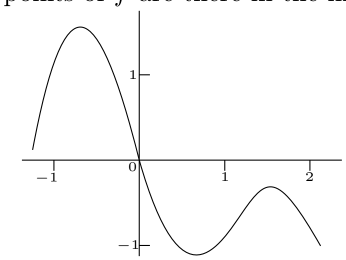
(A) 1 (B) 2 (C) 3 (D) 4 (E) 0

Problem 61. The following depicts the graph of f'' for some real-valued function f. How many inflection points of f are there in the interval $(-1,2)$?



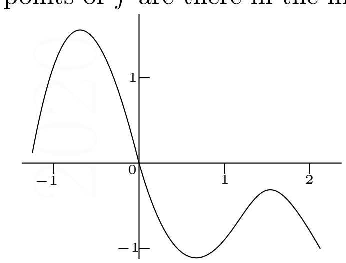
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 62. The following depicts the graph of f' for some real-valued function f. How many inflection points of f are there in the interval $(-1,2)$?



(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 63. The following depicts the graph of f for some real-valued function f. How many inflection points of f are there in the interval $(-1,2)$?



(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 64. Consider the polynomial $p(x)=(x-1)(x-2)^2(x-3)^4(x-4)^3$. How many inflection points does p have?

(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

Problem 65. How many inflection points are there for the function $f(x) = 2\cos(2x)$ on the interval $(0, 10)$?
(A) 0 (B) 2 (C) 4 (D) 6 (E) 8

Problem 66. Let f be a differentiable function defined on $\mathbb{R}$ whose derivative is strictly positive. Suppose the line $y = 2x + 5$ is tangent to the graph of f at $x = 3$. Which of the following is NOT true?
(A) $f(3) = 11$ (B) $f'(3) = 2$ (C) $\left(f^{-1}\right)'(11) = \dfrac{1}{2}$
(D) f must be injective (E) f must be surjective

Problem 67. Let f be a differentiable function defined on $\mathbb{R}$ that is tangent to the curve $g(x) = e^{x^2}$ at $x = 1$. What is $(g \circ f)'(1)$?
(A) $4e^{e^2} + 2$ (B) $4e^{e^2}$ (C) $4e^2$ (D) $4e^{e^2+2}$ (E) $4e^2 + 2$

Problem 68. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and bijective with $f(0) = \pi$, and for each $x \in \mathbb{R}$,

$$f'(x) = 3 - e^{-x^2} - \cos(f(x)).$$

Find $\left(f^{-1}\right)'(\pi)$.
(A) 1 (B) $\dfrac{1}{2}$ (C) $\dfrac{1}{3}$ (D) $\dfrac{\pi}{2}$ (E) $\dfrac{\pi}{3}$

Problem 69. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and the line $y = 3x + 2$ is tangent to the graph of f at $x = 5$. What is $(\log(f(x)))'(5)$?
(A) $\dfrac{3}{5}$ (B) $\dfrac{3}{13}$ (C) $\dfrac{3}{15}$ (D) $\dfrac{3}{17}$ (E) $\dfrac{3}{19}$

Problem 70. Suppose $f(x)$ is a differentiable real-valued function and is defined implicitly by

$$f(x) = e^x \log(f(x)) - 1.$$

What is $\left(f^{-1}\right)'(e)$?
(A) $\dfrac{1}{e^2+1}$ (B) $\dfrac{1}{e^2+e}$ (C) $\dfrac{-1}{e^2+e}$ (D) $e^2 + 1$ (E) $e^2 + e$

Problem 71. For what value of c does the equation $e^x = c\sqrt{x}$ have exactly one real solution for x?
(A) $\sqrt{2}$ (B) $\sqrt{e}$ (C) $\sqrt{2e}$ (D) $2\sqrt{e}$ (E) $e\sqrt{2}$

Problem 72. Let c be some positive number such that the graph of ce^x intersects the unit circle $x^2 + y^2 = 1$ at exactly one point. What is the x-coordinate of this point of intersection?
(A) 0 (B) $\dfrac{2-\sqrt{5}}{2}$ (C) $\dfrac{2+\sqrt{5}}{2}$ (D) $\dfrac{1-\sqrt{5}}{2}$ (E) $\dfrac{1+\sqrt{5}}{2}$

Problem 73. For what positive value of c does the graph of $y = \dfrac{c}{x}$ intersects the unit circle $x^2 + y^2 = 1$ at exactly two points?
(A) 1 (B) $\dfrac{1}{2}$ (C) $\dfrac{1}{3}$ (D) $\dfrac{1}{3}$ (E) $\dfrac{1}{5}$

Problem 74. The graph of $y = cx^2 + 1$ intersects the unit circle $x^2 + y^2 = 1$ at $(0, 1)$. What is the minimum value of c such that $(0, 1)$ is the only such intersection point?
(A) 1 (B) $\dfrac{1}{2}$ (C) 0 (D) $-\dfrac{1}{2}$ (E) -1

Problem 75. For which value(s) of c does the equation $c(x + 2) = 1 - x^2$ have exactly one solution?
(A) $-2 + \sqrt{3}$ and $-2 - \sqrt{3}$ only (B) $4 + 2\sqrt{3}$ and $4 - 2\sqrt{3}$ only
(C) $-2 + \sqrt{3}$ only (D) $-2 - \sqrt{3}$ only (E) $4 + 2\sqrt{3}$ only

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Problem 76. What is the derivative of the function $f(x) = \displaystyle\int_{2x}^{3x} \frac{t^2-1}{t^2+1}\,dt$?
(A) $\dfrac{9x^2-1}{9x^2+1} - \dfrac{4x^2-1}{4x^2+1}$ (B) $\dfrac{3(9x^2-1)}{9x^2+1} - \dfrac{2(4x^2-1)}{4x^2+1}$ (C) $\dfrac{x^2-1}{x^2+1} - \dfrac{x^2-1}{x^2+1}$
(D) $\dfrac{3(x^2-1)}{x^2+1} - \dfrac{2(x^2-1)}{x^2+1}$ (E) $\dfrac{4(3x)^2}{(9x^2+1)^2} - \dfrac{4(2x)^2}{(4x^2+1)^2}$

Problem 77. $\dfrac{d}{dx}\displaystyle\int_{1-2x}^{1+2x} t\sin t\,dt =$
(A) $2(1 + 2x)\sin(1 + 2x) + 2(1 - 2x)\sin(1 - 2x)$
(B) $2(1 + 2x)\sin(1 + 2x) - 2(1 - 2x)\sin(1 - 2x)$
(C) $2(1 + 2x)\cos(1 + 2x) + 2(1 - 2x)\cos(1 - 2x)$
(D) $2(1 + 2x)\cos(1 + 2x) + 2(1 - 2x)\cos(1 - 2x)$
(E) None of the above

Problem 78. $\dfrac{d}{dx}\displaystyle\int_{\sqrt{x}}^{x^2} \arctan t\, dt =$
(A) $\dfrac{1}{x^4+1} - \dfrac{1}{x+1}$
(B) $\dfrac{2x}{x^4+1} - \dfrac{1}{2\sqrt{x}\,(x+1)}$
(C) $\arctan x^2 - \arctan\sqrt{x}$
(D) $2x\arctan x^2 - \dfrac{1}{2\sqrt{x}}\arctan\sqrt{x}$
(E) $2x\arctan x^2 + \dfrac{1}{2\sqrt{x}}\arctan\sqrt{x}$

Problem 79. Let $f(x) = \displaystyle\int_x^1 (1-t^2)e^{t^2}\,dt$. Which of the following intervals is f increasing?
(A) $(0,\infty)$ (B) $(-\infty,0)$ (C) $(-1,1)$ (D) $(-\infty,1)$ (E) $(1,\infty)$

Problem 80. If $f(x) = \displaystyle\int_1^{\sin(x)} \sqrt{1+t^2}dt$ and $g(x) = \displaystyle\int_x^1 f(t)dt$, then what is $g''(\pi/3)$?
(A) $-\dfrac{\sqrt{7}}{4}$ (B) $\dfrac{\sqrt{7}}{4}$ (C) $-\dfrac{1+\sqrt{3}}{2}$ (D) $\dfrac{1+\sqrt{3}}{2}$ (E) $\sqrt{1+\dfrac{3}{2}}$

Problem 81. Let $f(x) = (x-1)\sin x$. What is the 10-th derivative of f evaluated at $x=0$? Namely, what is $f^{(10)}(0)$?
(A) 10 (B) $\dfrac{1}{9}$ (C) $\dfrac{1}{9!}$ (D) 9 (E) 9!

Problem 82. What is the 10-th derivative of $\dfrac{x}{e^{2x}}$?
(A) $1024(x-5)e^{-2x}$ (B) $(-1024x+10)e^{-2x}$ (C) $-1024xe^{-2x}$
(D) $1024xe^{-2x}$ (E) $(1024x-10)e^{-2x}$

Problem 83. What is the 20-th derivative of $(x^2+x)\cos(x)$?
(A) $(2x+1)\cos(x) - (x^2+x)\sin(x)$
(B) $(2x+1)\cos(x) + (x^2+x)\sin(x)$
(C) $(x^2+x)\cos(x) - (2x+1)\sin(x)$
(D) $(x^2+x-380)\cos(x) + (40x+20)\sin(x)$
(E) $(x^2+x-380)\cos(x) - (40x+20)\sin(x)$

Problem 84. Let $f(x) = (x^2+x+1)e^x$. What is the 19-th derivative of f evaluated at $x=0$? Namely, what is $f^{(19)}(0)$?
(A) 342 (B) 352 (C) 362 (D) 372 (E) 382

Problem 85. Which of the following series converges?
I. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$
II. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{1+1/n}}$
III. $\displaystyle\sum_{n=1}^{\infty}\left(\sqrt[n]{2}-1\right)^n$
(A) I only (B) II only (C) III only
(D) I and III only (E) I, II, and III all converges

Problem 86. How many positive numbers x satisfy the equation $\sin(50x) = x$?
(A) 10 (B) 15 (C) 20 (D) 25 (E) 30

Problem 87. How many real numbers x satisfy the equation $x^{18} = e^{-x}$?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 88. Which of the following has the greatest number of real solutions?
(A) $x^3 + x = 0$
(B) $x^9 + x = 0$
(C) $x^{12} + x = 0$
(D) $x^{15} + x = 0$
(E) $x^{21} + x = 0$

Problem 89. How many real numbers x satisfy the equation $\sinh(x) = \cosh(x)$?
(A) 0 (B) 1 (C) 2 (D) 4 (E) Infinitely many

Problem 90. How many real numbers x satisfy the equation $e^{\sin(x)} = 1 - x^2$
(A) 0 (B) 1 (C) 2 (D) 4 (E) Infinitely many

Problem 91. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?
(A) 0 cm/min (B) 1.4 cm/min (C) -1.4 cm/min
(D) 1.6 cm/min (E) -1.6 cm/min

Problem 92. A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?
(A) $\frac{7}{4}\sqrt{15}$ m/s (B) $\frac{7}{2}\sqrt{15}$ m/s (C) $\frac{7}{2}\sqrt{5}$ m/s
(D) $\frac{7}{4}\sqrt{5}$ m/s (E) $\frac{7}{4}\sqrt{25}$ m/s

Problem 93. The top of a ladder slides down a vertical wall at a rate of 0.15 m/s. At the moment when the bottom of the ladder is 3 m from the wall, it slides away from the wall at a rate of 0.2 m/s. How long is the ladder?
(A) 2 m (B) 3 m (C) 4 m (D) 5 m (E) 6 m

Problem 94. The volume of a cube is increasing at a rate of 10 cm^3/min. How fast is the surface area increasing when the length of an edge is 30 cm?
(A) $\frac{1}{3}$ cm^2/min (B) $\frac{2}{3}$ cm^2/min (C) 1 cm^2/min
(D) $\frac{4}{3}$ cm^2/min (E) $\frac{5}{3}$ cm^2/min

Problem 95. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.
(A) $\frac{1}{10\pi}$ cm/min (B) $\frac{1}{20\pi}$ cm/min (C) $\frac{4}{30\pi}$ cm/min
(D) $\frac{4}{3\pi}$ cm/min (E) $\frac{3}{4\pi}$ cm/min

Problem 96. Let $g(x) = x|x|$ defined on the real line. Which of the following is true?
I. The graph of g has an inflection point
II. $g'(0)$ exists
III. $g''(0)$ exists
(A) I only (B) II only (C) III only
(D) I and II only (E) None of them

Problem 97. $\lim_{n\to\infty} (n+4)^{2/3} - n^{2/3} =$
(A) 0 (B) 1 (C) 4 (D) $\frac{2}{3}$ (E) ∞

Problem 98. Let f be a real-valued continuously differentiable functions on $\mathbb{R}$, define
$$g(x) = \int_0^x f(t)(t-x)dt.$$
Then $g'(x) =$
(A) 0 (B) $f(x)$ (C) $-xf(x)$ (D) $\int_0^x f(t)dt$ (E) $-\int_0^x f(t)dt$

Problem 99. For any two real-valued infinitely differentiable functions f and g on $\mathbb{R}$, define
$$(f*g)(x) = \int_0^x f(t)g(x-t)dt.$$
Then $(f*g)''(x) =$
(A) $(f*g'')(x)$ (B) $(f*g'')(x)+f'(x)g(0)$ (C) $(f*g'')(x)-f'(x)g(0)$
(D) $(f*g'')(x)+f(x)g'(0)$ (E) $(f*g'')(x)+f(x)g'(0)+f'(x)g(0)$

Problem 100. Let f be a real-valued continuously differentiable function on $\mathbb{R}$, define
$$g(x) = \int_0^x f(x-t)(x-t)dt.$$
Then $g'(x) =$
(A) 0 (B) $f(x)$ (C) $\int_0^x f'(t)(x-t)dt$
(D) $\int_0^x f'(x-t)(x-t)dt$ (E) $\int_0^x [f'(x-t)(x-t)+f(x-t)]dt$

Problem 101. $\displaystyle\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}dx =$

(A) 0 (B) 1 (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$ (E) ∞

Problem 102. For each non-negative integer n, we have $\displaystyle\int_0^{\infty} x^n e^{-x} dx =$

(A) $n+1$ (B) $2n-1$ (C) $n!$ (D) $(n+1)!$ (E) n

Problem 103. $\displaystyle\int_8^{27} \frac{1}{\sqrt{x}-\sqrt[3]{x}}dx =$

(A) 1 (B) $6\log\sqrt{\frac{3}{2}}$ (C) $6\log\sqrt{\frac{3}{2}}-10\sqrt{2}+12\sqrt{3}$

(D) $6\log\frac{\sqrt{3}-1}{\sqrt{2}-1}$ (E) $3-10\sqrt{2}+12\sqrt{3}+6\log\frac{\sqrt{3}-1}{\sqrt{2}-1}$

Problem 104. $\displaystyle\int_0^{\pi} \frac{x\sin(x)}{1+\cos^2 x}dx =$

(A) $\frac{\pi}{4}$ (B) $\frac{\pi^2}{4}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi^2}{2}$ (E) 1

Problem 105. Let f be a continuous real-valued function on $\mathbb{R}$, then $\displaystyle\int_0^{\pi/2} f(\cos(x))dx =$

(A) $\displaystyle\int_0^{\pi/2} f(\sin(x))dx$ (B) $\displaystyle-\int_0^{\pi/2} f(\cos(x))dx$

(C) $\displaystyle-\int_{-\pi/2}^{0} f(\cos(x))dx$ (D) $\displaystyle-\int_0^{\pi/2} f(\sin(x))dx$ (E) π

2. Multivariable calculus.

Problem 106. What is the volume of the solid in xyz-space bounded by the surfaces $x=0$, $x=\pi$, $y=0$, $y=\sqrt{\sin(x)}$, $z=0$, $z=xy$?

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\pi-1$ (D) $1-\frac{\pi}{2}$ (E) 2π

Problem 107. What is the volume of the solid in xyz-space bounded by the surfaces $y=16x^2$, $z=3y$, $z=2+y$

(A) 2 (B) $\frac{16}{15}$ (C) 1 (D) $\frac{8}{15}$ (E) $\frac{4}{15}$

Problem 108. What is the volume of the solid in xyz-space bounded by the surfaces $z=1-x^2$, $z=x^2-1$ and the planes $x+y+z=2$, $x-2y-2z-10=0$.

(A) $\frac{56}{3}$ (B) $\frac{56}{5}$ (C) $\frac{56}{9}$ (D) $\frac{56}{25}$ (E) 1

Problem 109. What is the volume of solid in xyz-space under the surface $z=xy$ and above the triangle with vertices $(1,1)$, $(4,1)$, and $(1,2)$?

(A) 1 (B) $\frac{16}{3}$ (C) $\frac{31}{8}$ (D) $\frac{45}{8}$ (E) 2

Problem 110. What is the volume of the solid in xyz-space bounded by the coordinate planes and the plane $3x+2y+z=6$?

(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Problem 111. In xyz-space, what is the shortest distance between a point on the plane $P: x+2y+3z=2$ and a point on the plane $Q: x+2y+3z=7$?

(A) $\frac{5}{14}$ (B) $\frac{5}{\sqrt{14}}$ (C) $5\sqrt{14}$ (D) $\frac{\sqrt{14}}{5}$ (E) $\frac{14}{\sqrt{5}}$

Problem 112. In xyz-space, what is the shortest distance between a point on the parabolic surface $P: z=x^2+y^2$ and a point on the plane $Q: x+y+z=-10$?

(A) $\frac{19}{2\sqrt{3}}$ (B) $\frac{19}{2}$ (C) $\frac{19}{\sqrt{3}}$ (D) $\frac{\sqrt{3}}{19}$ (E) $\frac{2\sqrt{3}}{19}$

Problem 113. In xyz-space, consider the lines $L_1: t(0,0,1)$ and $L_2: (2,2,2)+t(1,2,3)$ given parametrically. What is the shortest distance between a point on L_1 and a point on L_2?

(A) $\frac{6}{\sqrt{5}}$ (B) $\frac{6}{5}$ (C) $\frac{\sqrt{6}}{5}$ (D) $\frac{36}{5}$ (E) $\sqrt{12}$

Problem 114. In xyz-space, what is the shortest distance between the line $L_1: (0,1,0)+t(1,1,1)$ and the origin?

(A) 0 (B) 1 (C) 2 (D) $\sqrt{2}$ (E) $\sqrt{3}$

Problem 115. In xyz-space, what is the shortest distance between the plane $P: 2x+y-z=2$ and the origin?

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{\frac{3}{2}}$ (D) $\sqrt{\frac{1}{3}}$ (E) $\sqrt{\frac{2}{3}}$

Problem 116. Let ℓ be the line that is the intersection of the planes $x+2y+3z=1$ and $x-y-z=2$. An equation that contains the point $(1,1,1)$ and is perpendicular to the line l is
(A) $-x-4y+3z=-2$ (B) $-x-2y+4z=0$ (C) $-2x-4y+z=-5$
(D) $-4x-y+3z=-2$ (E) $-3x-4y+z=-6$

Problem 117. Let P be the plane containing the lines ℓ_1 and ℓ_2, given parametrically as $(3+2t, 1-t, 2t)$ and $(-t, 2+2t, t)$. What is an equation for this plane P?
(A) $5x+4y-3z=0$ (B) $5x+4y-3z=8$ (C) $2x-y+2z=0$
(D) $2x-y+2z=-2$ (E) $-x+2y+z=0$

Problem 118. Let P be the plane tangent to the surface $x^2+y^2-2z=1$ at the point $(1,1,\frac{1}{2})$. What is an equation for this plane P?
(A) $x+y-z=\frac{3}{2}$ (B) $x-y-z=-\frac{1}{2}$
(C) $x-y+z=\frac{1}{2}$ (D) $2x+2y=4$ (E) $2x+2y-z=\frac{7}{2}$

Problem 119. Let P be the plane $3x-4y+z=10$, and let q be the point on the plane P that is closest to the origin. Which of the following is describes the line joining q and the origin parametrically in parameter t?
(A) $(3t,4t,t)$ (B) $(3t,4t,t)+(0,0,10)$ (C) (t,t,t)
(D) $(3t,-4t,t)$ (E) $(3t,-4t,t)+(0,0,10)$

Problem 120. Let S_1 be the sphere $(x+1)^2+(y-1)^2+z^2=4$ and S_2 be the sphere $(x+2)^2+(y+1)^2+(z+1)^2=4$. What is an equation of the plane that contains the intersection $S_1\cap S_2$?
(A) $x-y+z=0$ (B) $x-2y+z=2$ (C) $x-2y+z=-1$
(D) $x+2y+z=-2$ (E) $x+2y+z=-1$

Problem 121. Consider in the xy-plane the parametric curve $x(t)=3+t^2, y(t)=t+t^3$. Find $\frac{d^2y}{dx^2}$ at the point $(4,2)$.
(A) 1 (B) $\frac{1}{2}$ (C) 2 (D) 4 (E) $\frac{1}{4}$

Problem 122. Consider in the xy-plane the parametric curve $x(t)=e^t, y(t)=te^{-t}$. For which values of t is the curve concave up?
(A) $t>0$ (B) $t>\frac{1}{3}$ (C) $t>\frac{1}{2}$ (D) $t>1$ (E) $t>\frac{3}{2}$

Problem 123. In the xy-plane, at how many distinct points on the parametric curve $x(t)=2t^3, y(t)=1+4t-t^2$ does the tangent line have slope 1?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 124. In the xy-plane, find the length of the parametric curve $x(t)=1+3t^2, y(t)=4+2t^3$ where $0\leqslant t\leqslant 1$.
(A) $2\sqrt{2}-1$ (B) $2\sqrt{2}-2$ (C) $4\sqrt{2}$ (D) $4\sqrt{2}-1$ (E) $4\sqrt{2}-2$

Problem 125. In the xy-plane, at how many distinct points on the parametric curve $x(t)=e^{\cos t}, y(t)=e^{\sin t}$ does the tangent line have slope 0?
(A) 0 (B) 1 (C) 2 (D) 3 (E) Infinitely many

Problem 126. What is the maximum value of $F(x,y)=x^2+y^2$ on the circle of radius 2 with center $(x,y)=(1,-1)$?
(A) $1+\sqrt{2}$ (B) $3+2\sqrt{2}$ (C) $6+4\sqrt{2}$
(D) $\sqrt{2}-1$ (E) $3-2\sqrt{2}$

Problem 127. Find all points at which the direction of fastest change of the function $f(x,y)=x^2+y^2-2x-4y$ is parallel to $(1,1)$.
(A) $(\frac{3}{2},\frac{5}{2})$ only (B) $(-\frac{3}{2},-\frac{5}{2})$ only (C) $(\frac{3}{2},\frac{5}{2})$ and $(-\frac{3}{2},-\frac{5}{2})$ only
(D) All points on the line $y=x+1$ (E) All points on the line $y=x-1$

Problem 128. Let $u,v:\mathbb{R}^2\to\mathbb{R}$ be differentiable real-valued functions on $\mathbb{R}^2$. Which of the following is true?
I. $\nabla(uv)=u\nabla v+v\nabla u$
II. $\nabla(u^n)=nu^{n-1}\nabla u$, for integers $n\geqslant 1$
III. $\nabla\left(\frac{u}{v}\right)=\frac{v\nabla u-u\nabla v}{v^2}$, wherever $v\neq 0$.
(A) I only (B) II only (C) I and II only
(D) I and III only (E) I, II, and III

Problem 129. If $f(x,y)$ is a real-valued function on $\mathbb{R}^2$ that is differentiable at $\vec{x}_0 = (x_0, y_0)$, then

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - \nabla f(x_0,y_0)\cdot((x,y)-(x_0,y_0))}{|(x,y)|} =$$

(A) 0 (B) 1 (C) -1 (D) ∞ (E) Undefined

Problem 130. What is the number of points on the surface $x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$?
(A) 0 (B) 1 (C) 2 (D) 3 (E) Infinitely many

Problem 131. What is the arclength of the parametric curve $x(t) = t\sin t, y(t) = t\cos t$, with $0 \leqslant t \leqslant 1$?
(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{2}}{2} + \log\sqrt{1+\sqrt{2}}$ (C) $\frac{\sqrt{2}}{2} + \sqrt{1+\sqrt{2}}$
(D) $\frac{\pi}{2} + \log\sqrt{1+\sqrt{2}}$ (E) $\log\sqrt{1+\sqrt{2}}$

Problem 132. Consider the parametric curve $x(t) = a\cos^3 t, y(t) = a\sin^3 t$, with $0 \leqslant t \leqslant \frac{\pi}{2}$. What is the area of the surface obtained by rotating this curve about the x-axis?
(A) $\frac{2}{5}\pi a^2$ (B) $\frac{3}{5}\pi a^2$ (C) $\frac{4}{5}\pi a^2$ (D) πa^2 (E) $\frac{6}{5}\pi a^2$

Problem 133. Find the area enclosed by the astroid curve $x(t) = a\cos^3 t, y(t) = a\sin^3 t$ in the plane.
(A) $\frac{7}{8}\pi a^2$ (B) $\frac{2}{3}\pi a^2$ (C) $\frac{5}{8}\pi a^2$ (D) $\frac{1}{2}\pi a^2$ (E) $\frac{3}{8}\pi a^2$

Problem 134. In polar form, consider the circle $r = 3\sin\theta$ and the cardiod $r = 1 + \sin\theta$. What is the area inside the circle and outside the cardiod?
(A) $\frac{1}{3}\pi$ (B) $\frac{1}{2}\pi$ (C) π (D) $\frac{3}{2}\pi$ (E) 2π

Problem 135. In polar form, the curve $r = \sin 2\theta$ has a graph that is a four-leaf clover in the plane. Find the total area it enclosed for all four-leaves.
(A) $\frac{\pi}{8}$ (B) $\frac{3\pi}{8}$ (C) $\frac{\pi}{2}$ (D) π (E) 2π

Problem 136. Compute $\int_C xy^2dx + 2x^2ydy$ where C is the triangle curve with vertices $(0,0),(2,2),(2,4)$ that is positively oriented.
(A) 10 (B) 12 (C) 14 (D) 16 (E) 20

Problem 137. Compute $\iint_S (2x+2y+z^2)dS$ where S is the outward oriented sphere $x^2+y^2+z^2=1$
(A) π (B) $\frac{4}{3}\pi$ (C) 4π (D) $\frac{16}{3}\pi$ (E) $\frac{20}{3}\pi$

Problem 138. Let C be the oriented curve of intersection of the plane $x+z=5$ and the cylinder $x^2+y^2=9$, with orientation counterclockwise when viewed from above, and the vector field $\boldsymbol{F}(x,y,z) = \langle xy, 2z, 3y\rangle$. Find the line integral $\int_C \boldsymbol{F}\cdot d\boldsymbol{r}$.
(A) π (B) 2π (C) 4π (D) 8π (E) 9π

Problem 139. Let E be the cone in xyz-space given by $z^2 \geqslant x^2+y^2, z \geqslant 0, z \leqslant 1$, and let f be a continuous real-valued function on E. Which of the following expression is equal to $\iiint_E fdV$?

I. $\displaystyle\int_{z=0}^{1}\int_{y=-z}^{z}\int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f dxdydz$

II. $\displaystyle\int_{y=-1}^{1}\int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{z=\sqrt{x^2+y^2}}^{1} f dzdxdy$

III. $\displaystyle\int_{x=0}^{1}\int_{z=x}^{1}\int_{y=-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f dydzdx$

(A) I only (B) II only (C) I and II only
(D) I and III only (E) I, II, and III

Problem 140. Let $\boldsymbol{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$ be a vector field defined on $\mathbb{R}^2 \backslash \{(0,0)\}$. Let C_0 be the unit circle $x^2+y^2=1$ with clockwise orientation, and C_2 be the unit circle $(x-2)^2+(y-2)^2=1$ with clockwise orientation. Which of the following statements is true?

(A) $\int_{C_0} \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{C_0} \boldsymbol{F}\cdot d\boldsymbol{r} = 0$

(B) $\int_{C_0} \boldsymbol{F}\cdot d\boldsymbol{r} > \int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{r}$ (C) $\int_{C_0} \boldsymbol{F}\cdot d\boldsymbol{r} < \int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{r}$

(D) $\int_{C_0} \boldsymbol{F}\cdot d\boldsymbol{r} + \int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{r} = 0$ (E) $\int_{C_0} \boldsymbol{F}\cdot d\boldsymbol{r} - \int_{C_2} \boldsymbol{F}\cdot d\boldsymbol{r} = 0$

3. Differential equations.

Problem 141. If y is a real-valued function defined on the real line and satisfying the initial value problem

$$y'(x) = xe^{x^2 - \ln y}$$
$$y(0) = 1,$$

then $y(1) =$

(A) e (B) e^2 (C) $\sqrt{e}$ (D) $-e$ (E) $-e^2$

Problem 142. If y is a real-valued function defined on the real line and satisfying the initial value problem

$$y'(x) + y(x) = \cos(e^x)$$
$$y(0) = 0,$$

then $\lim_{x\to\infty} y(x) =$

(A) 0 (B) 1 (C) $+\infty$ (D) $-\infty$ (E) Undefined

Problem 143. If y is a real-valued function defined on the positive real line and satisfying the initial value problem

$$xy' + y = \sqrt{x}$$
$$y(1) = 1$$

then $\lim_{x\to\infty} y(x) =$

(A) 0 (B) 1 (C) $+\infty$ (D) $-\infty$ (E) Undefined

Problem 144. If y is a real-valued function defined on the real line and satisfying the initial value problem

$$y'' - y' + 2y = 0 = 0$$
$$y(0) = 1$$
$$y'(0) = 0$$

then $\lim_{x\to-\infty} y(x) =$

(A) 0 (B) 1 (C) $+\infty$ (D) $-\infty$ (E) Undefined

Problem 145. Consider the differential equation given as

$$4e^x \sin y\, dx + 3xy\, dy = 0$$

where y is some function of x. Which of the following is an integrating factor such that the above equation is exact?

(A) 1 (B) $\frac{1}{x\sin y}$ (C) $x \sin y$ (D) $\frac{1}{e^x y}$ (E) $e^x y$

Problem 146. A tank initially (at time $t=0$) holds 10 liters of salt water, at a concentration of 0.5 kg/liter. A salt solution of concentration 0.2 kg/liter is added to the tank at a rate of 3 liter/min. The tank also drains at a rate of 3 liter/min. Assume mixing occurs instantly. Let $M(t)$ be the mass of the salt in the tank at time t minutes. What is the concentration in kg/liter of the salt in the tank at 10 min?

(A) $2e^{-10}+3$ (B) $2e^{-3}+3$ (C) $3e^{-3}+2$
(D) $0.3e^{-3}+0.2$ (E) $0.3e^{-10}+0.2$

Problem 147. What is a particular solution to the differential equation $y''+2y'-3y = e^x + e^{-x}$?

(A) $\frac{1}{4}e^x$ (B) $\frac{1}{4}e^{-x}$ (C) $\frac{1}{4}xe^x$
(D) $\frac{1}{4}xe^x + \frac{1}{4}e^{-x}$ (E) $\frac{1}{4}xe^x - \frac{1}{4}e^{-x}$

Problem 148. The Wronskian function $W_{y_1,y_2}(x)$ of two differentiable functions y_1, y_2 is defined to be

$$W_{y_1,y_2}(x) = \det\begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}$$

Which of the following statements is necessarily true?
I. If $y_1(x)$ and $y_2(x)$ are differentiable functions with Wronskian $W_{y_1,y_2}(x) = 0$ for all x in some open interval I, then y_1 and y_2 are linearly dependent on the interval I.
II. If $y_1(x)$ and $y_2(x)$ are differentiable functions that are linearly dependent on some open interval I, then their Wronskian $W_{y_1,y_2}(x) = 0$ for all $x \in I$.
III. If $y_1(x)$ and $y_2(x)$ are solutions to $y'' + p(x)y' + q(x)y = 0$ on the interval $(-1, 1)$ for some continuous functions p, q, then it is possible for their Wronskian to be $W_{y_1,y_2}(x) = xe^x$.
(A) I only (B) II only (C) III only
(D) I and II only (E) II and III only

Problem 149. Let y_1 and y_2 be two solutions to the differential equation $y' = x^2y + \sin(x)$. Which of the following statements is true?
I. $y_1 + y_2$ is a solution to the differential equation $y' = x^2y + \sin(x)$
II. cy_1 is a solution to the differential equation $y' = x^2y + \sin(x)$, for all $c \in \mathbb{R}$
III. $y_1 - y_2$ is a solution to the differential equation $y' = x^2y$
(A) I only (B) II only (C) III only
(D) I and II only (E) II and III only

Problem 150. Consider the matrix differential equation $y' = \begin{pmatrix} 6 & 10 \\ -5 & -9 \end{pmatrix} y$.
What is a fundamental solution set to this differential equation?
(A) $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ (B) $x^4\begin{pmatrix} 1 \\ -1 \end{pmatrix}, x^2\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ (C) $e^x\begin{pmatrix} 1 \\ -1 \end{pmatrix}, e^x\begin{pmatrix} 2 \\ -1 \end{pmatrix}$
(D) $e^{-4x}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, e^x\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ (E) $e^x\begin{pmatrix} 1 \\ -1 \end{pmatrix}, e^{-4x}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

4. Linear algebra.

Problem 151. Two subspaces V and W in $\mathbb{R}^{10}$ have the same dimension. Suppose their intersection $V \cap W$ has dimension 3. Which of the following CANNOT be the dimension of V?
(A) 3 (B) 4 (C) 5 (D) 6 (E) 7

Problem 152. Let W_1, W_2, W_3 be three distinct subspaces in $\mathbb{R}^{10}$, where each W_i has dimension 9. Let $W = W_1 \cap W_2 \cap W_3$ be their common intersection. Which of the following must be true?
(A) $\dim W \leqslant 3$ (B) $\dim W \leqslant 6$ (C) $\dim W \leqslant 7$
(D) $\dim W \geqslant 7$ (E) $\dim W \geqslant 8$

Problem 153. What is the dimension of the subspace spanned by the vectors $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} -8 \\ 6 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 7 \end{pmatrix}$ in $\mathbb{R}^3$?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 154. Let V be the set of all 5×5 real symmetric matrices. As a subspace of the usual vector space of all 5×5 real matrices, what is the dimension of V?
(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

Problem 155. Let V be the set of all real polynomials p satisfying all the following conditions: (i) $p(4) = 0$, (ii) $p(1) = p'(2)$, and (ii) $p'''(x) = 0$ for all $x \in \mathbb{R}$. As a subspace of the usual vector space of all real polynomials, what is the dimension of V?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 156. Consider the system of linear equations

$$\begin{aligned} a + 3b - c &= 0 \\ 2a - b + c &= 0 \\ a + 10b - 2c &= 0 \end{aligned}$$

with solutions of the form (a, b, c), where a, b, c are real. Which of the following statements is FALSE?
(A) The system is consistent.
(B) Every solution is scalar multiple of some single solution (a_0, b_0, c_0).
(C) The sum of any two solutions is a solution.
(D) $(0, 0, 0)$ is a solution.
(E) The system has infinitely many solutions.

Problem 157. Consider the system of linear equation

$$\begin{aligned} a+3b-c&=1\\ a-b+c&=0\\ a+7b-3c&=2 \end{aligned}$$

with solutions of the form (a,b,c), where a,b,c are real. Which of the following statements is FALSE?
(A) The system is consistent.
(B) $(\frac{1}{2}, 0, \frac{-1}{2})$ is a solution.
(C) The sum of any two solutions is a solution.
(D) The set of solutions form line in $\mathbb{R}^3$.
(E) The system has infinitely many solutions.

Problem 158. Consider the system of linear equation in real variables x,y,z and k some real number parameter:

$$\begin{aligned} x+y+z&=k\\ y+z&=k-1\\ 2x+y+z&=2 \end{aligned}$$

For which value of k is the system consistent?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 159. Consider the system of linear equation in real variables x,y,z and k some real number parameter:

$$\begin{aligned} x+y+z&=1\\ x+y&=1\\ y+kz&=1 \end{aligned}$$

For which value of k will the system have infinitely many solutions?
(A) Only $k=0$ (B) Only $k=1$ (C) Only for $k=-1$
(D) For all real number k (E) For no real number k

Problem 160. Consider the system of linear equations

$$\begin{aligned} x+y+z+w&=0\\ x-y-z-w&=0\\ x-y+z-w&=0 \end{aligned}$$

with solutions of the form (x,y,z,w), where x,y,z,w are real. Which of the following statements is TRUE?
(A) The set of solutions is a single point in $\mathbb{R}^4$.
(B) The set of solutions form a line in $\mathbb{R}^4$.
(C) The set of solutions form a plane in $\mathbb{R}^4$.
(D) The set of solutions is $\mathbb{R}$.
(E) The set of solutions is $\mathbb{R}^2$.

Problem 161. Of the numbers 0, 3, and 6, which are eigenvalues of the matrix $\begin{pmatrix}1&1&1\\2&2&2\\3&3&3\end{pmatrix}$?
(A) None (B) 0 and 3 only (C) 3 and 6 only
(D) 0 and 6 only (E) 0, 3, and 6

Problem 162. Consider the matrix $A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$. Which of the following statements is true?
(A) A is invertible
(B) 1 is an eigenvalue of A
(C) If $p(x)$ is the characteristic polynomial of A, then $p(0)\neq 0$
(D) All eigenvalues of A are rational numbers
(E) A has three distinct eigenvalues

Problem 163. Consider the matrix $\begin{pmatrix}2&4&3&2\\1&5&3&2\\1&1&7&2\\1&1&8&1\end{pmatrix}$. Which of the following is NOT an eigenvalue of this matrix?
(A) 1 (B) -1 (C) 4 (D) -4 (E) 11

Problem 164. Let A be a 2×2 real matrix. If $A\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}2\\6\end{pmatrix}$ and the trace of A is 7, what is the determinant of A?
(A) 0 (B) 4 (C) 6 (D) 8 (E) 10

Problem 165. Let $A = \begin{pmatrix} 3 & 2 \\ 1 & k \end{pmatrix}$ be a real matrix, where k is some real number. For which real number k will the matrix A have two distinct real eigenvalues?
(A) Only for $k = 0$
(B) Only for $k < 6$
(C) Only for $k > 6$
(D) For all real number k
(E) For no real number k

Problem 166. Consider the matrix $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$. Which of the following is NOT true?
(A) A is not invertible
(B) $A^n = 2^{n-1}A$ for each positive integer n
(C) A has three distinct eigenvalues
(D) If p is the characteristic polynomial of A, then $p'(0) = 0$
(E) All eigenvalues of A are real

Problem 167. Consider the matrix $\begin{pmatrix} 1 & 2 & 1 \\ k & 2 & 2 \\ 1 & 0 & k \end{pmatrix}$ for some real number k. For which k is A invertible?
(A) For all real number k
(B) For no real number k
(C) For $k = \dfrac{1+\sqrt{5}}{2}$ and $\dfrac{1-\sqrt{5}}{2}$
(D) For all $k \neq \dfrac{1 \pm \sqrt{5}}{2}$
(E) For $k = 1 + \sqrt{5}$ and $1 - \sqrt{5}$

Problem 168. Consider the 6×6 matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & & \\ 1 & 1 & 1 & & & \\ 1 & 1 & & & & \\ 1 & & & & & \end{pmatrix}$. Which of the following statements is NOT true?
(A) $\det A = 1$
(B) Every eigenvalue of A is real
(C) The first row of A^2 is $\begin{bmatrix} 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}$
(D) $\ker A = \left\{\vec{0}\right\}$, where $\vec{0}$ is the zero vector of $\mathbb{R}^6$.
(E) For every $\vec{y} \in \mathbb{R}^6$, there exists a unique $\vec{x} \in \mathbb{R}^6$ such that $A\vec{x} = \vec{y}$.

Problem 169. Among the numbers $0, 1, 2$, which ones are eigenvalues of the 4×4 matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$?
(A) 0 only (B) 1 only (C) 0 and 1 only
(D) 0 and 2 only (E) 0, 1, and 2 are all eigenvalues of A

Problem 170. Let A be a 3×3 real matrix where some of its eigenvalues are 1 and 2. If A has trace 3, which of the following statements must be true?
(A) $\det A \neq 0$
(B) A is symmetric
(C) A is diagonalizable
(D) A is invertible
(E) A is an orthogonal projection

Problem 171. Let A be an $n \times n$ real matrix satisfying $A^2 + A = I$. Which of the following is necessarily true?
I. A is invertible
II. A is diagonalizable
III. A has only irrational eigenvalues.
(A) I only (B) II only (C) III only
(D) I and II only (E) I, II, and III

Problem 172. Let A be an $n \times n$ real matrix satisfying $A^2 = 2A$. Which of the following is necessarily true?
I. A is invertible
II. A is diagonalizable
III. 0 is an eigenvalue of A
(A) I only (B) II only (C) III only
(D) I and II only (E) I, II, and III

Problem 173. If A is an $n \times n$ real matrix such that A is diagonalizable and $A^2 + 2A = -I$, which of the following is necessarily true?
I. If $v_1, \dots, v_k$ are linearly independent vectors in $\mathbb{R}^n$, then so is $Av_1, \dots, Av_k$
II. A is the $n \times n$ identity matrix
III. A is symmetric
(A) I only (B) II only (C) III only
(D) I and III only (E) I, II, and III

Problem 174. Let A be a 4×4 real matrix where $A + 2I$ has rank 1. Which of the following statements is necessarily true?
I. A is invertible
II. If A is not invertible, then A is diagonalizable
III. If A is diagonalizable, then A is not invertible
(A) I only (B) II only (C) III only
(D) I and II only (E) I, II, and III

Problem 175. Let A be an $n \times n$ real matrix. Which of the following statements is necessarily true?
(A) If A is invertible then A is diagonalizable
(B) If A is diagonalizable then A is invertible
(C) If A is not invertible then A is diagonalizable
(D) If A is not diagonalizable then A is invertible
(E) None of the above

Problem 176. Which of the following is the result of perform Gram-Schmidt to obtain an orthonormal set from the ordered list of vectors

$$\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}?$$

(A) $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ (B) $\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}2\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}$
(C) $\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}2\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}$ (D) $\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\-1\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}$
(E) $\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\1\end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\-1\end{pmatrix}$

Problem 177. What is the orthogonal projection of the vector $\begin{pmatrix}1\\1\\1\end{pmatrix}$ onto the linear subspace spanned by the vectors $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$?

(A) $\begin{pmatrix}1\\1\\1\end{pmatrix}$ (B) $\begin{pmatrix}1\\1\\0\end{pmatrix}$ (C) $\begin{pmatrix}4/3\\2/3\\2/3\end{pmatrix}$ (D) $\begin{pmatrix}2/\sqrt{3}\\2/\sqrt{3}\\\sqrt{2/3}\end{pmatrix}$ (E) $\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\\sqrt{2/3}\end{pmatrix}$

Problem 178. Let A be any $n \times k$ real matrix (not necessarily square). Which of the following statements is true?
I. AA^T is diagonalizable
II. A^TA has only non-negative real eigenvalues
III. A^TA is an orthogonal transformation
(A) I only (B) II only (C) III only
(D) I and II only (E) I and III only

Problem 179. Find real ordered pair (a, b) such that the line $y = ax + b$ best fits the points $(x_1, y_1) = (1, 0)$, $(x_2, y_2) = (1, 1)$, $(x_3, y_3) = (2, 1)$. Namely the quantity $\sum_{i=1}^{3} |y(x_i) - y_i|^2$ is minimized.

(A) $(0, 0)$ (B) $(1, 0)$ (C) $(\frac{1}{2}, 0)$ (D) $(0, \frac{1}{2})$ (E) $(1, 1)$

Problem 180. Let A be a 4×6 real matrix with rank 3. What is $\dim \ker A^T - \dim \ker A$?
(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

Problem 181. Let A and B both be $n \times n$ matrices over $\mathbb{C}$. Suppose A is invertible. Which of the following is NOT necessarily true?
I. $rank(B) = rank(AB)$
II. $nullity(B) = nullity(AB)$
III. $image(B) = image(AB)$
IV. $kernel(B) = kernel(AB)$.
(A) I only (B) II only (C) III only
(D) IV only (E) More than one statements are not necessarily true

Problem 182. Let A and B both be $n \times n$ matrices over $\mathbb{C}$. Suppose B is invertible. Which of the following is NOT necessarily true?
I. $rank(A) = rank(AB)$
II. $nullity(A) = nullity(AB)$
III. $image(A) = image(AB)$
IV. $kernel(A) = kernel(AB)$.
(A) I only (B) II only (C) III only
(D) IV only (E) More than one statements are not necessarily true

Problem 183. Let A be a 3×3 real matrix with eigenvalues $-2, 1, 2$. Among the following five matrices, how many of them are invertible:

$$A\ ,\ A^2\ ,\ A - I\ ,\ A + I\ ,\ A^T$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 184. Let A be a 3×3 real matrix. Which of the following does NOT necessarily imply A is invertible?
(A) $\ker(A + I)$ has dimension 2 and $trace(A) = 4$
(B) For all nonzero vectors $v \in \mathbb{R}^3 \setminus \{0\}$, $Av \neq 0$
(C) For any two vectors $v, w \in \mathbb{R}^3$ whenever $v \cdot w = 0$, we have $Av \cdot Aw = 0$
(D) $A^3 = I$
(E) A has distinct eigenvalues

Problem 185. If A is a 4×6 matrix, and B is a 7×4 matrix, what is the smallest possible dimension for the kernel of the matrix BA?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

5. Geometry.

Problem 186. What is the area of the circle inscribed in an equilateral triangle with side length 5?
(A) $\frac{5\pi}{12}$ (B) $\frac{25\pi}{12}$ (C) $\frac{5\pi}{4}$ (D) $\frac{25\pi}{4}$ (E) $\frac{25\pi}{16}$

Problem 187. A circle is inscribed in a regular n-gon, where $n \geqslant 3$. If the inscribed circle has radius r, what is the area of this regular n-gon?
(A) $nr^2 \tan \frac{\pi}{n}$ (B) $2nr^2 \tan \frac{\pi}{n}$ (C) $\frac{1}{2} nr^2 \tan \frac{\pi}{n}$
(D) $nr^2 \sin \frac{\pi}{n}$ (E) $2nr^2 \sin \frac{\pi}{n}$

Problem 188. What is the area of the common region of unit circles with centers at $(0,0), (0,1), (1,0), (1,1)$?
(A) $\frac{\pi}{3}\sqrt{3}$ (B) $\sqrt{3} - \frac{\pi}{3}$ (C) $1 + \sqrt{3} - \frac{\pi}{3}$
(D) $\frac{\pi}{3} + \sqrt{3}$ (E) $1 + \frac{\pi}{3} - \sqrt{3}$

Problem 189. A rectangle R is circumscribed by a circle with area 10. If R has area 5, what is the length of the shorter side of R?
(A) $\sqrt{\frac{40}{\pi} + 10} + \sqrt{\frac{40}{\pi} - 10}$ (B) $\sqrt{\frac{40}{\pi} + 10} - \sqrt{\frac{40}{\pi} - 10}$
(C) $\frac{\sqrt{\frac{40}{\pi} + 10} - \sqrt{\frac{40}{\pi} - 10}}{2}$ (D) $\frac{\sqrt{\frac{40}{\pi} + 10} + \sqrt{\frac{40}{\pi} - 10}}{2}$
(E) $\frac{\sqrt{\frac{40}{\pi} + 10} - \sqrt{\frac{40}{\pi} - 10}}{2}$

Problem 190. What is the area of the largest possible square inscribed in an equilateral triangle with side lengths 1, such that one of the sides of the square lie entirely on one of the sides of the equilateral triangle?
(A) $\frac{\sqrt{3}}{2 + \sqrt{3}}$ (B) $\frac{2 + \sqrt{3}}{\sqrt{3}}$ (C) $\frac{\sqrt{3}}{1 + \sqrt{3}}$
(D) $\frac{1 + \sqrt{3}}{\sqrt{3}}$ (E) $\frac{1}{2 + \sqrt{3}}$

Problem 191. Consider the equation

$$x^2 + (y+1)^2 = (x+2)^2 + (y+3)^2$$

In the xy-plane the set of points whose coordinates satisfy the equation above is
(A) a line (B) a circle (C) an ellipse
(D) a parabola (E) a hyperbola

Problem 192. Consider the equation

$$\sqrt{x^2 + (y+1)^2} + \sqrt{(x+2)^2 + (y+3)^2} = 10$$

In the xy-plane the set of points whose coordinates satisfy the equation above is
(A) a line (B) a pair of lines (C) an ellipse
(D) a parabola (E) a hyperbola

Problem 193. Consider the equation

$$x^2 - y + xy = 9$$

In the xy-plane the set of points whose coordinates satisfy the equation above is
(A) a line (B) a pair of lines (C) an ellipse
(D) a parabola (E) a hyperbola

Problem 194. Consider the equation

$$x^2 - y^2 + xy = 9$$

In the xy-plane the set of points whose coordinates satisfy the equation above is
(A) a line (B) a pair of lines (C) an ellipse
(D) a parabola (E) a hyperbola

Problem 195. Consider the equation

$$x^2 + y^2 + 2xy = 9$$

In the xy-plane the set of points whose coordinates satisfy the equation above is
(A) a line (B) a pair of lines (C) an ellipse
(D) a parabola (E) a hyperbola

Problem 196. Let P be a convex n-gon where $n \geqslant 4$, let $a(P)$ denote the number of acute angles P has. What is $\max a(P)$, where the maximum is taken over all possible convex n-gon P?
(A) 4 (B) 5 (C) n (D) $n+1$ (E) $n+2$

Problem 197. Let H be a convex hexagon where opposite sides are pairwise parallel. Which of the following are necessarily true?
I. There exists a diagonal that divides H into two equal area halves.
II. There can be at most two acute angles in H.
III. There can be at most three distinct different side lengths for H.
(A) I only (B) I and II only (C) I and III only
(D) I, II, and III (E) II and III only

Problem 198. Which of the following statements is true?
I. There exists a triangle ABC that does not have a circumscribing circle.
II. There exists a quadrilateral $ABCD$ that does not have a circumscribing circle.
III. If a convex quadrilateral $ABCD$ has a circumscribing circle, then its opposite angles are supplementary, namely $\angle A + \angle C = \angle B + \angle D = \pi$.
(A) I only (B) II only (C) I and II only
(D) II and III only (E) I, II, and III

Problem 199. Which of the following statements is true?
I. Two distinct circles can intersect at most at 2 points.
II. Two distinct regular n-gons can intersect at most at 2 points, where $n \geqslant 3$
III. A regular n-gon and a circle can intersect at most at n points, where $n \geqslant 3$
(A) I only (B) II only (C) I and II only
(D) II and III only (E) I, II, and III

Problem 200. A convex 10-gon has five of its interior angles no greater than $140°$. Which of the following must be true?
(A) One of the angles must be at least $160°$
(B) One of the angles must be at least $155°$
(C) One of the angles must be at least $150°$
(D) One of the angles must be at least $145°$
(E) One of the angles must be acute

6. Combinatorics.

Problem 201. If a tree has 12 vertices, how many edges does it have?
(A) 10 (B) 11 (C) 12 (D) 13 (E) 14

Problem 202. A tree is a connected graph with no cycles. How many nonisomorphic trees with 6 vertices exist?
(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Problem 203. How many trees of 7 vertices are there such that the degrees of each vertex in increasing order are

$$1, 1, 1, 1, 2, 2, 2$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 204. A labeled tree is a tree where each vertex is labeled with a distinct label. How many nonisomorphic labeled trees with 4 vertices exist?
(A) 8 (B) 10 (C) 16 (D) 20 (E) 28

Problem 205. Denote K_6 to be the complete graph of 6 vertices, where each vertex is connected to every other vertex. How many edges do you need to remove from K_6 to form a tree?
(A) 5 (B) 36 (C) 125 (D) 515 (E) 715

Problem 206. Consider the following algorithm, which takes an input integer $n \geqslant 1$ and prints one or more integers.

```
input(n)
set i = 2
while i < n
  replace i by i*i
  print(k-i)
end
```

If the input integer is 20, what integers will be printed and in what order?
(A) $16, 4$ (B) $18, 16, 4$ (C) $18, 14$ (D) $16, 0$ (E) $20, 18, 14$

Problem 207. Consider the following algorithm, which takes an input integer $n \geqslant 1$ and prints one or more integers.

```
input(n)
set i = 0
set k = n
while i < k
  replace i by i+1
  replace k by k-2
end
print(i)
```

If the input integer is 20, what integer will be printed?
(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

Problem 208. Consider the following algorithm, which takes an input integer $n \geqslant 1$ and prints one or more integers.

```
input(n)
set i = 0
set k = n
while i < k
  replace i by i+2
  replace k by k+1
end
print(i)
```

If the input integer is 20, what integer will be printed?
(A) 36 (B) 38 (C) 40 (D) 42 (E) 44

Problem 209. Consider the following algorithm, which takes an input integer $n \geqslant 1$ and prints one or more integers.

```
input(n)
set i = 0
set k = n
while i*i < k
  replace i by i+1
  replace k by k-1
  print(k)
end
print(i)
```

If the input integer is 20, what integers will be printed and in what order?
(A) 19, 18, 17, 16 (B) 19, 18, 17, 16, 4
(C) 16, 4 (D) 4 (E) 16

Problem 210. Consider the following algorithm, which takes an input integer $n \geqslant 1$ and prints one or more integers.

```
input(n)
set i = 2
set k = n
while i < k
  replace i by i+1
    while k > i*i
      replace k by k-i
      print(k)
    end
end
print(i)
```

If the input integer is 10, what integers will be printed and in what order?
(A) 9 (B) 9, 9 (C) 9, 8, 7, 7 (D) 3, 3 (E) 3, 3, 3, 9

Problem 211. Nine different toys are to distributed to 3 children, Alice, Belle, and Cindy. Alice gets 4 of them, Belle gets 3, and Cindy gets 2. How many different ways can you distribute the toys?
(A) $9! - 4!3!2!$ (B) $\dfrac{9!}{4!3!2!}$ (C) $4!3!2!$
(D) $9! - (4)(3)(2)$ (E) $\dfrac{9!}{(4)(3)(2)}$

Problem 212. How many positive integer quadruples (x_1, x_2, x_3, x_4) are there satisfying the equation

$$x_1 + x_2 + x_3 + x_4 = 20$$

(A) $\binom{20}{3}$ (B) $\binom{19}{3}$ (C) $\binom{20}{4}$
(D) $\binom{23}{3}$ (E) $\binom{24}{4}$

Problem 213. How many positive integer quadruples (x_1, x_2, x_3, x_4) are there satisfying the equation

$$x_1 + x_2 + x_3 + x_4 = 20$$

and the constraints $x_1 \geqslant 2$ and $x_2 \leqslant 4$?
(A) $\binom{18}{3}$ (B) $\binom{23}{3}$ (C) $\binom{18}{3} - \binom{14}{3}$
(D) $\binom{18}{3} - \binom{16}{3}$ (E) $\binom{20}{3} - \binom{18}{3}$

Problem 214. Two points $A = (1, 3)$ and $B = (7, 10)$ lies on the integer lattice $\mathbb{Z}^2$. How many paths can you draw from A to B using only vector steps $(1, 0)$ or $(0, 1)$? (Only north and east steps)
(A) $\binom{7}{2} + \binom{6}{2}$ (B) $\binom{7}{2}\binom{6}{2}$ (C) $\frac{1}{2}\binom{13}{6}$
(D) $\binom{13}{6}$ (E) $\binom{14}{7}$

Problem 215. Two points $A = (1, 3)$ and $B = (7, 10)$ lies on the integer lattice $\mathbb{Z}^2$. How many paths can you draw from A to B using only vector steps $(1, 0)$ or $(0, 1)$, while avoiding the point $(4, 5)$? (Only north and east steps)
(A) $\frac{3}{13}\binom{13}{6}$ (B) $\binom{13}{6} - \binom{8}{3}$ (C) $\binom{13}{6} - \binom{5}{3}$
(D) $\binom{13}{6} - \binom{5}{3} - \binom{8}{3}$ (E) $\binom{13}{6} - \binom{5}{3}\binom{8}{3}$

7. Probability.

Problem 216. Toss a fair two-sided coin 5 times in a row. What is the probability that we see exactly 2 heads total?
(A) 1/2 (B) 1/4 (C) 5/16 (D) 5/32 (E) 13/32

Problem 217. What is the probability that choosing a positive integer not exceeding 100 at random will NOT be divisible by 5 or 7?
(A) 17/25 (B) 16/25 (C) 8/25 (D) 17/50 (E) 33/50

Problem 218. Toss a fair two-sided coin five times in a row. What is the probability that we DO NOT see two heads consecutively among the five tosses?
(A) 1/4 (B) 3/8 (C) 7/16 (D) 13/32 (E) 1/2

Problem 219. Two players each choose one of the 20 integer at random from 1 to 20. What is the probability that the difference between the two numbers is NO GREATER than 5?
(A) 17/40 (B) 18/40 (C) 19/40 (D) 20/40 (E) 21/40

Problem 220. Two players each choose one of the 10 integers at random from 1 to 10. What is the probability that the two number chosen are relatively prime?
(A) 0.66 (B) 0.63 (C) 0.60 (D) 0.57 (E) 0.50

Problem 221. If a real number x is chosen at random uniformly in the interval $[0, 10]$ and a real number y chosen at random uniformly in the interval $[5, 15]$, what is the probability that $x < y$?
(A) $\frac{1}{2}$ (B) $\frac{5}{8}$ (C) $\frac{3}{4}$ (D) $\frac{7}{8}$ (E) $\frac{15}{16}$

Problem 222. If a real number x is chosen at random uniformly in the interval $[0, 10]$ and a real number y chosen at random uniformly in the interval $[5, 15]$, what is the probability that $|x - y| \leqslant 2$?
(A) $\frac{1}{3}$ (B) $\frac{1}{4}$ (C) $\frac{2}{5}$ (D) $\frac{1}{5}$ (E) $\frac{2}{7}$

Problem 223. Toss a fair two-sided coin 10 times in a row. Given that there are 5 heads and 5 tails, what is the probability no two heads showed up consecutively?
(A) $\frac{1}{2^{10}}$ (B) $\frac{1}{\binom{10}{5}}$ (C) $\frac{6}{2^{10}}$ (D) $\frac{6}{\binom{10}{5}}$ (E) $\frac{6}{\binom{10}{5}}\frac{1}{2^{10}}$

Problem 224. Seven people each chooses an integer independently from 1 to 7 uniformly at random. What is the probability that no two person chose the same number?
(A) $\frac{7}{7^7}$ (B) $\frac{1}{7}$ (C) $\frac{7!}{7^7}$ (D) $\frac{1}{7^7}$ (E) $\frac{1}{7!}$

Problem 225. Seven people each chooses an integer from 1 to 7 uniformly at random. What is the expected number of distinct numbers chosen?
(A) $7\left(\frac{6}{7}\right)^7$ (B) $7\left(\frac{28}{7^7}\right)$ (C) $7\left(1 - \left(\frac{6}{7}\right)^7\right)$
(D) $\frac{7}{2}$ (E) 5

8. Real analysis and basic topology.

Problem 226. Let S be a nonempty subset of $\mathbb{R}$. Which of the following statements is NOT necessarily true?
(A) $\sup S$ exists and is a finite real number
(B) There exists a continuous function $f : S \to \mathbb{R}$.
(C) There exists a continuous function $g : \mathbb{R} \to S$.
(D) There exists a closed set containing S.
(E) There exists an open set containing S.

Problem 227. Let $\{A_i\}_{i=1}^{\infty}$ be a collection of open sets and $\{B_i\}_{i=1}^{\infty}$ be a collection of closed set in $\mathbb{R}$. Which of the following statements must be true?
I. $\bigcup_{i=1}^{\infty} A_i$ is open.
II. $\bigcup_{i=1}^{\infty} B_i$ is closed.
III. $\bigcap_{i=1}^{\infty} (B_i \backslash A_i)$ is closed.
(A) I only (B) II only (C) III only
(D) I and III only (E) II and III only

Problem 228. Let A denote some subset of $\mathbb{R}$. Which of the following statements must be true?
I. If A is infinite and bounded, then there exists a point $a \in A$, such that every open set U with $a \in U$, we have $U \cap A \neq \{a\}$.
II. If A is a subset where for each point $a \in A$ there exists an open set U such that $U \cap A = \{a\}$, then A is at most countable.
III. If A is at most countable, then for each point $a \in A$, there exists an open set U such that $U \cap A = \{a\}$.
(A) I only (B) II only (C) III only
(D) I and II only (E) II and III only

Problem 229. Which of the following sets is open in $\mathbb{R}$?
I. $\bigcup_{i=1}^{\infty}(i, i+\frac{1}{2})$
II. $\bigcup_{i=1}^{\infty}[\frac{1}{i}, 3-\frac{1}{i}]$
III. $\{x \in \mathbb{R} : f(x) \neq 0\}$, where $f : \mathbb{R} \to \mathbb{R}$ is some continuous function.
(A) I only (B) II only (C) III only
(D) I and II only (E) I, II, and III

Problem 230. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Which of the following statements is true?
I. If $\{a_n\}$ is a sequence converging to a in $\mathbb{R}$, then $\{f(a_n)\}$ is a sequence converging to $f(a)$ in $\mathbb{R}$.
II. The image of f is open.
III. The image of f is connected.
(A) I only (B) II only (C) III only
(D) I and II only (E) I and III only

Problem 231. Let (X, d) be a metric space, and define a function $f : X \times X \to \mathbb{R}$ by $f(x, y) = k \cdot d(x, y)$ where k is some real number. For which values of k is f also a metric on X?
(A) $k = 1$ only (B) All $k \neq 0$ (C) All $k \geqslant 0$
(D) All $k > 0$ (E) All real values k

Problem 232. Let (X, d) be a metric space, and define a function $f : X \times X \to \mathbb{R}$ by $f(x, y) = k + d(x, y)$ where k is some real number. For which values of k is f also a metric on X?
(A) $k = 0$ only (B) All $k \neq 0$ (C) All $k \geqslant 0$
(D) All $k > 0$ (E) All real values k

Problem 233. Which of the following functions defines a metric on the real line $\mathbb{R}$?
I. $d_1(x, y) = |x - y|$, where $|\cdot|$ is the usual absolute value function.
II. $d_2(x, y) = (x - y)^2$
III. $d_3(x, y) = \sqrt{|x - y|}$
(A) I only (B) I and II only (C) I and III only
(D) I, II, and III (E) None of the above

Problem 234. Let X be the set of all binary ordered triples (x_1, x_2, x_3) where each $x_i \in \{0, 1\}$. Consider a metric defined on X as follows:

$$d((x_1, x_2, x_3), (y_1, y_2, y_3)) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|$$

Which of the following statements is true regarding the metric space (X, d)?
I. Every subset of X is open.
II. Every subset of X is closed.
III. Any convergent sequence is eventually constant.
(A) I only (B) II only (C) III only
(D) I and III only (E) I, II, and III

Problem 235. Let (X, d) be a metric with the property that every real-valued function $f : X \to \mathbb{R}$ is continuous. Which of the following statements must be true?
I. Every subset of X is open.
II. Every subset of X is closed.
III. Every sequence in X is convergent.
(A) I only (B) I and II only (C) I and III only
(D) I, II, and III (E) None of them above

Problem 236. Consider the real-valued function

$$f(x) = \begin{cases} 5x^3 & x \in \mathbb{Q} \\ kx^2 & x \notin \mathbb{Q} \end{cases}$$

where k is some real number. For how many values of k will f be differentiable at exactly 2 points?
(A) 0 (B) 1 (C) 2 (D) 3 (E) Infinitely many

Problem 237. Consider the real-valued function
$$f(x) = \begin{cases} kx+1 & x \in \mathbb{Q} \\ \dfrac{1}{x} & x \notin \mathbb{Q} \end{cases}$$
where k is some real number. Which of the following statements is true?
I. There exists a unique real number k such that f is continuous at exactly one point and differentiable nowhere.
II. There exists a unique real number k such that f is continuous at exactly one point and differentiable at exactly one point.
III. There exists some real number k such that f is continuous at exactly two points and differentiable at exactly one point.
(A) I only (B) II only (C) III only
(D) I and II (E) I, II, and III

Problem 238. Consider the real-valued function
$$f(x) = \begin{cases} x^n \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$
for some integer n. For which integer values of n is f differentiable at 0?
(A) All integers n (B) $n \geqslant 0$ only (C) $n \geqslant 1$ only
(D) $n \geqslant 2$ only (E) $n \geqslant 3$ only

Problem 239. Consider the real-valued function
$$f(x) = \begin{cases} x^m \sin \dfrac{1}{x^n} & x \neq 0 \\ 0 & x = 0 \end{cases}$$
for some positive integer n and positive integer m. Which of the following statements is true?
I. At $x = 0$, f' is continuous whenever $m > n$
II. At $x = 0$, f' exists whenever $m \geqslant 2$
III. At $x = 0$, f' does not exist whenever $m < n$
(A) I only (B) II only (C) III only
(D) I and II only (E) I and III only

Problem 240. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $a \in \mathbb{R}$. Then $\displaystyle\lim_{x\to a} \frac{xf(a) - af(x)}{x-a} =$
(A) $f(a)$ (B) $f'(a)$ (C) $f(a) - f'(a)$
(D) $f(a) - af'(a)$ (E) $af(a) - f'(a)$

Problem 241. Which of the following statements is true?
I. $\displaystyle\sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \leqslant 1$ for all positive integer n
II. $e^x \geqslant x^5$ for all real number x
III. $\sin(x) \leqslant x$ for all positive x
(A) I only (B) II only (C) III only
(D) I and III only (E) I, II and III

Problem 242. Which of the following statements is true?
I. There exists constant C such that $n^5 \leqslant Ce^n$ for all positive integer n
II. $\displaystyle\sum_{k=1}^{n} \frac{1}{k} \leqslant \log n$ for all positive integer n
III. $(1+x)^n \geqslant 1+nx$ for all non-negative integer n and non-negative real number $x \geqslant 0$
(A) I only (B) II only (C) III only
(D) I and II only (E) I and III only

Problem 243. Which of the following statements is true?
I. There exists a constant C such that $\log n! - n\log n - n \leqslant C \log n$ for all positive integer n
II. There exists a constant C such that $\displaystyle\sum_{k=1}^{n} k^3 \leqslant Cn^4$
III. There exists a constant C such that $|x^2 - \cos x| \leqslant Cx^4$ for all real x
(A) I only (B) II only (C) III only
(D) I and II only (E) I, II and III

Problem 244. Which of the following statements is true?
I. There exists constant C such that $2^n \leqslant Cn!$
II. There exists constant C such that $\log(n!) \leqslant Cn \log n$
III. There exists constant C such that $n \log n \leqslant C \log(n!)$
(A) I only (B) II only (C) III only
(D) I and III only (E) I, II and III

Problem 245. Let $f : [0,1] \to \mathbb{R}^3$ be continuous. Suppose $f(0) = (1,0,1)$, $f(\frac{1}{3}) = (1,0,2)$, $f(\frac{1}{2}) = (0,0,0)$, $f(1) = (2,2,0)$. At least how many $t \in [0,1]$ such that $|f(t)| = 1$?
(A) No such t (B) At least 2 (C) At least 3
(D) At least 4 (E) Infinitely many

Problem 246. Suppose $f : X \to \mathbb{R}$ is uniformly continuous on some metric space (X, d). Which of the following statements must be true?
I. f is continuous on X.
II. If $\{a_n\}$ is a Cauchy sequence in X, then $\{f(a_n)\}$ is Cauchy in $\mathbb{R}$.
III. There exists some $\delta > 0$ such that for any $x_1, x_2 \in X$ where $d(x_1, x_2) < \delta$, we have $|f(x_1) - f(x_2)| < 1$.
(A) I only (B) II only (C) III only
(D) I and III only (E) I, II, and III

Problem 247. Let (X, d) be some metric space, and let $\{a_n\}, \{b_n\}$ be two Cauchy sequences in X. Which of the following statements must be true?
I. $\{a_n\}$ converges in X.
II. $\{d(a_n, b_n)\}$ converges in $\mathbb{R}$.
III. If $f : X \to \mathbb{R}$ is continuous, then $\{f(a_n)\}$ is Cauchy in $\mathbb{R}$.
(A) I only (B) II only (C) III only
(D) I and II only (E) I, II, and III

Problem 248. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and let $\{a_n\}$ be a Cauchy sequence in $\mathbb{R}$. Which of the following statements is true?
I. $\{f(a_n)\}$ is Cauchy.
II. If $\{f(x_n)\}$ is Cauchy whenever $\{x_n\}$ is Cauchy in $\mathbb{R}$, then f is uniformly continuous on $\mathbb{R}$.
III. The sequence $\{a_n - f(a_n)\}$ is Cauchy.
(A) I only (B) II only (C) III only
(D) I and III only (E) I, II, and III

Problem 249. Which of the following real-valued functions is uniformly continuous on the interval indicated?
I. $f(x) = x^2$ on $(0, 1)$
II. $f(x) = x^2$ on $(1, \infty)$
III. $g(x) = \dfrac{1}{x}$ on $(0, 1)$
IV. $h(x) = \dfrac{1}{x}$ on $(1, \infty)$
(A) I and II only (B) III and IV only (C) I and III only
(D) II and IV only (E) I and IV only

Problem 250. Let $f : I \to \mathbb{R}$ be a differentiable function on some open interval I. Which of the following statements is true?
I. If f' is bounded on I, then f is uniformly continuous on I.
II. If f is uniformly continuous on I, then f' is bounded on I.
III. If I is a bounded interval and f is uniformly continuous on I, then f is bounded.
(A) I and II only (B) II and III only (C) I and III only
(D) I, II, and III (E) None of them is true

Problem 251. Let f be a real-valued function defined on the real line that has the following property:
For all $\epsilon > 0$, there exists an x such that $|x - 1| < 1$ and $|f(x)| > \epsilon$.
Which of the following statements is necessarily true?
(A) f is continuous at $x = 1$
(B) f is not continuous at $x = 1$
(C) If $\lim x_n = 1$, then $\lim |f(x_n)| = \infty$
(D) The Riemann integral $\displaystyle\int_0^2 f(x)dx$ does not exist or is infinite.
(E) There exists a convergent sequence $\{x_n\}$ such that $\lim |f(x_n)| = \infty$

Problem 252. Let f be a real-valued function defined on the real line that has the following property:
For all $\epsilon > 0$, whenever $|x - y| < \epsilon$, then $|f(x) - f(y)| < \epsilon$.
Which of the following statements is necessarily true?
(A) f is bounded
(B) f has graph that is a line
(C) f is differentiable on $\mathbb{R}$
(D) f is uniformly continuous on $\mathbb{R}$
(E) f is unbounded

Problem 253. Let f be a real-valued function defined on the real line that has the following property:
For all $\epsilon > 0$, whenever $|x - y| > \epsilon$, then $|f(x) - f(y)| < 1$.
Which of the following is necessarily true?
(A) f is a constant function
(B) f is uniformly continuous on $\mathbb{R}$
(C) If f is differentiable, then f' is bounded
(D) There exists a sequence of strictly increasing integers $\{a_n\}$ such that $\{f(a_n)\}$ converges.
(E) For every convergent sequence $\{x_n\}$, the sequence $\{f(x_n)\}$ is Cauchy.

Problem 254. Which of the following functions are uniformly continuous on $\mathbb{R}$?
I. $f(x) = x \sin x$
II. $f(x) = x + \sin(x^{1/3})$
III. $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$
(A) I only (B) II only (C) III only
(D) I, II, and III (E) None of them

Problem 255. Let f be a real-valued function, which of the following statements must be true?
I. If f is uniformly continuous on open interval (a, b), then there exists a function g that is uniformly continuous on $[a, b]$ with $g|_{(a,b)} = f$
II. If f is uniformly continuous on a set S, then f is uniformly continuous on any subset $T \subset S$
III. If f is uniformly continuous on $(1, 5)$, and f is also uniformly continuous on $(4, 10)$, then f is uniformly continuous on $(1, 10)$
(A) I only (B) II only (C) III only
(D) I and III only (E) I, II, and III

Problem 256. Which of the following sets is connected in $\mathbb{R}^3$ with the usual topology?
I. The graph of the function $z = f(x, y) = \dfrac{1}{\log{(x + y)}}$ over the domain $[1, 2] \times [1, 2]$
II. $\mathbb{Q}^3$ in $\mathbb{R}^3$
III. $\mathbb{R}^3 \backslash C$ in $\mathbb{R}^3$, where C is any countable set
(A) I only (B) II only (C) III only
(D) I and III only (E) I, II, and III

Problem 257. Let A and B be two connected sets in $\mathbb{R}^n$, which of the following statements is true?
I. If $A \cap B \neq \varnothing$, then $A \cap B$ is connected
II. If $A \cap B \neq \varnothing$, then $A \cup B$ is connected
III. $A \backslash B$ is connected
(A) I only (B) II only (C) III only
(D) I and II only (E) II and III only

Problem 258. Which of the following sets is compact in $\mathbb{R}^3$ with the usual topology?
I. The image of the sequence $(\cos(\frac{1}{n}), \sin(\frac{1}{n}), e^{-n})$, where n is a positive integer
II. The image of the sequence $(\cos(\frac{n\pi}{3}), \sin(\frac{n\pi}{5}), \sin(\frac{n\pi}{7}))$, where n is a positive integer
III. The union of the sets I and II
(A) I only (B) II only (C) I and II only
(D) II and III only (E) I, II, and III

Problem 259. Which of the following sets is a complete metric space, with inherited metric from $\mathbb{R}^2$?
I. The set of all integer order pairs $\mathbb{Z}^2$
II. The graph of $y = e^x \sin x$, $x \in \mathbb{R}$
III. The union of all the lines $y = ax + b$ where $a, b \in \mathbb{Q}$
(A) I only (B) II only (C) I and II only
(D) II and III only (E) I, II, and III

Problem 260. Let X and Y be two metric spaces that are homeomorphic to each other, namely there exists a continuous bijection $\varphi : X \to Y$ with $\varphi^{-1} : Y \to X$ also continuous. Which of the following is true?
I. X is compact if and only if Y is compact.
II. X is connected if and only if Y is connected.
III. X is complete if and only if Y is complete.
(A) I only (B) II only (C) I and II only
(D) II and III only (E) I, II, and III

Problem 261. Consider two intervals in the real line $I = (a, b), J = (c, d)$. Denote $W = \{xy : x \in I, y \in J\}$. If $d < b$, what is $\sup W$?
(A) ac (B) bd (C) $\max\{ac, bd\}$
(D) $\max\{ac, ad, bd\}$ (E) $\max\{ac, bc, bd\}$

Problem 262. Consider two intervals in the real line $I = (a, b), J = (c, d)$. Denote $W = \{xy : x \in I, y \in J\}$. If $a < c$, what is $\sup W$?
(A) ac (B) bd (C) $\max\{ac, bd\}$
(D) $\max\{ac, ad, bd\}$ (E) $\max\{ac, bc, bd\}$

Problem 263. Let A be a nonempty bounded subset of $\mathbb{R}$. Consider the three real numbers x_1, x_2, x_3 where
I. If y is such that $y \geqslant a$ for all $a \in A$, then $x_1 \leqslant y$.
II. $x_2 = \sup A$
III. For all $a \in A$ we have $x_3 \geqslant a$, and for all $\epsilon > 0$, there exists $a' \in A$ such that $x_3 - \epsilon < a'$.
Which of the following statements is necessarily true?
(A) $x_1 = x_2$ (B) $x_1 = x_3$ (C) $x_2 = x_3$
(D) $x_1 = x_2 = x_3$ (E) $x_1 \neq x_2$

Problem 264. Which of the following statements is true?
I. There exists a real sequence $\{a_n\}$ whose set of subsequential limits is exactly the set $\mathbb{Z}$.
II. There exists a real sequence $\{a_n\}$ whose set of subsequential limits is exactly the set $\mathbb{Q}$.
III. There exists a real sequence $\{a_n\}$ whose set of subsequential limits is exactly the set $\mathbb{R}$.
(A) I only (B) I and II only (C) I and III only
(D) I, II and III (E) None of these

Problem 265. Let $f, g : [0,1] \to \mathbb{R}$ be two bounded real-valued function on $[0,1]$. Which of the following statements is necessarily true?
I. $\left| \sup_{x\in[0,1]} f(x) - \sup_{x\in[0,1]} g(x) \right| \leqslant \sup_{x\in[0,1]} |f(x) + g(x)|$
II. $\left| \inf_{x\in[0,1]} f(x) - \inf_{x\in[0,1]} g(x) \right| \leqslant \inf_{x\in[0,1]} |f(x) + g(x)|$
III. $\left| \inf_{x\in[0,1]} f(x) - \inf_{x\in[0,1]} g(x) \right| \leqslant \sup_{x\in[0,1]} |f(x) + g(x)|$
(A) I only (B) I and II only (C) I and III only
(D) I, II and III (E) None of these

9. Complex analysis.

Problem 266. If $z = x + iy$ is a complex variable and $Re(z) = x$ denotes its real part. What is $\lim_{z\to 0} \frac{Re(z)}{z}$?
(A) 0 (B) 1 (C) i (D) ∞ (E) The limit does not exist.

Problem 267. If $z = x + iy$ is a complex variable and $Re(z) = x$ denotes its real part. What is $\lim_{z\to 0} \frac{Re(z^2)}{z}$?
(A) 0 (B) 1 (C) i (D) ∞ (E) The limit does not exist.

Problem 268. If $z = x + iy$ is a complex variable and $\overline{z} = x - iy$ denotes its complex conjugate. What is $\lim_{z\to 0} \frac{\overline{z}}{z}$?
(A) 1 (B) -1 (C) 0 (D) ∞ (E) The limit does not exist.

Problem 269. If $z = x + iy$ is a complex variable and $Re(z) = x$ denotes its real part and $Im(z) = y$ denotes is imaginary part. What is $\lim_{z\to 0} \frac{Re(z)^2}{Im(z)}$?
(A) 1 (B) -1 (C) 0 (D) ∞ (E) The limit does not exist.

Problem 270. If $z = x + iy$ is a complex variable and $Re(z) = x$ denotes its real part. What is $\lim_{z\to 0} \frac{Re(z)^2}{Re(z^2)}$?
(A) 1 (B) -1 (C) 0 (D) ∞ (E) The limit does not exist.

Problem 271. Consider the power series $\sum_{n=0}^{\infty} a_n z^n$, where $a_n \in \{0, 2\}$ and the set $\{n : a_n = 2\}$ is infinite. What is the radius of converge of this power series?
(A) 0 (B) $\frac{1}{2}$ (C) 1 (D) 2 (E) ∞

Problem 272. What is the radius of converge of the power series $\sum_{n=0}^{\infty} n! z^{n!}$?
(A) 0 (B) 1 (C) e (D) e^e (E) ∞

Problem 273. $(1 + i\sqrt{3})^{10} =$
(A) $512(1 + i\sqrt{3})$ (B) $-512(1 + i\sqrt{3})$ (C) $512(1 - i\sqrt{3})$
(D) $512(\sqrt{3} + i)$ (E) $-512(\sqrt{3} + i)$

Problem 274. Consider the complex number $z = 1 - i$. Which of the following statement is true?
(A) z^{123} is strictly in the first quadrant of the complex plane.
(B) z^{123} is strictly in the second quadrant of the complex plane.
(C) z^{123} is strictly in the third quadrant of the complex plane.
(D) z^{123} is strictly in the fourth quadrant of the complex plane.
(E) z^{123} is on one of the coordinate axes of complex plane.

Problem 275. Let z be a complex variable. Consider S, the set of complex satisfying $|Re(z)| < |z|$. Which of the following best describe S in the complex plane?
(A) S is the entire complex plane.
(B) S is a line.
(C) S is the upper half plane.
(D) S is entire complex plane except a point.
(E) S is the entire complex plane except a line.

Problem 276. For how many real values $c \in \mathbb{R}$ will the function $u(x, y) = e^y \cos(cx)$ be the real part of an analytic function on $\mathbb{C}$?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 277. For what real value k is the function $f(x + iy) = kx - 2xy + i(x^2 - y^2 + 3y)$ analytic everywhere on the complex plane?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 278. Suppose $f : \mathbb{C} \to \mathbb{C}$ is an analytic function such that $|f(z)| > 2$ for all $z \in \mathbb{C}$. Which of the following statements is true?
I. f may be a function of the form $az + b$ for some $a, b \in \mathbb{C}$
II. f may be a function of the form ae^{bz} for some $a, b \in \mathbb{C}$
III. $\lim_{z\to\infty} |f(z)| = \infty$
(A) I only (B) II only (C) I and II only
(D) I and III only (E) II and III only

Problem 279. Consider the rational function $f(z) = \dfrac{1}{1+z^2}$ defined on points $z \in \mathbb{C}$ whenever $f(z)$ exists. What is the radius of convergence of $\sum_{n=0}^{\infty} a_n(z-1)^n$, the power series of f centered at $z = 1$?
(A) 0 (B) 1 (C) 2 (D) $\sqrt{2}$ (E) ∞

Problem 280. Let C be the circle on the complex plane centered at i with radius 1, oriented counterclockwise. Then $\oint_C \dfrac{1}{1+z^2} dz =$
(A) 0 (B) 1 (C) -1 (D) π (E) $-\pi$

10. Group theory.

Problem 281. Which of the following does NOT form a group?
(A) The set of 2×2 matrices under multiplication.
(B) The set of 2×2 matrices of the form

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$$

where $k \in \mathbb{Z}$ under multiplication.
(C) The set of 3×3 matrices of the form

$$\begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix}$$

where $a, b, c \in \mathbb{Z}$ under multiplication.
(D) The set of functions $f : \mathbb{R} \to \mathbb{R}$ under pointwise addition.
(E) The set of injective functions $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ under function composition.

Problem 282. Which of the following forms a group?
(A) The set of 5×5 real symmetric matrices under multiplication.
(B) The set of 5×5 real matrices whose diagonal entries sum to 0, under matrix addition.
(C) The set of 5×5 real diagonalizable matrices under matrix addition.
(D) The set of 5×5 real diagonalizable matrices under matrix multiplication.
(E) The set of 5×5 real invertible matrices under addition.

Problem 283. Up to group isomorphism, how many distinct groups are there with 4 elements?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 284. Which of the following statements is NOT true?
(A) For every positive integer n, there exists a group G of order n.
(B) The smallest group that is not abelian has order 6.
(C) There exists a group G (written multiplicatively) such that for each $x \in G$ we have $x^2 = e$, the identity element of G.
(D) There exists a group G (written multiplicatively) such that for each $x \in G$ we have $x^3 = e$.
(E) Every group of order 24 is abelian.

Problem 285. Let G be a finite cyclic group, which of the following statements is NOT true?
(A) G is abelian.
(B) Every subgroup H of G has an order that divides the order of G.
(C) For every positive integer k that divides the order of G, there exists a subgroup H of G of order k.
(D) Every element in G except the identity is a generator for the group G.
(E) If n is the order of G, then $x^n = e$ for all $x \in G$.

Problem 286. For which integers n such that $3 \leqslant n \leqslant 12$ is there only abelian groups of order n?
(A) For no such integer n
(B) For $3, 4, 5, 6, 9$,11 only
(C) For $3, 5, 7, 11$ only
(D) For $4, 6, 8, 10$ only
(E) For all such integers n

Problem 287. Which of the following statements must be true?
I. For every integer $n \geqslant 3$, there exists a non-abelian group of order $2n$.
II. For every positive integer n, there exists a cyclic group of order n.
III. If n is an integer such that all groups of order n is abelian, then n must be prime.
(A) I only (B) II only (C) III only
(D) I and II only (E) I, II and III

Problem 288. How many abelian groups (up to isomorphism) are there with order 16?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 289. How many abelian groups (up to isomorphism) are there with order 100?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 290. For which integers n such that $10 \leqslant n \leqslant 20$ is there only one group of order n (up to isomorphism)?
(A) For no such integer n
(B) For 11, 13, 15, 17, 19 only
(C) For 11, 13, 17, 19 only
(D) For 12, 14, 16, 18 only
(E) For all such integers n

Problem 291. Let a_n denote the largest order of an element in the group of permutations of n objects. What is (a_6, a_7, a_8)?
(A) $(6, 7, 8)$ (B) $(6, 6, 8)$ (C) $(6, 10, 12)$
(D) $(6, 12, 15)$ (E) $(6!, 7!, 8!)$

Problem 292. What is the least n such that S_n has an element of order 24? Here S_n is the group of permutations of n objects.
(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

Problem 293. What is the order of the following element in the group of permutation of 10 objects:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 4 & 10 & 1 & 8 & 9 & 6 & 5 & 7 & 3 \end{pmatrix}$$

(A) 2 (B) 3 (C) 6 (D) 12 (E) 36

Problem 294. Which of the following groups does not have an element of order 30?
(A) $\mathbb{Z}_{30}$, group of integers modulo 30 under addition
(B) $GL_2(\mathbb{R})$, group of all 2×2 invertible real matrices under multiplication
(C) S_{30}, group of permutations of 30 objects
(D) S_{10}, group of permutations of 10 objects
(E) A_{30}, group of even permutations of 30 objects

Problem 295. Let C_n denote a cyclic group of order n. Which of the following direct products of groups is not cyclic?
(A) $C_1 \times C_2 \times C_3$
(B) $C_{15} \times C_{16}$
(C) $C_{37} \times C_{137}$
(D) $C_{201} \times C_{501}$
(E) $C_{999} \times C_{1000}$

Problem 296. Let $\mathbb{Z}_{25}$ be the ring of integers modulo 25, and let $\mathbb{Z}_{25}^{\times}$ be the group of units. Which of the following is a generator for $\mathbb{Z}_{25}^{\times}$?
I. 2
II. 3
III. 12
(A) I only (B) II only (C) III only
(D) I and II only (E) I, II, and III

Problem 297. How many group homomorphisms are there from $\mathbb{Z}_3 \to \mathbb{Z}_7$?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 298. How many group homomorphisms are there from $\mathbb{Z}_8 \to \mathbb{Z}_6$?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 299. Let $f : \mathbb{Z}_{28} \to \mathbb{Z}_{12}$ such that $f(1) = 3$. What is $\ker f$?
(A) $\{0\}$ (B) $\{0, 14\}$ (C) $\{0, 7, 14, 21\}$
(D) $\{0, 4, 8, 12, 16, 20, 24\}$ (E) $\mathbb{Z}_{28}$

Problem 300. Which of the following indicated subgroup of a group is a normal subgroup?
I. The subgroup A_5 in S_5
II. The subgroup $\{0, 4, 8, 12\}$ in $\mathbb{Z}_{12}$, under addition
III. The subgroup of rotations in dihedral group D_8
(A) I only (B) II only (C) III only
(D) I and II only (E) I, II, and III

11. Ring theory.

Problem 301. Let $(\mathbb{Z}_{24}, +, \cdot)$ be the ring of integers modulo 24. Which of the following statements is FALSE?
(A) $\{0, 6, 12, 18\}$ is closed under addition modulo 24.
(B) $\{0, 6, 12, 18\}$ is closed under multiplication modulo 24.
(C) $\{0, 6, 12, 18\}$ has an identity under addition modulo 24
(D) $\{0, 6, 12, 18\}$ has an identity under multiplication modulo 24
(E) $\{0, 6, 12, 18\}$ is a subgroup of $(\mathbb{Z}_{24}, +)$, under addition modulo 24.

Problem 302. Consider the ring of polynomials $\mathbb{R}[x]$ with real coefficients, under the usual operations of polynomial addition and multiplication. Let $S = \{p \in \mathbb{R}[x] : p(1) = 0\}$, which of the following statements is FALSE?
(A) S is closed under addition.
(B) S is closed under multiplication.
(C) S has an identity under addition.
(D) S has an identity under multiplication.
(E) S is commutative under multiplication.

Problem 303. Consider $M_{2\times 2}(\mathbb{Z})$ the ring of 2×2 matrices with integer coefficients, with usual matrix addition and multiplication. Let $S = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Z} \right\}$, which of the following statements is TRUE?
(A) S is closed under addition.
(B) S has contains an additive inverse
(C) S is not commutative under addition.
(D) S is not commutative under multiplication.
(E) S with multiplication is isomorphic to the group of integers $\mathbb{Z}$ under addition.

Problem 304. Consider $\mathbb{R}^{\{1,2,3\}}$ the ring of all functions $f : \{1, 2, 3\} \to \mathbb{R}$, with usual pointwise addition and multiplication for functions. Let S be the subset of functions where $f(1) = f(3)$, which of the following statements is FALSE?
(A) S is closed under addition.
(B) S has an additive identity.
(C) S is closed under multiplication
(D) S is has a multiplicative identity.
(E) S is an ideal of $\mathbb{R}^{\{1,2,3\}}$.

Problem 305. Let $R = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ be a ring equipped with usual matrix addition and multiplication, and $f : R \to \mathbb{Z}$ be a map such that $f \begin{pmatrix} a & b \\ b & a \end{pmatrix} = a - b$. Which of the following statements is true?
I. R is a commutative ring
II. f is a ring homomorphism
III. $\ker f$ is a maximal ideal of R
(A) I only (B) II only (C) I and II only
(D) I, II, and III (E) None of the above

Problem 306. Let x, y, z be integers. Suppose

$$\begin{aligned} 3x &\equiv 1 \mod 10 \\ 7y &\equiv 2 \mod 10 \\ 9z &\equiv 3 \mod 10 \end{aligned}$$

What is $x + y + z$ congruent to modulo 10?
(A) 0 (B) 1 (C) 2 (D) 4 (E) 7

Problem 307. Let x and y be integers. Suppose

$$\begin{aligned} 4x &\equiv y \mod 17 \\ 4y &\equiv x \mod 17 \end{aligned}$$

What is $x + y$ congruent to modulo 17?
(A) 0 (B) 1 (C) 13 (D) 14 (E) 15

Problem 308. Let x and y be integers. Suppose

$$\begin{aligned} 4x &\equiv y \mod 10 \\ 4y &\equiv x \mod 10 \end{aligned}$$

What is xy congruent to modulo 10?
(A) 0 (B) 1 (C) 13 (D) 14 (E) 15

Problem 309. Let x be an integer. Suppose

$$\begin{aligned} 3x &\equiv 2 \mod 7 \\ 2x &\equiv 3 \mod 11 \end{aligned}$$

What is x congruent to modulo 77?
(A) 17 (B) 31 (C) 59 (D) 66 (E) 71

Problem 310. Consider the equations

$$\begin{aligned} x + y &\equiv 2 \mod 10 \\ x - y &\equiv 4 \mod 10 \end{aligned}$$

How many solutions of the form (x, y) modulo 10 are there?
(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 311. Let S be the set of all bijective functions $f : \mathbb{Z}_5 \to \mathbb{Z}_5$, where $\mathbb{Z}_5$ is the ring of integer addition and multiplication modulo 5. Consider the binary operations $+$ and $\circ$ defined on S as follows

$$\begin{aligned} (f + g)(x) &= f(x) + g(x) \\ (f \circ g)(x) &= f(g(x)) \end{aligned}$$

Which of the following statements is true?
I. $\circ$ is closed in S.
II. $+$ is closed in S.
III. For each $f \in S$, there exists $g \in S$ such that $(f \circ g)(x) = x$
(A) I only (B) II only (C) III only
(D) I and II only (E) I and III only

Problem 312. How many elements in $\mathbb{Z}_{20}$, the ring of integers modulo 20, have a multiplicative inverse?
(A) 1 (B) 2 (C) 4 (D) 8 (E) 16

Problem 313. How many elements in $\mathbb{Z}[x]$, the ring of polynomials with integer coefficients, have a multiplicative inverse?
(A) 0 (B) 1 (C) 2 (D) 4 (E) Infinitely many

Problem 314. How many elements in $M_{2\times 2}(\mathbb{Z})$, the ring of 2×2 integer matrices, have a multiplicative inverse?
(A) 0 (B) 1 (C) 2 (D) 4 (E) Infinitely many

Problem 315. How many elements in $\mathbb{C}[x]$, the ring of polynomials with integer coefficients, have a multiplicative inverse?
(A) 0 (B) 1 (C) 2 (D) 4 (E) Infinitely many

Problem 316. How many ideals are there in the ring $\mathbb{Z}_{20}$, the ring of integers modulo 20?
(A) 1 (B) 2 (C) 4 (D) 6 (E) 8

Problem 317. Let R be a ring, and let $R^\times$ be the set of multiplicatively invertible elements in R. Which of the following statements must be true?
I. $R^\times$ is an ideal of R
II. $R^\times$ is closed under multiplication.
III. If J is an ideal of R, and $J \cap R^\times \neq \varnothing$, then $J = R$.
(A) I only (B) II only (C) III only
(D) I and II only (E) II and III only

Problem 318. Let R be a ring, and let I and J be (two-sided) ideals of R. Which of the following statements must be true?
I. $I \cup J$ is an ideal of R
II. $A = \{x \in R : \forall j \in J, xj \in I\}$ is an ideal of R
III. $B = \{x \in R : \exists j \in J, xj \in I\}$ is an ideal of R
(A) I only (B) II only (C) III only
(D) II and III only (E) I, II, and III

Problem 319. Let R be a ring and fix $a \in R$. Let $A = \{x \in R : ax = 0\}$. Which of the following statements must be true?
I. A is closed under addition.
II. Every element in A is multiplicatively invertible.
III. A is an ideal
(A) I only (B) II only (C) III only
(D) I and III only (E) I, II, and III

Problem 320. Consider the ring of polynomials with real coefficients $\mathbb{R}[x]$. Which of the following sets form an ideal of $\mathbb{R}[x]$?
I. $\{p \in \mathbb{R}[x] : p(1) = 0\}$
II. $\{p \in \mathbb{R}[x] : p(0) = 1\}$
III. $\{p \in \mathbb{R}[x] : p'(0) = 0\}$
(A) I only (B) II only (C) III only
(D) I and II only (E) I, II, and III

12. Set and logic.

Problem 321. Let X, Y, Z be nonempty sets, and let $f : X \to Y$ and $g : Y \to Z$ be two functions. If $g \circ f$ is surjective, which of the following must be true?
(A) g is injective (B) g is surjective (C) f is injective
(D) f is surjective (E) $f \circ g$ is surjective

Problem 322. Let X, Y, Z be nonempty sets, and let $f : X \to Y$ and $g : Y \to Z$ be two functions. If $g \circ f$ is injective, which of the following must be true?
(A) g is injective (B) g is surjective (C) f is injective
(D) f is surjective (E) $g \circ f$ is surjective

Problem 323. Let X, Y, Z be nonempty sets, and let $f : X \to Y$ and $g : Y \to Z$ be two functions. If $g \circ f$ is bijective, which of the following statements must be true?
I. g is bijective
II. f is bijective
III. f is injective
(A) I only (B) II only (C) III only
(D) I and II only (E) I, II, and III

Problem 324. Let X, Y, Z be finite nonempty sets, and let $f : X \to Y$ and $g : Y \to Z$ be two functions. If $g \circ f$ is bijective, which of the following statements must be true?
I. $|X| \leqslant |Y|$
II. $|Y| \leqslant |Z|$
III. $|X| = |Z|$
(A) I only (B) II only (C) III only
(D) I and III only (E) II and III only

Problem 325. Let X be a finite nonempty set, and let f_1, f_2, f_3, f_4 be four functions $X \to X$. Suppose $f_1 \circ f_2 \circ f_3 \circ f_4$ is surjective, how many functions f_i must be injective?
(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 326. Suppose A, B, C, D are statements such that D is true if no more than one of A, B , or C is true. If D is false, which of the following statements must be true?
(A) A and B are true, and C is false.
(B) If B is true, then both A and C are true.
(C) If B if false, then both A and C are true.
(D) A, B, C are all true.
(E) A, B, C are all false.

Problem 327. Suppose A, B, C, D are statements such that A or B is true if exactly one of C or D is true. If C is false, which of the following statements must be true?
(A) A or B is false.
(B) If D is true, then A is true.
(C) If D is false, then both A and B are false.
(D) If D is true and A is false, then B is true.
(E) If D is false and A is true, then B is false.

Problem 328. Suppose A, B, C are statements such that A implies B if C is true. If B is false, which of the following statements must be true?
(A) If C is false, then A is true.
(B) If C is false, then A is false.
(C) If A is false, then C is false.
(D) If A is false, then C is true.
(E) If A is true then C is false.

Problem 329. Suppose A, B, C are statements such that A is true whenever one of B or C is false. Which of the following statements must be true?
(A) If A is true, then B or C is false.
(B) If C is false and A is true, then B is true.
(C) If B is true and A is false, then C is false.
(D) A, B, C are all false.
(E) A, B, C cannot all be false.

Problem 330. Suppose A, B, C, D are statements that D is true if no more than 2 of A, B, C are false. Which of the following statements must be true?
(A) If D is true then at least one of A, B, C is true
(B) If D is true then at least one of A, B, or C is false.
(C) If A is true, then D is true.
(D) If A is false then D is false.
(E) A, B, C, D cannot all be false.

13. References.

Some material adopted from various sources, such as the calculus and the multivariable calculus portion from Stewart's.