Practice Quiz 2 Solutions

Quiz 2 covers all material, as covered in Lectures 8 through 11, and Homework 4 and 5. It is open book, open notes, and online, but web search is prohibited. ***No collaboration or communication in any form is allowed***, except for questions to the instructor and TAs.

Quiz 2 will be posted online, and is due over Canvas as scheduled. Do not discuss any class-related topics among yourselves before or after you have completed your quiz, and until the submission deadline has passed.

References

Tranquillo JV. *Biomedical Signals and Systems*, Morgan & Claypool Publishers, Dec. 2013. Ch. 8, Ch. 9, Ch. 11.

PRACTICE QUIZ SOLUTIONS

Problem 1

Consider the following linear time-invariant (LTI) biosystem:

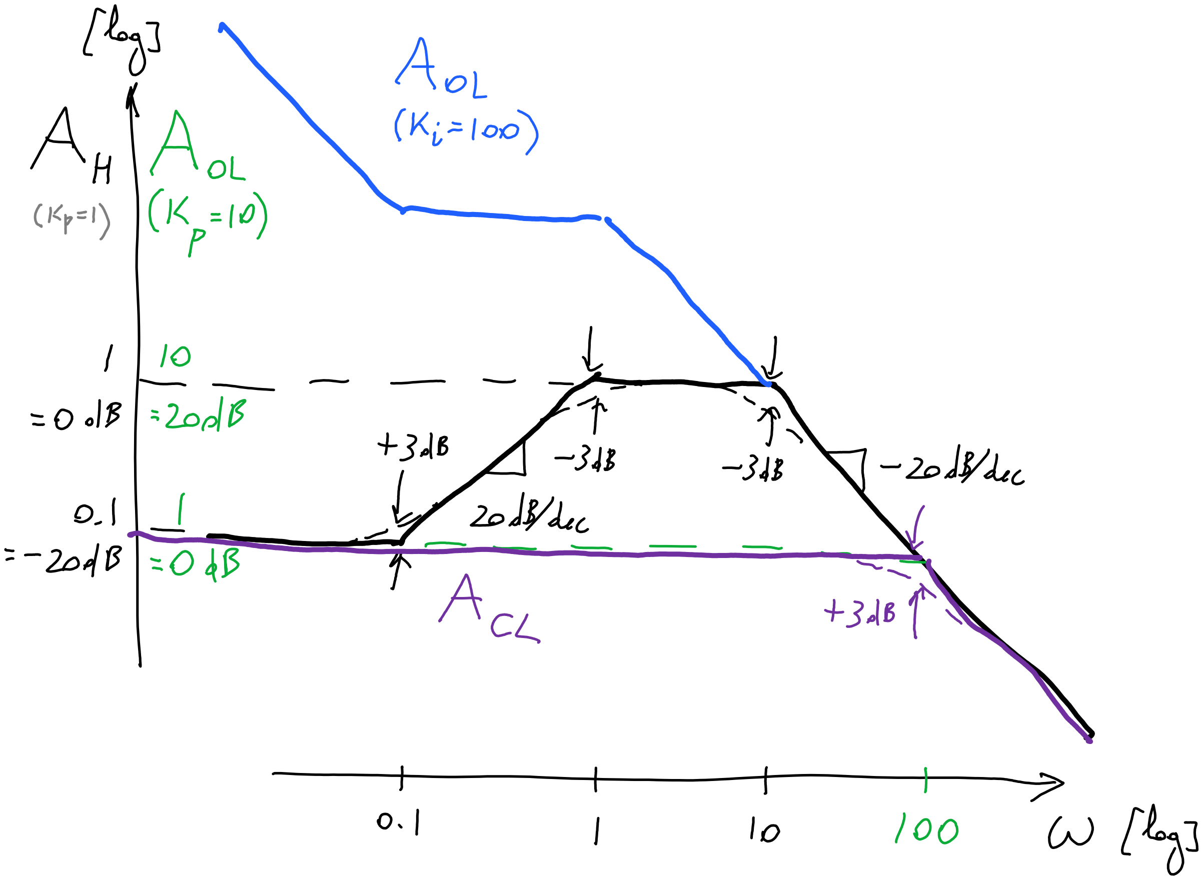
$$H(s) = \frac{10s + 1}{s^2 + 11s + 10}$$

1. Sketch the Bode plot.

$$H(s) = 10\,\frac{s + 0.1}{(s+1)(s+10)}$$

ZERO @ $s = z_1 = -0.1$

POLES @ $s = \begin{cases} p_1 = -1 \\ p_2 = -10 \end{cases}$



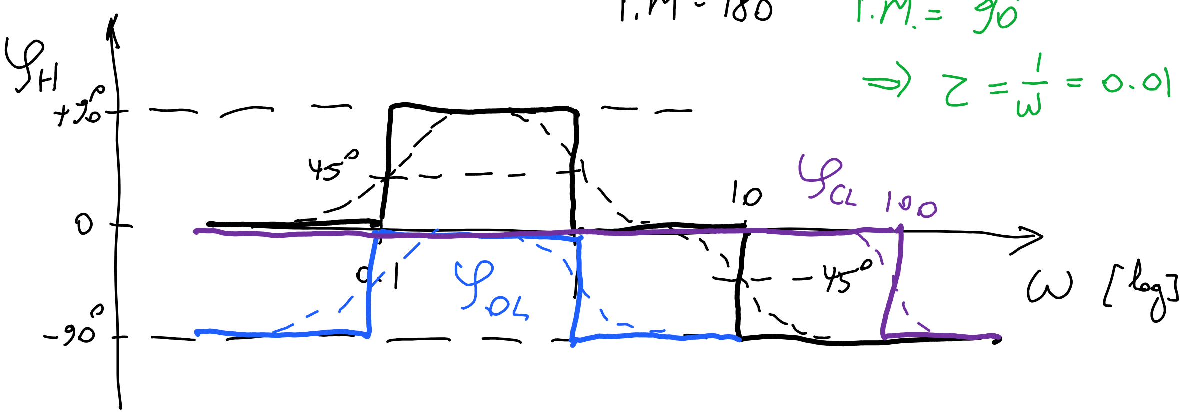
$K_p = 1$

$P.M = 180^\circ$

$K_p = 10$

$P.M. = 90^\circ$

$\Rightarrow \tau = \frac{1}{\omega} = 0.01$



2. Find the closed-loop DC error, and phase margin, for proportional control with 20 dB gain.

$$K_p = 1 \quad (0\,\text{dB}) \Rightarrow$$

$$e = t - u \qquad u = OL(s) \cdot e$$

$$u = \frac{OL}{1+OL}\, t$$

$$e = \left(1 - \frac{OL}{1+OL}\right) t = \frac{1}{1+OL}\, t \approx \frac{1}{1+0.1}\, t$$

$$\approx 0.9\, t \Rightarrow 90\% \text{ error (for 0 dB gain)}$$

$$P.M. = 180^\circ$$

$$K_p = 10 \quad (20\,\text{dB}) \Rightarrow$$

$$e = \frac{1}{1+1}\, t \approx 0.5\, t \Rightarrow 50\% \text{ error}$$

$$P.M. = 90^\circ$$

3. Now add integral control, keeping the proportional control with 20 dB gain. Maximize the value of integral gain to maintain the same value of phase margin as without integral control. Find the closed-loop DC error.

$$z_I = p_2 = -10$$

Since integral control decreases phase margin by 90°, the largest possible z_I is the one where phase drops to -90°.

$$F(s) = K_p + \frac{K_i}{s}$$

$$= \frac{K_p s + K_i}{s} \qquad \text{ZERO @ } s = z_I = -\frac{K_i}{K_p} \qquad \Rightarrow K_i = 10 K_p = 100$$

$$\text{POLE @ } s = 0$$

$$= 10 \frac{s+10}{s}$$

$$OL(0) = \infty \quad \Rightarrow \quad e = 0 \qquad 0\% \text{ error}$$

$$P.M. = 90^0$$

4. Find the closed-loop transfer function for these values of proportional and integral gain, and sketch the Bode plot. Validate that the closed-loop DC error and high-frequency dynamics are consistent with those predicted by the above open-loop analysis.

$$OL(s) = 10 \frac{\cancel{s+10}}{s} \; 10 \cdot \frac{s+0.1}{(s+1)\cancel{(s+10)}} = 100 \frac{s+0.1}{s\,(s+1)}$$

$$\text{ZERO @ } z_{OL} = -0.1$$

$$\text{POLES @ } \begin{cases} p_{OL1} = 0 \\ p_{OL2} = -1 \end{cases}$$

$$CL(s) = \frac{OL(s)}{1 + OL(s)} = \frac{100(s+0.1)}{s(s+1) + 100(s+0.1)}$$

$$= 100 \frac{s+0.1}{s^2 + 101 s + 10} \approx 100 \frac{1}{s+101}$$

$$\text{ZERO @ } \cancel{z_{CL} = -0.1}$$

$$\text{POLES @ } \cancel{p_{CL1}} = -\frac{101}{2} + \frac{\sqrt{101^2 - 80}}{2} \approx \cancel{-0.1}$$

$$p_{CL2} = -\frac{101}{2} - \frac{\sqrt{101^2 - 80}}{2} \approx -101$$

First-order response with $\omega = 101$ cutoff frequency. This is consistent with the 90^o phase margin at $\omega = 100$

5. Is it helpful to add derivative control for this biosystem? Explain.

   The phase margin is already 90^o, and so derivative control is not strictly necessary, although there is still benefit in adding derivative control to further increase phase margin.

   Adding derivative control with a second zero at $s = p_2 = -10$ will increase phase margin from 90^o to almost 180^o, and increase the frequency range beyond $\omega = 100$.

Problem 2

Here we consider the dynamics of a coupled set of three ordinary differential equations describing population dynamics of three interacting species x_1, x_2 and x_3:

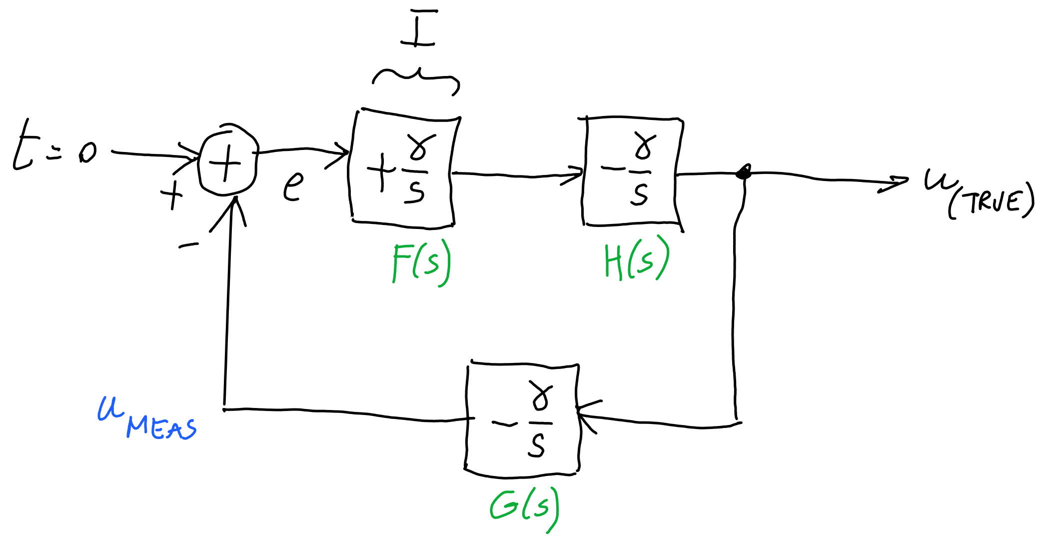
$$\frac{dx_1}{dt} = -\gamma x_3$$

$$\frac{dx_2}{dt} = -\gamma x_1 \qquad (\gamma > 0)$$

$$\frac{dx_3}{dt} = -\gamma x_2$$

1. Show a block diagram of this closed-loop system interconnecting three blocks: a "controller" generating x_1 from x_3 and zero target; a "plant" generating x_2 from x_1; and a "measurement system" generating x_3 from x_2.

Each ODE implements an integrator with gain $-\gamma$. The minus sign for the first integrator (integral control) is absorbed by that in the error, $e = 0 - u_{MEAS}$:



2. Find the open-loop transfer function, and find the phase margin. What does it imply about the stability of the closed-loop system?

$$OL(s) = \frac{\gamma^3}{s^3} \Rightarrow$$

-60 dB/dec

$\varphi = -270^\circ$

$P.M. = -90^\circ$

UNSTABLE

3. Find the closed-loop transfer function, and find the poles. Is the system stable or unstable, and in what form? Check the consistency of your answer with the above open-loop analysis.

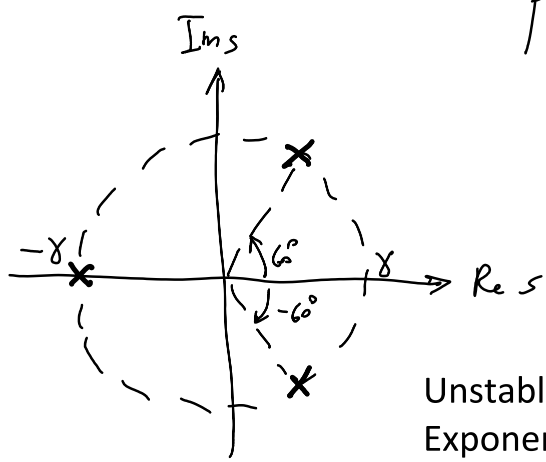
$$CL(s) = \frac{u(s)}{t(s)} = \frac{F(s)\,H(s)}{1 + F(s)H(s)G(s)} = \frac{-\frac{\gamma^2}{s^2}}{1 + \frac{\gamma^3}{s^3}}$$

Alternatively (absorbing measurement system as part of biosystem):

$$CL(s) = \frac{u_{MEAS}(s)}{t(s)} = \frac{\overbrace{F(s)\,H(s)\,G(s)}^{OL}}{1 + \underbrace{F(s)H(s)G(s)}_{OL}} = \frac{\frac{\gamma^3}{s^3}}{1 + \frac{\gamma^3}{s^3}}$$

Either way (the poles are the same):

$$\text{POLES}: \quad s^3 = -\gamma^3 \quad \Rightarrow \quad s = \begin{cases} p_1 = -\gamma \\ p_2 = \gamma e^{j\frac{\pi}{3}} \\ p_3 = \gamma e^{-j\frac{\pi}{3}} \end{cases}$$



$$e^{+\frac{\gamma}{2}t} \; e^{\pm j\frac{\sqrt{3}}{2}\gamma t}$$

Unstable (consistent with 2.)
Exponentially rising oscillations.

Problem 3

1. Find an expression for the closed-loop gain $A_{CL}(\omega)$ as a function of the open-loop gain $A_{OL}(\omega)$ and the open-loop phase $\varphi_{OL}(\omega)$.

$$OL(j\omega) = A_{OL}(\omega)\, e^{j\varphi_{OL}(\omega)}$$
$$CL(j\omega) = A_{CL}(\omega)\, e^{j\varphi_{CL}(\omega)}$$

$$CL(j\omega) = \frac{OL(j\omega)}{1 + OL(j\omega)} \Rightarrow$$

$$A_{CL}(\omega) = \frac{A_{OL}(\omega)}{\left\| 1 + A_{OL}(\omega) e^{j\varphi_{OL}(\omega)} \right\|}$$

$$= \frac{A_{OL}(\omega)}{\sqrt{\left(1 + A_{OL}(\omega)\cos(\varphi_{OL}(\omega))\right)^2 + \left(A_{OL}(\omega)\sin(\varphi_{OL}(\omega))\right)^2}}$$

$$= \frac{A_{OL}(\omega)}{\sqrt{1 + 2A_{OL}(\omega)\cos(\varphi_{OL}(\omega)) + A_{OL}^2(\omega)}}$$

2. Validate that for zero phase margin, the closed-loop gain goes to infinity. Explain why.

$$A_{OL}(\omega) = 1$$

$$\varphi_{OL}(\omega) = -180^\circ \Rightarrow \cos(\varphi_{OL}(\omega)) = -1$$

$$\Rightarrow A_{CL}(\omega) = \infty$$

At zero phase margin, any signal present in the loop at that frequency recirculates indefinitely. Hence even in the absence of any input, the output is non-zero, which implies that the gain is infinite.

3. Show that for small values of phase margin, the closed-loop gain at resonance is approximately given by the reciprocal of the phase margin (in radians):

$$A_{CL}(\omega_{res}) \approx \frac{1}{\varphi_{OL}(\omega_{res}) + \pi}$$

where ω_{res} is the resonant frequency at which $A_{OL}(\omega_{res}) = 1$.

$$A_{CL}(\omega_{res}) = \frac{1}{\sqrt{2\left(1 + \cos\left(\varphi_{OL}(\omega_{res})\right)\right)}}$$

Small phase margins PM << 1 (in radians):

$$PM = \varphi_{OL}(\omega_{res}) + \pi$$

$$\varphi_{OL}(\omega_{res}) = -\pi + PM$$

Taylor expansion of cosine to second order in PM:

$$\cos\left(\varphi_{OL}(\omega_{res})\right) \underset{PM \ll 1}{\approx} \underbrace{\cos(-\pi)}_{-1} + \cancel{PM \cdot \underbrace{\frac{d}{d\varphi}\cos(-\pi)}_{0}} + \frac{PM^2}{2} \cdot \underbrace{\frac{d^2}{d\varphi^2}\cos(-\pi)}_{+1}$$

$$A_{CL}(\omega_{res}) \approx \frac{1}{\sqrt{2\left(\cancel{1 - 1} + \frac{PM^2}{2}\right)}} = \frac{1}{|PM|}$$