Prandtl-Meyer Reflection Configurations, Transonic Shocks, and Free Boundary Problems

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Abstract

We are concerned with the Prandtl-Meyer reflection configurations of unsteady global solutions for supersonic flow impinging upon a symmetric solid wedge. Prandtl (1936) first employed the shock polar analysis to show that there are two possible steady configurations: the steady weak shock solution and the strong shock solution, when a steady supersonic flow impinges upon the solid wedge – the half-angle of which is less than a critical angle (*i.e.*, the detachment angle), and then conjectured that the steady weak shock solution is physically admissible since it is the one observed experimentally. The fundamental issue of whether one or both of the steady weak and strong shocks are physically admissible has been vigorously debated over the past eight decades and has not yet been settled in a definitive manner. On the other hand, the Prandtl-Meyer reflection configurations are core configurations in the structure of global entropy solutions of the two-dimensional Riemann problem, while the Riemann solutions themselves are local building blocks and determine local structures, global attractors, and large-time asymptotic states of general entropy solutions of multidimensional hyperbolic systems of conservation laws. In this sense, we have to understand the reflection configurations in order to understand fully the global entropy solutions of two-dimensional hyperbolic systems of conservation laws, including the admissibility issue for the entropy solutions. In this monograph, we address this longstanding open issue and present our analysis to establish the stability theorem for the steady weak shock solutions as the long-time asymptotics of the Prandtl-Meyer reflection configurations for unsteady potential flow for all the physical parameters up to the detachment angle. To achieve these, we first reformulate the problem as a free boundary problem involving transonic shocks and then obtain appropriate monotonicity properties and uniform *a priori* estimates for admissible solutions, which allow us to employ the Leray-Schauder degree argument to complete the theory for all the physical parameters up to the detachment angle.

CHAPTER 1

Introduction

We are concerned with unsteady global solutions for supersonic flow impinging upon a solid ramp, which can equivalently be regarded as portraying the symmetric gas flow impinging upon a solid wedge (by symmetry). When a steady supersonic flow impinges upon the solid wedge – the half-angle θ_{w} of which is less than a critical angle (*i.e.*, the detachment angle θ_{d}), Prandtl first employed the shock polar analysis to show that there are two possible steady configurations: the steady weak shock reflection with supersonic or subsonic downstream flow (determined by the wedge angle that is less or larger than the sonic angle $\theta_{\mathrm{s}} < \theta_{\mathrm{d}}$) and the steady strong shock reflection with subsonic downstream flow, both of which satisfy the entropy conditions, provided that no additional conditions are assigned downstream; see Courant-Friedrichs [**22**], von Neumann [**41**], and Prandtl [**42**].

A fundamental issue is whether one or both of the steady weak and strong shocks are physically admissible. This has been debated vigorously over the past eight decades and has not yet been settled in a definitive manner (*cf.* [**22, 23, 39, 41, 44**]). On the basis of experimental and numerical evidence, there are strong indications to show, as Prandtl conjectured (see [**3, 40, 42**]), that it is the steady weak shock solution that is physically admissible as the long-time asymptotics of the Prandtl-Meyer reflection configurations.

Furthermore, the Prandtl-Meyer reflection configurations are solutions of the lateral Riemann problem (Problem 2.6 below), and are core configurations in the structure of global entropy solutions of the two-dimensional Riemann problem for hyperbolic conservation laws. On the other hand, the Riemann solutions are building blocks and determine local structures, global attractors, and large-time asymptotic states of general entropy solutions of multidimensional hyperbolic systems of conservation laws (see [**4, 5, 6, 11, 32, 35, 36, 37, 43, 49**] and the references cited therein). Consequently, we have to understand the reflection configurations in order to fully understand global entropy solutions of the two-dimensional hyperbolic systems of conservation laws, including the admissibility issue for the entropy solutions.

A natural mathematical approach is to single out steady shock reflections by the stability analysis – the stable ones are physically admissible. It has been shown in the steady regime that the steady (supersonic or transonic) weak reflection is always structurally stable in Chen-Chen-Feldman [**8**] and Chen-Zhang-Zhu [**17**] with respect to the steady perturbation of both the wedge slope and the incoming steady upstream flow (even L^1–stable for the supersonic weak reflection with respect to the BV–perturbation of both the wedge slope and the incoming steady upstream flow as shown in Chen-Li [**15**]), while the strong reflection is also structurally stable under conditional perturbations (*cf.* Chen-Chen-Feldman [**8, 9**] and Chen-Fang [**19**]). The first rigorous unsteady analysis of the steady supersonic weak shock solution as the long-time behavior of an unsteady potential flow was due to Elling-Liu [**27**], who dealt with a class of physical parameters determined by an assumption for angle θ_{w} less than the sonic angle $\theta_{\mathrm{s}} \in (0, \theta_{\mathrm{d}})$ (see Chapter 3).

The purpose of this monograph is to establish the stability theorem for the steady (supersonic or transonic) weak shock solutions as the long-time asymptotics of the global Prandtl-Meyer reflection configurations for unsteady potential flow for all the admissible physical parameters, even beyond the sonic angle θ_{s}, up to the detachment angle $\theta_{\mathrm{d}} > \theta_{\mathrm{s}}$. As a corollary, the assumption in Elling-Liu's theorem [27] for the case that $\theta_{\mathrm{w}} \in (0, \theta_{\mathrm{s}})$ is no longer required. The global Prandtl-Meyer reflection configurations involve two types of transonic flow boundaries: discontinuous and continuous hyperbolic-elliptic phase transition boundaries for the fluid fields (transonic shocks and sonic arcs). To establish this theorem, we first reformulate the problem as a free boundary problem involving transonic shocks and then carefully establish the required appropriate monotonicity properties and uniform *a priori* estimates for admissible solutions so that the approach developed in Chen-Feldman [11] can be employed. This involves several core difficulties in the theory of the underlying nonlinear PDEs: optimal estimates of solutions of nonlinear degenerate PDEs and corner singularities (at the corners between the transonic shock as a free boundary and the sonic arcs, and between the transonic shock and the wedge when the wedge angle θ_{w} across the sonic angle θ_{s}), in addition to the involved nonlinear PDEs of mixed elliptic-hyperbolic type and free boundary problems. Some parts of the results have been announced in Bae-Chen-Feldman [2].

More precisely, in Chapter 2, we first formulate the physical problem of supersonic flow impinging upon the solid wedge as an initial-boundary value problem. By using the invariance under a self-similar scaling and the physical structure of the problem (see Fig. 1.1), the initial-boundary value problem is reformulated as a boundary value problem in an unbounded domain (Problem 2.9), and further as a free boundary problem (Problem 2.34) for a pseudo-steady potential flow in a bounded domain in the self-similar coordinates $\boldsymbol{\xi} = (\xi_1, \xi_2) = \frac{\mathbf{x}}{t}$ for $t > 0$. Next, we introduce the notion of admissible solutions that we seek in this monograph for all the admissible physical parameters $(u_\infty, u_0) \in \mathfrak{P}_{\mathrm{weak}}$, where u_∞ represents the speed of the incoming supersonic flow, and u_0 represents the horizontal speed of downstream flow behind a steady weak shock which is uniquely determined by u_∞ and angle θ_{w}. For simplicity, the density of incoming supersonic flow is normalized to be 1 without loss of generality. In §2.3, the existence of admissible solutions for all $(u_\infty, u_0) \in \mathfrak{P}_{\mathrm{weak}}$ is stated as one of the main theorems.
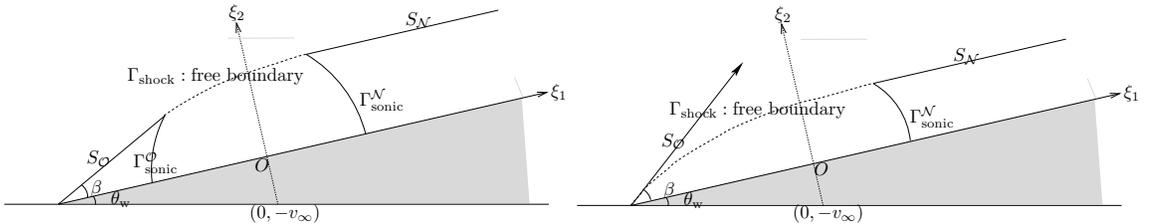
FIGURE 1.1. Admissible solutions in the (v_∞, β)–parameters in the rotated coordinates (ξ_1, ξ_2) by angle θ_{w} counterclockwise (Left: $0 < \beta < \beta_{\mathrm{s}}^{(v_\infty)}$; Right: $\beta_{\mathrm{s}}^{(v_\infty)} \leq \beta < \beta_{\mathrm{d}}^{(v_\infty)}$).

In order to prove the existence of admissible solutions for all $(u_\infty, u_0) \in \mathfrak{P}_{\mathrm{weak}}$ by employing the Leray-Schauder degree argument, the first essential step is to introduce a new parameter set $\mathfrak{R}_{\mathrm{weak}}$ in §2.4. Given $(u_\infty, u_0) \in \mathfrak{P}_{\mathrm{weak}}$, the half-angle θ_{w} of the symmetric solid wedge is uniquely determined. Define $v_\infty := u_\infty \sin \theta_{\mathrm{w}}$. As we will discuss later, $u_0 > 0$ represents the horizontal speed of the downstream flow behind the weak oblique shock $S_{\mathcal{O}}$. Then we define $\beta \in (0, \frac{\pi}{2})$ as the

angle between the wedge boundary and $S_{\mathcal{O}}$. Parameters (v_∞, β) were first introduced in [**27**]. In Lemma 2.19, we show that there exists a homeomorphism $\mathcal{T} : \mathfrak{P}_{\text{weak}} \to \mathcal{T}(\mathfrak{P}_{\text{weak}}) =: \mathfrak{R}_{\text{weak}}$. More importantly, we show that $\mathfrak{R}_{\text{weak}}$ is in the form of

$$\mathfrak{R}_{\text{weak}} = \bigcup_{v_\infty > 0} \{v_\infty\} \times (0, \beta_{\text{d}}^{(v_\infty)}).$$

This structure of $\mathfrak{R}_{\text{weak}}$ enables us to prove the existence of admissible solutions for all $\beta \in (0, \beta_{\text{d}}^{(v_\infty)})$ for any fixed $v_\infty > 0$ via the Leray-Schauder degree theorem. In particular, for each $v_\infty > 0$, there exists an admissible solution for $\beta = 0$ and, in §5.3, we prove that the Leray-Schauder fixed point index of this solution is 1. We also show that, for each $v_\infty > 0$, there exists a unique $\beta_{\text{s}}^{(v_\infty)} \in (0, \beta_{\text{d}}^{(v_\infty)})$, called the *sonic angle*, so that the structure of admissible solutions becomes different as β increases across $\beta = \beta_{\text{s}}^{(v_\infty)}$ (see Fig. 1.1). Finally, we restate both the definition and existence of admissible solutions for $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ in §2.5.

In Chapter 3, we establish all the *a priori* estimates which are essential for solving the free boundary problem introduced in Chapter 2. Furthermore, the *a priori* estimates are achieved uniformly on parameters (v_∞, β). In particular, this chapter contains the following estimates:

(i) Strict directional monotonicity properties of $\varphi_\infty - \varphi$;
(ii) Strict directional monotonicity properties of $\varphi - \varphi_{\mathcal{N}}$ and $\varphi - \varphi_{\mathcal{O}}$;
(iii) Uniform positive lower bound of the distance between Γ_{shock} and Γ_{wedge} away from the wedge vertex;
(iv) Uniform positive lower bound of $\text{dist}(\Gamma_{\text{shock}}, \partial B_1(0, -v_\infty))$;
(v) Uniform estimates of the ellipticity of equation $N(\varphi) = 0$ in Ω, given in (1.1) below;
(vi) Uniform weighted $C^{2,\alpha}$ estimates of admissible solutions in Ω.

In the above, φ_∞, $\varphi_{\mathcal{O}}$, and $\varphi_{\mathcal{N}}$ represent the pseudo-velocity potential functions for the state of incoming supersonic flow, the state behind the oblique shock $S_{\mathcal{O}}$, and the state behind the normal shock $S_{\mathcal{N}}$, respectively. Moreover, $\partial B_1(0, -v_\infty)$ is the *sonic circle* of the incoming supersonic flow:

$$\partial B_1(0, -v_\infty) := \{\boldsymbol{\xi} \in \mathbb{R}^2 : |D\varphi_\infty(\boldsymbol{\xi})| = 1\}.$$

For fixed $v_\infty > 0$ and $0 < \beta < \beta_{\text{s}}^{(v_\infty)}$, let Ω be the bounded region enclosed by $\Gamma_{\text{sonic}}^{\mathcal{O}}$, Γ_{shock}, $\Gamma_{\text{sonic}}^{\mathcal{N}}$, and $\xi_2 = 0$ in Fig. 1.1. In order to find an admissible solution in the sense of Definition 2.24, we need to solve the following free boundary problem for $(\varphi, \Gamma_{\text{shock}})$:

(1.1)
$$\begin{aligned}
N(\varphi) := \text{div}\big(\rho(|D\varphi|^2, \varphi)D\varphi\big) + 2\rho(|D\varphi|^2, \varphi) = 0 \qquad &\text{in } \Omega, \\
\varphi = \varphi_\infty, \quad \rho(|D\varphi|^2, \varphi)D\varphi \cdot \boldsymbol{\nu} = D\varphi_\infty \cdot \boldsymbol{\nu} \qquad &\text{on } \Gamma_{\text{shock}}, \\
\varphi = \varphi_{\mathcal{O}} \qquad &\text{on } \Gamma_{\text{sonic}}^{\mathcal{O}}, \\
\varphi = \varphi_{\mathcal{N}} \qquad &\text{on } \Gamma_{\text{sonic}}^{\mathcal{N}}, \\
\partial_{\xi_2}\varphi = 0 \qquad &\text{on } \partial\Omega \cap \{\xi_2 = 0\},
\end{aligned}$$

where $\rho = \rho(|\mathbf{q}|^2, z)$ is smooth with respect to $(\mathbf{q}, z) \in \mathbb{R}^2 \times \mathbb{R}$ for $|\mathbf{q}| \leq R_0$ and $|z| \leq R_1$ for some positive constants R_0 and R_1. Moreover, $\boldsymbol{\nu}$ is the inward unit normal vector to Γ_{shock}. In particular, we seek a solution so that equation $N(\varphi) = 0$ is strictly elliptic in Ω, but its ellipticity degenerates on $\Gamma_{\text{sonic}}^{\mathcal{O}} \cup \Gamma_{\text{sonic}}^{\mathcal{N}}$. As $\beta \in (0, \beta_{\text{s}}^{(v_\infty)})$ tends to $\beta_{\text{s}}^{(v_\infty)}$, $\Gamma_{\text{sonic}}^{\mathcal{O}}$ shrinks to the wedge vertex P_β, and the ellipticity of $N(\varphi) = 0$ degenerates at P_β for $\beta = \beta_{\text{s}}^{(v_\infty)}$. For $\beta > \beta_{\text{s}}^{(v_\infty)}$, $N(\varphi) = 0$ is strictly elliptic at P_β. For $\beta \geq \beta_{\text{s}}^{(v_\infty)}$, the boundary condition $\varphi = \varphi_{\mathcal{O}}$ on $\Gamma_{\text{sonic}}^{\mathcal{O}}$ given in (1.1) becomes a one-point

Dirichlet boundary condition. Therefore, it is crucial to achieve estimate (v) and then employ the result to establish uniform *a priori* estimates of admissible solutions in Ω by estimate (vi).

Once estimates (i)–(ii) are established, we adjust the argument in [**11**] to achieve estimates (iii)–(vi), although there are several technical differences, due to the structural differences of the solutions constructed in this monograph compared to those in [**11**]. We also point out that estimate (iv) is the key to achieving estimates (v)–(vi). Using the argument in [**11**], for any fixed $v_\infty > 0$, we are able to establish a uniform estimate of positive lower bound of $\mathrm{dist}(\Gamma_{\mathrm{shock}}, \partial B_1(0, -v_\infty))$ for all admissible solutions corresponding to $\beta \in (0, \beta_*]$ whenever $\beta_* \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$. Owing to this property, we prove the existence of admissible solutions for all the admissible physical parameters $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$, even beyond the sonic angle $\beta_{\mathrm{s}}^{(v_\infty)}$.
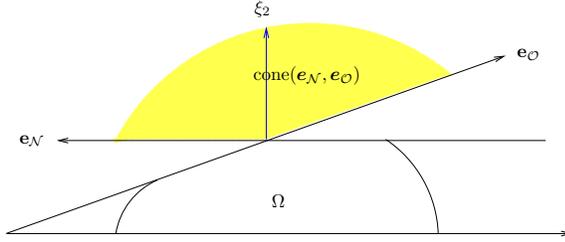
FIGURE 1.2. The cone of monotonicity

Even though the overall argument follows [**11**], there are several significant differences from [**11**]. One of them is the choice of directions for the monotonicity properties of $\varphi_\infty - \varphi$, $\varphi - \varphi_{\mathcal{O}}$, and $\varphi - \varphi_{\mathcal{N}}$. For fixed $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$, define $\mathbf{e}_{\mathcal{N}} := (0, -1)$ and $\mathbf{e}_{\mathcal{O}} := (\cos\beta, \sin\beta)$. Then $\mathbf{e}_{\mathcal{N}}$ is the unit tangent vector to the normal shock $S_{\mathcal{N}}$, and $\mathbf{e}_{\mathcal{O}}$ is the unit tangent vector to the oblique shock $S_{\mathcal{O}}$. Moreover, we define the cone of monotonicity as shown in Fig. 1.2 by

$$\mathrm{Cone}^0(\mathbf{e}_{S_{\mathcal{O}}}, \mathbf{e}_{S_{\mathcal{N}}}) := \{\alpha_1 \mathbf{e}_{S_{\mathcal{O}}} + \alpha_2 \mathbf{e}_{S_{\mathcal{N}}} : \alpha_1, \alpha_2 > 0\}.$$

In §3.1, we show that any admissible solution φ satisfies

$$(1.2) \qquad\qquad \partial_{\mathbf{e}}(\varphi_\infty - \varphi) < 0 \qquad \text{in } \overline{\Omega} \text{ for all } \mathbf{e} \in \mathrm{Cone}^0(\mathbf{e}_{S_{\mathcal{O}}}, \mathbf{e}_{S_{\mathcal{N}}}),$$

from which many essential estimates of admissible solutions can be further obtained. For example, (1.2), combined with the Rankine-Hugoniot conditions on Γ_{shock}, implies that Γ_{shock} is represented as a graph of a function $\xi_2 = f_{\mathrm{sh}}(\xi_1)$ with $f_{\mathrm{sh}}'(\xi_1) > 0$. This property is a key ingredient in the proof of the separation of Γ_{shock} from *the sonic circle* $\partial B_1(0, -v_\infty)$ of the incoming supersonic flow. Notice that this separation property is crucial for establishing the uniform estimate of the ellipticity of equation $N(\varphi) = 0$ in Ω. In addition, further monotonicity properties of $\varphi - \varphi_{\mathcal{O}}$ and $\varphi - \varphi_{\mathcal{N}}$ in $\mathrm{Cone}^0(\mathbf{e}_{S_{\mathcal{O}}}, \mathbf{e}_{S_{\mathcal{N}}})$ are achieved, which play important roles in the *a priori* estimates of admissible solutions near $\Gamma_{\mathrm{sonic}}^{\mathcal{O}} \cup \Gamma_{\mathrm{sonic}}^{\mathcal{N}}$.

In Chapter 4, we define the iteration set $\mathcal{K}$ consisting of approximate admissible solutions. Note that the pseudo-subsonic region Ω of each admissible solution is different. Furthermore, as β increases across $\beta_{\mathrm{s}}^{(v_\infty)}$, the shape of Ω changes from a rectangular domain to a triangular domain. This is because the sonic arc $\Gamma_{\mathrm{sonic}}^{\mathcal{O}}$ corresponding to the oblique shock $S_{\mathcal{O}}$ shrinks to the wedge vertex P_β as $\beta \in (0, \beta_{\mathrm{s}}^{(v_\infty)})$ tends to $\beta_{\mathrm{s}}^{(v_\infty)}$, and $\Gamma_{\mathrm{sonic}}^{\mathcal{O}} = \{P_\beta\}$ for $\beta \geq \beta_{\mathrm{s}}^{(v_\infty)}$. For this reason, it is necessary to introduce a diffeomorphism $\mathfrak{F}$ so that $\mathfrak{F}^{-1}(\Omega)$ is the fixed domain $\mathcal{Q}^{\mathrm{iter}} :=$

$(-1, 1) \times (0, 1)$. Moreover, $\mathfrak{F}$ should be defined so that $\mathfrak{F}$ depends continuously on $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)})$ and on admissible solutions in an appropriately chosen norm. In §4.1, we define a mapping $\mathfrak{F}$ for each admissible solution such that

$$\mathfrak{F}(\mathcal{Q}^{\mathrm{iter}}) = \Omega, \qquad \mathfrak{F}(\Gamma_{\mathrm{shock}}) = \{(s, 1) : -1 < s < 1\},$$
$$\mathfrak{F}(\Gamma_{\mathrm{sonic}}^{\mathcal{O}}) = \{(-1, t) : 0 < t < 1\}, \quad \mathfrak{F}(\Gamma_{\mathrm{sonic}}^{\mathcal{N}}) = \{(1, t) : 0 < t < 1\}.$$

Since the sonic arc $\Gamma_{\mathrm{sonic}}^{\mathcal{N}}$ corresponding to the normal shock $S_{\mathcal{N}}$ is fixed so as to be the same for all $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)})$ (see Fig. 1.1), the definition of $\mathfrak{F}$ in this monograph can be given more explicitly than the one given in [**11**]; see Definition 4.15. In §4.2, the definition of $\mathfrak{F}$ is extended to a class of approximate admissible solutions. Then we set up the iteration set $\mathcal{K}$ and analyze its properties in §4.3–§4.5. The iteration set $\mathcal{K}$ is given in the form

$$\mathcal{K} := \bigcup_{\beta \in [0, \beta_*]} \{\beta\} \times \mathcal{K}(\beta) \qquad \text{for fixed } v_\infty > 0 \text{ and } \beta_* \in (0, \beta_{\mathrm{d}}^{(v_\infty)}),$$

where each $\mathcal{K}(\beta)$ is a subset of $C^{1,\alpha}(\overline{\mathcal{Q}^{\mathrm{iter}}})$ for some $\alpha \in (0, 1)$.

In Chapter 5, for fixed $v_\infty > 0$, we define an iteration map

$$\mathcal{I}(\cdot, \beta) : \mathcal{K}(\beta) \to C_{(*, \alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}}) \qquad \text{for } \mathcal{Q}^{\mathrm{iter}} := (-1, 1) \times (0, 1) \subset \mathbb{R}^2,$$

where $C_{(*, \alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}})$ is a weighted $C^{2,\alpha}$ space. The iteration mapping $\mathcal{I}$ is defined so that, if $\mathcal{I}(u_*, \beta) = u_*$ for $u_* \in \mathcal{K}(\beta)$, then $(\varphi, \Gamma_{\mathrm{shock}})$, given by

$$\varphi = u_* \circ \mathfrak{F}_{(u_*, \beta)}^{-1} + \varphi_\beta^* \quad \text{in } \Omega = \mathfrak{F}_{(u_*, \beta)}(\mathcal{Q}^{\mathrm{iter}}), \qquad \Gamma_{\mathrm{shock}} = \mathfrak{F}_{(u_*, \beta)}(\{(s, 1) : -1 < s < 1\}),$$

solves the free boundary problem (1.1). In the above, φ_β^* is a smooth interpolation of $\varphi_{\mathcal{O}}$ and $\varphi_{\mathcal{N}}$. The precise definition of φ_β^* is given by (4.1.42). Finally, the existence of a fixed point of $\mathcal{I}(\cdot, \beta)$ in $\mathcal{K}(\beta)$ for all $\beta \in (0, \beta_*]$ is proved by employing the Leray-Schauder degree argument in §5.3. In this way, we establish the existence of admissible solutions for all $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$ (Theorem 2.31), hence the existence of admissible solutions for all $(u_\infty, u_0) \in \mathfrak{P}_{\mathrm{weak}}$ (Theorem 2.31).

Theorem 2.16, or equivalently, Theorem 2.33, which pertains to the optimal regularity of admissible solutions, is established in Chapter 6.

To make the monograph self-contained, we also include Appendices A–C, which include some results required for establishing the main theorems and a proof of the non-existence of self-similar strong shock solutions.

A closely related problem to the one we have solved here is the shock reflection-diffraction problem which was addressed in Chen-Feldman [**11**]. Even though the two problems are two different lateral Riemann problems and have different issues and features, the approach developed in Chen-Feldman [**11**] for the shock reflection-diffraction problem has been adopted for solving our Prandtl-Meyer reflection problem in this monograph. As discussed earlier, one of the main contributions of this monograph is to identify appropriate monotonicity properties and establish suitable uniform *a priori* estimates for admissible solutions, based on the new and careful choice of the directions for the monotonicity properties; as a result, the Chen-Feldman approach in [**11**] can be employed.

In this monograph, we have solved the Prandtl-Meyer reflection problem up to the detachment angle in the framework of the potential flow equation, which has been widely used for discontinuous flows in applications in aerodynamics, especially when the amount of vorticity is relatively small in the region of interest. When the flow regions of interest have large amounts of vorticity, the

full compressible Euler equations are usually required. Nevertheless, for the solutions containing a shock of small strength, the potential flow equation and the full Euler equations match each other well, right up to the third-order of the shock strength. Furthermore, for the problem analyzed in this monograph, the Euler equations for potential flow is actually *exact* in two important regions of the solutions near the two sonic arcs in the subsonic domain Ω. Even in the other part of domain Ω, under the Helmholtz-Hodge decomposition for the velocity field, the full Euler equations in the self-similar coordinates can be decomposed as the potential flow type equation, coupled with the incompressible Euler type equations plus a transport equation for the entropy function. These can be shown by directly following the arguments in §18.7 in Chen-Feldman [**11**]. In this sense, the analysis and related methods/techniques developed in this monograph could also play an essential role in finding a solution of the problem in the framework of the full Euler equations. In particular, our results for the potential flow equation have provided useful insights on what will happen for the case of the full Euler equations.

Finally, we remark in passing that, for the uniqueness/stability problems, it is necessary to consider solutions in a restricted class. Recent results [**20, 21, 29, 34**] show the non-uniqueness of solutions with flat shocks in the class of entropy solutions of the Cauchy problem (initial value problem) for the multidimensional compressible Euler equations (isentropic and full). The Prandtl-Meyer reflection problem under consideration in this monograph is different – the problem for solutions with non-flat shocks for potential flow on the domain with boundaries, so these non-uniqueness results do not apply directly. However, these results indicate that it is natural to study the uniqueness and stability problems in a more restricted class of solutions. Since the completion of this monograph, some progress on the uniqueness in the class of self-similar solutions of regular shock reflection-diffraction configurations with convex transonic shocks (which are called admissible solutions) has been made, as announced recently in [**13**]. A similar uniqueness result can also be obtained by combining the approach in [**13, 14**] with the estimate techniques developed in this monograph. Technically, restricting the uniqueness to the class of admissible solutions allows us to reduce the problem to a corresponding uniqueness problem for solutions of a free boundary problem for a nonlinear elliptic equation, which is degenerate for the supersonic case.

CHAPTER 2

Mathematical Problems and Main Theorems

In this chapter, we first formulate the physical problem of a supersonic flow impinging upon the solid wedge into an initial-boundary value problem. Then, based on the invariance of both the problem and the governing equations under the self-similar scaling, we reformulate the initial-boundary value problem as a boundary value problem in an unbounded domain (Problem 2.9), and further as a free boundary problem in a bounded domain (Problem 2.34) for the existence of Prandtl-Meyer reflection configurations involving two types of transonic flow boundaries: discontinuous and continuous hyperbolic-elliptic phase transition boundaries for the fluid fields (transonic shocks and sonic arcs). The main theorems of this monograph are presented in §2.3 and §2.5.

2.1. Mathematical Problems

The compressible potential flow is governed by the conservation law of mass and the Bernoulli law:

$$(2.1.1) \qquad \partial_t \rho + \nabla_{\mathbf{x}} \cdot (\rho \nabla_{\mathbf{x}} \Phi) = 0,$$

$$(2.1.2) \qquad \partial_t \Phi + \frac{1}{2}|\nabla_{\mathbf{x}} \Phi|^2 + h(\rho) = B,$$

where ρ is the density, Φ is the velocity potential, B is the Bernoulli constant determined by the incoming flow and/or boundary conditions, and $h(\rho)$ is given by

$$h(\rho) = \int_1^\rho \frac{p'(\varrho)}{\varrho}\, \mathrm{d}\varrho = \int_1^\rho \frac{c^2(\varrho)}{\varrho}\, \mathrm{d}\varrho$$

for the sound speed $c(\rho)$ and pressure p. For an ideal polytropic gas, the sound speed c and pressure p are given by

$$(2.1.3) \qquad c^2(\rho) = \kappa\gamma\rho^{\gamma-1}, \qquad p(\rho) = \kappa\rho^\gamma$$

for constants $\gamma > 1$ and $\kappa > 0$. If $(\rho, \Phi)(t, \mathbf{x})$ solves (2.1.1)–(2.1.2) with (2.1.3), then $(\tilde{\rho}, \tilde{\Phi})(t, \mathbf{x}) = (\rho, \Phi)(\alpha^2 t, \alpha\mathbf{x})$ with $\alpha := \frac{1}{\sqrt{\kappa\gamma}}$ solves

$$\partial_t \tilde{\rho} + \nabla_{\mathbf{x}} \cdot (\tilde{\rho}\nabla_{\mathbf{x}}\tilde{\Phi}) = 0,$$

$$\partial_t \tilde{\Phi} + \frac{1}{2}|\nabla_{\mathbf{x}}\tilde{\Phi}|^2 + \frac{\tilde{\rho}^{\gamma-1} - 1}{\gamma - 1} = \alpha^2 B.$$

Therefore, we choose $\kappa = \frac{1}{\gamma}$ without loss of generality so that

$$(2.1.4) \qquad h(\rho) = \int_1^\rho h'(\varrho)\mathrm{d}\varrho = \frac{\rho^{\gamma-1} - 1}{\gamma - 1}, \qquad c^2(\rho) = \rho^{\gamma-1}.$$

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The case of the isothermal flow can be included as the isothermal limit $\gamma \to 1+$ in (2.1.4). Therefore, we define $(h, c^2)(\rho)$ by

(2.1.5)
$$(h, c^2)(\rho) = \begin{cases} (\frac{\rho^{\gamma-1}-1}{\gamma-1}, \rho^{\gamma-1}) & \text{for } \gamma > 1, \\ (\ln \rho, 1) & \text{for } \gamma = 1. \end{cases}$$

By (2.1.2), ρ can be expressed as

(2.1.6)
$$\rho(\partial_t \Phi, \nabla_{\mathbf{x}} \Phi) = h^{-1}(B - \partial_t \Phi - \frac{1}{2}|\nabla_{\mathbf{x}} \Phi|^2).$$

Then system (2.1.1)–(2.1.2) can be rewritten as

(2.1.7)
$$\partial_t \rho(\partial_t \Phi, \nabla_{\mathbf{x}} \Phi) + \nabla_{\mathbf{x}} \cdot (\rho(\partial_t \Phi, \nabla_{\mathbf{x}} \Phi) \nabla_{\mathbf{x}} \Phi) = 0,$$

with $\rho(\partial_t \Phi, \nabla_{\mathbf{x}} \Phi)$ determined by (2.1.6).

A steady state solution $\bar{\Phi}(\mathbf{x})$ to (2.1.1)–(2.1.2) yields the steady potential flow equations

(2.1.8)
$$\nabla_{\mathbf{x}} \cdot (\bar{\rho} \nabla_{\mathbf{x}} \bar{\Phi}) = 0,$$
$$\frac{1}{2}|\nabla_{\mathbf{x}} \bar{\Phi}|^2 + h(\bar{\rho}) = B.$$

A symmetric wedge W of half-angle $\theta_{\mathrm{w}} \in (0, \frac{\pi}{2})$ in $\mathbb{R}^2$ (Fig. 2.1) is defined by

(2.1.9)
$$W := \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : |x_2| < x_1 \tan \theta_{\mathrm{w}}, x_1 > 0\}.$$

On the wedge boundary ∂W, $\bar{\Phi}$ must satisfy the slip boundary condition $\partial_{\mathbf{n}_{\mathrm{w}}} \bar{\Phi} = 0$ on ∂W, where

$$\rho_\infty > 0, \ u_\infty > \rho_\infty^{(\gamma-1)/2}$$
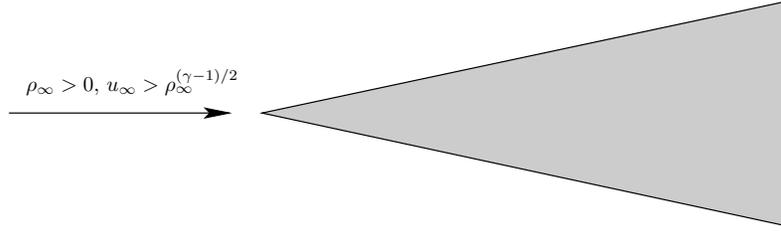
FIGURE 2.1. Supersonic flow impinging upon a solid wedge

$\mathbf{n}_{\mathrm{w}}$ indicates the outward unit normal vector to ∂W. Denote $D := \mathbb{R}^2 \setminus W$, and consider the boundary value problem for (2.1.8) in D with

(2.1.10)
$$\partial_{\mathbf{n}_{\mathrm{w}}} \bar{\Phi} = 0 \qquad \text{on } \partial D = \partial W.$$

If a supersonic flow with a constant density $\rho_\infty > 0$ and a velocity $\mathbf{u}_\infty = (u_\infty, 0)$, $u_\infty > \rho_\infty^{(\gamma-1)/2}$, moves towards wedge W, and if θ_{w} is less than a critical angle called the *detachment angle*, then the well-known *shock polar analysis* shows that there are two different steady weak solutions to the boundary value problem (2.1.8)–(2.1.10): *the steady weak shock solution* and *the steady strong shock solution*. For more precise arguments, we first define a class of weak solutions of the boundary value problem (2.1.8)–(2.1.10).

DEFINITION 2.1. Let Γ_{sh} be a C^1–curve that lies in D and divides D into two open subsets D^- and D^+. We say that $\bar{\Phi} \in W^{1,\infty}(D)$ is a steady entropy solution with a shock Γ_{sh} of the boundary value problem (2.1.8)–(2.1.10) if $\bar{\Phi}$ satisfies the following properties:

(i) $B - \frac{1}{2}|\nabla_{\mathbf{x}}\bar{\Phi}|^2 > h(0+)$ *a.e.* in D;

(ii) For each $\zeta \in C_0^\infty(\mathbb{R}^2)$,

$$\int_D \bar{\rho}(|\nabla_{\mathbf{x}}\bar{\Phi}|^2)\nabla_{\mathbf{x}}\bar{\Phi} \cdot \nabla_{\mathbf{x}}\zeta \, d\mathbf{x} = 0;$$

(iii) $\bar{\Phi} \in C^1(\overline{D^\pm}) \cap C^2(D^\pm)$;

(iv) Entropy condition: for $\bar{\Phi}^\pm := \bar{\Phi}|_{D^\pm \cup \Gamma_{\mathrm{sh}}}$,

$$\partial_{\mathbf{n}_{\mathrm{sh}}}\bar{\Phi}^- > \partial_{\mathbf{n}_{\mathrm{sh}}}\bar{\Phi}^+ > 0 \qquad \text{on } \Gamma_{\mathrm{sh}},$$

or equivalently, $\bar{\rho}(\nabla_{\mathbf{x}}\bar{\Phi}^-) < \bar{\rho}(\nabla_{\mathbf{x}}\bar{\Phi}^+)$ along the flow direction, where $\mathbf{n}_{\mathrm{sh}}$ represents the unit normal vector to Γ_{sh} pointing from D^- towards D^+.

REMARK 2.2. By performing integration by parts, condition (ii) of Definition 2.1 implies that any entropy solution with a shock Γ_{shock} of problem (2.1.8)–(2.1.10) in the sense of Definition 2.1 satisfies the conormal boundary condition:

$$\bar{\rho}(|\nabla_{\mathbf{x}}\bar{\Phi}|^2)\nabla_{\mathbf{x}}\bar{\Phi} \cdot \mathbf{n}_{\mathrm{w}} = 0 \qquad \text{on } \partial W.$$

Furthermore, combining conditions (i) and (iii) of Definition 2.1 with the conormal boundary condition stated immediately above yields that the entropy solution $\bar{\Phi}$ indeed satisfies the boundary condition (2.1.10) if $\bar{\rho}(|\nabla_{\mathbf{x}}\bar{\Phi}|^2) > 0$ holds on ∂W.

In particular, Definition 2.1, via integration by parts, leads to the following Rankine-Hugoniot jump conditions for the steady potential flow equations (2.1.8):

$$(2.1.11) \qquad [\bar{\Phi}]_{\Gamma_{\mathrm{sh}}} = [\bar{\rho}(|\nabla_{\mathbf{x}}\bar{\Phi}|^2)\nabla\bar{\Phi} \cdot \mathbf{n}_{\mathrm{sh}}]_{\Gamma_{\mathrm{sh}}} = 0,$$

where $[F(\mathbf{x})]_{\Gamma_{\mathrm{sh}}} := F^+(\mathbf{x}) - F^-(\mathbf{x})$ for $\mathbf{x} \in \Gamma_{\mathrm{sh}}$.

DEFINITION 2.3 (The steady Prandtl-Meyer reflection solution). *The steady Prandtl-Meyer reflection solution* for potential flow is an entropy solution $\bar{\Phi}$ with a shock Γ_{sh} of the boundary value problem (2.1.8)–(2.1.10) in the sense of Definition 2.1 with the following additional features:

(i) $\Gamma_{\mathrm{sh}} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \setminus W : |x_2| = x_1 \tan\theta_{\mathrm{sh}}, x_1 \geq 0\}$ for some $\theta_{\mathrm{sh}} \in (\theta_{\mathrm{w}}, \frac{\pi}{2})$;

(ii) For some constants $u_0, v_0 > 0$,

$$\bar{\Phi}(\mathbf{x}) = \begin{cases} u_\infty x_1 & \text{in } D^- = \{\mathbf{x} \in D : x_1 < |x_2|\cot\theta_{\mathrm{sh}}\}, \\ u_0 x_1 + v_0 x_2 & \text{in } D^+ := D \setminus \overline{D^-}; \end{cases}$$

(iii) $\tan\theta_{\mathrm{sh}} = \frac{u_\infty - u_0}{v_0}$;

(iv) Entropy condition: for the unit normal vector $\mathbf{n}_{\mathrm{sh}}$ to Γ_{sh} pointing from D^- towards D^+,

$$\nabla\bar{\Phi}^- \cdot \mathbf{n}_{\mathrm{sh}} > \nabla_{\mathbf{x}}\bar{\Phi}^+ \cdot \mathbf{n}_{\mathrm{sh}} > 0 \qquad \text{on } \Gamma_{\mathrm{sh}},$$

or equivalently, $\bar{\rho}(|\nabla_{\mathbf{x}}\bar{\Phi}^-|^2) < \bar{\rho}(|\nabla_{\mathbf{x}}\bar{\Phi}^+|^2)$.

LEMMA 2.4. *Given any $\gamma \geq 1$ and (ρ_∞, u_∞) with $u_\infty > c_\infty = \rho_\infty^{(\gamma-1)/2} > 0$, there exist unique $\underline{u}^{(\rho_\infty, u_\infty)} \in (0, u_\infty)$ and $\theta_{\mathrm{d}}^{(\rho_\infty, u_\infty)} \in (0, \frac{\pi}{2})$ such that the following properties hold:*

(a) *For each $\theta_{\mathrm{w}} \in (0, \theta_{\mathrm{d}}^{(\rho_\infty, u_\infty)})$, there are exactly two constants u_{st} and u_{wk} with $\underline{u}^{(\rho_\infty, u_\infty)} < u_{\mathrm{st}} < u_{\mathrm{wk}} < u_\infty$ yielding two steady Prandtl-Meyer reflection configurations in the sense that, if $(u_0, v_0) = u_{\mathrm{st}}(1, \tan \theta_{\mathrm{w}})$ or $u_{\mathrm{wk}}(1, \tan \theta_{\mathrm{w}})$ in Definition 2.3, then the corresponding $\bar{\Phi}$ is an entropy solution of the boundary value problem (2.1.8)–(2.1.10) with shock Γ_{sh} given by Definition 2.3(i) with θ_{sh} being determined by Definition 2.3(iii);*

(b) *u_{st} and u_{wk} depend continuously on $(\rho_\infty, u_\infty, \gamma)$ and $\theta_{\mathrm{w}} \in (0, \theta_{\mathrm{d}}^{(\rho_\infty, u_\infty)})$, and $u_{\mathrm{st}} = u_{\mathrm{wk}}$ at $\theta_{\mathrm{w}} = \theta_{\mathrm{d}}^{(\rho_\infty, u_\infty)}$;*

(c) *For each $\theta_{\mathrm{w}} \in (0, \theta_{\mathrm{d}}^{(\rho_\infty, u_\infty)})$, let $u_{\mathrm{wk}}^{(\theta_{\mathrm{w}})}$ denote the value of u_{wk} corresponding to θ_{w}. Then there exists a unique $\theta_{\mathrm{s}}^{(\rho_\infty, u_\infty)} \in (0, \theta_{\mathrm{d}}^{(\rho_\infty, u_\infty)})$ such that*

$$|u_{\mathrm{wk}}^{(\theta_{\mathrm{s}}^{(\rho_\infty, u_\infty)})}|\,|(1, \tan \theta_{\mathrm{s}}^{(\rho_\infty, u_\infty)})| = \left(\bar{\rho}(|u_{\mathrm{wk}}^{(\theta_{\mathrm{s}}^{(\rho_\infty, u_\infty)})}|^2 |(1, \tan \theta_{\mathrm{s}}^{(\rho_\infty, u_\infty)})|^2) \right)^{(\gamma-1)/2}.$$

In other words, the flow behind the weak shock corresponding to $\theta_{\mathrm{s}}^{(\rho_\infty, u_\infty)}$ is sonic.
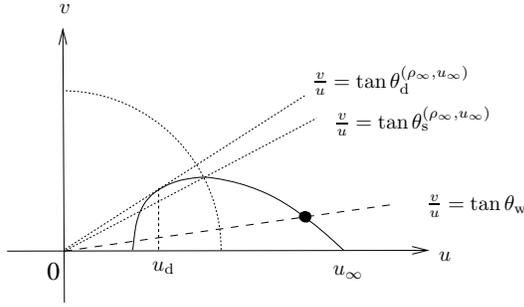
FIGURE 2.2. Shock polars in the (u, v)–plane

PROOF. (a) and (b) can be checked directly from Lemmas A.1 and A.3.

Define $q(\theta_{\mathrm{w}}) := |u_{\mathrm{wk}}^{(\theta_{\mathrm{w}})}|\,|(1, \tan \theta_{\mathrm{w}})|$. We first observe that $|q(\theta_{\mathrm{w}})|^2 = \left(\bar{\rho}(|q(\theta_{\mathrm{w}})|^2) \right)^{\gamma-1}$ if and only if $|q(\theta_{\mathrm{w}})|^2 = \frac{2}{\gamma+1}\left(1 + (\gamma - 1)B \right) =: \mathcal{K}_0$. To prove (c), it suffices to show that there exists a unique $\theta_* \in (0, \theta_{\mathrm{d}}^{(\rho_\infty, u_\infty)})$ satisfying $|q(\theta_*)|^2 = \mathcal{K}_0$.

Condition $u_\infty^2 > \rho_\infty^{\gamma-1}$ implies that $|q(0)|^2 > \mathcal{K}_0$. This can also be checked from the Bernoulli law (*i.e.*, $\frac{1}{2}|\nabla_{\mathbf{x}} \bar{\Phi}|^2 + h(\bar{\rho}) = B$) and the conservation law of mass (*i.e.*, $\bar{\rho}(u_{\mathrm{st}}^{(0)})u_{\mathrm{st}}^{(0)} = \rho_\infty u_\infty$ so that $|u_{\mathrm{st}}^{(0)}|^2 < \mathcal{K}_0$). Then there exists a unique point $P_* = u_*(1, \tan \theta_*)$ on the shock polar $\Upsilon^{(\rho_\infty, u_\infty)}$ satisfying $|P_*|^2 = \mathcal{K}_0$ (see Lemma A.3). It remains to verify that $u_* = u_{\mathrm{wk}}^{(\theta_*)}$; that is, P_* is the weak shock point corresponding to $\theta_* \in (0, \theta_{\mathrm{d}}^{(\rho_\infty, u_\infty)})$.

In Lemmas A.1 and A.3, it is shown that the shock polar curve $\Upsilon^{(\rho_\infty, u_\infty)}$, as shown in Fig. 2.2, is given as the zero-level curve of $g(\mathbf{u})$ in the first quadrant of the (u, v)–plane and that $\Upsilon^{(\rho_\infty, u_\infty)}$ is convex. Furthermore, $g_{\mathbf{u}}(\mathbf{u})$ is a normal vector to $\Upsilon^{(\rho_\infty, u_\infty)}$ at $\mathbf{u} \in \Upsilon^{(\rho_\infty, u_\infty)}$ towards the u–axis.

From this observation, we see that

$$g_{\mathbf{u}}(P_*) \cdot P_* > 0 \qquad \text{if and only if} \qquad u_* = u_{\text{st}}^{(\theta_*)},$$

$$g_{\mathbf{u}}(P_*) \cdot P_* = 0 \qquad \text{if and only if} \qquad \theta_* = \theta_{\text{d}}^{(\rho_\infty, u_\infty)},$$

$$g_{\mathbf{u}}(P_*) \cdot P_* < 0 \qquad \text{if and only if} \qquad u_* = u_{\text{wk}}^{(\theta_*)}.$$

Now we compute $g_{\mathbf{u}}(P_*) \cdot P_*$. A direct computation by using (A.7) gives that

$$g_{\mathbf{u}}(\mathbf{u}) = \frac{1}{\bar{\rho}^{\gamma-2}} \left(\bar{c}^2 \frac{\mathbf{u}_\infty - \mathbf{u}}{|\mathbf{u}_\infty - \mathbf{u}|} - \left(\mathbf{u} \cdot \frac{\mathbf{u}_\infty - \mathbf{u}}{|\mathbf{u}_\infty - \mathbf{u}|}\right)\mathbf{u} \right) - \frac{\bar{\rho}\mathbf{u} - \rho_\infty \mathbf{u}_\infty}{|\mathbf{u}_\infty - \mathbf{u}|},$$

where $\bar{\rho} = \bar{\rho}(|\mathbf{u}|^2)$, $\bar{c}^2 = \bar{\rho}^{\gamma-1}$, and $\mathbf{u}_\infty = (u_\infty, 0)$. Combining (2.1.11) with $|P_*|^2 = \mathcal{K}_0$ yields

$$g_{\mathbf{u}}(P_*) \cdot P_* = -\big(\bar{\rho}(|P_*|^2) - \rho_\infty\big)(P_* \cdot \boldsymbol{\tau}_{\text{s}})^2,$$

where $\boldsymbol{\tau}_{\text{s}}$ represents a unit tangent vector to shock S_0 corresponding to state P_*. Since $P_* \cdot \boldsymbol{\tau}_{\text{s}} \neq 0$, we obtain from the entropy condition $\bar{\rho}(P_*) - \rho_\infty > 0$ that $g_{\mathbf{u}}(P_*) \cdot P_* < 0$. From this, we conclude that $u_* = u_{\text{wk}}^{(\theta_*)}$. Choosing $\theta_{\text{s}}^{(\rho_\infty, u_\infty)} = \theta_*$, we complete the proof. $\square$

DEFINITION 2.5. Fix parameters $(\rho_\infty, u_\infty, \gamma, \theta_{\text{w}})$. In Lemma 2.4, $\bar{\Phi}$ with $(u_0, v_0) = u_{\text{st}}(1, \tan\theta_{\text{w}})$ is called a steady Prandtl-Meyer strong reflection solution, and $\bar{\Phi}$ with $(u_0, v_0) = u_{\text{wk}}(1, \tan\theta_{\text{w}})$ is called a steady Prandtl-Meyer weak reflection solution in the sense that

$$|(u_\infty, 0) - u_{\text{st}}(1, \tan\theta_{\text{w}})| > |(u_\infty, 0) - u_{\text{wk}}(1, \tan\theta_{\text{w}})| \qquad \text{for } 0 < \theta_{\text{w}} < \theta_{\text{d}}^{(\rho_\infty, u_\infty)};$$

that is, the shock strength of a steady Prandtl-Meyer weak reflection solution is weaker than the steady strong one.

The goal of this work is to prove the existence of global unsteady Prandtl-Meyer reflection configurations for unsteady potential flow, determined by Eq. (2.1.7), which converge to the steady Prandtl-Meyer weak reflection solution as t tends to infinity for all possible physical parameters $\gamma \geq 1$, $u_\infty > c_\infty$, and $\theta_{\text{w}} \in (0, \theta_{\text{d}}^{(\rho_\infty, u_\infty)})$. Therefore, we consider the following initial-boundary value problem for (2.1.7):

PROBLEM 2.6 (Initial-boundary value problem). Given $\gamma \geq 1$, fix (ρ_∞, u_∞) with $u_\infty > c_\infty$. For a fixed $\theta_{\text{w}} \in (0, \theta_{\text{d}}^{(\rho_\infty, u_\infty)})$, let W be given by (2.1.9). Find a global weak solution $\Phi \in W_{\text{loc}}^{1,\infty}(\mathbb{R}_+ \times (\mathbb{R}^2 \setminus W))$ of Eq. (2.1.7) with ρ determined by (2.1.6) and

$$(2.1.12) \qquad\qquad B = \frac{u_\infty^2}{2} + h(\rho_\infty)$$

so that Φ satisfies both the initial condition at $t = 0$:

$$(2.1.13) \qquad\qquad (\rho, \Phi)|_{t=0} = (\rho_\infty, u_\infty x_1) \qquad \text{for } (x_1, x_2) \in \mathbb{R}^2 \setminus W,$$

and the slip boundary condition along the wedge boundary ∂W:

$$(2.1.14) \qquad\qquad \nabla_{\mathbf{x}}\Phi \cdot \mathbf{n}_{\text{w}}|_{\partial W} = 0 \qquad \text{for } t > 0,$$

where $\mathbf{n}_{\text{w}}$ is the exterior unit normal vector to ∂W.

REMARK 2.7. In particular, we seek a solution $\Phi \in W_{\text{loc}}^{1,\infty}(\mathbb{R}_+ \times (\mathbb{R}^2 \setminus W))$ that converges to the steady Prandtl-Meyer weak reflection solution $\bar{\Phi}$ when t tends to infinity in the following sense: if $\bar{\Phi}$ is the steady Prandtl-Meyer weak reflection solution corresponding to the fixed parameters

$(\rho_\infty, u_\infty, \gamma, \theta_{\mathrm{w}})$ in the sense of Definition 2.5 with $\bar{\rho} = h^{-1}(B - \frac{1}{2}|\nabla\bar{\Phi}|^2)$, then, for any $R > 0$, Φ satisfies

$$(2.1.15) \qquad \lim_{t\to\infty} \left(\|\nabla_{\mathbf{x}}\Phi(t, \cdot) - \nabla_{\mathbf{x}}\bar{\Phi}\|_{L^1(B_R(\mathbf{0})\setminus W)} + \|\rho(t, \cdot) - \bar{\rho}\|_{L^1(B_R(\mathbf{0})\setminus W)} \right) = 0$$

for $\rho(t, \mathbf{x})$ given by (2.1.6).

The definition of a weak solution of Problem 2.6 is given as follows:

DEFINITION 2.8. A function $\Phi \in W_{\mathrm{loc}}^{1,\infty}(\mathbb{R}_+ \times (\mathbb{R}^2 \setminus W))$ is called a weak solution of Problem 2.6 if Φ satisfies the following properties:

(i) $B - \partial_t\Phi - \frac{1}{2}|\nabla_{\mathbf{x}}\Phi|^2 > h(0+)$ *a.e.* in $\mathbb{R}_+ \times (\mathbb{R}^2 \setminus W)$;

(ii) $(\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi), \rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi)|\nabla_{\mathbf{x}}\Phi|) \in \left(L_{\mathrm{loc}}^1(\mathbb{R}_+ \times (\mathbb{R}^2 \setminus W)) \right)^2$;

(iii) For every $\zeta \in C_{\mathrm{c}}^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$,

$$\int_0^\infty \int_{\mathbb{R}^2\setminus W} \left(\rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi)\partial_t\zeta + \rho(\partial_t\Phi, \nabla_{\mathbf{x}}\Phi)\nabla_{\mathbf{x}}\Phi \cdot \nabla_{\mathbf{x}}\zeta \right) \, \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\mathbb{R}^2\setminus W} \rho_\infty\zeta(0, \mathbf{x}) \, \mathrm{d}\mathbf{x} = 0.$$

Since the initial data (2.1.13) does not satisfy the boundary condition (2.1.14), a boundary layer is generated along the wedge boundary starting at $t = 0$, which is proved to form the Prandtl-Meyer reflection configuration in this monograph.

Notice that the initial-boundary value problem, Problem 2.6, is invariant under the scaling

$$(t, \mathbf{x}) \to (\alpha t, \alpha \mathbf{x}), \quad (\rho, \Phi) \to (\rho, \frac{\Phi}{\alpha}) \qquad \text{for } \alpha \neq 0,$$

in the sense that, if $(\rho, \Phi)(t, \mathbf{x})$ is a solution, then so is $(\tilde{\rho}, \tilde{\Phi})(t, \mathbf{x}) = (\rho, \frac{\Phi}{\alpha})(\alpha t, \alpha \mathbf{x})$. Based on this observation, we look for *self-similar solutions* of Problem 2.6 in the form

$$(2.1.16) \qquad \rho(t, \mathbf{x}) = \rho(\boldsymbol{\xi}), \quad \Phi(t, \mathbf{x}) = t\phi(\boldsymbol{\xi}) \qquad \text{with } \boldsymbol{\xi} = (\xi_1, \xi_2) = \frac{\mathbf{x}}{t} \text{ for } t > 0.$$

For such ϕ, introduce *the pseudo-potential function* φ given by

$$\varphi = \phi - \frac{1}{2}|\boldsymbol{\xi}|^2.$$

If Φ solves (2.1.7) with (2.1.6), then φ satisfies the following *Euler equations for the self-similar solutions*:

$$(2.1.17) \qquad\qquad \mathrm{div}(\rho D\varphi) + 2\rho = 0,$$

$$(2.1.18) \qquad\qquad \frac{1}{2}|D\varphi|^2 + \varphi + h(\rho) = B,$$

where the *divergence* div and *gradient* D are with respect to the self-similar variables $\boldsymbol{\xi} \in \mathbb{R}^2$. Solve (2.1.18) first for ρ and then substitute the result into (2.1.17) to obtain

$$(2.1.19) \qquad\qquad N(\varphi) := \mathrm{div}\left(\rho(|D\varphi|^2, \varphi)D\varphi \right) + 2\rho(|D\varphi|^2, \varphi) = 0$$

for

$$(2.1.20) \qquad \rho(|D\varphi|^2, \varphi) = \begin{cases} \left(1 + (\gamma - 1)(B - \frac{1}{2}|D\varphi|^2 - \varphi) \right)^{\frac{1}{\gamma-1}} & \text{if } \gamma > 1, \\ \exp(B - \frac{1}{2}|\nabla\varphi|^2 - \varphi) & \text{if } \gamma = 1. \end{cases}$$

Note that the Bernoulli constant B is given by (2.1.12).

The local sound speed $c = c(|D\varphi|^2, \varphi) > 0$ for the pseudo-steady potential flow equation (2.1.19) is given by

$$(2.1.21) \qquad c^2(|D\varphi|^2, \varphi) = 1 + (\gamma - 1)\big(B - \frac{1}{2}|D\varphi|^2 - \varphi\big).$$

Eq. (2.1.19) is a second-order nonlinear equation of mixed elliptic-hyperbolic type. It is elliptic if and only if

$$(2.1.22) \quad |D\varphi| < c(|D\varphi|^2, \varphi) \iff |D\varphi| < \sqrt{\frac{2}{\gamma + 1}\big(1 + (\gamma - 1)(B - \varphi)\big)} \quad \text{(pseudo-subsonic)},$$

and (2.1.19) is hyperbolic if and only if

$$|D\varphi| > c(|D\varphi|^2, \varphi) \iff |D\varphi| > \sqrt{\frac{2}{\gamma + 1}\big(1 + (\gamma - 1)(B - \varphi)\big)} \quad \text{(pseudo-supersonic)}.$$

In order to find a function $\varphi(\boldsymbol{\xi})$ such that $\Phi(t, \mathbf{x})$ with $\rho(t, \mathbf{x})$ given by (2.1.16) is a solution of Problem 2.6 satisfying (2.1.15), we make the following additional observations:

- (i) *Symmetric domain*: Since the solid wedge W is symmetric with respect to the axis $x_2 = 0$, it suffices to consider Problem 2.6 in the upper half-plane $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$. In the self-similar plane, define

$$(2.1.23) \qquad \mathfrak{D}_{\theta_{\mathrm{w}}} := \{\boldsymbol{\xi} \in \mathbb{R}^2 : \xi_2 > 0\} \setminus \{\boldsymbol{\xi} : \xi_2 \leq \xi_1 \tan\theta_{\mathrm{w}}, \xi_1 \geq 0\}.$$

    Then Problem 2.6 is reformulated as a boundary value problem in $\mathfrak{D}_{\theta_{\mathrm{w}}}$.

- (ii) *Initial condition*: For each $\mathbf{x} \in \mathbb{R}^2 \setminus (W \cup \{\mathbf{0}\})$, $|\boldsymbol{\xi}| = |\frac{\mathbf{x}}{t}| \to \infty$ as $t \to 0+$. This means that the initial condition (2.1.13) in Problem 2.6 becomes an asymptotic boundary condition in the self-similar variables.

- (iii) *Time-asymptotic limit*: For each $\mathbf{x} \in \mathbb{R}^2 \setminus W$, $|\boldsymbol{\xi}| = |\frac{\mathbf{x}}{t}| \to 0$ as $t \to \infty$. To find a global weak solution of Problem 2.6 satisfying (2.1.15), we seek a self-similar weak solution $\varphi(\boldsymbol{\xi})$ satisfying

$$\lim_{R \to 0+} \frac{1}{|B_R(\mathbf{0}) \cap \mathfrak{D}_{\theta_{\mathrm{w}}}|} \int_{B_R(\mathbf{0}) \cap \mathfrak{D}_{\theta_{\mathrm{w}}}} |\nabla_{\boldsymbol{\xi}}\varphi - \nabla_{\mathbf{x}}\bar{\Phi}| \, \mathrm{d}\boldsymbol{\xi} = 0,$$

    where $\bar{\Phi}$ is the steady Prandtl-Meyer weak reflection solution of problem (2.1.8)–(2.1.10), and $|B_R(\mathbf{0}) \cap \mathfrak{D}_{\theta_{\mathrm{w}}}|$ is the area of $B_R(\mathbf{0}) \cap \mathfrak{D}_{\theta_{\mathrm{w}}}$.

- (iv) *Constant density state*: If $\rho > 0$ is a constant in (2.1.17)–(2.1.18), then the corresponding pseudo-potential φ is given in the form

$$(2.1.24) \qquad \varphi(\boldsymbol{\xi}) = -\frac{1}{2}|\boldsymbol{\xi}|^2 + (u, v) \cdot \boldsymbol{\xi} + k$$

    for some constant state (u, v) and a constant k. In Problem 2.6, the initial state has a constant density $\rho_\infty > 0$ and a constant velocity $(u_\infty, 0)$. Then the corresponding pseudo-potential φ_∞ in the self-similar variables is given by

$$(2.1.25) \qquad \varphi_\infty = -\frac{1}{2}|\boldsymbol{\xi}|^2 + (u_\infty, 0) \cdot \boldsymbol{\xi} + k_\infty$$

    for a constant k_∞. It follows from (2.1.12) that $k_\infty = 0$.

Hereafter, we assume without loss of generality that $\rho_\infty = 1$, so that $c_\infty = 1$. This can be achieved by the scaling

$$\boldsymbol{\xi} \mapsto c_\infty \boldsymbol{\xi}, \qquad (\rho, \varphi, u_\infty) \to (\frac{\rho}{\rho_\infty}, \frac{\varphi}{c_\infty^2}, \frac{u_\infty}{c_\infty})$$

for any $\gamma \geq 1$.

Given $\gamma \geq 1$, $\rho_\infty = 1$, and $u_\infty > 1$, we now reformulate Problem 2.6 in the self-similar variables. Hereafter, we denote $(\theta_{\mathrm{d}}^{(\rho_\infty, u_\infty)}, \theta_{\mathrm{s}}^{(\rho_\infty, u_\infty)})$ by $(\theta_{\mathrm{d}}^{(u_\infty)}, \theta_{\mathrm{s}}^{(u_\infty)})$, since ρ_∞ is fixed as 1.

Taking into account the additional observations stated above, we reformulate Problem 2.6 as a boundary value problem in the self-similar variables.

PROBLEM 2.9 (Boundary value problem in the self-similar variables $\boldsymbol{\xi}$). Given $\gamma \geq 1$, $u_\infty > 1$, and $\theta_{\mathrm{w}} \in (0, \theta_{\mathrm{d}}^{(u_\infty)})$, find a weak solution $\varphi \in W^{1,\infty}(\mathfrak{D}_{\theta_{\mathrm{w}}})$ of Eq. (2.1.19) in $\mathfrak{D}_{\theta_{\mathrm{w}}}$ satisfying the following conditions:

(i) *Slip boundary condition on* Γ_{wedge}:

(2.1.26) $D\varphi \cdot \mathbf{n}_{\mathrm{w}} = 0$ on $\Gamma_{\mathrm{wedge}} = \{\boldsymbol{\xi} : \xi_2 = \xi_1 \tan\theta_{\mathrm{w}}, \xi_1 > 0\}$,

where $\mathbf{n}_{\mathrm{w}}$ represents the exterior unit normal vector to the wedge boundary Γ_{wedge};

(ii) *Time-asymptotic limit condition in the self-similar variables*:

(2.1.27) $\displaystyle \lim_{R \to 0+} \frac{1}{|B_R(\mathbf{0}) \cap \mathfrak{D}_{\theta_{\mathrm{w}}}|} \int_{B_R(\mathbf{0}) \cap \mathfrak{D}_{\theta_{\mathrm{w}}}} |\nabla_{\boldsymbol{\xi}} \varphi - \nabla_{\mathbf{x}} \bar{\Phi}| \, \mathrm{d}\boldsymbol{\xi} = 0$,

where $\bar{\Phi}$ is the steady Prandtl-Meyer weak reflection solution corresponding to θ_{w};

(iii) *Asymptotic boundary condition at infinity*: For each $\theta \in (\theta_{\mathrm{w}}, \pi]$,

(2.1.28) $\displaystyle \lim_{r \to \infty} \|\varphi - \varphi_\infty\|_{C(R_\theta \setminus B_r(\mathbf{0}))} = 0$

for each ray $R_\theta := \{\xi_1 = \xi_2 \cot\theta, \xi_2 > 0\}$; see Fig. 2.3.
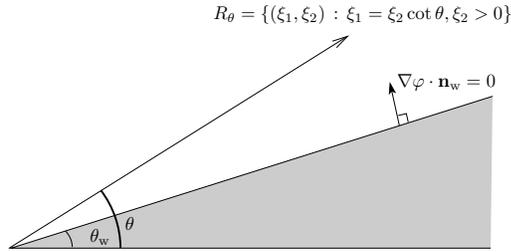
FIGURE 2.3. Asymptotic boundary condition at infinity

DEFINITION 2.10. A function $\varphi \in W^{1,1}_{\mathrm{loc}}(\mathfrak{D}_{\theta_{\mathrm{w}}})$ is called a weak solution of Problem 2.9 if φ satisfies conditions (i)–(iii) of Problem 2.9 and the following additional properties:

(i) $\rho(|D\varphi|^2, \varphi) > 0$ *a.e. in* $\mathfrak{D}_{\theta_{\mathrm{w}}}$;

(ii) $(\rho(|D\varphi|^2, \varphi), \rho(|D\varphi|^2, \varphi)|D\varphi|) \in L^1_{\mathrm{loc}}(\mathfrak{D}_{\theta_{\mathrm{w}}})$;

(iii) For every $\zeta \in C_c^\infty(\mathbb{R}^2)$,

$$(2.1.29) \qquad \int_{\mathfrak{D}_{\theta_w}} \left(\rho(|D\varphi|^2, \varphi) D\varphi \cdot D\zeta - 2\rho(|D\varphi|^2, \varphi)\zeta \right) \, d\boldsymbol{\xi} = 0.$$

For $\rho > 0$, note that (2.1.26) is equivalent to the conormal boundary condition:

$$(2.1.30) \qquad\qquad\qquad \rho D\varphi \cdot \mathbf{n}_w = 0 \qquad \text{on } \Gamma_{\text{wedge}}.$$

Condition (ii) of Problem 2.9 indicates that a solution of Problem 2.9 converges to a steady potential flow with a shock near the wedge vertex. To find such a solution, we define an entropy solution of Problem 2.9 with a shock. The definition is given in a way similar to Definition 2.1.

DEFINITION 2.11. Let Γ_{sh} be a C^1–curve that lies in $\mathfrak{D}_{\theta_w}$ and divides $\mathfrak{D}_{\theta_w}$ into two subdomains: $\mathfrak{D}_{\theta_w}^-$ and $\mathfrak{D}_{\theta_w}^+$. A weak solution φ of Problem 2.9 is an *entropy solution* with a shock Γ_{sh} if φ satisfies the following properties:

(i) $\varphi \in W_{\text{loc}}^{1,\infty}(\mathfrak{D}_{\theta_w})$;

(ii) $\varphi \in C_{\text{loc}}^1(\overline{\mathfrak{D}_{\theta_w}^\pm}) \cap C^2(\mathfrak{D}_{\theta_w}^\pm)$;

(iii) For $\varphi^+ := \varphi|_{\mathfrak{D}_{\theta_w}^+ \cup \Gamma_{\text{sh}}}$ and $\varphi^- := \varphi|_{\mathfrak{D}_{\theta_w}^- \cup \Gamma_{\text{sh}}}$,

$$\partial_{\mathbf{n}_{\text{sh}}} \varphi^- > \partial_{\mathbf{n}_{\text{sh}}} \varphi^+ > 0 \qquad \text{on } \Gamma_{\text{sh}},$$

where $\mathbf{n}_{\text{sh}}$ represents a unit normal vector to Γ_{sh} pointing from $\mathfrak{D}_{\theta_w}^-$ towards $\mathfrak{D}_{\theta_w}^+$;

(iv) φ satisfies the Rankine-Hugoniot jump conditions on Γ_{sh}:

$$(2.1.31) \qquad\qquad [\varphi]_{\Gamma_{\text{sh}}} = [\rho(|D\varphi|^2, \varphi) D\varphi \cdot \mathbf{n}_{\text{sh}}]_{\Gamma_{\text{sh}}} = 0,$$

which is similar to the steady case (2.1.8).

If $\mathbf{n}_{\text{sh}} = \frac{D\varphi^- - D\varphi^+}{|D\varphi^- - D\varphi^+|}$ is oriented so that $\partial_{\mathbf{n}_{\text{sh}}} \varphi^\pm > 0$, and if $\partial_{\mathbf{n}_{\text{sh}}} \varphi^- > \partial_{\mathbf{n}_{\text{sh}}} \varphi^+$ holds on Γ_{sh}, the shock solution is said to satisfy the entropy condition. By (2.1.31), the entropy condition is equivalent to $\rho(|D\varphi^-|^2, \varphi^-) < \rho(|D\varphi^+|^2, \varphi^+)$ on Γ_{sh}.

2.2. Structure of Solutions of Problem 2.9

Given $\gamma \geq 1$, $\rho_\infty = 1$, and $u_\infty > 1$, fix $\theta_w \in (0, \theta_d^{(u_\infty)})$.

2.2.1. Near the origin. We seek a solution φ of Problem 2.9 so that the solution at the origin coincides with the steady Prandtl-Meyer weak reflection solution corresponding to parameters $(1, u_\infty, \gamma, \theta_w)$ in the sense of Definition 2.5. For φ_∞ given by (2.1.25), define

$$(2.2.1) \qquad \varphi_0 = -\tfrac{1}{2}|\boldsymbol{\xi}|^2 + (u_0, v_0) \cdot \boldsymbol{\xi}, \qquad S_0 = \{\boldsymbol{\xi} \in \mathfrak{D}_{\theta_w} : \varphi_0(\boldsymbol{\xi}) = \varphi_\infty(\boldsymbol{\xi})\}.$$

Choose the constant vector (u_0, v_0) as

$$(2.2.2) \qquad\qquad\qquad (u_0, v_0) = u_{\text{wk}}^{(\theta_w)}(1, \tan\theta_w),$$

and define

$$\bar{\varphi}(\boldsymbol{\xi}) := \max\{\varphi_\infty(\boldsymbol{\xi}), \varphi_0(\boldsymbol{\xi})\}.$$

Then $\varphi := \bar{\varphi}$ satisfies (2.1.26)–(2.1.27) and (2.1.31) with $\Gamma_{\text{shock}} = S_0$.

For the nonlinear differential operator N defined by (2.1.19), equation $N(\varphi_0) = 0$ introduces *the pseudo-sonic circle* $\partial B_{c_0}(u_0, v_0)$ with $c_0^2 = \rho_0^{\gamma-1}$ for $\rho_0 = \rho(|D\varphi_0|^2, \varphi_0)$ in the following-sense:

- $N(\varphi_0) = 0$ is elliptic in $B_{c_0}(u_0, v_0)$,

- $N(\varphi_0) = 0$ is hyperbolic in $\mathbb{R}^2 \setminus \overline{B_{c_0}(u_0, v_0)}$.

REMARK 2.12. Let $\theta_{\mathrm{s}}^{(u_\infty)}$ be from Lemma 2.4(c). Then wedge vertex $O = (0, 0)$ satisfies the following:

- $O \in \mathbb{R}^2 \setminus \overline{B_{c_0}(u_0, v_0)}$ for $0 < \theta_{\mathrm{w}} < \theta_{\mathrm{s}}^{(u_\infty)}$,
- $O \in \partial B_{c_0}(u_0, v_0)$ at $\theta_{\mathrm{w}} = \theta_{\mathrm{s}}^{(u_\infty)}$,
- $O \in B_{c_0}(u_0, v_0)$ for $\theta_{\mathrm{s}}^{(u_\infty)} < \theta_{\mathrm{w}} < \theta_{\mathrm{d}}^{(u_\infty)}$.

2.2.2. Away from the origin. To determine a solution φ of Problem 2.9, we look for a solution φ with a piecewise constant density $\rho(|D\varphi|^2, \varphi)$, defined by (2.1.20) in $\mathfrak{D}_{\theta_{\mathrm{w}}} \setminus B_R(O)$ for some sufficiently large $R > 0$, so that such a solution φ satisfies the asymptotic boundary condition (iii) of Problem 2.9. For this purpose, we introduce a straight shock solution in $\mathfrak{D}_{\theta_{\mathrm{w}}} \setminus B_R(O)$. In fact, the only straight shock solution that satisfies (2.1.28) is a normal shock solution. This can be seen more clearly in §2.4. We now compute the normal shock solution and discuss its useful properties.

To compute the normal shock, denoted by S_1, and the corresponding pseudo-potential φ_1 below S_1, it is convenient to rotate the self-similar plane by angle θ_{w} clockwise. In the rotated self-similar
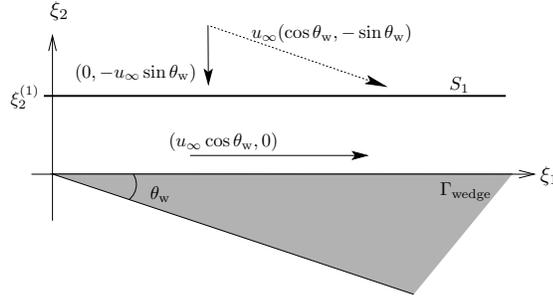
FIGURE 2.4. The normal shock

plane, φ_∞ in (2.1.25) is written as

$$\varphi_\infty = -\frac{1}{2}|\boldsymbol{\xi}|^2 + u_\infty(\cos\theta_{\mathrm{w}}, -\sin\theta_{\mathrm{w}}) \cdot \boldsymbol{\xi}.$$

Then φ_1 is in the form

$$\varphi_1 = -\frac{1}{2}|\boldsymbol{\xi}|^2 + u_\infty(\cos\theta_{\mathrm{w}}, -\sin\theta_{\mathrm{w}}) \cdot (\xi_1, \xi_2^{(1)}),$$

where $\xi_2^{(1)}$ is the distance of S_1 from Γ_{wedge}. Denote

$$(2.2.3) \qquad\qquad v_\infty := u_\infty \sin\theta_{\mathrm{w}}.$$

It follows from (2.1.20)–(2.1.31) that density ρ_1 and distance $\xi_2^{(1)}$ satisfy

$$(2.2.4) \qquad\qquad \xi_2^{(1)} = \frac{v_\infty}{\rho_1 - 1},$$

$$(2.2.5) \qquad\qquad h(\rho_1) - h(1) = \frac{1}{2}v_\infty^2 + \xi_2^{(1)} v_\infty,$$

where $h(\rho)$ is defined by (2.1.5).

Consider

$$F(\rho) := \big(h(\rho) - h(1)\big)(\rho - 1) - \frac{1}{2}(\rho - 1)v_\infty^2 - v_\infty^2.$$

A direct computation shows that $F(1) = -v_\infty^2 < 0$, $\lim_{\rho \to \infty} F(\rho) = \infty$, $F'(1) = -\frac{1}{2}v_\infty^2 < 0$, and $F''(\rho) > 0$ whenever $\rho \geq 1$. This implies that there exists a unique $\rho_1 \in (1, \infty)$ such that $F(\rho_1) = 0$. Then (2.2.4) yields that $\xi_2^{(1)} > 0$. Rotating the self-similar plane back by angle θ_{w} counterclockwise, we find that φ_1 is given by

$$(2.2.6) \qquad \varphi_1 = -\frac{1}{2}|\boldsymbol{\xi}|^2 + u_\infty \cos\theta_{\mathrm{w}}(\cos\theta_{\mathrm{w}}, \sin\theta_{\mathrm{w}}) \cdot \boldsymbol{\xi} - u_\infty \xi_2^{(1)} \sin\theta_{\mathrm{w}},$$

and the normal shock S_1 by

$$S_1 = \{\boldsymbol{\xi} : \varphi_\infty(\boldsymbol{\xi}) = \varphi_1(\boldsymbol{\xi})\} = \{\boldsymbol{\xi} : \xi_2 = \xi_1 \tan\theta_{\mathrm{w}} + \xi_2^{(1)} \sec\theta_{\mathrm{w}}\}.$$

LEMMA 2.13. *For any given $u_\infty > 1$ and the wedge angle $\theta_{\mathrm{w}} \in (0, \theta_{\mathrm{d}}^{(u_\infty)})$,*

$$\mathrm{dist}(S_1, \Gamma_{\mathrm{wedge}}) < c_1 := \rho_1^{(\gamma-1)/2}.$$

PROOF. By the mean value theorem, there exists a constant $\rho_* \in (1, \rho_1)$ satisfying

$$h(\rho_1) - h(1) = \mu(\rho_1 - 1) \qquad \text{for } \mu = \rho_*^{\gamma-2}.$$

Then $F(\rho_1) = 0$ implies that

$$\mu(\rho_1 - 1)^2 - \frac{1}{2}v_\infty^2(\rho_1 - 1) - v_\infty^2 = 0 \quad \Longrightarrow \quad \rho_1 - 1 = \frac{\frac{1}{2}v_\infty^2 + \sqrt{v_\infty^2(\frac{1}{4}v_\infty^2 + 4\mu)}}{2\mu}.$$

Since $v_\infty > 0$, (2.2.4) yields that

$$\xi_2^{(1)} = \frac{4\mu}{\sqrt{16\mu + v_\infty^2} + v_\infty} \leq \sqrt{\mu}.$$

By the definition of μ above, it can directly be checked that

$$\sqrt{\mu} < \begin{cases} \sqrt{\rho_1^{\gamma-2}} < \sqrt{\rho_1^{\gamma-1}} = c_1 & \text{if } \gamma \geq 2, \\ 1 < \sqrt{\rho_1^{\gamma-1}} = c_1 & \text{if } 1 < \gamma < 2, \\ 1 = c_1 & \text{if } \gamma = 1, \end{cases}$$

which implies that $\xi_2^{(1)} < c_1$. $\square$

2.2.3. Global configurations of the solutions of Problem 2.9. Following Remark 2.12, our desired solution of Problem 2.9 has two different configurations depending on the two different intervals of the wedge angle: $\theta_{\mathrm{w}} \in (0, \theta_{\mathrm{s}}^{(u_\infty)})$ and $\theta_{\mathrm{w}} \in [\theta_{\mathrm{s}}^{(u_\infty)}, \theta_{\mathrm{d}}^{(u_\infty)})$.

Case I. Fix $\theta_{\mathrm{w}} \in (0, \theta_{\mathrm{s}}^{(u_\infty)})$. Let φ_0 and φ_1 be defined by (2.2.1) and (2.2.6), respectively. Define $Q_0 := D\varphi_0(O)$ and $Q_1 := D\varphi_1(O)$. Consider two sonic circles $\partial B_{c_0}(Q_0)$ and $\partial B_{c_1}(Q_1)$.

The left sonic arc: The sonic circle $\partial B_{c_0}(Q_0)$ and the straight oblique shock $S_0 := \{\boldsymbol{\xi} : \varphi_0(\boldsymbol{\xi}) = \varphi_\infty(\boldsymbol{\xi})\}$ intersect at two points in $\mathfrak{D}_{\theta_{\mathrm{w}}}$, which will be verified in detail in §2.4. Let P_1 be the intersection with the smaller ξ_2–coordinate. Also, $\partial B_{c_0}(Q_0)$ intersects with Γ_{wedge} at two points;

FIGURE 2.5. Admissible solutions for $0 < \theta_w < \theta_s^{(u_\infty)}$

let P_4 be the intersection point with the smaller ξ_2–coordinate. Denote $\omega_0 := \angle P_4 Q_0 P_1 \in (0, \pi)$. We define

$$\Gamma^0_{\text{sonic}} := \{P \in \partial B_{c_0}(Q_0) \,:\, 0 \leq \angle P_4 Q_0 P \leq \omega_0\},$$

which is a closed subset of $\partial B_{c_0}(Q_0)$. We call Γ^0_{sonic} *the sonic arc corresponding to φ_0*.

The right sonic arc: By Lemma 2.13, the sonic circle $\partial B_{c_1}(Q_1)$ and the normal shock $S_1 = \{\boldsymbol{\xi} \,:\, \varphi_1(\boldsymbol{\xi}) = \varphi_\infty(\boldsymbol{\xi})\}$ intersect at two distinct points; let P_2 be the intersection point with the larger ξ_2–coordinate. Also, $\partial B_{c_1}(Q_1)$ intersects with Γ_{wedge} at two distinct points; let P_3 be the intersection point with the larger ξ_2–coordinate. Denote $\omega_1 := \angle P_3 Q_1 P_2 \in (0, \pi)$. We define

$$\Gamma^1_{\text{sonic}} := \{P' \in \partial B_{c_1}(Q_1) \,:\, 0 \leq \angle P_3 Q_1 P' \leq \omega_1\},$$

which is a closed subset of $\partial B_{c_1}(Q_1)$, similar to Γ^0_{sonic}. We call Γ^1_{sonic} *the sonic arc corresponding to φ_1*.

For each $j = 1, \cdots, 4$, let $\boldsymbol{\xi}^{P_j} = (\xi_1^{P_j}, \xi_2^{P_j})$ denote the $\boldsymbol{\xi}$–coordinate of point P_j. Let $S_{0,\text{seg}}$ be the line segment $\overline{OP_1}$, and let $\Omega_0 \subset \mathfrak{D}_{\theta_w}$ be the open set enclosed by $S_{0,\text{seg}}$, Γ^0_{sonic}, and the line segment $\overline{OP_4}$. Next, let $S_{1,\text{seg}}$ be the portion of S_1 with the left endpoint P_2, and let $\Omega_1 \subset \mathfrak{D}_{\theta_w}$ be the unbounded open set enclosed by $S_{1,\text{seg}}$, Γ^1_{sonic}, and $\Gamma_{\text{wedge}} \cap \{\xi_2 \geq \xi_2^{P_3}\}$.

Our goal is to find a curved shock Γ_{shock} that connects P_1 with P_2 and a solution φ of Problem 2.9 to satisfy both (2.1.22) in the open region Ω (enclosed by Γ_{shock}, Γ^1_{sonic}, $\overline{P_4 P_3}$, and Γ^0_{sonic}) and

$$\varphi = \begin{cases} \varphi_0 & \text{in } \Omega_0, \\ \varphi_1 & \text{in } \Omega_1, \\ \varphi_\infty & \text{in } \mathfrak{D}_{\theta_w} \setminus \overline{\Omega_0 \cup \Omega \cup \Omega_1}. \end{cases}$$

Problem 2.9 is now a free boundary problem given in a bounded region Ω with a free boundary Γ_{shock} to be determined simultaneously with φ.

Case 2. Fix $\theta_w \in [\theta_s^{(u_\infty)}, \theta_d^{(u_\infty)})$. The right sonic arc Γ^1_{sonic} is given in the same way as **Case 1**. By Remark 2.12, since the triangular region Ω_0 in Fig. 2.5 shrinks to the origin as $\theta_w \in (0, \theta_s^{(u_\infty)})$ increases up to $\theta_s^{(u_\infty)}$, we look for a curved shock Γ_{shock} that connects origin O with P_2 for $\theta_w \geq \theta_s^{(u_\infty)}$ and a solution φ to satisfy both (2.1.22) in the triangular domain Ω (enclosed by Γ_{shock}, Γ^1_{sonic}, and the line segment $\overline{OP_3}$) and

$$\varphi = \begin{cases} \varphi_1 & \text{in } \Omega_1, \\ \varphi_\infty & \text{in } \mathfrak{D}_{\theta_w} \setminus \overline{\Omega \cup \Omega_1}, \end{cases}$$

FIGURE 2.6. Admissible solutions for $\theta_{\mathrm{w}} \geq \theta_{\mathrm{s}}^{(u_\infty)}$

with

(2.2.7) $$\lim_{\substack{|P| \to 0 \\ P \in \Omega}} \varphi(P) = \varphi_0(O), \qquad \lim_{\substack{|P| \to 0 \\ P \in \Omega}} D\varphi(P) = D\varphi_0(O).$$

The condition that $\varphi = \varphi_0$ in Ω_0 for $\theta_{\mathrm{w}} < \theta_{\mathrm{s}}^{(u_\infty)}$ is replaced by (2.2.7) so that our desired solution still satisfies (2.1.27).

2.3. Main Theorems

Fix $\gamma \geq 1$ and $u_\infty > 1$. For each $\theta_{\mathrm{w}} \in (0, \theta_{\mathrm{d}}^{(u_\infty)})$, let u_0 be given by (2.2.2). By Lemmas A.1 and A.3, u_0 decreases with respect to θ_{w}. Define

$$u_{\mathrm{d}}^{(u_\infty)} := \lim_{\theta_{\mathrm{w}} \to \theta_{\mathrm{d}}^{(u_\infty)}-} u_0, \qquad u_{\mathrm{s}}^{(u_\infty)} := \lim_{\theta_{\mathrm{w}} \to \theta_{\mathrm{s}}^{(u_\infty)}} u_0.$$

For each $u_\infty > 1$, define an open interval $I^{(u_\infty)} = (u_N^{(u_\infty)}, u_\infty)$, where $u_N^{(u_\infty)}$ is from Lemma A.3. Given $\gamma \geq 1$, we introduce a set of parameters

$$\mathfrak{P} = \bigcup_{u_\infty > 1} \{u_\infty\} \times I^{(u_\infty)}.$$

Then $\mathfrak{P}$ consists of three disjoint sets $\mathfrak{P}_{\mathrm{weak}}$, $\mathfrak{P}_{\mathrm{detach}}$, and $\mathfrak{P}_{\mathrm{strong}}$:

(2.3.1)
$$\mathfrak{P}_{\mathrm{weak}} = \bigcup_{u_\infty > 1} \{u_\infty\} \times (u_{\mathrm{d}}^{(u_\infty)}, u_\infty),$$
$$\mathfrak{P}_{\mathrm{detach}} = \{(u_\infty, u_{\mathrm{d}}^{(u_\infty)}) : u_\infty > 1\},$$
$$\mathfrak{P}_{\mathrm{strong}} = \bigcup_{u_\infty > 1} \{u_\infty\} \times (u_N^{(u_\infty)}, u_{\mathrm{d}}^{(u_\infty)}).$$

Our goal is to prove the existence of a global weak solution of Problem 2.9, satisfying the entropy condition, for each $(u_\infty, u_0) \in \mathfrak{P}_{\mathrm{weak}}$ so that, if $\theta_{\mathrm{w}} < \theta_{\mathrm{s}}^{(u_\infty)}$, the solution has the configuration of Fig. 2.5 and, if $\theta_{\mathrm{w}} \geq \theta_{\mathrm{s}}^{(u_\infty)}$, the solution has the configuration of Fig. 2.6. We first give a definition of *admissible solutions* of Problem 2.9 for $(u_\infty, u_0) \in \mathfrak{P}_{\mathrm{weak}}$.

DEFINITION 2.14 (Admissible solutions). Given $\gamma \geq 1$, $u_\infty > 1$, and $(u_\infty, u_0) \in \mathfrak{P}_{\mathrm{weak}}$, define θ_{w} as

(2.3.2) $$\tan \theta_{\mathrm{w}} = \frac{f_{\mathrm{polar}}(u_0)}{u_0},$$

where f_{polar} is determined in Lemma A.3. Let $\mathfrak{D}_{\theta_{\text{w}}}$ be the domain defined by (2.1.23), and let φ_0 and φ_1 be defined by (2.2.1) and (2.2.6), respectively. A weak solution $\varphi \in C^{0,1}(\mathfrak{D}_{\theta_{\text{w}}})$ of Problem 2.9 is called *an admissible solution of Problem* 2.9 if φ satisfies the following properties:

Case I. $u_0 > u_{\text{s}}^{(u_\infty)}$, or equivalently, $\theta_{\text{w}} \in (0, \theta_{\text{s}}^{(u_\infty)})$:

  (i) There exists a shock curve Γ_{shock} with endpoints $P_1 = (\xi_1^{P_1}, \xi_2^{P_1})$ and $P_2 = (\xi_1^{P_2}, \xi_2^{P_2})$ such that the following properties hold:

    (i-1) Curve Γ_{shock} satisfies

$$(2.3.3) \qquad\qquad \Gamma_{\text{shock}} \subset \mathfrak{D}_{\theta_{\text{w}}} \setminus \overline{B_1(u_\infty, 0)},$$

      where $\partial B_1(u_\infty, 0)$ is the sonic circle of the state in $\Omega^\infty := \mathfrak{D}_{\theta_{\text{w}}} \setminus \overline{\Omega_0 \cup \Omega_1 \cup \Omega}$;

    (i-2) Curve Γ_{shock} is C^2 in its relative interior. That is, for any $P \in \Gamma_{\text{shock}} \setminus \{P_1, P_2\}$, there exist $r > 0$, $f \in C^2$, and an orthogonal coordinate system (S, T) in $\mathbb{R}^2$ such that $\Gamma_{\text{shock}} \cap B_r(P) = \{S = f(T)\} \cap B_r(P)$;

    (i-3) Curve $\overline{S_{0,\text{seg}} \cup \Gamma_{\text{shock}} \cup S_{1,\text{seg}}}$ is C^1, including at points P_1 and P_2;

    (i-4) $\Gamma_{\text{shock}}, \Gamma_{\text{sonic}}^1, \Gamma_{\text{sonic}}^0$, and $\Gamma_{\text{wedge}} := \{\xi_2 = \xi_1 \tan\theta_{\text{w}}, \xi_2 \geq 0\} \cap \{\boldsymbol{\xi} : \xi_1^{P_4} \leq \xi_1 \leq \xi_1^{P_3}\}$ do not have common points except for P_1, P_2, P_3, and P_4. Thus, $\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}^1 \cup \Gamma_{\text{sonic}}^0 \cup \Gamma_{\text{wedge}}$ is a closed curve without self-intersection. Denote by Ω the bounded domain enclosed by this closed curve.

  (ii) φ satisfies the following properties:

    (ii-1) $\varphi \in C_{\text{loc}}^{0,1}(\mathfrak{D}_{\theta_{\text{w}}}) \cap C_{\text{loc}}^1(\overline{\mathfrak{D}_{\theta_{\text{w}}}} \setminus \overline{S_{0,\text{seg}} \cup \Gamma_{\text{shock}} \cup S_{1,\text{seg}}})$;

    (ii-2) $\varphi \in C^3(\Omega) \cap C^2(\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}^0 \cup \Gamma_{\text{sonic}}^1})) \cap C^1(\overline{\Omega})$;

    (ii-3)

$$(2.3.4) \qquad\qquad \varphi = \begin{cases} \varphi_\infty & \text{in } \overline{\mathfrak{D}_{\theta_{\text{w}}}} \setminus \overline{\Omega_0 \cup \Omega \cup \Omega_1}, \\ \varphi_0 & \text{in } \Omega_0, \\ \varphi_1 & \text{in } \Omega_1, \end{cases}$$

      where $\overline{\Omega_0}$ shrinks to $\{O\} = \{P_1\} = \{P_4\}$ when $\theta_{\text{w}} = \theta_{\text{s}}^{(u_\infty)}$;

    (ii-4) φ satisfies

      - Eq. (2.1.19) in Ω with $\rho(|D\varphi|^2, \varphi)$ defined by (2.1.20),
      - the slip boundary condition: $\partial_{\xi_2}\varphi = 0$ on $\Gamma_{\text{wedge}} \cap \partial\Omega$,
      - the Rankine-Hugoniot conditions: $[\varphi]_{\Gamma_{\text{shock}}} = [\rho(|D\varphi|^2, \varphi)D\varphi \cdot \mathbf{n}_{\text{sh}}]_{\Gamma_{\text{shock}}} = 0$ for the unit normal vector $\mathbf{n}_{\text{sh}}$ to Γ_{shock} towards the interior of Ω.

  (iii) Eq. (2.1.19) is strictly elliptic in $\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}^0 \cup \Gamma_{\text{sonic}}^1})$; that is,

$$|D\varphi| < c(|D\varphi|^2, \varphi) \qquad \text{in } \overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}^0 \cup \Gamma_{\text{sonic}}^1}).$$

  (iv) $\max\{\varphi_0, \varphi_1\} \leq \varphi \leq \varphi_\infty$ in Ω.

  (v) Let $\boldsymbol{\tau}_{\text{w}} = (\cos\theta_{\text{w}}, \sin\theta_{\text{w}})$, which is tangential to the wedge boundary Γ_{wedge}. Let $\boldsymbol{e}_{S_0}$ be the unit vector parallel to S_0 and oriented so that $\boldsymbol{e}_{S_0} \cdot \boldsymbol{\tau}_{\text{w}} > 0$, and let $\boldsymbol{e}_{S_1}$ be the unit vector parallel to S_1 and oriented so that $\boldsymbol{e}_{S_1} \cdot \boldsymbol{\tau}_{\text{w}} < 0$:

$$\boldsymbol{e}_{S_0} = \frac{OP_1}{|OP_1|} = \frac{(v_0, u_\infty - u_0)}{\sqrt{(u_0 - u_\infty)^2 + v_0^2}}, \qquad \boldsymbol{e}_{S_1} = -(\cos\theta_{\text{w}}, \sin\theta_{\text{w}}).$$

Then
$$\partial_{\boldsymbol{e}_{S_0}}(\varphi_\infty - \varphi) \le 0, \quad \partial_{\boldsymbol{e}_{S_1}}(\varphi_\infty - \varphi) \le 0 \qquad \text{in } \Omega.$$

Case II. $u_0 \le u_{\mathrm{s}}^{(u_\infty)}$, or equivalently, $\theta_{\mathrm{w}} \in [\theta_{\mathrm{s}}^{(u_\infty)}, \theta_{\mathrm{d}}^{(u_\infty)})$:

(i) There exists a shock curve Γ_{shock} with endpoints $O = (0,0)$ and $P_2 = (\xi_1^{P_2}, \xi_2^{P_2})$ such that the following properties hold:

(i-1) Curve Γ_{shock} satisfies

(2.3.5) $$\Gamma_{\mathrm{shock}} \subset (\mathfrak{D}_{\theta_{\mathrm{w}}} \setminus \overline{B_1(u_\infty, 0)}),$$

where $\partial B_1(u_\infty, 0)$ is the sonic circle of the state in $\Omega^\infty := \mathfrak{D}_{\theta_{\mathrm{w}}} \setminus \overline{\Omega \cup \Omega_1}$;

(i-2) Curve Γ_{shock} is C^2 in its relative interior. That is, for any $P \in \Gamma_{\mathrm{shock}} \setminus \{O, P_2\}$, there exist $r > 0$, $f \in C^2$, and an orthogonal coordinate system (S, T) in $\mathbb{R}^2$ such that $\Gamma_{\mathrm{shock}} \cap B_r(P) = \{S = f(T)\} \cap B_r(P)$;

(i-3) Curve $\overline{\Gamma_{\mathrm{shock}} \cup S_{1,\mathrm{seg}}}$ is C^1, including at point P_2;

(i-4) $\Gamma_{\mathrm{shock}}, \Gamma_{\mathrm{sonic}}^1$, and $\Gamma_{\mathrm{wedge}} := \{\xi_2 = \xi_1 \tan\theta_{\mathrm{w}}, \xi_2 \ge 0\} \cap \{\boldsymbol{\xi} : 0 \le \xi_1 \le \xi_1^{P_3}\}$ do not have common points except for O, P_2, and P_3. Thus, $\Gamma_{\mathrm{shock}} \cup \Gamma_{\mathrm{sonic}}^1 \cup \Gamma_{\mathrm{wedge}}$ is a closed curve without self-intersection. Denote by Ω the bounded domain enclosed by this closed curve.

(ii) φ satisfies the following properties:

(ii-1) $\varphi \in C_{\mathrm{loc}}^{0,1}(\mathfrak{D}_{\theta_{\mathrm{w}}}) \cap C_{\mathrm{loc}}^1(\overline{\mathfrak{D}_{\theta_{\mathrm{w}}}} \setminus \overline{\Gamma_{\mathrm{shock}} \cup S_{1,\mathrm{seg}}})$;

(ii-2) $\varphi \in C^3(\Omega) \cap C^2(\overline{\Omega} \setminus (\{O\} \cup \overline{\Gamma_{\mathrm{sonic}}^1})) \cap C^1(\overline{\Omega})$;

(ii-3) $D\varphi(O) = D\varphi_0(O)$ and

(2.3.6) $$\varphi = \begin{cases} \varphi_\infty & \text{in } \overline{\mathfrak{D}_{\theta_{\mathrm{w}}}} \setminus \overline{\Omega \cup \Omega_1}, \\ \varphi_0 & \text{at } O, \\ \varphi_1 & \text{in } \Omega_1; \end{cases}$$

(ii-4) φ satisfies
  - Eq. (2.1.19) in Ω with $\rho(|D\varphi|^2, \varphi)$ defined by (2.1.20),
  - the slip boundary condition: $\partial_{\xi_2}\varphi = 0$ on $\Gamma_{\mathrm{wedge}} \cap \partial\Omega$,
  - the Rankine-Hugoniot conditions: $[\varphi]_{\Gamma_{\mathrm{shock}}} = [\rho(|D\varphi|^2, \varphi)D\varphi \cdot \mathbf{n}_{\mathrm{sh}}]_{\Gamma_{\mathrm{shock}}} = 0$ for the unit normal vector $\mathbf{n}_{\mathrm{sh}}$ to Γ_{shock} towards the interior of Ω.

(iii) Eq. (2.1.19) is strictly elliptic in $\overline{\Omega} \setminus (\{O\} \cup \overline{\Gamma_{\mathrm{sonic}}^1})$; that is,
$$|D\varphi| < c(|D\varphi|^2, \varphi) \qquad \text{in } \overline{\Omega} \setminus (\{O\} \cup \overline{\Gamma_{\mathrm{sonic}}^1}).$$

(iv) $\max\{\varphi_0, \varphi_1\} \le \varphi \le \varphi_\infty$ in Ω.

(v) Let $\boldsymbol{\tau}_{\mathrm{w}} = (\cos\theta_{\mathrm{w}}, \sin\theta_{\mathrm{w}})$, which is tangential to the wedge boundary Γ_{wedge}. Let $\boldsymbol{e}_{S_0}$ be the unit vector parallel to S_0 and oriented so that $\boldsymbol{e}_{S_0} \cdot \boldsymbol{\tau}_{\mathrm{w}} > 0$, and let $\boldsymbol{e}_{S_1}$ be the unit vector parallel to S_1 and oriented so that $\boldsymbol{e}_{S_1} \cdot \boldsymbol{\tau}_{\mathrm{w}} < 0$. Then
$$\partial_{\boldsymbol{e}_{S_1}}(\varphi_\infty - \varphi) \le 0, \quad \partial_{\boldsymbol{e}_{S_0}}(\varphi_\infty - \varphi) \le 0 \qquad \text{in } \Omega.$$

Our two main theorems are as follows:

THEOREM 2.15. *Fix $\gamma \ge 1$ and $u_\infty > 1$. For any $(u_\infty, u_0) \in \mathfrak{P}_{\mathrm{weak}}$, there exists an admissible solution of Problem 2.9 in the sense of Definition 2.14.*

THEOREM 2.16. *Fix $\gamma \geq 1$ and $u_\infty > 1$. Given $(u_\infty, u_0) \in \mathfrak{P}_{\text{weak}}$, let φ be an admissible solution with the curved shock Γ_{shock} of Problem 2.9 in the sense of Definition 2.14. Then the following properties hold:*

Case I. $u_0 > u_s^{(u_\infty)}$, *or equivalently, $\theta_{\text{w}} \in (0, \theta_s^{(u_\infty)})$:*

(a) *The curved shock Γ_{shock} is C^∞ in its relative interior, and $\varphi \in C^\infty(\overline{\Omega} \setminus (\overline{\Gamma^0_{\text{sonic}}} \cup \overline{\Gamma^1_{\text{sonic}}})) \cap C^{1,1}(\overline{\Omega})$.*

(b) *For a constant $\sigma > 0$ and a set $\mathcal{D}$ given by*

$$\mathcal{D} = \left\{ \boldsymbol{\xi} \,:\, \max\{\varphi_0(\boldsymbol{\xi}), \varphi_1(\boldsymbol{\xi})\} < \varphi_\infty(\boldsymbol{\xi}) \right\} \cap \mathfrak{D}_{\theta_{\text{w}}},$$

define

(2.3.7)
$$\begin{aligned}
\mathcal{D}^0_\sigma &= \mathcal{D} \cap \{ \boldsymbol{\xi} \,:\, \text{dist}\{\boldsymbol{\xi}, \Gamma^0_{\text{sonic}}\} < \sigma \} \cap B_{c_0}(Q_0), \\
\mathcal{D}^1_\sigma &= \mathcal{D} \cap \{ \boldsymbol{\xi} \,:\, \text{dist}\{\boldsymbol{\xi}, \Gamma^1_{\text{sonic}}\} < \sigma \} \cap B_{c_1}(Q_1),
\end{aligned}$$

where $c_j = \rho_j^{(\gamma-1)/2}$ and $Q_j = D\varphi_j(O)$, $j = 0, 1$. Then, for any $\alpha \in (0, 1)$ and any $\boldsymbol{\xi}_0 \in (\overline{\Gamma^0_{\text{sonic}}} \cup \overline{\Gamma^1_{\text{sonic}}}) \setminus \{P_1, P_2\}$, there exist ε_0 depending on (γ, u_∞), and $K < \infty$ depending on $(u_\infty, \gamma, \theta_{\text{w}}, \varepsilon_0, \alpha)$, $\|\varphi\|_{C^{1,1}(\Omega \cap (\mathcal{D}^{\mathcal{O}}_{\varepsilon_0} \cup \mathcal{D}^{\mathcal{N}}_{\varepsilon_0}))}$, and $d = \text{dist}\{\boldsymbol{\xi}_0, \Gamma_{\text{shock}}\}$ such that

(2.3.8)
$$\|\varphi\|_{2,\alpha, \overline{\Omega \cap B_{d/2}(\boldsymbol{\xi}_0) \cap (\mathcal{D}^1_{\varepsilon_0/2} \cup \mathcal{D}^0_{\varepsilon_0/2})}} \leq K.$$

(c) *For any $\boldsymbol{\xi}_0 \in \overline{\Gamma^0_{\text{sonic}} \cup \Gamma^1_{\text{sonic}}} \setminus \{P_1, P_2\}$,*

(2.3.9)
$$\lim_{\substack{\boldsymbol{\xi} \to \boldsymbol{\xi}_0 \\ \boldsymbol{\xi} \in \Omega}} \left(D_{rr}\varphi - D_{rr} \max\{\varphi_1, \varphi_0\} \right)(\boldsymbol{\xi}) = \frac{1}{\gamma + 1},$$

where $r = |\boldsymbol{\xi} - Q_1|$ near Γ^1_{sonic} and $r = |\boldsymbol{\xi} - Q_0|$ near Γ^0_{sonic}.

(d) *Limits $\lim_{\substack{\boldsymbol{\xi} \to P_1 \\ \boldsymbol{\xi} \in \Omega}} D^2\varphi$ and $\lim_{\substack{\boldsymbol{\xi} \to P_2 \\ \boldsymbol{\xi} \in \Omega}} D^2\varphi$ do not exist.*

(e) *$\overline{S_{0,\text{seg}} \cup \Gamma_{\text{shock}} \cup S_{1,\text{seg}}}$ is a $C^{2,\alpha}$–curve for any $\alpha \in (0, 1)$, including at points P_1 and P_2.*

Case II. $u_0 \leq u_s^{(u_\infty)}$, *or equivalently, $\theta_{\text{w}} \in [\theta_s^{(u_\infty)}, \theta_d^{(u_\infty)})$:*

(a) *The curved shock Γ_{shock} is C^∞ in its relative interior, and $\varphi \in C^\infty(\overline{\Omega} \setminus (\{O\} \cup \overline{\Gamma^1_{\text{sonic}}})) \cap C^{1,1}(\overline{\Omega} \setminus \{O\}) \cap C^{1,\bar{\alpha}}(\overline{\Omega})$ for some $\bar{\alpha} \in (0, 1)$ that depends on u_∞ and θ_{w} and is non-increasing with respect to θ_{w}.*

(b) *For a constant $\sigma > 0$, define $\mathcal{D}^1_\sigma$ by (2.3.7). Then, for any $\alpha \in (0, 1)$ and any $\boldsymbol{\xi}_0 \in \overline{\Gamma^1_{\text{sonic}}} \setminus \{P_2\}$, there exist ε_0 depending on (γ, u_∞), and $K < \infty$ depending on $(u_\infty, \gamma, \theta_{\text{w}}, \varepsilon_0, \alpha)$, $\|\varphi\|_{C^{1,1}(\Omega \cap \mathcal{D}^1_{\varepsilon_0})}$, and $d = \text{dist}\{\boldsymbol{\xi}_0, \Gamma_{\text{shock}}\}$ such that*

(2.3.10)
$$\|\varphi\|_{2,\alpha, \overline{\Omega \cap B_{d/2}(\boldsymbol{\xi}_0) \cap \mathcal{D}^1_{\varepsilon_0/2}}} \leq K.$$

(c) *For any $\boldsymbol{\xi}_0 \in \overline{\Gamma^1_{\text{sonic}}} \setminus \{P_2\}$,*

(2.3.11)
$$\lim_{\substack{\boldsymbol{\xi} \to \boldsymbol{\xi}_0 \\ \boldsymbol{\xi} \in \Omega}} \left(D_{rr}\varphi - D_{rr}\varphi_1 \right)(\boldsymbol{\xi}) = \frac{1}{\gamma + 1},$$

where $r = |\boldsymbol{\xi} - Q_1|$.

(d) *Limit* $\lim\limits_{\substack{\boldsymbol{\xi}\to P_2 \\ \boldsymbol{\xi}\in\Omega}} D^2\varphi$ *does not exist.*

(e) $\overline{\Gamma_{\text{shock}} \cup S_{1,\text{seg}}}$ *is a* $C^{1,\bar\alpha}$*–curve for the same* $\bar\alpha$ *as in statement* (a). *Furthermore, curve* $\overline{\Gamma_{\text{shock}} \cup S_{1,\text{seg}}} \setminus \{O\}$ *is* $C^{2,\alpha}$ *for any* $\alpha \in (0,1)$, *including at point* P_2.

2.4. Change of the Parameters and Basic Properties

2.4.1. Straight oblique shocks in the self-similar plane. Given a constant $v_\infty > 0$, denote

$$\varphi_\infty := -\frac{1}{2}|\boldsymbol{\xi}|^2 - v_\infty\xi_2. \tag{2.4.1}$$

LEMMA 2.17. *For any given* $\beta \in [0, \frac{\pi}{2})$, *there exists a unique pseudo-potential function*

$$\varphi_{\mathcal{O}} = -\frac{1}{2}|\boldsymbol{\xi}|^2 + (u_{\mathcal{O}}, v_{\mathcal{O}}) \cdot \boldsymbol{\xi} + k_{\mathcal{O}}$$

satisfying the following properties:

($\mathcal{O}_1$) $S_{\mathcal{O}} := \{\boldsymbol{\xi} \in \mathbb{R}^2 : \varphi_\infty(\boldsymbol{\xi}) = \varphi_{\mathcal{O}}(\boldsymbol{\xi})\}$ *forms a line of angle* β *with the* ξ_1*–axis, as shown in Fig.* 2.7;
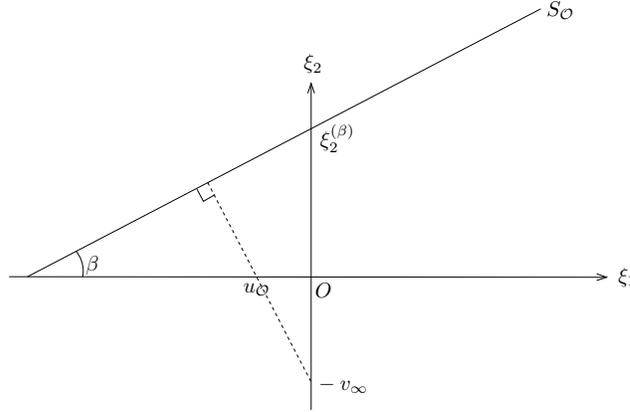
FIGURE 2.7. $S_{\mathcal{O}}$ is a line of angle β with the ξ_1–axis

($\mathcal{O}_2$) $\varphi_{\mathcal{O}}$ *satisfies the Rankine-Hugoniot conditions* (2.1.31) *on* $S_{\mathcal{O}}$:

$$\varphi_{\mathcal{O}} = \varphi_\infty, \quad \rho(|D\varphi_{\mathcal{O}}|^2, \varphi_{\mathcal{O}})D\varphi_{\mathcal{O}} \cdot \boldsymbol{\nu}_{\text{sh}} = D\varphi_\infty \cdot \boldsymbol{\nu}_{\text{sh}} \qquad \text{on } S_{\mathcal{O}}$$

for

$$\rho(|D\varphi|^2, \varphi) = \begin{cases} \left(1 + (\gamma-1)(\mathcal{B}_\infty - \frac{1}{2}|D\varphi|^2 - \varphi)\right)^{\frac{1}{\gamma-1}} & \text{for } \gamma > 1, \\ \exp\left(\mathcal{B}_\infty - \frac{1}{2}|D\varphi|^2 - \varphi\right) & \text{for } \gamma = 1, \end{cases} \tag{2.4.2}$$

with

$$\mathcal{B}_\infty = \frac{1}{2}|D\varphi_\infty|^2 + \varphi_\infty = \frac{v_\infty^2}{2},$$

where $\boldsymbol{\nu}_{\text{sh}} := \frac{D(\varphi_\infty - \varphi_0)}{|D(\varphi_\infty - \varphi_0)|}$;

($\mathcal{O}_3$) *Entropy condition*:

$$\rho(|D\varphi_{\mathcal{O}}|^2, \varphi_{\mathcal{O}}) > 1, \qquad 0 < D\varphi_{\mathcal{O}} \cdot \boldsymbol{\nu}_{\mathrm{sh}} < D\varphi_\infty \cdot \boldsymbol{\nu}_{\mathrm{sh}};$$

($\mathcal{O}_4$) $\varphi_{\mathcal{O}}$ *satisfies the slip boundary condition on the* ξ_1–*axis*:

$$\partial_{\xi_2}\varphi_{\mathcal{O}} = 0 \qquad on \ \{\xi_2 = 0\}.$$

PROOF. By choosing $(u_{\mathcal{O}}, v_{\mathcal{O}})$ as

(2.4.3) $(u_{\mathcal{O}}, v_{\mathcal{O}}) = (-v_\infty \tan\beta, 0),$

$\varphi_{\mathcal{O}}$ satisfies conditions ($\mathcal{O}_1$) and ($\mathcal{O}_4$). If line $S_{\mathcal{O}}$ has the ξ_2–intercept at $(0, \xi_2^{(\beta)})$, then $\varphi_{\mathcal{O}}$ can be written as

(2.4.4) $\varphi_{\mathcal{O}} = -\dfrac{1}{2}|\boldsymbol{\xi}|^2 - \xi_1 v_\infty \tan\beta - v_\infty \xi_2^{(\beta)}.$

It remains to find the ξ_2–intercept $\xi_2^{(\beta)}$ of $S_{\mathcal{O}}$ so that $\varphi_{\mathcal{O}}$ satisfies conditions ($\mathcal{O}_2$)–($\mathcal{O}_3$). Define

$$\rho_{\mathcal{O}} := \rho(|D\varphi_{\mathcal{O}}|^2, \varphi_{\mathcal{O}}).$$

Then $\rho_{\mathcal{O}}$ satisfies

(2.4.5) $h(\rho_{\mathcal{O}}) + \dfrac{1}{2}|D\varphi_{\mathcal{O}}|^2 + \varphi_{\mathcal{O}} = h(1) + \dfrac{1}{2}|D\varphi_\infty|^2 + \varphi_\infty,$

where $h(\rho)$ is defined by (2.1.4).

In order to determine $\xi_2^{(\beta)}$, we follow the idea from [**27**]. Define *the pseudo-Mach numbers* $M_{\mathcal{O}}$ and M_∞ by

(2.4.6) $M_{\mathcal{O}} := \dfrac{\partial_{\boldsymbol{\nu}_{\mathrm{sh}}}\varphi_{\mathcal{O}}}{c_{\mathcal{O}}} \qquad$ for $c_{\mathcal{O}} = \rho_{\mathcal{O}}^{\frac{\gamma-1}{2}}$ and $M_\infty := \partial_{\boldsymbol{\nu}_{\mathrm{sh}}}\varphi_\infty.$

Since $\partial_{\boldsymbol{\tau}_{\mathrm{sh}}}^k(\varphi_\infty - \varphi_{\mathcal{O}}) = 0$ on $S_{\mathcal{O}}$ for $k = 0, 1$, for a unit tangent vector $\boldsymbol{\tau}_{\mathrm{sh}}$ of $S_{\mathcal{O}}$, it follows from (2.4.5) that

(2.4.7) $h(\rho_{\mathcal{O}}) + \dfrac{1}{2}(\partial_{\boldsymbol{\nu}_{\mathrm{sh}}}\varphi_{\mathcal{O}})^2 = \underbrace{h(1)}_{(=0)} + \dfrac{1}{2}(\partial_{\boldsymbol{\nu}_{\mathrm{sh}}}\varphi_\infty)^2 \qquad$ on $S_{\mathcal{O}}.$

By (2.4.6), $\rho(|D\varphi_{\mathcal{O}}|^2, \varphi_{\mathcal{O}})D\varphi_{\mathcal{O}} \cdot \boldsymbol{\nu}_{\mathrm{sh}} = D\varphi_\infty \cdot \boldsymbol{\nu}_{\mathrm{sh}}$ can be rewritten as

(2.4.8) $\rho_{\mathcal{O}}^{\frac{\gamma+1}{2}} = \dfrac{M_\infty}{M_{\mathcal{O}}}.$

We substitute this expression into (2.4.7) to obtain

(2.4.9) $\left(1 + \dfrac{\gamma-1}{2}M_{\mathcal{O}}^2\right)M_{\mathcal{O}}^{-\frac{2(\gamma-1)}{\gamma+1}} = \left(1 + \dfrac{\gamma-1}{2}M_\infty^2\right)M_\infty^{-\frac{2(\gamma-1)}{\gamma+1}}.$

Notice that $f(M) := (1 + \frac{\gamma-1}{2}M^2)M^{-\frac{2(\gamma-1)}{\gamma+1}}$ satisfies

$$\lim_{M\to 0+} f(M) = \infty, \quad \lim_{M\to\infty} f(M) = \infty, \quad f'(M) = \dfrac{2(\gamma-1)}{\gamma+1}M^{-\frac{2(\gamma-1)}{\gamma+1}-1}(M^2 - 1).$$

Therefore, if $M_\infty = 1$, then $M_{\mathcal{O}} = 1$ is the only solution of (2.4.9). If $M_\infty \in (0, \infty) \setminus \{1\}$, then (2.4.9) has a unique nontrivial solution $M_{\mathcal{O}}$ in $(0, \infty) \setminus \{1\}$ with $M_{\mathcal{O}} \neq M_\infty$. Furthermore, a direct computation from (2.4.9) shows that

(2.4.10) $\dfrac{\mathrm{d}M_{\mathcal{O}}}{\mathrm{d}M_\infty} < 0 \qquad$ for all $M_\infty \in (0, \infty) \setminus \{1\}.$

It follows from (2.4.8) that conditions $(\mathcal{O}_2)$–$(\mathcal{O}_3)$ are satisfied if there exists $\xi_2^{(\beta)}$ so that $M_\infty > 1$ holds.

Denote $q_\infty := M_\infty$ and $q_{\mathcal{O}} := c_{\mathcal{O}} M_{\mathcal{O}}$. Note that $q_{\mathcal{O}} = \mathrm{dist}(S_{\mathcal{O}}, (u_{\mathcal{O}}, 0))$ and $q_\infty = \mathrm{dist}(S_{\mathcal{O}}, (0, -v_\infty))$ for $u_{\mathcal{O}}$ given by (2.4.3). Then

$$(2.4.11) \qquad q_\infty - q_{\mathcal{O}} = v_\infty \sec \beta.$$

We substitute the representations of $q_\infty = M_\infty$ and $q_{\mathcal{O}} = M_{\mathcal{O}} c_{\mathcal{O}} = M_{\mathcal{O}} \left(\frac{M_\infty}{M_{\mathcal{O}}} \right)^{\frac{\gamma-1}{\gamma+1}}$ into (2.4.11) to obtain

$$(2.4.12) \qquad M_\infty^{\frac{\gamma-1}{\gamma+1}} \left(M_\infty^{\frac{2\gamma}{\gamma+1}} - M_{\mathcal{O}}^{\frac{2\gamma}{\gamma+1}} \right) = v_\infty \sec \beta,$$

where $M_{\mathcal{O}} \le 1$ solves (2.4.9) for $M_\infty \ge 1$. As a function of $M_\infty \ge 1$, the left-hand side of (2.4.12) is strictly increasing for $M_\infty > 1$, and its value at $M_\infty = 1$ is 0. Therefore, for given constants $v_\infty > 0$ and $\beta \in [0, \frac{\pi}{2})$, there exists a unique

$$(2.4.13) \qquad M_\infty > 1$$

satisfying equation (2.4.12). Once $M_\infty > 1$ is decided, it follows from (2.4.3) and (2.4.11) that

$$(2.4.14) \qquad \xi_2^{(\beta)} = M_\infty \sec \beta - v_\infty.$$

It can be seen from $0 < D\varphi_{\mathcal{O}} \cdot \boldsymbol{\nu}_{\mathrm{sh}} < D\varphi_\infty \cdot \boldsymbol{\nu}_{\mathrm{sh}}$ that the ξ_2–intercept $\xi_2^{(\beta)}$ given by (2.4.14) satisfies

$$\xi_2^{(\beta)} > 0.$$

Case $\gamma = 1$ can be proved similarly. $\qquad\qquad\square$

2.4.2. New parameters (v_∞, β). We define $\boldsymbol{\xi}' = (\xi_1', \xi_2')$ by

$$(2.4.15) \qquad \begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} := \begin{pmatrix} \cos\theta_{\mathrm{w}} & \sin\theta_{\mathrm{w}} \\ -\sin\theta_{\mathrm{w}} & \cos\theta_{\mathrm{w}} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} - \begin{pmatrix} u_\infty \cos\theta_{\mathrm{w}} \\ 0 \end{pmatrix}.$$

In the new coordinates (ξ_1', ξ_2'), center Q_1 of the sonic circle $\partial B_{c_1}(Q_1)$ becomes the origin, and Γ_{wedge} lies on the horizontal axis $\xi_2' = 0$.

Hereafter, for simplicity of notation, we denote $\boldsymbol{\xi} = (\xi_1, \xi_2)$ as the new coordinates (ξ_1', ξ_2') given by (2.4.15). In the new coordinate system, φ_∞, φ_0, and φ_1, defined by (2.1.25), (2.2.1), and (2.2.6), are expressed respectively as

$$\varphi_\infty^{\mathrm{op}}(\boldsymbol{\xi}) = -\frac{1}{2} |\boldsymbol{\xi}|^2 - \xi_2 u_\infty \sin\theta_{\mathrm{w}} + \frac{1}{2} u_\infty^2 \cos^2\theta_{\mathrm{w}},$$

$$(2.4.16) \qquad \varphi_{\mathcal{O}}^{\mathrm{op}}(\boldsymbol{\xi}) = -\frac{1}{2} |\boldsymbol{\xi}|^2 + (\xi_1 + u_\infty \cos\theta_{\mathrm{w}})(u_0 \sec\theta_{\mathrm{w}} - u_\infty \cos\theta_{\mathrm{w}}) + \frac{1}{2} u_\infty^2 \cos^2\theta_{\mathrm{w}},$$

$$\varphi_{\mathcal{N}}^{\mathrm{op}}(\boldsymbol{\xi}) = -\frac{1}{2} |\boldsymbol{\xi}|^2 - u_\infty \xi_2^{(1)} \sin\theta_{\mathrm{w}} + \frac{1}{2} u_\infty^2 \cos^2\theta_{\mathrm{w}}.$$

We define $(\varphi_\infty, \varphi_{\mathcal{O}}, \varphi_{\mathcal{N}})$ in the new coordinates by

$$\varphi_\infty(\boldsymbol{\xi}) = \varphi_\infty^{\mathrm{op}}(\boldsymbol{\xi}) - \frac{1}{2} u_\infty^2 \cos^2\theta_{\mathrm{w}},$$

$$(2.4.17) \qquad \varphi_{\mathcal{O}}(\boldsymbol{\xi}) = \varphi_{\mathcal{O}}^{\mathrm{op}}(\boldsymbol{\xi}) - \frac{1}{2} u_\infty^2 \cos^2\theta_{\mathrm{w}},$$

$$\varphi_{\mathcal{N}}(\boldsymbol{\xi}) = \varphi_{\mathcal{N}}^{\mathrm{op}}(\boldsymbol{\xi}) - \frac{1}{2} u_\infty^2 \cos^2\theta_{\mathrm{w}}.$$

In the new coordinate system, $S_0, S_1, \Gamma^0_{\text{sonic}}$, and Γ^1_{sonic} are denoted as $S_{\mathcal{O}}, S_{\mathcal{N}}, \Gamma^{\mathcal{O}}_{\text{sonic}}$, and $\Gamma^{\mathcal{N}}_{\text{sonic}}$, respectively.

DEFINITION 2.18 (New parameters (v_∞, β)). For each $(u_\infty, u_0) \in \mathfrak{P}$, we introduce new parameters $(v_\infty, \beta) \in (0, \infty) \times (0, \frac{\pi}{2})$ as follows:

(i) For $\theta_{\text{w}} \in (0, \theta_{\text{d}}^{(u_\infty)})$ given by (2.3.2), define v_∞ by

$$v_\infty = u_\infty \sin \theta_{\text{w}};$$

(ii) Let S_0 be the straight oblique shock corresponding to point $u_0(1, \tan \theta_{\text{w}})$ on the shock polar (Fig. 2.2) with the incoming state $(u_\infty, 0)$. For such S_0, let θ_{S_0} be the angle of S_0 from the horizontal ground (*i.e.*, $\xi_2 = 0$ in the coordinates $\boldsymbol{\xi}$ before (2.4.15)). Define $\beta \in (0, \frac{\pi}{2})$ by

$$(2.4.18) \qquad\qquad\qquad \beta := \theta_{S_0} - \theta_{\text{w}}.$$

Note that the definition of v_∞ stated in (i) coincides with (2.2.3).

The weak shock configuration in the new self-similar plane is shown in Figs. 2.8–2.9 for $(v_\infty, \beta) \in (0, \infty) \times (0, \frac{\pi}{2})$.
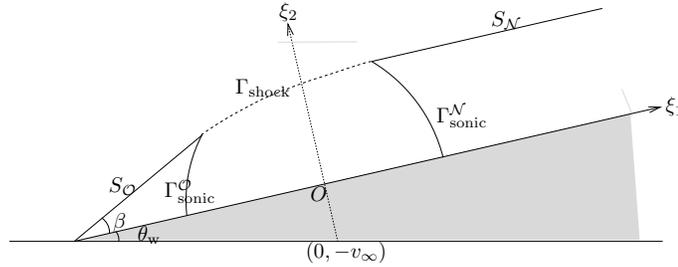
FIGURE 2.8. Weak shock solutions in the new self-similar plane when $\theta_{\text{w}} < \theta_{\text{s}}^{(u_\infty)}$
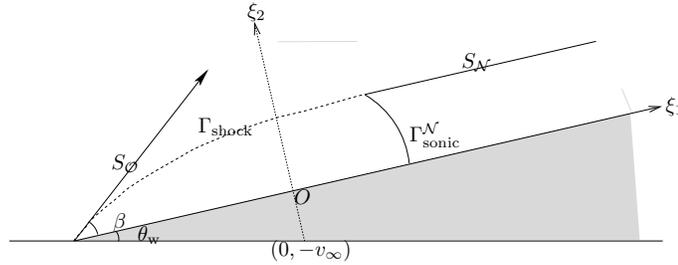
FIGURE 2.9. Weak shock solutions in the new self-similar plane when $\theta_{\text{s}}^{(u_\infty)} \le \theta_{\text{w}} < \theta_{\text{d}}^{(u_\infty)}$

We define a parameter set $\mathfrak{R}$ by

$$(2.4.19) \qquad\qquad \mathfrak{R} := \{(v_\infty, \beta) : v_\infty > 0, \ 0 < \beta < \frac{\pi}{2}\},$$

and define a mapping $\mathcal{T} : \mathfrak{P} \to \mathfrak{R}$ by

$$(2.4.20) \qquad \mathcal{T}(u_\infty, u_0) = (v_\infty, \beta) \qquad \text{for } (v_\infty, \beta) \text{ given by Definition 2.18.}$$

LEMMA 2.19. *For any given $\gamma \geq 1$, mapping $\mathcal{T} : \mathfrak{P} \to \mathfrak{R}$ is a homeomorphism.*

PROOF. Fix $(u_\infty, u_0) \in \mathfrak{P}$. By Definition 2.18(i), the corresponding half-wedge angle θ_w is given by

$$\theta_\mathrm{w} = \arctan\left(\frac{f_\mathrm{polar}(u_0)}{u_0}\right), \tag{2.4.21}$$

where f_polar is the function introduced in Lemma A.3.

By Definition 2.18(ii), a unit tangent vector $\boldsymbol{\tau}_{S_0}$ of the straight oblique shock S_0 corresponding to (u_∞, u_0) is $\boldsymbol{\tau}_{S_0} = (\cos\theta_{S_0}, \sin\theta_{S_0})$ in the coordinate system introduced right before transformation (2.4.15). Substituting this expression of $\boldsymbol{\tau}_{S_0}$ into one of the Rankine-Hugoniot conditions:

$$(u_\infty, 0) \cdot \boldsymbol{\tau}_{S_0} = (u_0, f_\mathrm{polar}(u_0)) \cdot \boldsymbol{\tau}_{S_0},$$

we have

$$\tan\theta_{S_0} = \frac{u_\infty - u_0}{f_\mathrm{polar}(u_0)}. \tag{2.4.22}$$

From (2.4.18) and (2.4.21)–(2.4.22), we obtain

$$\tan\beta = \frac{\tan\theta_{S_0} - \tan\theta_\mathrm{w}}{1 + \tan\theta_{S_0}\tan\theta_\mathrm{w}} = \frac{u_0(u_\infty - u_0) - \left(f_\mathrm{polar}(u_0)\right)^2}{u_0 f_\mathrm{polar}(u_0)} > 0.$$

By Definition 2.18(i) and (2.4.21), we can express v_∞ as

$$v_\infty = u_\infty \sin\left(\arctan\left(\frac{f_\mathrm{polar}(u_0)}{u_0}\right)\right).$$

Therefore, mapping $\mathcal{T} : \mathfrak{P} \to \mathfrak{R}$ is continuous.

In order to show that $\mathcal{T} : \mathfrak{P} \to \mathfrak{R}$ is invertible and its inverse is continuous, for fixed $(v_\infty, \beta) \in \mathfrak{R}$, we find a solution $(u_\infty, u_0) \in \mathfrak{P}$ of the following equations:

$$u_\infty \sin\theta_\mathrm{w} = v_\infty, \tag{2.4.23}$$

$$u_\infty \cos\theta_\mathrm{w} = \xi_2^{(\beta)} \cot\beta, \tag{2.4.24}$$

$$u_0 \sec\theta_\mathrm{w} = \xi_2^{(\beta)} \cot\beta - v_\infty \tan\beta, \tag{2.4.25}$$

so that the definitions of $\varphi_\mathcal{O}$ in (2.4.4) and (2.4.17) coincide. Combining (2.4.23) with (2.4.24), we have

$$u_\infty = \sqrt{v_\infty^2 + (\xi_2^{(\beta)})^2 \cot^2\beta} =: T_1(v_\infty, \beta). \tag{2.4.26}$$

Using (2.4.1), we can rewrite (2.4.26) as

$$u_\infty = |D\varphi_\infty(-\xi_2^{(\beta)}\cot\beta, 0)|.$$

Then we obtain from (2.4.13) that $u_\infty \geq M_\infty > 1$.

Once u_∞ is given by (2.4.26), we combine it with (2.4.24)–(2.4.25) to obtain u_0 as

$$u_0 = \frac{\left(\xi_2^{(\beta)}\cot\beta - v_\infty\tan\beta\right)\xi_2^{(\beta)}\cot\beta}{T_1(v_\infty, \beta)} =: T_2(v_\infty, \beta). \tag{2.4.27}$$

Note that $(-\xi_2^{(\beta)}\cot\beta, 0)$ is the ξ_1–intercept of line $S_\mathcal{O}$ from Lemma 2.17. Therefore, it can be seen from Fig. 2.7 that $\xi_2^{(\beta)}\cot\beta + u_\mathcal{O} = \xi_2^{(\beta)}\cot\beta - v_\infty\tan\beta > 0$. This implies that $u_0 > 0$.

Since $\tan\theta_{\mathrm{w}} = \frac{v_\infty}{\xi_2^{(\beta)}\cot\beta} > 0$ is obtained from (2.4.23)–(2.4.24), we conclude that (u_∞, u_0) given by (2.4.26)–(2.4.27) is contained in $\mathfrak{P}$.

Finally, the continuity of $\mathcal{T}^{-1}$ follows directly from the definitions of (T_1, T_2).   □

For any given $(v_\infty, \beta) \in \mathfrak{R}$, the ξ_2–intercept $\xi_2^{(\beta)} > 0$ of the oblique shock $S_\mathcal{O}$ of angle β from the ξ_1–axis is uniquely defined. Moreover, $\xi_2^{(\beta)}$ varies continuously on $\beta \in (0, \frac{\pi}{2})$, and $\lim_{\beta\to 0+}\xi_2^{(\beta)}$ exists, which is positive. Denote $\xi_2^\mathcal{N} := \xi_2^{(\beta)}|_{\beta=0}$. Let $\varphi_\mathcal{N}$ denote $\varphi_\mathcal{O}$ corresponding to $\beta = 0$. Then $\varphi_\mathcal{N}$ is given by

$$(2.4.28) \qquad \varphi_\mathcal{N}(\boldsymbol{\xi}) = -\frac{1}{2}|\boldsymbol{\xi}|^2 - v_\infty\xi_2^\mathcal{N}.$$

REMARK 2.20 (The normal shock: Case $\beta = 0$). For fixed $\gamma \geq 1$ and $v_\infty > 0$, the straight shock of angle $\beta = 0$ from the horizontal ground (*i.e.*, $\xi_2 = 0$ in the new coordinates $\boldsymbol{\xi}$ given by (2.4.15)) can be considered by taking the limit $\beta \to 0+$ in the argument above. The state of $\beta = 0$ is that of a normal shock, which corresponds to the state of $\frac{u_0}{u_\infty} = 1$ with $\theta_{\mathrm{w}} = 0$. Even though the case of $\beta = 0$ is not physical because $u_\infty = \infty$, we still put this case under our consideration as it is useful in applying the Leray-Schauder degree argument to prove the existence of admissible solutions of Problem 2.9 for all $(u_\infty, u_0) \in \mathfrak{P}_{\mathrm{weak}}$.

REMARK 2.21. According to Lemma A.4, for each $v_\infty > 0$, there exists $\beta_{\mathrm{d}}^{(v_\infty)} \in (0, \frac{\pi}{2})$ such that, if the parameter sets $\mathfrak{R}_{\mathrm{weak}}, \mathfrak{R}_{\mathrm{detach}}$, and $\mathfrak{R}_{\mathrm{strong}}$ are defined by

$$(2.4.29) \qquad \begin{aligned} \mathfrak{R}_{\mathrm{weak}} &= \bigcup_{v_\infty>0}\{v_\infty\} \times (0, \beta_{\mathrm{d}}^{(v_\infty)}), \\ \mathfrak{R}_{\mathrm{detach}} &= \bigcup_{v_\infty>0}\{v_\infty\} \times \{\beta_{\mathrm{d}}^{(v_\infty)}\}, \\ \mathfrak{R}_{\mathrm{strong}} &= \bigcup_{v_\infty>0}\{v_\infty\} \times (\beta_{\mathrm{d}}^{(v_\infty)}, \frac{\pi}{2}), \end{aligned}$$

then

$$(2.4.30) \qquad \mathcal{T}^{-1}(\mathfrak{R}_{\mathrm{weak}}) = \mathfrak{P}_{\mathrm{weak}}, \quad \mathcal{T}^{-1}(\mathfrak{R}_{\mathrm{detach}}) = \mathfrak{P}_{\mathrm{detach}}, \quad \mathcal{T}^{-1}(\mathfrak{R}_{\mathrm{strong}}) = \mathfrak{P}_{\mathrm{strong}},$$

for $\mathfrak{P}_{\mathrm{weak}}, \mathfrak{P}_{\mathrm{detach}}$, and $\mathfrak{P}_{\mathrm{strong}}$ defined by (2.3.1). In Lemma 2.22, we will also show that, for any $v_\infty > 0$, there exists a unique $\beta_{\mathrm{s}}^{(v_\infty)} \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$ such that $T_2(v_\infty, \beta) > u_{\mathrm{s}}^{(u_\infty)}$ if and only if $\beta < \beta_{\mathrm{s}}^{(v_\infty)}$ for $u_\infty = T_1(v_\infty, \beta)$, where $u_{\mathrm{s}}^{(u_\infty)}$ denotes the value of $u_{\mathrm{wk}}^{(\theta_{\mathrm{w}})}$ for $\theta_{\mathrm{w}} = \theta_{\mathrm{s}}^{(u_\infty)}$.

For fixed $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$, let $M_\mathcal{O}$ be defined by (2.4.6). In the proof of Lemma 2.17, it is shown that $0 < M_\mathcal{O} < 1$. This implies that the corresponding straight oblique shock $S_\mathcal{O}$ intersects with *the sonic circle* $\partial B_{c_\mathcal{O}}(u_\mathcal{O}, 0) = \{\boldsymbol{\xi} : |D\varphi_\mathcal{O}(\boldsymbol{\xi})| = c_\mathcal{O}\}$ at two distinct points. For each $\beta \in [0, \frac{\pi}{2})$, let $\boldsymbol{\xi}^\mathcal{O} := (\xi_1^\mathcal{O}, \xi_2^\mathcal{O})$ be the intersection point P_1 with the smaller ξ_1–coordinate (see Fig. 2.10). Moreover, let $(\xi_1^{(\beta)}, 0)$ be the ξ_1–intercept of $S_\mathcal{O}$. If $\xi_2^\mathcal{O} > 0$, then $|D\varphi_\mathcal{O}| > c_\mathcal{O}$ at $(\xi_1^{(\beta)}, 0)$, which means that an admissible solution in the sense of Definition 2.14 for $(u_\infty, u_0) = \mathcal{T}^{-1}(v_\infty, \beta)$ has the structure shown in Fig. 2.8. On the other hand, if $\xi_2^\mathcal{O} \leq 0$, then an admissible solution for $(u_\infty, u_0) = \mathcal{T}^{-1}(v_\infty, \beta)$ has the structure shown in Fig. 2.9.

LEMMA 2.22. *Fix $\gamma \geq 1$ and $v_\infty > 0$. The ξ_2–coordinate $\xi_2^\mathcal{O}$ of point P_1 satisfies*

$$\frac{\mathrm{d}\xi_2^\mathcal{O}}{\mathrm{d}\beta} < 0 \qquad \text{for all } \beta \in (0, \frac{\pi}{2}) \text{ and } \lim_{\beta\to\frac{\pi}{2}-}\xi_2^\mathcal{O} = -\infty.$$
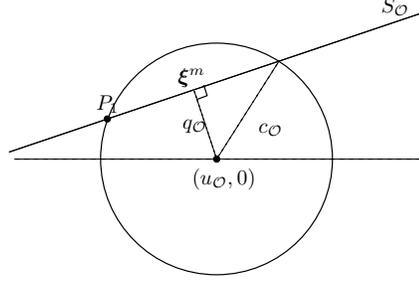
FIGURE 2.10. Two intersection points of $S_{\mathcal{O}}$ with the sonic circle $\partial B_{c_{\mathcal{O}}}(u_{\mathcal{O}},0)$

Therefore, there exists $\beta_{\mathrm{s}}^{(v_{\infty})} \in (0, \frac{\pi}{2})$ such that

$$(2.4.31) \qquad \begin{cases} \xi_2^{\mathcal{O}} > 0 \iff \frac{|D\varphi_{\mathcal{O}}(\xi_1^{(\beta)},0)|}{c_{\mathcal{O}}} > 1 & \text{for } \beta \in [0, \beta_{\mathrm{s}}^{(v_{\infty})}), \\ \xi_2^{\mathcal{O}} = 0 \iff \frac{|D\varphi_{\mathcal{O}}(\xi_1^{(\beta)},0)|}{c_{\mathcal{O}}} = 1 & \text{for } \beta = \beta_{\mathrm{s}}^{(v_{\infty})}, \\ \xi_2^{\mathcal{O}} < 0 \iff \frac{|D\varphi_{\mathcal{O}}(\xi_1^{(\beta)},0)|}{c_{\mathcal{O}}} < 1 & \text{for } \beta \in (\beta_{\mathrm{s}}^{(v_{\infty})}, \frac{\pi}{2}). \end{cases}$$

In addition, $\beta_{\mathrm{s}}^{(v_{\infty})}$ satisfies the inequality:

$$(2.4.32) \qquad \beta_{\mathrm{s}}^{(v_{\infty})} < \beta_{\mathrm{d}}^{(v_{\infty})}.$$

PROOF. For M_{∞} and $M_{\mathcal{O}}$ given by (2.4.6), define

$$(2.4.33) \qquad (q_{\infty}, q_{\mathcal{O}}) = (M_{\infty}, M_{\mathcal{O}}c_{\mathcal{O}}).$$

For each $\beta \in (0, \frac{\pi}{2})$, let $\boldsymbol{\xi}^m = (\xi_1^m, \xi_2^m)$ be the midpoint of two intersections of $S_{\mathcal{O}}$ with $\partial B_{c_{\mathcal{O}}}(u_{\mathcal{O}},0)$. By (2.4.6), we have

$$(2.4.34) \qquad \xi_2^{\mathcal{O}} = \xi_2^m - c_{\mathcal{O}}\sqrt{1 - M_{\mathcal{O}}^2}\sin\beta.$$

Since $(\xi_1^m - u_{\mathcal{O}}, \xi_2^m)$ is perpendicular to $S_{\mathcal{O}}$,

$$\partial_{\boldsymbol{\tau}_{\mathrm{sh}}}\varphi_{\mathcal{O}}(\boldsymbol{\xi}^m) = 0 = \partial_{\boldsymbol{\tau}_{\mathrm{sh}}}\varphi_{\infty}(\boldsymbol{\xi}^m) = (-\xi_1^m, -\xi_2^m - v_{\infty}) \cdot \boldsymbol{\tau}_{\mathrm{sh}}$$

for a unit tangent vector $\boldsymbol{\tau}_{\mathrm{sh}} = (\cos\beta, \sin\beta)$ to $S_{\mathcal{O}}$. Then we have

$$\boldsymbol{\xi}^m = (0, -v_{\infty}) - q_{\infty}\boldsymbol{\nu}_{\mathrm{sh}} = (0, -v_{\infty}) - q_{\infty}(\sin\beta, -\cos\beta)$$

for the unit normal vector $\boldsymbol{\nu}_{\mathrm{sh}}$ to $S_{\mathcal{O}}$ pointing towards the ξ_1–axis. This yields that

$$(2.4.35) \qquad \xi_2^m = -v_{\infty} + q_{\infty}\cos\beta.$$

We differentiate (2.4.11) and (2.4.35) with respect to β to obtain

$$(2.4.36) \qquad \frac{\mathrm{d}\xi_2^m}{\mathrm{d}\beta} = -q_{\infty}\sin\beta + \frac{\mathrm{d}q_{\infty}}{\mathrm{d}\beta}\cos\beta, \qquad \frac{\mathrm{d}q_{\infty}}{\mathrm{d}\beta} = \frac{q_{\infty} - q_{\mathcal{O}}}{1 - \frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}q_{\infty}}}\tan\beta,$$

and combine the results to obtain

$$(2.4.37) \qquad \frac{\mathrm{d}\xi_2^m}{\mathrm{d}\beta} = -\frac{1 - \frac{q_{\infty}}{q_{\mathcal{O}}}\frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}q_{\infty}}}{1 - \frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}q_{\infty}}}\xi_2^m\tan\beta.$$

If $\frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}q_\infty} \leq 0$, then

$$\frac{1 - \frac{q_\infty}{q_{\mathcal{O}}}\frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}q_\infty}}{1 - \frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}q_\infty}} > 1 \geq \frac{2}{\gamma + 1}.$$

A direct computation by using (2.4.6)–(2.4.9) shows that

$$
\begin{aligned}
(2.4.38) \qquad \frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}q_\infty} &= \Big(\frac{M_{\mathcal{O}}}{M_\infty}\Big)^{\frac{2}{\gamma+1}}\Big(\frac{\gamma-1}{\gamma+1} + \frac{2}{\gamma+1}\frac{M_\infty}{M_{\mathcal{O}}}\frac{\mathrm{d}M_{\mathcal{O}}}{\mathrm{d}M_\infty}\Big) \\
&= \Big(\frac{q_\infty}{\rho_{\mathcal{O}} q_{\mathcal{O}}}\Big)^{\frac{1}{\gamma+1}}\frac{q_{\mathcal{O}}}{q_\infty}\Big(\frac{\gamma-1}{\gamma+1} + \frac{2M_\infty}{(\gamma+1)M_{\mathcal{O}}}\frac{\mathrm{d}M_{\mathcal{O}}}{\mathrm{d}M_\infty}\Big) \\
&\leq \frac{\gamma-1}{\gamma+1}\frac{q_{\mathcal{O}}}{q_\infty}.
\end{aligned}
$$

If $\frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}q_\infty} > 0$, it follows from $0 < 1 - \frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}q_\infty} < 1$ that

$$\frac{1 - \frac{q_\infty}{q_{\mathcal{O}}}\frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}q_\infty}}{1 - \frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}q_\infty}} > 1 - \frac{q_\infty}{q_{\mathcal{O}}}\frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}q_\infty} \geq \frac{2}{\gamma + 1}.$$

We apply inequality $\dfrac{1 - \frac{q_\infty}{q_{\mathcal{O}}}\frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}q_\infty}}{1 - \frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}q_\infty}} > \dfrac{2}{\gamma + 1}$ to derive from (2.4.37) that

$$(2.4.39) \qquad \frac{\mathrm{d}\xi_2^m}{\mathrm{d}\beta} \leq -\frac{2}{\gamma+1}\xi_2^m \tan\beta \qquad \text{for all } \beta \in \Big(0, \frac{\pi}{2}\Big).$$

Next, we differentiate $c_{\mathcal{O}}^2 = 1 + \frac{\gamma-1}{2}(q_\infty^2 - q^2)$ with respect to β and use (2.4.11) to obtain

$$
\begin{aligned}
(2.4.40) \qquad \frac{\mathrm{d}c_{\mathcal{O}}^2}{\mathrm{d}\beta} &= (\gamma-1)q_\infty\Big(1 - \frac{q_{\mathcal{O}}}{q_\infty}\frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}q_\infty}\Big)\frac{\mathrm{d}q_\infty}{\mathrm{d}\beta} \\
&\geq \frac{2(\gamma-1)}{\gamma+1}v_\infty \sec\beta \tan\beta \qquad \text{for all } \beta \in \Big(0, \frac{\pi}{2}\Big).
\end{aligned}
$$

From this, we have

$$(2.4.41) \qquad \lim_{\beta \to \frac{\pi}{2}-}\xi_2^m = 0, \qquad \lim_{\beta \to \frac{\pi}{2}-}c_{\mathcal{O}} = \infty, \qquad \lim_{\beta \to \frac{\pi}{2}-}\xi_2^{\mathcal{O}} = -\infty.$$

Notice that

$$(2.4.42) \qquad \frac{\mathrm{d}q_\infty}{\mathrm{d}\beta} > 0,$$

which can be obtained from differentiating (2.4.12) with respect to β, where $0 < M_{\mathcal{O}} < 1 < M_\infty$ is used. From (2.4.10), we obtain

$$(2.4.43) \qquad \frac{\mathrm{d}M_{\mathcal{O}}}{\mathrm{d}\beta} = \frac{\mathrm{d}M_{\mathcal{O}}}{\mathrm{d}M_\infty}\frac{\mathrm{d}M_\infty}{\mathrm{d}\beta} = \frac{\mathrm{d}M_{\mathcal{O}}}{\mathrm{d}M_\infty}\frac{\mathrm{d}q_\infty}{\mathrm{d}\beta} < 0.$$

Therefore, we conclude from (2.4.34) and the monotonicity properties of $(\xi_2^m, c_{\mathcal{O}}^2, M_{\mathcal{O}})$ with respect to β that $\frac{\partial \xi_2^{\mathcal{O}}}{\partial \beta} < 0$ for all $\beta \in (0, \frac{\pi}{2})$. $\qquad\square$

2.5. Main Theorems in the (v_∞, β)–Parameters

With Lemma 2.19 and Remark 2.21, we can restate Theorems 2.15–2.16 by using parameters $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$.

For fixed $\gamma \geq 1$ and $(v_\infty, \beta) \in \mathfrak{R}$, we recall the definitions of $(\varphi_\infty, \varphi_\mathcal{O}, \varphi_\mathcal{N})$ given by (2.4.1), (2.4.4), and (2.4.28) as follows:

$$(2.5.1) \qquad \varphi_\infty = -\frac{1}{2}|\boldsymbol{\xi}|^2 - v_\infty \xi_2, \quad \varphi_\mathcal{O} = -\frac{1}{2}|\boldsymbol{\xi}|^2 + u_\mathcal{O}\xi_1 - v_\infty \xi_2^{(\beta)}, \quad \varphi_\mathcal{N} = -\frac{1}{2}|\boldsymbol{\xi}|^2 - v_\infty \xi_2^\mathcal{N},$$

for $\xi_2^{(\beta)}$ given by (2.4.14).

Let
$$\rho_\mathcal{O} = \rho(|D\varphi_\mathcal{O}|^2, \varphi_\mathcal{O}), \qquad \rho_\mathcal{N} = \rho(|D\varphi_\mathcal{N}|^2, \varphi_\mathcal{N})$$

for $\rho(|D\varphi|^2, \varphi)$ defined by (2.4.2). Note that $\xi_2^\mathcal{N}$ satisfies that $\xi_2^\mathcal{N} < c_\mathcal{N}$ for $c_\mathcal{N} = \rho_\mathcal{N}^{\frac{\gamma-1}{2}}$. Define
$$O_\mathcal{O} := (u_\mathcal{O}, 0), \qquad O_\mathcal{N} = (0, 0).$$

Since $\xi_2^\mathcal{N} < c_\mathcal{N}$, $\partial B_{c_\mathcal{N}}(O_\mathcal{N})$ intersects with $S_\mathcal{N} = \{\xi_2 = \xi_2^\mathcal{N}\}$ at two distinct points. For each $\beta \in [0, \frac{\pi}{2})$, $\xi_2 = f_\mathcal{O}(\xi_1)$, obtained by solving the equation $\varphi_\infty(\xi_1, \xi_2) - \varphi_\mathcal{O}(\xi_1, \xi_2) = 0$ for ξ_2, is given by

$$(2.5.2) \qquad\qquad\qquad f_\mathcal{O}(\xi_1) := \xi_1 \tan\beta + \xi_2^{(\beta)}.$$

Note that $S_\mathcal{O} = \{\xi_2 = f_\mathcal{O}(\xi_1)\}$ intersects with $\partial B_{c_\mathcal{O}}(O_\mathcal{O})$ at two distinct points. The ξ_1–intercept of $S_\mathcal{O}$ is

$$(2.5.3) \qquad\qquad\qquad P_\beta = (-\xi_2^{(\beta)}\cot\beta, 0) =: (\xi_1^{(\beta)}, 0).$$

The line passing through P_β and $O_\infty = (0, -v_\infty)$ is given by

$$(2.5.4) \qquad\qquad L_\text{w} := \{\boldsymbol{\xi} : \xi_2 = f_\text{w}(\xi_1) := \tan\theta_\infty(\xi_1 - \xi_1^{(\beta)})\}$$

for
$$\tan\theta_\infty = \frac{v_\infty}{\xi_1^{(\beta)}} \qquad \text{with } \theta_\infty \in (\frac{\pi}{2}, \pi).$$

Then L_w represents the horizontal ground in the self-similar plane before the linear transformation (2.4.15) of the self-similar variables (ξ_1, ξ_2). Moreover, $\tan\theta_\infty$ and L_w depend continuously on (v_∞, β).

DEFINITION 2.23. For each $v_\infty > 0$ and $\beta \in [0, \frac{\pi}{2})$, define

$$(2.5.5) \qquad \begin{aligned} &O_\infty := (0, -v_\infty), \quad O_\mathcal{O} := (u_\mathcal{O}, 0) = (-v_\infty \tan\beta, 0), \quad O_\mathcal{N} := (0, 0), \\ &\Lambda_\beta := \mathbb{R}_+^2 \setminus \{\boldsymbol{\xi} \in \mathbb{R}^2 : \xi_2 \leq f_\text{w}(\xi_1)\}, \\ &\Gamma_\text{sonic}^\mathcal{N} := \partial B_{c_\mathcal{N}}(O_\mathcal{N}) \cap \{\xi_1 > 0, 0 \leq \xi_2 \leq \xi_2^\mathcal{N}\}, \\ &\boldsymbol{e}_{S_\mathcal{O}} = (\cos\beta, \sin\beta). \end{aligned}$$

For φ_∞, $\varphi_\mathcal{O}$, and $\varphi_\mathcal{N}$ given by (2.5.1), define
$$S_\mathcal{N} = \{\boldsymbol{\xi} : \varphi_\infty(\boldsymbol{\xi}) = \varphi_\mathcal{N}(\boldsymbol{\xi})\}, \quad S_\mathcal{O} = \{\boldsymbol{\xi} : \varphi_\infty(\boldsymbol{\xi}) = \varphi_\mathcal{O}(\boldsymbol{\xi})\}.$$

Let $\Omega^\mathcal{N}$ be the unbounded open region enclosed by $S_\mathcal{N}$, $\Gamma_\text{sonic}^\mathcal{N}$, and line $\{(\xi_1, 0) : \xi_1 \geq \xi_1^{P_3}\}$ so that $\Omega^\mathcal{N}$ is a fixed domain for all $\beta \in [0, \beta_\text{d}^{(v_\infty)})$ for fixed $v_\infty > 0$. Denote the two points P_2 and P_3 by:

- P_2 – the intersection point of line $\xi_2 = \xi_2^\mathcal{N}$ and $\Gamma_\text{sonic}^\mathcal{N}$,
- P_3 – the intersection point of the ξ_1–axis and $\Gamma_\text{sonic}^\mathcal{N}$.
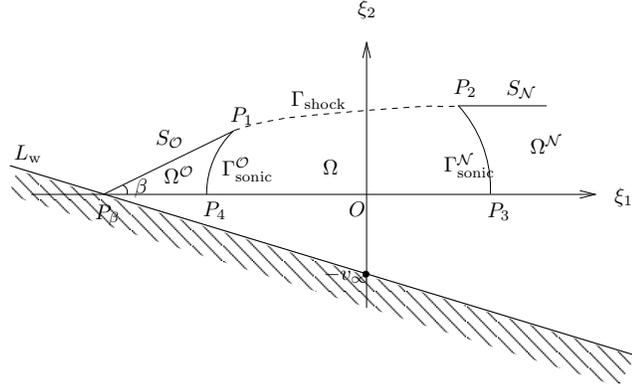
FIGURE 2.11. Admissible solutions for $\beta < \beta_{\mathrm{s}}^{(v_\infty)}$
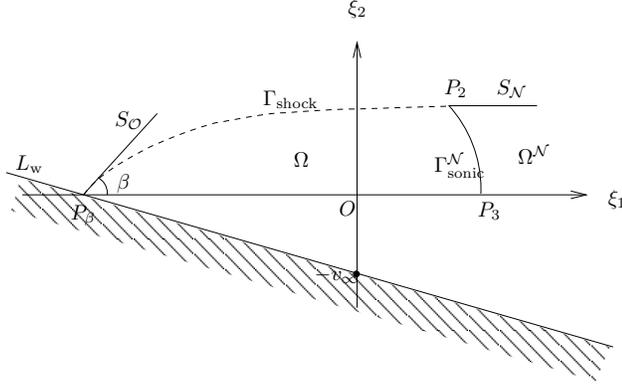
FIGURE 2.12. Admissible solutions for $\beta_{\mathrm{s}}^{(v_\infty)} \le \beta < \beta_{\mathrm{d}}^{(v_\infty)}$

For each $v_\infty > 0$ and $\beta \in [0, \beta_{\mathrm{s}}^{(v_\infty)})$, define

$$\Gamma_{\mathrm{sonic}}^{\mathcal{O}} := \partial B_{c_{\mathcal{O}}}(O_{\mathcal{O}}) \cap \{\xi_1 < 0, 0 \le \xi_2 \le f_{\mathcal{O}}(\xi_1)\}.$$

Set the two points P_1 and P_4 as

- $\{P_1\} = \Gamma_{\mathrm{sonic}}^{\mathcal{O}} \cap \{\xi_2 = f_{\mathcal{O}}(\xi_1)\}$,
- $\{P_4\} = \Gamma_{\mathrm{sonic}}^{\mathcal{O}} \cap \{\xi_2 = 0\}$.

Let $\Omega^{\mathcal{O}}$ be the bounded open region enclosed by $S_{\mathcal{O}}$, $\Gamma_{\mathrm{sonic}}^{\mathcal{O}}$, and the line segment $\overline{P_\beta P_4}$.
By Lemma 2.22, we have

$$\lim_{\beta \to \beta_{\mathrm{s}}^{(v_\infty)}-} |P_1 - P_\beta| = \lim_{\beta \to \beta_{\mathrm{s}}^{(v_\infty)}-} |P_1 - P_4| = 0.$$

This implies that, as β tends to $\beta_{\mathrm{s}}^{(v_\infty)}$ from the left, $\Gamma_{\mathrm{sonic}}^{\mathcal{O}}$ and $\overline{\Omega^{\mathcal{O}}}$ shrink to a single point $P_\beta = P_1 = P_4$. Therefore, the definitions of $\Gamma_{\mathrm{sonic}}^{\mathcal{O}}$, P_1, and P_4 for $\beta \in [\beta_{\mathrm{s}}^{(v_\infty)}, \frac{\pi}{2})$ are given by

(2.5.6) $\Gamma_{\mathrm{sonic}}^{\mathcal{O}} = \{P_1\} = \{P_4\} := \{P_\beta\}.$

DEFINITION 2.24 (Admissible solutions with parameters $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$). Fix $\gamma \geq 1$ and $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$, and let $(\varphi_\infty, \varphi_\mathcal{O}, \varphi_\mathcal{N})$ be defined by (2.5.1). For $S_\mathcal{O}$ and $S_\mathcal{N}$ given in Definition 2.23, define

$$S_{\mathcal{O}, \text{seg}} := S_\mathcal{O} \cap \{ -\xi_2^{(\beta)} \cot \beta \leq \xi_1 \leq \xi_1^{P_1} \}, \qquad S_{\mathcal{N}, \text{seg}} := S_\mathcal{N} \cap \{ \xi_1 \geq \xi_1^{P_2} \}.$$

A function $\varphi \in C_{\text{loc}}^{0,1}(\Lambda_\beta)$ is called *an admissible solution* corresponding to (v_∞, β) if φ satisfies the following properties:

Case I. $\beta \in (0, \beta_s^{(v_\infty)})$:

(i) There exists a shock curve Γ_{shock} with endpoints $P_1 = (\xi_1^\mathcal{O}, \xi_2^\mathcal{O})$ and $P_2 = (\xi_1^\mathcal{N}, \xi_2^\mathcal{N})$ such that

(i-1) Curve Γ_{shock} satisfies

$$(2.5.7) \qquad\qquad\qquad \Gamma_{\text{shock}} \subset \Lambda_\beta \setminus \overline{B_1(O_\infty)},$$

where $\partial B_1(0, -v_\infty)$ is the sonic circle of the state in $\Omega^\infty := \Lambda_\beta \setminus \overline{\Omega^\mathcal{O} \cup \Omega \cup \Omega^\mathcal{N}}$;

(i-2) Curve Γ_{shock} is C^2 in its relative interior. That is, for any $P \in \Gamma_{\text{shock}} \setminus \{P_1, P_2\}$, there exist a constant $r > 0$, a function $f \in C^2$, and an orthogonal coordinate system (S, T) in $\mathbb{R}^2$ such that $\Gamma_{\text{shock}} \cap B_r(P) = \{S = f(T)\} \cap B_r(P)$;

(i-3) Curve $\overline{S_{\mathcal{O}, \text{seg}} \cup \Gamma_{\text{shock}} \cup S_{\mathcal{N}, \text{seg}}}$ is C^1, including at points P_1 and P_2;

(i-4) $\Gamma_{\text{shock}}, \Gamma_{\text{sonic}}^\mathcal{N}, \Gamma_{\text{sonic}}^\mathcal{O}$, and $\Gamma_{\text{wedge}} := \{ \xi_2 = 0, u_\mathcal{O} - c_\mathcal{O} \leq \xi_1 \leq c_\mathcal{N} \}$ do not have common points except for P_1, P_2, P_3, and P_4. Thus, $\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}^\mathcal{N} \cup \Gamma_{\text{sonic}}^\mathcal{O} \cup \Gamma_{\text{wedge}}$ is a closed curve without self-intersection. Denote by Ω the bounded domain enclosed by this closed curve.

(ii) φ satisfies the following properties:

(ii-1) $\varphi \in C_{\text{loc}}^{0,1}(\Lambda_\beta) \cap C_{\text{loc}}^1 \big(\Lambda_\beta \setminus \overline{S_{\mathcal{O}, \text{seg}} \cup \Gamma_{\text{shock}} \cup S_{\mathcal{N}, \text{seg}}} \big)$;

(ii-2) $\varphi \in C^3(\Omega) \cap C^2(\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}^\mathcal{O}} \cup \overline{\Gamma_{\text{sonic}}^\mathcal{N}})) \cap C^1(\overline{\Omega})$;

(ii-3) For $\Omega^\mathcal{O}$ defined in Definition 2.23,

$$(2.5.8) \qquad\qquad \varphi = \begin{cases} \varphi_\infty & \text{in } \Lambda_\beta \setminus \overline{\Omega^\mathcal{O} \cup \Omega \cup \Omega^\mathcal{N}}, \\ \varphi_\mathcal{O} & \text{in } \Omega^\mathcal{O}, \\ \varphi_\mathcal{N} & \text{in } \Omega^\mathcal{N}, \end{cases}$$

where $\overline{\Omega^\mathcal{O}}$ shrinks to $\{P_\beta\} = \{P_1\} = \{P_4\}$ when $\beta = \beta_s^{(v_\infty)}$;

(ii-4) φ satisfies

- Eq. (2.1.19) in Ω with $\rho(|D\varphi|^2, \varphi)$ defined by (2.4.2),
- the slip boundary condition $\varphi_{\xi_2} = 0$ on Γ_{wedge},
- the Rankine-Hugoniot conditions: $[\varphi]_{\Gamma_{\text{shock}}} = [\rho(|D\varphi|^2, \varphi)D\varphi \cdot \mathbf{n}_{\text{sh}}]_{\Gamma_{\text{shock}}} = 0$ for the unit normal vector $\mathbf{n}_{\text{sh}}$ to Γ_{shock} towards the interior of Ω.

(iii) Eq. (2.1.19) is strictly elliptic in $\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}^\mathcal{O}} \cup \overline{\Gamma_{\text{sonic}}^\mathcal{N}})$.

(iv) $\max\{\varphi_\mathcal{O}, \varphi_\mathcal{N}\} \leq \varphi \leq \varphi_\infty$ in Ω.

(v) Let $\boldsymbol{e}_{S_\mathcal{O}}$ be the unit vector parallel to $S_\mathcal{O}$ and oriented so that $\boldsymbol{e}_{S_\mathcal{O}} \cdot \boldsymbol{e}_1 > 0$, and let $\boldsymbol{e}_{S_\mathcal{N}}$ be the unit vector parallel to $S_\mathcal{N}$ and oriented so that $\boldsymbol{e}_{S_\mathcal{N}} \cdot \boldsymbol{e}_1 < 0$, where $\boldsymbol{e}_1$ is the unit vector in the ξ_1–direction, *i.e.*, $\boldsymbol{e}_1 = (1, 0)$. That is,

$$(2.5.9) \qquad\qquad \boldsymbol{e}_{S_\mathcal{O}} = (\cos \beta, \sin \beta), \qquad \boldsymbol{e}_{S_\mathcal{N}} = (-1, 0).$$

Then

(2.5.10) $$\partial_{\boldsymbol{e}_{S_\mathcal{O}}}(\varphi_\infty - \varphi) \leq 0, \quad \partial_{\boldsymbol{e}_{S_\mathcal{N}}}(\varphi_\infty - \varphi) \leq 0 \qquad \text{in } \Omega.$$

Case II. $\beta \in [\beta_{\mathrm{s}}^{(v_\infty)}, \beta_{\mathrm{d}}^{(v_\infty)})$:

(i) There exists a shock curve Γ_{shock} with endpoints $P_\beta = (-\xi_2^{(\beta)} \cot\beta, 0)$ and $P_2 = (\xi_1^\mathcal{N}, \xi_2^\mathcal{N})$ such that

   (i-1) Curve Γ_{shock} satisfies

(2.5.11) $$\Gamma_{\mathrm{shock}} \subset (\Lambda_\beta \setminus \overline{B_1(O_\infty)}),$$

       where $\partial B_1(0, -v_\infty)$ is the sonic circle of the state in $\Omega^\infty := \Lambda_\beta \setminus \overline{\Omega \cup \Omega^\mathcal{N}}$;

   (i-2) Curve Γ_{shock} is C^2 in its relative interior: for any $P \in \Gamma_{\mathrm{shock}} \setminus \{P_\beta, P_2\}$, there exist $r > 0$, $f \in C^2$, and an orthogonal coordinate system (S, T) in $\mathbb{R}^2$ so that $\Gamma_{\mathrm{shock}} \cap B_r(P) = \{S = f(T)\} \cap B_r(P)$;

   (i-3) Curve $\overline{\Gamma_{\mathrm{shock}} \cup S_\mathcal{N}}$ is C^1, including at point P_2;

   (i-4) $\Gamma_{\mathrm{shock}}, \Gamma_{\mathrm{sonic}}^\mathcal{N}$, and $\Gamma_{\mathrm{wedge}} := \{\xi_2 = 0, -\xi_2^{(\beta)} \cot\beta \leq \xi_1 \leq c_\mathcal{N}\}$ do not have common points except for P_β, P_2, and P_3. Thus, $\Gamma_{\mathrm{shock}} \cup \Gamma_{\mathrm{sonic}}^\mathcal{N} \cup \Gamma_{\mathrm{wedge}}$ is a closed curve without self-intersection. Denote by Ω the bounded domain enclosed by this closed curve.

(ii) φ satisfies the following properties:

   (ii-1) $\varphi \in C_{\mathrm{loc}}^{0,1}(\Lambda_\beta) \cap C_{\mathrm{loc}}^1(\Lambda_\beta \setminus \overline{\Gamma_{\mathrm{shock}} \cup S_{\mathcal{N},\mathrm{seg}}})$;

   (ii-2) $\varphi \in C^3(\Omega) \cap C^2(\overline{\Omega} \setminus (\{P_\beta\} \cup \overline{\Gamma_{\mathrm{sonic}}^\mathcal{N}})) \cap C^1(\overline{\Omega})$;

   (ii-3) $D\varphi(P_\beta) = D\varphi_\mathcal{O}(P_\beta)$ and

(2.5.12) $$\varphi = \begin{cases} \varphi_\infty & \text{in } \Lambda_\beta \setminus \overline{\Omega \cup \Omega^\mathcal{N}}, \\ \varphi_\mathcal{O} & \text{at } P_\beta, \\ \varphi_\mathcal{N} & \text{in } \Omega^\mathcal{N}; \end{cases}$$

   (ii-4) φ satisfies
       - Eq. (2.1.19) in Ω with $\rho(|D\varphi|^2, \varphi)$ defined by (2.4.2),
       - the slip boundary condition $\varphi_{\xi_2} = 0$ on Γ_{wedge},
       - the Rankine-Hugoniot conditions: $[\varphi]_{\Gamma_{\mathrm{shock}}} = [\rho(|D\varphi|^2, \varphi)D\varphi \cdot \mathbf{n}_{\mathrm{sh}}]_{\Gamma_{\mathrm{shock}}} = 0$ for the unit normal vector $\mathbf{n}_{\mathrm{sh}}$ to Γ_{shock} towards the interior of Ω.

(iii) Eq. (2.1.19) is strictly elliptic in $\overline{\Omega} \setminus (\{P_\beta\} \cup \overline{\Gamma_{\mathrm{sonic}}^\mathcal{N}})$.

(iv) $\max\{\varphi_\mathcal{O}, \varphi_\mathcal{N}\} \leq \varphi \leq \varphi_\infty$ in Ω.

(v) φ satisfies (2.5.10).

REMARK 2.25. The inequalities in (2.5.10) for two directions $\boldsymbol{e}_{S_\mathcal{O}}$ and $\boldsymbol{e}_{S_\mathcal{N}}$ imply that

(2.5.13) $$\partial_{\mathbf{e}}(\varphi_\infty - \varphi) \leq 0 \qquad \text{in } \Omega \text{ for all } \mathbf{e} \in \mathrm{Cone}(\boldsymbol{e}_{S_\mathcal{O}}, \boldsymbol{e}_{S_\mathcal{N}}),$$

where

(2.5.14) $$\mathrm{Cone}(\boldsymbol{e}_{S_\mathcal{O}}, \boldsymbol{e}_{S_\mathcal{N}}) := \{a_1 \boldsymbol{e}_{S_\mathcal{O}} + a_2 \boldsymbol{e}_{S_\mathcal{N}} : a_1 \geq 0, \ a_2 \geq 0\}.$$

LEMMA 2.26 (Entropy condition of admissible solutions). *Let φ be an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ in the sense of Definition 2.24, and let Γ_{shock} be the curved shock satisfying condition* (i) *of Definition 2.24. Let $\boldsymbol{\nu}$ be the unit normal vector to Γ_{shock} towards the interior of Ω. Then the following properties hold:*

(a) $\partial_{\boldsymbol{\nu}} \varphi_\infty > \partial_{\boldsymbol{\nu}} \varphi > 0$ *on* $\overline{\Gamma_{\text{shock}}}$;

(b) *Let*

$$M_{\infty, \boldsymbol{\nu}} := \frac{\partial_{\boldsymbol{\nu}} \varphi_\infty}{c(|D\varphi_\infty|^2, \varphi_\infty)} = \partial_{\boldsymbol{\nu}} \varphi_\infty, \quad M_{\boldsymbol{\nu}} := \frac{\partial_{\boldsymbol{\nu}} \varphi}{c(|D\varphi|^2, \varphi)}$$

for

(2.5.15)
$$c(|\mathbf{q}|^2, z) = \rho^{\frac{\gamma-1}{2}}(|\mathbf{q}|^2, z),$$

where $\rho(|\mathbf{q}|^2, z)$ is defined by (2.4.2). *Then*

$$0 < M_{\boldsymbol{\nu}} < 1 < M_{\infty, \boldsymbol{\nu}} \qquad \text{on } \overline{\Gamma_{\text{shock}}}.$$

PROOF. Denote $w := \varphi_\infty - \varphi$. From (2.1.19), (2.4.2), and (2.4.5), it can directly be checked that

$$(c^2 - \varphi_{\xi_1}^2) w_{\xi_1 \xi_1} - 2\varphi_{\xi_1} \varphi_{\xi_2} w_{\xi_1 \xi_2} + (c^2 - \varphi_{\xi_2}^2) w_{\xi_2 \xi_2} = 0 \qquad \text{in } \Omega$$

for $c^2 = \rho^{\gamma-1}(|D\varphi|^2, \varphi)$, where $\rho(|D\varphi|^2, \varphi)$ is given by (2.4.2). By condition (iii) of Definition 2.24, the minimum principle applies to w so that w cannot attain its minimum in Ω, unless it is a constant in $\overline{\Omega}$. By conditions (ii) and (iv) of Definition 2.24, we see that $w \geq 0$ in $\overline{\Omega}$, and $w = 0$ on $\overline{\Gamma_{\text{shock}}}$. Furthermore, w is not a constant in $\overline{\Omega}$, because $\partial_{\xi_2} w = -v_\infty$ on Γ_{wedge} by (2.4.1) and the slip boundary condition $\partial_{\xi_2} \varphi = 0$ on Γ_{wedge}, stated in (ii-4) of Definition 2.24. Then it follows from Hopf's lemma that $\partial_{\boldsymbol{\nu}} w > 0$ on Γ_{shock}. This implies that

(2.5.16)
$$\partial_{\boldsymbol{\nu}} \varphi_\infty > \partial_{\boldsymbol{\nu}} \varphi \qquad \text{on } \Gamma_{\text{shock}}.$$

If $\partial_{\boldsymbol{\nu}} \varphi(P) = 0$ for some $P \in \Gamma_{\text{shock}}$, then it follows from the condition: $\rho(|D\varphi|^2, \varphi) \partial_{\boldsymbol{\nu}} \varphi(P) = \partial_{\boldsymbol{\nu}} \varphi_\infty(P)$ stated in (ii-4) of Definition 2.24 that $\partial_{\boldsymbol{\nu}} \varphi_\infty(P) = 0$, which is impossible, due to (2.5.16). Therefore, we have

(2.5.17)
$$|\partial_{\boldsymbol{\nu}} \varphi| > 0 \qquad \text{on } \Gamma_{\text{shock}}.$$

By conditions (ii-2)–(ii-3) of Definition 2.24, we have

$$D\varphi(P_2) = D\varphi_{\mathcal{N}}(P_2).$$

Then it follows from the definitions of $(\varphi_\infty, \varphi_{\mathcal{N}})$ given in (2.5.1) and conditions (ii-4) and (iv) of Definition 2.24 that

(2.5.18)
$$\boldsymbol{\nu}(P_2) = \frac{D\varphi_\infty - D\varphi_{\mathcal{N}}}{|D\varphi_\infty - D\varphi_{\mathcal{N}}|} = (0, -1),$$

(2.5.19)
$$\partial_{\boldsymbol{\nu}}(\varphi_\infty - \varphi)(P_2) = |D\varphi_\infty - D\varphi_{\mathcal{N}}| = v_\infty > 0, \quad \partial_{\boldsymbol{\nu}} \varphi(P_2) = \partial_{\boldsymbol{\nu}} \varphi_{\mathcal{N}}(P_2) = \xi_2^{P_2} > 0.$$

Similarly, at P_1, we have

$$D\varphi(P_1) = D\varphi_{\mathcal{O}}(P_1),$$

so that (2.4.3), (2.4.35), (2.4.39), and (2.4.41) yield that

$$\boldsymbol{\nu}(P_1) = \frac{D\varphi_\infty - D\varphi_\mathcal{O}}{|D\varphi_\infty - D\varphi_\mathcal{O}|} = (\sin\beta, -\cos\beta),$$

(2.5.20) $$\partial_{\boldsymbol{\nu}}(\varphi_\infty - \varphi)(P_1) = |D\varphi_\infty - D\varphi_\mathcal{O}| = v_\infty\sec\beta > 0,$$

(2.5.21) $$\partial_{\boldsymbol{\nu}}\varphi(P_1) = \partial_{\boldsymbol{\nu}}\varphi_\mathcal{O}(P_1) = \partial_{\boldsymbol{\nu}}\varphi_\infty(P_1) - v_\infty\sec\beta = \xi_2^m > 0.$$

Then statement (a) follows directly from (2.5.16)–(2.5.21) and the continuity of $\partial_{\boldsymbol{\nu}}\varphi$ along Γ_{shock} up to its endpoints P_1 and P_2.

Note that the calculations given in (2.4.8)–(2.4.9) are still valid when $(\rho_\mathcal{O}, M_\mathcal{O}, M_\infty)$ are replaced by $(\rho, M_{\boldsymbol{\nu}}, M_{\infty,\boldsymbol{\nu}})$ on $\overline{\Gamma_{\text{shock}}}$. Then we see that, on $\overline{\Gamma_{\text{shock}}}$,

(2.5.22) $$\rho^{\frac{\gamma+1}{2}} = \frac{M_{\infty,\boldsymbol{\nu}}}{M_{\boldsymbol{\nu}}},$$

(2.5.23) $$\big(1 + \frac{\gamma-1}{2}M_{\boldsymbol{\nu}}^2\big)|M_{\boldsymbol{\nu}}|^{\frac{-2(\gamma-1)}{\gamma+1}} = \big(1 + \frac{\gamma-1}{2}M_{\infty,\boldsymbol{\nu}}^2\big)|M_{\infty,\boldsymbol{\nu}}|^{\frac{-2(\gamma-1)}{\gamma+1}}.$$

This is because (2.4.8)–(2.4.9) are all derived from the Rankine-Hugoniot conditions stated in Definition 2.24(ii-4). By the result obtained in statement (a) and the Rankine-Hugoniot condition $\rho\partial_{\boldsymbol{\nu}}\varphi = \partial_{\boldsymbol{\nu}}\varphi_\infty$ on $\overline{\Gamma_{\text{shock}}}$, (2.5.22) implies that $\frac{M_{\infty,\boldsymbol{\nu}}}{M_{\boldsymbol{\nu}}} > 1$ on $\overline{\Gamma_{\text{shock}}}$. Since $(M_{\boldsymbol{\nu}}, M_{\infty,\boldsymbol{\nu}})$ satisfy (2.5.23) and $M_{\infty,\boldsymbol{\nu}} \neq M_{\boldsymbol{\nu}}$ on $\overline{\Gamma_{\text{shock}}}$, it follows from the observation right after (2.4.9) that

$$0 < M_{\boldsymbol{\nu}} < 1 < M_{\infty,\boldsymbol{\nu}} \qquad \text{on } \overline{\Gamma_{\text{shock}}}.$$

This completes the proof of statement (b). $\square$

In (2.5.2)–(2.5.4) and Definition 2.24, the values of $\xi_1^{(\beta)}$, $\xi_2^{(\beta)}$, θ_∞, $c_\mathcal{O}$, and $u_\mathcal{O}$ depend continuously on $\beta \in (0, \frac{\pi}{2})$ with

$$\lim_{\beta\to 0+}(\xi_1^{(\beta)}, \xi_2^{(\beta)}, \theta_\infty, c_\mathcal{O}, u_\mathcal{O}) = (-\infty, \xi_2^\mathcal{N}, \pi, c_\mathcal{N}, 0).$$

As a result, we obtain

$$\lim_{\beta\to 0+}|P_1 - (-\xi_1^{P_2}, \xi_2^\mathcal{N})| = 0 = \lim_{\beta\to 0+}|P_4 - (-c_\mathcal{N}, 0)|,$$

$$\lim_{\beta\to 0+}\|\varphi_\mathcal{O} - \varphi_\mathcal{N}\|_{C^3(B_R(\mathbf{0}))} = 0 \qquad \text{for any } R > 0.$$

For $\beta = 0$, we define P_1, P_4, $\Lambda_\beta|_{\beta=0}$, and $S_{\mathcal{O},\text{seg}}|_{\beta=0}$ by

(2.5.24) $$P_1 = (-\xi_1^{P_2}, \xi_2^\mathcal{N}), \quad P_4 = (-c_\mathcal{N}, 0),$$

$$\Lambda_\beta|_{\beta=0} := \mathbb{R} \times \mathbb{R}_+, \quad S_{\mathcal{O},\text{seg}}|_{\beta=0} = \{(\xi_1, \xi_2^\mathcal{N}) : \xi_1 \leq -\xi_1^{P_2}\}.$$

Then two points P_1 and P_4 depend continuously on $\beta \in [0, \frac{\pi}{2})$, so that Λ_β and $S_{\mathcal{O},\text{seg}}$ depend continuously on $\beta \in [0, \frac{\pi}{2})$. Using this, we extend Definition 2.24 up to $\beta = 0$.

DEFINITION 2.27 (Admissible solutions when $\beta = 0$). Given $\gamma \geq 1$ and $v_\infty > 0$, a function $\varphi \in C^{0,1}(\mathbb{R}\times\mathbb{R}_+)$ is called *an admissible solution* corresponding to $(v_\infty, 0)$ if φ satisfies the following properties:

(i) There exists a shock Γ_{shock} with endpoints $P_1 = (-\xi_1^\mathcal{N}, \xi_2^\mathcal{N})$ and $P_2 = (\xi_1^\mathcal{N}, \xi_2^\mathcal{N})$ such that

(i-1) Curve Γ_{shock} satisfies

$$(2.5.25) \hspace{3cm} \Gamma_{\text{shock}} \subset (\mathbb{R} \times \mathbb{R}_+) \setminus \overline{B_1(O_\infty)},$$

where $\partial B_1(O_\infty)$ is the sonic circle of state $O_\infty = (0, -v_\infty)$ in $\Omega^\infty := (\mathbb{R} \times \mathbb{R}_+) \setminus \overline{\Omega^{\mathcal{O}} \cup \Omega \cup \Omega^{\mathcal{N}}}$;

(i-2) Curve Γ_{shock} is C^2 in its relative interior; that is, for any $P \in \Gamma_{\text{shock}} \setminus \{P_1, P_2\}$, there exist $r > 0$, $f \in C^2$, and an orthogonal coordinate system (S, T) in $\mathbb{R}^2$ so that $\Gamma_{\text{shock}} \cap B_r(P) = \{S = f(T)\} \cap B_r(P)$;

(i-3) Curve $\overline{S_{\mathcal{O},\text{seg}} \cup \Gamma_{\text{shock}} \cup S_{\mathcal{N},\text{seg}}}$ is C^1, including at points P_1 and P_2;

(i-4) $\Gamma_{\text{shock}}, \Gamma_{\text{sonic}}^{\mathcal{N}}, \Gamma_{\text{sonic}}^{\mathcal{O}}$, and $\Gamma_{\text{wedge}} := \{(\xi_1, 0) : -c_{\mathcal{N}} < \xi_1 < c_{\mathcal{N}}\}$ do not have common points, and $\Gamma_{\text{shock}} \cup \Gamma_{\text{sonic}}^{\mathcal{N}} \cup \Gamma_{\text{sonic}}^{\mathcal{O}} \cup \Gamma_{\text{wedge}}$ is a closed curve without self-intersection. Denote by Ω the bounded domain enclosed by this closed curve.

(ii) φ satisfies the following properties:

(ii-1) $\varphi \in C^{0,1}(\mathbb{R} \times \mathbb{R}_+) \cap C^1\big((\mathbb{R} \times \mathbb{R}_+) \setminus \overline{S_{\mathcal{O},\text{seg}} \cup \Gamma_{\text{shock}} \cup S_{\mathcal{N},\text{seg}}}\big)$;

(ii-2) $\varphi \in C^3(\Omega) \cap C^2(\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}^{\mathcal{O}}} \cup \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}})) \cap C^1(\overline{\Omega})$;

(ii-3)

$$\varphi = \begin{cases} \varphi_\infty & \text{in } (\mathbb{R} \times \mathbb{R}_+) \setminus \overline{\Omega^{\mathcal{O}} \cup \Omega \cup \Omega^{\mathcal{N}}}, \\ \varphi_{\mathcal{N}} & \text{in } \Omega^{\mathcal{O}} \cup \Omega^{\mathcal{N}}; \end{cases}$$

(ii-4) φ satisfies

- Eq. (2.1.19) in Ω with $\rho(|D\varphi|^2, \varphi)$ defined by (2.4.2),
- the slip boundary condition $\varphi_{\xi_2} = 0$ on Γ_{wedge},
- the Rankine-Hugoniot conditions: $[\varphi]_{\Gamma_{\text{shock}}} = [\rho(|D\varphi|^2, \varphi)D\varphi \cdot \mathbf{n}_{\text{sh}}]_{\Gamma_{\text{shock}}} = 0$ for the unit normal vector $\mathbf{n}_{\text{sh}}$ to Γ_{shock} towards the interior of Ω.

(iii) Eq. (2.1.19) is strictly elliptic in $\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}^{\mathcal{O}}} \cup \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}})$.

(iv) $\varphi_{\mathcal{N}} \leq \varphi \leq \varphi_\infty$ in Ω.

(v) $\partial_{\mathbf{e}}(\varphi_\infty - \varphi) \leq 0$ in Ω for all $\mathbf{e} \in \mathbb{R} \times \mathbb{R}^+$.

REMARK 2.28. Condition (v) of Definition 2.27 is a continuous extension of condition (v) of Definition 2.24 in the sense that

(i) Cone$(\boldsymbol{e}_{S_{\mathcal{O}}}, \boldsymbol{e}_{S_{\mathcal{N}}})$ for $\beta > 0$ defined by (2.5.14) monotonically increases as $\beta > 0$ decreases in the sense that, if $0 < \beta_1 < \beta_2 < \frac{\pi}{2}$, then

$$\text{Cone}(\boldsymbol{e}_{S_{\mathcal{O}}}, \boldsymbol{e}_{S_{\mathcal{N}}})|_{\beta_2} \subset \text{Cone}(\boldsymbol{e}_{S_{\mathcal{O}}}, \boldsymbol{e}_{S_{\mathcal{N}}})|_{\beta_1};$$

(ii) $\overline{\cup_{0 < \beta < \frac{\pi}{2}} \text{Cone}(\boldsymbol{e}_{S_{\mathcal{O}}}, \boldsymbol{e}_{S_{\mathcal{N}}})|_\beta} = \mathbb{R} \times \mathbb{R}^+$.

REMARK 2.29. Similarly to Definition 2.10, it can directly be checked that any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cup \{(v_\infty, 0) : v_\infty > 0\}$ in the sense of Definitions 2.24 or 2.27 satisfies the following properties:

(i) $\varphi \in W_{\text{loc}}^{1,1}(\Lambda_\beta)$;

(ii) $\rho(|D\varphi|^2, \varphi) > 0$ in Λ_β for $\rho(|D\varphi|^2, \varphi)$ defined by (2.4.2);

(iii) $\rho(|D\varphi|^2, \varphi), \ \rho(|D\varphi|^2, \varphi)|D\varphi| \in L_{\text{loc}}^1(\Lambda_\beta)$;

(iv) For every $\zeta \in C_0^\infty(\mathbb{R}^2)$,

$$\int_{\Lambda_\beta} \left(\rho(|D\varphi|^2, \varphi) D\varphi \cdot D\zeta - 2\rho(|D\varphi|^2, \varphi)\zeta \right) \, d\boldsymbol{\xi} = 0.$$

Specifically, property (iv) here is obtained by condition (ii) of Definitions 2.24 and 2.27, and via integration by parts. Property (iv) indicates that any admissible solution φ is a *weak solution* of the boundary value problem rewritten from Problem 2.9 with respect to parameters (v_∞, β).

LEMMA 2.30. *For any given $\gamma \geq 1$ and $v_\infty > 0$, there exists at least one admissible solution corresponding to $(v_\infty, 0)$ in the sense of Definition* 2.27.

PROOF. The conditions stated in (ii-4) and (v) of Definition 2.27 imply that

$$\Gamma_{\text{shock}} = \{(\xi_1, \xi_2^{\mathcal{N}}) \, : \, -\xi_1^{\mathcal{N}} < \xi_1 < \xi_1^{\mathcal{N}}\};$$

that is, $\overline{S_{\mathcal{O},\text{seg}} \cup \Gamma_{\text{shock}} \cup S_{\mathcal{N},\text{seg}}}$ is a normal shock. Therefore, the pseudo-subsonic region Ω is enclosed by $\Gamma_{\text{sonic}}^{\mathcal{O}}, \Gamma_{\text{sonic}}^{\mathcal{N}}, \Gamma_{\text{wedge}}$, and the line segment $(-\xi_1^{\mathcal{N}}, \xi_1^{\mathcal{N}}) \times \{\xi_2^{\mathcal{N}}\}$. It can directly be checked that a function $\varphi_{\text{norm}} \in C^{0,1}(\Lambda_\beta|_{\beta=0})$ defined by

$$\varphi_{\text{norm}} = \begin{cases} \varphi_\infty & \text{in } (\mathbb{R} \times \mathbb{R}_+) \setminus \overline{\Omega^{\mathcal{O}} \cup \Omega \cup \Omega^{\mathcal{N}}}, \\ \varphi_{\mathcal{N}} & \text{in } \Omega^{\mathcal{O}} \cup \Omega \cup \Omega^{\mathcal{N}} \end{cases}$$

is an admissible solution corresponding to $(v_\infty, 0)$ in the sense of Definition 2.27.   □

For a fixed $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$, let φ be an admissible solution corresponding to (v_∞, β) in the sense of Definition 2.24. Let (u_∞, u_0) be given by $(u_\infty, u_0) = \mathcal{T}^{-1}(v_\infty, \beta) \in \mathfrak{P}_{\text{weak}}$ for mapping $\mathcal{T}$ from Lemma 2.19. Let θ_{w} be given by (2.4.21). For each $\boldsymbol{\xi}' = (\xi_1', \xi_2') \in \Lambda_\beta$, let $\boldsymbol{\xi} = (\xi_1, \xi_2)$ be given by

$$\boldsymbol{\xi}^\top = \begin{pmatrix} \cos\theta_{\text{w}} & \sin\theta_{\text{w}} \\ -\sin\theta_{\text{w}} & \cos\theta_{\text{w}} \end{pmatrix}^{-1} \left((\boldsymbol{\xi}')^\top + \begin{pmatrix} u_\infty \cos\theta_{\text{w}} \\ 0 \end{pmatrix} \right).$$

This is the inverse transformation of (2.4.15). Finally, let a function $\tilde{\varphi}$ be given by

$$(2.5.26) \qquad\qquad \tilde{\varphi}(\boldsymbol{\xi}) = \varphi(\boldsymbol{\xi}') + \frac{1}{2}(u_\infty \cos\theta_{\text{w}})^2 \qquad \text{for } \boldsymbol{\xi}' \in \Lambda_\beta.$$

Then $\tilde{\varphi}$ is an admissible solution corresponding to $(u_\infty, u_0) \in \mathfrak{P}_{\text{weak}}$ in the sense of Definition 2.14. From this perspective, Theorem 2.15 is equivalent to the following theorem:

THEOREM 2.31 (Existence of admissible solutions). *For any given $\gamma \geq 1$ and $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$, there exists an admissible solution in the sense of Definition* 2.24.

REMARK 2.32 (Non-existence of self-similar strong shock solutions). Fix $\gamma \geq 1$. For $(v_\infty, \beta) \in \mathfrak{R}_{\text{detach}} \cup \mathfrak{R}_{\text{strong}}$, let $(\Lambda_\beta, \varphi_\infty, \varphi_{\mathcal{O}}, \varphi_{\mathcal{N}})$ be defined as in Definition 2.23. We call $\varphi \in C^{0,1}(\Lambda_\beta)$ an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{detach}} \cup \mathfrak{R}_{\text{strong}}$ if it satisfies conditions (i)–(v) stated in Definition 2.24 for **Case** II. By the convexity of the shock polar for steady potential flow, which is shown in *Appendix* A, and condition (iv) of Definition 2.24, it follows from the non-existence result as proved in *Appendix* B (see also [**25**]) that there exists no admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{strong}}$ in the sense of Definition 2.24.

The existence of admissible solutions corresponding to $(v_\infty, \beta_{\text{d}}^{(v_\infty)})$ is still an open question.

THEOREM 2.33 (Regularity of admissible solutions). *Given $\gamma \geq 1$ and $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$, let φ be a corresponding admissible solution with the curved shock Γ_{shock} in the sense of Definition 2.24. Then the following properties hold:*

***Case* I.** $\beta \in (0, \beta_s^{(v_\infty)})$:

    (a) *Γ_{shock} is C^∞ in its relative interior, and $\varphi \in C^\infty(\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}^{\mathcal{O}}} \cup \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}})) \cap C^{1,1}(\overline{\Omega})$;*

    (b) *Define a set $\mathcal{D}$ by*

$$(2.5.27) \qquad \mathcal{D} = \Lambda_\beta \cap \{\boldsymbol{\xi} \,:\, \max\{\varphi_{\mathcal{O}}(\boldsymbol{\xi}), \varphi_{\mathcal{N}}(\boldsymbol{\xi})\} < \varphi_\infty(\boldsymbol{\xi})\}.$$

    For a constant $\sigma > 0$, define $\mathcal{D}_\sigma^{\mathcal{O}}$ and $\mathcal{D}_\sigma^{\mathcal{N}}$ by

$$(2.5.28) \qquad \begin{aligned} \mathcal{D}_\sigma^{\mathcal{O}} &= \mathcal{D} \cap \{\boldsymbol{\xi} \,:\, \text{dist}\{\boldsymbol{\xi}, \Gamma_{\text{sonic}}^{\mathcal{O}}\} < \sigma\} \cap B_{c_{\mathcal{O}}}(O_{\mathcal{O}}), \\ \mathcal{D}_\sigma^{\mathcal{N}} &= \mathcal{D} \cap \{\boldsymbol{\xi} \,:\, \text{dist}\{\boldsymbol{\xi}, \Gamma_{\text{sonic}}^{\mathcal{N}}\} < \sigma\} \cap B_{c_{\mathcal{N}}}(O_{\mathcal{N}}) \end{aligned}$$

    for $c_{\mathcal{N}} = \rho_{\mathcal{N}}^{(\gamma-1)/2}$, $c_{\mathcal{O}} = \rho_{\mathcal{O}}^{(\gamma-1)/2}$, $O_{\mathcal{O}} = (u_{\mathcal{O}}, 0)$, and $O_{\mathcal{N}} := (0, 0)$. Fix any point $\boldsymbol{\xi}_0 \in (\overline{\Gamma_{\text{sonic}}^{\mathcal{O}}} \cup \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}}) \setminus \{P_1, P_2\}$, and denote $d := \text{dist}\{\boldsymbol{\xi}_0, \Gamma_{\text{shock}}\}$. Then, for any $\alpha \in (0, 1)$, there exists a constant $K < \infty$ depending on $(v_\infty, \gamma, \varepsilon_0, \alpha, d)$ and $\|\varphi\|_{C^{1,1}(\Omega \cap (\mathcal{D}_{\varepsilon_0}^{\mathcal{O}} \cup \mathcal{D}_{\varepsilon_0}^{\mathcal{N}}))}$ such that

$$(2.5.29) \qquad \|\varphi\|_{2,\alpha,\overline{\Omega \cap B_{d/2}(\boldsymbol{\xi}_0) \cap (\mathcal{D}_{\varepsilon_0/2}^{\mathcal{O}} \cup \mathcal{D}_{\varepsilon_0/2}^{\mathcal{N}})}} \leq K;$$

    (c) *For any $\boldsymbol{\xi}_0 \in (\overline{\Gamma_{\text{sonic}}^{\mathcal{O}}} \cup \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}}) \setminus \{P_1, P_2\}$,*

$$(2.5.30) \qquad \lim_{\substack{\boldsymbol{\xi} \to \boldsymbol{\xi}_0 \\ \boldsymbol{\xi} \in \Omega}} \big(D_{rr}\varphi - D_{rr}\max\{\varphi_{\mathcal{O}}, \varphi_{\mathcal{N}}\}\big)(\boldsymbol{\xi}) = \frac{1}{\gamma + 1},$$

    where $r = |\boldsymbol{\xi}|$ near $\Gamma_{\text{sonic}}^{\mathcal{N}}$ and $r = |\boldsymbol{\xi} - (u_{\mathcal{O}}, 0)|$ near $\Gamma_{\text{sonic}}^{\mathcal{O}}$;

    (d) *$\lim_{\substack{\boldsymbol{\xi} \to P_1 \\ \boldsymbol{\xi} \in \Omega}} D^2\varphi$ and $\lim_{\substack{\boldsymbol{\xi} \to P_2 \\ \boldsymbol{\xi} \in \Omega}} D^2\varphi$ do not exist;*

    (e) *$\overline{S_{\mathcal{O},\text{seg}} \cup \Gamma_{\text{shock}} \cup S_{\mathcal{N},\text{seg}}}$ is a $C^{2,\alpha}$–curve for any $\alpha \in (0, 1)$, including at points P_1 and P_2.*

***Case* II.** $\beta \in [\beta_s^{(v_\infty)}, \beta_d^{(v_\infty)})$:

    (a) *Γ_{shock} is C^∞ in its relative interior, and $\varphi \in C^\infty(\overline{\Omega} \setminus (\{P_\beta\} \cup \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}})) \cap C^{1,1}(\overline{\Omega} \setminus \{P_\beta\}) \cap C^{1,\bar{\alpha}}(\overline{\Omega})$ for some $\bar{\alpha} \in (0, 1)$;*

    (b) *For a constant $\sigma > 0$, let $\mathcal{D}_\sigma^{\mathcal{N}}$ be defined by (2.5.28). Fix any point $\boldsymbol{\xi}_0 \in \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}} \setminus \{P_2\}$, and denote $d := \text{dist}\{\boldsymbol{\xi}_0, \Gamma_{\text{shock}}\}$. Then, for any $\alpha \in (0, 1)$, there exists a constant $K < \infty$ depending on $(v_\infty, \gamma, \varepsilon_0, \alpha, d)$ and $\|\varphi\|_{C^{1,1}(\Omega \cap \mathcal{D}_{\varepsilon_0}^{\mathcal{N}})}$ such that*

$$(2.5.31) \qquad \|\varphi\|_{2,\alpha,\overline{\Omega \cap B_{d/2}(\boldsymbol{\xi}_0) \cap \mathcal{D}_{\varepsilon_0/2}^{\mathcal{N}}}} \leq K;$$

    (c) *For any $\boldsymbol{\xi}_0 \in \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}} \setminus \{P_2\}$,*

$$(2.5.32) \qquad \lim_{\substack{\boldsymbol{\xi} \to \boldsymbol{\xi}_0 \\ \boldsymbol{\xi} \in \Omega}} \big(D_{rr}\varphi - D_{rr}\varphi_{\mathcal{N}}\big)(\boldsymbol{\xi}) = \frac{1}{\gamma + 1},$$

    where $r = |\boldsymbol{\xi}|$;

    (d) *$\lim_{\substack{\boldsymbol{\xi} \to P_2 \\ \boldsymbol{\xi} \in \Omega}} D^2\varphi$ does not exist;*

(e) $\overline{\Gamma_{\text{shock}} \cup S_{\mathcal{N},\text{seg}}}$ is a $C^{1,\bar{\alpha}}$–curve for the same $\bar{\alpha}$ as in statement (a). Furthermore, curve $\overline{\Gamma_{\text{shock}} \cup S_{\mathcal{N},\text{seg}}} \setminus \{P_\beta\}$ is $C^{2,\alpha}$ for any $\alpha \in (0,1)$, including at point P_2.

Since Theorems 2.15–2.16 follow directly from Theorems 2.31 and 2.33 through (2.5.26), the rest of the monograph is devoted to establishing Theorems 2.31 and 2.33.

We will prove Theorem 2.31 by solving the following free boundary problem:

PROBLEM 2.34 (Free boundary problem). Given $\gamma \geq 1$ and $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$, define φ_β and Γ_{sonic} by

$$(2.5.33) \qquad \varphi_\beta := \max\{\varphi_{\mathcal{O}}, \varphi_{\mathcal{N}}\}, \qquad \Gamma_{\text{sonic}} := \Gamma_{\text{sonic}}^{\mathcal{O}} \cup \Gamma_{\text{sonic}}^{\mathcal{N}}.$$

Find a curved shock Γ_{shock} and a function $\varphi \in C^3(\Omega) \cap C^2(\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}^{\mathcal{O}}} \cup \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}})) \cap C^1(\overline{\Omega})$ satisfying the following:

$$(2.5.34) \qquad\qquad\qquad\qquad \text{Eq. } (2.1.19) \qquad \text{in } \Omega,$$
$$(2.5.35) \qquad\qquad\qquad \varphi = \varphi_\beta, \;\; D\varphi = D\varphi_\beta \qquad \text{on } \Gamma_{\text{sonic}},$$
$$(2.5.36) \qquad\qquad\qquad\qquad\qquad \partial_{\xi_2}\varphi = 0 \qquad \text{on } \Gamma_{\text{wedge}},$$
$$(2.5.37) \qquad\qquad \varphi = \varphi_\infty, \;\; \rho D\varphi \cdot \boldsymbol{\nu}_{\text{sh}} = D\varphi_\infty \cdot \boldsymbol{\nu}_{\text{sh}} \qquad \text{on } \Gamma_{\text{shock}},$$

where $\boldsymbol{\nu}_{\text{sh}}$ is the unit normal vector to Γ_{shock} towards the interior of Ω, and ρ is defined by (2.4.2). Note that $\Gamma_{\text{sonic}}^{\mathcal{O}}$ is a closed portion of a circle, which becomes one point for $\beta \geq \beta_{\text{s}}^{(v_\infty)}$. Therefore, the boundary condition (2.5.35) on $\Gamma_{\text{sonic}}^{\mathcal{O}}$ becomes a one-point boundary condition for $\beta \geq \beta_{\text{s}}^{(v_\infty)}$.

REMARK 2.35. It can be checked from the definitions of $(\varphi_{\mathcal{O}}, \varphi_{\mathcal{N}})$ given in (2.5.1) that, for each $\beta \in (0, \frac{\pi}{2})$, there exists a unique ξ_1^* such that

$$\varphi_\beta(\xi_1, \xi_2) = \begin{cases} \varphi_{\mathcal{O}} & \text{for } \xi_1 < \xi_1^*, \\ \varphi_{\mathcal{O}} = \varphi_{\mathcal{N}} & \text{at } \xi_1 = \xi_1^*, \\ \varphi_{\mathcal{N}} & \text{for } \xi_1 > \xi_1^*. \end{cases}$$

Moreover, ξ_1^* satisfies that $f_{\mathcal{O}}(\xi_1^*) = \xi_2^{\mathcal{N}}$ and $\xi_1^{P_\beta} < \xi_1^* < 0$. In particular, $\varphi_\beta = \varphi_{\mathcal{O}}$ on $\Gamma_{\text{sonic}}^{\mathcal{O}}$ and $\varphi_\beta = \varphi_{\mathcal{N}}$ on $\Gamma_{\text{sonic}}^{\mathcal{N}}$.

2.6. Further Features of Problem 2.34

Fix $\gamma \geq 1$. For $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ with $\beta < \beta_{\text{s}}^{(v_\infty)}$, let P_1 and P_2 be the points as defined in Definition 2.23. Let $L_{\mathcal{O}}$ be the line segment connecting P_1 with P_2. For $0 < v_\infty < 1$, there exists a unique line L_∞ that passes through P_2 and is tangential to $\partial B_1(O_\infty)$ so that the intersection point of L_∞ with $\partial B_1(O_\infty)$ has a negative ξ_1–coordinate; see Fig. 2.13. Let $\tan\theta_{\mathcal{O}}$ and $\tan\theta_\infty$ be the slopes of $L_{\mathcal{O}}$ and L_∞, respectively. Then

$$\text{dist}(L_{\mathcal{O}}, O_\infty)) \begin{cases} > 1 & \text{iff } \theta_{\mathcal{O}} < \theta_\infty, \\ < 1 & \text{iff } \theta_{\mathcal{O}} > \theta_\infty. \end{cases}$$

Note that $\tan\theta_\infty$ is independent of $\beta \in (0, \beta_{\text{s}}^{(v_\infty)})$, and $O_\infty = (0, -v_\infty)$.
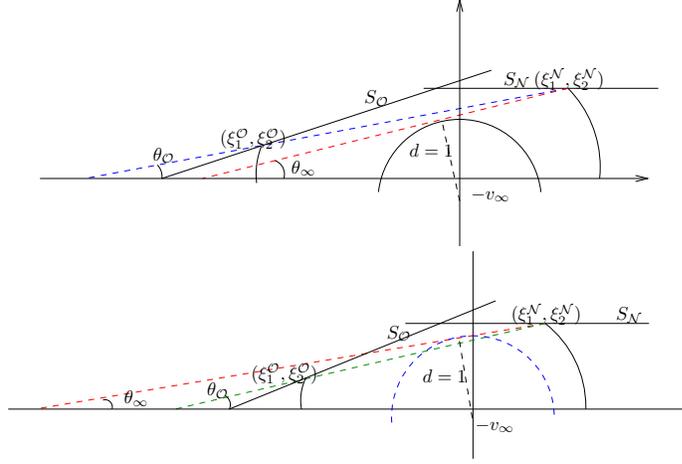
FIGURE 2.13. Top: $\theta_{\mathcal{O}} < \theta_{\infty}$; Bottom: $\theta_{\mathcal{O}} > \theta_{\infty}$

PROPOSITION 2.36. *For any given $\gamma \geq 1$, there exists a constant $v_* \in (0,1)$ so that, if $0 < v_{\infty} < v_*$, then there is $\hat{\beta}^{(v_{\infty})} \in (0, \beta_{\mathrm{s}}^{(v_{\infty})})$ such that*

$$\text{(2.6.1)} \qquad \text{dist}(L_{\mathcal{O}}, O_{\infty}) > 1 \qquad \text{for } \beta \in (0, \hat{\beta}^{(v_{\infty})}),$$

$$\text{(2.6.2)} \qquad \text{dist}(L_{\mathcal{O}}, O_{\infty}) < 1 \qquad \text{for } \beta \in (\hat{\beta}^{(v_{\infty})}, \beta_{\mathrm{s}}^{(v_{\infty})}).$$

PROOF. In this proof, we consider only case $\gamma > 1$. Case $\gamma = 1$ can be handled similarly. The proof is divided into seven steps.

1. *Claim: For each $\gamma > 1$, $P_2 = (\xi_1^{\mathcal{N}}, \xi_2^{\mathcal{N}})$ and $\rho_{\mathcal{N}}$ depend continuously on $v_{\infty} > 0$ and*

$$\text{(2.6.3)} \qquad \lim_{v_{\infty} \to 0+} \xi_1^{\mathcal{N}} = 0, \quad \lim_{v_{\infty} \to 0+} \rho_{\mathcal{N}} = \lim_{v_{\infty} \to 0+} \xi_2^{\mathcal{N}} = 1.$$

Substituting $\rho_{\mathcal{O}} = \rho_{\mathcal{N}}$ into (2.4.5), we have

$$\text{(2.6.4)} \qquad F_1(\rho_{\mathcal{N}}, v_{\infty}) := \frac{\rho_{\mathcal{N}}^{\gamma-1} - 1}{\gamma - 1}(\rho_{\mathcal{N}} - 1) - \frac{1}{2}v_{\infty}^2(\rho_{\mathcal{N}} - 1) - v_{\infty}^2 = 0.$$

We differentiate F_1 with respect to $\rho_{\mathcal{N}}$ to obtain

$$\text{(2.6.5)} \qquad \partial_{\rho_{\mathcal{N}}} F_1 = \rho_{\mathcal{N}}^{\gamma-2}(\rho_{\mathcal{N}} - 1) + \frac{\rho_{\mathcal{N}}^{\gamma-1} - 1}{\gamma - 1} - \frac{1}{2}v_{\infty}^2.$$

Using (2.6.4) to obtain that $\dfrac{\rho_{\mathcal{N}}^{\gamma-1} - 1}{\gamma - 1} = \dfrac{1}{2}v_{\infty}^2 + \dfrac{v_{\infty}^2}{\rho_{\mathcal{N}} - 1}$, substituting this expression into (2.6.5), and then applying $\rho_{\mathcal{N}} > 1$, we have

$$\partial_{\rho_{\mathcal{N}}} F_1 = \rho_{\mathcal{N}}^{\gamma-2}(\rho_{\mathcal{N}} - 1) + \frac{v_{\infty}^2}{\rho_{\mathcal{N}} - 1} > 0.$$

Then the implicit function theorem implies that $\rho_{\mathcal{N}}$ is of the C^1–dependence on $v_{\infty} > 0$.

The C^1–dependence of P_2 on v_{∞} follows directly from (2.2.4) and $\xi_1^{\mathcal{N}} = \sqrt{c_{\mathcal{N}}^2 - (\xi_2^{\mathcal{N}})^2}$.

By the C^1–dependence of $\rho_\mathcal{N}$ on v_∞, we have

$$0 = \frac{\mathrm{d}F_1(\rho_\mathcal{N}(v_\infty), v_\infty)}{\mathrm{d}v_\infty} = \partial_{\rho_\mathcal{N}} F_1 \frac{\mathrm{d}\rho_\mathcal{N}}{\mathrm{d}v_\infty} - (\rho_\mathcal{N} - 1)v_\infty.$$

Since $\partial_{\rho_\mathcal{N}} F_1 > 0$ is shown above, then $\frac{\mathrm{d}\rho_\mathcal{N}}{\mathrm{d}v_\infty} > 0$. This implies that $\rho_\mathcal{N}(v_\infty)$ is bounded above by a finite constant for $v_\infty > 0$ sufficiently small so that it follows directly from (2.6.4) that

$$(2.6.6) \qquad \lim_{v_\infty \to 0+} \rho_\mathcal{N} = 1.$$

By (2.4.8) and (2.6.6), we find that $\lim_{v_\infty \to 0+} M_\infty(P_2) = 1$. We combine this limit with (2.4.14) to obtain

$$(2.6.7) \qquad \lim_{v_\infty \to 0+} \xi_2^\mathcal{N} = 1.$$

Finally, $\lim_{v_\infty \to 0+} \xi_1^\mathcal{N} = 0$ is obtained from $\xi_1^\mathcal{N} = \sqrt{c_\mathcal{N}^2 - (\xi_2^\mathcal{N})^2}$, and the limit of $\xi_2^\mathcal{N}$ is given in (2.6.7). The claim is verified.

2. For each $\gamma > 1$, there exists a small constant $\sigma > 0$ so that $\xi_1^\mathcal{N} < 1$ whenever $0 < v_\infty \leq \sigma$. Fix $\gamma > 1$. For $0 < v_\infty \leq \sigma$, define a function $F : (0, \beta_\mathrm{s}^{(v_\infty)}) \to \mathbb{R}$ by

$$(2.6.8) \qquad F(\beta) := \tan \theta_\mathcal{O} - \tan \theta_\infty.$$

Claim: For any given $\gamma > 1$, there exists a constant $v_* \in (0, \sigma]$ so that, if $0 < v_\infty < v_*$, there is $\hat{\beta}^{(v_\infty)} \in (0, \beta_\mathrm{s}^{(v_\infty)})$ such that

$$(2.6.9) \qquad \begin{aligned} F(\beta) &< 0 \qquad \text{for all } \beta \in (0, \hat{\beta}^{(v_\infty)}), \\ F(\beta) &> 0 \qquad \text{for all } \beta \in (\hat{\beta}^{(v_\infty)}, \beta_\mathrm{s}^{(v_\infty)}). \end{aligned}$$

Once the claim is verified, then (2.6.1) directly follows.

3. We first show that, for each $v_\infty \in (0, \sigma]$, $F'(\beta) \geq 0$ holds for all $\beta \in (0, \beta_\mathrm{s}^{(v_\infty)})$. Fix $v_\infty \in (0, \sigma]$.

We use the equation of line L_∞:

$$(\xi_1 - \xi_1^\mathcal{N}) \tan \theta_\infty - (\xi_2 - \xi_2^\mathcal{N}) = 0$$

to see

$$\mathrm{dist}(L_\infty, (0, -v_\infty)) = \frac{|(\tan \theta_\infty, -1, -\xi_1^\mathcal{N} \tan \theta_\infty + \xi_2^\mathcal{N}) \cdot (0, -v_\infty, 1)|}{\sqrt{1 + \tan^2 \theta_\infty}} = 1,$$

and then solve it for $\tan \theta_\infty$ to obtain

$$\tan \theta_\infty = \frac{\sqrt{((v_\infty + \xi_2^\mathcal{N})^2 - 1) + (\xi_1^\mathcal{N})^2} - (v_\infty + \xi_2^\mathcal{N})\xi_1^\mathcal{N}}{1 - (\xi_1^\mathcal{N})^2}.$$

Let $(q_\infty, q_\mathcal{O})$ be given by (2.4.33). By (2.4.34)–(2.4.35) in the proof of Lemma 2.22, we have shown that $\xi_2^\mathcal{O} = -v_\infty + q_\infty \cos \beta - \sin \beta \sqrt{\Delta}$ with $\Delta := c_\mathcal{O}^2 - q_\mathcal{O}^2$. Substituting this expression into $\xi_1^\mathcal{O} = u_\mathcal{O} - \sqrt{c_\mathcal{O}^2 - (\xi_2^\mathcal{O})^2}$ and then using (2.4.11) and (2.5.5), we have

$$\xi_1^\mathcal{O} = -v_\infty \tan \beta - \left(\cos \beta \sqrt{c_\mathcal{O}^2 - q_\mathcal{O}^2} + q_\mathcal{O} \sin \beta \right),$$

so that

$$\tan\theta_{\mathcal{O}} = \frac{\xi_2^{\mathcal{N}} - \xi_2^{\mathcal{O}}}{\xi_1^{\mathcal{N}} - \xi_1^{\mathcal{O}}} = \frac{v_\infty - q_\infty\cos\beta + \sin\beta\sqrt{\Delta} + \xi_2^{\mathcal{N}}}{\cos\beta\sqrt{\Delta} + q_\infty\sin\beta + \xi_1^{\mathcal{N}}}.$$

Since $\tan\theta_\infty$ is independent of β, we have

$$F'(\beta) = \frac{G(\beta)}{(\xi_1^{\mathcal{N}} + q_\infty\sin\beta + \cos\beta\sqrt{\Delta})^2},$$

where

$$G(\beta) = \left(q_\infty + \frac{1}{2\sqrt{\Delta}}\frac{\mathrm{d}\,\Delta}{\mathrm{d}\beta}\right)\left(q_\infty + \xi_1^{\mathcal{N}}\sin\beta - (v_\infty + \xi_2^{\mathcal{N}})\cos\beta\right)$$
$$+ \left(\sqrt{\Delta} - \frac{\mathrm{d}q_\infty}{\mathrm{d}\beta}\right)\left(\sqrt{\Delta} + \xi_1^{\mathcal{N}}\cos\beta + (v_\infty + \xi_2^{\mathcal{N}})\sin\beta\right).$$

By (2.4.33), (2.4.40), and (2.4.43), we obtain

$$\frac{\mathrm{d}\,\Delta}{\mathrm{d}\beta} = \frac{\mathrm{d}c_{\mathcal{O}}^2(1 - M_{\mathcal{O}}^2)}{\mathrm{d}\beta} > 0 \qquad \text{for all } \beta \in (0, \frac{\pi}{2}).$$

A direct computation yields that

$$q_\infty + \xi_1^{\mathcal{N}}\sin\beta - (v_\infty + \xi_2^{\mathcal{N}})\cos\beta = (P_2 - P_1)\cdot \mathbf{n}_{S_{\mathcal{O}}} > 0$$

for the unit normal vector $\mathbf{n}_{S_{\mathcal{O}}}$ to $S_{\mathcal{O}}$ pointing towards $O_\infty = (0, -v_\infty)$ for all $\beta \in (0, \beta_{\mathrm{s}}^{(v_\infty)})$. Combining the two previous inequalities, we have

$$G(\beta) > \left(\sqrt{\Delta} - \frac{\mathrm{d}q_\infty}{\mathrm{d}\beta}\right)\left(\sqrt{\Delta} + \xi_1^{\mathcal{N}}\cos\beta + (v_\infty + \xi_2^{\mathcal{N}})\sin\beta\right).$$

Therefore, we can conclude that $F'(\beta) > 0$, provided that $\sqrt{\Delta} - \frac{\mathrm{d}q_\infty}{\mathrm{d}\beta} > 0$ for $0 < \beta < \beta_{\mathrm{s}}^{(v_\infty)}$ can be proved.

A straightforward computation by using (2.4.9), (2.4.36), and (2.4.38) yields that

$$\frac{\mathrm{d}q_\infty}{\mathrm{d}\beta} = \frac{(q_\infty^{\gamma-1} - q_{\mathcal{O}}^{\gamma+1})\tan\beta}{q_\infty^{\gamma-2} + q_{\mathcal{O}}^{\gamma}}.$$

Using (2.4.8) and (2.4.33), we obtain that $c_{\mathcal{O}}^2 = \left(\frac{q_\infty}{q_{\mathcal{O}}}\right)^{\gamma-1}$. Then

$$\Delta - \left(\frac{\mathrm{d}q_\infty}{\mathrm{d}\beta}\right)^2 = \frac{q_\infty^{\gamma-1} - q_{\mathcal{O}}^{\gamma+1}}{q_{\mathcal{O}}^{\gamma-1}}\left(1 - \underbrace{\frac{q_{\mathcal{O}}^{\gamma-1}(q_\infty^{\gamma-1} - q_{\mathcal{O}}^{\gamma+1})\tan^2\beta}{(q_\infty^{\gamma-2} + q_{\mathcal{O}}^{\gamma})^2}}_{(=:\sigma)}\right).$$

It can be checked directly that $\frac{\mathrm{d}q_\infty}{\mathrm{d}\beta} > 0$, by differentiating (2.4.12) with respect to β and applying (2.4.10). Then we have

$$(2.6.10) \qquad q_\infty^{\gamma-1} - q_{\mathcal{O}}^{\gamma+1} = \frac{q_\infty^{\gamma-2} + q_{\mathcal{O}}^{\gamma}}{\tan\beta}\frac{\mathrm{d}q_\infty}{\mathrm{d}\beta} > 0.$$

Since $\xi_2^{\mathcal{O}} = -v_\infty + q_\infty\cos\beta - \sin\beta\sqrt{\Delta} > 0$ for $\beta < \beta_{\mathrm{s}}^{(v_\infty)}$, (2.4.11) implies that $q_{\mathcal{O}}^2 > c_{\mathcal{O}}^2\sin^2\beta$. Substituting $c_{\mathcal{O}}^2 = \rho_{\mathcal{O}}^{\gamma-1} = \left(\frac{q_\infty}{q_{\mathcal{O}}}\right)^{\gamma-1}$ into this inequality, we find that $q_\infty^{\gamma-1} < \frac{q_{\mathcal{O}}^{\gamma+1}}{\sin^2\beta}$, which implies

that

$$\sigma = \frac{(\frac{1}{K} - 1)\tan^2\beta}{(\frac{q_{\mathcal{O}}}{Kq_{\infty}} + 1)^2} < 1 \qquad \text{for all } \beta \in (0, \beta_{\mathrm{s}}^{(v_{\infty})}),$$

where $K = \frac{q_{\mathcal{O}}^{\gamma+1}}{q_{\infty}^{\gamma-1}}$. This implies that $\sqrt{\Delta} - \frac{\mathrm{d}q_{\infty}}{\mathrm{d}\beta} > 0$ for $0 < \beta < \beta_{\mathrm{s}}^{(v_{\infty})}$.

Therefore, $F'(\beta) > 0$ for all $\beta \in (0, \beta_{\mathrm{s}}^{(v_{\infty})})$.

4. At $\beta = 0$, $\xi_2^{\mathcal{O}} = \xi_2^{\mathcal{N}}$. This directly yields that $F(0) = -\tan\theta_{\infty} < 0$.

5. Fix $v_{\infty} \in (0, \sigma]$. At $\beta = \beta_{\mathrm{s}}^{(v_{\infty})}$, $\xi_2^{\mathcal{O}} = 0$. Let $\xi_1^{\mathcal{O}*}$ denote the ξ_1–coordinate of point P_1 at $\beta = \beta_{\mathrm{s}}^{(v_{\infty})}$. Then we have

$$F(\beta_{\mathrm{s}}^{(v_{\infty})}) = \frac{a - b}{(1 - (\xi_1^{\mathcal{N}})^2)(\xi_1^{\mathcal{N}} - \xi_1^{\mathcal{O}*})},$$

where

(2.6.11)
$$a := \left(\xi_1^{\mathcal{N}} - \xi_1^{\mathcal{O}*}\right)\left(v_{\infty} + \xi_2^{\mathcal{N}}\right)\xi_1^{\mathcal{N}} - \xi_2^{\mathcal{N}}\left((\xi_1^{\mathcal{N}})^2 - 1\right),$$
$$b := \left(\xi_1^{\mathcal{N}} - \xi_1^{\mathcal{O}*}\right)\sqrt{(v_{\infty} + \xi_2^{\mathcal{N}})^2 + ((\xi_1^{\mathcal{N}})^2 - 1)}.$$

Claim: $\xi_1^{\mathcal{O}*}$ depends continuously on $v_{\infty} \in (0, \sigma]$.

This can be seen as follows: fix $\beta = \beta_{\mathrm{s}}^{(v_{\infty})}$, then

(5-1) Since $\xi_2^{\mathcal{O}} = 0$ at $\beta = \beta_{\mathrm{s}}^{(v_{\infty})}$, we derive from (2.4.34)–(2.4.35) that

$$c_{\mathcal{O}}\sqrt{1 - M_{\mathcal{O}}^2}\sin\beta_{\mathrm{s}}^{(v_{\infty})} = -v_{\infty} + q_{\infty}\cos\beta_{\mathrm{s}}^{(v_{\infty})}.$$

We combine this equation with (2.4.11) to yield that $M_{\mathcal{O}} = \sin\beta_{\mathrm{s}}^{(v_{\infty})}$ and substitute this into (2.4.8) to obtain

(2.6.12)
$$\frac{q_{\mathcal{O}}^{\gamma+1}}{q_{\infty}^{\gamma-1}} = \sin^2\beta_{\mathrm{s}}^{(v_{\infty})}.$$

(5-2) By (2.4.7) and the Rankine-Hugoniot jump condition: $\rho_{\mathcal{O}}q_{\mathcal{O}} = q_{\infty}$, we have

$$F_2(\rho_{\mathcal{O}}, q_{\infty}) := \frac{\rho_{\mathcal{O}}^{\gamma-1} - 1}{\gamma - 1} + \frac{1}{2}\left(\frac{q_{\infty}}{\rho_{\mathcal{O}}}\right)^2 - \frac{1}{2}q_{\infty}^2 = 0.$$

The fact that $\partial_{\rho}F_2(\rho_{\mathcal{O}}, q_{\infty}) = \frac{1}{\rho_{\mathcal{O}}}(c_{\mathcal{O}}^2 - q_{\mathcal{O}}^2) > 0$ implies that $\rho_{\mathcal{O}}$ is of the C^1–dependence on q_{∞}, so that $q_{\mathcal{O}} = \frac{q_{\infty}}{\rho_{\mathcal{O}}}$ is of the C^1–dependence on q_{∞}.

(5-3) It can be derived directly from (2.4.11) and (2.6.12) that

(2.6.13)
$$F_3(q_{\infty}, v_{\infty}) := (q_{\infty} - q_{\mathcal{O}})^2\left(1 - \frac{q_{\mathcal{O}}^{\gamma+1}}{q_{\infty}^{\gamma-1}}\right) - v_{\infty}^2 = 0,$$

where $q_{\mathcal{O}}$ is regarded as a C^1–function of q_{∞} by (5-2). A direct computation by using (2.4.11), (2.4.38), and (2.6.12) shows that $\partial_{q_{\infty}}F_2(q_{\infty}, v_{\infty}) \geq \frac{4v_{\infty}\cos\beta_{\mathrm{s}}^{(v_{\infty})}}{\gamma+1} > 0$. This implies that q_{∞} is of the C^1–dependence on v_{∞}.

(5-4) $\xi_1^{\mathcal{O}*}$ is the ξ_1–intercept of $S_{\mathcal{O}}$ so that $\xi_1^{\mathcal{O}*} = -v_{\infty}\tan\beta_{\mathrm{s}}^{(v_{\infty})} - q_{\mathcal{O}}\csc\beta_{\mathrm{s}}^{(v_{\infty})}$. By the C^1–dependence of $\beta_{\mathrm{s}}^{(v_{\infty})}$ and $q_{\mathcal{O}}$ on v_{∞}, we conclude that $\xi_1^{\mathcal{O}*}$ is of the C^1–dependence on v_{∞}. The claim is verified.

6. *Claim*: For a and b defined in (2.6.11), $\lim_{v_\infty \to 0+} (a^2 - b^2) = 1$.

It suffices to show that $\sup_{v_\infty \in (0,\sigma]} |\xi_1^{\mathcal{O}*}|$ is bounded, due to (2.6.3). From (2.6.13), we have two cases: $\lim_{v_\infty \to 0+} \dfrac{q_{\mathcal{O}}}{q_\infty} = 1$ and $\lim_{v_\infty \to 0+} \dfrac{q_{\mathcal{O}}^{\gamma+1}}{q_\infty^{\gamma-1}} = 1$.

For the case that $\lim_{v_\infty \to 0+} \dfrac{q_{\mathcal{O}}}{q_\infty} = 1$, (2.6.12) implies that $\sup_{(0,\sigma]} q_\infty$ is finite. Then it follows from $q_\infty = D\varphi_\infty(P_1) \cdot \mathbf{n}_{S_{\mathcal{O}}} = -\xi_1^{\mathcal{O}*} \sin \beta_s^{(v_\infty)} + v_\infty \cos \beta_s^{(v_\infty)}$ that $\sup_{v_\infty \in (0,\sigma]} |\xi_1^{\mathcal{O}*} \sin \beta_s^{(v_\infty)}|$ is finite. We multiply (2.6.12) by $(\xi_1^{\mathcal{O}*})^2$ to obtain

$$\sup_{v_\infty \in (0,\sigma]} \left(\xi_1^{\mathcal{O}*}\right)^2 \leq \sup_{v_\infty \in (0,\sigma]} \left(\xi_1^{\mathcal{O}*} \sin \beta_s^{(v_\infty)}\right)^2 \frac{q_\infty^{\gamma+1}}{q_{\mathcal{O}}^{\gamma+1}} < \infty,$$

where we have used the fact that $q_\infty > 1$ for each $v_\infty > 0$.

For the case that $\lim_{v_\infty \to 0+} \dfrac{q_{\mathcal{O}}^{\gamma+1}}{q_\infty^{\gamma-1}} = 1$, we substitute $\rho_{\mathcal{O}} = \frac{q_\infty}{q_{\mathcal{O}}}$ into $F_2(\rho_{\mathcal{O}}, q_\infty) = 0$ to obtain

$$\frac{1}{2}\left(\frac{q_\infty}{q_{\mathcal{O}}}\right)^2 = \frac{1}{\gamma - 1}\left(\frac{q_\infty^{\gamma-1}}{q_{\mathcal{O}}^{\gamma+1}} - \frac{1}{q_{\mathcal{O}}^2}\right) + \frac{1}{2} \leq \frac{1}{\gamma - 1}\frac{q_\infty^{\gamma-1}}{q_{\mathcal{O}}^{\gamma+1}} + \frac{1}{2}.$$

From this, it follows that $\sup_{v_\infty \in (0,\sigma]} |\frac{q_\infty}{q_{\mathcal{O}}}|$ is finite. Then we use (2.6.12) to see that $\sup_{v_\infty \in (0,\sigma]} q_\infty$ is finite. Finally, we repeat the argument for the case that $\lim_{v_\infty \to 0+} \dfrac{q_{\mathcal{O}}}{q_\infty} = 1$ to conclude that $\sup_{v_\infty \in (0,\sigma]} \left(\xi_1^{\mathcal{O}*}\right)^2$ is finite, which implies the claim.

7. By the result obtained from Step 6, there exists a constant $v_* \in (0, \sigma]$ such that $F(0) < 0 < F(\beta_s^{(v_\infty)})$ for all $v_\infty \in (0, v_*]$. Finally, the monotonicity of $F(\beta)$, proved in Step 3, yields Proposition 2.36. $\square$

When (2.6.1) holds, the existence of a solution of Problem 2.34 has been proved in [**26**]. This implies the global existence of a weak solution of Problem 2.9 with the structure of Fig. 2.5, provided that (2.6.1) holds. In this monograph, we establish the global existence of admissible solutions for all $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ (*i.e.*, the global existence of weak solutions to *Problem* 2.9 for all $(u_\infty, u_0) \in \mathfrak{P}_{\text{weak}}$), which includes the case that (2.6.2) holds, or the case that $\beta \geq \beta_s^{(v_\infty)}$.

CHAPTER 3

Uniform Estimates of Admissible Solutions

As in [**11**], we employ the Leray-Schauder degree to prove Theorem 2.31. In order to construct an iteration set (as a subset of a properly defined Banach space) and an iteration mapping, we first establish uniform estimates of admissible solutions corresponding to (v_∞, β) with respect to $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)} - \varepsilon]$ in the sense of Definitions 2.24 and 2.27 for each $v_\infty > 0$ and small $\varepsilon > 0$. In particular, it is crucial to establish the uniform estimates of the size of the pseudo-subsonic region Ω, and the pseudo-potential function φ restricted to Ω in properly chosen norms. Following the approach of [**11**], we establish various uniform estimates of admissible solutions in the following order:

- Strict directional monotonicity properties of $\varphi_\infty - \varphi$,
- Strict directional monotonicity properties of $\varphi - \varphi_\mathcal{N}$ and $\varphi - \varphi_\mathcal{O}$,
- Uniform positive lower bound of the distance between Γ_{shock} and Γ_{wedge} away from P_β,
- Uniform positive lower bound of $\mathrm{dist}(\Gamma_{\mathrm{shock}}, \partial B_1(O_\infty))$,
- Uniform estimates of the ellipticity of Eq. (2.1.19) in Ω,
- Uniform weighted $C^{2,\alpha}$ estimates of admissible solutions in Ω.

Fix $\gamma \geq 1$ and $v_\infty > 0$. For each $\beta \in [0, \frac{\pi}{2})$, let $(\varphi_\infty, \varphi_\mathcal{N}, \varphi_\mathcal{O})$ and $(O_\infty, O_\mathcal{O}, O_\mathcal{N})$ be defined by Definition 2.23. We also follow Definition 2.23 for the notations of $(\Gamma_{\mathrm{sonic}}^{\mathcal{N}}, \Gamma_{\mathrm{sonic}}^{\mathcal{O}})$ and (P_1, P_2, P_3, P_4).

Note that the definitions of $(\Gamma_{\mathrm{sonic}}^{\mathcal{O}}, P_1, P_4)$ are different for the respective cases $\beta \in [0, \beta_{\mathrm{s}}^{(v_\infty)})$ and $\beta \in [\beta_{\mathrm{s}}^{(v_\infty)}, \frac{\pi}{2})$, but they depend continuously on $\beta \in (0, \frac{\pi}{2})$.

3.1. Directional Monotonicity Properties of Admissible Solutions

In this section, we establish directional monotonicity properties of $\varphi_\infty - \varphi$, $\varphi - \varphi_\mathcal{N}$, and $\varphi - \varphi_\mathcal{O}$ for admissible solutions φ in the sense of Definition 2.24.

3.1.1. Strict directional monotonicity of $\varphi_\infty - \varphi$**.** For an admissible solution φ in the sense of Definition 2.24 for $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$, define

$$(3.1.1) \qquad\qquad \phi := \varphi - \varphi_\mathcal{N} \qquad \text{in } \Omega.$$

Then ϕ satisfies the equation:

$$(3.1.2) \qquad\qquad (c^2 - \varphi_{\xi_1}^2)\phi_{\xi_1\xi_1} - 2\varphi_{\xi_1}\varphi_{\xi_2}\phi_{\xi_1\xi_2} + (c^2 - \varphi_{\xi_2}^2)\phi_{\xi_2\xi_2} = 0$$

in the pseudo-subsonic region Ω for $c^2 = c^2(|D\varphi|^2, \varphi, \boldsymbol{\xi})$ given by

$$(3.1.3) \qquad\qquad c^2(|\boldsymbol{p}|^2, z, \boldsymbol{\xi}) := \rho^{\gamma-1}(|\boldsymbol{p}|^2, z, \boldsymbol{\xi}),$$

where $\rho(|\boldsymbol{p}|^2, z, \boldsymbol{\xi})$ is defined by (2.4.2).

LEMMA 3.1. *Fix $\gamma \geq 1$ and $v_\infty > 0$. Let φ be an admissible solution in the sense of Definition 2.24 for $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ with $\beta > 0$, and let ϕ be given by (3.1.1). Then, for any given unit vector $\boldsymbol{e} \in \mathbb{R}^2$, $\partial_{\boldsymbol{e}} \phi$ is not a constant in Ω.*

PROOF. By condition (ii) of Definition 2.24, ϕ satisfies

$$(3.1.4) \qquad \partial_{\boldsymbol{e}} \phi = 0 \qquad\qquad\qquad \text{on } \Gamma_{\text{sonic}}^{\mathcal{N}},$$

$$(3.1.5) \qquad \partial_{\boldsymbol{e}} \phi = \partial_{\boldsymbol{e}}(\varphi_{\mathcal{O}} - \varphi_{\mathcal{N}}) = \boldsymbol{e} \cdot (u_{\mathcal{O}}, 0) \qquad\qquad \text{on } \Gamma_{\text{sonic}}^{\mathcal{O}}$$

for each unit vector $\boldsymbol{e}$ in $\mathbb{R}^2$.

Suppose that $\partial_{\boldsymbol{e}} \phi$ is a constant in Ω. Then (3.1.4)–(3.1.5) imply that $\boldsymbol{e}$ must be parallel to $\boldsymbol{e}_2 = (0, 1)$, because $u_{\mathcal{O}} \neq 0$, by Definition 2.23. Then $\partial_{\xi_2} \phi \equiv 0$ in Ω, which implies that $\partial_{\xi_1 \xi_2} \phi = \partial_{\xi_2 \xi_2} \phi \equiv 0$ in Ω. Since Eq. (3.1.2) is strictly elliptic in Ω, it follows that $\partial_{\xi_1 \xi_1} \phi \equiv 0$ in Ω. Thus, there exist constants (u, v, k) such that $\phi(\xi_1, \xi_2) = u\xi_1 + v\xi_2 + k$ in Ω. Since the length of $\Gamma_{\text{sonic}}^{\mathcal{N}}$ is nonzero, we obtain from the boundary condition $\phi \equiv 0$ on $\Gamma_{\text{sonic}}^{\mathcal{N}}$ that $D\phi \equiv 0$ in Ω, so that $\phi \equiv 0$ in Ω. However, this contradicts the boundary condition (2.5.35) on $\Gamma_{\text{sonic}}^{\mathcal{O}}$, because $\phi = \varphi_{\mathcal{O}} - \varphi_{\mathcal{N}} = u_{\mathcal{O}} \xi_1 - v_\infty \xi_2^{(\beta)} + v_\infty \xi_2^{\mathcal{N}}$ on $\Gamma_{\text{sonic}}^{\mathcal{O}}$, by Remark 2.35. $\square$

LEMMA 3.2. *Fix $\gamma \geq 1$ and $v_\infty > 0$. Let φ be an admissible solution in the sense of Definition 2.24 for $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ with $\beta > 0$. For vectors $\boldsymbol{e}_{S_{\mathcal{O}}}$ and $\boldsymbol{e}_{S_{\mathcal{N}}}$ given by Definition 2.23, φ satisfies*

$$(3.1.6) \qquad \partial_{\boldsymbol{e}_{S_{\mathcal{O}}}}(\varphi_\infty - \varphi) < 0 \qquad \text{in } \overline{\Omega} \setminus \overline{\Gamma_{\text{sonic}}^{\mathcal{O}}},$$

$$(3.1.7) \qquad \partial_{\boldsymbol{e}_{S_{\mathcal{N}}}}(\varphi_\infty - \varphi) < 0 \qquad \text{in } \overline{\Omega} \setminus \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}}.$$

PROOF. By Definition 2.24(v), any admissible solution φ satisfies that $\partial_{\boldsymbol{e}_{S_{\mathcal{N}}}}(\varphi_\infty - \varphi) \leq 0$ and $\partial_{\boldsymbol{e}_{S_{\mathcal{O}}}}(\varphi_\infty - \varphi) \leq 0$ in Ω. Therefore, it suffices to prove the strict inequalities.

For $\boldsymbol{e} = \boldsymbol{e}_{S_{\mathcal{O}}}$ or $\boldsymbol{e}_{S_{\mathcal{N}}}$, we introduce a coordinate system (S, T) so that $\boldsymbol{e} = (1, 0)$ and $\boldsymbol{e}^\perp = (0, 1)$ in the (S, T)–coordinates. We note that Eq. (2.1.19) is invariant under a coordinate rotation. Also, $D^2(\varphi_\infty - \varphi) = -D^2\phi$ for ϕ given by (3.1.1). Then $\varphi_\infty - \varphi$ satisfies

$$(3.1.8) \qquad (c^2 - \varphi_S^2)(\varphi_\infty - \varphi)_{SS} - 2\varphi_S \varphi_T (\varphi_\infty - \varphi)_{ST} + (c^2 - \varphi_T^2)(\varphi_\infty - \varphi)_{TT} = 0 \qquad \text{in } \Omega.$$

Denote $v := \partial_S(\varphi_\infty - \varphi)$. Then v satisfies the following properties:

(i) $v < 0$ *in Ω.* We differentiate (3.1.8) with respect to S and use the expression:

$$(\varphi_\infty - \varphi)_{TT} = -\frac{(c^2 - \varphi_S^2)(\varphi_\infty - \varphi)_{SS} - 2\varphi_S \varphi_T (\varphi_\infty - \varphi)_{ST}}{c^2 - \varphi_T^2}$$

to obtain a second-order equation for v. Since Eq. (3.1.8) is strictly elliptic in $\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}^{\mathcal{O}}} \cup \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}})$ by Definition 2.24(iii), the equation for v is strictly elliptic in $\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}^{\mathcal{O}}} \cup \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}})$, because the coefficients of the principal part of the equation for v are the same as those in Eq. (3.1.8). Moreover, v is not a constant in Ω by Lemma 3.1, so v cannot attain its maximum in Ω by the strong maximum principle. Thus, $v < 0$ holds in Ω.

(ii) $v < 0$ *on Γ_{wedge}.* On Γ_{wedge}, the slip boundary condition (2.5.36) for φ implies that $\partial_{\xi_2}(\varphi_\infty - \varphi) = -v_\infty$, so that $\partial_{\xi_1 \xi_2}(\varphi_\infty - \varphi) = 0$. In Eq. (3.1.8), we replace (S, T) by (ξ_1, ξ_2) to obtain

$$(3.1.9) \qquad (c^2 - \varphi_{\xi_1}^2)\partial_{\xi_1 \xi_1}(\varphi_\infty - \varphi) + (c^2 - \varphi_{\xi_2}^2)\partial_{\xi_2 \xi_2}(\varphi_\infty - \varphi) = 0 \qquad \text{on } \Gamma_{\text{wedge}}.$$

Let $\{\mathbf{e}_{\xi_1}, \mathbf{e}_{\xi_2}\}$ form an orthonormal basis for coordinates $\boldsymbol{\xi} = (\xi_1, \xi_2)$. By setting $a_1 := \boldsymbol{e} \cdot \mathbf{e}_{\xi_1}$ and $a_2 := \boldsymbol{e} \cdot \mathbf{e}_{\xi_2}$, v is expressed as $v = a_1 \partial_{\xi_1}(\varphi_\infty - \varphi) + a_2 \partial_{\xi_2}(\varphi_\infty - \varphi)$ so that $v_{\xi_1} = a_1 \partial_{\xi_1 \xi_1}(\varphi_\infty - \varphi)$ and $v_{\xi_2} = a_2 \partial_{\xi_2 \xi_2}(\varphi_\infty - \varphi)$ on Γ_{wedge}.

Substituting these expressions into (3.1.9), we obtain the following boundary condition for v:

$$(3.1.10) \qquad \partial_{\xi_2} v + \frac{a_2(c^2 - \varphi_{\xi_1}^2)}{a_1(c^2 - \varphi_{\xi_2}^2)} \partial_{\xi_1} v = 0 \qquad \text{on } \Gamma_{\text{wedge}}.$$

Since $\boldsymbol{e} \cdot \mathbf{e}_{\xi_1} \neq 0$, i.e., $a_1 \neq 0$, (3.1.10) is an oblique boundary condition for v on Γ_{wedge}. Thus, Hopf's lemma applies. Therefore, v cannot attain its maximum on Γ_{wedge}, which implies that $v < 0$ on Γ_{wedge}.

(iii) $v < 0$ *on* Γ_{shock}. Suppose that $v(\hat{P}) = 0$ for some $\hat{P} \in \Gamma_{\text{shock}}$. Let $\mathbf{n}_{\text{sh}}$ be the unit normal vector to Γ_{shock} towards the interior of Ω, and let $\boldsymbol{\tau}_{\text{sh}}$ be the unit tangent vector to Γ_{shock} with $\boldsymbol{\tau}_{\text{sh}} \cdot \boldsymbol{e}_{S_{\mathcal{N}}} < 0$. Differentiating the Rankine-Hugoniot jump condition: $\left[\rho(|D\varphi|^2, \varphi)D\varphi \cdot \mathbf{n}_{\text{sh}}\right]_{\Gamma_{\text{shock}}} = 0$ in the direction of $\boldsymbol{\tau}_{\text{sh}}$, we have

$$(3.1.11) \qquad D^2(\varphi_\infty - \varphi)[\boldsymbol{\tau}_{\text{sh}}, \boldsymbol{h}] := \boldsymbol{\tau}_{\text{sh}} \cdot D^2(\varphi_\infty - \varphi)\boldsymbol{h} = 0 \qquad \text{on } \Gamma_{\text{shock}},$$

where $\boldsymbol{h} = h_{\text{n}} \mathbf{n}_{\text{sh}} + h_{\text{t}} \boldsymbol{\tau}_{\text{sh}}$ with

$$(3.1.12) \qquad h_{\text{n}} = -\rho \varphi_{\mathbf{n}_{\text{sh}}}(c^2 - \varphi_{\mathbf{n}_{\text{sh}}}^2), \qquad h_{\text{t}} = (c^2 + \rho \varphi_{\mathbf{n}_{\text{sh}}}^2)\varphi_{\boldsymbol{\tau}_{\text{sh}}}.$$

We refer to §5.1.3 of [**11**] for the verification of (3.1.11).

It follows from Lemma 2.26(a) and the ellipticity of (2.1.19) in $\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}^{\mathcal{O}}} \cup \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}})$ that

$$(3.1.13) \qquad h_{\text{n}} < 0 \qquad \text{on } \Gamma_{\text{shock}}.$$

Since it is assumed that $v = \partial_{\boldsymbol{e}}(\varphi_\infty - \varphi)$ has a local extremum at $\hat{P} \in \Gamma_{\text{shock}}$, we have

$$(3.1.14) \qquad D^2(\varphi_\infty - \varphi)[\boldsymbol{\tau}_{\text{sh}}, \boldsymbol{e}] = 0 \qquad \text{at } \hat{P}.$$

We express $\boldsymbol{e} = b_1 \mathbf{n}_{\text{sh}} + b_2 \boldsymbol{\tau}_{\text{sh}}$. Then we rewrite (3.1.11) restricted at $\hat{P}$ and (3.1.14) as a linear system for $(\varphi_\infty - \varphi)_{\boldsymbol{\tau}_{\text{sh}} \mathbf{n}_{\text{sh}}}(\hat{P})$ and $(\varphi_\infty - \varphi)_{\boldsymbol{\tau}_{\text{sh}} \boldsymbol{\tau}_{\text{sh}}}(\hat{P})$. By this linear system and (3.1.8), we find that $D^2(\varphi_\infty - \varphi)(\hat{P}) = \mathbf{0}$, unless

$$(3.1.15) \qquad \det \begin{pmatrix} h_{\text{n}} & h_{\text{t}} \\ b_1 & b_2 \end{pmatrix} = 0 \qquad \text{at } \hat{P}.$$

On the other hand, v is not a constant in Ω by Lemma 3.1, so that $D^2(\varphi_\infty - \varphi)(\hat{P}) = \mathbf{0}$ is, by Hopf's lemma, impossible. Therefore, (3.1.15) must hold, so that $\boldsymbol{e} = k\boldsymbol{h}$ at $\hat{P}$ for some constant $k \neq 0$. This yields that

$$|v(\hat{P})| = |k h_{\text{n}}(\hat{P}) D(\varphi_\infty - \varphi)(\hat{P})| > 0.$$

This contradicts the fact that $v(\hat{P}) = 0$. Therefore, we conclude that $v < 0$ on Γ_{shock}.

(iv) $v < 0$ *on the sonic arcs*. If $\boldsymbol{e} = \boldsymbol{e}_{S_{\mathcal{O}}}$, then $v = \partial_{\boldsymbol{e}_{S_{\mathcal{O}}}}(\varphi_\infty - \varphi_{\mathcal{N}}) = \frac{(v_\infty, -u_{\mathcal{O}}) \cdot (0, -v_\infty)}{\sqrt{u_{\mathcal{O}}^2 + v_\infty^2}} < 0$ on $\Gamma_{\text{sonic}}^{\mathcal{N}}$. This proves (3.1.6).

If $\boldsymbol{e} = \boldsymbol{e}_{S_{\mathcal{N}}}$, then $v = \partial_{\boldsymbol{e}_{S_{\mathcal{N}}}}(\varphi_\infty - \varphi_{\mathcal{O}}) = -(u_{\mathcal{O}}, v_\infty) \cdot (-1, 0) < 0$ on $\Gamma_{\text{sonic}}^{\mathcal{O}}$. This proves (3.1.7). This computation holds even for the case that $\Gamma_{\text{sonic}}^{\mathcal{O}} = \{P_\beta\}$, i.e., $\beta \geq \beta_{\text{s}}^{(v_\infty)}$ by the condition stated in (ii-3) for Case II. $\qquad \square$

Define the following set:

$$(3.1.16) \qquad \text{Cone}(\boldsymbol{e}_{S_\mathcal{O}}, \boldsymbol{e}_{S_\mathcal{N}}) := \{\alpha_1 \boldsymbol{e}_{S_\mathcal{O}} + \alpha_2 \boldsymbol{e}_{S_\mathcal{N}} \ : \ \alpha_1, \alpha_2 \geq 0\},$$

and let $\text{Cone}^0(\boldsymbol{e}_{S_\mathcal{O}}, \boldsymbol{e}_{S_\mathcal{N}})$ be the interior of $\text{Cone}(\boldsymbol{e}_{S_\mathcal{O}}, \boldsymbol{e}_{S_\mathcal{N}})$. By Lemma 3.2, if φ is an admissible solution corresponding to (v_∞, β), then φ satisfies

$$(3.1.17) \qquad \partial_{\boldsymbol{e}}(\varphi_\infty - \varphi) < 0 \qquad \text{in } \overline{\Omega} \text{ for all } \boldsymbol{e} \in \text{Cone}^0(\boldsymbol{e}_{S_\mathcal{O}}, \boldsymbol{e}_{S_\mathcal{N}}).$$

REMARK 3.3. By (2.5.9), $\text{Cone}^0(\boldsymbol{e}_{S_\mathcal{O}}, \boldsymbol{e}_{S_\mathcal{N}})$ can be represented as

$$\text{Cone}^0(\boldsymbol{e}_{S_\mathcal{O}}, \boldsymbol{e}_{S_\mathcal{N}}) = \{re^{i\theta} \ : \ r > 0, \ \beta < \theta < \pi\}.$$

Note that the unit normal vector $\mathbf{n}_{\text{sh}}$ to Γ_{shock} is expressed as $\mathbf{n}_{\text{sh}} = \dfrac{D(\varphi_\infty - \varphi)}{|D(\varphi_\infty - \varphi)|}$. It follows from (3.1.6) that $-\mathbf{n}_{\text{sh}}(P) \in \{e^{i\theta} \ : \ \beta - \frac{\pi}{2} < \theta < \beta + \frac{\pi}{2}\}$ for all $P \in \Gamma_{\text{shock}}$. Moreover, it follows from (3.1.7) that $-\mathbf{n}_{\text{sh}}(P) \in \{e^{i\theta} \ : \ \frac{\pi}{2} < \theta < \frac{3\pi}{2}\}$ for all $P \in \Gamma_{\text{shock}}$. Therefore, we have

$$(3.1.18) \qquad -\mathbf{n}_{\text{sh}}(P) \in \{e^{i\theta} \ : \ \frac{\pi}{2} < \theta < \beta + \frac{\pi}{2}\} \subset \text{Cone}^0(\boldsymbol{e}_{S_\mathcal{O}}, \boldsymbol{e}_{S_\mathcal{N}}) \qquad \text{for all } P \in \Gamma_{\text{shock}},$$

since $\beta \in (0, \beta_{\text{d}}^{(v_\infty)}) \subset (0, \frac{\pi}{2})$.

PROPOSITION 3.4. Given $\gamma \geq 1$ and $v_\infty > 0$, let φ be an admissible solution in the sense of Definition 2.24 for $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$. Then there exists a function $\xi_2 = f_{\text{sh}}(\xi_1)$ such that

(i) $\Gamma_{\text{shock}} = \{\boldsymbol{\xi} \ : \ \xi_2 = f_{\text{sh}}(\xi_1), \ \xi_1^{P_1} < \xi_1 < \xi_1^{P_2}\}$, where $\xi_1^{P_j}$ is the ξ_1–coordinate of point P_j for $j = 1, 2$;

(ii) f_{sh} satisfies

$$(3.1.19) \qquad 0 = f'_{\text{sh}}(\xi_1^{P_2}) < f'_{\text{sh}}(\xi_1) < f'_{\text{sh}}(\xi_1^{P_1}) = \tan\beta \qquad \text{for } \xi_1^{P_1} < \xi_1 < \xi_1^{P_2}.$$

PROOF. Note that $\boldsymbol{e}_{\xi_2} \in \text{Cone}^0(\boldsymbol{e}_{S_\mathcal{O}}, \boldsymbol{e}_{S_\mathcal{N}})$. By (3.1.17), we have

$$(3.1.20) \qquad \partial_{\xi_2}(\varphi_\infty - \varphi) < 0 \qquad \text{on } \overline{\Gamma_{\text{shock}}}.$$

This, combined with Definition 2.24(i), implies that there exists a unique C^1–function f_{sh} satisfying statement (i) above.

Since $\varphi_\infty - \varphi = 0$ holds on Γ_{shock}, f_{sh} satisfies that $(\varphi_\infty - \varphi)(\xi_1, f_{\text{sh}}(\xi_1)) = 0$ for $\xi_1^{P_1} < \xi_1 < \xi_1^{P_2}$. We differentiate this expression with respect to ξ_1 to obtain

$$f'_{\text{sh}}(\xi_1) = -\frac{\partial_{\xi_1}(\varphi_\infty - \varphi)(\xi_1, f_{\text{sh}}(\xi_1))}{\partial_{\xi_2}(\varphi_\infty - \varphi)(\xi_1, f_{\text{sh}}(\xi_1))}.$$

By condition (i-3) of Definition 2.24, we have

$$(3.1.21) \qquad f'_{\text{sh}}(\xi_1^{P_1}) = \tan\beta, \qquad f'_{\text{sh}}(\xi_1^{P_2}) = 0.$$

By conditions (ii-3) and (iv) of Definition 2.24, the unit normal vector $\mathbf{n}_{\text{sh}}$ to Γ_{shock} towards the interior of Ω can be expressed as

$$\mathbf{n}_{\text{sh}}(P) = \frac{D(\varphi_\infty - \varphi)(P)}{|D(\varphi_\infty - \varphi)(P)|} = \frac{(f'_{\text{sh}}(\xi_1), -1)}{\sqrt{1 + (f'_{\text{sh}}(\xi_1))^2}} \qquad \text{at } P = (\xi_1, f_{\text{sh}}(\xi_1)).$$

By Lemma 3.2 and the definition of $(\boldsymbol{e}_{S_\mathcal{O}}, \boldsymbol{e}_{S_\mathcal{N}})$ given in Definition 2.23, we have

$$a_1 \cos\beta(-f'_{\mathrm{sh}}(\xi_1) + \tan\beta) - a_2 f'_{\mathrm{sh}}(\xi_1)$$

(3.1.22)
$$= \sqrt{1 + (f'_{\mathrm{sh}}(\xi_1))^2} \mathbf{n}_{\mathrm{sh}}(P) \cdot (a_1 \boldsymbol{e}_{S_\mathcal{O}} + a_2 \boldsymbol{e}_{S_\mathcal{N}})$$

$$= \sqrt{1 + (f'_{\mathrm{sh}}(\xi_1))^2} \frac{D(\varphi_\infty - \varphi)(P) \cdot (a_1 \boldsymbol{e}_{S_\mathcal{O}} + a_2 \boldsymbol{e}_{S_\mathcal{N}})}{|D(\varphi_\infty - \varphi)(P)|} < 0 \qquad \text{for } \xi_1^{P_1} < \xi_1 < \xi_1^{P_2}$$

for any constants $a_1 \geq 0$ and $a_2 \geq 0$.

If we choose $(a_1, a_2) = (1, 0)$, then (3.1.22) yields

$$f'_{\mathrm{sh}}(\xi_1) < \tan\beta \qquad \text{for } \xi_1^{P_1} < \xi_1 < \xi_1^{P_2}.$$

Choosing $(a_1, a_2) = (0, 1)$, then we have

$$f'_{\mathrm{sh}}(\xi_1) > 0 \qquad \text{for } \xi_1^{P_1} < \xi_1 < \xi_1^{P_2}.$$

Finally, (3.1.19) is obtained by combining the previous two inequalities with (3.1.21). $\qquad\square$

Given $\gamma \geq 1$ and $v_\infty > 0$, if $\beta_* \in \left(0, \beta_{\mathrm{s}}^{(v_\infty)}\right)$ is fixed, then Proposition 3.4 directly implies that

(3.1.23)
$$\inf_{\beta \in (0, \beta_*]} \mathrm{dist}\{\Gamma_{\mathrm{shock}}, \Gamma_{\mathrm{wedge}}\} \geq \inf_{(0, \beta_*]} \xi_2^{P_1} > 0.$$

LEMMA 3.5. *Fix $\gamma \geq 1$ and $v_\infty > 0$. Let φ be an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$ in the sense of Definition 2.24, and let Ω be its pseudo-subsonic region. Then there exists a constant $C > 0$ depending only on (v_∞, γ) such that the following properties hold:*

(3.1.24)
$$\Omega \subset B_C(\mathbf{0}),$$

(3.1.25)
$$\max_{\overline{\Omega}} |\varphi| \leq C, \quad \|\varphi\|_{C^{0,1}(\overline{\Omega})} \leq C,$$

(3.1.26)
$$\rho^*(\gamma) \leq \rho \leq C \ \ \text{in } \Omega, \qquad 1 < \rho \leq C \ \ \text{on } \Gamma_{\mathrm{shock}},$$

where

$$\rho^*(\gamma) = \begin{cases} \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} & \text{for } \gamma > 1, \\ e^{-\frac{1}{2}} = \lim_{\gamma \to 1+} \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} & \text{for } \gamma = 1. \end{cases}$$

PROOF. To prove this lemma, we follow the ideas in the proofs for [**11**, Proposition 9.1.2, Corollary 9.1.3, Lemma 9.1.4].

1. *Proof of* (3.1.24). For an admissible solution φ, let f_{sh} be as in Proposition 3.4. From (3.1.19), it follows that $0 \leq \xi_2^{P_1} \leq f_{\mathrm{sh}}(\xi_1) \leq \xi_2^{P_2}$ on $\xi_1^{P_1}, \xi_1^{P_2}]$. Then

$$\Omega \subset \{\boldsymbol{\xi} = (\xi_1, \xi_2) : u_\mathcal{O} - c_\mathcal{O} < \xi_1 < c_\mathcal{N}, 0 < \xi_2 < \xi_2^{P_2}\}.$$

For any given $v_\infty > 0$, $c_\mathcal{O}$ and $u_\mathcal{O}$ depend continuously on $\beta \in [0, \frac{\pi}{2})$, and $\beta_{\mathrm{d}}^{(v_\infty)}$ depends continuously on $v_\infty > 0$. Therefore, there exists a constant $C_1 > 0$ depending only on (v_∞, γ) such that

$$\sup_{\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]} \left(|u_\mathcal{O}| + |c_\mathcal{O}|\right) \leq C_1.$$

This proves (3.1.24).

2. *Proof of* (3.1.25). By Definition 2.24(iv), we have

$$\inf_\Omega \max\{\varphi_\mathcal{O}, \varphi_\mathcal{N}\} \leq \varphi \leq \sup_\Omega \varphi_\infty.$$

By (3.1.24) and the definition of $(\varphi_\infty, \varphi_\mathcal{O}, \varphi_\mathcal{N})$ given in Definition 2.23, there exists a constant $C_2 > 0$ depending only on (v_∞, γ) such that $-C_2 \leq \min_{\overline{\Omega}} \max\{\varphi_\mathcal{O}, \varphi_\mathcal{N}\} < \max_{\overline{\Omega}} \varphi_\infty \leq C_2$. Then condition (iv) of Definition 2.24 implies that

$$(3.1.27) \qquad\qquad\qquad\qquad \max_{\overline{\Omega}} |\varphi| \leq C_2.$$

By conditions (ii)–(iii) of Definition 2.24, (2.1.22), and (3.1.27), we can choose a constant $\hat{C}_2 > 0$ depending only on (v_∞, γ) such that $\max_{\overline{\Omega}} |D\varphi| \leq \hat{C}_2$ holds for each admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$. This, combined with (3.1.27), yields (3.1.25).

3. *Proof of* (3.1.26). A uniform upper bound of ρ in (3.1.26) is obtained directly from (3.1.25) and (2.4.2).

By condition (iii) of Definition 2.24, any admissible solution φ satisfies

$$h(\rho) + \frac{c^2}{2} \geq h(\rho) + \frac{1}{2}|D\varphi|^2 \qquad \text{in } \overline{\Omega}.$$

Moreover, by (2.1.18) and condition (iv) of Definition 2.24,

$$h(\rho) + \frac{1}{2}|D\varphi|^2 \geq \underbrace{h(1)}_{(=0)} + \frac{1}{2}|D\varphi_\infty|^2 \geq 0 \qquad \text{in } \overline{\Omega}.$$

Then we have

$$h(\rho) + \frac{c^2}{2} \geq 0 \qquad \text{in } \overline{\Omega},$$

so that the first inequality in (3.1.26) is proved.

By Definition 2.11 and condition (iv) of Definition 2.24, any admissible solution satisfies that $\partial_{\boldsymbol{\nu}}\varphi_\infty > \partial_{\boldsymbol{\nu}}\varphi$ on Γ_{shock} for the unit normal vector $\boldsymbol{\nu}$ to Γ_{shock} towards the interior of Ω. Then the Rankine-Hugoniot jump condition stated in Definition 2.24(ii-4) implies that $\rho > 1$ holds on Γ_{shock}, because $\rho_\infty = 1$ is the density of the incoming state corresponding to φ_∞. This verifies the second inequality in (3.1.26). $\qquad\qquad\square$

3.1.2. Directional monotonicity of $\varphi - \varphi_\mathcal{N}$ and $\varphi - \varphi_\mathcal{O}$. Let φ be an admissible solution, and let $\boldsymbol{\nu}$ be the unit normal vector to Γ_{shock} towards the interior of Ω. For each point $P \in \Gamma_{\text{shock}}$, define

$$d(P) := \partial_{\boldsymbol{\nu}}\varphi_\infty(P), \quad \omega(P) := \partial_{\boldsymbol{\nu}}(\varphi_\infty - \varphi)(P)$$

so that

$$\partial_{\boldsymbol{\nu}}\varphi(P) = d(P) - \omega(P).$$

By Lemma 2.26, $d(P) > 1$ and $\omega(P) < d(P)$ on Γ_{shock}. By the Rankine-Hugoniot conditions stated in Definition 2.24(ii-4), $\rho(|D\varphi|^2, \varphi) = \frac{d}{d-\omega}$ on Γ_{shock}. Then it can be derived from (2.4.2) and $\varphi_\infty - \varphi = 0$ on Γ_{shock} that

$$G(\omega, d) := h\left(\frac{d}{d-\omega}\right) + \frac{1}{2}\left((d-\omega)^2 - d^2\right) = 0 \qquad \text{on } \Gamma_{\text{shock}},$$

where $h(\rho)$ is defined by (2.1.5). For a fixed constant $d > 0$, it is direct to see that

$$G(0, d) = 0, \quad \lim_{\omega \to d-} G(\omega, d) = \infty,$$

$$G_\omega(\omega, d) = \frac{d^{\gamma-1}}{(d-\omega)^\gamma} - (d-\omega) \begin{cases} \leq 0 & \text{for } 0 \leq \omega \leq d(1 - d^{-\frac{2}{\gamma+1}}), \\ > 0 & \text{for } \omega > d(1 - d^{-\frac{2}{\gamma+1}}). \end{cases}$$

Therefore, for each $d > 0$, there exists a unique $\omega_d \in (0, d)$ satisfying that $G(\omega_d, d) = 0$. Define a function $H : (1, \infty) \to \mathbb{R}^+$ by

$$(3.1.28) \qquad\qquad H(d) := \omega_d.$$

By continuation, H can be defined up to $d = 1$ with $H(1) = \lim_{d \to 1+} H(d) = 0$. It is shown in [**11**, Lemma 6.1.3] that

$$(3.1.29) \qquad H \in C([1, \infty)) \cap C^\infty((1, \infty)), \qquad H'(d) > 0 \text{ for all } d \in (1, \infty).$$

Therefore, we have

$$(3.1.30) \qquad\qquad H(1) = 0, \qquad H(d) > 0 \text{ if and only if } d > 1.$$

For each $P \in \Gamma_{\text{shock}}$, we have

$$(3.1.31) \qquad\qquad \partial_{\boldsymbol{\nu}}(\varphi_\infty - \varphi)(P) = H(\partial_{\boldsymbol{\nu}}\varphi_\infty(P)).$$

The function, H, is useful in proving several properties of admissible solutions, which include the lemma stated below. The lemma is essential to obtain uniform *a priori* estimates of admissible solutions near $\Gamma^{\mathcal{O}}_{\text{sonic}} \cup \Gamma^{\mathcal{N}}_{\text{sonic}}$.

LEMMA 3.6. *Fix $\gamma \geq 1$ and $v_\infty > 0$. For vectors $(\boldsymbol{e}_{S_{\mathcal{O}}}, \boldsymbol{e}_{S_{\mathcal{N}}})$ given by Definition 2.23, any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ with $\beta > 0$ satisfies*

$$(3.1.32) \qquad\qquad \partial_{\boldsymbol{e}_{S_{\mathcal{N}}}}(\varphi - \varphi_{\mathcal{N}}), \ \ \partial_{\boldsymbol{e}_{S_{\mathcal{O}}}}(\varphi - \varphi_{\mathcal{O}}) \geq 0 \qquad in \ \overline{\Omega},$$

$$(3.1.33) \qquad\qquad -\partial_{\xi_2}(\varphi - \varphi_{\mathcal{N}}), \ \ -\partial_{\xi_2}(\varphi - \varphi_{\mathcal{O}}) \geq 0 \qquad in \ \overline{\Omega}.$$

PROOF. Since $\varphi_\infty - \varphi_{\mathcal{N}}$ is a linear function that vanishes on $S_{\mathcal{N}}$, $\partial_{\boldsymbol{e}_{S_{\mathcal{N}}}}(\varphi - \varphi_{\mathcal{N}}) = \partial_{\boldsymbol{e}_{S_{\mathcal{N}}}}(\varphi - \varphi_\infty)$ in $\overline{\Omega}$. Then (2.5.10) yields that $\partial_{\boldsymbol{e}_{S_{\mathcal{N}}}}(\varphi - \varphi_{\mathcal{N}}) \geq 0$ in $\overline{\Omega}$. Similarly, (2.5.10) also implies that $\partial_{\boldsymbol{e}_{S_{\mathcal{O}}}}(\varphi - \varphi_{\mathcal{O}}) \geq 0$ in $\overline{\Omega}$. This proves (3.1.32).

Define

$$w := \partial_{\xi_2}(\varphi - \varphi_{\mathcal{N}}).$$

We first differentiate Eq. (3.1.2) for $\phi = \varphi - \varphi_{\mathcal{N}}$ with respect to ξ_2 to obtain

$$(3.1.34) \qquad \begin{aligned} &(c^2 - \varphi_{\xi_1}^2)w_{\xi_1\xi_1} - 2\varphi_{\xi_1}\varphi_{\xi_2}w_{\xi_1\xi_2} + (c^2 - \varphi_{\xi_2}^2)w_{\xi_2\xi_2} \\ &+ (c^2 - \varphi_{\xi_1}^2)_{\xi_2}\phi_{\xi_1\xi_1} - 2(\varphi_{\xi_1}\varphi_{\xi_2})_{\xi_2}w_{\xi_1} + (c^2 - \varphi_{\xi_2}^2)_{\xi_2}w_{\xi_2} = 0 \qquad \text{in } \Omega. \end{aligned}$$

Since $c^2 - \varphi_{\xi_1}^2 > 0$ from condition (iii) of Definition 2.24, we use Eq. (3.1.2) to express $\phi_{\xi_1\xi_1}$ as

$$\phi_{\xi_1\xi_1} = \frac{2\varphi_{\xi_1}\varphi_{\xi_2}w_{\xi_1} - (c^2 - \varphi_{\xi_2}^2)w_{\xi_2}}{c^2 - \varphi_{\xi_1}^2}.$$

A direct computation by using (2.4.2) yields that $c_{\xi_2}^2 = -(\gamma - 1)(\varphi_{\xi_1}w_{\xi_1} + \varphi_{\xi_2}w_{\xi_2})$. Finally, $(\varphi_{\xi_i}\varphi_{\xi_j})_{\xi_2}$, $i, j = 1, 2$, can be expressed in terms of $(\varphi_{\xi_1}, \varphi_{\xi_2}, w, w_{\xi_1}, w_{\xi_2})$. Therefore, Eq. (3.1.34) can be rewritten as

$$(c^2 - \varphi_{\xi_1}^2)w_{\xi_1\xi_1} - 2\varphi_{\xi_1}\varphi_{\xi_2}w_{\xi_1\xi_2} + (c^2 - \varphi_{\xi_2}^2)w_{\xi_2\xi_2} + \sum_{j=1}^{2} a_j(\varphi_{\xi_1}, \varphi_{\xi_2}, w, w_{\xi_1}, w_{\xi_2})w_{\xi_j} = 0 \qquad \text{in } \Omega.$$

This equation is strictly elliptic in Ω, and w is not a constant whenever $\beta > 0$, due to Lemma 3.1. Then the maximum principle implies that $\max_{\overline{\Omega}} w = \max_{\partial\Omega} w$.

On $\Gamma^{\mathcal{O}}_{\text{sonic}} \cup \Gamma^{\mathcal{N}}_{\text{sonic}}$, it follows from the definition of $(\varphi_{\mathcal{O}}, \varphi_{\mathcal{N}})$ given in Definition 2.23 and conditions (ii-1) and (ii-3) of Definition 2.24 that

$$(3.1.35) \qquad w = \begin{cases} \partial_{\xi_2}(\varphi_{\mathcal{O}} - \varphi_{\mathcal{N}}) = 0 & \text{on } \Gamma^{\mathcal{O}}_{\text{sonic}}, \\ \partial_{\xi_2}(\varphi_{\mathcal{N}} - \varphi_{\mathcal{N}}) = 0 & \text{on } \Gamma^{\mathcal{N}}_{\text{sonic}}. \end{cases}$$

Using the slip boundary condition: $\partial_{\xi_2}\varphi = 0$ on Γ_{wedge}, stated in Definition 2.24(ii-4), we have

$$w = 0 \qquad \text{on } \Gamma_{\text{wedge}},$$

since $\partial_{\xi_2}\varphi_{\mathcal{N}} = 0$ holds on Γ_{wedge}.

Suppose that there exists a point $\hat{P} \in \Gamma_{\text{shock}}$ such that

$$w(\hat{P}) = \max_{\overline{\Omega}} w, \qquad w(\hat{P}) > 0.$$

Let $\boldsymbol{\nu}$ be the unit normal vector to Γ_{shock} towards the interior of Ω, and let $\boldsymbol{\tau}$ be a tangent vector to Γ_{shock}. Since $D^2\varphi_\infty = D^2\varphi_{\mathcal{N}} = -\mathbf{I}_2$, we can rewrite (3.1.11) as

$$(3.1.36) \qquad D^2(\varphi - \varphi_{\mathcal{N}})[\boldsymbol{\tau}, \boldsymbol{h}] = 0 \qquad \text{on } \Gamma_{\text{shock}},$$

with $\boldsymbol{h} = h_\nu \boldsymbol{\nu} + h_\tau \boldsymbol{\tau}$ for (h_ν, h_τ) given by (3.1.12).

From the assumption that $w(\hat{P}) = \max_{\overline{\Omega}} w$, it follows that $\partial_{\boldsymbol{\tau}} w(\hat{P}) = D^2(\varphi - \varphi_{\mathcal{N}})[\boldsymbol{\tau}, \boldsymbol{e}_{\xi_2}] = 0$ at $\hat{P}$. Also, by Hopf's lemma, w satisfies

$$(3.1.37) \qquad \partial_{\boldsymbol{\nu}} w(\hat{P}) = D^2(\varphi - \varphi_{\mathcal{N}})[\boldsymbol{\nu}, \boldsymbol{e}_{\xi_2}] < 0 \qquad \text{at } \hat{P}.$$

Then we can use similar arguments as to those for the proof of Lemma 3.2 to obtain

$$(3.1.38) \qquad \boldsymbol{e}_{\xi_2} = k\boldsymbol{h}(\hat{P})$$

with some constant $k \neq 0$. By Remark 2.20, $\boldsymbol{e}_{\xi_2} \in \text{Cone}^0(\boldsymbol{e}_{S_{\mathcal{O}}}, \boldsymbol{e}_{S_{\mathcal{N}}})$, so that (3.1.17) implies that $\boldsymbol{e}_{\xi_2} \cdot \boldsymbol{\nu} < 0$ on Γ_{shock}. Then, at point $\hat{P}$, it follows from (3.1.12) and (3.1.38) that

$$kh_\nu(\hat{P}) = k\boldsymbol{h}(\hat{P}) \cdot \boldsymbol{\nu}(\hat{P}) = \boldsymbol{e}_{\xi_2} \cdot \boldsymbol{\nu}(\hat{P}) < 0.$$

Then we obtain from (3.1.13) that $k > 0$.

By the invariance of Eq. (3.1.2) under a coordinate rotation and condition (ii) of Definition 2.24, $\phi = \varphi - \varphi_{\mathcal{N}}$ satisfies

$$(3.1.39) \qquad (c^2 - \varphi_\nu^2)\phi_{\boldsymbol{\nu}\boldsymbol{\nu}} - 2\varphi_{\boldsymbol{\tau}}\varphi_{\boldsymbol{\nu}}\phi_{\boldsymbol{\nu}\boldsymbol{\tau}} + (c^2 - \varphi_\tau^2)\phi_{\boldsymbol{\tau}\boldsymbol{\tau}} = 0 \qquad \text{at } \hat{P}.$$

Here and hereafter, we denote $\varphi_{\boldsymbol{\nu}} = \partial_{\boldsymbol{\nu}}\varphi = D\varphi \cdot \boldsymbol{\nu}$ and $\varphi_{\boldsymbol{\tau}} = \partial_{\boldsymbol{\tau}}\varphi = D\varphi \cdot \boldsymbol{\tau}$ for any function φ,

Using (3.1.36), (3.1.39), and Definition 2.24(iii), we have

$$(3.1.40) \qquad (\phi_{\boldsymbol{\nu}\boldsymbol{\tau}}, \phi_{\boldsymbol{\nu}\boldsymbol{\nu}}) = -\Big(\frac{h_\tau}{h_\nu}, \frac{2\varphi_{\boldsymbol{\nu}}\varphi_{\boldsymbol{\tau}}\frac{h_\tau}{h_\nu} + (c^2 - \varphi_\tau^2)}{c^2 - \varphi_\nu^2}\Big)\phi_{\boldsymbol{\tau}\boldsymbol{\tau}} \qquad \text{at } \hat{P}.$$

Substituting $\boldsymbol{e}_{\xi_2} = k\boldsymbol{h}(\hat{P})$ into (3.1.37), we obtain

$$(3.1.41) \qquad D^2\phi[\boldsymbol{\nu}, \boldsymbol{h}] < 0 \qquad \text{at } \hat{P}.$$

Using (3.1.40), we rewrite (3.1.41) as

$$A\phi_{\boldsymbol{\tau}\boldsymbol{\tau}}(\hat{P}) < 0 \qquad \text{for } A = \frac{c^4\varphi_\tau^2 + \rho^2 c^2 \varphi_\nu^2(c^2 - |D\varphi|^2)}{\rho\varphi_{\boldsymbol{\nu}}} \quad \text{at } \hat{P}.$$

Then it follows from Definition 2.24(iii) and Lemma 2.26 that $A > 0$. Thus, we conclude that $\phi_{\boldsymbol{\tau\tau}}(\hat{P}) < 0$. This implies that
$$(\varphi - \varphi_\infty)_{\boldsymbol{\tau\tau}}(\hat{P}) < 0.$$

Let $f := f_{\mathrm{sh}}$ be from Proposition 3.4. Then, using $(\varphi - \varphi_\infty)_{\boldsymbol{\tau\tau}}(\hat{P}) < 0$ and (3.1.17), we have

$$(3.1.42) \qquad f''(\xi_1^{\hat{P}}) = \frac{(\varphi - \varphi_\infty)_{\boldsymbol{\tau\tau}}\left(1 + (f')^2\right)}{\partial_{\xi_2}(\varphi_\infty - \varphi)} > 0 \qquad \text{at } \hat{P},$$

since $\boldsymbol{e}_{\xi_2} \in \mathrm{Cone}^0(\boldsymbol{e}_{S_\mathcal{O}}, \boldsymbol{e}_{S_\mathcal{N}})$ implies that $\partial_{\xi_2}(\varphi_\infty - \varphi) < 0$ at $\hat{P} \in \Gamma_{\mathrm{shock}}$, due to (3.1.17).

Let $\xi_2 = L(\xi_1)$ be the equation of the tangent line to Γ_{shock} at $\hat{P}$. Denote $F(\xi_1) := f(\xi_1) - L(\xi_1)$. Then there exists a point $P_* \neq \hat{P}$ on $\mathrm{int}\,\Gamma_{\mathrm{shock}}$ such that $F(\xi_1^{P_*}) = \max_{[\xi_1^{P_1}, \xi_1^{P_2}]} F(\xi_1)$, due to (3.1.42).

Note that $P_* \notin \{P_1, P_2\}$, due to (3.1.19) in Proposition 3.4. If $P_* = P_1$, then $F'(\xi_1^{P_1}) \leq 0$ must hold, but this is impossible because $f'(\xi_1^{P_*}) = \tan\beta > f'(\xi_1^{\hat{P}}) = L'(\xi_1^{P_*})$. Similarly, if $P_* = P_2$, then $F'(\xi_1^{P_2}) \geq 0$ must hold, but this is also impossible because $f'(\xi_1^{P_*}) = 0 < f'(\xi_1^{\hat{P}}) = L'(\xi_1^{\hat{P}})$. Therefore, we conclude that $f'(\xi_1^{P_*}) = L'(\xi_1^{P_*}) = f'(\xi_1^{\hat{P}})$. This implies that $\boldsymbol{\nu}(P_*) = \boldsymbol{\nu}(\hat{P})$. Denoting $\boldsymbol{\nu} := \boldsymbol{\nu}(P_*) = \boldsymbol{\nu}(\hat{P})$ by $\boldsymbol{\nu}$, we use the definition of φ_∞ given in Definition 2.23 to obtain

$$(3.1.43) \qquad \partial_{\boldsymbol{\nu}}\varphi_\infty(P_*) = \partial_{\boldsymbol{\nu}}\varphi_\infty(\hat{P}) - \left(\partial_{\boldsymbol{\nu}}\varphi_\infty(\hat{P}) - \partial_{\boldsymbol{\nu}}\varphi_\infty(P_*)\right) = \partial_{\boldsymbol{\nu}}\varphi_\infty(\hat{P}) - (P_* - \hat{P}) \cdot \boldsymbol{\nu}.$$

For each point $P \in \Gamma_{\mathrm{shock}}$, we represent P as $(\xi_1, f_{\mathrm{sh}}(\xi_1))$ and rewrite the expression as
$$P = (\xi_1, f_{\mathrm{sh}}(\xi_1)) = (\xi_1, F(\xi_1) + L(\xi_1)) = (\xi_1, L(\xi_1)) + (0, F(\xi_1)).$$

By using this expression, $P_* - \hat{P}$ is represented as
$$P_* - \hat{P} = (\xi_1^{P_*} - \xi_1^{\hat{P}})(1, L'(\xi_1^{\hat{P}})) + \left(F(T_{P_*}) - F(T_{\hat{P}})\right)\boldsymbol{e}_{\xi_2}.$$

Since $L'(\xi_1^{\hat{P}}) = f'(\xi_1^{\hat{P}})$, $(1, L'(\xi_1^{\hat{P}})) \cdot \boldsymbol{\nu} = (1, f'(\xi_1^{\hat{P}})) \cdot \boldsymbol{\nu}(\hat{P}) = 0$. This yields that
$$(P_* - \hat{P}) \cdot \boldsymbol{\nu} = \left(F(T_{P_*}) - F(T_{\hat{P}})\right)\boldsymbol{e}_{\xi_2} \cdot \boldsymbol{\nu}.$$

By substituting this expression into (3.1.43), $\partial_{\boldsymbol{\nu}}\varphi_\infty(P_*)$ is represented as
$$\partial_{\boldsymbol{\nu}}\varphi_\infty(P_*) = \partial_{\boldsymbol{\nu}}\varphi_\infty(\hat{P}) - \left(F(T_{P_*}) - F(T_{\hat{P}})\right)\boldsymbol{e}_{\xi_2} \cdot \boldsymbol{\nu}(P_*).$$

By (3.1.17) and the definition of P_*, $\left(F(T_{P_*}) - F(T_{\hat{P}})\right)\boldsymbol{e}_{\xi_2} \cdot \boldsymbol{\nu}(P_*) < 0$, which implies that
$$\partial_{\boldsymbol{\nu}}\varphi_\infty(P_*) > \partial_{\boldsymbol{\nu}}\varphi_\infty(\hat{P}).$$

This, combined with (3.1.29) and (3.1.31), leads to
$$(3.1.44) \qquad \partial_{\boldsymbol{\nu}}(\varphi_\infty - \varphi)(P_*) > \partial_{\boldsymbol{\nu}}(\varphi_\infty - \varphi)(\hat{P}).$$

We rewrite $w(P_*)$ as
$$w(P_*) = \partial_{\xi_2}(\varphi - \varphi_\infty)(P_*) + \underbrace{\partial_{\xi_2}(\varphi_\infty - \varphi_\mathcal{N})(P_*)}_{(\equiv -v_\infty)},$$

and further express $\partial_{\xi_2}(\varphi - \varphi_\infty)(P_*) = (\boldsymbol{\nu}(P_*) \cdot \boldsymbol{e}_{\xi_2})\partial_{\boldsymbol{\nu}}(\varphi - \varphi_\infty)(P_*)$, where we have used that $\partial_{\boldsymbol{\tau}}(\varphi - \varphi_\infty) = 0$ holds on Γ_{shock}. Note that $\boldsymbol{\nu}(P_*) \cdot \boldsymbol{e}_{\xi_2} = \boldsymbol{\nu}(\hat{P}) \cdot \boldsymbol{e}_{\xi_2} < 0$, by (3.1.17). Then it follows from (3.1.44) that
$$w(P_*) = \left(\boldsymbol{\nu}(P_*) \cdot \boldsymbol{e}_{\xi_2}\right)\partial_{\boldsymbol{\nu}}(\varphi - \varphi_\infty)(P_*) + \partial_{\xi_2}(\varphi_\infty - \varphi_\mathcal{N})(P_*)$$
$$> \left(\boldsymbol{\nu}(\hat{P}) \cdot \boldsymbol{e}_{\xi_2}\right)\partial_{\boldsymbol{\nu}}(\varphi - \varphi_\infty)(\hat{P}) + \partial_{\xi_2}(\varphi_\infty - \varphi_\mathcal{N})(\hat{P}) = w(\hat{P}).$$

However, this contradicts the assumption that $w(\hat{P}) = \max\limits_{\overline{\Omega}} w$.

Therefore, we conclude that

$$\partial_{\xi_2}(\varphi - \varphi_{\mathcal{N}}) \leq 0 \qquad \text{in } \overline{\Omega}.$$

Since $\partial_{\xi_2}(\varphi_{\mathcal{N}} - \varphi_{\mathcal{O}}) \equiv 0$, we also obtain that $\partial_{\xi_2}(\varphi - \varphi_{\mathcal{O}}) \leq 0$ in $\overline{\Omega}$. This proves (3.1.33). $\qquad\square$

3.2. Uniform Positive Lower Bound of $\text{dist}(\Gamma_{\text{shock}}, \partial B_1(O_\infty))$

In order to obtain a uniform estimate of the ellipticity of Eq. (2.1.19) in the pseudo-subsonic regions of admissible solutions, it is essential to make a uniform estimate of positive lower bound of $\text{dist}(\Gamma_{\text{shock}}, \partial B_1(O_\infty))$ for admissible solutions. Once the estimate of $\text{dist}(\Gamma_{\text{shock}}, \partial B_1(O_\infty))$ is achieved, the ellipticity of Eq. (2.1.19) at each point $\boldsymbol{\xi} \in \Omega$ is uniformly controlled by $\text{dist}(\boldsymbol{\xi}, \Gamma_{\text{sonic}}^{\mathcal{O}} \cup \Gamma_{\text{sonic}}^{\mathcal{N}})$.

PROPOSITION 3.7. Fix $\gamma \geq 1$ and $v_\infty > 0$. Then there exists a constant $C > 0$ depending only on (v_∞, γ) such that any admissible solution corresponding to (v_∞, β) satisfies

$$(3.2.1) \qquad\qquad \text{dist}(\Gamma_{\text{shock}}, \partial B_1(O_\infty)) \geq \frac{1}{C}.$$

To prove Proposition 3.7, some preliminary properties are first required, as shown in Lemmas 3.8–3.13 below.

We rewrite Eq. (2.1.19) as

$$(3.2.2) \qquad\qquad \text{div}\mathcal{A}(D\varphi, \varphi) + \mathcal{B}(D\varphi, \varphi) = 0,$$

with $\mathbf{p} = (p_1, p_2) \in \mathbb{R}^2$ and $z \in \mathbb{R}$, where

$$(3.2.3) \qquad\qquad \mathcal{A}(\mathbf{p}, z) := \rho(|\mathbf{p}|^2, z)\mathbf{p}, \qquad \mathcal{B}(\mathbf{p}, z) := 2\rho(|\mathbf{p}|^2, z)$$

for $\rho(|\mathbf{p}|^2, z)$, given by

$$(3.2.4) \qquad\qquad \rho(|\mathbf{p}|^2, z) = \left(1 + (\gamma - 1)(\frac{v_\infty^2}{2} - \frac{1}{2}|\mathbf{p}|^2 - z)\right)^{\frac{1}{\gamma - 1}}.$$

We also need the definition of $c(|\mathbf{p}|^2, z)$:

$$(3.2.5) \qquad\qquad c(|\mathbf{p}|^2, z) := \rho^{\frac{\gamma - 1}{2}}(|\mathbf{p}|^2, z).$$

For a constant $R > 1$, define

$$(3.2.6) \qquad \mathcal{K}_R = \left\{(\mathbf{p}, z) \in \mathbb{R}^2 \times \mathbb{R} : |\mathbf{p}| + |z| \leq R, \ \rho(|\mathbf{p}|^2, z) \geq R^{-1}, \ \frac{|\mathbf{p}|^2}{c^2(|\mathbf{p}|^2, z)} \leq 1 - R^{-1}\right\}.$$

For each $R > 1$, there exists a constant $\lambda_R > 0$ depending only on (v_∞, γ, R) such that

$$\sum_{i,j=1}^{2} \partial_{p_j}\mathcal{A}_i(\mathbf{p}, z)\kappa_i\kappa_j \geq \lambda_R|\boldsymbol{\kappa}|^2 \qquad \text{for any } (\mathbf{p}, z) \in \mathcal{K}_R \text{ and } \boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2.$$

LEMMA 3.8 ([**11**, Lemma 9.2.1]). *For $R > 2$, let $\mathcal{K}_R$ be given by* (3.2.6). *Then there exist functions $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})(\mathbf{p}, z)$ in $\mathbb{R}^2 \times \mathbb{R}$ satisfying the following properties:*

(i) *If $|(\mathbf{p}, z) - (\tilde{\mathbf{p}}, \tilde{z})| < \varepsilon$ for some $(\tilde{\mathbf{p}}, \tilde{z}) \in \mathcal{K}_R$, then*

$$(3.2.7) \qquad\qquad (\tilde{\mathcal{A}}, \tilde{\mathcal{B}})(\mathbf{p}, z) = (\mathcal{A}, \mathcal{B})(\mathbf{p}, z);$$

(ii) *For any* $(\mathbf{p}, z) \in \mathbb{R}^2 \times \mathbb{R}$ *and* $\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2$,

$$(3.2.8) \qquad \sum_{i,j=1}^{2} \partial_{p_j} \tilde{\mathcal{A}}_i(\mathbf{p}, z) \kappa_i \kappa_j \geq \lambda |\boldsymbol{\kappa}|^2;$$

(iii) *For each* $k = 1, 2, \cdots$,

$$(3.2.9) \qquad |\tilde{\mathcal{B}}(\mathbf{p}, z)| \leq C_0, \quad |D_{(\mathbf{p},z)}^k (\tilde{\mathcal{A}}, \tilde{\mathcal{B}})(\mathbf{p}, z)| \leq C_k \qquad in \ \mathbb{R}^2 \times \mathbb{R},$$

where the positive constants ε, λ, *and* C_k *with* $k = 0, 1, 2, \cdots$, *depend only on* (v_∞, γ, R).

For $\alpha \in (0, 1)$ and $m \in \mathbb{Z}^+$, we now define the standard Hölder norms by

$$(3.2.10) \quad \|u\|_{m,0,U} := \sum_{0 \leq |\boldsymbol{\beta}| \leq m} \sup_{\mathbf{x} \in U} |D^{\boldsymbol{\beta}} u(\mathbf{x})|, \quad [u]_{m,\alpha,U} := \sum_{|\boldsymbol{\beta}| = m} \sup_{\mathbf{x},\mathbf{y} \in U, \mathbf{x} \neq \mathbf{y}} \frac{|D^{\boldsymbol{\beta}} u(\mathbf{x}) - D^{\boldsymbol{\beta}} u(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\alpha},$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2)$ with $\beta_j \geq 0$ for $j = 1, 2$, $D^{\boldsymbol{\beta}} = \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2}$, and $|\boldsymbol{\beta}| = \beta_1 + \beta_2$.

LEMMA 3.9. *Fix* $\gamma \geq 1$ *and* $v_\infty > 0$. *For any given constants* $\alpha \in (0, 1)$, $k \in \mathbb{N}$, *and* $r > 0$, *there exist constants* $C, C_k > 0$ *depending only on* $(v_\infty, \gamma, \alpha, r)$ *with* C_k *depending additionally on* k *such that any admissible solution* φ *corresponding to* $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$ *satisfies the following estimates*:

(i) *For any* $B_{4r}(P) \subset \Omega$,

$$(3.2.11) \qquad \|\varphi\|_{2,\alpha,\overline{B_{2r}(P)}} \leq C,$$

$$(3.2.12) \qquad \|\varphi\|_{k,\overline{B_r(P)}} \leq C_k.$$

(ii) *If* $P \in \Gamma_{\mathrm{wedge}}$, *and* $B_{4r}(P) \cap \Omega$ *is the half-ball* $B_{4r}^+(P) = B_{4r}(P) \cap \{\xi_2 > 0\}$, *then*

$$(3.2.13) \qquad \|\varphi\|_{2,\alpha,\overline{B_{2r}(P)} \cap \Omega} \leq C,$$

$$(3.2.14) \qquad \|\varphi\|_{k,\overline{B_r(P)} \cap \Omega} \leq C_k.$$

PROOF. Fix $\beta \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$, and let φ be an admissible solution corresponding to (v_∞, β) with the pseudo-subsonic region Ω. Using Definition 2.24(iii) and Lemma 3.5, we can apply Lemmas C.1–C.2 to estimate the ellipticity of Eq. (2.1.19).

Suppose that $B_{4r}(P) \subset \Omega$ for some constant $r \in (0, 1)$. By (3.1.26), there exists a constant $\hat{c} > 0$ depending only on (v_∞, γ) such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$ in the sense of Definition 2.24 satisfies

$$0 < \sup_{\Omega} c(|D\varphi|^2, \varphi) \leq \hat{c}.$$

One can choose a smooth function $\tilde{b}(\boldsymbol{\xi})$ satisfying the following properties:

$$\tilde{b} = 1 \quad \text{in } B_{3r}(P), \qquad \tilde{b} = 0 \quad \text{on } \partial B_{4r}(P), \qquad |D^k \tilde{b}| \leq \frac{C_k}{r^k} \quad \text{in } B_{4r}(P),$$

for constants $C_k > 0$ depending only on k for each $k = 1, 2, \cdots$. For a constant $\delta_r > 0$ to be determined later, we define $b(\boldsymbol{\xi}) := \delta_r \tilde{b}(\boldsymbol{\xi})$. Then b satisfies

$$(3.2.15) \qquad |Db| + \hat{c}|D^2 b| \leq \frac{C_*}{r^2} \delta_r \qquad \text{in } B_{4r}(P)$$

for some constant C_*.

Since $\mathrm{diam}(\Omega) \leq \bar{d}$ for some constant $\bar{d} > 0$ depending only on (v_∞, γ) due to Lemma 3.5, it follows from Lemma C.1(b) that there exists a constant $C_0 > 0$ depending on (v_∞, γ) such that,

for any given $\delta \in (0,1)$, if $|Db| + \hat{c}|D^2 b| \leq \frac{\delta}{\hat{c}}$ in $B_{4r}(P)$, then either the pseudo-Mach number $M = \frac{|D\varphi|}{c(|D\varphi|^2, \varphi)}$ satisfies that $M^2 \leq C_0 \delta$ in $B_{4r}(P)$ or $M^2 + b$ does not attain its maximum in $B_{4r}(P)$.

Now we fix δ_r in the definition of b as $\delta_r = \frac{r^2}{8(C_0+1)(C_*+1)\hat{c}}$. Then (3.2.15) leads to

$$|Db| + \hat{c}|D^2 b| \leq \frac{1}{8(C_0+1)\hat{c}},$$

which implies that $M = \frac{|D\varphi|}{c(|D\varphi|^2, \varphi)}$ satisfies

$$\text{either } M^2 \leq \frac{1}{8} \text{ in } B_{4r}(P) \qquad \text{or} \qquad \max_{\overline{B_{4r}(P)}} M^2 + b = \max_{\partial B_{4r}(P)} M^2 < 1.$$

Therefore, there exists a constant $\sigma_r \in (0,1)$ depending on (v_∞, γ, r) such that φ satisfies

$$(3.2.16) \qquad\qquad \frac{|D\varphi|^2}{c^2(|D\varphi|^2, \varphi)} \leq 1 - \sigma_r \qquad \text{in } B_{3r}(P).$$

For a C^1–function ϕ defined in $U \subset \mathbb{R}^2$, denote $\mathcal{E}(\phi, U)$ as

$$(3.2.17) \qquad\qquad \mathcal{E}(\phi, U) := \{(\mathbf{p}, z) \,:\, z = \phi(\boldsymbol{\xi}), \mathbf{p} = D\phi(\boldsymbol{\xi}), \boldsymbol{\xi} \in U\}.$$

By (3.2.16) and Lemma 3.5, there exists a constant $R_r > 2$ depending only on (v_∞, γ, r) so that $\mathcal{E}(\varphi, B_{3r}(P)) \subset \mathcal{K}_{R_r}$. Let $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})(\mathbf{p}, z)$ be the extensions given by Lemma 3.8 for $R = R_r$.

In order to prove (3.2.11) by applying Theorem C.3, we rewrite Eq. (2.1.19) as

$$\sum_{i,j=1}^{2} \underbrace{\partial_{p_j}\tilde{\mathcal{A}}_i(D\varphi, \varphi)}_{\left(=:A_{ij}(D\varphi, \varphi)\right)} \partial_{ij}\varphi + \underbrace{\sum_{i=1}^{2} \partial_z \tilde{\mathcal{A}}_i(D\varphi, \varphi)\partial_i \varphi + 2\left(\tilde{\mathcal{B}}(D\varphi, \varphi) - \tilde{\mathcal{B}}(\mathbf{0}, 0)\right)}_{\left(=:A(D\varphi, \varphi)\right)} = -2\tilde{\mathcal{B}}(\mathbf{0}, 0).$$

By Lemma 3.8 , $(A_{ij}, A)(D\varphi, \varphi)$ satisfy (C.2.2)–(C.2.5). Then (3.2.11) is obtained from Lemma 3.5 and Corollary C.4.

Also, (3.2.13) is similarly obtained from Lemma C.2 and Theorem C.7.

Once we have (3.2.11) and (3.2.13), estimates (3.2.12) and (3.2.14) can be obtained by a bootstrap argument and [**30**, Theorem 6.2, Lemma 6.29]. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

For an admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$, we define an extension φ^{ext} into $\mathbb{R}_+^2$ by

$$(3.2.18) \qquad\qquad \varphi^{\text{ext}}(\boldsymbol{\xi}) := \begin{cases} \varphi(\boldsymbol{\xi}) & \text{if } \boldsymbol{\xi} \in \Lambda_\beta, \\ \varphi_\infty(\boldsymbol{\xi}) & \text{otherwise.} \end{cases}$$

For $S_{\mathcal{O},\text{seg}}$ and $S_{\mathcal{N},\text{seg}}$ defined by Definition 2.24, denote $\Gamma_{\text{shock}}^{\text{ext}}$ as

$$\Gamma_{\text{shock}}^{\text{ext}} = \begin{cases} S_{\mathcal{O},\text{seg}} \cup \Gamma_{\text{shock}} \cup S_{\mathcal{N},\text{seg}} & \text{if } \beta < \beta_s^{(v_\infty)}, \\ \Gamma_{\text{shock}} \cup S_{\mathcal{N},\text{seg}} & \text{otherwise.} \end{cases}$$

By (2.5.12) and the Rankine-Hugoniot condition: $\varphi = \varphi_\infty$ on $\overline{\Gamma_{\text{shock}}^{\text{ext}}}$, the extension function φ^{ext} satisfies the following:

(i) $\varphi^{\text{ext}} \in C_{\text{loc}}^{0,1}(\mathbb{R}_+^2) \cap C_{\text{loc}}^1(\mathbb{R}_+^2 \setminus \overline{\Gamma_{\text{shock}}^{\text{ext}}})$;

(ii) $\phi^{\text{ext}}(\boldsymbol{\xi}) = \varphi^{\text{ext}}(\boldsymbol{\xi}) + \frac{1}{2}|\boldsymbol{\xi}|^2$ satisfies $\|D\phi^{\text{ext}}\|_{L^\infty(\mathbb{R}^2_+)} = \|D\phi\|_{L^\infty(\Lambda_\beta)}$ for $\phi(\boldsymbol{\xi}) := \varphi(\boldsymbol{\xi}) + \frac{1}{2}|\boldsymbol{\xi}|^2$.

In the following corollary, we regard each admissible solution φ as its extension φ^{ext} given by (3.2.18):

COROLLARY 3.10. *Let $\{\varphi^{(k)}\}$ be a sequence of admissible solutions corresponding to $(v_\infty, \beta^{(k)}) \in \mathfrak{R}_{\text{weak}}$ in the sense of Definition 2.24 with*

$$\lim_{k\to\infty} \beta^{(k)} = \beta^* \qquad \text{for some } \beta^* \in [0, \beta_{\text{d}}^{(v_\infty)}].$$

Then there exists a subsequence $\{\varphi^{(k_j)}\}$ converging to a function $\varphi^* \in C_{\text{loc}}^{0,1}(\overline{\Lambda_{\beta^*}})$ uniformly in any compact subset of $\overline{\Lambda_{\beta^*}}$, where Λ_{β^*} is defined by Definition 2.23 for $\beta^* > 0$ and by (2.5.24) for $\beta^* = 0$. Moreover, φ^* is a weak solution of (2.1.19) in Λ_{β^*} in the sense of Remark 2.29(iv). For the rest of the statement, let superscripts (k) and $*$ indicate that each object is related to $\beta^{(k)}$ and β^*, respectively. Then we have the following properties:

(a) *For P_l, $l = 1, 2, 3, 4$, defined by Definition 2.23,*

$$\lim_{j\to\infty} P_l^{(k_j)} = P_l^* \qquad \text{for } l = 1, 4.$$

Note that P_2 and P_3 are fixed to be the same for all $\beta \in [0, \beta_{\text{d}}^{(v_\infty)}]$.

(b) *Let $f_{\text{sh}}^{(k_j)}$ be the functions from Proposition 3.4. Extend $f_{\text{sh}}^{(k_j)}$ by*

$$f_{\text{sh}}^{(k_j)}(\xi_1) = \begin{cases} f_{\mathcal{O}}^{(k_j)}(\xi_1) & \text{for } \xi_1 \leq \xi_1^{P_1^{(k_j)}}, \\ \xi_2^{\mathcal{N}} & \text{for } \xi_1 \geq \xi_1^{P_2}, \end{cases}$$

where $f_{\mathcal{O}}^{(k_j)}(\xi_1)$ is given by (2.5.2) with $\beta = \beta^{(k_j)}$. Then sequence $\{f_{\text{sh}}^{(k_j)}\}$ is uniformly bounded in $C^{0,1}([\xi_1^{P_{\beta^*}}, \xi_1^{P_2}])$ and converges uniformly on $[\xi_1^{P_{\beta^*}}, \xi_1^{P_2}]$, where P_β denotes the ξ_1–intercept of the straight oblique shock $S_{\mathcal{O}}$ of angle β with the ξ_1–axis. Denoting the limit function by f_{sh}^*, we see that $f_{\text{sh}}^* \in C^{0,1}([\xi_1^{P_{\beta^*}}, \xi_1^{P_2}])$.

(c) *For each k_j, the sonic arcs $\Gamma_{\text{sonic}}^{\mathcal{O},(k_j)}$ and $\Gamma_{\text{sonic}}^{\mathcal{N}}$, defined by Definition 2.23 corresponding to $(v_\infty, \beta^{(k_j)}) \in \mathfrak{R}_{\text{weak}}$, can be represented as*

$$\Gamma_{\text{sonic}}^{\mathcal{N}} = \{(\xi_1, g_{\mathcal{N},\text{so}}(\xi_1)) : \xi_1^{P_2} \leq \xi_1 \leq \xi_1^{P_3}\},$$
$$\Gamma_{\text{sonic}}^{\mathcal{O},(k_j)} = \{(\xi_1, g_{\mathcal{O},\text{so}}^{(k_j)}(\xi_1)) : \xi_1^{P_4^{(k_j)}} \leq \xi_1 \leq \xi_1^{P_1^{(k_j)}}\},$$

for smooth functions $g_{\mathcal{N},\text{so}}$ and $g_{\mathcal{O},\text{so}}^{(k_j)}$. Note that $g_{\mathcal{N},\text{so}}$ is fixed to be the same for all $\beta \in [0, \beta_{\text{d}}^{(v_\infty)}]$ and that $g_{\mathcal{O},\text{so}}^{(k_j)}$ depends continuously on $\beta \in [0, \beta_{\text{d}}^{(v_\infty)}]$. Therefore, $g_{\mathcal{O},\text{so}}^{(k_j)}$ converges to $g_{\mathcal{O},\text{so}}^*$ on $(\xi_1^{P_4^*}, \xi_1^{P_1^*})$ as $k_j \to \infty$. If $\beta^* \geq \beta_{\text{s}}^{(v_\infty)}$, then it follows from (2.5.6) that $\Gamma_{\text{sonic}}^{\mathcal{O},*}$ is a point set.
Define

$$\widehat{\Omega^*} := \{(\xi_1, \xi_2) \in [\xi_1^{P_4^*}, \xi_1^{P_3}] \times \mathbb{R}^+ : 0 \leq \xi_2 \leq f_{\text{bd}}^*(\xi_1)\}$$

for a function f_{bd}^* given by

$$f_{\mathrm{bd}}^*(\xi_1) = \begin{cases} g_{\mathcal{O},\mathrm{so}}^*(\xi_1) & \text{for } \xi_1^{P_4^*} \leq \xi_1 \leq \xi_1^{P_1^*}, \\ f_{\mathrm{sh}}^*(\xi_1) & \text{for } \xi_1^{P_1^*} < \xi_1 \leq \xi_1^{P_2}, \\ g_{\mathcal{N},\mathrm{so}}^*(\xi_1) & \text{for } \xi_1^{P_2} < \xi_1 \leq \xi_1^{P_3}. \end{cases}$$

Denote by Ω^* the interior of $\widehat{\Omega^*}$. Define $\Gamma_{\mathrm{shock}}^* := \{\xi_2 = f_{\mathrm{sh}}^*(\xi_1) : \xi_1 \in (\xi_1^{P_1^*}, \xi_1^{P_2})\}$ and $\Gamma_{\mathrm{wedge}}^* := \{(\xi_1, 0) : \xi_1 \in (\xi_1^{P_4^*}, \xi_1^{P_3})\}$. Denote by $\Gamma_{\mathrm{wedge}}^{*,0}$ the relative interior of $\Gamma_{\mathrm{wedge}}^* \setminus \Gamma_{\mathrm{shock}}^*$. Then φ^* satisfies the following properties:

(c-1) $\varphi^* = \varphi_\infty$ on $\Gamma_{\mathrm{shock}}^*$,

(c-2) $\varphi^* \in C^\infty(\Omega^* \cup \Gamma_{\mathrm{wedge}}^{*,0})$,

(c-3) $\varphi^{(k_j)} \to \varphi^*$ in C^2 on any compact subset of $\Omega^* \cup \Gamma_{\mathrm{wedge}}^{*,0}$,

(c-4) $\partial_{\mathbf{e}}(\varphi_\infty - \varphi^*) \leq 0$ in Ω^* for all $\mathbf{e} \in \mathrm{Cone}^0(\mathbf{e}_{S_{\mathcal{O}}^*}, \mathbf{e}_{S_{\mathcal{N}}})$,

(c-5)   Eq. (2.1.19) is strictly elliptic in $\Omega^* \cup \Gamma_{\mathrm{wedge}}^{*,0}$,

where we have followed Definition 2.23 for $(O_\infty, \Gamma_{\mathrm{sonic}}^{\mathcal{O}}, \mathbf{e}_{S_{\mathcal{O}}})$. If $\beta^* = 0$, $\mathrm{Cone}^0(\mathbf{e}_{S_{\mathcal{O}}^*}, \mathbf{e}_{S_{\mathcal{N}}})$ is understood in the sense of Remark 2.28.

(d) In $\Lambda_{\beta^*} \setminus \Omega^*$, φ^* is equal to the constant density states $\varphi_{\mathcal{O}}^*$, $\varphi_{\mathcal{N}}$, and φ_∞ in their respective domains as in (2.5.8), where $\varphi_{\mathcal{O}}^*$ is defined by (2.4.4) corresponding to β^*.

(e) $f_{\mathrm{sh}}^*(\xi_1) > 0$ for all $\xi_1 \in (\xi_1^{P_1^*}, \xi_1^{P_2})$.

PROOF. We divide the proof into four steps.

1. Statement (a) directly follows from Definition 2.23 and the continuous dependence of $(O_{\mathcal{O}}, c_{\mathcal{O}})$ on (v_∞, β). Statement (b) directly follows from Proposition 3.4.

2. Statement (c-1) directly follows from Definition 2.24(ii-4), Corollary 3.10(a), and the uniform convergence of $(\varphi^{(k_j)}, f_{\mathrm{sh}}^{(k_j)})$ to $(\varphi^*, f_{\mathrm{sh}}^*)$. For a point $P \in \Omega^*$, there are constants $r_P > 0$ and $N \in \mathbb{N}$ such that $B_{3r}(P) \subset \Omega^{(k_j)}$ for all $k_j \geq N$. Then it follows from Lemma 3.9(i) and the Arzelà-Ascoli theorem that $\varphi^* \in C^\infty(B_{3r}(P))$, which implies that $\varphi^* \in C^\infty(\Omega^*)$. We can similarly check from Lemma 3.9(ii) that $\varphi^* \in C^\infty(\Omega^* \cup \Gamma_{\mathrm{wedge}}^{*,0})$, which proves (c-2).

For a fixed compact set $K \subset \Omega^* \cup \Gamma_{\mathrm{wedge}}^{*,0}$, there exists a constant $N_K \in \mathbb{N}$ so that K is contained in $\Omega^{(k_j)} \cap \Gamma_{\mathrm{wedge}}^{(k_j)}$ for any $k_j \geq N_K$. By Lemma 3.9 and the compactness of K, $\{\varphi^{(k_j)}\}_{k_j \geq N_K}$ is sequentially compact in $C^2(K)$. Then the uniform convergence of $\{\varphi^{(k_j)}\}$ to φ^* in K implies that the subsequence converges to φ^* in $C^2(K)$. This proves (c-3).

For any $\mathbf{e} \in \mathrm{Cone}^0(\mathbf{e}_{S_{\mathcal{O}}^*}, \mathbf{e}_{S_{\mathcal{N}}})$, there exists $N_{\mathbf{e}} \in \mathbb{N}$ such that $\mathbf{e} \in \mathrm{Cone}^0(\mathbf{e}_{S_{\mathcal{O}}^{(k_j)}}, \mathbf{e}_{S_{\mathcal{N}}})$ for any $k_j \geq N_{\mathbf{e}}$. Then (c-4) follows from Lemma 3.2 and (c-3).

For a point $P \in \Omega^*$, we choose $r_P > 0$ small so that $B_{r_P}(P) \subset \Omega^*$. Then we fix $N_P \in \mathbb{N}$ sufficiently large so that $B_{r_P}(P) \subset \Omega^{(k_j)}$ for all $k_j \geq N_P$. Since $\sigma_r \in (0, 1)$ in (3.2.16) is a given constant independent of admissible solutions corresponding to $\beta \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$, we can fix a constant $\sigma_P \in (0, 1)$ such that

$$\frac{|D\varphi^{(k_j)}|^2}{c^2(|D\varphi^{(k_j)}|^2, \varphi^{(k_j)})} \leq 1 - \sigma_P \qquad \text{in } B_{r_P}(P) \text{ for all } k_j \geq N_P.$$

This estimate, combined with statement (c-3), implies that Eq. (2.1.19) for $\varphi = \varphi^*$ is strictly elliptic in Ω^*. We can use similar arguments by using Lemma C.2 to conclude that Eq. (2.1.19) for $\varphi = \varphi^*$ is strictly elliptic on $\Gamma^{*,0}_{\mathrm{wedge}}$, which implies (c-5).

Statement (d) follows directly from statements (a)–(c) and Definition 2.23.

3. Observe that

- f_{w} given by (2.5.4), P_1, P_4, $\Gamma^{\mathcal{O}}_{\mathrm{sonic}}$, and $S_{\mathcal{O},\mathrm{seg}}$ depend continuously on $\beta \in [0, \frac{\pi}{2})$;
- P_2, P_3, $S_{\mathcal{N},\mathrm{seg}}$, and $\Gamma^{\mathcal{N}}_{\mathrm{sonic}}$ are fixed to be the same for all $\beta \in [0, \frac{\pi}{2})$.

Combining this observation with statements (b), (c-3), and (d) implies that, for any compact set $K \subset \mathbb{R}^2$,

(i) $K \cap \Lambda_{\beta^{(k_j)}}$ converges to $K \cap \Lambda_{\beta^*}$ in the Hausdorff metric;

(ii) $D\varphi^{(k_j)}$ converges to $D\varphi^*$ almost everywhere in $K \cap \Lambda_{\beta^*}$.

Then it follows from Definition 2.24 that

$$\int_{\Lambda_{\beta^*}} \left(\rho(|D\varphi^*|^2, \varphi^*) D\varphi^* \cdot D\zeta - 2\rho(|D\varphi^*|^2, \varphi^*)\zeta \right) \mathrm{d}\boldsymbol{\xi} = 0 \qquad \text{for all } \zeta \in C_0^\infty(\mathbb{R}^2).$$

In other words, φ^* is a weak solution of (2.1.19) in Λ_{β^*} in the sense of Remark 2.29(iv).

4. To prove statement (e), we consider two cases separately: $\beta < \beta_{\mathrm{s}}^{(v_\infty)}$ and $\beta \geq \beta_{\mathrm{s}}^{(v_\infty)}$.

By Proposition 3.4 and statement (b), f_{sh}^* increases monotonically on $[\xi_1^{P_1^*}, \xi_1^{P_2}]$.

If $\beta^* < \beta_{\mathrm{s}}^{(v_\infty)}$, then it follows from statement (a) and the monotonicity of f_{sh}^* that

$$f_{\mathrm{sh}}^*(\xi_1) \geq f_{\mathrm{sh}}^*(\xi_1^{P_1^*}) \geq \xi_2^{P_1^*} > 0 \qquad \text{for all } \xi_1 \in [\xi_1^{P_1^*}, \xi_1^{P_2}].$$

If $\beta^* \geq \beta_{\mathrm{s}}^{(v_\infty)}$, it follows from statement (a) and Definition 2.23 that $f_{\mathrm{sh}}^*(\xi_1^{P_1^*}) = 0$. Suppose that $f_{\mathrm{sh}}^*(\xi_1) = 0$ for some $\xi_1 \in (\xi_1^{P_1^*}, \xi_1^{P_2})$, and define

$$\xi_1^* := \sup\{\xi_1 \in (\xi_1^{P_1^*}, \xi_1^{P_2}) \ : \ f_{\mathrm{sh}}^*(\xi_1) = 0\}.$$

Since $f_{\mathrm{sh}}^*(\xi_1^{P_2}) = \xi_2 > 0$, then $\xi_1^* \in (\xi_1^{P_{\beta^*}}, \xi_1^{P_2})$. Note that $\xi_1^{P_{\beta^*}} = \xi_1^{P_1^*} = \xi_1^{P_4^*}$ for $\beta^* \geq \beta_{\mathrm{s}}^{(v_\infty)}$. By the monotonicity of f_{sh}^* with respect to ξ_1, we have

(3.2.19)
$$f_{\mathrm{sh}}^*(\xi_1) = 0 \qquad \text{for all } \xi_1 \in [\xi_1^{P_{\beta^*}}, \xi_1^*].$$

Let Q be the midpoint of P_{β^*} and $(\xi_1^*, 0)$. Then Q lies on Γ_{wedge}. Denote $d_* := \frac{\xi_1^{P_{\beta^*}} + \xi_1^*}{4}$. Then it follows from (3.2.19) that

$$\varphi^* = \varphi_\infty \qquad \text{in } B_{d_*}(Q) \cap \Lambda_{\beta^*} = B_{d_*}(Q) \cap \{\xi_2 \geq 0\}.$$

However, this contradicts the fact that φ^* is a weak solution of (2.1.19) in Λ_{β^*} in the sense of Remark 2.29(iv), because a direct computation by using the definition of φ_∞ given by Definition 2.23 shows that a test function $\zeta \in C_0^\infty(B_{d_*}(Q))$ can be chosen so that

$$\int_{B_{d_*}(Q) \cap \{\xi_2 \geq 0\}} \left(\rho(|D\varphi_\infty|^2, \varphi_\infty) D\varphi_\infty \cdot D\zeta - 2\rho(|D\varphi_\infty|^2, \varphi_\infty)\zeta \right) \mathrm{d}\boldsymbol{\xi}$$

$$= v_\infty \int_{\Gamma^*_{\mathrm{wedge}} \cap B_{d_*}(Q)} \zeta \, \mathrm{d}\xi_1 \neq 0.$$

Therefore, we conclude that $f_{\mathrm{sh}}^*(\xi_1) > 0$ holds for any $\xi_1 \in (\xi_1^{P_1^*}, \xi_1^{P_2})$, which implies statement (e). This completes the proof. $\square$

Define

$$(3.2.20) \qquad\qquad r_1 := \min_{\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]} |P_\beta|.$$

For each $\beta \in [0, \beta_{\mathrm{s}}^{(v_\infty)}]$, we know that $|P_\beta| \geq c_{\mathcal{O}} \geq c_{\mathcal{N}}$, by (2.4.31). For $\beta \in [\beta_{\mathrm{s}}^{(v_\infty)}, \beta_{\mathrm{d}}^{(v_\infty)}]$, (2.4.3) implies that $|P_\beta| > v_\infty \tan \beta \geq v_\infty \tan \beta_{\mathrm{s}}^{(v_\infty)}$. Therefore, we have

$$r_1 \geq \min\{c_{\mathcal{N}}, v_\infty \tan \beta_{\mathrm{s}}^{(v_\infty)}\} > 0.$$

PROPOSITION 3.11. *For every* $r \in (0, \frac{r_1}{2})$, *there exists a constant* $\mathcal{C}_r > 0$ *depending only on* (v_∞, γ, r) *such that any admissible solution corresponding to* $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$ *satisfies*

$$(3.2.21) \qquad\qquad \mathrm{dist}(\Gamma_{\mathrm{shock}} \setminus B_r(P_\beta), \Gamma_{\mathrm{wedge}}) > \mathcal{C}_r^{-1}.$$

PROOF. This proposition is proved for two cases separately: (i) $P_4 \notin B_{\frac{r}{2}}(P_\beta)$, and (ii) $P_4 \in B_{\frac{r}{2}}(P_\beta)$ for P_4 defined by Definition 2.23 depending on $\beta \in [0, \frac{\pi}{2})$. Fix $r \in (0, \frac{r_1}{2})$.

1. We first consider the case that $P_4 \notin B_{\frac{r}{2}}(P_\beta)$.

Define

$$I_r := \{\beta \in (0, \beta_{\mathrm{d}}^{(v_\infty)}) \ : \ |P_4 - P_\beta| \geq \frac{r}{2}\}.$$

Then $I_r \subset (0, \beta_{\mathrm{s}}^{(v_\infty)})$. Since P_β and P_4 depend continuously on $\beta \in (0, \beta_{\mathrm{s}}^{(v_\infty)})$, I_r is relatively closed in $(0, \beta_{\mathrm{s}}^{(v_\infty)})$. Then there exists $\delta_0 > 0$ such that, for any $\beta \in I_r$, $\varphi_{\mathcal{O}}$ given by (2.4.4) satisfies that $\frac{|D\varphi_{\mathcal{O}}(P_\beta)|}{c_{\mathcal{O}}(\beta)} \geq 1 + \delta_0$. By Lemma 2.22, there exists a constant $\sigma_r \in (0, \frac{\beta_{\mathrm{s}}^{(v_\infty)}}{2})$ satisfying that $I_r \subset [0, \beta_{\mathrm{s}}^{(v_\infty)} - \sigma_r]$. Then Proposition 3.4 implies that

$$(3.2.22) \qquad\qquad \inf_{\beta \in I_r} \mathrm{dist}(\Gamma_{\mathrm{shock}}, \Gamma_{\mathrm{wedge}}) \geq \inf_{\beta \in [0, \beta_{\mathrm{s}}^{(v_\infty)} - \sigma_r]} \xi_2^{P_1} > 0.$$

2. Now consider the case that $P_4 \in B_{\frac{r}{2}}(P_\beta)$.

For an admissible solution φ, define

$$J_d^\varphi := \{P \in \Gamma_{\mathrm{shock}} \ : \ |\xi_1^P - \xi_1^{P_4}| < d\}.$$

Claim: For any $r \in (0, \frac{r_1}{2})$, *there exists a constant* $C_r > 0$ *such that any admissible solution corresponding to* $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$ *satisfies*

$$(3.2.23) \qquad\qquad \sup_{P \in J_{r/2}^\varphi} \mathrm{dist}(P, \Gamma_{\mathrm{wedge}}) > C_r^{-1}.$$

This claim is proved by deriving a contradiction. On the contrary, the claim is false. Then there exists a sequence $\{\beta^{(k)}\} \subset (0, \beta_{\mathrm{d}}^{(v_\infty)})$ such that, for each $k \in \mathbb{N}$, there exists an admissible solution $\varphi^{(k)}$ corresponding to $(v_\infty, \beta^{(k)})$ in the sense of Definition 2.24 with

$$(3.2.24) \qquad\qquad \sup_{P \in J_{r/2}^{\varphi^{(k)}}} \mathrm{dist}(P, \Gamma_{\mathrm{wedge}}^{(k)}) \leq \frac{1}{k}.$$

By Corollary 3.10, such a sequence $\{\beta^{(k)}\}$ can be chosen so that it converges to $\beta^* \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]$ and the corresponding solution sequence $\varphi^{(k)}$ uniformly converges in any compact subset of $\overline{\Lambda_{\beta^*}}$ to

a function $\varphi^* \in C^{0,1}_{\mathrm{loc}}(\overline{\Lambda_{\beta^*}})$ satisfying all the properties described in Corollary 3.10. Furthermore, (3.2.24) implies that

$$\max_{P \in J^{\varphi^*}_{r/4}} \mathrm{dist}(P, \Gamma^*_{\mathrm{wedge}}) = 0.$$

This contradicts Corollary 3.10(e). Thus, the claim is verified.

For each admissible solution φ, let f_{sh} be given as an extension defined by Corollary 3.10(b). Then

$$\mathrm{dist}(\Gamma_{\mathrm{shock}} \setminus B_r(P_\beta), \Gamma_{\mathrm{wedge}}) \geq f_{\mathrm{sh}}(\xi_1^{P_\beta} + r) \geq \sup_{P \in J^{\varphi}_{r/2}} \mathrm{dist}(P, \Gamma_{\mathrm{wedge}}),$$

where we have used the assumption that $|P_4 - P_\beta| < \frac{r}{2}$ in the second inequality. Finally, (3.2.21) is directly obtained from this inequality, combined with (3.2.23). $\qquad\square$

For $0 < v_\infty \leq 1$, define $B_1^+(O_\infty) := B_1(O_\infty) \cap \{\xi_2 \geq 0\}$. Following Definition 2.23, for each $\beta \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$, $\rho_{\mathcal{O}} > \rho_{\mathcal{N}} > 1$ by (2.4.40). Moreover, the entropy condition yields that $|D\varphi_\infty(P_\beta)| > 1$. By combining these properties with condition (i-1) of Definition 2.24, any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$ satisfies

$$(3.2.25) \qquad\qquad B_1^+(O_\infty) \subset \overline{\Omega} \setminus \overline{\Gamma^{\mathcal{O}}_{\mathrm{sonic}} \cup \Gamma_{\mathrm{shock}} \cup \Gamma^{\mathcal{N}}_{\mathrm{sonic}}}.$$

For $v_\infty > 1$, (3.2.25) still holds, because $B_1^+(O_\infty) = \emptyset$. Therefore, any compact set $K \subset B_1^+(O_\infty)$ is contained in the pseudo-subsonic region Ω.

LEMMA 3.12. *Fix $\gamma \geq 1$ and $v_\infty \in (0, 1)$. For every compact set $K \subset B_1^+(O_\infty)$, there exists a constant $C_K > 0$ depending only on (v_∞, γ, K) such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$ satisfies*

$$(3.2.26) \qquad\qquad\qquad \inf_K (\varphi_\infty - \varphi) \geq C_K^{-1}.$$

PROOF. Suppose that this lemma is false. By Definition 2.24(iv), there exist a compact set $K \subset B_1^+(O_\infty)$, a sequence $\{\beta_j\} \subset (0, \beta_{\mathrm{d}}^{(v_\infty)})$, and a sequence of points $\{Q_j\} \subset K$ so that

$$(\varphi_\infty - \varphi^{(j)})(Q_j) \to 0 \qquad \text{as } j \to \infty,$$

where $\varphi^{(j)}$ is an admissible solution for each β_j in the sense of Definition 2.24. By passing to a subsequence (without changing index notation), there exist $\beta_\flat \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]$ and $Q_\flat \in K$ so that

$$\beta_j \to \beta_\flat, \quad Q_j \to Q_\flat \qquad \text{as } j \to \infty.$$

By (2.5.8) and (3.1.25), for any compact set $L \subset \mathbb{R}^2_+ := \{\boldsymbol{\xi} \in \mathbb{R}^2 : \xi_2 \geq 0\}$, each $\varphi^{(j)}$ satisfies that $\|\varphi^{(j)}\|_{C^{0,1}(L \cap \overline{\Lambda_{\beta_j}})} \leq C_L$ for a positive constant C_L depending only on (v_∞, γ, L). Therefore, passing to a further subsequence, we conclude that $\varphi^{(j)}$ converges uniformly to a function $\varphi_\flat \in C^{0,1}(L \cap \overline{\Lambda_{\beta_\flat}})$ in $L \cap \Lambda_{\beta_\flat}$ for a continuous function $\varphi_\flat$ defined in $\Lambda_{\beta_\flat}$, where $\Lambda_{\beta_\flat}$ is given by Definition 2.23. This yields that $(\varphi_\infty - \varphi_\flat)(Q_\flat) = 0$.

Since K is compact, there exists a small constant $\epsilon \in (0, \frac{1}{10})$ such that $K \subset B_{1-2\epsilon}^+(O_\infty)$. By Corollary 3.10, sequence $\{\varphi^{(j)}\}$ of admissible solutions is uniformly bounded in $C^3(\overline{B_{1-\epsilon/2}^+(O_\infty)})$. By the Arzelá-Ascoli theorem, there exists a subsequence (still denoted by) $\{\varphi^{(j)}\}$ that converges to a function $\varphi_\flat \in C^3(\overline{B_{1-\epsilon/2}^+(O_\infty)})$. Then $\varphi_\flat$ satisfies Eq. (2.1.19) in $B_{1-\frac{\epsilon}{2}}^+(O_\infty)$, where the equation is strictly elliptic by Definition 2.24(iii). Moreover, $\varphi_\flat$ satisfies the boundary condition $\partial_{\xi_2}(\varphi_\infty - \varphi) = -v_\infty < 0$ on $B_{1-\frac{\epsilon}{2}}^+(O_\infty) \cap \{\xi_2 = 0\}$. Note that condition (iv) of Definition 2.24

implies that $\varphi_\infty - \varphi_\flat \geq 0$ in $B^+_{1-\frac{\varepsilon}{2}}(O_\infty)$. By Hopf's lemma, $Q_\flat$ cannot lie on $B^+_{1-\frac{\varepsilon}{2}}(O_\infty) \cap \{\xi_2 = 0\}$. Thus, $Q_\flat$ must lie in $B^+_{1-\frac{\varepsilon}{2}}(O_\infty)$. However, by the strong maximum principle, this is impossible since $\varphi_\infty - \varphi_\flat$ cannot be a constant in $B^+_{1-\frac{\varepsilon}{2}}(O_\infty)$, owing to $\partial_{\xi_2}(\varphi_\infty - \varphi_\flat) = -v_\infty$ on $B^+_{1-\frac{\varepsilon}{2}}(O_\infty) \cap \{\xi_2 = 0\}$. This completes the proof. $\qquad\square$

Let (r, θ) be the polar coordinates centered at O_∞:

$$(3.2.27) \qquad\qquad r(\cos\theta, \sin\theta) = (\xi_1, \xi_2) - O_\infty.$$

In $\mathbb{R}^2_+ \setminus \{O_\infty\}$, define the (x, y)–coordinates by

$$(3.2.28) \qquad\qquad (x, y) = (c_\infty - r, \theta) \qquad \text{with } c_\infty = 1.$$

Suppose that a C^2–function φ satisfies Eq. (2.1.19), and define $w := \varphi_\infty - \varphi$. Then Eq. (2.1.19) can be written as an equation for w in the (x, y)–coordinates:

$$\mathcal{N}_p(w) := \big(2x + (\gamma + 1)w_x + O_1^-\big)w_{xx} + O_2^- w_{xy} + \big(\frac{1}{c_\infty} + O_3^-\big)w_{yy} - (1 + O_4^-)w_x + O_5^- w_y = 0,$$

with $O_j^-(Dw, w, x) = O_j(-Dw, -w, x, c_\infty)$ for $j = 1, \cdots, 5$, where $O_j(\mathbf{p}, z, x, c)$ for $j = 1, \cdots, 5$, with $\mathbf{p} = (p_1, p_2)$, are given by

$$O_1(\mathbf{p}, z, x, c) = -\frac{x^2}{c} + \frac{\gamma + 1}{2c}(2x - p_1)p_1 - \frac{\gamma - 1}{c}\Big(z + \frac{p_2^2}{2(c - x)^2}\Big),$$

$$O_2(\mathbf{p}, z, x, c) = -\frac{2}{c(c - x)^2}(p_1 + c - x)p_2,$$

$$(3.2.29) \qquad O_3(\mathbf{p}, z, x, c) = \frac{1}{c(c - x)^2}\Big(x(2c - x) - (\gamma - 1)\big(z + (c - x)p_1 + \frac{1}{2}p_1^2\big) - \frac{(\gamma + 1)p_2^2}{2(c - x)^2}\Big),$$

$$O_4(\mathbf{p}, z, x, c) = \frac{1}{c - x}\Big(x - \frac{\gamma - 1}{c}\big(z + (c - x)p_1 + \frac{1}{2}p_1^2 + \frac{(\gamma + 1)p_2^2}{2(\gamma - 1)(c - x)^2}\big)\Big),$$

$$O_5(\mathbf{p}, z, x, c) = -\frac{2(p_1 + c - 2x)p_2}{c(c - x)^3}.$$

LEMMA 3.13. *For constants $\delta, \varepsilon \geq 0$, define*

$$\mathcal{D}^\varepsilon_{-\delta} := B^+_{1+\delta}(O_\infty) \setminus \overline{B_{1-\varepsilon}(O_\infty)}.$$

Suppose that $v_\infty \in (0, 1)$ so that $\mathcal{D}^\varepsilon_{-\delta} \neq \emptyset$ for $\varepsilon > 0$. Then, for any $\alpha \in (\frac{1}{2}, 1)$, there exist constants $A, \varepsilon_0 > 0$ depending only on $(v_\infty, \gamma, \alpha)$ such that, if φ is an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ with $v_\infty \in (0, 1)$, then $w := \varphi_\infty - \varphi$ satisfies

$$w(x, y) \geq Ax^{1+\alpha} \qquad \text{in } \mathcal{D}^{\varepsilon_0}_0.$$

PROOF. The proof is divided into three steps.

1. Define $\widehat{O_1^-}(Dw, x) := O_1^-(Dw, w, x) - (\gamma - 1)w$ and

$$(3.2.30) \qquad \begin{aligned} N_1(v) := &\big(2x + (\gamma + 1)v_x + \widehat{O_1^-} + (\gamma - 1)w\big)v_{xx} + O_2^- v_{xy} + (1 + O_3^-)v_{yy} \\ &- (1 + O_4^-)v_x + O_5^- v_y, \end{aligned}$$

with $\widehat{O_1^-} = \widehat{O_1^-}(Dv, x)$ and $O_j^- = O_j^-(Dv, v, x)$ for $j = 2, \cdots, 5$.

Fix $\alpha \in (\frac{1}{2}, 1)$, and define a function

$$U(x) := Ax^{1+\alpha}$$

for a constant $A \in (0, 1)$ to be determined later. For each $\varepsilon_0 > 0$, U satisfies

$$N_1(U) \geq \left(2x + (\gamma + 1)U_x + \widehat{O_1^-}(DU, x)\right)U_{xx} - \left(1 + O_4^-(DU, U, x)\right)U_x$$

$$\geq (1 + \alpha)Ax^\alpha \left(2\alpha - 1 + \frac{\widehat{O_1^-}}{x} - O_4^-\right) \qquad \text{in } \mathcal{D}_0^{\varepsilon_0},$$

where we have applied the fact that $w \geq 0$ in Ω by Definition 2.24(iv). Using the definitions of $\widehat{O}_1$ and O_4, we can choose $\varepsilon_0 > 0$ sufficiently small depending only on $(v_\infty, \gamma, \alpha)$ such that

$$\frac{|\widehat{O_1^-}(DU, x)|}{x} \leq \frac{2\alpha - 1}{4}, \quad |O_4^-(DU, U, x)| \leq \frac{2\alpha - 1}{4} \qquad \text{in } \mathcal{D}_0^{\varepsilon_0}.$$

Under the choice of ε_0 above,

$$(3.2.31) \qquad\qquad N_1(U) - N_1(w) > 0 \qquad \text{in } \mathcal{D}_0^{\varepsilon_0}.$$

2. *Claim: There exists a constant $A > 0$ depending only on $(v_\infty, \gamma, \alpha)$ such that $U - w$ cannot attain its nonnegative maximum on $\partial \mathcal{D}_0^{\varepsilon_0}$.*

On $\partial \mathcal{D}_0^{\varepsilon_0} \cap \{x = 0\}$, condition (iv) of Definition 2.24 implies that $U - w = -w \leq 0$. By Lemma 3.12, there exists a constant C_{ε_0} depending only on $(v_\infty, \gamma, \alpha)$ such that

$$U - w \leq A\varepsilon_0^{1+\alpha} - C_{\varepsilon_0} \qquad \text{on } \partial \mathcal{D}_0^{\varepsilon_0} \cap \{x = \varepsilon_0\}.$$

Thus, a constant $A \in (0, 1)$ can be chosen sufficiently small to satisfy that $A\varepsilon_0^{1+\alpha} \leq \frac{1}{2}C_{\varepsilon_0}$. Then we have

$$U - w \leq 0 \qquad \text{on } \partial \mathcal{D}_0^{\varepsilon_0} \cap \{x = \varepsilon_0\}.$$

Since φ satisfies the slip boundary condition on Γ_{wedge}, w satisfies that $w_{\xi_2} = -v_\infty$ on $\partial \mathcal{D}_0^{\varepsilon_0} \cap \Gamma_{\mathrm{wedge}}$ so that

$$\partial_{\xi_2}(U - w) = A(1 + \alpha)x^\alpha \frac{\partial x}{\partial \xi_2} + v_\infty \qquad \text{on } \partial \mathcal{D}_0^{\varepsilon_0} \cap \Gamma_{\mathrm{wedge}}.$$

Therefore, we can reduce $A > 0$ depending only on $(v_\infty, \gamma, \alpha)$ so that

$$\partial_{\xi_2}(U - w) \geq \frac{v_\infty}{2} \qquad \text{on } \partial \mathcal{D}_0^{\varepsilon_0} \cap \Gamma_{\mathrm{wedge}},$$

which implies the claim.

3. Suppose that $\max_{\overline{\mathcal{D}_0^{\varepsilon_0}}}(U - w) > 0$. Then there exists a point $P_0 \in \mathrm{int}\, \mathcal{D}_0^{\varepsilon_0}$ such that

$$(U - w)(P_0) = \max_{\overline{\mathcal{D}_0^{\varepsilon_0}}}(U - w).$$

At P_0, we have

$$(U - w)_x(P_0) = (U - w)_y(P_0) = 0,$$

$$(3.2.32) \qquad (U - w)_{xx}(P_0) \leq 0, \quad (U - w)_{yy}(P_0) \leq 0,$$

$$U_y(P_0) = w_y(P_0) = 0, \quad -w_{yy}(P_0) = (U - w)_{yy}(P_0) \leq 0.$$

A direct computation by using (3.2.29)–(3.2.30) and (3.2.32) gives that

$$N_1(U) - N_1(w)$$
$$= \big(2x + (\gamma+1)U_x + \widehat{O_1^-}(DU, x) + (\gamma-1)w\big)(U-w)_{xx} - \frac{\gamma-1}{1-x}(U-w)U_x$$

(3.2.33) $$- \big(1 + O_3^-(DU, w)\big)w_{yy} \qquad \text{at } P_0.$$

Note that $w(P_0) > 0$, by Definition 2.24(iv). Since $|\widehat{O_1^-}(DU, x)| \leq C_{O_1}A\varepsilon_0^{2\alpha}$ for some constant $C_{O_1} > 0$ depending only on γ, and constant A depends only on $(\gamma, v_\infty, \alpha)$, we can choose $\varepsilon_0 > 0$ sufficiently small depending on $(\gamma, v_\infty, \alpha)$ such that $2x + (\gamma+1)U_x + \widehat{O_1^-}(DU, x) + (\gamma-1)w > 0$ at P_0. Moreover, $(U-w)U_x > 0$ at P_0. Therefore, we obtain from (3.2.33) that

$$N_1(U) - N_1(w) \leq -\big(1 + O_3^-(DU, w)\big)w_{yy} \qquad \text{at } P_0.$$

By Definition 2.24(iv) and (3.2.29), there exists a constant $C_* > 0$ depending only on γ such that $1 + O_3^-(DU, w) \geq 1 - C_*\varepsilon_0^\alpha$ at P_0. Reducing ε_0 further, depending only on (γ, α), to satisfy that $1 - C_*\varepsilon_0^\alpha \geq \frac{1}{2}$, we obtain that $N_1(U) - N_1(w) \leq 0$ at P_0. This contradicts (3.2.31). Therefore, we conclude that there exist constants (A, ε_0) depending on $(\gamma, v_\infty, \alpha)$ such that $w \geq Ax^{1+\alpha}$ in $\mathcal{D}_0^{\varepsilon_0}$. $\square$

Now we are ready to prove Proposition 3.7.

PROOF OF PROPOSITION 3.7. Let φ be an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$. Define

$$d_\varphi := \text{dist}\{B_1(O_\infty), \Gamma_{\text{shock}}\}.$$

We consider two separate cases: $v_\infty \geq 1$ and $0 < v_\infty < 1$.

1. We first consider the case that $v_\infty \geq 1$. Then $B_1(O_\infty) \subset \mathbb{R} \times \mathbb{R}^-$. By (2.4.42) and Lemma 2.26, there exists a constant $d_0 > 0$ depending only on (v_∞, γ) such that, for any $\beta \in (0, \beta_{\text{d}}^{(v_\infty)})$,

$$\text{dist}(P_\beta, B_1(O_\infty)) = |P_\beta O_\infty| - 1 = |D\varphi_\infty(P_\beta)| - 1 \geq M_{\infty,\nu}(P_\beta) - 1 \geq d_0.$$

Denote $\bar{r} := \frac{1}{4}\min\{r_1, d_0\}$ for r_1 from (3.2.20). By Proposition 3.11, there exists a constant $\mathcal{C}_{\bar{r}} > 0$ depending only on (v_∞, γ) such that any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ satisfies

$$\text{dist}(\Gamma_{\text{shock}} \setminus \overline{B_{\frac{\bar{r}}{2}}(P_\beta)}, B_1(O_\infty)) \geq \text{dist}(\Gamma_{\text{shock}} \setminus \overline{B_{\frac{\bar{r}}{2}}(P_\beta)}, \Gamma_{\text{wedge}}) \geq \mathcal{C}_{\bar{r}}^{-1} > 0.$$

By the definition of $\bar{r}$ above, $\text{dist}(\Gamma_{\text{shock}} \cap \overline{B_{\bar{r}}(P_\beta)}, B_1(O_\infty)) \geq \frac{d_0}{4} > 0$. Then

$$d_\varphi \geq \min\{\mathcal{C}_{\bar{r}}^{-1}, \frac{d_0}{4}\} > 0$$

for any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ with $v_\infty \geq 1$.

2. Now we consider the second case that $0 < v_\infty < 1$. Let $P_* \in \Gamma_{\text{shock}}$ be a point such that

$$d_\varphi = \text{dist}(P_*, B_1(O_\infty)).$$

At point P_*, we have

(3.2.34) $$d_\varphi = \partial_{\boldsymbol{\nu}}\varphi_\infty(P_*) - 1$$

for the unit normal vector $\boldsymbol{\nu}$ to Γ_{shock} at P_* towards the interior of Ω. Denote

(3.2.35) $$\omega_\varphi := \partial_{\boldsymbol{\nu}}(\varphi_\infty - \varphi)(P_*).$$

Claim: There exist two positive constants d_0 and d_1 depending only on (v_∞, γ) such that, if $d_\varphi > d_0$ does not hold, then $\omega_\varphi \geq d_1$ holds.

Fix an admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$. For the (x, y)–coordinates defined by (3.2.28), let $\varepsilon_0 > 0$ be the constant from Lemma 3.13 with $\alpha = \frac{3}{4}$. In other words, $w := \varphi_\infty - \varphi$ satisfies

$$w(x, y) \geq A x^{\frac{7}{4}} \qquad \text{in } \mathcal{D}_0^{\varepsilon_0}$$

for some constant $A > 0$ chosen depending only on (v_∞, γ). For constants k and $\varepsilon \in (0, \varepsilon_0)$, to be determined later, define a function V in $\mathcal{D}_{-d_\varphi}^\varepsilon$ by

$$(3.2.36) \qquad\qquad V := (x + d_\varphi)^2 + k(x + d_\varphi).$$

For a constant $d_0 > 0$ to be specified later, assume that $d_\varphi \leq d_0$. Then a direct computation by using (3.2.28)–(3.2.29) and Definition 2.24(iv) shows that V satisfies

$$(3.2.37) \qquad
\begin{aligned}
N_1(V) &\geq 3k - 4d_0 - C(\varepsilon + d_0 + k)^2 & &\text{in } \mathcal{D}_{-d_\varphi}^\varepsilon, \\
V &= 0 & &\text{on } \partial \mathcal{D}_{-d_\varphi}^\varepsilon \cap \{x = -d_\varphi\}, \\
V &\leq (\varepsilon + d_0)^2 + k(\varepsilon + d_0) & &\text{on } \partial \mathcal{D}_{-d_\varphi}^\varepsilon \cap \{x = \varepsilon\}, \\
V_{\xi_2} &\geq \frac{-v_\infty}{1 - \varepsilon}\left(2(\varepsilon + d_0) + k\right) & &\text{on } \partial \mathcal{D}_{-d_\varphi}^\varepsilon \cap \Gamma_{\mathrm{wedge}},
\end{aligned}$$

for a constant $C > 0$ chosen depending only on (γ, v_∞). Choosing

$$k = 2\varepsilon, \qquad d_0 = \varepsilon,$$

we obtain from (3.2.37), $w \geq 0$ in $\overline{\Omega}$, and (2.4.1) that

$$(3.2.38) \qquad
\begin{aligned}
N_1(V) - N_1(w) &\geq 2\varepsilon - 16 C \varepsilon^2 & &\text{in } \mathcal{D}_{-d_\varphi}^\varepsilon, \\
V - w &\leq 0 & &\text{on } \partial \mathcal{D}_{-d_\varphi}^\varepsilon \cap \{x = -d_\varphi\}, \\
V - w &\leq 10\varepsilon^2 - A\varepsilon^{\frac{7}{4}} & &\text{on } \partial \mathcal{D}_{-d_\varphi}^\varepsilon \cap \{x = \varepsilon\}, \\
(V - w)_{\xi_2} &\geq v_\infty - \frac{6 v_\infty \varepsilon}{1 - \varepsilon} & &\text{on } \partial \mathcal{D}_{-d_\varphi}^\varepsilon \cap \Gamma_{\mathrm{wedge}}.
\end{aligned}$$

Then we can fix a small constant $\varepsilon \in (0, \varepsilon_0)$ depending only on (v_∞, γ) such that, by (3.2.38), $N_1(V) - N_1(w) \geq 0$ in $\mathcal{D}_{-d_\varphi}^\varepsilon$, $V - w \leq 0$ on $\partial \mathcal{D}_{-d_\varphi}^\varepsilon \cap \{x = -d_\varphi \text{ or } \varepsilon\}$, and $(V - w)_{\xi_2} \geq 0$ on $\partial \mathcal{D}_{-d_\varphi}^\varepsilon \cap \Gamma_{\mathrm{wedge}}$. Thus, the maximum principle yields that

$$(3.2.39) \qquad\qquad V - w \leq 0 \qquad \text{in } \overline{\mathcal{D}_{-d_\varphi}^\varepsilon}.$$

Since $P_* \in \partial \mathcal{D}_{-d_\varphi}^\varepsilon \cap \{x = -d_\varphi\}$, $(V - w)(P_*) = \max_{\overline{\mathcal{D}_{-d_\varphi}^\varepsilon}}(V - w) = 0$. Note that Γ_{shock} is tangential to $\partial \mathcal{D}_{-d_\varphi}^\varepsilon \cap \{x = -d_\varphi\}$ at P_* so that $(V - w)_x(P_*) = \partial_{\boldsymbol{\nu}}(V - w)(P_*)$. Then (3.2.39) implies that $(V - w)_x(P_*) = \partial_{\boldsymbol{\nu}}(V - w)(P_*) \leq 0$. Combining this with (3.2.35)–(3.2.36) implies that

$$\omega_\varphi \geq V_x(P_*) = 2\varepsilon.$$

Therefore, the claim is verified by choosing $(d_0, d_1) := (\varepsilon, 2\varepsilon)$.

According to the claim, either d_φ is bounded below by ε or ω_φ is bounded below by 2ε. By (3.1.31) and (3.2.34), $\omega_\varphi = H(d_\varphi + 1)$ for H defined by (3.1.28). Then it follows from (3.1.30) that d_φ is uniformly bounded below by a positive constant if and only if ω_φ is uniformly bounded below

by a positive constant. Therefore, the claim implies that there exists a constant $\delta > 0$ depending only on (v_∞, γ) such that

$$d_\varphi \geq \min\{\varepsilon, \delta\} > 0$$

for any admissible solution φ corresponding to $(v_\infty, \gamma) \in \mathfrak{R}_{\mathrm{weak}}$ with $0 < v_\infty < 1$.

The proof of Proposition 3.7 is now completed. $\square$

3.3. Uniform Estimates for the Ellipticity of Eq. (2.1.19)

Given $\gamma \geq 1$ and $v_\infty > 0$, let φ be an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$. A direct computation by using (3.2.3) shows that Eq. (3.2.2) (the same as Eq. (2.1.19)) satisfies

$$(3.3.1)\quad \rho(1 - \frac{|D\varphi|^2}{c^2})|\boldsymbol{\kappa}|^2 \leq \sum_{i,j=1}^{2} \partial_{\boldsymbol{p}_i} \mathcal{A}_j(D\varphi, \varphi)\kappa_i\kappa_j \leq 2\rho|\boldsymbol{\kappa}|^2 \qquad \text{in } \Omega \text{ for any } \boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2.$$

Fix a function $h \in C^\infty(\mathbb{R}_+)$ such that

$$(3.3.2)\qquad h(s) = \begin{cases} s & \text{if } s \in [0, \frac{1}{2}], \\ 1 & \text{if } s \geq 1, \end{cases} \quad \text{and} \quad 0 \leq h' \leq 2 \quad \text{on } \mathbb{R}_+.$$

For each $\beta \in (0, \frac{\pi}{2})$, let $O_\mathcal{O}$ be defined by Definition 2.23, and denote

$$r_\beta := \min\{c_\mathcal{O}, |O_\mathcal{O} P_\beta|\} = \begin{cases} c_\mathcal{O} & \text{if } \beta < \beta_{\mathrm{s}}^{(v_\infty)}, \\ |O_\mathcal{O} P_\beta| & \text{if } \beta \geq \beta_{\mathrm{s}}^{(v_\infty)}. \end{cases}$$

Let $Q_\mathcal{O} \in S_\mathcal{O} \cap \{\xi_2 \geq 0\}$ be the midpoint of the two intersections of circle $|\boldsymbol{\xi} - O_\mathcal{O}| = r_\beta$ and $S_\mathcal{O} \cap \{\xi_2 \geq 0\}$, and let

$$\hat{r}_\beta := |O_\mathcal{O} Q_\mathcal{O}| = \begin{cases} r_\beta M_\mathcal{O} & \text{for } \beta < \beta_{\mathrm{s}}^{(v_\infty)}, \\ r_\beta \sin\beta & \text{for } \beta \geq \beta_{\mathrm{s}}^{(v_\infty)}, \end{cases}$$

for $M_\mathcal{O}$ defined by (2.4.6). Note that r_β and $\hat{r}_\beta$ depend continuously on $\beta \in (0, \frac{\pi}{2})$. It follows from (2.4.43) and the definitions of $(r_\beta, \hat{r}_\beta)$ stated above that $r_\beta - \hat{r}_\beta > 0$ for all $\beta \in [0, \frac{\pi}{2})$. Therefore, there exists a constant $\delta_0 > 0$ depending only on (v_∞, γ) so that $r_\beta - \hat{r}_\beta \geq \delta_0$ for all $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]$.

We define $(g_\mathcal{O}, g_\mathcal{N}, Q_\mathcal{N})$ by

$$g_\mathcal{O}(\boldsymbol{\xi}) := \frac{1}{2}(r_\beta - \hat{r}_\beta)\, h\big(\frac{\mathrm{dist}(\boldsymbol{\xi}, \partial B_{r_\beta}(O_\mathcal{O}))}{r_\beta - \hat{r}_\beta}\big),$$

$$g_\mathcal{N}(\boldsymbol{\xi}) := \lim_{\beta \to 0+} g_\mathcal{O}(\boldsymbol{\xi}) = \frac{1}{2}(c_\mathcal{N} - \xi_2^\mathcal{N})\, h\big(\frac{\mathrm{dist}(\boldsymbol{\xi}, \partial B_{c_\mathcal{N}}(O_\mathcal{N}))}{c_\mathcal{N} - \xi_2^\mathcal{N}}\big),$$

$$Q_\mathcal{N} := \lim_{\beta \to 0+} Q_\mathcal{O}.$$

Let $Q^* = (\xi_1^*, \xi_2^\mathcal{N})$ be the midpoint of $Q_\mathcal{N}$ and P_2 for point P_2 given by Definition 2.23. Moreover, we fix a function $\chi = \chi(\xi_1) \in C^\infty(\mathbb{R})$ such that

$$\chi(\xi_1) = \begin{cases} 1 & \text{for } \xi_1 \leq \frac{\xi_1^*}{10}, \\ 0 & \text{for } \xi_1 \geq \frac{\xi_1^*}{2}, \end{cases} \qquad -\frac{5}{\xi_1^*} \leq \chi'(\xi_1) \leq 0 \ \text{ for all } \xi_1 \in \mathbb{R}.$$

Finally, we define a function $g_\beta : \mathbb{R}^2 \to \mathbb{R}_+$ by

$$(3.3.3)\qquad g_\beta(\boldsymbol{\xi}) := \chi(\xi_1)\big(g_\mathcal{O}(\boldsymbol{\xi}) + \max\big\{1 - \frac{|D\varphi_\mathcal{O}(P_\beta)|^2}{c_\mathcal{O}^2}, 0\big\}\big) + \big(1 - \chi(\xi_1)\big)g_\mathcal{N}(\boldsymbol{\xi}).$$

REMARK 3.14. By Definition 2.24 and Lemma 3.5, there exist constants $d > 0$ and $C > 1$ depending only on (v_∞, γ) such that, if φ is an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$, and if Ω is its pseudo-subsonic region, then g_β satisfies the following properties:

(i) For $\boldsymbol{\xi} \in \Omega$ satisfying $\text{dist}(\boldsymbol{\xi}, \Gamma^{\mathcal{N}}_{\text{sonic}}) < d$,

$$C^{-1}\text{dist}(\boldsymbol{\xi}, \Gamma^{\mathcal{N}}_{\text{sonic}}) \leq g_\beta(\boldsymbol{\xi}) \leq C\text{dist}(\boldsymbol{\xi}, \Gamma^{\mathcal{N}}_{\text{sonic}});$$

(ii) For $\boldsymbol{\xi} \in \Omega$ satisfying $\text{dist}(\boldsymbol{\xi}, \Gamma^{\mathcal{O}}_{\text{sonic}}) < d$,

$$C^{-1}\text{dist}_\beta(\boldsymbol{\xi}, \Gamma^{\mathcal{O}}_{\text{sonic}}) \leq g_\beta(\boldsymbol{\xi}) \leq C\text{dist}_\beta(\boldsymbol{\xi}, \Gamma^{\mathcal{O}}_{\text{sonic}}),$$

where $\text{dist}_\beta(\boldsymbol{\xi}, \Gamma^{\mathcal{O}}_{\text{sonic}})$ is given by

$$(3.3.4) \qquad \text{dist}_\beta(\boldsymbol{\xi}, \Gamma^{\mathcal{O}}_{\text{sonic}}) := \text{dist}(\boldsymbol{\xi}, \Gamma^{\mathcal{O}}_{\text{sonic}}) + (c_{\mathcal{O}} - |D\varphi_{\mathcal{O}}(P_1)|);$$

(iii) Furthermore, for each $\varepsilon > 0$, there exists a constant $C_\varepsilon > 1$ depending only on $(v_\infty, \gamma, \varepsilon)$ such that, if a point $\boldsymbol{\xi} \in \overline{\Omega}$ satisfies $\text{dist}(\boldsymbol{\xi}, \Gamma^{\mathcal{O}}_{\text{sonic}} \cup \Gamma^{\mathcal{N}}_{\text{sonic}}) > \varepsilon$, then g_β satisfies

$$C_\varepsilon^{-1} \leq g_\beta(\boldsymbol{\xi}) \leq C_\varepsilon.$$

In (i)–(iii), $\Gamma^{\mathcal{N}}_{\text{sonic}}$, $\Gamma^{\mathcal{O}}_{\text{sonic}}$, and $\varphi_{\mathcal{O}}$ are defined by Definition 2.23.

For a constant $\hat{\zeta} > 0$, let us define

$$(3.3.5) \qquad \text{dist}^\flat(\boldsymbol{\xi}, \Gamma^{\mathcal{O}}_{\text{sonic}} \cup \Gamma^{\mathcal{N}}_{\text{sonic}}) := \min\left\{\hat{\zeta}, \text{dist}(\boldsymbol{\xi}, \Gamma^{\mathcal{N}}_{\text{sonic}}), \text{dist}_\beta(\boldsymbol{\xi}, \Gamma^{\mathcal{O}}_{\text{sonic}})\right\}.$$

Using properties (i)–(iii) stated in Remark 3.14, we can find constants $\mathcal{C} > 1$ and $\hat{\zeta} \in (0, 1)$ depending only on (v_∞, γ) such that each g_β for $\beta \in (0, \beta_{\text{d}}^{(v_\infty)})$ satisfies

$$\mathcal{C}^{-1}\text{dist}^\flat(\boldsymbol{\xi}, \Gamma^{\mathcal{O}}_{\text{sonic}} \cup \Gamma^{\mathcal{N}}_{\text{sonic}}) \leq g_\beta(\boldsymbol{\xi}) \leq \mathcal{C}\text{dist}^\flat(\boldsymbol{\xi}, \Gamma^{\mathcal{O}}_{\text{sonic}} \cup \Gamma^{\mathcal{N}}_{\text{sonic}}) \qquad \text{for all } \boldsymbol{\xi} \in \overline{\Omega},$$

where Ω is the pseudo-subsonic region of an admissible solution φ corresponding to (v_∞, β).

Let $\mathcal{A}(\mathbf{p}, z)$ be given by (3.2.3). The following proposition is essential to establish *a priori* weighted $C^{2,\alpha}$ estimates of admissible solutions:

PROPOSITION 3.15. There exists a constant $\mu > 0$ such that, if φ is an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ and Ω is its pseudo-subsonic region, then the pseudo-Mach number given by

$$(3.3.6) \qquad M(\boldsymbol{\xi}) := \frac{|D\varphi(\boldsymbol{\xi})|}{c(|D\varphi|^2(\boldsymbol{\xi}), \varphi(\boldsymbol{\xi}))}$$

satisfies

$$(3.3.7) \qquad M^2(\boldsymbol{\xi}) \leq 1 - \mu g_\beta(\boldsymbol{\xi}) \qquad \text{in } \overline{\Omega},$$

and there exists a constant $C > 1$ such that

$$(3.3.8) \qquad C^{-1}\text{dist}^\flat(\boldsymbol{\xi}, \Gamma^{\mathcal{O}}_{\text{sonic}} \cup \Gamma^{\mathcal{N}}_{\text{sonic}})|\boldsymbol{\kappa}|^2 \leq \sum_{i,j=1}^{2} \mathcal{A}^i_{p_j}(D\varphi(\boldsymbol{\xi}), \varphi(\boldsymbol{\xi}))\kappa_i\kappa_j \leq C|\boldsymbol{\kappa}|^2$$

for all $\boldsymbol{\xi} \in \overline{\Omega}$ and $\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2$, where constants μ and C are chosen depending only on (v_∞, γ). On the left-hand side of (3.3.8), $\text{dist}^\flat(\cdot, \cdot)$ is given by (3.3.5).

PROOF. Once (3.3.7) is proved, (3.3.8) is obtained directly from (3.3.7), Lemma 3.5, (3.3.1), and Remark 3.14. Therefore, it now suffices to prove (3.3.7).

In this proof, φ represents any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ with Ω and Γ_{shock} being its pseudo-subsonic region and the curved transonic shock, respectively. Unless otherwise specified, all the constants appearing in the proof are chosen depending only on (v_∞, γ). The proof is divided into four steps.

1. By Lemma 3.5, there exist constants $R > 1$ and $\hat{c} > 1$ such that

$$\Omega \subset B_{R/2}(\mathbf{0}), \quad \|c(|D\varphi|^2, \varphi)\|_{C^0(\overline{\Omega})} \leq \hat{c}, \quad \|g_\beta\|_{C^2(\overline{\Omega})} \leq \hat{c}$$

for g_β given by (3.3.3). Since $O_{\mathcal{O}} \in \{\xi_2 = 0\}$, $\partial_{\xi_2} g_\beta = 0$ on $\{\xi_2 = 0\}$. By Lemmas C.1–C.2, we can choose constants $C_0 > 0$, $\delta \in (0, \frac{3}{4}C_0)$, and $\mu_1 \in (0, 1)$ so that, whenever $\mu \in (0, \mu_1]$, either the inequality: $M^2 + \mu g_\beta \leq C_0\delta < 1$ holds in Ω, or the maximum of $M^2 + \mu g_\beta$ over $\overline{\Omega}$ cannot be attained in $\Omega \cup \Gamma_{\text{wedge}}$.

Since $M^2 + \mu g_\beta = 1$ on $\Gamma_{\text{sonic}}^{\mathcal{N}}$, the maximum of $M^2 + \mu g_\beta$ must be attained on $\partial\Omega \setminus \Gamma_{\text{wedge}}$.

2. Let $\boldsymbol{\nu}$ be the unit normal vector to Γ_{shock} towards the interior of Ω, and let $\boldsymbol{\tau}$ be a unit tangent vector to Γ_{shock}.

Claim: There exist constants $\alpha \in (0, \frac{1}{2})$ and $\zeta \in (0, 1)$ such that $M^2(P) \leq 1 - \zeta$ when $|\varphi_{\boldsymbol{\tau}}|^2 \leq \alpha|\varphi_{\boldsymbol{\nu}}|^2$ at $P \in \Gamma_{\text{shock}}$.

This claim is verified by adjusting the proof of [**11**, Lemma 9.6.2]. For a constant $\alpha \in (0, \frac{1}{2})$ to be specified later, assume that $|\varphi_{\boldsymbol{\tau}}|^2 \leq \alpha|\varphi_{\boldsymbol{\nu}}|^2$ holds at $P \in \Gamma_{\text{shock}}$. Since $\rho\varphi_{\boldsymbol{\nu}} = \partial_{\boldsymbol{\nu}}\varphi_\infty$ and $\varphi_{\boldsymbol{\tau}} = \partial_{\boldsymbol{\tau}}\varphi_\infty$ hold along Γ_{shock}, we have

$$|D\varphi_\infty|^2 - |\partial_{\boldsymbol{\nu}}\varphi_\infty|^2 = |\varphi_{\boldsymbol{\tau}}|^2 \leq \alpha|\varphi_{\boldsymbol{\nu}}|^2 \leq \alpha\Big(\frac{\partial_{\boldsymbol{\nu}}\varphi_\infty}{\rho}\Big)^2,$$

which yields that

$$|D\varphi_\infty|^2 \leq \big(1 + \frac{\alpha}{\rho^2}\big)|\partial_{\boldsymbol{\nu}}\varphi_\infty|^2 \qquad \text{at } P \in \Gamma_{\text{shock}}.$$

We combine this inequality with Lemma 3.5 and Proposition 3.7 to obtain

$$|\partial_{\boldsymbol{\nu}}\varphi_\infty(P)|^2 \geq \frac{1 + d_0}{1 + \alpha/C}$$

for some constants $d_0 > 0$ and $C > 1$. Therefore, we can fix constants $\bar{\alpha} \in (0, \frac{1}{2})$ and $d_1 > 0$ small so that $|\partial_{\boldsymbol{\nu}}\varphi_\infty(P)| \geq 1 + d_1$ when $\alpha \in [0, \bar{\alpha}]$.

Define $M_{\infty, \boldsymbol{\nu}} := |\partial_{\boldsymbol{\nu}}\varphi_\infty(P)|$ and $M_{\boldsymbol{\nu}} := \frac{|\varphi_{\boldsymbol{\nu}}(P)|}{c(|D\varphi|^2(P), \varphi(P))}$. Then it follows from (2.4.9) that

$$\big(1 + \frac{\gamma - 1}{2}M_{\boldsymbol{\nu}}^2\big)M_{\boldsymbol{\nu}}^{-\frac{2(\gamma-1)}{\gamma+1}} = \big(1 + \frac{\gamma - 1}{2}(M_{\infty, \boldsymbol{\nu}})^2\big)|M_{\infty, \boldsymbol{\nu}}|^{-\frac{2(\gamma-1)}{\gamma+1}}.$$

Owing to $M_{\infty, \boldsymbol{\nu}} = |\partial_{\boldsymbol{\nu}}\varphi_\infty(P)| \geq 1 + d_1$, there exists a constant $\zeta_* \in (0, 1)$ satisfying that $M_{\boldsymbol{\nu}}^2 \leq 1 - \zeta_*$ at $P \in \Gamma_{\text{shock}}$. By the assumption that $|\varphi_{\boldsymbol{\tau}}|^2 \leq \alpha|\varphi_{\boldsymbol{\nu}}|^2$ at $P \in \Gamma_{\text{shock}}$, we have

$$M^2 \leq (1 + \alpha)M_{\boldsymbol{\nu}}^2 \leq (1 + \alpha)(1 - \zeta_*) \qquad \text{at } P \in \Gamma_{\text{shock}}.$$

Therefore, we can further reduce $\alpha \in (0, \bar{\alpha}]$ so that the inequality right above implies that

$$M^2 \leq 1 - \frac{\zeta_*}{2} =: 1 - \zeta \qquad \text{at } P \in \Gamma_{\text{shock}}.$$

The claim is verified.

3. Let μ_1 be the constant from Step 1. In this step, we follow the approach of [**11**, Steps 2–3 in the proof of Proposition 9.6.3] to find a constant $\mu \in (0, \mu_1]$ so that $M^2 + \mu g_\beta$ cannot attain its maximum on Γ_{shock}. Here, we give an outline to see how such a constant μ is chosen. We refer to [**11**, Proposition 9.6.3] for further details.

3-1. Suppose that the maximum of $M^2 + \mu g_\beta$ over $\overline{\Omega}$ is attained at $P_{\max} \in \Gamma_{\text{shock}}$. Then $(M^2 + \mu g_\beta)(P_{\max}) \geq 1$, which implies that

$$(3.3.9) \qquad\qquad M^2(P_{\max}) \geq 1 - C_* \mu$$

for some constant $C_* > 0$. Moreover, we have

$$(3.3.10) \qquad\qquad \partial_{\boldsymbol{\tau}}(M^2 + \mu g_\beta)(P_{\max}) = 0,$$

$$(3.3.11) \qquad\qquad \partial_{\boldsymbol{\nu}}(M^2 + \mu g_\beta)(P_{\max}) \leq 0.$$

For simplicity of notation, denote

$$(3.3.12) \qquad\qquad k(\boldsymbol{\xi}) := \mu g_\beta(\boldsymbol{\xi}) \qquad \text{for } \boldsymbol{\xi} \in \mathbb{R}^2.$$

By using (2.4.2) and (2.5.15), a direct computation yields that, for each unit vector $\mathbf{w}$,

$$(3.3.13) \qquad\qquad (M^2)_{\mathbf{w}} = \frac{\left(2 + (\gamma - 1)M^2\right) D^2 \varphi[\mathbf{w}, D\varphi] + (\gamma - 1)M^2 \varphi_{\mathbf{w}}}{c^2},$$

where we have defined

$$D^2 \varphi[\mathbf{q}_1, \mathbf{q}_2] := (D^2 \varphi\, \mathbf{q}_1) \cdot \mathbf{q}_2 \qquad \text{for } \mathbf{q}_1, \mathbf{q}_2 \in \mathbb{R}^2.$$

By (3.3.13), we obtain from (3.3.10) that

$$(3.3.14) \qquad\qquad D^2 \varphi[\boldsymbol{\tau}, D\varphi] = -\frac{(\gamma - 1)M^2 \varphi_{\boldsymbol{\tau}} + c^2 k_{\boldsymbol{\tau}}}{2 + (\gamma - 1)M^2} =: B_1 \qquad \text{at } P_{\max}.$$

3-2. Next, we differentiate the Rankine-Hugoniot condition:

$$(3.3.15) \qquad\qquad (\rho D\varphi - D\varphi_\infty) \cdot D(\varphi_\infty - \varphi) = 0 \qquad \text{on } \Gamma_{\text{shock}}$$

in the tangential direction $\boldsymbol{\tau}$ of Γ_{shock}, and then use (2.4.1)–(2.4.2) and $(\varphi_\infty - \varphi)_{\boldsymbol{\tau}} = 0$ on Γ_{shock} to obtain

$$(3.3.16) \qquad \begin{aligned} &\left(\rho D^2 \varphi\, \boldsymbol{\tau} - \frac{\rho}{c^2}(D\varphi \cdot (D^2 \varphi\, \boldsymbol{\tau}) + \varphi_{\boldsymbol{\tau}})D\varphi\right) \cdot (D\varphi_\infty - D\varphi) \\ &\quad - (\rho D\varphi - D\varphi_\infty) \cdot (D^2 \varphi\, \boldsymbol{\tau} + \boldsymbol{\tau}) = 0 \qquad \text{on } \Gamma_{\text{shock}}. \end{aligned}$$

Using the Rankine-Hugoniot conditions (3.3.15) and $(\varphi_\infty - \varphi)_{\boldsymbol{\tau}} = 0$ on Γ_{shock}, we see that $D(\varphi_\infty - \varphi) = \partial_{\boldsymbol{\nu}}(\varphi_\infty - \varphi)\boldsymbol{\nu} = (\rho - 1)\varphi_{\boldsymbol{\nu}}\boldsymbol{\nu}$. Then we obtain

$$D\varphi \cdot D(\varphi_\infty - \varphi) = (\rho - 1)\varphi_{\boldsymbol{\nu}}^2 \qquad \text{on } \Gamma_{\text{shock}}.$$

Owing to the condition that $(\varphi_\infty - \varphi)_{\boldsymbol{\tau}} = 0$ on Γ_{shock} again, we have

$$(\rho D\varphi - D\varphi_\infty) \cdot \boldsymbol{\tau} = (\rho - 1)\varphi_{\boldsymbol{\tau}} \qquad \text{on } \Gamma_{\text{shock}}.$$

We substitute the expressions of $D\varphi \cdot D(\varphi_\infty - \varphi)$ and $(\rho D\varphi - D\varphi_\infty) \cdot \boldsymbol{\tau}$ given above into (3.3.16) to obtain

$$(3.3.17) \qquad \begin{aligned} &D^2 \varphi[\boldsymbol{\tau}, \rho D(\varphi_\infty - \varphi) + D\varphi_\infty] \\ &= \rho(1 + \frac{\rho - 1}{c^2}\varphi_{\boldsymbol{\nu}}^2)D^2 \varphi[\boldsymbol{\tau}, D\varphi] + \frac{\rho}{c^2}(\rho - 1)\varphi_{\boldsymbol{\nu}}^2 \varphi_{\boldsymbol{\tau}} + (\rho - 1)\varphi_{\boldsymbol{\tau}} \qquad \text{on } \Gamma_{\text{shock}}. \end{aligned}$$

3-3. Define
$$M_1 := \frac{|\varphi_{\boldsymbol{\nu}}|}{c(|D\varphi|^2, \varphi)}, \quad M_2 := \frac{|\varphi_{\boldsymbol{\tau}}|}{c(|D\varphi|^2, \varphi)}.$$

We substitute the expression of $D^2\varphi[\boldsymbol{\tau}, D\varphi]$ given by (3.3.14) into the right-hand side of (3.3.17) to obtain

(3.3.18)
$$\begin{aligned} &D^2\varphi[\boldsymbol{\tau}, \rho D(\varphi_\infty - \varphi) + D\varphi_\infty] \\ &= \rho(1 + (\rho - 1)M_1^2)B_1 + \rho(\rho - 1)M_1^2\varphi_{\boldsymbol{\tau}} + (\rho - 1)\varphi_{\boldsymbol{\tau}} =: B_2 \qquad \text{at } P_{\max}. \end{aligned}$$

A direct computation shows that

(3.3.19)
$$B_2 = \frac{\big(2(\rho - 1)(1 + \rho M_1^2) - (\gamma - 1)M^2\big)\varphi_{\boldsymbol{\tau}} - c^2\rho\big(1 + (\rho - 1)M_1^2\big)k_{\boldsymbol{\tau}}}{2 + (\gamma - 1)M^2}.$$

Apply α and ζ from Step 2, and assume that

(3.3.20)
$$0 < \mu \le \min\big\{\mu_1, \frac{\zeta}{2C_*}\big\}.$$

Then it follows from (3.3.9) and Step 2 that

(3.3.21) $0 < \alpha|\varphi_{\boldsymbol{\nu}}(P_{\max})|^2 < |\varphi_{\boldsymbol{\tau}}(P_{\max})|^2$,   or equivalently, $0 < \alpha M_1^2(P_{\max}) < M_2^2(P_{\max})$.

Using (3.3.9), (3.3.21), and $\alpha \in (0, \frac{1}{2})$, we have

(3.3.22)
$$M_2^2(P_{\max}) > \frac{\alpha}{2}(1 - C_*\mu).$$

We rewrite (3.3.14) and (3.3.18) as the following linear system for $(\varphi_{\boldsymbol{\nu}\boldsymbol{\tau}}, \varphi_{\boldsymbol{\tau}\boldsymbol{\tau}})$:

$$\mathbf{A}\begin{pmatrix}\varphi_{\boldsymbol{\nu}\boldsymbol{\tau}} \\ \varphi_{\boldsymbol{\nu}\boldsymbol{\tau}}\end{pmatrix} = \begin{pmatrix}B_1 \\ B_2\end{pmatrix} \qquad \text{at } P_{\max} \text{ for } \mathbf{A} = \begin{pmatrix}\varphi_{\boldsymbol{\nu}} & \varphi_{\boldsymbol{\tau}} \\ \rho^2\varphi_{\boldsymbol{\nu}} & \varphi_{\boldsymbol{\tau}}\end{pmatrix}.$$

By (3.1.26) and (3.3.21), $|\det \mathbf{A}| = |(\rho^2 - 1)\varphi_{\boldsymbol{\nu}}\varphi_{\boldsymbol{\tau}}| > 0$ at $P_{\max}$. Thus, $(\varphi_{\boldsymbol{\nu}\boldsymbol{\tau}}, \varphi_{\boldsymbol{\nu}\boldsymbol{\tau}})$ can be written as

(3.3.23)
$$\varphi_{\boldsymbol{\nu}\boldsymbol{\tau}} = \frac{B_1 - B_2}{(1 - \rho^2)\varphi_{\boldsymbol{\nu}}}, \quad \varphi_{\boldsymbol{\tau}\boldsymbol{\tau}} = \frac{\rho^2 B_1 - B_2}{(\rho^2 - 1)\varphi_{\boldsymbol{\tau}}} \qquad \text{at } P_{\max}.$$

Note that Eq. (2.1.19) is invariant under a coordinate rotation. We rewrite Eq. (2.1.19) as

(3.3.24) $(c^2 - \varphi_{\boldsymbol{\nu}}^2)\varphi_{\boldsymbol{\nu}\boldsymbol{\nu}} - 2\varphi_{\boldsymbol{\nu}}\varphi_{\boldsymbol{\tau}}\varphi_{\boldsymbol{\nu}\boldsymbol{\tau}} + (c^2 - \varphi_{\boldsymbol{\tau}}^2)\varphi_{\boldsymbol{\tau}\boldsymbol{\tau}} = |D\varphi|^2 - 2c^2$ in $\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}^{\mathcal{O}}} \cup \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}})$,

and use this to express $\varphi_{\boldsymbol{\nu}\boldsymbol{\nu}}$ in terms of $(M_{\boldsymbol{\nu}}, M_{\boldsymbol{\tau}}, M, \rho, \varphi_{\boldsymbol{\nu}\boldsymbol{\tau}}, \varphi_{\boldsymbol{\tau}\boldsymbol{\tau}})$. Then we use (3.3.23) to obtain

(3.3.25)
$$\begin{aligned} \varphi_{\boldsymbol{\nu}\boldsymbol{\nu}} =\; &\frac{M^2 - 2}{1 - M_1^2} - \frac{1}{1 - M_1^2}\left(\frac{2M_1M_2}{(\rho^2 - 1)\varphi_{\boldsymbol{\nu}}} + \frac{\rho^2(1 - M_2^2)}{(\rho^2 - 1)\varphi_{\boldsymbol{\tau}}}\right)B_1 \\ &+ \frac{1}{1 - M_1^2}\left(\frac{1 - M_2^2}{(\rho^2 - 1)\varphi_{\boldsymbol{\tau}}} + \frac{2M_1M_2}{(\rho^2 - 1)\varphi_{\boldsymbol{\nu}}}\right)B_2 \qquad \text{at } P_{\max}. \end{aligned}$$

Using (3.3.14), (3.3.19)–(3.3.23), and (3.3.25), we can also express $(\varphi_{\boldsymbol{\nu}\boldsymbol{\tau}}, \varphi_{\boldsymbol{\nu}\boldsymbol{\nu}})$ in terms of M, M_1, $M_2, \rho, \varphi_{\boldsymbol{\tau}}, \varphi_{\boldsymbol{\nu}}, c$, and $k_{\boldsymbol{\tau}}$ at $P_{\max} \in \Gamma_{\text{shock}}$.

3-4. Now we choose a constant $\mu \in (0, \mu_1]$ sufficiently small so that a contradiction is derived. By (3.3.13), (3.3.11) can be written as

$$\big(2 + (\gamma - 1)M^2\big)(\varphi_{\boldsymbol{\tau}}\varphi_{\boldsymbol{\nu}\boldsymbol{\tau}} + \varphi_{\boldsymbol{\nu}}\varphi_{\boldsymbol{\nu}\boldsymbol{\nu}}) + (\gamma - 1)M^2\varphi_{\boldsymbol{\nu}} + c^2k_{\boldsymbol{\nu}} \le 0 \qquad \text{at } P_{\max}.$$

Using (3.3.12) and the expressions of $(\varphi_{\boldsymbol{\nu\tau}}, \varphi_{\boldsymbol{\nu\nu}})$ in terms of $M, M_1, M_2, \rho, \varphi_{\boldsymbol{\tau}}, \varphi_{\boldsymbol{\nu}}, c$, and $k_{\boldsymbol{\tau}}$, we can further rewrite the inequality stated above as

(3.3.26) $\qquad \Delta := 2M_2^2 + 2(2\rho+1)M_1^2(M^2-1) + \mu\left(l_1\partial_{\boldsymbol{\nu}}g_\beta - l_2\partial_{\boldsymbol{\tau}}g_\beta\right) \leq 0 \qquad$ at $P_{\max}$

for

$$l_1 = c^2\varphi_{\boldsymbol{\tau}}(\rho+1)(1-M_1^2), \qquad l_2 = c^2\frac{(1-\rho^2)M_1^2M_2^2 + \rho^2M_1^2 + M_2^2}{(\rho+1)\varphi_{\boldsymbol{\tau}}}.$$

By (3.3.9), Lemma 3.5, and the definition of g_β given in (3.3.3), there exists a constant $C > 0$ such that

(3.3.27)
$$2(2\rho+1)M_1^2(M^2-1) \geq -C\mu \qquad \text{at } P_{\max},$$
$$|l_1| \leq C \qquad \text{on } \Gamma_{\text{shock}},$$
$$\|Dg_\beta\|_{C^0(\mathbb{R}^2)} \leq C \qquad \text{for all } \beta \in [0, \beta_{\text{d}}^{(v_\infty)}].$$

Moreover, by Lemma 3.5 and (3.3.22), we have

(3.3.28) $\qquad\qquad\qquad |l_2| \leq \dfrac{1}{\sqrt{\alpha(1-C_*\mu)}} \qquad$ at $P_{\max}$.

From (3.3.22)–(3.3.28), we obtain

$$\Delta \geq \alpha(1-C_*\mu) - C\mu\left(1 + \frac{1}{\sqrt{\alpha(1-C_*\mu)}}\right) \qquad \text{at } P_{\max}$$

for some constant $C > 0$, provided that μ satisfies (3.3.20). Therefore, there exists a constant $\mu_2 \in (0, \mu_1^*]$ for $\mu_1^* = \min\{\mu_1, \frac{\zeta}{2C_*}\}$ such that, if $0 < \mu \leq \mu_2$, then $\Delta > \frac{\alpha}{8} > 0$ holds at $P_{\max}$, which contradicts (3.3.26). Therefore, we conclude that the maximum of $M_\varphi^2 + \mu g_\beta$ over $\overline{\Omega}$ must be attained on $\partial\Omega \setminus (\Gamma_{\text{wedge}} \cup \Gamma_{\text{shock}})$, provided that $\mu > 0$ is chosen sufficiently small, depending only on (v_∞, γ).

4. For constant μ_2 given in Step 3, we fix a constant $\mu \in (0, \mu_2]$. Then $M_\varphi^2 + \mu g_\beta$ satisfies

$$\sup_{\overline{\Omega}} \left(M_\varphi^2 + \mu g_\beta\right) = \sup_{\Gamma_{\text{sonic}}^{\mathcal{O}} \cup \Gamma_{\text{sonic}}^{\mathcal{N}}} \left(M_\varphi^2 + \mu g_\beta\right) = 1.$$

This proves (3.3.7). $\hfill\square$

REMARK 3.16. By Remark 3.14 and (3.3.7) in Proposition 3.15, there exists a constant $\mu_{\text{el}} > 0$ depending only on (v_∞, γ) such that, if φ is an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$,

(3.3.29) $\qquad\qquad M_\varphi^2(\boldsymbol{\xi}) \leq 1 - \mu_{\text{el}}\text{dist}^\flat(\boldsymbol{\xi}, \Gamma_{\text{sonic}}^{\mathcal{O}} \cup \Gamma_{\text{sonic}}^{\mathcal{N}}) \qquad$ in $\overline{\Omega}$.

3.4. Uniform Weighted $C^{2,\alpha}$–Estimates Away From $\Gamma_{\text{sonic}}^{\mathcal{O}}$

According to Proposition 3.15, the ellipticity of Eq. (3.2.2) (or equivalently, Eq. (2.1.19)) depends on $\text{dist}(\boldsymbol{\xi}, \Gamma_{\text{sonic}}^{\mathcal{O}} \cup \Gamma_{\text{sonic}}^{\mathcal{N}})$. In particular, (3.3.5) indicates that the ellipticity of (3.2.2) depends continuously on $\beta \in (0, \beta_{\text{d}}^{(v_\infty)})$, even across $\beta_{\text{s}}^{(v_\infty)}$ up to $\beta_{\text{d}}^{(v_\infty)}$. For this reason, we can establish uniform weighted $C^{2,\alpha}$–estimates of admissible solutions.

We first estimate (weighted) $C^{2,\alpha}$–norms of admissible solutions away from $\Gamma_{\text{sonic}}^{\mathcal{O}}$. We will discuss the uniform (weighted) $C^{2,\alpha}$–estimates of admissible solutions near $\Gamma_{\text{sonic}}^{\mathcal{O}}$ in §3.5.

3.4.1. $C^{2,\alpha}$–estimates away from $\Gamma_{\text{sonic}}^{\mathcal{O}} \cup \Gamma_{\text{sonic}}^{\mathcal{N}}$. Fix $\gamma \geq 1$ and $v_\infty > 0$.

For a set $U \subset \mathbb{R}^2$ and a constant $\varepsilon > 0$, define $\mathcal{N}_\varepsilon(U) := \{\boldsymbol{\xi} \in \mathbb{R}^2 : \text{dist}(\boldsymbol{\xi}, U) < \varepsilon\}$.

Let $C > 0$ be the constant from Proposition 3.7. Then there exists a constant $d_0 > 0$ depending only on (v_∞, γ) such that

$$(3.4.1) \qquad\qquad |D\varphi_\infty|^2 \geq 1 + d_0 \qquad \text{on } \mathcal{N}_{\frac{1}{2C}}(\Gamma_{\text{shock}}).$$

(i) If $\gamma = 1$, then it follows directly from Definition 2.24 that any admissible solution φ satisfies that $|D\varphi| \leq 1$ in $\overline{\Omega}$. Thus, it follows from (3.4.1) that

$$(3.4.2) \qquad\qquad |D\varphi_\infty|^2 - |D\varphi|^2 \geq d_0 \qquad \text{on } \mathcal{N}_{\frac{1}{2C}}(\Gamma_{\text{shock}}) \cap \overline{\Omega}.$$

(ii) If $\gamma > 1$, then we can rewrite the Bernoulli law (2.4.2) as

$$(3.4.3) \qquad \rho^{\gamma-1} + \frac{\gamma-1}{2}\big(|D\varphi|^2 + \varphi\big) = 1 + \frac{\gamma-1}{2}\big(|D\varphi_\infty|^2 + \varphi_\infty\big).$$

Let φ be an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$. Since $|D\varphi|^2 \leq \rho^{\gamma-1}$ and $\varphi_\infty - \varphi \geq 0$ hold in $\overline{\Omega}$, we obtain from (3.4.1) and (3.4.3) that

$$\frac{\gamma+1}{2}\rho^{\gamma-1} \geq \rho^{\gamma-1} + \frac{\gamma-1}{2}|D\varphi|^2 \geq 1 + \frac{\gamma-1}{2}(1 + d_0) \qquad \text{on } \mathcal{N}_{\frac{1}{2C}}(\Gamma_{\text{shock}}) \cap \overline{\Omega}.$$

This implies that $\rho^{\gamma-1} - 1 \geq \delta_0$ for some constant $\delta_0 > 0$ depending only on (v_∞, γ). Then

$$|D\varphi_\infty|^2 - |D\varphi|^2 = \frac{2(\rho^{\gamma-1} - 1)}{\gamma-1} + (\varphi_\infty - \varphi) \geq \frac{2\delta_0}{\gamma-1} + (\varphi_\infty - \varphi) \qquad \text{on } \mathcal{N}_{\frac{1}{2C}}(\Gamma_{\text{shock}}) \cap \overline{\Omega}.$$

Since $\varphi_\infty - \varphi = 0$ on Γ_{shock}, it follows from (3.1.25) in Lemma 3.5 that there exist small constants $\varepsilon \in (0, \frac{1}{4C})$ and $\delta_0' > 0$ depending only on (γ, v_∞) such that

$$(3.4.4) \qquad\qquad |D\varphi_\infty| - |D\varphi| \geq \delta_0' \qquad \text{on } \mathcal{N}_\varepsilon(\Gamma_{\text{shock}}) \cap \overline{\Omega}.$$

Let (r, θ) be the polar coordinates defined by (3.2.27). Note that $|D\varphi_\infty| = -\partial_r \varphi_\infty$. Then (3.4.2) and (3.4.4) imply that there exists a constant $d_1 > 0$ depending only on (v_∞, γ) such that

$$(3.4.5) \qquad \partial_r(\varphi_\infty - \varphi) \leq -(|D\varphi_\infty| - |D\varphi|) \leq -d_1 \qquad \text{on } \mathcal{N}_\varepsilon(\Gamma_{\text{shock}}) \cap \overline{\Omega}.$$

Therefore, by the implicit function theorem, there exists a unique function $f_{O_\infty, \text{sh}}(\theta)$ such that

$$(3.4.6) \qquad\qquad \Gamma_{\text{shock}} = \{r = f_{O_\infty, \text{sh}}(\theta), \ \theta_{P_2} < \theta < \theta_{P_1}\},$$

where $(f_{O_\infty, \text{sh}}(\theta_{P_j}), \theta_{P_j})$ represent the (r, θ)–coordinates of points P_j for $j = 1, 2$, given by Definition 2.23. By Lemma 3.5 and (3.4.5), there exists a constant C_1 depending only on (v_∞, γ) such that

$$(3.4.7) \qquad\qquad \|f_{O_\infty, \text{sh}}\|_{C^{0,1}([\theta_{P_2}, \theta_{P_1}])} \leq C_1.$$

LEMMA 3.17. *Fix $\gamma \geq 1$ and $v_\infty > 0$. There exists a constant $\delta_1 > 0$ depending only on (v_∞, γ) such that, if φ is an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$, then*

$$(3.4.8) \qquad\qquad \partial_{\boldsymbol{\nu}}(\varphi_\infty - \varphi) > \delta_1 \qquad \text{on } \overline{\Gamma_{\text{shock}}},$$

$$(3.4.9) \qquad\qquad \partial_{\boldsymbol{\nu}} \varphi_\infty > \partial_{\boldsymbol{\nu}} \varphi \geq \delta_1 \qquad \text{on } \overline{\Gamma_{\text{shock}}}$$

for the unit normal vector $\boldsymbol{\nu} = \frac{D(\varphi_\infty - \varphi)}{|D(\varphi_\infty - \varphi)|}$ to Γ_{shock} towards the interior of Ω.

PROOF. If φ is an admissible solution corresponding to (v_∞, β), then it follows from (3.4.5) and $\varphi_\infty - \varphi = 0$ on Γ_{shock} that

$$(3.4.10) \qquad \partial_{\boldsymbol{\nu}}(\varphi_\infty - \varphi) = |D(\varphi_\infty - \varphi)| \geq |D\varphi_\infty| - |D\varphi| \geq d_1 \qquad \text{on } \overline{\Gamma_{\text{shock}}}.$$

Since $\partial_{\boldsymbol{\nu}}\varphi = \frac{\partial_{\boldsymbol{\nu}}\varphi_\infty}{\rho(|D\varphi|^2, \varphi)}$, $\partial_{\boldsymbol{\nu}}\varphi_\infty > 1$, and $\rho(|D\varphi|^2, \varphi) > 1$ on $\overline{\Gamma_{\text{shock}}}$, Lemma 3.5 yields that $\partial_{\boldsymbol{\nu}}\varphi_\infty > \partial_{\boldsymbol{\nu}}\varphi \geq C^{-1}$ for a constant $C > 0$ depending only on (v_∞, γ). The proof is completed by choosing δ_1 as

$$\delta_1 = \min\{d_1, C^{-1}\}.$$

$\square$

LEMMA 3.18. *Fix $\gamma \geq 1$ and $v_\infty > 0$. Let φ be an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$. Then, for each $d > 0$ and $k = 2, 3, \cdots$, there exist constants $s, C_k > 0$ depending only on (v_∞, γ, d) such that, if $P = (r_P, \theta_P) \in \Gamma_{\text{shock}}$ in the (r, θ)–coordinates, defined by (3.2.27), satisfies that $\text{dist}(P, \Gamma^{\mathcal{O}}_{\text{sonic}} \cup \Gamma^{\mathcal{N}}_{\text{sonic}}) \geq d$, then*

$$(3.4.11) \qquad |D^k f_{O_\infty, \text{sh}}(\theta_P)| \leq C_k, \qquad |D^k \varphi| \leq C_k \quad \text{in } B_s(P) \cap \Omega.$$

PROOF. The proof is divided into three steps.

1. Let φ be an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$, and let Ω be its pseudo-subsonic region. For a constant $d > 0$, define

$$\overline{\Omega}_d := \{\boldsymbol{\xi} \in \overline{\Omega} \,:\, \text{dist}(\boldsymbol{\xi}, \Gamma^{\mathcal{O}}_{\text{sonic}} \cup \Gamma^{\mathcal{N}}_{\text{sonic}}) > \frac{d}{2}\}.$$

Let $\mathcal{E}(\varphi, \overline{\Omega}_d)$ be defined by (3.2.17). Moreover, for a constant R, let $\mathcal{K}_R$ be given by (3.2.6). By Lemma 3.5 and Proposition 3.15, there exists a constant $M_d > 0$ depending only on (v_∞, γ, d) such that $\mathcal{E}(\varphi, \overline{\Omega}_d)$ is contained in $\mathcal{K}_{M_d}$.

Let $\mathcal{A}(\mathbf{p}, z) = (\mathcal{A}_1, \mathcal{A}_2)(\mathbf{p}, z)$ and $\mathcal{B}(\mathbf{p}, z)$ be defined by (3.2.3), and let $(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})(\mathbf{p}, z)$ be the extensions of $(\mathcal{A}, \mathcal{B})(\mathbf{p}, z)$ onto $\mathbb{R}^2 \times \mathbb{R}$ described in Lemma 3.8 with $M = M_d$.

2. We express the Rankine-Hugoniot jump condition: $\rho D\varphi \cdot \boldsymbol{\nu} = D\varphi_\infty \cdot \boldsymbol{\nu}$ as

$$(3.4.12) \qquad g^{\text{sh}}(D\varphi, \varphi, \boldsymbol{\xi}) = 0 \qquad \text{on } \Gamma_{\text{shock}}$$

for $g^{\text{sh}}(\mathbf{p}, z, \boldsymbol{\xi})$ defined by

$$(3.4.13) \qquad g^{\text{sh}}(\mathbf{p}, z, \boldsymbol{\xi}) = \big(\mathcal{A}(\mathbf{p}, z) - D\varphi_\infty(\boldsymbol{\xi})\big) \cdot \frac{D\varphi_\infty(\boldsymbol{\xi}) - \mathbf{p}}{|D\varphi_\infty(\boldsymbol{\xi}) - \mathbf{p}|}.$$

For $\delta_1 > 0$ from Lemma 3.17, define a smooth function $\zeta \in C^\infty(\mathbb{R})$ by

$$\zeta(t) = \begin{cases} t & \text{on } t \geq \frac{3}{4}\delta_1, \\ \frac{\delta_1}{2} & \text{for } t < \frac{\delta_1}{2}, \end{cases} \qquad \zeta'(t) \geq 0 \text{ on } \mathbb{R}.$$

Also, we define an extension of $g^{\text{sh}}_{\text{mod}}(\mathbf{p}, z, \boldsymbol{\xi})$ onto $\mathbb{R}^2 \times \mathbb{R} \times \overline{\Omega}_d$ by

$$(3.4.14) \qquad g^{\text{sh}}_{\text{mod}}(\mathbf{p}, z, \boldsymbol{\xi}) = \big(\tilde{\mathcal{A}}(\mathbf{p}, z) - D\varphi_\infty(\boldsymbol{\xi})\big) \cdot \frac{D\varphi_\infty(\boldsymbol{\xi}) - \mathbf{p}}{\zeta(|D\varphi_\infty(\boldsymbol{\xi}) - \mathbf{p}|)}.$$

Fix a point $P \in \Gamma_{\text{shock}}$ with $\text{dist}(P, \Gamma^{\mathcal{O}}_{\text{sonic}} \cup \Gamma^{\mathcal{N}}_{\text{sonic}}) > 2d$ for $d > 0$. Then φ satisfies

$$(3.4.15) \qquad \begin{aligned} &\text{div}\tilde{\mathcal{A}}(D\varphi, \varphi) + \tilde{\mathcal{B}}(D\varphi, \varphi) = 0 &&\text{in } B_{d/2}(P) \cap \Omega, \\ &g^{\text{sh}}_{\text{mod}}(D\varphi, \varphi, \boldsymbol{\xi}) = 0 &&\text{on } B_{d/2}(P) \cap \Gamma_{\text{shock}}. \end{aligned}$$

For $\varepsilon > 0$ from (3.4.5), define

$$R := \min\{\frac{d}{2}, \varepsilon\}.$$

Note that such a constant $R > 0$ is given depending only on (v_∞, γ, d), but independent of φ and P. By (3.4.5), we can write $D_{\mathbf{p}} g_{\text{mod}}^{\text{sh}}(D\varphi, \varphi, \boldsymbol{\xi})$ as

$$D_{\mathbf{p}} g_{\text{mod}}^{\text{sh}}(D\varphi, \varphi, \boldsymbol{\xi}) = D_{\mathbf{p}}\big((\mathcal{A}(\mathbf{p}, z, \boldsymbol{\xi}) - D\varphi_\infty(\boldsymbol{\xi})) \cdot \hat{\mathbf{n}}(\mathbf{p}, \boldsymbol{\xi})\big) \qquad \text{in } B_R(P) \cap \overline{\Omega}$$

for

$$\hat{\mathbf{n}}(\mathbf{p}, \boldsymbol{\xi}) = \frac{D\varphi_\infty(\boldsymbol{\xi}) - \mathbf{p}}{|D\varphi_\infty(\boldsymbol{\xi}) - \mathbf{p}|}.$$

Since

$$\hat{\mathbf{n}}(\mathbf{p}, \boldsymbol{\xi}) \cdot \big((\mathcal{A}(\mathbf{p}, z, \boldsymbol{\xi}) - D\varphi_\infty(\boldsymbol{\xi})) D_{\mathbf{p}} \hat{\mathbf{n}}(\mathbf{p}, \boldsymbol{\xi})\big) = \frac{1}{2}(\mathcal{A}(\mathbf{p}, z, \boldsymbol{\xi}) - D\varphi_\infty(\boldsymbol{\xi})) \cdot D_{\mathbf{p}}(|\hat{\mathbf{n}}(\mathbf{p}, \boldsymbol{\xi})|^2) = 0,$$

a direct computation yields that

$$D_{\mathbf{p}} g_{\text{mod}}^{\text{sh}}(D\varphi, \varphi, \boldsymbol{\xi}) \cdot \hat{\mathbf{n}}(D\varphi, \boldsymbol{\xi}) = \sum_{i,j=1}^{2} \partial_{p_i} \tilde{\mathcal{A}}_j(D\varphi, \varphi, \boldsymbol{\xi}) \hat{n}_i \hat{n}_j =: \hat{\lambda}(D\varphi, \boldsymbol{\xi}) \qquad \text{in } B_R(P) \cap \overline{\Omega}$$

for $\hat{n}_i = \hat{\mathbf{e}}_i \cdot \hat{\mathbf{n}}(D\varphi, \boldsymbol{\xi})$.

By Lemma 3.8(ii), there exists a constant $\lambda_d > 0$ depending only on (v_∞, γ, d) such that

$$D_{\mathbf{p}} g_{\text{mod}}^{\text{sh}}(D\varphi, \varphi, \boldsymbol{\xi}) \cdot \hat{\mathbf{n}}(D\varphi, \boldsymbol{\xi}) \geq \lambda_d > 0 \qquad \text{in } B_R(P) \cap \overline{\Omega}.$$

This implies that

$$(3.4.16) \qquad |D_{\mathbf{p}} g_{\text{mod}}^{\text{sh}}(D\varphi, \varphi, \boldsymbol{\xi})| \geq D_{\mathbf{p}} g_{\text{mod}}^{\text{sh}}(D\varphi, \varphi, \boldsymbol{\xi}) \cdot \hat{\mathbf{n}}(D\varphi, \boldsymbol{\xi}) \geq \lambda_d > 0 \qquad \text{in } B_R(P) \cap \overline{\Omega}.$$

3. By estimate (3.1.25) of Lemma 3.5, (3.4.7), Lemma 3.8, and (3.4.16), the boundary value problem (3.4.15) satisfies all the conditions necessary to apply Theorem C.8. Therefore, there exist $\beta \in (0, 1)$ and $C > 0$ depending only on (v_∞, γ, d) such that

$$\|\varphi\|_{1,\beta, B_{d/4}(P) \cap \Omega} \leq C \qquad \text{for all } P \in \Gamma_{\text{shock}} \cap \overline{\Omega}_d.$$

Combining the $C^{1,\beta}$–estimate of φ with (3.4.5) implies that $f_{O_\infty, \text{sh}}$ is $C^{1,\beta}$ away from $\theta = \theta_{P_1}, \theta_{P_2}$. Then we apply Theorem C.9 to the boundary value problem (3.4.15) to obtain the estimate:

$$\|\varphi\|_{2,\beta, B_{d/8}(P) \cap \Omega} \leq C \qquad \text{for all } P \in \Gamma_{\text{shock}} \cap \overline{\Omega}_d$$

for some constant $C > 0$ depending only on (v_∞, γ, d). This implies that $f_{O_\infty, \text{sh}}$ is $C^{1,\alpha}$ for any $\alpha \in (0, 1)$ away from $\theta = \theta_{P_1}, \theta_{P_2}$, so that φ is $C^{2,\alpha}$ for any $\alpha \in (0, 1)$ on Γ_{shock} away from $\overline{\Gamma_{\text{sonic}}^{\mathcal{O}} \cup \Gamma_{\text{sonic}}^{\mathcal{N}}}$ by Theorem C.9.

Finally, the C^k–estimates, $k = 2, 3, \cdots$, are obtained by a bootstrap argument via application of Theorem C.9 and Corollary C.10. $\qquad \square$

As a result, directly following from Lemmas 3.9 and 3.18, we conclude the following uniform C^k–estimates of admissible solutions:

COROLLARY 3.19. Fix $\gamma \geq 1$ and $v_\infty > 0$. For each $d > 0$ and $k = 2, 3, \cdots$, there exists a constant $C_{k,d} > 0$ depending only on (v_∞, γ, k, d) such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ satisfies

$$\|\varphi\|_{k, \overline{\Omega \cap \{\text{dist}(\boldsymbol{\xi}, \Gamma_{\text{sonic}}^{\mathcal{O}} \cup \Gamma_{\text{sonic}}^{\mathcal{N}}) > d\}}} \leq C_{k,d}.$$

3.4.2. $C^{2,\alpha}$**–estimates near** $\Gamma_{\text{sonic}}^{\mathcal{N}}$**.** For fixed $\gamma \geq 1$ and $v_\infty > 0$, the sonic arc $\Gamma_{\text{sonic}}^{\mathcal{N}}$, defined by Definition 2.23 corresponding to the normal shock part of each admissible solution, is fixed to be the same for all $\beta \in (0, \frac{\pi}{2})$. By Definition 2.24(ii) and Proposition 3.15, the ellipticity of Eq. (3.2.2) (or equivalently, Eq. (2.1.19)) degenerates near $\Gamma_{\text{sonic}}^{\mathcal{N}}$. In order to establish a uniform weighted $C^{2,\alpha}$–estimate of admissible solutions up to $\Gamma_{\text{sonic}}^{\mathcal{N}}$, the method of parabolic scaling is employed. We keep following Definition 2.23 for the notations used hereafter.

Define

$$(3.4.17) \qquad \hat{c}_{\mathcal{N}} := \frac{c_{\mathcal{N}} + \xi_2^{\mathcal{N}}}{2}.$$

Note that $\hat{c}_{\mathcal{N}}$ is the same for all $\beta \in [0, \frac{\pi}{2})$. In $U_{\mathcal{N}} := \left(B_{\frac{3c_{\mathcal{N}}}{2}}(O_{\mathcal{N}}) \setminus B_{\hat{c}_{\mathcal{N}}}(O_{\mathcal{N}}) \right) \cap \{\boldsymbol{\xi} : \xi_1 > 0\}$, let (r, θ) be the polar coordinates with respect to $O_{\mathcal{N}} = (0, 0)$. Define

$$(3.4.18) \qquad (x, y) := (c_{\mathcal{N}} - r, \theta).$$

Let φ be an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$, and let Ω be its pseudo-subsonic region. Define

$$(3.4.19) \qquad \Omega^{\mathcal{N}} := \left(\Omega \cap \{\xi_1 > 0\} \right) \setminus B_{\hat{c}_{\mathcal{N}}}(O_{\mathcal{N}}).$$

Then $\Omega^{\mathcal{N}} \subset B_{c_{\mathcal{N}}}(O_{\mathcal{N}})$ and $\Omega^{\mathcal{N}} \subset \{(x, y) : x > 0\}$.

In $\Omega^{\mathcal{N}}$, we define a function ψ by

$$(3.4.20) \qquad \psi := \varphi - \varphi_{\mathcal{N}} \qquad \text{in } \Omega^{\mathcal{N}}.$$

We rewrite Eq. (2.1.19) and the boundary conditions (2.5.35)–(2.5.37) in the (x, y)–coordinates as follows:

(i) *Equation for ψ in $\Omega^{\mathcal{N}}$*: For each $j = 1, \cdots, 5$, define $O_j^{\mathcal{N}}(\mathbf{p}, z, x)$ by

$$O_j^{\mathcal{N}}(\mathbf{p}, z, x) := O_j(\mathbf{p}, z, x, c_{\mathcal{N}})$$

for $O_j(\mathbf{p}, z, x, c)$ given by (3.2.29). Then Eq. (2.1.19) is written as

$$(3.4.21) \quad \left(2x - (\gamma + 1)\psi_x + O_1^{\mathcal{N}} \right)\psi_{xx} + O_2^{\mathcal{N}}\psi_{xy} + \left(\frac{1}{c_{\mathcal{N}}} + O_3^{\mathcal{N}} \right)\psi_{yy} - \left(1 + O_4^{\mathcal{N}} \right)\psi_x + O_5^{\mathcal{N}}\psi_y = 0,$$

with $O_j^{\mathcal{N}} = O_j^{\mathcal{N}}(D\psi, \psi, x)$ for $j = 1, \cdots, 5$.

(ii) *Boundary condition for ψ on $\Gamma_{\text{shock}} \cap \partial\Omega^{\mathcal{N}}$*: By the definitions of $(\varphi_\infty, \varphi_{\mathcal{N}})$ given in Definition 2.23, we rewrite the condition that $\varphi_\infty - \varphi = 0$ on $\Gamma_{\text{shock}} \cap \Omega^{\mathcal{N}}$ as

$$\xi_2 = \xi_2^{\mathcal{N}} - \frac{\psi}{v_\infty} \qquad \text{on } \Gamma_{\text{shock}} \cap \Omega^{\mathcal{N}}.$$

For $g_{\text{mod}}^{\text{sh}}(\mathbf{p}, z, \boldsymbol{\xi})$ given by (3.4.14), we define

$$(3.4.22) \qquad M(\mathbf{p}, z, \xi_1) := g_{\text{mod}}^{\text{sh}}\left(\mathbf{p} + D\varphi_{\mathcal{N}}, z + \varphi_{\mathcal{N}}, \xi_1, \xi_2^{\mathcal{N}} - \frac{z}{v_\infty} \right)$$

with $(D\varphi_{\mathcal{N}}, \varphi_{\mathcal{N}})$ evaluated at $(\xi_1, \xi_2^{\mathcal{N}} - \frac{z}{v_\infty})$. Then the boundary condition (2.5.37) is written as $M(D\psi, \psi, \xi_1) = 0$ on Γ_{shock}. Denote

$$\phi_\infty^{\mathcal{N}} := \varphi_\infty - \varphi_{\mathcal{N}}.$$

Then $|D(\phi_\infty^{\mathcal{N}} - \psi)| = |\partial_{\boldsymbol{\nu}}(\varphi_\infty - \varphi)| > 0$ on Γ_{shock}. Rewriting the boundary condition $|D(\phi_\infty^{\mathcal{N}} - \psi)| M(D\psi, \psi, \xi_1) = 0$ on $\Gamma_{\text{shock}} \cap \partial\Omega^{\mathcal{N}}$ in the (x, y)–coordinates, we obtain

$$(3.4.23) \qquad \mathcal{B}_1^{\mathcal{N}}(\psi_x, \psi_y, \psi, x, y) = 0 \qquad \text{on } \Gamma_{\text{shock}} \cap \partial\Omega^{\mathcal{N}}$$

for $\mathcal{B}_1^{\mathcal{N}}(p_x, p_y, z, x, y)$ defined by

$$(3.4.24) \qquad \mathcal{B}_1^{\mathcal{N}}(p_x, p_y, z, x, y) := |D\phi_\infty^{\mathcal{N}} - \mathbf{p}| M(\mathbf{p}, z, \xi_1)$$

with

$$(3.4.25) \qquad \xi_1 = (c_{\mathcal{N}} - x)\cos y, \qquad \mathbf{p} = \begin{pmatrix} -\cos y & -\sin y \\ -\sin y & \cos y \end{pmatrix} \begin{pmatrix} p_x \\ \frac{p_y}{c_{\mathcal{N}} - x} \end{pmatrix}.$$

(iii) *Other properties of ψ*: By (2.1.30) and Definition 2.24(ii)–(iv), ψ satisfies

$$(3.4.26) \qquad \begin{aligned} \psi &\geq 0 && \text{in } \Omega^{\mathcal{N}}, \\ \psi &= 0 && \text{on } \Gamma_{\text{sonic}}^{\mathcal{N}}, \\ \psi_y &= 0 && \text{on } \Gamma_{\text{wedge}} \cap \partial\Omega^{\mathcal{N}}. \end{aligned}$$

For each $\beta \in [0, \frac{\pi}{2})$, let $\mathcal{D}$ be defined by (2.5.27), and define

$$\Lambda^{\mathcal{N}} := \mathcal{D} \cap \left(B_{\frac{3c_{\mathcal{N}}}{2}}(O_{\mathcal{N}}) \setminus B_{\hat{c}_{\mathcal{N}}}(O_{\mathcal{N}}) \right) \cap \{\xi_1 > 0\}.$$

Note that $\Lambda^{\mathcal{N}}$ is the same for all $\beta \in [0, \frac{\pi}{2})$, and $\Lambda^{\mathcal{N}} \subset \{\xi_2 < \xi_2^{\mathcal{N}}\}$.

By using the definitions of $(\Gamma_{\text{sonic}}^{\mathcal{N}}, \varphi_\infty, \varphi_{\mathcal{N}})$ given in Definition 2.23, the following lemma can be directly verified:

LEMMA 3.20. *Fix $\gamma \geq 1$ and $v_\infty > 0$. There exist positive constants ε_1, ε_0, δ_0, ω_0, C, and $\mathfrak{M}$ depending only on (v_∞, γ) with $\varepsilon_1 > \varepsilon_0$ and $\mathfrak{M} \geq 2$ so that the following properties hold:*

(a) $\{\varphi_{\mathcal{N}} < \varphi_\infty\} \cap \Lambda^{\mathcal{N}} \cap \mathcal{N}_{\varepsilon_1}(\Gamma_{\text{sonic}}^{\mathcal{N}}) \subset \{0 < y < \frac{\pi}{2} - \delta_0\}$, *where $\mathcal{N}_\varepsilon(\Gamma)$ denotes the ε–neighborhood of a set Γ in the $\boldsymbol{\xi}$–coordinates;*

(b) $\{\varphi_{\mathcal{N}} < \varphi_\infty\} \cap \mathcal{N}_{\varepsilon_1}(\Gamma_{\text{sonic}}^{\mathcal{N}}) \cap \{y > y_{P_2}\} \subset \{x > 0\}$;

(c) *In $\{(x,y) : |x| < \varepsilon_1, \ 0 < y < \frac{\pi}{2} - \delta_0\}$, $\phi_\infty^{\mathcal{N}} = \varphi_\infty - \varphi_{\mathcal{N}}$ satisfies*

$$(3.4.27) \qquad \frac{2}{\mathfrak{M}} y \leq \partial_x \phi_\infty^{\mathcal{N}}(x, y) \leq \frac{\mathfrak{M}}{2}, \qquad \frac{2}{\mathfrak{M}} \leq -\partial_y \phi_\infty^{\mathcal{N}} \leq \frac{\mathfrak{M}}{2};$$

(d) $|(D_{(x,y)}^2, D_{(x,y)}^3)\phi_\infty^{\mathcal{N}}| \leq C$ *in $\{|x| < \varepsilon_1\}$;*

(e) *There exists a unique function $\hat{f}_{\mathcal{N},0} \in C^\infty([-\varepsilon_0, \varepsilon_0])$ such that*

$$(3.4.28)$$
$$\begin{cases} \{\varphi_{\mathcal{N}} < \varphi_\infty\} \cap \Lambda^{\mathcal{N}} \cap \mathcal{N}_{\varepsilon_1}(\Gamma_{\text{sonic}}^{\mathcal{N}}) \cap \{|x| < \varepsilon_0\} = \{(x,y) : |x| < \varepsilon_0, 0 < y < \hat{f}_{\mathcal{N},0}(x)\}, \\ S_{\mathcal{N}} \cap \mathcal{N}_{\varepsilon_1}(\Gamma_{\text{sonic}}^{\mathcal{N}}) \cap \{|x| < \varepsilon_0\} = \{(x,y) : x \in (-\varepsilon_0, \varepsilon_0), y = \hat{f}_{\mathcal{N},0}(x)\}; \end{cases}$$

(f) $\hat{f}_{\mathcal{N},0}$ *in (e) satisfies*

$$2\omega_0 \leq \hat{f}_{\mathcal{N},0}' \leq C \qquad \text{on } (-\varepsilon_0, \varepsilon_0).$$

Let Ω be the pseudo-subsonic region of an admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$. For $\varepsilon \in (0, \varepsilon_1]$, define a set $\Omega_\varepsilon^{\mathcal{N}}$ by

$$(3.4.29) \qquad \Omega_\varepsilon^{\mathcal{N}} := \Omega \cap \mathcal{N}_{\hat{\varepsilon}}(\Gamma_{\text{sonic}}^{\mathcal{N}}) \cap \{x < \varepsilon\}$$

for some $\hat{\varepsilon} = \hat{\varepsilon}(\varepsilon, \omega_0) > \varepsilon$.

Note that $\Omega_\varepsilon^{\mathcal{N}} \subset \{0 < x < \varepsilon\}$.

LEMMA 3.21. *Let ε_0, ω_0, and $\mathfrak{M}$ be from Lemma 3.20. Then there exist constants $\bar{\varepsilon} \in (0, \varepsilon_0]$, $L \geq 1$, $\delta \in (0, \frac{1}{2})$, and $\omega \in (0, \omega_0] \cap (0,1)$ depending only on (v_∞, γ) such that, whenever $\varepsilon \in (0, \bar{\varepsilon}]$, any admissible solution $\varphi = \psi + \varphi_{\mathcal{N}}$ satisfies the following properties in $\Omega^{\mathcal{N}}_\varepsilon$:*

(a) $\psi_x(x,y) \leq \frac{2-\delta}{1+\gamma} x \leq Lx$;

(b) $\psi_x \geq 0$ *and* $|\psi_y(x,y)| \leq Lx$;

(c) $\frac{2}{\mathfrak{M}} y - \frac{2-\delta}{1+\gamma} x \leq \partial_x(\varphi_\infty - \varphi)(x,y) \leq \mathfrak{M}$ *and* $\frac{1}{\mathfrak{M}} \leq -\partial_y(\varphi_\infty - \varphi) \leq \mathfrak{M}$;

(d) *there exists a unique function $\hat{f}_{\mathcal{N},\text{sh}} \in C^1([0,\varepsilon])$ such that*

$$\Omega^{\mathcal{N}}_\varepsilon = \{(x,y) \,:\, x \in (0,\varepsilon), 0 < y < \hat{f}_{\mathcal{N},\text{sh}}(x)\},$$

$$\Gamma_{\text{shock}} \cap \partial\Omega^{\mathcal{N}}_\varepsilon = \{(x,y) \,:\, x \in (0,\varepsilon), \; y = \hat{f}_{\mathcal{N},\text{sh}}(x)\},$$

$$\omega \leq \hat{f}'_{\mathcal{N},\text{sh}}(x) \leq L \qquad \text{for } 0 < x < \varepsilon;$$

(e) $0 \leq \psi(x,y) \leq Lx^2$.

PROOF. We divide the proof into four steps.

1. By (3.3.8) and (3.4.21), there exists a constant $\bar{\delta} \in (0, \frac{1}{4})$ depending only on (v_∞, γ) such that

$$(3.4.30) \qquad 2x - (\gamma+1)\psi_x + O^{\mathcal{N}}_1(D\psi(x,y), \psi(x,y), x) \geq 2\bar{\delta}x \qquad \text{in } \Omega^{\mathcal{N}}$$

for $\Omega^{\mathcal{N}}$ defined by (3.4.19). Since $O^{\mathcal{N}}_1(D\psi(x,y), \psi(x,y), x) \leq \frac{(\gamma+1)}{c_{\mathcal{N}}} x\psi_x$ by (3.2.29) and (3.4.26), we obtain from (3.4.30) that

$$\psi_x(x,y) \leq \frac{2 - 2\bar{\delta}}{(1+\gamma)(1 - \frac{\bar{\varepsilon}_0}{c_{\mathcal{N}}})} x \qquad \text{in } \Omega^{\mathcal{N}}_{\bar{\varepsilon}_0}$$

for

$$\bar{\varepsilon}_0 = \min\{c_{\mathcal{N}} - \hat{c}_{\mathcal{N}}, \varepsilon_0\},$$

where $\hat{c}_{\mathcal{N}}$ is given by (3.4.17). Then $\bar{\varepsilon} \in (0, \varepsilon_0]$ can be chosen, depending only on (v_∞, γ), so that ψ satisfies

$$\psi_x(x,y) \leq \frac{2-\bar{\delta}}{1+\gamma} x \qquad \text{in } \Omega^{\mathcal{N}}_{\bar{\varepsilon}}.$$

This proves statement (a).

By Lemma 3.6, (3.4.18), and (3.4.25), we have

$$(3.4.31) \qquad \psi_x \cos y + \frac{\psi_y}{c_{\mathcal{N}} - x} \sin y \geq 0, \quad \psi_x \sin y - \frac{\psi_y}{c_{\mathcal{N}} - x} \cos y \geq 0 \qquad \text{in } \Omega^{\mathcal{N}}.$$

By property (f) of Lemma 3.20, there exists a constant $\delta_1 \in (0, \frac{\pi}{10})$ depending only on (v_∞, γ) such that

$$(3.4.32) \qquad \Omega^{\mathcal{N}} \subset \{0 < y < \frac{\pi}{2} - \delta_1\}.$$

Then (3.4.31), combined with statement (a), yields that

$$(3.4.33) \qquad 0 \leq \psi_x(x,y) \leq \frac{2-\bar{\delta}}{1+\gamma} x \qquad \text{in } \Omega^{\mathcal{N}}_{\bar{\varepsilon}}.$$

Owing to (3.4.32), the second inequality in (3.4.31) is equivalent to

$$\psi_y(x,y) \leq (c_{\mathcal{N}} - x)\psi_x(x,y) \tan y \qquad \text{in } \Omega^{\mathcal{N}}.$$

Then it follows directly from (3.4.33) that

$$(3.4.34) \qquad\qquad \psi_y \leq Cx \qquad \text{in } \Omega^{\mathcal{N}}_{\bar{\varepsilon}}$$

for a constant $C > 0$ chosen depending only on (v_∞, γ).

2. In order to obtain a lower bound of ψ_y by a linear function of x near $\Gamma^{\mathcal{N}}_{\text{sonic}}$, a different approach is used.

By Proposition 3.11 and (3.4.32), there exists $\delta_1' \in (0, \frac{\pi}{10})$ depending only on (v_∞, γ) such that

$$(3.4.35) \qquad\qquad \partial\Omega^{\mathcal{N}} \cap \Gamma_{\text{shock}} \subset \{\delta_1' \leq y \leq \frac{\pi}{2} - \delta_1'\},$$

where Γ_{shock} denotes the curved pseudo-transonic shock of φ. Thus, the first inequality in (3.4.31) is equivalent to $\psi_y(x, y) \geq -(c_{\mathcal{N}} - x)\psi_x(x, y) \cot y$ on $\Omega^{\mathcal{N}} \cap \Gamma_{\text{shock}}$. Then (3.4.33) implies that there exists a constant $C_{\text{sh}} > 0$ depending only on (v_∞, γ) such that

$$(3.4.36) \qquad\qquad \psi_y \geq -C_{\text{sh}}\, x \qquad \text{on } \Gamma_{\text{shock}} \cap \partial\Omega^{\mathcal{N}}_{\bar{\varepsilon}}.$$

By (3.4.26), we have

$$(3.4.37) \qquad\qquad \psi_y = 0 \qquad \text{on } \Gamma^{\mathcal{N}}_{\text{sonic}} \cup (\Gamma_{\text{wedge}} \cap \partial\Omega^{\mathcal{N}}_{\bar{\varepsilon}}).$$

By (3.1.25) in Lemma 3.5, there exists a constant $C_{\text{in}} > 0$ depending only on (v_∞, γ) such that ψ satisfies

$$(3.4.38) \qquad\qquad \psi_y \geq -C_{\text{in}} \qquad \text{on } \Omega^{\mathcal{N}}.$$

3. By adjusting Step 3 in the proof of [**11**, Lemma 11.2.6], the following lemma holds:

LEMMA 3.22. *Fix constants $\gamma \geq 1$, $c > 0$, and $r_0 \in (0, \frac{c}{2}]$. Given an open set*

$$U \subset \{(x, y) \in \mathbb{R}^2 : 0 < x < r_0\},$$

assume that a function $\psi \in C^3(U)$ satisfies the equation:

$$\mathcal{N}_{pl}(\psi) := \left(2x - (\gamma + 1)\psi_x + O_1\right)\psi_{xx} + O_2\psi_{xy} + \left(\frac{1}{c} + O_3\right)\psi_{yy} - (1 + O_4)\psi_x + O_5\psi_y = 0 \quad \text{in } U,$$

with $O_j = O_j(D\psi(x, y), \psi(x, y), x, c)$ for $j = 1 \cdots, 5$, where each $O_j(p_x, p_y, z, x, c)$ is defined by (3.2.29). Moreover, let ψ satisfy the following inequalities:

$$\psi \geq 0, \quad 0 \leq \psi_x \leq \frac{2 - \delta_0}{1 + \gamma}x \qquad \text{in } U,$$

for some constant $\delta_0 \in (0, 1)$. Then there exists a constant $\varepsilon \in (0, r_0)$ depending only on (γ, c, δ_0) so that $\partial_y\mathcal{N}_{pl}(\psi) = 0$ is rewritten as a linear equation for $w := \psi_y$ in the following form:

$$(3.4.39) \qquad \begin{aligned} \mathcal{L}_\psi(w) &:= \left(2x - (\gamma + 1)O_1\right)w_{xx} + O_2 w_{xy} + \left(\frac{1}{c} + O_3\right)w_{yy} \\ &\quad + b_1^{(\psi)}w_x + b_2^{(\psi)}w_y + b_0^{(\psi)}w = 0 \qquad \text{in } U \cap \{x < \varepsilon\}, \end{aligned}$$

with

$$(3.4.40) \qquad\qquad b_1^{(\psi)} \leq 0, \quad b_0^{(\psi)} \leq 0 \qquad \text{in } U \cap \{x < \varepsilon\}.$$

By Definition 2.24(iv) and (3.4.33), we can apply Lemma 3.22 to $\psi = \varphi - \varphi_{\mathcal{N}}$. Therefore, we can further reduce constant $\bar{\varepsilon} \in (0, \varepsilon_0]$ depending only on (v_∞, γ) so that ψ_y satisfies the elliptic equation:

$$\mathcal{L}_\psi(\psi_y) = 0 \qquad \text{in } \Omega^{\mathcal{N}}_{\bar{\varepsilon}}.$$

For constants C_{sh} and C_{in} from (3.4.36) and (3.4.38), respectively, we choose $M := \max\{C_{\text{sh}}, \frac{C_{\text{in}}}{\bar{\varepsilon}}\}$. Then $w = \psi_y$ satisfies

$$w + Mx \geq 0 \qquad \text{on } \partial\Omega^{\mathcal{N}}_{\bar{\varepsilon}},$$

$$\mathcal{L}_\psi(w + Mx) = \mathcal{L}_\psi(Mx) = M\big(b_1^{(\psi)} + b_0^{(\psi)}x\big) \leq 0 \qquad \text{in } \Omega^{\mathcal{N}}_{\bar{\varepsilon}}.$$

The second inequality stated above is obtained from (3.4.40). Note that constant M is chosen depending only on (v_∞, γ). By the maximum principle, we obtain

$$w(x, y) \geq -Mx \qquad \text{in } \Omega^{\mathcal{N}}_{\bar{\varepsilon}}.$$

Combining this with (3.4.33)–(3.4.34) yields statement (b) of Lemma 3.21.

4. By Lemma 3.20(c) and Lemma 3.21(b), we have

$$\partial_x(\varphi_\infty - \varphi) \leq \partial_x\phi^{\mathcal{N}}_\infty \leq \frac{\mathfrak{M}}{2} \qquad \text{in } \Omega^{\mathcal{N}}_{\bar{\varepsilon}}.$$

By Lemma 3.20(c) and Lemma 3.21(a), we obtain

$$\partial_x(\varphi_\infty - \varphi)(x, y) = \partial_x\phi^{\mathcal{N}}_\infty(x, y) - \psi_x \geq \frac{2y}{\mathfrak{M}} - \frac{2 - \delta}{1 + \gamma}x \qquad \text{in } \Omega^{\mathcal{N}}_{\bar{\varepsilon}}.$$

The estimate of $\partial_y(\varphi_\infty - \varphi)$ stated in statement (c) of Lemma 3.21 is similarly obtained.

The existence of a function $\hat{f}_{\mathcal{N},\text{sh}} : [0, \bar{\varepsilon}] \to \mathbb{R}^+$ satisfying statement (d) directly follows from $\varphi_\infty - \varphi = 0$ on Γ_{shock}, Lemma 3.21(c), and the implicit function theorem.

Finally, statement (e) directly follows from statements (a)–(b) and (d) of Lemma 3.21, and Definition 2.24(iv). □

LEMMA 3.23. *Write Eq. (3.4.21) in $\Omega^{\mathcal{N}}$ as*

$$\sum_{i,j=1}^2 \hat{A}^{\mathcal{N}}_{ij}(D\psi, \psi, x)D^2_{ij}\psi + \sum_{i=1}^2 \hat{A}^{\mathcal{N}}_i(D\psi, \psi, x)D_i\psi = 0,$$

with $(D_1, D_2) = (D_x, D_y)$ and $\hat{A}^{\mathcal{N}}_{21} = \hat{A}^{\mathcal{N}}_{12}$. Then there exist $\varepsilon_{\mathcal{N}} \in (0, \frac{\bar{\varepsilon}}{4}]$ and $\lambda_{\mathcal{N}} > 0$ depending only on (v_∞, γ) such that, for any admissible solution $\varphi = \psi + \varphi_{\mathcal{N}}$ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$, if $(x, y) \in \overline{\Omega^{\mathcal{N}}_{4\varepsilon_{\mathcal{N}}}}$, then

$$(3.4.41) \quad \frac{\lambda_{\mathcal{N}}}{2}|\boldsymbol{\kappa}|^2 \leq \sum_{i,j=1}^2 \hat{A}^{\mathcal{N}}_{ij}(D\psi(x, y), \psi(x, y), x)\frac{\kappa_i\kappa_j}{x^{2 - \frac{i+j}{2}}} \leq \frac{2}{\lambda_{\mathcal{N}}}|\boldsymbol{\kappa}|^2 \qquad \text{for all } \boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2.$$

Moreover, $\mathcal{B}^{\mathcal{N}}_1$ defined by (3.4.24) satisfies the following properties:

(a) $\mathcal{B}^{\mathcal{N}}_1(\mathbf{0}, 0, x, y) = 0$ *for all $(x, y) \in \mathbb{R}^2$;*

(b) *For each $k = 2, 3, \cdots$, there exist constants $\delta_{\text{bc}} > 0$ and $C > 1$ depending only on (v_∞, γ, k) such that, whenever $|(p_x, p_y, z, x)| \leq \delta_{\text{bc}}$ and $|y - y_{P_2}| \leq \delta_{\text{bc}}$,*

$$|D^k_{(p_x, p_y, z, x, y)}\mathcal{B}^{\mathcal{N}}_1(p_x, p_y, z, x, y)| \leq C;$$

(c) *There exist constants $\hat{\delta}_{bc} > 0$ and $C > 1$ depending only on (v_∞, γ) such that, whenever $|(p_x, p_y, z, x)| \leq \hat{\delta}_{bc}$ and $|y - y_{P_2}| \leq \hat{\delta}_{bc}$,*

$$D_{(p_x, p_y, z)} B_1^{\mathcal{N}}(p_x, p_y, z, x, y) \leq -C^{-1}.$$

In (b) *and* (c) *above, y_{P_2} represents the y–coordinate of point P_2, defined by Definition 2.23.*

PROOF. (3.4.41) can be checked directly from (3.2.29). Properties (a)–(b) of $\mathcal{B}_1^{\mathcal{N}}$ are the results directly following from the definition of $\varphi_{\mathcal{N}}$, (3.4.13), (3.4.22), and (3.4.24).

A direct calculation by using the definition of $\varphi_{\mathcal{N}}$ in Definition 2.23, (3.2.3)–(3.2.4), (3.4.13), (3.4.22), and (3.4.24) yields that

$$\partial_z \mathcal{B}_1^{\mathcal{N}}(\mathbf{0}, 0, 0, y_{P_2}) = -\frac{\rho_{\mathcal{N}} v_\infty \xi_2^{\mathcal{N}}}{c_{\mathcal{N}}^2},$$

$$\partial_{p_x} \mathcal{B}_1^{\mathcal{N}}(\mathbf{0}, 0, 0, y_{P_2}) = -\frac{\rho_{\mathcal{N}} - 1}{c_{\mathcal{N}}}(\xi_1^{\mathcal{N}})^2,$$

$$\partial_{p_y} \mathcal{B}_1^{\mathcal{N}}(\mathbf{0}, 0, 0, y_{P_2}) = -\frac{\xi_1^{\mathcal{N}}}{c_{\mathcal{N}}^2}\big(\rho_{\mathcal{N}} v_\infty + (\rho_{\mathcal{N}} - 1)\xi_2^{\mathcal{N}}\big).$$

Then property (c) is obtained by combining the results stated immediately above with property (b). $\qquad\square$

LEMMA 3.24. *Let $\varepsilon_0 > 0$ and $L \geq 1$ be the constants from Lemma 3.20 and Lemma 3.21, respectively. Then there exist constants $\varepsilon \in (0, \frac{\varepsilon_0}{2}]$ and $C > 0$ depending only on (v_∞, γ) such that any admissible solution $\varphi = \varphi_{\mathcal{N}} + \psi$ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{weak}$ satisfies the following equation:*

$$\sum_{i,j=1}^{2} \hat{A}_{ij}^{(mod)}(D\psi, \psi, x) D_{ij}\psi + \sum_{i=1}^{2} \hat{A}_i^{(mod)}(D\psi, \psi, x) D_i\psi = 0 \qquad in \ \Omega_\varepsilon^{\mathcal{N}},$$

with coefficients $(\hat{A}_{ij}^{(mod)}, \hat{A}_i^{(mod)})$ satisfying the following properties:

(a) $(\hat{A}_{ij}^{(mod)}, \hat{A}_i^{(mod)}) = (\hat{A}_{ij}^{\mathcal{N}}, \hat{A}_i^{\mathcal{N}})$ *in* $\{(p_x, p_y, z, x) : |(p_x, p_y)| \leq Lx, \ |z| \leq Lx^2, \ x \in (0, \varepsilon)\}$,

(b) $|(\hat{A}_{11}^{(mod)}, \hat{A}_{12}^{(mod)}, \hat{A}_2^{(mod)})(p_x, p_y, z, x)| \leq Cx$ *in* $\mathbb{R}^2 \times \mathbb{R} \times (0, \varepsilon)$,

(c) $\|(\hat{A}_{22}^{(mod)}, \hat{A}_1^{(mod)})\|_{0, \mathbb{R}^2 \times \mathbb{R} \times (0, \varepsilon)} \leq C$,

(d) $\|D_{(p_x, p_y, z, x)}(\hat{A}_{ij}^{(mod)}, \hat{A}_i^{(mod)})\|_{0, \mathbb{R}^2 \times \mathbb{R} \times (0, \varepsilon)} \leq C$.

PROOF. This lemma can be proved by adjusting the proof of [**11**, Corollary 11.2.12].

Choose a function $\eta \in C^\infty(\mathbb{R})$ such that $0 \leq \eta \leq 1$ with $\eta(t) = 1$ for $|t| \leq L$ and $\eta(t) = 0$ for $|t| \geq 2L$. For such a function η, we define $(\hat{A}_{ij}^{(mod)}, \hat{A}_i^{(mod)})$ by

(3.4.42) $(\hat{A}_{ij}^{(mod)}, \hat{A}_i^{(mod)})(p_x, p_y, z, x) = (\hat{A}_{ij}^{\mathcal{N}}, \hat{A}_i^{\mathcal{N}})(x\eta(\frac{p_x}{x}), x\eta(\frac{p_y}{x}), x^2\eta(\frac{z}{x^2}), x).$

Then Lemma 3.24 directly follows from (3.4.21) and Lemma 3.21. $\qquad\square$

For the uniform weighted $C^{2,\alpha}$–estimates of admissible solutions near $\Gamma_{sonic}^{\mathcal{N}}$, we recall the definition of the norm introduced in [**10**].

DEFINITION 3.25 (Parabolic norms). Fix a constant $\alpha \in (0, 1)$.

(i) For $z = (x, y), \tilde{z} = (\tilde{x}, \tilde{y}) \in \mathbb{R}^2 \cap \{x > 0\}$, define

$$\delta^{(\text{par})}_\alpha(z, \tilde{z}) := \left(|x - \tilde{x}|^2 + \max\{x, \tilde{x}\}|y - \tilde{y}|^2\right)^{\frac{\alpha}{2}}.$$

(ii) Let $\mathcal{D}$ be an open set in $\mathbb{R}^2 \cap \{x > 0\}$. For a function $u \in C^2(\mathcal{D})$ in the (x, y)–coordinates, define

$$\|u\|^{(\text{par})}_{2,0,\mathcal{D}} := \sum_{0 \leq k+l \leq 2} \sup_{z \in \mathcal{D}} \left(x^{k+\frac{l}{2}-2}|\partial^k_x \partial^l_y u(z)|\right),$$

$$[u]^{(\text{par})}_{2,\alpha,\mathcal{D}} := \sum_{k+l=2} \sup_{z,\tilde{z} \in \mathcal{D}, z \neq \tilde{z}} \left(\min\left\{x^{\alpha+k+\frac{l}{2}-2}, \tilde{x}^{\alpha+k+\frac{l}{2}-2}\right\} \frac{|\partial^k_x \partial^l_y u(z) - \partial^k_x \partial^l_y u(\tilde{z})|}{\delta^{(\text{par})}_\alpha(z, \tilde{z})}\right),$$

$$\|u\|^{(\text{par})}_{2,\alpha,\mathcal{D}} := \|u\|^{(\text{par})}_{2,0,\mathcal{D}} + [u]^{(\text{par})}_{2,\alpha,\mathcal{D}}.$$

(iii) Fix an open interval $I := (0, a)$. For a function $f \in C^2(I)$, define

$$\|f\|^{(\text{par})}_{2,0,I} := \sum_{k=0}^{2} \sup_{x \in I} \left(x^{k-2}|\partial^k_x f(x)|\right),$$

$$[f]^{(\text{par})}_{2,\alpha,I} := \sup_{x,\tilde{x} \in I, x \neq \tilde{x}} \left(\min\{x^\alpha, \tilde{x}^\alpha\} \frac{|\partial^2_x f(x) - \partial^2_x f(\tilde{x})|}{|x - \tilde{x}|^\alpha}\right),$$

$$\|f\|^{(\text{par})}_{2,\alpha,I} := \|f\|^{(\text{par})}_{2,0,I} + [f]^{(\text{par})}_{2,\alpha,I}.$$

(iv) Given constants $\sigma > 0$, $\alpha \in (0, 1)$, and $m \in \mathbb{Z}_+$, define

$$\|u\|^{(\sigma),(\text{par})}_{m,0,\mathcal{D}} := \sum_{0 \leq k+l \leq m} \sup_{z \in \mathcal{D}} \left(x^{k+\frac{l}{2}-\sigma}|\partial^k_x \partial^l_y u(z)|\right),$$

$$[u]^{(\sigma),(\text{par})}_{m,\alpha,\mathcal{D}} := \sum_{k+l=m} \sup_{z,\tilde{z} \in \mathcal{D}, z \neq \tilde{z}} \left(\min\left\{x^{\alpha+k+\frac{l}{2}-\sigma}, \tilde{x}^{\alpha+k+\frac{l}{2}-\sigma}\right\} \frac{|\partial^k_x \partial^l_y u(z) - \partial^k_x \partial^l_y u(\tilde{z})|}{\delta^{(\text{par})}_\alpha(z, \tilde{z})}\right),$$

$$\|f\|^{(\sigma),(\text{par})}_{m,0,I} := \sum_{k=0}^{m} \sup_{x \in I} \left(x^{k-\sigma}|\partial^k_x f(x)|\right),$$

$$[f]^{(\sigma),(\text{par})}_{m,\alpha,I} := \sup_{x,\tilde{x} \in I, x \neq \tilde{x}} \left(\min\left\{x^{\alpha+m-\sigma}, \tilde{x}^{\alpha+m-\sigma}\right\} \frac{|\partial^m_x f(x) - \partial^m_x f(\tilde{x})|}{|x - \tilde{x}|^\alpha}\right),$$

$$\|u\|^{(\sigma),(\text{par})}_{m,\alpha,\mathcal{D}} := \|u\|^{(\sigma),(\text{par})}_{m,0,\mathcal{D}} + [u]^{(\sigma),(\text{par})}_{m,\alpha,\mathcal{D}}, \qquad \|f\|^{(\sigma),(\text{par})}_{m,\alpha,I} := \|f\|^{(\sigma),(\text{par})}_{m,0,I} + [f]^{(\sigma),(\text{par})}_{m,\alpha,I}.$$

Note that norm $\|\cdot\|^{(\text{par})}_{2,\alpha,\mathcal{D}}$ in (ii) is norm $\|\cdot\|^{(2),(\text{par})}_{2,\alpha,\mathcal{D}}$ above here.

(v) Denote by $C^{m,\alpha}_{(\sigma),(\text{par})}(\mathcal{D})$ the completion of set $\{u \in C^\infty(\mathcal{D}) : \|u\|^{(\sigma),(\text{par})}_{m,\alpha,\mathcal{D}} < \infty\}$ under norm $\|\cdot\|^{(\sigma),(\text{par})}_{m,\alpha,\mathcal{D}}$.

PROPOSITION 3.26. Let $\varepsilon_\mathcal{N} > 0$ be from Lemma 3.23. For each $\alpha \in (0, 1)$, there exists $C > 0$ depending only on $(v_\infty, \gamma, \alpha)$ such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ satisfies

$$(3.4.43) \qquad \|\varphi - \varphi_\mathcal{N}\|^{(\text{par})}_{2,\alpha,\Omega^\mathcal{N}_{\varepsilon_\mathcal{N}}} + \|\hat{f}_{\mathcal{N},\text{sh}} - \hat{f}_{\mathcal{N},0}\|^{(\text{par})}_{2,\alpha,(0,\varepsilon_\mathcal{N})} \leq C.$$

PROOF. The proof is divided into six steps.

1. *Re-scaling coordinates.* Fix $\varepsilon \in (0, \frac{\varepsilon_{\mathcal{N}}}{2}]$. For $z_0 := (x_0, y_0) \in \overline{\Omega^{\mathcal{N}}_{\varepsilon}} \setminus \overline{\Gamma^{\mathcal{N}}_{\text{sonic}}}$ and $r \in (0, 1]$, define

$$\tilde{R}_{z_0, r} := \{(x, y) : |x - x_0| < \frac{r}{4}x_0, \ |y - y_0| < \frac{r}{4}\sqrt{x_0}\}, \qquad R_{z_0, r} := \tilde{R}_{z_0, r} \cap \Omega^{\mathcal{N}}_{2\varepsilon}.$$

If $\varepsilon \leq y_{P_2}^2$ and $z_0 \in \Gamma_{\text{shock}} \cap \overline{\Omega^{\mathcal{N}}_{\varepsilon}}$, then it follows from Lemma 3.21(d) that

$$(3.4.44) \qquad\qquad R_{z_0, 1} \subset \{(x, y) : \frac{3}{4}x_0 < x < \frac{5}{4}x_0\}.$$

For $r > 0$, define the sets:

$$Q_r := (-r, r)^2, \qquad Q_r^{(z_0)} := \{(S, T) \in Q_r : z_0 + \frac{1}{4}(x_0 S, \sqrt{x_0} \, T) \in R_{z_0, r}\}.$$

2. *Re-scaled function $\psi^{(z_0)}$.* Let ψ be given by (3.4.20). For $z_0 \in \overline{\Omega^{\mathcal{N}}} \cap \Gamma_{\text{shock}}$, define a function $\psi^{(z_0)}(S, T)$ by

$$\psi^{(z_0)}(S, T) = \frac{1}{x_0^2} \psi(x_0 + \frac{x_0}{4}S, y_0 + \frac{\sqrt{x_0}}{4}T) \qquad \text{for } (S, T) \in Q_1^{(z_0)}.$$

By Lemma 3.21 and (3.4.44), we have

$$|\psi^{(z_0)}| \leq L, \quad |\psi_S^{(z_0)}| \leq L, \quad |\psi_T^{(z_0)}| \leq L x_0^{-1/2} \qquad \text{in } Q_{1/2}^{(z_0)}.$$

Moreover, Lemma 3.24 implies that $\psi^{(z_0)}$ satisfies the equation:

$$\sum_{i,j=1}^{2} A_{ij}^{(z_0)}(D\psi^{(z_0)}, \psi^{(z_0)}, S) D_{ij}\psi^{(z_0)} + \sum_{i=1}^{2} A_i^{(z_0)}(D\psi^{(z_0)}, \psi^{(z_0)}, S) D_i\psi^{(z_0)} = 0 \qquad \text{in } Q_{1/2}^{(z_0)},$$

where $(D_1, D_2) = (D_S, D_T)$, $D_{ij} = D_i D_j$, and

$$A_{ij}^{(z_0)}(p_1, p_2, z, S) := x_0^{\frac{i+j}{2} - 2} A_{ij}^{(\text{mod})}(4x_0 p_1, 4x_0^{3/2} p_2, x_0^2 z, x_0(1 + \frac{S}{4})),$$

$$A_i^{(z_0)}(p_1, p_2, z, S) := \frac{1}{4}x_0^{\frac{i-1}{2}} A_i^{(\text{mod})}(4x_0 p_1, 4x_0^{3/2} p_2, x_0^2 z, x_0(1 + \frac{S}{4})).$$

For $\hat{f}_{\mathcal{N}, \text{sh}}$ given in Lemma 3.21(d), we define

$$(3.4.45) \qquad F^{(z_0)}(S) := \frac{4}{\sqrt{x_0}} \left(\hat{f}_{\mathcal{N}, \text{sh}}(x_0 + \frac{x_0}{4}S) - \hat{f}_{\mathcal{N}, \text{sh}}(x_0) \right) \qquad \text{for } -1 < S < 1.$$

It follows directly from Lemma 3.21(d) and (3.4.45) that $F^{(z_0)}$ satisfies

$$(3.4.46) \qquad F^{(z_0)}(0) = 0, \qquad \|F^{(z_0)}\|_{C^1([-1,1])} \leq C\sqrt{x_0}$$

for some constant $C > 0$ depending only on (v_∞, γ). Therefore, there exists $\varepsilon_* \in (0, \frac{\bar{\varepsilon}}{2}]$ depending only on (v_∞, γ) such that $F^{(z_0)}(S) > -\frac{r}{2}$ for $S \in (-r, r)$, whenever $r \in (0, 1)$ and $z_0 \in \overline{\Omega^{\mathcal{N}}_{\varepsilon_*}} \cap \Gamma_{\text{shock}}$.

For $z_0 \in \overline{\Omega^{\mathcal{N}}_{\varepsilon_*}} \cap \Gamma_{\text{shock}}$, define

$$\Gamma_{\text{shock}}^{(z_0)} := \{(S, T) : S \in (-1, 1), T = F^{(z_0)}(S)\} \subset \partial Q_1^{(z_0)}.$$

Then $\text{dist}(\Gamma_{\text{shock}}^{(z_0)}, \partial Q_1^{(z_0)} \cap \{T = -1\}) \geq \frac{1}{2}$.

By Lemma 3.21(a)–(b) and (e), we can fix a small constant $\varepsilon_* \in (0, \frac{\bar{\varepsilon}}{2}]$ depending only on (v_∞, γ) so that any admissible solution satisfies

$$|(\psi_x, \psi_y, \psi, y - y_{P_2})| \leq \frac{1}{4} \min\{\delta_{\text{bc}}, \hat{\delta}_{\text{bc}}\} \qquad \text{in } \Omega^{\mathcal{N}}_{2\varepsilon_*}$$

for constants $(\delta_{\text{bc}}, \hat{\delta}_{\text{bc}})$ from Lemma 3.23. Then we apply Lemma 3.23(c) and the implicit function theorem to rewrite the boundary condition (3.4.23) as

$$(3.4.47) \qquad \psi_x = b_{\mathcal{N}}(\psi_y, \psi, x, y) \qquad \text{on } \Gamma_{\text{shock}} \cap \overline{\Omega_{2\varepsilon_*}^{\mathcal{N}}}.$$

By Lemma 3.23(a)–(b), we have

$$(3.4.48) \qquad \begin{aligned} b_{\mathcal{N}}(0,0,x,y) &= 0 \qquad \text{in } \overline{\Omega_{2\varepsilon_*}^{\mathcal{N}}}, \\ |D^k b_{\mathcal{N}}(p_y, z, x, y)| &\leq C_k \qquad \text{in } \mathbb{R} \times \mathbb{R} \times \overline{\Omega_{2\varepsilon_*}^{\mathcal{N}}} \text{ for } k = 1, 2, 3, \cdots, \end{aligned}$$

where constants $C_k > 0$ depend only on (v_∞, γ, k).

For each $z_0 \in \Gamma_{\text{shock}} \cap \overline{\Omega_{\varepsilon_*}^{\mathcal{N}}}$, denote

$$(3.4.49) \qquad B_{\mathcal{N}}^{(z_0)}(p_T, z, S, T) := \frac{1}{4x_0} b_{\mathcal{N}}(4x_0^{3/2} p_T, x_0^2 z, x, y) \qquad \text{for } (x, y) = z_0 + (\tfrac{x_0}{4}S, \tfrac{\sqrt{x_0}}{4}T).$$

It follows directly from (3.4.48) that there exists a constant $m_1 > 0$ depending only on (v_∞, γ) such that

$$(3.4.50) \qquad \begin{aligned} B_{\mathcal{N}}^{(z_0)}(0,0,S,T) &= 0 \qquad \text{in } \overline{Q_1^{(z_0)}}, \\ \|\partial_{p_T} B_{\mathcal{N}}^{(z_0)}(p_T, z, \cdot)\|_{0, \overline{Q_1^{(z_0)}}} &\leq m_1 \sqrt{x_0} \qquad \text{for all } (p_T, z) \in \mathbb{R} \times \mathbb{R}, \\ \|D_{(p_T, z)} B_{\mathcal{N}}^{(z_0)}(p_T, z, \cdot)\|_{1, \overline{Q_1^{(z_0)}}} &\leq m_1 \sqrt{x_0} \qquad \text{for all } (p_T, z) \in \mathbb{R} \times \mathbb{R}. \end{aligned}$$

By (3.4.47), $\psi^{(z_0)}$ satisfies

$$(3.4.51) \qquad \psi_S^{(z_0)} = B_{\mathcal{N}}^{(z_0)}(\psi_T^{(z_0)}, \psi^{(z_0)}, S, T) \qquad \text{on } \Gamma_{\text{shock}}^{(z_0)}.$$

3. *Uniform estimates of $\psi^{(z_0)}$ for $z_0 \in \Gamma_{\text{shock}}$.* By (3.4.46) and (3.4.50), we can apply Theorem C.5 to find constants $(\varepsilon, \delta, C) \in (0, \varepsilon_*) \times (0, 1) \times (0, \infty)$ depending only on (v_∞, γ) so that, for any $z_0 \in \overline{\Omega_\varepsilon^{\mathcal{N}}} \cap \Gamma_{\text{shock}}$, we have

$$(3.4.52) \qquad \|\psi^{(z_0)}\|_{1, \delta, \overline{Q_{3/4}^{(z_0)}}} \leq C.$$

By (3.4.45), for each $z_0 \in \overline{\Omega_\varepsilon^{\mathcal{N}}} \cap \Gamma_{\text{shock}}$, $\phi_\infty^{\mathcal{N}} = \varphi_\infty - \varphi_{\mathcal{N}}$ satisfies

$$(3.4.53) \quad \phi_\infty^{\mathcal{N}}(x_0 + \frac{x_0}{4}S, \hat{f}_{\mathcal{N}, \text{sh}}(x_0) + \frac{\sqrt{x_0}}{4} F^{(z_0)}(S)) - x_0^2 \psi^{(z_0)}(S, F^{(z_0)}(S)) = 0 \qquad \text{for } -1 < S < 1.$$

Differentiating (3.4.53) with respect to S, we have

$$(3.4.54) \qquad (F^{(z_0)})' = -\frac{\sqrt{x_0}(\partial_x \phi_\infty^{\mathcal{N}} - 4x_0 \partial_S \psi^{(z_0)})}{\partial_y \phi_\infty^{\mathcal{N}} - 4x_0^{3/2} \partial_T \psi^{(z_0)}}.$$

By combining this expression with Lemma 3.20(c) and (3.4.52), a direct computation shows that there exists a small constant $\varepsilon \in (0, \varepsilon_*]$ depending on (v_∞, γ) such that $F^{(z_0)}$ satisfies the estimate:

$$(3.4.55) \qquad \|F^{(z_0)}\|_{1, \delta, [-3/4, 3/4]} \leq C\sqrt{x_0} \qquad \text{for all } z_0 = (x_0, y_0) \in \Gamma_{\text{shock}} \cap \overline{\Omega_\varepsilon^{\mathcal{N}}}$$

for some constant $C > 0$ depending only on (v_∞, γ).

This result, combined with Lemma 3.18, yields that Γ_{shock} is $C^{1,\delta}$ up to $\Gamma_{\text{sonic}}^{\mathcal{N}}$ away from $\Gamma_{\text{sonic}}^{\mathcal{O}}$.

Next, it follows directly from (3.4.55) and a direct computation by using (3.4.48)–(3.4.49) that the boundary condition (3.4.51) satisfies all the conditions stated in Theorem C.6 with $(\alpha, \Phi, W) = (\delta, \frac{1}{\sqrt{x_0}} F^{(z_0)}, B_{\mathcal{N}}^{(z_0)})$ for all $z_0 \in \Gamma_{\text{shock}} \cap \partial\Omega_\varepsilon^{\mathcal{N}}$, where $\varepsilon > 0$ is the constant in (3.4.55). Therefore,

we can further reduce $\varepsilon \in (0, \varepsilon_*]$ depending on (v_∞, γ) so that, for each $z_0 \in \Gamma_{\text{shock}} \cap \partial\Omega_\varepsilon^{\mathcal{N}}$, the re-scaled function $\psi^{(z_0)}$ satisfies the estimate:

$$(3.4.56) \qquad \left\|\psi^{(z_0)}\right\|_{2,\delta,Q_{1/2}^{(z_0)}} \leq C,$$

where C depends only on (v_∞, γ).

We combine estimate (3.4.56) with (3.4.54) to see that $F^{(z_0)} \in C^{1,\alpha}([-\frac{1}{2}, \frac{1}{2}])$ for any $\alpha \in (0,1)$. Furthermore, we have

$$\sup_{z_0 \in \Gamma_{\text{shock}} \cap \partial\Omega_\varepsilon^{\mathcal{N}}} \frac{1}{\sqrt{x_0}} \left\|F^{(z_0)}\right\|_{1,\alpha,[-\frac{1}{2},\frac{1}{2}]} \leq C,$$

where $C > 0$ depends only on (v_∞, γ). Then we can repeat the previous argument by applying Theorem C.6 to conclude that, for each $\alpha \in (0,1)$, the small constant $\varepsilon \in (0, \varepsilon_*]$ can be further reduced so that

$$\sup_{z_0 \in \Gamma_{\text{shock}} \cap \partial\Omega_\varepsilon^{\mathcal{N}}} \left\|\psi^{(z_0)}\right\|_{2,\alpha,\overline{Q_{1/4}^{(z_0)}}} + \frac{1}{\sqrt{x_0}} \left\|F^{(z_0)}\right\|_{2,\alpha,[-\frac{1}{4},\frac{1}{4}]} \leq C,$$

where $C > 0$ is a constant depending only on $(v_\infty, \gamma, \alpha)$.

4. *Uniform estimates of $\psi^{(z_0)}$ for $z_0 \notin \Gamma_{\text{shock}}$.* If $Q_1^{(z_0)} = Q_1$, we apply Theorem C.3 to obtain that, for each $\alpha \in (0,1)$, $\left\|\psi^{(z_0)}\right\|_{2,\alpha,\overline{Q_{1/2}^{(z_0)}}}$ is uniformly bounded above by a constant depending only on $(v_\infty, \gamma, \alpha)$. If $z_0 \in \Gamma_{\text{wedge}} \cap \partial\Omega_\varepsilon^{\mathcal{N}}$, then $Q_1^{(z_0)} = Q_1 \cap \{T > 0\}$, and $\psi^{(z_0)}$ satisfies that $\psi_T^{(z_0)}(S,0) = 0$ for all $-1 < S < 1$. This is owing to the slip boundary condition (3.4.37). In this case, we apply Theorem C.7 to obtain a uniform estimate of $\left\|\psi^{(z_0)}\right\|_{2,\alpha,\overline{Q_{1/2}^{(z_0)}}}$ for all $z_0 \in \Gamma_{\text{wedge}} \cap \partial\Omega_\varepsilon^{\mathcal{N}}$.

5. *Estimate for $\|\varphi - \varphi_{\mathcal{N}}\|_{2,\alpha,\Omega_{\varepsilon_{\mathcal{N}}}^{\mathcal{N}}}^{(\text{par})}$.* Since the estimates of $\left\|\psi^{(z_0)}\right\|_{2,\alpha,\overline{Q_{1/8}^{(z_0)}}}$ are given independently of $z_0 \in \overline{\Omega_\varepsilon^{\mathcal{N}}} \setminus \Gamma_{\text{sonic}}^{\mathcal{N}}$ and $\beta \in [0, \beta_{\text{d}}^{(v_\infty)})$, the estimate of $\|\varphi - \varphi_{\mathcal{N}}\|_{2,\alpha,\Omega_{\varepsilon_{\mathcal{N}}}^{\mathcal{N}}}^{(\text{par})}$ in (3.4.43) is finally obtained by combining the uniform C^k–estimate of admissible solutions given in Corollary 3.19 and all the estimates of $\left\|\psi^{(z_0)}\right\|_{2,\alpha,\overline{Q_{1/8}^{(z_0)}}}$ from Steps 3–4, and by scaling back to the (x,y)–coordinates. For the details, we refer to [**1**, Steps 3–4 in the proof of Theorem 3.1] or [**11**, Lemma 4.6.1].

6. *Estimate for $\|\hat{f}_{\mathcal{N},\text{sh}} - \hat{f}_{\mathcal{N},0}\|_{2,\alpha,(0,\varepsilon_{\mathcal{N}})}^{(\text{par})}$.* By Lemma 3.20(e) and Lemma 3.21(d), we have

$$\phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},0}(x)) = 0, \quad (\phi_\infty^{\mathcal{N}} - \psi)(x, \hat{f}_{\mathcal{N},\text{sh}}(x)) = (\varphi_\infty - \varphi)(x, \hat{f}_{\mathcal{N},\text{sh}}(x)) = 0 \qquad \text{for all } x \in [0, \varepsilon_{\mathcal{N}}].$$

This yields that

$$(3.4.57) \qquad \phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},\text{sh}}(x)) - \phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},0}(x)) = \psi(x, \hat{f}_{\mathcal{N},\text{sh}}(x)) \qquad \text{for all } x \in [0, \varepsilon_{\mathcal{N}}].$$

Since $|\partial_y \phi_\infty^{\mathcal{N}}| > 0$ from Lemma 3.20(c), we can rewrite (3.4.57) as

$$\hat{f}_{\mathcal{N},\text{sh}}(x) - \hat{f}_{\mathcal{N},0}(x) = \frac{\psi(x, \hat{f}_{\mathcal{N},\text{sh}}(x))}{\int_0^1 \partial_y \phi_\infty^{\mathcal{N}}(x, t\hat{f}_{\mathcal{N},\text{sh}}(x) + (1-t)\hat{f}_{\mathcal{N},0}(x))\, \mathrm{d}t}.$$

Then a direct computation by using Lemma 3.20 and the estimate of $\|\psi\|_{2,\alpha,\Omega_{\varepsilon_{\mathcal{N}}}^{\mathcal{N}}}^{(\text{par})} \leq C$ achieved in Step 5 implies that

$$\|\hat{f}_{\mathcal{N},\text{sh}} - \hat{f}_{\mathcal{N},0}\|_{2,\alpha,(0,\varepsilon_{\mathcal{N}})}^{(\text{par})} \leq C,$$

where $C > 0$ is a constant depending only on $(v_\infty, \gamma, \alpha)$. This completes the proof. $\qquad \square$

3.5. Weighted $C^{2,\alpha}$–Estimates Near $\Gamma^{\mathcal{O}}_{\text{sonic}}$

According to Definition 2.23, $\Gamma^{\mathcal{O}}_{\text{sonic}}$ depends continuously on $\beta \in [0, \frac{\pi}{2})$. In particular, the sonic arc $\Gamma^{\mathcal{O}}_{\text{sonic}}$ shrinks to a point when β increases up to $\beta_{\text{s}}^{(v_\infty)}$, and becomes a point P_β for all $\beta \geq \beta_{\text{s}}^{(v_\infty)}$, although the location of P_β changes continuously on $\beta \in [\beta_{\text{s}}^{(v_\infty)}, \frac{\pi}{2})$. Furthermore, the ellipticity of Eq. (3.2.2) on $\Gamma^{\mathcal{O}}_{\text{sonic}}$ also changes. According to Proposition 3.15, the ellipticity of (3.2.2) degenerates on $\Gamma^{\mathcal{O}}_{\text{sonic}}$ for $\beta \leq \beta_{\text{s}}^{(v_\infty)}$. On the other hand, for $\beta > \beta_{\text{s}}^{(v_\infty)}$, Eq. (3.2.2) (or equivalently Eq. (2.1.19)) is uniformly elliptic up to $\Gamma^{\mathcal{O}}_{\text{sonic}}$ away from $\Gamma^{\mathcal{N}}_{\text{sonic}}$. For that reason, the weighted $C^{2,\alpha}$–estimates of admissible solutions near $\Gamma^{\mathcal{O}}_{\text{sonic}}$ are given for the following four cases separately:

1. $\beta < \beta_{\text{s}}^{(v_\infty)}$ away from $\beta_{\text{s}}^{(v_\infty)}$,
2. $\beta < \beta_{\text{s}}^{(v_\infty)}$ close to $\beta_{\text{s}}^{(v_\infty)}$,
3. $\beta \geq \beta_{\text{s}}^{(v_\infty)}$ close to $\beta_{\text{s}}^{(v_\infty)}$,
4. $\beta \in (\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{d}}^{(v_\infty)})$ away from $\beta_{\text{s}}^{(v_\infty)}$.

3.5.1. Case 1: Admissible solutions for $\beta < \beta_{\text{s}}^{(v_\infty)}$ away from $\beta_{\text{s}}^{(v_\infty)}$. For

$$(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta \,:\, 0 < \beta < \beta_{\text{s}}^{(v_\infty)}\},$$

let $O_{\mathcal{O}}$ and P_1 be given by Definition 2.23. For each $\beta > 0$, let $M_{\mathcal{O}}$ be defined by (2.4.6). Define

$$(3.5.1) \qquad c_{\mathcal{O}}^* := \frac{|P_1 O_{\mathcal{O}}| + c_{\mathcal{O}} M_{\mathcal{O}}}{2} \begin{cases} = \frac{c_{\mathcal{O}}(1 + M_{\mathcal{O}})}{2} & \text{for } \beta \leq \beta_{\text{s}}^{(v_\infty)}, \\ < \frac{c_{\mathcal{O}}(1 + M_{\mathcal{O}})}{2} & \text{for } \beta \geq \beta_{\text{s}}^{(v_\infty)}. \end{cases}$$

In $U_{\mathcal{O}} := \big(B_{\frac{3c_{\mathcal{O}}}{2}}(O_{\mathcal{O}}) \setminus B_{c_{\mathcal{O}}^*}(O_{\mathcal{O}})\big) \cap \{\boldsymbol{\xi} : \xi_1 < u_{\mathcal{O}}\}$, use (r, θ) as the polar coordinates with respect to $O_{\mathcal{O}} = (u_{\mathcal{O}}, 0)$ and define

$$(3.5.2) \qquad (x, y) := (c_{\mathcal{O}} - r, \pi - \theta).$$

Also, define a set $\Omega^{\mathcal{O}}$ by

$$\Omega^{\mathcal{O}} := \big(\Omega \cap \{\xi_1 < u_{\mathcal{O}}\}\big) \setminus B_{c_{\mathcal{O}}^*}(O_{\mathcal{O}}).$$

Since $\Omega^{\mathcal{O}} \subset B_{c_{\mathcal{O}}}(O_{\mathcal{O}})$, $\Omega^{\mathcal{O}} \subset \{(x, y) : x > 0\}$. In the (x, y)–coordinates defined by (3.5.2), $\varphi_{\mathcal{O}}$ given by Definition 2.23 is written as

$$(3.5.3) \qquad \varphi_{\mathcal{O}} = -\frac{1}{2}(c_{\mathcal{O}} - x)^2 + \frac{1}{2}u_{\mathcal{O}}^2 - v_\infty \xi_2^{(\beta)} \qquad \text{in } U_{\mathcal{O}}.$$

For an admissible solution φ corresponding to (v_∞, β), let ψ be given by

$$(3.5.4) \qquad \psi = \varphi - \varphi_{\mathcal{O}} \qquad \text{in } \Omega^{\mathcal{O}}.$$

(i) *Equation for ψ in $\Omega^{\mathcal{O}}$*: Similarly to (3.4.21), we rewrite Eq. (3.2.2) for ψ in the (x, y)–coordinates given by (3.5.2). For each $j = 1, \cdots, 5$, let $O_j^{\mathcal{O}}(\mathbf{p}, z, x)$ be given by

$$O_j^{\mathcal{O}}(\mathbf{p}, z, x) = O_j(\mathbf{p}, z, x, c_{\mathcal{O}})$$

for $O_j(\mathbf{p}, z, x, c)$ given by (3.2.29). Then Eq. (2.1.19) is written as

$$(3.5.5) \qquad \big(2x - (\gamma + 1)\psi_x + O_1^{\mathcal{O}}\big)\psi_{xx} + O_2^{\mathcal{O}}\psi_{xy} + \big(\frac{1}{c_{\mathcal{O}}} + O_3^{\mathcal{O}}\big)\psi_{yy} - \big(1 + O_4^{\mathcal{O}}\big)\psi_x + O_5^{\mathcal{O}}\psi_y = 0,$$

with $O_j^{\mathcal{O}} = O_j^{\mathcal{O}}(D\psi, \psi, x)$ for $j = 1, \cdots, 5$.

(ii) *Boundary condition for ψ on $\Gamma_{\text{shock}} \cap \partial\Omega^{\mathcal{O}}$*: Similarly to (3.4.22), we define

(3.5.6)
$$M_\beta(\mathbf{p}, z, \xi_1) = g^{\text{sh}}_{\text{mod}}(\mathbf{p} + D\varphi_{\mathcal{O}}, z + \varphi_{\mathcal{O}}, \xi_1, \xi_2^{(\beta)} - \frac{u_{\mathcal{O}}\xi_1 + z}{v_\infty})$$

for $g^{\text{sh}}_{\text{mod}}$ given by (3.4.14), where $(D\varphi_{\mathcal{O}}, \varphi_{\mathcal{O}})$ are evaluated at $(\xi_1, \xi_2^{(\beta)} - \frac{u_{\mathcal{O}}\xi_1 + z}{v_\infty})$. Note that $(u_{\mathcal{O}}, \xi_2^{(\beta)})$ depend continuously on $\beta \in (0, \frac{\pi}{2})$ and that

$$\lim_{\beta \to 0+}(u_{\mathcal{O}}, \xi_2^{(\beta)}) = (0, \xi_2^{\mathcal{N}}).$$

Define

(3.5.7)
$$\phi_\infty^{\mathcal{O}} = \varphi_\infty - \varphi_{\mathcal{O}}.$$

Rewriting the boundary condition: $|D(\phi_\infty^{\mathcal{O}} - \psi)|M_\beta(D\psi, \psi, \xi_1) = 0$ on $\Gamma_{\text{shock}} \cap \partial\Omega^{\mathcal{O}}$ in the (x, y)–coordinates given by (3.5.2), we have

(3.5.8)
$$\mathcal{B}_1^{\mathcal{O}}(\psi_x, \psi_y, \psi, x, y) = 0 \qquad \text{on } \Gamma_{\text{shock}} \cap \partial\Omega^{\mathcal{O}}$$

for $\mathcal{B}_1^{\mathcal{O}}(p_x, p_y, z, x, y)$ given by

(3.5.9)
$$\mathcal{B}_1^{\mathcal{O}}(p_x, p_y, z, x, y) = |D\phi_\infty^{\mathcal{O}} - (p_1, p_2)|M_\beta(p_1, p_2, z, \xi_1)$$

with

(3.5.10)
$$\xi_1 = u_{\mathcal{O}} - (c_{\mathcal{O}} - x)\cos y, \qquad \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \cos y & \sin y \\ -\sin y & \cos y \end{pmatrix} \begin{pmatrix} p_x \\ \frac{p_y}{c_{\mathcal{O}} - x} \end{pmatrix}.$$

(iii) *Other properties of ψ*: By (2.1.30) and conditions (ii) and (iv) of Definition 2.24, ψ satisfies

(3.5.11)
$$\begin{aligned} \psi &\geq 0 \qquad \text{in } \Omega, \\ \psi &= 0 \qquad \text{on } \Gamma_{\text{sonic}}^{\mathcal{O}}, \\ \psi_y &= 0 \qquad \text{on } \Gamma_{\text{wedge}} \cap \partial\Omega^{\mathcal{O}}. \end{aligned}$$

For set $\mathcal{D}$ defined by (2.5.27), let an open subset $\Lambda_\beta^{\mathcal{O}}$ of $\mathcal{D}$ be given by

(3.5.12)
$$\Lambda_\beta^{\mathcal{O}} := \mathcal{D} \cap \left(B_{\frac{3c_{\mathcal{O}}}{2}}(O_{\mathcal{O}}) \setminus B_{c_{\mathcal{O}}^*}(O_{\mathcal{O}})\right) \cap \{\xi_1 < u_{\mathcal{O}}\}$$

for $c_{\mathcal{O}}^*$ defined by (3.5.1).

LEMMA 3.27. *Fix $\gamma \geq 1$ and $v_\infty > 0$. There exist positive constants ε_1, ε_0, δ_0, ω_0, C, and $\mathfrak{M}$ depending only on (v_∞, γ) with $\varepsilon_1 > \varepsilon_0$ and $\mathfrak{M} \geq 2$ such that, for each $\beta \in (0, \beta_s^{(v_\infty)}]$, the following properties hold:*

(a) $\{\varphi_{\mathcal{O}} < \varphi_\infty\} \cap \Lambda_\beta^{\mathcal{O}} \cap \mathcal{N}_{\varepsilon_1}(\Gamma_{\text{sonic}}^{\mathcal{O}}) \subset \{0 < y < \frac{\pi}{2} - \beta - \delta_0\}$;

(b) $\{\varphi_{\mathcal{O}} < \varphi_\infty\} \cap \mathcal{N}_{\varepsilon_1}(\Gamma_{\text{sonic}}^{\mathcal{O}}) \cap \{y > y_{P_1}\} \subset \{x > 0\}$;

(c) *In $\{(x, y) : |x| < \varepsilon, 0 < y < \frac{\pi}{2} - \beta - \delta_0\}$, $\phi_\infty^{\mathcal{O}}$ given by (3.5.7) satisfies*

(3.5.13)
$$\frac{2}{\mathfrak{M}}(y + \tan\beta) \leq \partial_x \phi_\infty^{\mathcal{O}} \leq \frac{\mathfrak{M}}{2}, \qquad \frac{2}{\mathfrak{M}} \leq -\partial_y \phi_\infty^{\mathcal{O}} \leq \frac{\mathfrak{M}}{2};$$

(d) $|(D_{(x,y)}^2, D_{(x,y)}^3)\phi_\infty^{\mathcal{O}}| \leq C \quad \text{in } \{|x| < \varepsilon_1\}$;

(e) *There exists a unique function $\hat{f}_{\mathcal{O},0} \in C^\infty([-\varepsilon_0, \varepsilon_0])$ such that*

(3.5.14)
$$\begin{cases} \{\varphi_{\mathcal{O}} < \varphi_\infty\} \cap \Lambda_\beta^{\mathcal{O}} \cap \mathcal{N}_{\varepsilon_1}(\Gamma_{\text{sonic}}^{\mathcal{O}}) \cap \{|x| < \varepsilon_0\} = \{(x, y) : |x| < \varepsilon_0, 0 < y < \hat{f}_{\mathcal{O},0}(x)\}, \\ S_{\mathcal{O}} \cap \mathcal{N}_{\varepsilon_1}(\Gamma_{\text{sonic}}^{\mathcal{O}}) \cap \{|x| < \varepsilon_0\} = \{(x, y) : x \in (-\varepsilon_0, \varepsilon_0), y = \hat{f}_{\mathcal{O},0}(x)\}; \end{cases}$$

(f) $\hat{f}_{\mathcal{O},0}$ *given in* (e) *satisfies*

$$2\omega_0 \leq \hat{f}'_{\mathcal{O},0} \leq C \qquad on \ (-\varepsilon_0, \varepsilon_0).$$

PROOF. Note that line $S_{\mathcal{O}}$ intersects with circle $\partial B_{c_{\mathcal{O}}}(O_{\mathcal{O}})$ at two different points, due to (2.4.43) for any $(v_\infty, \gamma) \in \mathfrak{R}_{\text{weak}}$. Point P_1 is an intersection point of $S_{\mathcal{O}} = \{\boldsymbol{\xi} \ : \ \varphi_\infty = \varphi_{\mathcal{O}}\}$ with $\partial B_{c_{\mathcal{O}}}(O_{\mathcal{O}})$. Let P'_1 be the other intersection point of $S_{\mathcal{O}}$ and $\partial B_{c_{\mathcal{O}}}(O_{\mathcal{O}})$, and let $Q_{\mathcal{O}}$ be the midpoint of the line segment $\overline{P_1 P'_1}$. Then $\angle Q_{\mathcal{O}} O_{\mathcal{O}} P_4 = \frac{\pi}{2} - \beta$. Since $|\overline{P_1 Q_{\mathcal{O}}}|$ depends continuously on $\beta \in [0, \frac{\pi}{2})$, there exists $\varepsilon_1 > 0$ depending only on (v_∞, γ) such that $|\overline{P_1 Q_{\mathcal{O}}}| \geq 2\varepsilon_1$ for all $\beta \in [0, \beta_s^{(v_\infty)}]$. Let $Q'_{\mathcal{O}}$ be the midpoint of $\overline{P_1 Q_{\mathcal{O}}}$, and let $(x_{Q'_{\mathcal{O}}}, y_{Q'_{\mathcal{O}}})$ denote the (x,y)–coordinates of $Q'_{\mathcal{O}}$. Then there exists a constant $\delta_0 > 0$ depending only on (v_∞, γ) such that

$$(3.5.15) \qquad\qquad y_{Q'_{\mathcal{O}}} < \frac{\pi}{2} - \beta - \delta_0.$$

Moreover, it follows directly from (3.5.7) that

$$\partial_x \phi_\infty^{\mathcal{O}} = v_\infty(\sin y + \cos y \tan \beta), \ \ \partial_y \phi_\infty^{\mathcal{O}} = -v_\infty(c_{\mathcal{O}} - x)(\cos y + \sin y \tan \beta) \qquad \text{in } \Lambda_\beta^{\mathcal{O}}.$$

Then statements (a)–(e) can be verified by performing a direct computation and using the observation obtained above.

Since $\phi_\infty^{\mathcal{O}} = 0$ on $S_{\mathcal{O}}$, we have

$$\phi_\infty^{\mathcal{O}}(x, \hat{f}_{\mathcal{O},0}(x)) = 0 \qquad \text{for } |x| < \varepsilon_0,$$

so that $\hat{f}'_{\mathcal{O},0}(x) = -\frac{\partial_x \phi_\infty^{\mathcal{O}}}{\partial_y \phi_\infty^{\mathcal{O}}}(x, \hat{f}'_{\mathcal{O},0}(x))$ holds. This expression, combined with (3.5.13), yields statement (f). $\qquad\square$

Similarly to (3.4.29), for an admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta \leq \beta_s^{(v_\infty)}\}$, let Ω be its pseudo-subsonic region. Let ε_1 be the constant given in Lemma 3.27. For $\varepsilon \in (0, \varepsilon_1]$, define

$$(3.5.16) \qquad\qquad \Omega_\varepsilon^{\mathcal{O}} := \Omega \cap \mathcal{N}_{\varepsilon_1}(\Gamma_{\text{sonic}}^{\mathcal{O}}) \cap \{x < \varepsilon\}.$$

Then $\Omega_\varepsilon^{\mathcal{O}} = \Omega_\varepsilon^{\mathcal{O}} \cap \{x > 0\}$.

Adjusting the proof of Lemma 3.21 by using Lemma 3.27 instead of Lemma 3.20, we have the following lemma:

LEMMA 3.28. *Let ε_0, ω_0, and $\mathfrak{M}$ be three constants in Lemma 3.27. Then there exist $\bar{\varepsilon} \in (0, \varepsilon_0]$, $L \geq 1$, $\delta \in (0, \frac{1}{2})$, and $\omega \in (0, \omega_0] \cap (0,1)$ depending only on (v_∞, γ) such that any admissible solution $\varphi = \psi + \varphi_{\mathcal{O}}$ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta \leq \beta_s^{(v_\infty)}\}$ satisfies the following properties in $\Omega_{\bar{\varepsilon}}^{\mathcal{O}}$:*

(a) $\psi_x(x,y) \leq \frac{2-\delta}{1+\gamma} x \leq Lx$;

(b) $\psi_x \geq 0$ *and* $|\psi_y(x,y)| \leq Lx$;

(c) $\frac{2}{\mathfrak{M}}(y + \tan \beta) - \frac{2-\delta}{1+\gamma} x \leq \partial_x(\varphi_\infty - \varphi)(x,y) \leq \mathfrak{M}$ *and* $\frac{1}{\mathfrak{M}} \leq -\partial_y(\varphi_\infty - \varphi) \leq \mathfrak{M}$;

(d) *There exists a function $\hat{f}_{\mathcal{O},\text{sh}} \in C^1([0, \bar{\varepsilon}])$ such that*

$$\Omega_{\bar{\varepsilon}}^{\mathcal{O}} = \{(x,y) \ : \ x \in (0, \bar{\varepsilon}), 0 < y < \hat{f}_{\mathcal{O},\text{sh}}(x)\},$$

$$\Gamma_{\text{shock}} \cap \partial \Omega_{\bar{\varepsilon}}^{\mathcal{O}} = \{(x,y) \ : \ x \in (0, \bar{\varepsilon}), \ y = \hat{f}_{\mathcal{O},\text{sh}}(x)\},$$

$$\omega \leq \hat{f}'_{\mathcal{O},\text{sh}}(x) \leq L \qquad \text{for } 0 < x < \bar{\varepsilon};$$

(e) $0 \leq \psi(x, y) \leq Lx^2$.

LEMMA 3.29. *Let φ be an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta \leq \beta_{\text{s}}^{(v_\infty)}\}$. Let Eq.* (3.5.5) *in $\Omega^{\mathcal{O}}$ be expressed as*

$$(3.5.17) \qquad \sum_{i,j=1}^{2} \hat{A}_{ij}^{\mathcal{O}}(D\psi, \psi, x) D_{ij}\psi + \sum_{i=1}^{2} \hat{A}_i^{\mathcal{O}}(D\psi, \psi, x) D_i\psi = 0,$$

with $(D_1, D_2) = (D_x, D_y)$, $D_{ij} = D_i D_j$, and $\hat{A}_{21}^{\mathcal{O}} = \hat{A}_{12}^{\mathcal{O}}$. Then there exist $\varepsilon_{\mathcal{O}} \in (0, \frac{\bar{\varepsilon}}{4}]$ and $\lambda_{\mathcal{O}} > 0$ depending only on (v_∞, γ) such that, if $(x, y) \in \overline{\Omega_{4\varepsilon_{\mathcal{O}}}^{\mathcal{O}}}$,

$$(3.5.18) \qquad \frac{\lambda_{\mathcal{O}}}{2}|\boldsymbol{\kappa}|^2 \leq \sum_{i,j=1}^{2} \hat{A}_{ij}^{\mathcal{O}}(D\psi(x,y), \psi(x,y), x) \frac{\kappa_i \kappa_j}{x^{2-\frac{i+j}{2}}} \leq \frac{2}{\lambda_{\mathcal{O}}}|\boldsymbol{\kappa}|^2 \qquad \text{for all } \boldsymbol{\kappa} \in \mathbb{R}^2.$$

Moreover, $\mathcal{B}_1^{\mathcal{O}}$ defined by (3.5.9) *satisfies the following properties:*

(a) *$\mathcal{B}_1^{\mathcal{O}}(\mathbf{0}, 0, x, y) = 0$ holds for all $(x, y) \in \mathbb{R}^2$;*

(b) *For each $k = 2, 3, \cdots$, there exist constants $\delta_{\text{bc}} > 0$ and $C > 1$ depending only on (v_∞, γ, k) such that, whenever $|(p_x, p_y, z, x)| \leq \delta_{\text{bc}}$ and $|y - y_{P_1}| \leq \delta_{\text{bc}}$,*

$$|D^k_{(p_x, p_y, z, x, y)} \mathcal{B}_1^{\mathcal{O}}(p_x, p_y, z, x, y)| \leq C;$$

(c) *There exist constants $\hat{\delta}_{\text{bc}} > 0$ and $C > 1$ depending only on (v_∞, γ) such that, whenever $|(p_x, p_y, z, x)| \leq \hat{\delta}_{\text{bc}}$ and $|y - y_{P_1}| \leq \hat{\delta}_{\text{bc}}$,*

$$D_{(p_x, p_y, z)} \mathcal{B}_1^{\mathcal{O}}(p_x, p_y, z, x, y) \leq -C^{-1};$$

(d) *There exists a constant $\varepsilon' > 0$ depending only on (v_∞, γ), and constants $\hat{\delta}_{\text{bc}} > 0$ and $C > 1$ in property (c) can be further reduced depending only on (v_∞, γ) such that, whenever $|(p_x, p_y, z, x)| \leq \hat{\delta}_{\text{bc}}$ and $|y - y_{P_1}| \leq \hat{\delta}_{\text{bc}}$,*

$$D_{(p_x, p_y)} \mathcal{B}_1^{\mathcal{O}}(p_x, p_y, z, x, y) \cdot \boldsymbol{\nu}_{\text{sh}}^{(x,y)} \geq C^{-1} \qquad \text{on } \Gamma_{\text{shock}} \cap \partial \Omega_{\varepsilon'}^{\mathcal{O}},$$

where Γ_{shock} represents the curved shock of the admissible solution, and $\boldsymbol{\nu}_{\text{sh}}^{(x,y)}$ is the unit normal vector to Γ_{shock}. The vector field $\boldsymbol{\nu}_{\text{sh}}^{(x,y)}$ is expressed in the (x, y)–coordinates and oriented towards the interior of Ω.

In properties (b)–(d)*, y_{P_1} represents the y–coordinate of point P_1, defined by Definition 2.23.*

Even though this lemma is similar to Lemma 3.23, the proof is more complicated, because $u_{\mathcal{O}}, c_{\mathcal{O}}, \varphi_{\mathcal{O}}$, and $S_{\mathcal{O}}$ depend on $\beta \in (0, \beta_{\text{s}}^{(v_\infty)}]$.

PROOF. We divide the proof into three steps.

1. As just mentioned above, $(u_{\mathcal{O}}, c_{\mathcal{O}})$ depend continuously on $\beta \in (0, \frac{\pi}{2})$. In particular, $|u_{\mathcal{O}}|$ and $c_{\mathcal{O}}$ increase with respect to β. Therefore, there exists a constant $\bar{c} > 1$ depending only on (v_∞, γ) such that

$$|u_{\mathcal{O}}| \leq \bar{c}, \quad 1 \leq c_{\mathcal{O}} \leq \bar{c} \qquad \text{for all } \beta \in [0, \beta_{\text{d}}^{(v_\infty)}].$$

Then inequality (3.5.18) and properties (a)–(b) can be directly checked from (2.4.4), (3.2.29), (3.4.14), (3.5.6), (3.5.9), and Lemma 3.28.

2. A direct computation by using (2.4.3)–(2.4.4), (3.2.4), (3.4.13), (3.5.6), and (3.5.9) yields that

$$\partial_z \mathcal{B}_1^{\mathcal{O}}(\mathbf{0}, 0, 0, y_{P_1}) = -\frac{c_{\mathcal{O}} v_{\infty} \sec \beta}{\rho_{\mathcal{O}}^{\gamma-2}} \sin(y_{P_1} + \beta),$$

$$\partial_{p_x} \mathcal{B}_1^{\mathcal{O}}(\mathbf{0}, 0, 0, y_{P_1}) = -c_{\mathcal{O}}(\rho_{\mathcal{O}} - 1) \cos^2(y_{P_1} + \beta),$$

$$\partial_{p_y} \mathcal{B}_1^{\mathcal{N}}(\mathbf{0}, 0, 0, y_{P_1}) = -\left((\rho_{\mathcal{O}} - 1) \sin(y_{P_1} + \beta) + \frac{c_{\mathcal{O}} v_{\infty} \sec \beta}{c_{\mathcal{O}}}\right) \cos(y_{P_1} + \beta).$$

For $\beta \le \beta_{\mathrm{s}}^{(v_{\infty})}$, we have

$$\cos(\frac{\pi}{2} - \beta - y_{P_1}) = M_{\mathcal{O}}(\beta),$$

where $M_{\mathcal{O}}$ is defined by (2.4.6), which is a continuous function of $\beta \in [0, \frac{\pi}{2})$ that satisfies $M_{\mathcal{O}} < 1$. Then there exists a constant $\delta_0 \in (0, \frac{\pi}{2})$ depending only on (v_{∞}, γ) such that $y_{P_1} + \beta \le \frac{\pi}{2} - \delta_0$ for all $\beta \in [0, \beta_{\mathrm{s}}^{(v_{\infty})}]$. This implies that there exists a constant $m_0 > 0$ depending only on (v_{∞}, γ) such that

$$D_{(p_x, p_y, z)} \mathcal{B}_1^{\mathcal{O}}(\mathbf{0}, 0, 0, y_{P_1}) \le -m_0^{-1} \qquad \text{for all } \beta \in (0, \beta_{\mathrm{s}}^{(v_{\infty})}].$$

We combine this inequality with property (b) to obtain property (c).

3. By (2.4.6) and (A.18), we have

$$D_{\mathbf{p}} g_{\mathrm{mod}}^{\mathrm{sh}}(D\varphi_{\mathcal{O}}(P_1), \varphi_{\mathcal{O}}(P_1), P_1) \cdot \boldsymbol{\nu}_{\mathcal{O}} = \rho_{\mathcal{O}}(1 - M_{\mathcal{O}}^2)$$

for the unit normal vector $\boldsymbol{\nu}_{\mathcal{O}}$ to the straight oblique shock $S_{\mathcal{O}}$ pointing towards the ξ_1–axis. It is shown in the proof of Lemma 2.22 that

$$\frac{\mathrm{d}M_{\mathcal{O}}}{\mathrm{d}\beta} < 0 \qquad \text{for all } \beta \in (0, \frac{\pi}{2}).$$

Therefore, there exists a constant $m_1 > 0$ depending only on (v_{∞}, γ) such that

$$\boldsymbol{\nu}_{\mathcal{O}} \cdot D_{\mathbf{p}} g_{\mathrm{mod}}^{\mathrm{sh}}(D\varphi_{\mathcal{O}}(P_1), \varphi_{\mathcal{O}}(P_1), P_1) \ge m_1^{-1} \qquad \text{for all } \beta \in (0, \beta_{\mathrm{d}}^{(v_{\infty})}].$$

A direct computation by using (3.5.2), (3.5.6), and (3.5.10) leads to

$$(3.5.19) \quad D_{(p_x, p_y)} \mathcal{B}_1^{\mathcal{O}}(0, 0, 0, 0, y_{P_1}) \cdot \boldsymbol{\nu}_{\mathrm{sh}}^{(x,y)}(0, y_{P_1}) = \boldsymbol{\nu}_{\mathcal{O}} \cdot D_{\mathbf{p}} g_{\mathrm{mod}}^{\mathrm{sh}}(D\varphi_{\mathcal{O}}(P_1), \varphi_{\mathcal{O}}(P_1), P_1) \ge m_1^{-1}.$$

Owing to (3.5.19) and property (b), there exist small constants $\hat{\delta}_{\mathrm{bc}} > 0$ and $\hat{\delta}_{\boldsymbol{\nu}} > 0$ depending only on (v_{∞}, γ) such that, whenever

$$|(p_x, p_y, z, x)| \le \hat{\delta}_{\mathrm{bc}}, \quad |y - y_{P_1}| \le \hat{\delta}_{\mathrm{bc}}, \quad |\boldsymbol{\nu}_{\mathrm{sh}}^{(x,y)} - \boldsymbol{\nu}_{\mathrm{sh}}^{(x,y)}(0, y_{P_1})| \le \hat{\delta}_{\boldsymbol{\nu}},$$

we have

$$D_{(p_x, p_y)} \mathcal{B}_1^{\mathcal{O}}(p_x, p_y, z, x, y) \cdot \boldsymbol{\nu}_{\mathrm{sh}}^{(x,y)} \ge \frac{1}{4m_1}.$$

Note that $\boldsymbol{\nu}_{\mathrm{sh}}^{(x,y)}$ is represented as $\boldsymbol{\nu}_{\mathrm{sh}}^{(x,y)} = \dfrac{D_{(x,y)}(\varphi_{\infty} - \varphi_{\mathcal{O}} - \psi)}{|D_{(x,y)}(\varphi_{\infty} - \varphi_{\mathcal{O}} - \psi)|}$ on $\Gamma_{\mathrm{shock}} \cap \partial\Omega^{\mathcal{O}}$. Therefore, we can choose a small constant $\varepsilon' > 0$ depending only on (v_{∞}, γ) so that, by properties (a)–(b) of Lemma 3.28, $|\boldsymbol{\nu}_{\mathrm{sh}}^{(x,y)} - \boldsymbol{\nu}_{\mathrm{sh}}^{(x,y)}(0, y_{P_1})| \le \hat{\delta}_{\boldsymbol{\nu}}$ on $\Gamma_{\mathrm{shock}} \cap \partial\Omega_{\varepsilon'}^{\mathcal{O}}$. This completes the proof of property (d) of Lemma 3.29. $\qquad \square$

PROPOSITION 3.30. Let $\bar{\varepsilon} > 0$ be the constant introduced in Lemma 3.28. Fix $\sigma \in (0, \beta_{\mathrm{s}}^{(v_\infty)})$. For each $\alpha \in (0,1)$, there exist $\varepsilon \in (0, \bar{\varepsilon}]$ depending only on $(v_\infty, \gamma, \sigma)$, and $C > 0$ depending only on $(v_\infty, \gamma, \alpha)$ such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta \leq \beta_{\mathrm{s}}^{(v_\infty)} - \sigma\}$ satisfies

$$(3.5.20) \qquad \|\varphi - \varphi_{\mathcal{O}}\|_{2,\alpha,\Omega_{\bar{\varepsilon}}^{\mathcal{O}}}^{(\mathrm{par})} + \|\hat{f}_{\mathcal{O},\mathrm{sh}} - \hat{f}_{\mathcal{O},0}\|_{2,\alpha,(0,\varepsilon)}^{(\mathrm{par})} \leq C.$$

PROOF. For each $\beta \in (0, \beta_{\mathrm{s}}^{(v_\infty)}]$, point P_1 defined by Definition 2.23 satisfies

$$(3.5.21) \qquad \sin y_{P_1} = \frac{\xi_2^{P_1}}{c_{\mathcal{O}}}.$$

In the proof of Lemma 2.22, it is shown that $\xi_2^{P_1}$ is a decreasing function of $\beta \in (0, \beta_{\mathrm{s}}^{(v_\infty)}]$ with $\xi_2^{P_1} = 0$ at $\beta = \beta_{\mathrm{s}}^{(v_\infty)}$, and $c_{\mathcal{O}}$ is an increasing function of β. Therefore, for each $\sigma \in (0, \beta_{\mathrm{s}}^{(v_\infty)})$, there exists a constant $\delta_1 > 0$ depending only on $(\gamma, c_\infty, \sigma)$ such that $y_{P_1} \geq \delta_1$ for all $\beta \in (0, \beta_{\mathrm{s}}^{(v_\infty)} - \sigma]$. By combining this estimate with Lemma 3.28(d), we obtain a constant $l_{\mathrm{so}} > 0$ depending only on $(v_\infty, \gamma, \sigma)$ such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta \leq \beta_{\mathrm{s}}^{(v_\infty)} - \sigma\}$ satisfies

$$(3.5.22) \qquad \hat{f}_{\mathcal{O},\mathrm{sh}} \geq l_{\mathrm{so}} \qquad \mathrm{on} \ [0, \bar{\varepsilon}].$$

We choose

$$\varepsilon_* = \min\{\frac{\bar{\varepsilon}}{2}, l_{\mathrm{so}}^2\}.$$

Then we repeat the proof of Proposition 3.26 to find a constant $\varepsilon \in [0, \varepsilon_*]$ depending only on (v_∞, γ) such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta \leq \beta_{\mathrm{s}}^{(v_\infty)} - \sigma\}$ satisfies estimate (3.5.20) for a constant $C > 0$ depending only on $(v_\infty, \gamma, \alpha)$.

The main difference from the proof of Proposition 3.26 is that the uniform positive lower bound of $\hat{f}_{\mathcal{O},\mathrm{sh}}$ for admissible solutions corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta \leq \beta_{\mathrm{s}}^{(v_\infty)} - \sigma\}$ depends on $\sigma \in (0, \beta_{\mathrm{s}}^{(v_\infty)})$ so that the choice of ε to satisfy estimate (3.5.20) becomes dependent on σ as well, due to Theorem C.11. $\qquad \square$

REMARK 3.31. Note that $\xi_2^{P_1}$ depends on $\beta \in [0, \frac{\pi}{2})$ continuously. Furthermore, $\xi_2^{P_1} > 0$ for $\beta < \beta_{\mathrm{s}}^{(v_\infty)}$, and $\xi_2^{P_1} = 0$ for $\beta \geq \beta_{\mathrm{s}}^{(v_\infty)}$. Since

$$(3.5.23) \qquad \lim_{\beta \to \beta_{\mathrm{s}}^{(v_\infty)}} \xi_2^{P_1} = 0,$$

we have

$$l_{\mathrm{so}} = 0 \qquad \mathrm{at} \ \beta = \beta_{\mathrm{s}}^{(v_\infty)}$$

for constant l_{so} from (3.5.22).

3.5.2. Case 2: Admissible solutions for $\beta < \beta_{\mathrm{s}}^{(v_\infty)}$ close to $\beta_{\mathrm{s}}^{(v_\infty)}$. Now we extend Proposition 3.30 up to $\beta = \beta_{\mathrm{s}}^{(v_\infty)}$.

PROPOSITION 3.32. Let $\bar{\varepsilon} > 0$ be the constant introduced in Lemma 3.28. For each $\alpha \in (0,1)$, there exist $\varepsilon \in (0, \bar{\varepsilon}]$ and $\sigma_1 \in (0,1)$ depending only on (v_∞, γ), and $C > 0$ depending only on $(v_\infty, \gamma, \alpha)$, such that any admissible solution $\varphi = \psi + \varphi_{\mathcal{O}}$ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta_{\mathrm{s}}^{(v_\infty)} - \sigma_1 \leq \beta < \beta_{\mathrm{s}}^{(v_\infty)}\}$ satisfies estimate (3.5.20).

PROOF. We divide the proof into five steps.

1. Owing to Remark 3.31, we cannot apply Theorem C.11 directly to establish estimate (3.5.20) up to $\beta = \beta^{(v_\infty)}_{\text{s}}$. We first observe that there exists a constant $k > 1$ depending only on (v_∞, γ) such that, for any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \beta < \beta^{(v_\infty)}_{\text{s}}\}$,

$$(3.5.24) \qquad \{0 < x < 2\bar{\varepsilon}, \ 0 < y < y_{P_1} + \frac{x}{k}\} \subset \Omega^{\mathcal{O}}_{2\bar{\varepsilon}} \subset \{0 < x < 2\bar{\varepsilon}, \ 0 < y < y_{P_1} + kx\}.$$

Using (3.5.24) and Lemmas 2.22 and 3.28, we can adjust the proof of Proposition 3.26 to conclude that, for each $\alpha \in (0, 1)$, there exist a small constant $\sigma^* > 0$ depending on (v_∞, γ) and a constant $C > 0$ depending on $(v_\infty, \gamma, \alpha)$ such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta^{(v_\infty)}_{\text{s}} - \sigma^* \leq \beta < \beta^{(v_\infty)}_{\text{s}}\}$ satisfies

$$(3.5.25) \qquad \|\varphi - \varphi_{\mathcal{O}}\|^{(\text{par})}_{2,\alpha,\Omega^{\mathcal{O}}_{y^2_{P_1}}} \leq C.$$

2. *Claim: There exist $\hat{\varepsilon} \in (0, \frac{\bar{\varepsilon}}{2}]$, $\sigma' \in (0, \sigma^*]$, and $C^* > 0$ depending only on (v_∞, γ) such that any admissible solution $\varphi = \psi + \varphi_{\mathcal{O}}$ corresponding to $\mathfrak{R}_{\text{weak}} \cap \{\beta^{(v_\infty)}_{\text{s}} - \sigma' \leq \beta < \beta^{(v_\infty)}_{\text{s}}\}$ satisfies*

$$(3.5.26) \qquad 0 \leq \psi(x, y) \leq C^* x^4 \qquad \text{in } \Omega^{\mathcal{O}}_{2\hat{\varepsilon}} \cap \{x > \frac{y^2_{P_1}}{10}\}.$$

In what follows, unless otherwise specified, the universal constant C represents a positive constant depending only on (v_∞, γ), which may be different at each occurrence.

For an admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta < \beta^{(v_\infty)}_{\text{s}}\}$, let ψ be given by (3.5.4). We regard Eq. (3.5.17) (or equivalently, (3.5.17)) as a linear equation for ψ in $\Omega^{\mathcal{O}}_{\bar{\varepsilon}}$, and represent it as

$$(3.5.27) \qquad \mathcal{L}\psi := \sum_{i,j=1}^{2} a_{ij}(x, y)D_{ij}\psi + \sum_{i=1}^{2} a_i(x, y)D_i\psi = 0,$$

with $(a_{ij}, a_i)(x, y) = (\hat{A}^{\mathcal{O}}_{ij}, \hat{A}^{\mathcal{O}}_i)(D\psi(x, y), \psi(x, y), x)$ for $i, j = 1, 2$, where $\hat{A}^{\mathcal{O}}_{ij}$ and $\hat{A}^{\mathcal{O}}_i$ are from Lemma 3.29. By (3.2.29) and Lemma 3.28, there exists a constant $C > 0$ depending only on (v_∞, γ) such that $a_{ij}, i, j = 1, 2$, satisfy

$$(3.5.28) \qquad x \leq a_{11}(x, y) \leq 3x, \quad C^{-1} \leq a_{22}(x, y) \leq C, \quad |(a_{12}, a_{21})(x, y)| \leq Cx \qquad \text{in } \Omega^{\mathcal{O}}_{\bar{\varepsilon}},$$

$$(3.5.29) \qquad a_1(x, y) \leq 0, \quad |a_2(x, y)| \leq Cx \qquad \text{in } \Omega^{\mathcal{O}}_{\bar{\varepsilon}}.$$

By properties (a)–(b) and (e) of Lemma 3.28, there exists $\varepsilon_1 \in (0, \bar{\varepsilon}]$ such that ψ satisfies the estimates:

$$|(\psi_x, \psi_y, \psi, x)| \leq \frac{1}{2}\min\{\delta_{\text{bc}}, \hat{\delta}_{\text{bc}}\}, \quad |y - y_{P_1}| \leq \frac{1}{2}\min\{\delta_{\text{bc}}, \hat{\delta}_{\text{bc}}\} \qquad \text{in } \overline{\Omega^{\mathcal{O}}_{\varepsilon_1}}$$

for constants $(\delta_{\text{bc}}, \hat{\delta}_{\text{bc}})$ determined in Lemma 3.29. Then the boundary condition (3.5.8) can be written as a linear boundary condition:

$$(3.5.30) \qquad \mathcal{B}^L_1\psi := b_1(x, y)\psi_x + b_2(x, y)\psi_y + b_3(x, y)\psi = 0 \qquad \text{on } \Gamma_{\text{shock}} \cap \partial\Omega^{\mathcal{O}}_{\varepsilon_1},$$

and Lemma 3.29 implies

$$(3.5.31) \qquad -C \leq b_j \leq -C^{-1} \text{ for } j = 1, 2, \quad (b_1, b_2) \cdot \boldsymbol{\nu}^{(x,y)}_{\text{sh}} \geq C^{-1} \qquad \text{on } \Gamma_{\text{shock}} \cap \partial\Omega^{\mathcal{O}}_{\varepsilon_1}.$$

By (3.5.24), we have

(3.5.32) $$\Omega_{\bar{\varepsilon}}^{\mathcal{O}} \subset \{(x,y) \,:\, 0 < x < \bar{\varepsilon}, \ 0 < y < y_{P_1} + kx\}.$$

For constants $m, \mu > 1$ to be determined, define a function v by

$$v(x,y) := (x + m\mu y_{P_1}^2)^4 - m(x + m\mu y_{P_1}^2)^3 y^2.$$

Suppose that

(3.5.33) $$y_{P_1} \leq \frac{1}{(m\mu)^2}, \qquad \hat{\varepsilon} \leq \frac{1}{2} \min\{\varepsilon_1, \varepsilon_{\mathcal{O}}, \frac{1}{m\mu}\}$$

for $\varepsilon_{\mathcal{O}}$ from Lemma 3.29. Then a lengthy computation by using (3.5.28) and (3.5.32) shows that constants (m, μ) can be fixed sufficiently large depending only on (v_∞, γ) such that

(3.5.34)
$$v(x,y) \geq \frac{1}{2}(x + m\mu y_{P_1})^4 \qquad \text{in } \Omega_{2\hat{\varepsilon}}^{\mathcal{O}},$$
$$\mathcal{L}v < 0 \qquad \text{in } \Omega_{2\hat{\varepsilon}}^{\mathcal{O}},$$
$$\mathcal{B}_1^L v < 0 \qquad \text{on } \Gamma_{\text{shock}} \cap \partial\Omega_{2\hat{\varepsilon}}^{\mathcal{O}}.$$

For detailed calculations to obtain (3.5.34), we refer to [**11**, Lemma 16.4.1].

For $\hat{\varepsilon} := \frac{1}{2}\min\{\varepsilon_1, \varepsilon_{\mathcal{O}}, \frac{1}{m\mu}\}$, we define

$$a := \frac{1}{2\hat{\varepsilon}^2} \max_{\partial\Omega_{2\hat{\varepsilon}}^{\mathcal{O}} \cap \{x = 2\hat{\varepsilon}\}} \psi.$$

Note that, by the strong maximum principle, a is a positive constant. By Lemma 3.28(e), a is uniformly bounded above depending only on (v_∞, γ).

Note that ψ satisfies the boundary conditions (3.5.11) on $\partial\Omega_{2\hat{\varepsilon}}^{\mathcal{O}} \setminus (\{x = 2\hat{\varepsilon}\} \cup \Gamma_{\text{shock}})$. Since $|y| \leq y_{P_1}$ on $\Gamma_{\text{sonic}}^{\mathcal{O}}$ and $\mu > 1$, we have

$$av \geq 0 = \psi \qquad \text{on } \Gamma_{\text{sonic}}^{\mathcal{O}}.$$

On $\Gamma_{\text{wedge}} \cap \partial\Omega_{2\hat{\varepsilon}}^{\mathcal{O}}$, $v_y = 0 = \psi_y$.

By the maximum principle, we have

$$\psi \leq av \qquad \text{in } \Omega_{2\hat{\varepsilon}}^{\mathcal{O}},$$

provided that y_{P_1} satisfies the inequality that $y_{P_1} \leq (m\mu)^{-2}$.

By (3.5.21) and (3.5.23), there exists $\sigma' \in (0, \sigma^*]$ such that each y_{P_1} corresponding to $\beta \in [\beta_s^{(v_\infty)} - \sigma', \beta_s^{(v_\infty)})$ satisfies the inequality that $y_{P_1} \leq (m\mu)^{-2}$. This verifies the claim.

3. Let $\varphi = \psi + \varphi_{\mathcal{O}}$ be an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta_s^{(v_\infty)} - \sigma' \leq \beta < \beta_s^{(v_\infty)}\}$. For $z_0 = (x_0, y_0) \in \overline{\Omega_{\hat{\varepsilon}}^{\mathcal{O}}} \cap \{x > \frac{y_{P_1}^2}{5}\}$ and $r \in (0, 1]$, define the sets:

$$\tilde{R}_{z_0, r} := \{(x,y) \,:\, |x - x_0| < \frac{x_0^{3/2}}{10k} r, \ |y - y_0| < \frac{x_0}{10k} r\},$$
$$R_{z_0, r} := \tilde{R}_{z_0, r} \cap \Omega_{2\hat{\varepsilon}}^{\mathcal{O}}.$$

Here, $R_{z_0, 1}$ may intersect with $\Gamma_{\text{shock}} \cup \Gamma_{\text{wedge}}$. However, if $R_{z_0, 1} \cap \Gamma_{\text{shock}} \neq \emptyset$, then $R_{z_0, 1} \cap \Gamma_{\text{wedge}} = \emptyset$, and vice versa. Note that the dimensions of rectangle $\tilde{R}_{z_0, r}$ are given such that

    (i) the re-scaled function $\psi^{(z_0)}$ defined below satisfies a uniformly elliptic equation, due to (3.5.18) stated in Lemma 3.29;

    (ii) $\overline{R_{z_0, 1}}$ does not intersect with Γ_{shock} and Γ_{wedge} simultaneously.

For $r > 0$, define the sets:

$$Q_r := (-r, r)^2,$$

$$Q_r^{(z_0)} := \{(S, T) \in Q_r \ : \ z_0 + \frac{\sqrt{x_0}}{10k}(x_0 S, \sqrt{x_0}\, T) \in R_{z_0, r}\}.$$

For $z_0 \in \overline{\Omega^{\mathcal{O}}_{\hat{\varepsilon}}} \cap \{x > \frac{y_{P_1}^2}{5}\}$, define

$$\psi^{(z_0)}(S, T) = \frac{1}{x_0^4}\psi(x_0 + \frac{x_0^{3/2}}{10k}S, y_0 + \frac{x_0}{10k}T) \qquad \text{for } (S, T) \in Q_1^{(z_0)}.$$

For constant L from Lemma 3.28, choose a function $\eta \in C^{\infty}(\mathbb{R})$ such that $0 \leq \eta \leq 1$ with $\eta(t) = 1$ for $|t| \leq L$ and $\eta(t) = 0$ for $|t| \geq 2L$. For such a function η, we define

(3.5.35) $$(\hat{A}^{\mathcal{O},(\text{mod})}_{ij}, \hat{A}^{\mathcal{O},(\text{mod})}_i)(p_x, p_y, z, x) := (\hat{A}^{\mathcal{O}}_{ij}, \hat{A}^{\mathcal{O}}_i)(x\eta(\frac{p_x}{x}), x\eta(\frac{p_y}{x}), x^2\eta(\frac{z}{x^2}), x).$$

Then $(\hat{A}^{\mathcal{O},(\text{mod})}_{ij}, \hat{A}^{\mathcal{O},(\text{mod})}_i), i, j = 1, 2$, satisfy the following lemma, which is a generalization of Lemma 3.24:

LEMMA 3.33. *Let $\varepsilon_0 > 0$ and $L \geq 1$ be the constants from Lemmas 3.27–3.28, respectively. Then there exist constants $\varepsilon \in (0, \frac{\varepsilon_0}{2}]$ and $C > 0$ depending only on (v_∞, γ) such that any admissible solution $\varphi := \varphi_{\mathcal{O}} + \psi$ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ satisfies the following equation:*

(3.5.36) $$\sum_{i,j=1}^2 \hat{A}^{\mathcal{O},(\text{mod})}_{ij}(D\psi, \psi, x)D_{ij}\psi + \sum_{i=1}^2 \hat{A}^{\mathcal{O},(\text{mod})}_i(D\psi, \psi, x)D_i\psi = 0 \qquad \text{in } \Omega^{\mathcal{O}}_\varepsilon,$$

with coefficients $(\hat{A}^{\mathcal{O},(\text{mod})}_{ij}, \hat{A}^{\mathcal{O},(\text{mod})}_i)$ satisfying the following properties:

(a) $(\hat{A}^{\mathcal{O},(\text{mod})}_{ij}, \hat{A}^{\mathcal{O},(\text{mod})}_i) = (\hat{A}^{\mathcal{O}}_{ij}, \hat{A}^{\mathcal{O}}_i)$
    in $\{(p_x, p_y, z, x) \ : \ |(p_x, p_y)| \leq Lx, \ |z| \leq Lx^2, \ x \in (0, \varepsilon)\}$,

(b) $|(\hat{A}^{\mathcal{O},(\text{mod})}_{11}, \hat{A}^{\mathcal{O},(\text{mod})}_{12}, \hat{A}^{(\text{mod})}_2)(p_x, p_y, z, x)| \leq Cx$ *in* $\mathbb{R}^2 \times \mathbb{R} \times (0, \varepsilon)$,

(c) $\|(\hat{A}^{\mathcal{O},(\text{mod})}_{22}, \hat{A}^{\mathcal{O},(\text{mod})}_1)\|_{0,\mathbb{R}^2 \times \mathbb{R} \times (0,\varepsilon)} \leq C$,

(d) $\|D_{(p_x, p_y, z, x)}(\hat{A}^{\mathcal{O},(\text{mod})}_{ij}, \hat{A}^{\mathcal{O},(\text{mod})}_i)\|_{0,\mathbb{R}^2 \times \mathbb{R} \times (0,\varepsilon)} \leq C$.

Substituting the definition of $\psi^{(z_0)}$ into Eq. (3.5.36), we have

(3.5.37)
$$\sum_{i,j=1}^2 A^{(z_0)}_{ij}(D\psi^{(z_0)}, \psi^{(z_0)}, S, T)D_{ij}\psi^{(z_0)} + \sum_{i=1}^2 A^{(z_0)}_i(D\psi^{(z_0)}, \psi^{(z_0)}, S, T)D_i\psi^{(z_0)} = 0 \qquad \text{in } Q_1^{(z_0)},$$

with

$$A^{(z_0)}_{ij}(\mathbf{p}, z, S) = x_0^{\frac{i+j}{2}-2}\hat{A}^{\mathcal{O},\text{mod}}_{ij}(10kx_0^{4-\frac{3}{2}}p_1, 10kx_0^3 p_2, x_0^4 z, x_0 + \frac{x_0^{3/2}}{10k}S),$$

$$A^{(z_0)}_i(\mathbf{p}, z, S) = \frac{x_0^{\frac{i-1}{2}-1}}{10k}\hat{A}^{\mathcal{O},\text{mod}}_i(10kx_0^{4-\frac{3}{2}}p_1, 10kx_0^3 p_2, x_0^4 z, x_0 + \frac{x_0^{3/2}}{10k}S).$$

By (3.5.26), there exists a constant $C > 0$ depending only on (v_∞, γ) such that

(3.5.38) $$|\psi^{(z_0)}| \leq C \qquad \text{in } Q_1^{(z_0)}$$

for all $z_0 \in \overline{\Omega^{\mathcal{O}}_{\hat{\varepsilon}}} \cap \{x > \frac{y_{P_1}^2}{5}\}$.

For $\hat{f}_{\mathcal{O},\mathrm{sh}}$ from Lemma 3.28, define

$$(3.5.39) \qquad F^{(z_0)}(S) := \frac{10k}{x_0}\Big(\hat{f}_{\mathcal{O},\mathrm{sh}}(x_0 + \frac{x_0^{3/2}}{10k}S) - \hat{f}_{\mathcal{O},\mathrm{sh}}(x_0)\Big) \qquad \text{for } -1 < S < 1.$$

Similarly to (3.4.46), a direct computation by using (3.5.39) and Lemma 3.28(d) shows that there exists a constant $C > 0$ depending only on (v_∞, γ) so that, for each $z_0 = (x_0, \hat{f}_{\mathcal{O},\mathrm{sh}}(x_0)) \in \Gamma_{\mathrm{shock}} \cap \partial\Omega_{\hat{\varepsilon}}^{\mathcal{O}}$, $F^{(z_0)}$ satisfies

$$(3.5.40) \qquad F^{(z_0)}(0) = 0, \qquad \|F^{(z_0)}\|_{C^1([-1,1])} \leq C\sqrt{x_0}.$$

However, it follows from $\varphi_\infty - \varphi = 0$ on Γ_{shock} that

$$(3.5.41) \qquad \phi_\infty^{\mathcal{O}}(x_0 + \frac{x_0^{3/2}}{10k}S, \hat{f}_{\mathcal{O},\mathrm{sh}}(x_0) + \frac{x_0}{10k}F^{(z_0)}(S)) - x_0^4\psi^{(z_0)}(S, F^{(z_0)}(S)) = 0$$

for $\phi_\infty^{\mathcal{O}}$ given by (3.5.7).

Similarly to (3.4.47), by using Lemmas 3.28–3.29, we can further reduce $\hat{\varepsilon} \in (0, \frac{\bar{\varepsilon}}{2}]$ depending only on (v_∞, γ) so that the boundary condition (3.5.8) can be rewritten as

$$(3.5.42) \qquad \psi_x = b_{\mathcal{O}}(\psi_y, \psi, x, y) \qquad \text{on } \Gamma_{\mathrm{shock}} \cap \partial\Omega_{2\hat{\varepsilon}}^{\mathcal{O}},$$

where $b_{\mathcal{O}}$ satisfies the following properties:

$$(3.5.43) \qquad \begin{aligned} b_{\mathcal{O}}(0,0,x,y) &= 0 \qquad \text{in } \overline{\Omega_{2\hat{\varepsilon}}^{\mathcal{O}}}, \\ |D^l b_{\mathcal{O}}(p_y, z, x, y)| &\leq C_l \qquad \text{in } \mathbb{R} \times \mathbb{R} \times \overline{\Omega_{2\hat{\varepsilon}}^{\mathcal{O}}}, \text{ for } l = 1, 2, 3, \cdots, \end{aligned}$$

for $C_l > 0$ chosen depending only on (v_∞, γ, l).

For each $z_0 \in \Gamma_{\mathrm{shock}} \cap \partial\Omega_{\hat{\varepsilon}}^{\mathcal{O}}$, we substitute $\psi^{(z_0)}$ into (3.5.42) to obtain the following boundary condition on $\Gamma_{\mathrm{shock}}^{(z_0)} = \{T = F^{(z_0)}(S) : -1 < S < 1\}$:

$$(3.5.44) \qquad \psi_S^{(z_0)} = B_{\mathcal{O}}^{(z_0)}(\psi_T^{(z_0)}, \psi^{(z_0)}, S, T),$$

for $B_{\mathcal{O}}^{(z_0)}(\psi_T^{(z_0)}, \psi^{(z_0)}, S, T)$ given by

$$B_{\mathcal{O}}^{(z_0)}(\psi_T^{(z_0)}, \psi^{(z_0)}, S, T) := \frac{x_0^{-4+3/2}}{10k} b_{\mathcal{O}}(10kx_0^3\psi_T^{(z_0)}, x_0^4\psi^{(z_0)}, x_0 + \frac{x_0^{3/2}}{10k}S, y_0 + \frac{x_0}{10k}T).$$

It can be checked directly from (3.5.43) that, for each $z_0 \in \Gamma_{\mathrm{shock}} \cap \partial\Omega_{\hat{\varepsilon}}^{\mathcal{O}}$, $B_{\mathcal{O}}^{(z_0)}$ satisfies

$$\begin{aligned} B_{\mathcal{O}}^{(z_0)}(0,0,S,T) &= 0 \qquad \text{in } \overline{Q_1^{(z_0)}}, \end{aligned}$$

$$(3.5.45) \qquad \begin{aligned} \|\partial_{p_T} B_{\mathcal{O}}^{(z_0)}(p_T, z, \cdot)\|_{0, \overline{Q_1^{(z_0)}}} &\leq m_2\sqrt{x_0} \qquad \text{for all } (p_T, z) \in \mathbb{R} \times \mathbb{R}, \\ \|D_{(p_T, z)} B_{\mathcal{O}}^{(z_0)}(p_T, z, \cdot)\|_{1, \overline{Q_1^{(z_0)}}} &\leq m_2\sqrt{x_0} \qquad \text{for all } (p_T, z) \in \mathbb{R} \times \mathbb{R}, \end{aligned}$$

where $m_2 > 0$ is a constant depending only on (v_∞, γ).

4. Using (3.5.18), Lemma 3.33, (3.5.40), and (3.5.45), we see that Eq. (3.5.37) and the boundary condition (3.5.44) satisfy all the conditions required to apply Theorem C.5. Therefore, by (3.5.38) and Theorem C.5, there exist $\varepsilon \in (0, \hat{\varepsilon}]$, $\hat{\alpha} \in (0, 1)$, C, and $\sigma_1 \in (0, \sigma']$ depending only on (v_∞, γ)

such that any admissible solution $\varphi = \psi + \varphi_{\mathcal{O}}$ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta_{\text{s}}^{(v_\infty)} - \sigma_1 \leq \beta < \beta_{\text{s}}^{(v_\infty)}\}$ satisfies

$$(3.5.46) \qquad \|\psi^{(z_0)}\|_{1,\hat{\alpha},\overline{Q_{3/4}^{(z_0)}}} \leq C \qquad \text{for all } z_0 \in \Gamma_{\text{shock}} \cap \partial\Omega_\varepsilon^{\mathcal{O}} \cap \{x > \frac{y_{P_1}^2}{5}\}.$$

To obtain the $C^{1,\hat{\alpha}}$–estimate of $F^{(z_0)}$, we follow the approach given in the latter part of Step 3 in the proof of Proposition 3.26. Namely, we differentiate (3.5.41) with respect to S to obtain

$$(3.5.47) \qquad (F^{(z_0)})' = -\frac{\sqrt{x_0}\left(\partial_x \phi_\infty^{\mathcal{O}}(x_S, y_S) - 10kx_0^{5/2}\partial_S \psi^{(z_0)}(S,T)\right)}{\partial_y \phi_\infty^{\mathcal{O}}(x_S, y_S) - 10kx_0^3 \partial_T \psi^{(z_0)}(S,T)}$$

for $(x_S, y_S) := (x_0 + \frac{x_0^{3/2}}{10k}S, \hat{f}_{\mathcal{O},\text{sh}}(x_0) + \frac{x_0}{10k}F^{(z_0)}(S))$.

Then a direct computation by using Lemma 3.27(c), (3.5.46)–(3.5.47), and the smoothness of $\phi_\infty^{\mathcal{O}}$ yields that there exists a constant $C > 0$ depending only on (v_∞, γ) such that

$$(3.5.48) \qquad \frac{1}{\sqrt{x_0}}\|F^{(z_0)}\|_{1,\hat{\alpha},[-3/4,3/4]} \leq C \qquad \text{for all } z_0 \in \Gamma_{\text{shock}} \cap \partial\Omega_\varepsilon^{\mathcal{O}} \cap \{x > \frac{y_{P_1}^2}{5}\}.$$

For higher order derivative estimates of $\psi^{(z_0)}$ and $F^{(z_0)}$, we follow the bootstrap argument given in the latter part of Step 3 in the proof of Proposition 3.26 by using (3.5.46), (3.5.48), and Theorem C.6. As a result, we find constants $\varepsilon \in (0, \hat{\varepsilon}]$ and $\sigma_1 \in (0, \sigma']$ depending only on (v_∞, γ) such that, for each $\alpha \in (0,1)$, any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta_{\text{s}}^{(v_\infty)} - \sigma_1 \leq \beta < \beta_{\text{s}}^{(v_\infty)}\}$ satisfies

$$\|\psi^{(z_0)}\|_{2,\alpha,\overline{Q_{1/2}^{(z_0)}}} + \frac{1}{\sqrt{x_0}}\|F^{(z_0)}\|_{2,\alpha,[-1/2,1/2]} \leq C \qquad \text{for all } z_0 \in \Gamma_{\text{shock}} \cap \partial\Omega_\varepsilon^{\mathcal{O}} \cap \{x > \frac{y_{P_1}^2}{5}\},$$

where the estimate constant C depends only on $(v_\infty, \gamma, \alpha)$.

Furthermore, by repeating the argument of Step 4 in the proof of Proposition 3.26, it can be shown that, for each $\alpha \in (0,1)$, there exists a constant $C > 0$ depending only on $(v_\infty, \gamma, \alpha)$ such that any admissible solution $\varphi = \psi + \varphi_{\mathcal{O}}$ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta_{\text{s}}^{(v_\infty)} - \sigma_1 \leq \beta < \beta_{\text{s}}^{(v_\infty)}\}$ satisfies

$$\|\psi^{(z_0)}\|_{2,\alpha,\overline{Q_{1/2}^{(z_0)}}} + \frac{1}{\sqrt{x_0}}\|F^{(z_0)}\|_{2,\alpha,[-1/2,1/2]} \leq C \qquad \text{for all } z_0 \in \overline{\Omega_\varepsilon^{\mathcal{O}}} \cap \{x > \frac{y_{P_1}^2}{5}\}.$$

Denote $\mathcal{U}_\varepsilon := \Omega_\varepsilon^{\mathcal{O}} \cap \{x > \frac{y_{P_1}^2}{5}\}$. Collecting all the estimates of $\psi^{(z_0)}$ established above, scaling back to the (x,y)–coordinates, and following the argument of Step 3 in the proof of [**11**, Proposition 16.4.6], we have

$$\sum_{0 \leq k+l \leq 2} \sup_{z \in \mathcal{U}_\varepsilon} \left(x^{\frac{3k}{2}+l-4}|\partial_x^k \partial_y^l \psi(z)|\right)$$

$$+ \sum_{k+l=2} \sup_{\substack{z,\tilde{z}\in\mathcal{U}_\varepsilon, \\ z \neq \tilde{z}}} \left(\min\{x^{\frac{3}{2}(\alpha+k)+l-4}, \tilde{x}^{\frac{3}{2}(\alpha+k)+l-4}\}\frac{|\partial_x^k \partial_y^l \psi(z) - \partial_x^k \partial_y^l \psi(\tilde{z})|}{\delta_\alpha^{\text{par}}(z,\tilde{z})}\right) \leq C,$$

where k and l are nonnegative integers, C is a constant depending only on $(v_\infty, \gamma, \alpha)$, and we have used the notation that $z = (x,y)$ and $\tilde{z} = (\tilde{x}, \tilde{y})$. This implies that

$$(3.5.49) \qquad \|\psi\|_{2,\alpha,\Omega_\varepsilon^{\mathcal{O}} \cap \{x > y_{P_1}^2/5\}}^{(\text{par})} \leq C.$$

5. Combining estimates (3.5.25) and (3.5.49) together, we obtain

$$\|\varphi - \varphi_{\mathcal{O}}\|^{(\mathrm{par})}_{2,0,\Omega_{\varepsilon}^{\mathcal{O}}} \leq C,$$

where constant $C > 0$ depends only on $(v_\infty, \gamma, \alpha)$.

In order to estimate $[\varphi - \varphi_{\mathcal{O}}]^{(2),(\mathrm{par})}_{2,\alpha,\Omega_{\varepsilon}^{\mathcal{O}}}$, we consider two cases: (i) either $z = (x,y), \tilde{z} = (\tilde{x}, \tilde{y}) \in \Omega^{\mathcal{O}}_{y^2_{P_1}}$, or $z, \tilde{z} \in \Omega_{\varepsilon}^{\mathcal{O}} \cap \{x > \frac{y^2_{P_1}}{5}\}$, and (ii) $x > y^2_{P_1} > \frac{y^2_{P_1}}{5} > \tilde{x}$.

For $k + l = 2$, define

$$q_{k,l}(z,\tilde{z}) := \min\{x^{\alpha+k+\frac{l}{2}-2}, \tilde{x}^{\alpha+k+\frac{l}{2}-2}\} \frac{|\partial_x^k \partial_y^l \psi(z) - \partial_x^k \partial_y^l \psi(\tilde{z})|}{\delta_\alpha^{(\mathrm{par})}(z,\tilde{z})}.$$

For case (i), $q_{k,l}(z,\tilde{z})$ satisfies

$$\sum_{k+l=2} q_{k,l}(z,\tilde{z}) \leq 4\left(\|\psi\|^{(\mathrm{par})}_{2,\alpha,\Omega^{\mathcal{O}}_{y^2_{P_1}}} + \|\psi\|^{(\mathrm{par})}_{2,\alpha,\Omega_{\varepsilon}^{\mathcal{O}} \cap \{x > y^2_{P_1}/5\}}\right).$$

For case (ii), since $\delta_\alpha^{(\mathrm{par})}(z,\tilde{z}) \geq \frac{x^\alpha}{2^\alpha} \geq \frac{\tilde{x}^\alpha}{2^\alpha}$, we have

$$\sum_{k+l=2} q_{k,l}(z,\tilde{z}) \leq 2^{\alpha+2}\left(\|\psi\|^{(\mathrm{par})}_{2,0,\Omega^{\mathcal{O}}_{y^2_{P_1}}} + \|\psi\|^{(\mathrm{par})}_{2,0,\Omega_{\varepsilon}^{\mathcal{O}} \cap \{x > y^2_{P_1}/5\}}\right).$$

Therefore, we conclude that there exists a constant $C > 0$ depending only on $(v_\infty, \gamma, \alpha)$ such that

$$\|\varphi - \varphi_{\mathcal{O}}\|^{(\mathrm{par})}_{2,\alpha,\Omega_{\varepsilon}^{\mathcal{O}}} \leq C.$$

In order to estimate $\|\hat{f}_{\mathcal{O},\mathrm{sh}} - \hat{f}_{\mathcal{O},0}\|^{(\mathrm{par})}_{2,\alpha,(0,\varepsilon)}$, we adjust the argument of Step 6 in the proof of Proposition 3.26 by using Lemma 3.27, instead of Lemma 3.20. □

3.5.3. Case 3: Admissible solutions for $\beta \geq \beta_{\mathrm{s}}^{(v_\infty)}$ close to $\beta_{\mathrm{s}}^{(v_\infty)}$.

LEMMA 3.34 (Extension of Lemma 3.27 for all $\beta \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$). *For the (x,y)–coordinates given by* (3.5.2), *define*

(3.5.50) $$\hat{x} := x - x_{P_1}.$$

Then there exist positive constants $\varepsilon_1, \varepsilon_0, \delta_0, \omega_0, C$, and $\mathfrak{M}$ depending only on (v_∞, γ) with $\varepsilon_1 > \varepsilon_0$ and $\mathfrak{M} \geq 2$ such that Lemma 3.27 holds for any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$, where x is replaced by $\hat{x}$ in all the properties stated in Lemma 3.27.

PROOF. By the definition of P_1 given in Definition 2.23, $x_{P_1} = 0$ for $\beta \leq \beta_{\mathrm{s}}^{(v_\infty)}$, which implies that $\hat{x} = x$ for $\beta \leq \beta_{\mathrm{s}}^{(v_\infty)}$. Therefore, Lemma 3.34 coincides with Lemma 3.27 for $\beta \leq \beta_{\mathrm{s}}^{(v_\infty)}$.

For $\beta > \beta_{\mathrm{s}}^{(v_\infty)}$, $\hat{x} < x$, since $x_{P_1} > 0$.

For $\beta > \beta_{\mathrm{s}}^{(v_\infty)}$, we repeat the proof of Lemma 3.27, except for replacing $c_{\mathcal{O}}$ by $|P_1 O_{\mathcal{O}}| = c_{\mathcal{O}} M_{\mathcal{O}} \csc \beta$ for $M_{\mathcal{O}}$ defined by (2.4.6). Note that $\frac{|P_1 O_{\mathcal{O}}|}{c_{\mathcal{O}}} = M_{\mathcal{O}} \csc \beta = 1$ at $\beta = \beta_{\mathrm{s}}^{(v_\infty)}$. Since $M_{\mathcal{O}}$ is decreasing with respect to β by (2.4.43), we see that $\frac{\mathrm{d} M_{\mathcal{O}} \csc \beta}{\mathrm{d}\beta} \leq 0$ for $\beta \in (0, \frac{\pi}{2})$ as well. Then we conclude that $0 < M_{\mathcal{O}} \csc \beta|_{\beta = \beta_{\mathrm{d}}^{(v_\infty)}} \leq M_{\mathcal{O}} \csc \beta < 1$ for $\beta > \beta_{\mathrm{s}}^{(v_\infty)}$ with $M_{\mathcal{O}} \csc \beta = 1$

at $\beta = \beta_{\text{s}}^{(v_\infty)}$, and $|\overline{P_1 Q_{\mathcal{O}}}| > 0$ depends continuously on β for all $\beta \in (0, \beta_{\text{d}}^{(v_\infty)}]$. Therefore, there exists a constant $\varepsilon_1 > 0$ depending only on (v_∞, γ) such that

$$|\overline{P_1 Q_{\mathcal{O}}}| \geq 2\varepsilon_1 \qquad \text{for all } \beta \in (0, \beta_{\text{d}}^{(v_\infty)}].$$

Then we can also choose a constant $\delta_0 > 0$, depending only on (v_∞, γ), to satisfy (3.5.15) for all $\beta \in (0, \beta_{\text{d}}^{(v_\infty)})$. The rest of the proof is the same as for the case $\beta \leq \beta_{\text{s}}^{(v_\infty)}$. $\qquad \square$

LEMMA 3.35. *Let ε_1 be the constant introduced in Lemma 3.34. For $\varepsilon \in (0, \varepsilon_1)$, let $\Omega_\varepsilon^{\mathcal{O}}$ be given by (3.5.16). For each $\sigma \in (0, \beta_{\text{d}}^{(v_\infty)} - \beta_{\text{s}}^{(v_\infty)})$, define a half-open interval $I(\sigma)$ by*

$$(3.5.51) \qquad\qquad I(\sigma) := (0, \beta_{\text{s}}^{(v_\infty)} + \sigma].$$

Then, for any given $\varepsilon \in (0, \varepsilon_1)$, there exists $\sigma > 0$ depending only on $(v_\infty, \gamma, \varepsilon)$ such that, for any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta \in I(\sigma)\}$, $\Omega_\varepsilon^{\mathcal{O}}$ is nonempty.

PROOF. For $\beta \leq \beta_{\text{s}}^{(v_\infty)}$, $\Omega_\varepsilon^{\mathcal{O}}$ is always nonempty, owing to Proposition 3.11.

Suppose that $\beta > \beta_{\text{s}}^{(v_\infty)}$. It follows from Definition 2.24(i-4) of Case II, Proposition 3.11, and the definition of the (x, y)–coordinates given by (3.5.2) that $\Omega_\varepsilon^{\mathcal{O}}$ is nonempty if $x_{P_\beta} < \varepsilon$. From this perspective, we need to find a small constant $\sigma > 0$ so that $x_{P_\beta} < \varepsilon$ holds for all $\beta \in I(\sigma)$.

For each admissible solution φ, define $M(P) := \frac{|D\varphi(P)|}{c(|D\varphi(P)|^2, \varphi(P))}$; that is, $M(P)$ is the *pseudo-Mach number* of φ at point P. For each $\beta \in (0, \frac{\pi}{2})$, let P_β be the ξ_1–intercept P_β of the straight oblique shock $S_{\mathcal{O}}$. By Definition 2.24(ii-3), we have

$$M(P_\beta) = \frac{|D\varphi_{\mathcal{O}}(P_\beta)|}{c_{\mathcal{O}}} = M_{\mathcal{O}} \csc \beta$$

for $M_{\mathcal{O}}$ given by (2.4.6). According to the proof of Lemma 2.22, $M_{\mathcal{O}}$ is a decreasing function of $\beta \in (0, \frac{\pi}{2})$. This implies that

$$(3.5.52) \qquad\qquad \frac{\mathrm{d}M(P_\beta)}{\mathrm{d}\beta} \leq 0 \qquad \text{for all } \beta \in (0, \frac{\pi}{2}),$$

so that

$$(3.5.53) \qquad \inf_{\beta \in I(\sigma)} M(P_\beta) = M(P_{\beta_{\text{s}}^{(v_\infty)}+\sigma}) < 1, \qquad \lim_{\sigma \to 0+} \inf_{\beta \in I(\sigma)} M(P_\beta) = 1.$$

By (3.5.2), x_{P_β} can be expressed as

$$(3.5.54) \qquad\qquad x_{P_\beta} = c_{\mathcal{O}} - |D\varphi_{\mathcal{O}}(P_\beta)| = c_{\mathcal{O}}(1 - M(P_\beta)).$$

Moreover, we obtain from (2.4.40) and (3.5.52) that

$$(3.5.55) \qquad\qquad \frac{\mathrm{d}x_{P_\beta}}{\mathrm{d}\beta} > 0 \qquad \text{for } \beta \in (0, \frac{\pi}{2}).$$

Furthermore, (3.5.53) yields that

$$(3.5.56) \qquad \sup_{\beta \in I(\sigma)} x_{P_\beta} = x_{P_\beta}\big|_{\beta = \beta_{\text{s}}^{(v_\infty)}+\sigma} > 0, \qquad \lim_{\sigma \to 0+} \sup_{\beta \in I(\sigma)} x_{P_\beta} = 0.$$

Therefore, for any given $\varepsilon > 0$, we can choose $\sigma > 0$ depending only on $(v_\infty, \gamma, \varepsilon)$ so that $x_{P_\beta} < \varepsilon$ for all $\beta \in I(\sigma)$. $\qquad \square$

LEMMA 3.36 (Extension of Lemma 3.28 for $\beta > \beta_{\mathrm{s}}^{(v_\infty)}$). *Let ε_0, ω_0, and $\mathfrak{M}$ be from Lemma 3.34. Then there exist constants $\bar{\varepsilon} \in (0, \varepsilon_0]$, $\sigma_2 \in (0,1)$, $L \geq 1$, $\delta \in (0, \frac{1}{2})$, and $\omega \in (0, \omega_0] \cap (0,1)$ depending only on (v_∞, γ) such that any admissible solution $\varphi = \psi + \varphi_{\mathcal{O}}$ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta \in I(\sigma_2)\}$ satisfies properties* (a)–(e) *of Lemma 3.28 with the following changes:*

(i) *The definition of $\Omega_{\bar{\varepsilon}}^{\mathcal{O}}$ in* (3.5.16) *is replaced by*

$$(3.5.57) \qquad \Omega_{\bar{\varepsilon}}^{\mathcal{O}} = \Omega \cap \mathcal{N}_{\varepsilon}(\overline{\Gamma_{\mathrm{sonic}}^{\mathcal{O}}}) \cap \{x_{P_1} < x < x_{P_1} + \bar{\varepsilon}\},$$

(ii) $\Omega_{\bar{\varepsilon}}^{\mathcal{O}} = \{(x,y) \, : \, x \in (x_{P_1}, x_{P_1} + \bar{\varepsilon}), 0 < y < \hat{f}_{\mathcal{O},\mathrm{sh}}(x)\}$,

(iii) $\Gamma_{\mathrm{shock}} \cap \partial \Omega_{\bar{\varepsilon}}^{\mathcal{O}} = \{(x,y) \, : \, x \in (x_{P_1}, x_{P_1} + \bar{\varepsilon}), y = \hat{f}_{\mathcal{O},\mathrm{sh}}(x)\}$,

(iv) $\omega \leq \hat{f}'_{\mathcal{O},\mathrm{sh}}(x) \leq L$ *for* $x_{P_1} < x < x_{P_1} + \bar{\varepsilon}$,

where $I(\sigma_2)$ is given by (3.5.51).

PROOF. As in Lemma 3.28, this lemma is proved by adjusting the proof of Lemma 3.21.

Let $\hat{x}$ be given by (3.5.50). Since $\hat{x} = x$ holds for $\beta \leq \beta_{\mathrm{s}}^{(v_\infty)}$ so that Lemma 3.36 is the same as Lemma 3.28, it suffices to consider the case that $\beta > \beta_{\mathrm{s}}^{(v_\infty)}$.

By Definition 2.23, Remark 3.14, and Proposition 3.15, combined with (3.3.4)–(3.3.5), (3.5.5), and (3.5.54), there exist constants $\sigma' \in (0,1)$, $\varepsilon' \in (0, \varepsilon_0)$, and $\delta' \in (0, \frac{1}{2})$ depending only on (v_∞, γ) so that any admissible solution corresponding to $(v_\infty, \gamma) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta \in I(\sigma') \cap [\beta_{\mathrm{s}}^{(v_\infty)}, \frac{\pi}{2})\}$ satisfies

$$(3.5.58) \qquad \begin{aligned} 2x - (\gamma + 1)\psi_x + O_1^{\mathcal{O}}(D\psi, \psi, x) &\geq \delta'\Big(\mathrm{dist}(\boldsymbol{\xi}, \Gamma_{\mathrm{sonic}}^{\mathcal{O}}) + c_{\mathcal{O}}\big(1 - \frac{|D\varphi_{\mathcal{O}}(P_1)|}{c_{\mathcal{O}}}\big)\Big) \\ &= \delta'\big((x - x_{P_1}) + x_{P_\beta}\big) = \delta'x \qquad \text{in } \Omega_{\varepsilon'}^{\mathcal{O}}, \end{aligned}$$

where we have used $P_1 = P_\beta$ for $\beta \geq \beta_{\mathrm{s}}^{(v_\infty)}$, and (3.3.8) in Proposition 3.15.

Since $\psi \geq 0$ holds in $\Omega_{\varepsilon_0}^{\mathcal{O}}$ by Definition 2.24(iv), we use (3.2.29) to obtain

$$O_1^{\mathcal{O}}(D\psi, \psi, x) \leq \frac{\gamma + 1}{c_{\mathcal{O}}} x \psi_x \qquad \text{in } \Omega_{\varepsilon_0}^{\mathcal{O}}.$$

Then we can choose $\bar{\varepsilon} \in (0, \varepsilon']$ and $\delta \in (0, \frac{1}{2})$ depending only on (v_∞, β) so that, for any admissible solution $\varphi = \psi + \varphi_{\mathcal{O}}$ corresponding to $(v_\infty, \gamma) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta \in I(\sigma') \cap [\beta_{\mathrm{s}}^{(v_\infty)}, \frac{\pi}{2})\}$, (3.5.58) implies that

$$\psi_x(x,y) \leq \frac{2 - \delta}{1 + \gamma} x$$

in domain $\Omega_{\bar{\varepsilon}}^{\mathcal{O}}$ given by (3.5.57).

By Lemma 3.28, we can adjust $\delta \in (0, \delta']$ and $\bar{\varepsilon} \in (0, \varepsilon']$ depending only on (v_∞, γ) so that property (a) of Lemma 3.36 holds for any admissible solution corresponding to $(v_\infty, \gamma) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta \in I(\sigma')\}$.

Next, we choose a constant $\sigma_2 \in (0, \sigma']$ depending only on (v_∞, γ) so that $\Omega_{\bar{\varepsilon}}^{\mathcal{O}}$ is nonempty for any admissible solution corresponding to $(v_\infty, \gamma) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta \in I(\sigma_2)\}$. Such a constant σ_2 can be chosen due to Lemma 3.35. Then property (a) of Lemma 3.36 is verified.

The proofs of properties (b)–(e) of Lemma 3.36 for $\beta > \beta_{\mathrm{s}}^{(v_\infty)}$ are the same as for the case that $\beta \leq \beta_{\mathrm{s}}^{(v_\infty)}$, except that x is replaced by $\hat{x}$ for the range of variables for which the lemma holds, and Lemma 3.34 is applied instead of Lemma 3.27. More details for proving (b)–(e) of this lemma can be given by adjusting the proof of Lemma 3.21. $\qquad \square$

LEMMA 3.37. *For each $\sigma \in (0, \beta_{\text{d}}^{(v_\infty)} - \beta_{\text{s}}^{(v_\infty)})$, there exists a constant $\mu_0 > 0$ depending only on $(v_\infty, \gamma, \sigma)$ such that, for any $\beta \in [\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{d}}^{(v_\infty)} - \sigma]$, $g^{\text{sh}}_{\text{mod}}$ defined by (3.4.14) satisfies the following properties*:

$$\partial_{p_j} g^{\text{sh}}_{\text{mod}}(D\varphi_{\mathcal{O}}(P_1), \varphi_{\mathcal{O}}(P_1), P_1) \leq -\mu_0 \qquad \text{for } j = 1, 2,$$
$$\partial_z g^{\text{sh}}_{\text{mod}}(D\varphi_{\mathcal{O}}(P_1), \varphi_{\mathcal{O}}(P_1), P_1) \leq -\mu_0.$$

PROOF. Since $P_\beta = P_1$ for $\beta \geq \beta_{\text{s}}^{(v_\infty)}$ due to (2.5.6) in Definition 2.23, we apply Lemma A.4 to obtain

$$\partial_{p_1} g^{\text{sh}}_{\text{mod}}(D\varphi_{\mathcal{O}}(P_1), \varphi_{\mathcal{O}}(P_1), P_1) \leq -C^{-1} \qquad \text{for any } \beta \in [\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{d}}^{(v_\infty)} - \sigma],$$

with a constant $C > 1$ depending only on $(v_\infty, \gamma, \sigma)$.

A direct computation by using $\partial_{\xi_2} \varphi_{\mathcal{O}}(P_1) = \partial_{\xi_2} \varphi_{\mathcal{O}}(P_\beta) = 0$, (2.4.3), Definition 2.23, and (A.18) yields that

$$\partial_{p_2} g^{\text{sh}}_{\text{mod}}(D\varphi_{\mathcal{O}}(P_1), \varphi_{\mathcal{O}}(P_1), P_1) = -(\rho_{\mathcal{O}} + 1)\cos\beta.$$

By using (2.4.2), it can be directly checked that

$$\partial_z g^{\text{sh}}_{\text{mod}}(D\varphi_{\mathcal{O}}(P_1), \varphi_{\mathcal{O}}(P_1), P_1) = -\frac{c_{\mathcal{O}} M_{\mathcal{O}}}{\rho_{\mathcal{O}}^{\gamma-2}}$$

for $M_{\mathcal{O}} > 0$ given by (2.4.6).

Since $(\rho_{\mathcal{O}}, c_{\mathcal{O}}, M_{\mathcal{O}})$ depend continuously on $\beta \in [0, \frac{\pi}{2})$, we conclude that there exists a constant $C > 1$ depending only on (v_∞, γ) such that

$$(\partial_{p_2}, \partial_z) g^{\text{sh}}_{\text{mod}}(D\varphi_{\mathcal{O}}(P_1), \varphi_{\mathcal{O}}(P_1), P_1) \leq -C^{-1} \qquad \text{for all } \beta \in [\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{d}}^{(v_\infty)}].$$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

COROLLARY 3.38. *Let $\bar\varepsilon$ and σ_2 be the constants in Lemma 3.36. Then Lemma 3.29 holds for all $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ with $\beta \in [\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{s}}^{(v_\infty)} + \sigma_2]$.*

PROOF. It suffices to check property (c) of Lemma 3.29 for $\beta \geq \beta_{\text{s}}^{(v_\infty)}$, as the rest of the properties of Lemma 3.29 can be verified for $\beta \geq \beta_{\text{s}}^{(v_\infty)}$ in the same way as for the case that $\beta < \beta_{\text{s}}^{(v_\infty)}$. Since $P_1 = P_\beta$ for $\beta \geq \beta_{\text{s}}^{(v_\infty)}$, $y_{P_1} = 0$. From (2.4.3) and (3.5.9)–(3.5.10), we have

$$(D_{p_x}, D_{p_y})\mathcal{B}^{\mathcal{O}}_1(0,0,0,x_{P_1}, y_{P_1}) = v_\infty \sec\beta \left(D_{p_1}, \frac{1}{c_{\mathcal{O}}} D_{p_2}\right) g^{\text{sh}}_{\text{mod}}(D\varphi_{\mathcal{O}}(P_1), \varphi_{\mathcal{O}}(P_1), P_1).$$

Then property (c) of Lemma 3.29 is obtained for the case that $\beta_{\text{s}}^{(v_\infty)} \leq \beta \leq \beta_{\text{s}}^{(v_\infty)} + \sigma_2$ from Lemma 3.37 and the smoothness of $\mathcal{B}^{\mathcal{O}}_1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We now establish the uniform $C^{2,\alpha}$–estimate of the admissible solution $\varphi = \psi + \varphi_{\mathcal{O}}$ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ for $\beta \geq \beta_{\text{s}}^{(v_\infty)}$ close to $\beta_{\text{s}}^{(v_\infty)}$.

PROPOSITION 3.39. *Let $\bar\varepsilon$ and σ_2 be the constants in Lemma 3.36. Then, for each $\alpha \in (0,1)$, there exist constants $\varepsilon \in (0, \bar\varepsilon]$ and $\sigma_3 \in (0, \sigma_2]$ depending only on (v_∞, γ), and a constant $C > 0$ depending only on $(v_\infty, \gamma, \alpha)$, such that any admissible solution $\varphi = \psi + \varphi_{\mathcal{O}}$ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta_{\text{s}}^{(v_\infty)} \leq \beta \leq \beta_{\text{s}}^{(v_\infty)} + \sigma_3\}$ satisfies*

(3.5.59)
$$\|\psi\|_{C^{2,\alpha}(\overline{\Omega^{\mathcal{O}}_\varepsilon})} \leq C,$$
$$|D^m_{\boldsymbol\xi}\psi(P_1)| = |D^m_{\boldsymbol\xi}\psi(P_\beta)| = 0 \qquad \text{for } m = 0,1,2.$$

Moreover, $\hat{f}_{\mathcal{O},\mathrm{sh}}$ from Lemma 3.36 satisfies

$$
\|\hat{f}_{\mathcal{O},\mathrm{sh}} - \hat{f}_{\mathcal{O},0}\|_{2,\alpha,[x_{P_1},\varepsilon]} \le C,
$$
(3.5.60)
$$
\frac{\mathrm{d}^m}{\mathrm{d}x^m}(\hat{f}_{\mathcal{O},\mathrm{sh}} - \hat{f}_{\mathcal{O},0})(x_{P_1}) = \frac{\mathrm{d}^m}{\mathrm{d}x^m}(\hat{f}_{\mathcal{O},\mathrm{sh}} - \hat{f}_{\mathcal{O},0})(x_{P_\beta}) = 0 \qquad \text{for } m = 0, 1, 2.
$$

PROOF. In this proof, all the constants are chosen depending only on (v_∞, γ), unless otherwise specified.

1. For a fixed $\beta \in [\beta_{\mathrm{s}}^{(v_\infty)}, \beta_{\mathrm{s}}^{(v_\infty)} + \sigma_2]$, define

$$
d_{\mathrm{so}}(x) := x - x_{P_1}.
$$

If $\beta > \beta_{\mathrm{s}}^{(v_\infty)}$, then $d_{\mathrm{so}}(x) < x$ for all $x \in \Omega_{\bar{\varepsilon}}^{\mathcal{O}}$.

Claim: There exist constants $\varepsilon \in (0, \frac{\bar{\varepsilon}}{2}]$, $\sigma_3 \in (0, \sigma_2]$, and $m > 1$ such that any admissible solution $\varphi = \psi + \varphi_{\mathcal{O}}$ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$ with $\beta \in [\beta_{\mathrm{s}}^{(v_\infty)}, \beta_{\mathrm{s}}^{(v_\infty)} + \sigma_3]$ satisfies

(3.5.61)
$$
x_{P_1} \le \frac{\varepsilon}{10},
$$
$$
0 \le \psi(x,y) \le m\,(d_{\mathrm{so}}(x))^5 \qquad \text{in } \Omega_{2\varepsilon}^{\mathcal{O}}.
$$

A more general version of the claim stated immediately above can be found from [**11**, Lemma 16.5.1].

Note that $\psi \ge 0$ holds in Ω, due to Definition 2.24(iv).

For a large constant $M > 1$ to be determined later, define

$$
v(x,y) := (x - x_{P_1})^5 - \frac{1}{M}(x - x_{P_1})^3 y^2.
$$

By Lemma 3.36, there exists a constant $k > 1$ such that
(3.5.62)
$$
\{(x,y) : x_{P_1} < x < \bar{\varepsilon},\ 0 < y < \frac{1}{k}(x - x_{P_1})\} \subset \Omega_{\bar{\varepsilon}}^{\mathcal{O}} \subset \{(x,y) : x_{P_1} < x < \bar{\varepsilon},\ 0 < y < k(x - x_{P_1})\}.
$$

As in the proof of Proposition 3.32, we regard ψ as a solution of the linear boundary value problem:

$$
\begin{aligned}
\mathcal{L}\psi &= 0 && \text{in } \Omega_{\bar{\varepsilon}}^{\mathcal{O}}, \\
\mathcal{B}_1^L \psi &= 0 && \text{on } \Gamma_{\mathrm{shock}} \cap \partial\Omega_{\bar{\varepsilon}}^{\mathcal{O}}, \\
\psi_y &= 0 && \text{on } \Gamma_{\mathrm{wedge}} \cap \partial\Omega_{\bar{\varepsilon}}^{\mathcal{O}},
\end{aligned}
$$

where the linear operators $\mathcal{L}$ and $\mathcal{B}_1^L$ are given by (3.5.27) and (3.5.30), respectively.

It follows from (3.2.29) and Lemma 3.36 that there exist constants $\hat{\varepsilon}_1 \in (0, \bar{\varepsilon}]$ and C depending only on (v_∞, γ) so that the linear operator $\mathcal{L}$ satisfies properties (3.5.28)–(3.5.29) in $\Omega_{\hat{\varepsilon}_1}^{\mathcal{O}}$ for any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$ with $\beta_{\mathrm{s}}^{(v_\infty)} \le \beta \le \beta_{\mathrm{s}}^{(v_\infty)} + \sigma_2$.

From Corollary 3.38, there also exist constants $\hat{\varepsilon}_2 \in (0, \hat{\varepsilon}_1]$ and C depending only on (v_∞, γ) so that the boundary operator $\mathcal{B}_1^L$ satisfies (3.5.31) in $\Gamma_{\mathrm{shock}} \cap \partial\Omega_{\hat{\varepsilon}_2}^{\mathcal{O}}$ for any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$ with $\beta_{\mathrm{s}}^{(v_\infty)} \le \beta \le \beta_{\mathrm{s}}^{(v_\infty)} + \sigma_2$.

Similarly to Step 2 in the proof of Proposition 3.32, a lengthy computation by using (3.5.28)–(3.5.29) and (3.5.31) shows that there exist a sufficiently large constant $M > 1$, a sufficiently

small constant $\varepsilon \in (0, \frac{\hat{\varepsilon}_2}{2}]$, and a small constant $\sigma_3 \in (0, \sigma_2]$ such that, for any admissible solution $\varphi = \psi + \varphi_{\mathcal{O}}$ corresponding to $\beta \in \mathfrak{R}_{\text{weak}}$ with $\beta \in [\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{s}}^{(v_\infty)} + \sigma_3]$, we have

$$x_{P_1} \leq \frac{\varepsilon}{10},$$
$$\mathcal{L}v < 0 \qquad \text{on } \Omega^{\mathcal{O}}_{2\varepsilon},$$
$$\mathcal{B}_1^L v < 0 \qquad \text{on } \Gamma_{\text{shock}} \cap \partial\Omega^{\mathcal{O}}_{2\varepsilon},$$
$$v_y = 0 \qquad \text{on } \Gamma_{\text{wedge}} \cap \partial\Omega^{\mathcal{O}}_{2\varepsilon},$$
$$v(x, y) \geq \frac{1}{2}(x - x_{P_1})^5 \qquad \text{in } \Omega^{\mathcal{O}}_{2\varepsilon}.$$

Detailed calculations for the results stated above can be obtained by following the arguments in the proof of [**11**, Lemma 16.5.1].

Note that $\sigma_3 := \sigma_3(v_\infty, \gamma, \varepsilon) \in (0, \sigma_2]$ can be chosen sufficiently small so that $\Omega^{\mathcal{O}}_{2\varepsilon}$ is nonempty for any admissible solution $\varphi = \psi + \varphi_{\mathcal{O}}$ corresponding to $\beta \in \mathfrak{R}_{\text{weak}}$ with $\beta \in [\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{s}}^{(v_\infty)} + \sigma_3]$.

For $\varepsilon \in (0, \frac{\hat{\varepsilon}_2}{2}]$ fixed above, define m_ψ for (3.5.61) as

$$m_\psi := \frac{2}{\varepsilon^5} \max_{\partial\Omega^{\mathcal{O}}_{2\varepsilon} \cap \{x = 2\varepsilon\}} \psi(x, y).$$

By (3.1.25) stated in Lemma 3.5, there exists a constant $m > 0$ depending only on (γ, v_∞) such that

$$m_\psi \leq m$$

for any admissible solution $\varphi = \psi + \varphi_{\mathcal{O}}$ corresponding to $\beta \in \mathfrak{R}_{\text{weak}}$ with $\beta \in [\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{s}}^{(v_\infty)} + \sigma_3]$. Moreover, we have

$$\psi(x, y) \leq mv(x, y) \qquad \text{on } \partial\Omega^{\mathcal{O}}_{2\varepsilon} \cap \{x = 2\varepsilon\}.$$

Then the maximum principle implies that

$$\psi(x, y) \leq \frac{m}{2}(x - x_{P_1})^5 \qquad \text{in } \Omega^{\mathcal{O}}_{2\varepsilon}.$$

The claim is verified.

2. Take $\varepsilon > 0$ and $\sigma_3 > 0$ from Step 1. Let $\varphi = \psi + \varphi_{\mathcal{O}}$ be an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ with $\beta \in [\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{s}}^{(v_\infty)} + \sigma_3]$. For each $r \in (0, 1)$ and $z_0 = (x_0, y_0) \in \overline{\Omega^{\mathcal{O}}_{\varepsilon}} \setminus \{P_1\}$, we define Q_r and $Q_r^{(z_0)}$ by

$$Q_r := (-r, r)^2, \quad Q_r^{(z_0)} := \{(S, T) \in Q_r \, : \, z_0 + \frac{d_{\text{so}}(x_0)}{10k}(\sqrt{x_0}S, T) \in \Omega^{\mathcal{O}}_{2\varepsilon}\},$$

and a re-scaled function $\psi^{(z_0)}$ by

$$\psi^{(z_0)}(S, T) := \frac{1}{(d_{\text{so}}(x_0))^5}\psi(x_0 + \frac{d_{\text{so}}(x_0)}{10k}\sqrt{x_0}S, y_0 + \frac{d_{\text{so}}(x_0)}{10k}T) \qquad \text{for } (S, T) \in Q_1^{(z_0)},$$

where $k > 1$ is the constant from (3.5.62).

We repeat the arguments used in Steps 3–4 in the proof of Proposition 3.32 with some adjustments to obtain that, for each $\alpha \in (0, 1)$, there exists a constant $\mathfrak{C} > 0$ depending only on $(v_\infty, \gamma, \alpha)$ such that any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ with $\beta \in [\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{s}}^{(v_\infty)} + \sigma_3]$ satisfies

(3.5.63) $$\|\psi^{(z_0)}\|_{C^{2,\alpha}(\overline{Q_{1/10}^{(z_0)}})} \leq \mathfrak{C} \qquad \text{for all } z_0 \in \overline{\Omega^{\mathcal{O}}_{\varepsilon}} \setminus \{P_1\}.$$

Following the argument of Step 2 in the proof of [**11**, Proposition 16.5.3] and using estimate (3.5.63), we obtain

(3.5.64)
$$\sum_{0 \le k+l \le 2} \sup_{z \in \Omega_\varepsilon^{\mathcal{O}}} \left((x - x_{P_1})^{k+l-5} x^{\frac{k}{2}} |\partial_x^k \partial_y^l \psi(z)| \right)$$
$$+ \sum_{k+l=2} \sup_{z, \tilde{z} \in \Omega_\varepsilon^{\mathcal{O}}, z \ne \tilde{z}} \left(\left(\max\{x, \tilde{x}\} - x_{P_1} \right)^{k+l+\alpha-5} \left(\max\{x, \tilde{x}\} \right)^{\frac{k+\alpha}{2}} \frac{|\partial_x^k \partial_y^l \psi(z) - \partial_x^k \partial_y^l \psi(\tilde{z})|}{\delta_\alpha^{(\mathrm{par})}(z, \tilde{z})} \right)$$
$$\le C\mathfrak{C}$$

for $\delta_\alpha^{(\mathrm{par})}(z, \tilde{z})$ given by Definition 3.25, where we have used the notation that $z = (x, y)$ and $\tilde{z} = (\tilde{x}, \tilde{y})$.

We further follow the proof of [**11**, Proposition 16.5.3] to obtain that, for all $x, \tilde{x} \in (x_{P_1}, \varepsilon)$,

(3.5.65)
$$(x - x_{P_1})^{k+l-5} x^{\frac{k}{2}} \ge x^{\frac{3}{2}k+l-5} \quad \text{for } 0 \le k + l \le 2,$$
$$(\max\{x, \tilde{x}\} - x_{P_1})^{k+l+\alpha-5} (\max\{x, \tilde{x}\})^{\frac{k+\alpha}{2}} \ge (\max\{x, \tilde{x}\})^{\frac{3}{2}(k+\alpha)+l-5} \quad \text{for } k + l = 2.$$

This is because $k + l + \alpha - 5 < 0$ holds for $k, l \in \mathbb{Z}^+$ with $0 \le k + l \le 2$ and $\alpha \in (0, 1)$. Since $\frac{3}{2}(k + \alpha) + l - 5 < 0$ holds for $k, l \in \mathbb{Z}^+$ with $0 \le k + l \le 2$ and $\alpha \in (0, 1)$, it follows from (3.5.65) that

(3.5.66)
$$(x - x_{P_1})^{k+l-5} x^{\frac{k}{2}} \ge \varepsilon^{\frac{3}{2}k+l-5} \quad \text{for } 0 \le k + l \le 2,$$
$$(\max\{x, \tilde{x}\} - x_{P_1})^{k+l+\alpha-5} (\max\{x, \tilde{x}\})^{\frac{k+\alpha}{2}} \ge \varepsilon^{\frac{3}{2}(k+\alpha)+l-5} \quad \text{for } k + l = 2.$$

Assuming that $\varepsilon \le 1$ without loss of generality, we also have

(3.5.67)
$$\delta_\alpha^{(\mathrm{par})}(z, \tilde{z}) \le |z - \tilde{z}|^\alpha \quad \text{for } z, \tilde{z} \in \Omega_\varepsilon^{\mathcal{O}}.$$

Using (3.5.64) and (3.5.66)–(3.5.67), we obtain

$$\|\psi\|_{C^{2,\alpha}(\overline{\Omega_\varepsilon^{\mathcal{O}}})} \le C$$

for some constant $C > 0$ depending only on $(v_\infty, \gamma, \alpha)$, because the choice of ε given in Step 1 depends only on (v_∞, γ).

Furthermore, it follows directly from (3.5.64) that

$$|D_{(x,y)}^2 \psi(x, y)| \le C\mathfrak{C}(x - x_{P_1})^2 \quad \text{in } \Omega_\varepsilon^{\mathcal{O}},$$

which implies that

$$|D_{\boldsymbol{\xi}}^2 \psi(P_1)| = 0.$$

Note that $\psi(P_1) = |D_{\boldsymbol{\xi}} \psi(P_1)| = 0$, due to Definition 2.24(ii-3) for Case 2. Therefore, (3.5.59) is proved.

Finally, (3.5.60) can be proved by adjusting Step 6 in the proof of Proposition 3.26 and using (3.5.59). $\qquad\square$

3.5.4. Case 4: **Admissible solutions for** $\beta > \beta_{\text{s}}^{(v_\infty)}$ **away from** $\beta_{\text{s}}^{(v_\infty)}$. We first introduce a weighted Hölder space.

For a bounded connected open set $U \subset \mathbb{R}^2$, let Γ be a closed portion of ∂U. For $\mathbf{x}, \mathbf{y} \in U$, define
$$\delta_{\mathbf{x}} := \text{dist}(\mathbf{x}, \Gamma), \qquad \delta_{\mathbf{x},\mathbf{y}} := \min\{\delta_{\mathbf{x}}, \delta_{\mathbf{y}}\}.$$
For $k \in \mathbb{R}$, $\alpha \in (0,1)$, and $m \in \mathbb{Z}^+$, define the standard Hölder norms by

$$\|u\|_{m,0,U} := \sum_{0 \le |\boldsymbol{\beta}| \le m} \sup_{\mathbf{x} \in U} |D^{\boldsymbol{\beta}} u(\mathbf{x})|, \qquad [u]_{m,\alpha,U} := \sum_{|\boldsymbol{\beta}| = m} \sup_{\mathbf{x},\mathbf{y} \in U, \mathbf{x} \ne \mathbf{y}} \frac{|D^{\boldsymbol{\beta}} u(\mathbf{x}) - D^{\boldsymbol{\beta}} u(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\alpha},$$

and the weighted Hölder norms by

$$\|u\|_{m,0,U}^{(k),\Gamma} := \sum_{0 \le |\boldsymbol{\beta}| \le m} \sup_{\mathbf{x} \in U} \left(\delta_{\mathbf{x}}^{\max(|\boldsymbol{\beta}|+k,0)} |D^{\boldsymbol{\beta}} u(\mathbf{x})| \right),$$

$$[u]_{m,\alpha,U}^{(k),\Gamma} := \sum_{|\boldsymbol{\beta}| = m} \sup_{\mathbf{x},\mathbf{y} \in U, \mathbf{x} \ne \mathbf{y}} \left(\delta_{\mathbf{x},\mathbf{y}}^{\max\{m+\alpha+k,0\}} \frac{|D^{\boldsymbol{\beta}} u(\mathbf{x}) - D^{\boldsymbol{\beta}} u(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\alpha} \right),$$

$$\|u\|_{m,\alpha,U} := \|u\|_{m,0,U} + [u]_{m,\alpha,U}, \qquad \|u\|_{m,\alpha,U}^{(k),\Gamma} := \|u\|_{m,0,U}^{(k),\Gamma} + [u]_{m,\alpha,U}^{(k),\Gamma},$$

where $D^{\boldsymbol{\beta}} := \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2}$ for $\boldsymbol{\beta} = (\beta_1, \beta_2)$ with $\beta_j \in \mathbb{Z}_+$ and $|\boldsymbol{\beta}| = \beta_1 + \beta_2$. Denote $C_{(k),\Gamma}^{m,\alpha}(U)$ the completion space of the set of all smooth functions whose $\| \cdot \|_{m,\alpha,U}^{(k),\Gamma}$–norms are finite.

Let σ_3 be from Proposition 3.39. Then, by Proposition 3.15, there exists $\delta \in (0,1)$ depending only on (v_∞, γ) such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ with $\beta_{\text{s}}^{(v_\infty)} + \frac{\sigma_3}{2} \le \beta < \beta_{\text{d}}^{(v_\infty)}$ satisfies

$$(3.5.68) \qquad \frac{|D\varphi|}{c(|D\varphi|^2, \varphi)} \le 1 - \delta \qquad \text{in } \overline{\Omega} \cap \{\xi_1 \le 0\}$$

for $c(|\mathbf{p}|^2, z)$ defined by (3.2.5). By (3.5.68) and Lemma 3.5, there exists $M_* \ge 2$ depending only on (v_∞, γ) such that $(D\varphi(\boldsymbol{\xi}), \varphi(\boldsymbol{\xi})) \in \mathcal{K}_{M_*}$ for $\mathcal{K}_{M_*}$ defined by (3.2.6). In particular, there exist $\lambda_* > 0$ and $R_* > 0$ depending only on (v_∞, γ) such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ with $\beta_{\text{s}}^{(v_\infty)} + \frac{\sigma_3}{2} \le \beta < \beta_{\text{d}}^{(v_\infty)}$ satisfies

$$\sum_{i,j=1}^{2} \partial_{p_j} \mathcal{A}_i(D\varphi(\boldsymbol{\xi}), \varphi(\boldsymbol{\xi})) \kappa_i \kappa_j \ge \lambda_* |\boldsymbol{\kappa}|^2$$

for any $\boldsymbol{\xi} \in \overline{\Omega} \cap B_{R_*}(P_\beta)$ and any $\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2$.

According to Definition 2.23, $P_\beta = P_1$ for $\beta \ge \beta_{\text{s}}^{(v_\infty)}$. In this chapter, we use P_β, instead of P_1, to emphasize that P_β is the ξ_1–intercept of the straight oblique shock $S_{\mathcal{O}}$. In order to achieve the *a priori* estimates of admissible solutions for $\beta > \beta_{\text{s}}^{(v_\infty)}$ away from $\beta_{\text{s}}^{(v_\infty)}$, the convexity of the shock polar curves is heavily used, particularly in establishing the functional independence property of the boundary conditions for admissible solutions near P_β.

LEMMA 3.40. *For each small* $\bar{\sigma} \in (0, \frac{\beta_{\text{d}}^{(v_\infty)}}{10})$, *there exist positive constants* r *and* M *depending only on* $(v_\infty, \gamma, \bar{\sigma})$ *such that any admissible solution* φ *corresponding to* $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta_{\text{s}}^{(v_\infty)} \le \beta \le \beta_{\text{d}}^{(v_\infty)} - \bar{\sigma}\}$ *satisfies*

$$\partial_{p_1} g_{\text{mod}}^{\text{sh}}(D\varphi(\boldsymbol{\xi}), \varphi(\boldsymbol{\xi}), \boldsymbol{\xi}) \le -\frac{1}{M} \qquad \text{for all } \boldsymbol{\xi} \in \Gamma_{\text{shock}} \cap B_r(P_\beta),$$

where $g_{\mathrm{mod}}^{\mathrm{sh}}$ is given by (3.4.14).

PROOF. In this proof, all the constants are chosen depending only on (v_∞, γ), unless otherwise specified. The proof is divided into six steps.

1. For $\boldsymbol{\xi} \in \mathbb{R} \setminus B_1(O_\infty)$, denote $u_\infty^{(\boldsymbol{\xi})} := |D\varphi_\infty(\boldsymbol{\xi})|$, and denote $f_{\mathrm{polar}}^{(\boldsymbol{\xi})}$ as f_{polar} defined by Lemma A.3 corresponding to $(\rho_\infty, u_\infty) = (1, |D\varphi_\infty(\boldsymbol{\xi})|)$. Denote $(\hat{u}_0^{(\boldsymbol{\xi})}, u_{\mathrm{d}}^{(\boldsymbol{\xi})}, u_{\mathrm{s}}^{(\boldsymbol{\xi})})$ as $(\hat{u}_0, u_{\mathrm{d}}, u_{\mathrm{s}})$ corresponding to $(\rho_\infty, u_\infty) = (1, u_\infty^{(\boldsymbol{\xi})})$.

Fix $\bar{\sigma} \in (0, \frac{\beta_{\mathrm{d}}^{(v_\infty)}}{10})$. Let φ be an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta_{\mathrm{s}}^{(v_\infty)} \le \beta \le \beta_{\mathrm{d}}^{(v_\infty)} - \bar{\sigma}\}$, and let Γ_{shock} be its curved pseudo-transonic shock. By Proposition 3.7, $f_{\mathrm{polar}}^{(\boldsymbol{\xi})}$ is well defined for each $\boldsymbol{\xi} \in \overline{\Gamma_{\mathrm{shock}}}$. For $\boldsymbol{\xi} \in \mathbb{R}^2$, denote

$$(3.5.69) \qquad \mathbf{e}(\boldsymbol{\xi}) := \frac{D\varphi_\infty(\boldsymbol{\xi})}{|D\varphi_\infty(\boldsymbol{\xi})|},$$

and let $\mathbf{e}^\perp(\boldsymbol{\xi})$ be the unit vector obtained from rotating $\mathbf{e}(\boldsymbol{\xi})$ by $\frac{\pi}{2}$ counterclockwise. More generally, for each $\mathbf{e} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, let $\mathbf{e}^\perp$ denote the vector obtained from rotating $\mathbf{e}$ by $\frac{\pi}{2}$ counterclockwise.

The Rankine-Hugoniot condition (2.5.37) implies that $D\varphi(\boldsymbol{\xi})$ can be expressed as

$$(3.5.70) \qquad D\varphi(\boldsymbol{\xi}) = u\mathbf{e}(\boldsymbol{\xi}) + f_{\mathrm{polar}}^{(\boldsymbol{\xi})}(u)\mathbf{e}^\perp(\boldsymbol{\xi}) \qquad \text{for each } \boldsymbol{\xi} \in \overline{\Gamma_{\mathrm{shock}}},$$

with $u = u(D\varphi, \boldsymbol{\xi})$ given by

$$(3.5.71) \qquad u(D\varphi, \boldsymbol{\xi}) := D\varphi(\boldsymbol{\xi}) \cdot \mathbf{e}(\boldsymbol{\xi}).$$

By Proposition 3.15, we have

$$(3.5.72) \qquad u(D\varphi, \boldsymbol{\xi}) \le u_{\mathrm{s}}^{(\boldsymbol{\xi})} \qquad \text{for each } \boldsymbol{\xi} \in \overline{\Gamma_{\mathrm{shock}}}.$$

2. By (2.5.12) and Lemma A.4, there exists a constant $M_0 > 1$ depending only on $(v_\infty, \gamma, \bar{\sigma})$ such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta_{\mathrm{s}}^{(v_\infty)} \le \beta \le \beta_{\mathrm{d}}^{(v_\infty)} - \bar{\sigma}\}$ satisfies

$$(3.5.73) \qquad \partial_{p_1} g_{\mathrm{mod}}^{\mathrm{sh}}(D\varphi(P_\beta), \varphi(P_\beta), P_\beta) = \partial_{p_1} g^{\mathrm{sh}}(D\varphi_{\mathcal{O}}(P_\beta), \varphi_\infty(P_\beta), P_\beta) \le -M_0^{-1}.$$

Let (t_1, t_2)–coordinates be given so that $(1,0)_{(t_1,t_2)} = \mathbf{e}(P_\beta)$ and $(0,1)_{(t_1,t_2)} = \mathbf{e}^\perp(P_\beta)$. For $\boldsymbol{\xi} \in \mathbb{R}^2 \setminus B_1(O_\infty)$, we define a function $g^{(\boldsymbol{\xi})}(\mathbf{u})$ by

$$(3.5.74) \qquad g^{(\boldsymbol{\xi})}(\mathbf{u}) = g(\mathbf{u})$$

for $g(\mathbf{u})$ given by (A.7) with $\mathbf{u}_\infty = (|D\varphi_\infty(\boldsymbol{\xi})|, 0)$ (see Fig. 3.1). If we denote

$$u_* := \mathbf{e}(P_\beta) \cdot D\varphi_{\mathcal{O}}(P_\beta),$$

then

$$D\varphi_{\mathcal{O}}(P_\beta) = (u_*, f_{\mathrm{polar}}^{(P_\beta)}(u_*)), \qquad g^{(P_\beta)}(D\varphi_{\mathcal{O}}(P_\beta)) = 0.$$

Since $D\varphi_{\mathcal{O}}(P_\beta) \cdot \mathbf{e}_{\xi_2} = 0$, it can be checked directly from the definitions of g^{sh} and g given in (3.4.13) and (A.7), respectively, that

$$(3.5.75) \qquad g_{\mathbf{u}}^{(P_\beta)}(D\varphi_{\mathcal{O}}(P_\beta)) \cdot \mathbf{e}_1 = \partial_{p_1} g^{\mathrm{sh}}(D\varphi_{\mathcal{O}}(P_\beta), \varphi_\infty(P_\beta), P_\beta).$$

Moreover, from (3.5.73), we obtain

$$(3.5.76) \qquad g_{\mathbf{u}}^{(P_\beta)}(D\varphi_{\mathcal{O}}(P_\beta)) \cdot \mathbf{e}_1 \le -M_0^{-1}.$$
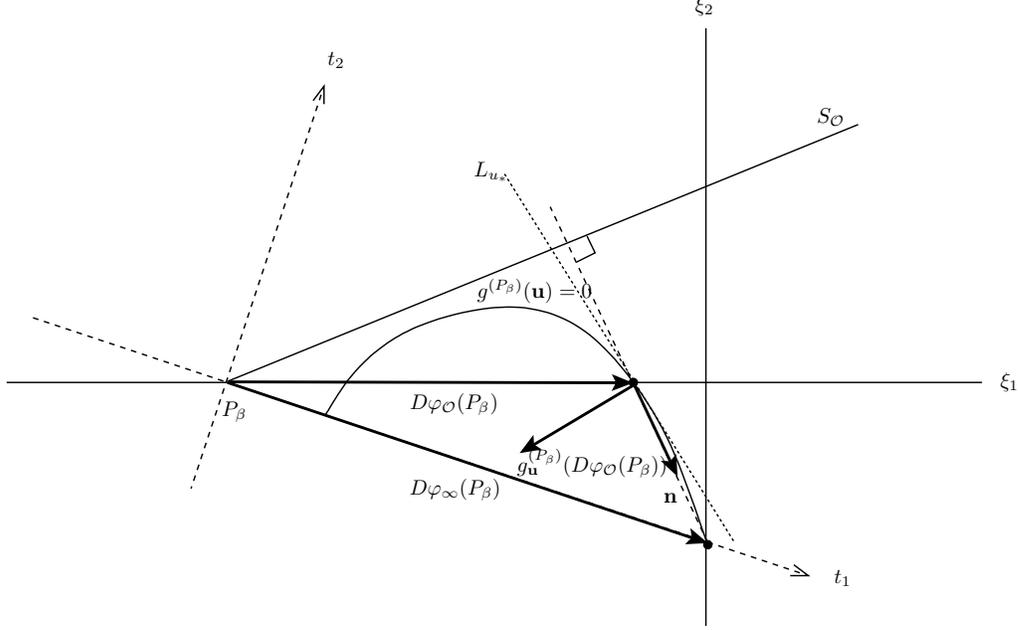
FIGURE 3.1. The graph of curve $g^{(P_\beta)}(\mathbf{u}) = 0$

Note that $g_{\mathbf{u}}^{(P_\beta)}(D\varphi_{\mathcal{O}}(P_\beta))$ is a normal vector of curve $(u, f_{\text{polar}}^{(P_\beta)}(u))$ at $u = u_*$. Let L_{u_*} be the tangent line of curve $(u, f_{\text{polar}}^{(P_\beta)}(u))$ at $u = u_*$. Then $g_{\mathbf{u}}^{(P_\beta)}(D\varphi_{\mathcal{O}}(P_\beta))$ is perpendicular to L_{u_*}. Let $\mathbf{n}_*$ be the unit normal vector to L_{u_*} with $\mathbf{n}_* \cdot \mathbf{e}^{\perp}(P_\beta) > 0$. Then $\mathbf{n}_* \cdot \mathbf{n} < 0$ for $\mathbf{n} = \frac{D\varphi_\infty - D\varphi_{\mathcal{O}}}{|D\varphi_\infty - D\varphi_{\mathcal{O}}|}$, owing to the convexity of curve $(u, f_{\text{polar}}^{(P_\beta)}(u))$.

It follows from (A.9) that $g_{\mathbf{u}}^{(P_\beta)}(D\varphi_{\mathcal{O}}(P_\beta)) \cdot \mathbf{n}_* = -|g_{\mathbf{u}}^{(P_\beta)}(D\varphi_{\mathcal{O}}(P_\beta))| < 0$ (see Fig. 3.1). This implies that

$$\frac{g_{\mathbf{u}}^{(P_\beta)}(u, f_{\text{polar}}^{(P_\beta)}(u))}{|g_{\mathbf{u}}^{(P_\beta)}(u, f_{\text{polar}}^{(P_\beta)}(u))|} = \frac{(\frac{\mathrm{d}}{\mathrm{d}u} f_{\text{polar}}^{(P_\beta)}(u), -1)}{\sqrt{1 + (\frac{\mathrm{d}}{\mathrm{d}u} f_{\text{polar}}^{(P_\beta)}(u))^2}},$$

and

$$(3.5.77) \qquad \text{sgn}\big(g_{\mathbf{u}}^{(P_\beta)}(u, f_{\text{polar}}^{(P_\beta)}(u)) \cdot \mathbf{e_1}\big) = \text{sgn}\big(\frac{\mathrm{d}}{\mathrm{d}u} f_{\text{polar}}^{(P_\beta)}(u)\big) \qquad \text{for } \hat{u}_0^{(P_\beta)} < u < u_\infty^{(P_\beta)},$$

where we have continued to work in the (t_1, t_2)–coordinates with basis $\{\mathbf{e}(P_\beta), \mathbf{e}^{\perp}(P_\beta)\}$.

By the convexity of curve $(u, f_{\text{polar}}^{(P_\beta)}(u))$, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2} f_{\text{polar}}^{(P_\beta)}(u) \leq 0 \qquad \text{for } \hat{u}_0^{(P_\beta)} < u < u_\infty^{(P_\beta)}.$$

Then, from (3.5.75)–(3.5.77), we obtain

$$g_{\mathbf{u}}^{(P_\beta)}(u, f_{\text{polar}}^{(P_\beta)}(u)) \cdot \mathbf{e_1} \leq -M_0^{-1} \qquad \text{for } \partial_{\mathbf{e}(P_\beta)} \varphi_{\mathcal{O}}(P_\beta) \leq u_\infty^{(P_\beta)}.$$

Note that $(P_\beta, D\varphi_{\mathcal{O}}(P_\beta), \mathbf{e}(P_\beta))$ and the shock polar curve $(u, f_{\mathrm{polar}}^{(P_\beta)}(u))$ depend smoothly on $\beta \in [\beta_{\mathrm{s}}^{(v_\infty)}, \beta_{\mathrm{d}}^{(v_\infty)}]$ (for further details, see Lemma A.3 or [**11**, Claim 16.6.7]). Therefore, there exists a small constant $\varepsilon_1 > 0$ depending only on $(\gamma, v_\infty, \bar{\sigma})$ so that

$$(3.5.78) \qquad g_{\mathbf{u}}^{(P_\beta)}(u, f_{\mathrm{polar}}^{(P_\beta)}(u)) \cdot \mathbf{e_1} \leq -\frac{1}{2M_0} \qquad \text{for } \partial_{\mathbf{e}(P_\beta)} \varphi_{\mathcal{O}}(P_\beta) - \varepsilon_1 \leq u < u_\infty^{(P_\beta)},$$

where $\beta \in [\beta_{\mathrm{s}}^{(v_\infty)}, \beta_{\mathrm{d}}^{(v_\infty)} - \bar{\sigma}]$.

3. For $u(D\varphi, \boldsymbol{\xi})$ given by (3.5.71), we define

$$(3.5.79) \qquad \mathbf{q}_\beta(u(D\varphi, \boldsymbol{\xi})) := u(D\varphi, \boldsymbol{\xi})\mathbf{e}(P_\beta) + f_{\mathrm{polar}}^{(P_\beta)}(u(D\varphi, \boldsymbol{\xi}))\mathbf{e}^\perp(P_\beta),$$

provided that $\hat{u}_0^{(P_\beta)} < u(D\varphi, \boldsymbol{\xi}) < u_\infty^{(P_\beta)}$ holds.

By the definitions of g^{sh} and $g^{(P_\beta)}$ given in (3.4.13) and (3.5.74), respectively, we have

$$(3.5.80) \qquad \partial_{p_1} g^{\mathrm{sh}}(\mathbf{q}_\beta(u), \varphi_\infty(P_\beta), P_\beta) = g_{\mathbf{u}}^{(P_\beta)}(\mathbf{q}_\beta(u(D\varphi, \boldsymbol{\xi}))) \cdot \mathbf{e}_1.$$

Since $\varphi - \varphi_\infty = 0$ holds on $\overline{\Gamma_{\mathrm{shock}}}$, we have

$$(3.5.81) \quad \begin{aligned} \partial_{p_1} g^{\mathrm{sh}}(D\varphi(\boldsymbol{\xi}), \varphi(\boldsymbol{\xi}), \boldsymbol{\xi}) &\leq \partial_{p_1} g^{\mathrm{sh}}(\mathbf{q}_\beta(u), \varphi_\infty(P_\beta), P_\beta) \\ &+ |\partial_{p_1} g_{\mathrm{mod}}^{\mathrm{sh}}(D\varphi(\boldsymbol{\xi}), \varphi_\infty(\boldsymbol{\xi}), \boldsymbol{\xi}) - \partial_{p_1} g_{\mathrm{mod}}^{\mathrm{sh}}(\mathbf{q}_\beta(u), \varphi_\infty(P_\beta), P_\beta))|, \end{aligned}$$

where $u = u(D\varphi, \boldsymbol{\xi})$ for $\boldsymbol{\xi} \in \Gamma_{\mathrm{shock}}$.

4. *Claim: There exist a small constant $r_1 > 0$ and a constant $C > 0$ so that, if $r \in (0, r_1]$ and φ is an admissible solution corresponding to $\beta \in [\beta_{\mathrm{s}}^{(v_\infty)}, \beta_{\mathrm{d}}^{(v_\infty)} - \bar{\sigma}]$, then*

$$(3.5.82) \qquad \partial_{\mathbf{e}(\boldsymbol{\xi})} \varphi(\boldsymbol{\xi}) \geq \partial_{\mathbf{e}(P_\beta)} \varphi_{\mathcal{O}}(P_\beta) - \varepsilon_1 \qquad \text{on } \overline{\Gamma_{\mathrm{shock}}} \cap B_{r_1}(P_\beta)$$

for constant $\varepsilon_1 > 0$ from (3.5.78).

Similarly to (3.1.16), define a cone generated by vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ by

$$\mathrm{cone}(\mathbf{u}, \mathbf{v}) := \{\alpha_1 \mathbf{u} + \alpha_2 \mathbf{v} : \alpha_1, \alpha_2 \geq 0\}.$$

For each $\beta \in [\beta_{\mathrm{s}}^{(v_\infty)}, \beta_{\mathrm{d}}^{(v_\infty)})$, it is clear that

$$(3.5.83) \qquad \mathbf{e}(P_\beta) \in \mathrm{cone}(\boldsymbol{e}_{S_{\mathcal{O}}}, -\mathbf{e}_2)$$

for $\boldsymbol{e}_{S_{\mathcal{O}}} = (\cos\beta, \sin\beta)$ and $\mathbf{e}_2 = (0, 1)$. We also find from (2.4.1) that

$$\boldsymbol{e}_{S_{\mathcal{O}}} \cdot \mathbf{e}(P_\beta) = \frac{c_{\mathcal{O}} M_{\mathcal{O}} \cot\beta}{|D\varphi_\infty(P_\beta)|} > 0, \qquad -\mathbf{e}_2 \cdot \mathbf{e}(P_\beta) = \frac{v_\infty}{|D\varphi_\infty(P_\beta)|} > 0$$

for $M_{\mathcal{O}}$ defined by (2.4.6). Moreover, $\boldsymbol{e}_{S_{\mathcal{O}}} \cdot \mathbf{e}(P_\beta)$ and $-\mathbf{e}_2 \cdot \mathbf{e}(P_\beta)$ depend continuously on β. Thus, there exists a constant $\kappa_0 > 0$ such that

$$\min_{\beta \in [\beta_{\mathrm{s}}^{(v_\infty)}, \beta_{\mathrm{d}}^{(v_\infty)}]} \{\boldsymbol{e}_{S_{\mathcal{O}}} \cdot \mathbf{e}(P_\beta), -\mathbf{e}_2 \cdot \mathbf{e}(P_\beta)\} \geq \kappa_0.$$

Therefore, we can fix a small constant $r_1 > 0$ so that

$$(3.5.84) \qquad \min_{\boldsymbol{\xi} \in \overline{B_{r_1}(P_\beta)}} \min\{\boldsymbol{e}_{S_{\mathcal{O}}} \cdot \mathbf{e}(\boldsymbol{\xi}), -\mathbf{e}_2 \cdot \mathbf{e}(\boldsymbol{\xi})\} \geq \frac{\kappa_0}{2} \qquad \text{for all } \beta \in [\beta_{\mathrm{s}}^{(v_\infty)}, \beta_{\mathrm{d}}^{(v_\infty)}].$$

By (3.5.83) and Lemmas 3.5–3.6, there exists a constant $C_\sharp > 0$ such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta_{\text{s}}^{(v_\infty)} \leq \beta \leq \beta_{\text{d}}^{(v_\infty)} - \bar{\sigma}\}$ satisfies

$$
\begin{aligned}
\partial_{\mathbf{e}(\boldsymbol{\xi})}\varphi(\boldsymbol{\xi}) &= \partial_{\mathbf{e}(\boldsymbol{\xi})}(\varphi - \varphi_{\mathcal{O}})(\boldsymbol{\xi}) + \partial_{\mathbf{e}(\boldsymbol{\xi})}\varphi_{\mathcal{O}}(\boldsymbol{\xi}) \\
&\geq \partial_{\mathbf{e}(P_\beta)}\varphi_{\mathcal{O}}(P_\beta) - C_\sharp|\boldsymbol{\xi} - P_\beta| \qquad \text{for } \boldsymbol{\xi} \in \overline{\Gamma_{\text{shock}}}.
\end{aligned}
\tag{3.5.85}
$$

We choose a constant $r_1 > 0$ depending only on $(v_\infty, \gamma, \bar{\sigma})$ to satisfy $C_\sharp r_1 \leq \frac{\varepsilon_1}{2}$ so that (3.5.82) follows directly from (3.5.85). The claim is verified.

5. *Claim: There exists a small constant* $r_2 \in (0, r_1]$ *depending only on* $(\gamma, v_\infty, \bar{\sigma})$ *so that, if* φ *is an admissible solution corresponding to* $\beta \in [\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{d}}^{(v_\infty)} - \bar{\sigma}]$, *then*

$$
|D\varphi(\boldsymbol{\xi}) - \mathbf{q}_\beta(u(D\varphi, \boldsymbol{\xi}))| \leq C|\boldsymbol{\xi} - P_\beta| \qquad \text{for all } \boldsymbol{\xi} \in \overline{\Gamma_{\text{shock}}} \cap B_{r_2}(P_\beta).
\tag{3.5.86}
$$

Define

$$
\mu_2 := \min_{\beta \in [\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{d}}^{(v_\infty)}]} \left(u_\infty^{(P_\beta)} - u_{\text{s}}^{(P_\beta)}\right).
$$

Such a constant μ_2 is positive, depending only on (v_∞, γ). Choose a small constant $\hat{r}_2 \in (0, r_1]$ so that $|u_{\text{s}}^{(\boldsymbol{\xi})} - u_{\text{s}}^{(P_\beta)}| \leq \frac{\mu_2}{4}$ for all $\boldsymbol{\xi} \in B_{\hat{r}_2}(P_\beta)$. Then we obtain from (3.5.72) and (3.5.82) that

$$
\partial_{\mathbf{e}(P_\beta)}\varphi_{\mathcal{O}}(P_\beta) - \varepsilon_1 \leq u(D\varphi, \boldsymbol{\xi}) \leq u_\infty^{(P_\beta)} - \frac{\mu_2}{2} \qquad \text{on } \overline{\Gamma_{\text{shock}}} \cap B_{\hat{r}_2}(P_\beta).
\tag{3.5.87}
$$

By Lemma 3.5, (3.5.70), and (3.5.79), we have

$$
|D\varphi(\boldsymbol{\xi}) - \mathbf{q}_\beta(u(D\varphi, \boldsymbol{\xi}))| \leq C\big(|\boldsymbol{\xi} - P_\beta| + |(f_{\text{polar}}^{(P_\beta)} - f_{\text{polar}}^{(\boldsymbol{\xi})})(u(D\varphi, \boldsymbol{\xi}))|\big)
\tag{3.5.88}
$$

on $\overline{\Gamma_{\text{shock}}} \cap B_{\hat{r}_2}(P_\beta)$.

By the continuous dependence of $(\hat{u}_0^{(\boldsymbol{\xi})}, u_\infty^{(\boldsymbol{\xi})})$ and the smooth dependence of $f_{\text{polar}}^{(\boldsymbol{\xi})}(u)$ on $\boldsymbol{\xi} \in \mathbb{R}^2 \setminus B_1(O_\infty)$ for $u \in (\hat{u}_0^{(\boldsymbol{\xi})}, u_\infty^{(\boldsymbol{\xi})})$ due to Lemma A.3, and by (3.5.87) and the continuous dependence of P_β on $\beta \in [\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{d}}^{(v_\infty)}]$, there exist $C > 0$ and $r_2 \in (0, \hat{r}_2]$ depending only on $(v_\infty, \gamma, \bar{\sigma})$ such that

$$
|(f_{\text{polar}}^{(P_\beta)} - f_{\text{polar}}^{(\boldsymbol{\xi})})(u(D\varphi, \boldsymbol{\xi}))| \leq C|\boldsymbol{\xi} - P_\beta| \qquad \text{on } \overline{\Gamma_{\text{shock}}} \cap B_{r_2}(P_\beta).
\tag{3.5.89}
$$

Then (3.5.86) follows directly from (3.5.88)–(3.5.89).

6. By (3.5.78), (3.5.80), and (3.5.87), we have

$$
\partial_{p_1} g^{\text{sh}}(\mathbf{q}_\beta(D\varphi, \boldsymbol{\xi}), \varphi_\infty(P_\beta), P_\beta) \leq -\frac{1}{2M_0} \qquad \text{for } \boldsymbol{\xi} \in \overline{\Gamma_{\text{shock}}} \cap B_{r_2}(P_\beta)
\tag{3.5.90}
$$

for any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta_{\text{s}}^{(v_\infty)} \leq \beta \leq \beta_{\text{d}}^{(v_\infty)} - \bar{\sigma}\}$.

By Lemma 3.8, (3.4.14), and (3.5.86), there exists a constant $C_{\text{polar}} > 0$ such that

$$
\big|\partial_{p_1} g_{\text{mod}}^{\text{sh}}(D\varphi(\boldsymbol{\xi}), \varphi_\infty(\boldsymbol{\xi}), \boldsymbol{\xi}) - \partial_{p_1} g_{\text{mod}}^{\text{sh}}(\mathbf{q}_\beta(u), \varphi_\infty(P_\beta), P_\beta))\big| \leq C_{\text{polar}}|\boldsymbol{\xi} - P_\beta|
\tag{3.5.91}
$$

for $\boldsymbol{\xi} \in \overline{\Gamma_{\text{shock}}} \cap B_{r_2}(P_\beta)$.

Choosing

$$
r := \min\{r_2, \frac{1}{4M_0 C_{\text{polar}}}\},
$$

we conclude from (3.5.81) and (3.5.90)–(3.5.91) that

$$\partial_{p_1} g^{\text{sh}}(D\varphi(\boldsymbol{\xi}), \varphi(\boldsymbol{\xi}), \boldsymbol{\xi}) \leq -\frac{1}{4M_0} \qquad \text{on } \overline{\Gamma_{\text{shock}}} \cap B_r(P_\beta)$$

for any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta_{\text{s}}^{(v_\infty)} \leq \beta \leq \beta_{\text{d}}^{(v_\infty)} - \bar{\sigma}\}$. This completes the proof. $\square$

To simplify notations, let $\mathbf{e}_\beta$ denote $\mathbf{e}(P_\beta)$ for each $\beta \in [\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{d}}^{(v_\infty)})$, and let $\mathbf{e}_\beta^\perp$ be the unit vector obtained from rotating $\mathbf{e}_\beta$ by $\frac{\pi}{2}$ counterclockwise. By (3.2.27), (3.4.5), and (3.5.69), we have

$$\partial_{\mathbf{e}_\beta}(\varphi_\infty - \varphi)(\boldsymbol{\xi}) \geq d_1 + (\mathbf{e}_\beta - \mathbf{e}(\boldsymbol{\xi})) \cdot D(\varphi_\infty - \varphi)(\boldsymbol{\xi}) \qquad \text{for all } \boldsymbol{\xi} \in \mathcal{N}_\varepsilon(\Gamma_{\text{shock}}) \cap \Omega,$$

where constants d_1 and ε are from (3.4.5). Therefore, we can apply Lemma 3.5 to choose a constant $s_* > 0$ depending only on (v_∞, γ) such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta_{\text{s}}^{(v_\infty)} \leq \beta < \beta_{\text{d}}^{(v_\infty)}\}$ satisfies

$$(3.5.92) \qquad\qquad \partial_{\mathbf{e}_\beta}(\varphi_\infty - \varphi) \geq \frac{d_1}{8} \qquad \text{in } B_{2s_*}(P_\beta) \cap \Omega.$$

DEFINITION 3.41. Introduce the (S, T)–coordinates so that

(i) P_β becomes the origin in the (S, T)–coordinates,

(ii) $\mathbf{e}_\beta = (1, 0)_{(S,T)}$ and $\mathbf{e}_\beta^\perp = (0, 1)_{(S,T)}$.

In fact, the (S, T)–coordinates are the same as the (t_1, t_2)–coordinates in Fig. 3.1.

In the (S, T)–coordinates given by Definition 3.41, $S_\mathcal{O}$, Γ_{shock}, Γ_{wedge}, and Ω near P_β can be represented as

$$S_\mathcal{O} \cap B_{s^*}(P_\beta) = \{S = a_{S_\mathcal{O}}(\beta)T \ : \ T > 0\} \cap B_{s^*}(P_\beta),$$

$$\Gamma_{\text{shock}} \cap B_{s^*}(P_\beta) = \{S = f_{\mathbf{e}}(T) \ : \ T > 0\} \cap B_{s^*}(P_\beta),$$

$$\Gamma_{\text{wedge}} \cap B_{s^*}(P_\beta) = \{S = a_{\text{w}}(\beta)T \ : \ T > 0\} \cap B_{s^*}(P_\beta),$$

$$\Omega \cap B_{s^*}(P_\beta) = \{(S, T) \ : \ a_{\mathbf{e}_{S_\mathcal{O}}}(\beta)T \leq f_{\mathbf{e}}(T) < S < a_{\text{w}}(\beta)T, \ T > 0\} \cap B_{s^*}(P_\beta),$$

where $a_{\text{w}}(\beta)$ depends continuously on $\beta \in (0, \frac{\pi}{2})$, and $a_{S_\mathcal{O}}(\beta) = \tan\theta_\beta$ with $\theta_\beta := \tan^{-1} a_{\text{w}}(\beta) - \beta > 0$ for each $\beta \in (0, \frac{\pi}{2})$. Note that there is a constant $C > 0$ depending only on (v_∞, γ) such that $C^{-1} \leq a_{\text{w}}(\beta) \leq C$ for all $\beta \in [\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{d}}^{(v_\infty)})$. The representation of $\Gamma_{\text{shock}} \cap B_{s^*}(P_\beta)$ as a graph of $S = f_{\mathbf{e}}(T)$ is obtained by the implicit function theorem, combined with (3.5.92).

PROPOSITION 3.42. Let positive constants σ_3 and r be from Proposition 3.39 and Lemma 3.40, respectively. For small constants $\sigma_{\text{s}} \in (0, \frac{\sigma_3}{2}]$ and $\sigma_{\text{d}} \in (0, \frac{\beta_{\text{d}}^{(v_\infty)}}{10})$, there exist constants $s \in (0, r)$, $\alpha \in (0, 1)$, and $C > 0$ depending only on $(v_\infty, \gamma, \sigma_{\text{s}}, \sigma_{\text{d}})$ such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta_{\text{s}}^{(v_\infty)} + \sigma_{\text{s}} \leq \beta \leq \beta_{\text{d}}^{(v_\infty)} - \sigma_{\text{d}}\}$ satisfies the estimates:

$$\|\varphi\|_{2,\alpha,\Omega\cap B_s(P_\beta)}^{(-1-\alpha),\{P_\beta\}} + \|f_{\mathbf{e}}\|_{2,\alpha,(0,s)}^{(-1-\alpha),\{0\}} \leq C,$$

$$|D_{\boldsymbol{\xi}}^m(\varphi - \varphi_\mathcal{O})(P_\beta)| = 0 \qquad \text{for } m = 0, 1.$$

PROOF. In this proof, all the estimate constants are chosen depending only on $(v_\infty, \gamma, \sigma_{\text{s}}, \sigma_{\text{d}})$, unless otherwise specified. For fixed $\sigma_{\text{s}} \in (0, \frac{\sigma_3}{2}]$ and $\sigma_{\text{d}} \in (0, \frac{\beta_{\text{d}}^{(v_\infty)}}{10})$, let φ be an admissible solution for $\beta \in [\beta_{\text{s}}^{(v_\infty)} + \sigma_{\text{s}}, \beta_{\text{d}}^{(v_\infty)} - \sigma_{\text{d}}]$.

1. Denote $\bar{\phi} := \varphi_\infty - \varphi$, and rewrite Eq. (2.1.19) and the derivative boundary conditions (3.4.12) and (2.5.36) in terms of $\bar{\phi}$ as follows:

(3.5.93)
$$\sum_{i,j=1}^{2} A_{ij}(D\bar{\phi}, \bar{\phi}, \boldsymbol{\xi}) D_{ij}\bar{\phi} = 0 \qquad \text{in } B_{s^*}(P_\beta) \cap \Omega,$$
$$\hat{g}^{\text{sh}}(D\bar{\phi}, \bar{\phi}, \boldsymbol{\xi}) = 0 \qquad \text{on } \Gamma_{\text{shock}},$$
$$\hat{g}^{\text{w}}(D\bar{\phi}, \bar{\phi}, \boldsymbol{\xi}) = 0 \qquad \text{on } \Gamma_{\text{wedge}},$$

where

(3.5.94)
$$A_{ij}(\mathbf{p}, z, \boldsymbol{\xi}) = \hat{c}^2(\mathbf{p}, z, \boldsymbol{\xi})\delta_{ij} - (\partial_i\varphi_\infty - p_i)(\partial_j\varphi_\infty - p_j) \qquad \text{for } i, j = 1, 2,$$
$$\hat{c}^2(\mathbf{p}, z, \boldsymbol{\xi}) = 1 - (\gamma - 1)\Big(\frac{1}{2}|D\varphi_\infty - \mathbf{p}|^2 + \varphi_\infty - z\Big),$$
$$\hat{g}^{\text{sh}}(\mathbf{p}, z, \boldsymbol{\xi}) = -g^{\text{sh}}(D\varphi_\infty(\boldsymbol{\xi}) - \mathbf{p}, \varphi_\infty(\boldsymbol{\xi}) - z, \boldsymbol{\xi}),$$
$$\hat{g}^{\text{w}}(\mathbf{p}, z, \boldsymbol{\xi}) = p_2 + (\xi_2 + v_\infty),$$

where g^{sh} is given by (3.4.13) and $s^* \in (0, r]$ is from (3.5.92).

Next, we apply a partial hodograph transform to $\bar{\phi}$ in $B_{s^*}(P_\beta) \cap \Omega$ in the direction of $\mathbf{e}_\beta$. For each $(S, T) \in B_{s^*}(P_\beta) \cap \Omega$, define $\mathbf{y} = (y_1, y_2) = (\bar{\phi}(S, T), T)$. By (3.5.92), there exists a unique function $v(\mathbf{y})$ such that

(3.5.95)
$$v(y_1, y_2) = S \qquad \text{if and only if} \qquad \bar{\phi}(S, y_2) = y_1$$

for $\mathbf{y} \in \mathcal{D}^\beta_{s^*} := \{\mathbf{y} = (\bar{\phi}(S, T), T) : (S, T) \in B_{s^*}(P_\beta) \cap \Omega\}$. By taking derivatives of $v(\bar{\phi}(S, y_2), y_2) = S$, it can be directly checked that

(3.5.96)
$$\partial_{y_1}v = \frac{1}{\partial_S\bar{\phi}}, \quad \partial_{y_2}v = -\frac{\partial_T\bar{\phi}}{\partial_S\bar{\phi}}.$$

By Lemma 3.5, (3.5.92), and (3.5.95)–(3.5.96), there exists a constant $K > 1$ depending only on (γ, v_∞) such that

(3.5.97)
$$\frac{1}{K} \leq \partial_{y_1}v \leq \frac{8}{d_1}, \quad |v| + |Dv| < 2K \qquad \text{in } \overline{\mathcal{D}^\beta_{s^*}}.$$

Using the definition of v, (3.5.93) can be written in terms of v:

(3.5.98)
$$\sum_{i,j=1}^{2} a_{ij}(Dv, v, \mathbf{y})\partial_{y_iy_j}v = 0 \qquad \text{in } \mathcal{D}^\beta_{s^*},$$
$$g_{\text{h}}^{\text{sh}}(Dv, v, \mathbf{y}) = 0 \qquad \text{on } \Gamma^{(\text{h})}_{\text{shock}} = \{\mathbf{y} = (0, T) : (S, T) \in B_{s^*}(P_\beta) \cap \Gamma_{\text{shock}}\},$$
$$g_{\text{h}}^{\text{w}}(Dv, v, \mathbf{y}) = 0 \qquad \text{on } \Gamma^{(\text{h})}_{\text{wedge}} = \{\mathbf{y} = (\bar{\phi}(S, T), T) : (S, T) \in B_{s^*}(P_\beta) \cap \Gamma_{\text{wedge}}\},$$

where $(a_{ij}, g_{\mathrm{h}}^{\mathrm{sh}}, g_{\mathrm{h}}^{\mathrm{w}})(\mathbf{p}, z, \mathbf{y})$ are directly computed by using (3.5.94) and the definition of v. More precisely, $(a_{ij}, g_{\mathrm{h}}^{\mathrm{sh}}, g_{\mathrm{h}}^{\mathrm{w}})(\mathbf{p}, z, \mathbf{y})$ are given by

$$a_{11}(\mathbf{p}, z, \mathbf{y}) = \frac{1}{p_1^3}(A_{11} - 2p_2 A_{12} + p_2^2 A_{22}),$$

$$a_{12}(\mathbf{p}, z, \mathbf{y}) = a_{21}(\mathbf{p}, z, \mathbf{y}) = \frac{1}{p_1^2}(A_{12} - p_2 A_{22}),$$

$$a_{22}(\mathbf{p}, z, \mathbf{y}) = \frac{p_2}{p_1}A_{22},$$

$$(g_{\mathrm{h}}^{\mathrm{sh}}, g_{\mathrm{h}}^{\mathrm{w}})(\mathbf{p}, z, \mathbf{y}) = -(\hat{g}^{\mathrm{sh}}, \hat{g}^{\mathrm{w}}),$$

with

$$(A_{11}, A_{12}, A_{22}, \hat{g}^{\mathrm{sh}}, \hat{g}^{\mathrm{w}}) = (A_{11}, A_{12}, A_{22}, \hat{g}^{\mathrm{sh}}, \hat{g}^{\mathrm{w}})((\frac{1}{p_1}, -\frac{p_2}{p_1}), y_1, (z, y_2)).$$

Define a set

$$U := \{(\mathbf{p}, z, \mathbf{y}) \in \mathbb{R}^2 \times \mathbb{R} \times \mathcal{D}_{s^*}^\beta\}.$$

We fix a cut-off function $\zeta \in C^\infty(\mathbb{R})$ satisfying that $\zeta(t) \equiv 0$ on $(-\infty, \frac{1}{10K})$ and $\zeta(t) \equiv 1$ on $(\frac{1}{4K}, \infty)$. Furthermore, we define

$$(a_{ij}^{\mathrm{mod}}, g_{\mathrm{h}}^{\mathrm{sh,mod}}, g_{\mathrm{h}}^{\mathrm{w,mod}})(\mathbf{p}, z, \mathbf{y}) = \zeta(p_1)(a_{ij}, g_{\mathrm{h}}^{\mathrm{sh}}, g_{\mathrm{h}}^{\mathrm{w}})(\mathbf{p}, z, \mathbf{y}) \qquad \text{for } i, j = 1, 2.$$

Then (3.5.98) can be rewritten as

$$\text{(3.5.99)} \qquad \begin{aligned} \sum_{i,j=1}^2 a_{ij}^{\mathrm{mod}}(Dv, v, \mathbf{y})\partial_{y_i y_j} v &= 0 \qquad \text{in } \mathcal{D}_{s^*}^\beta, \\ g_{\mathrm{h}}^{\mathrm{sh,mod}}(Dv, v, \mathbf{y}) &= 0 \qquad \text{on } \Gamma_{\mathrm{shock}}^{(\mathrm{h})}, \\ g_{\mathrm{h}}^{\mathrm{w,mod}}(Dv, v, \mathbf{y}) &= 0 \qquad \text{on } \Gamma_{\mathrm{wedge}}^{(\mathrm{h})}. \end{aligned}$$

Furthermore, for any $l = 0, 1, 2, \cdots$, there exists a constant $C_l > 0$ depending only on (γ, v_∞, l) such that

$$\text{(3.5.100)} \qquad |D_{(\mathbf{p}, z, \mathbf{y})}^l (a_{ij}^{\mathrm{mod}}, g_{\mathrm{h}}^{\mathrm{sh,mod}}, g_{\mathrm{h}}^{\mathrm{w,mod}})| \leq C_l \qquad \text{on } U.$$

2. In this step, we apply Proposition C.12 to obtain

$$\text{(3.5.101)} \qquad |g_{\mathrm{h}}^{\mathrm{w}}(Dv(\mathbf{y}), v(\mathbf{y}), \mathbf{y}) - g_{\mathrm{h}}^{\mathrm{w}}(Dv(\mathbf{0}), v(\mathbf{0}), \mathbf{0})| \leq C|\mathbf{y}|^{\alpha_1} \qquad \text{for } \mathbf{y} \in \overline{\mathcal{D}_{s^*}^\beta \cap B_{l^*}(\mathbf{0})}$$

for some $\alpha_1 \in (0, 1)$, $C > 0$, and $l^* > 0$.

$\Gamma_{\mathrm{shock}}^{(\mathrm{h})}$ is flat so that it is C^2 up to its endpoints, and $\Gamma_{\mathrm{wedge}}^{(\mathrm{h})}$ is Lipschitz continuous up to its endpoints. Then we regard $\Gamma_{\mathrm{wedge}}^{(\mathrm{h})}$ and $\Gamma_{\mathrm{shock}}^{(\mathrm{h})}$ as Γ^1 and Γ^2, respectively, in Proposition C.12. Then $(g_{\mathrm{h}}^{\mathrm{w,mod}}, g_{\mathrm{h}}^{\mathrm{sh,mod}}, 0)$ in (3.5.99) become $(b^{(1)}, b^{(2)}, h)$ in Proposition C.12. It follows directly from (3.5.100) that (3.5.99) satisfies conditions (C.4.5)–(C.4.8).

Also, (3.5.97) implies that v satisfies condition (C.4.1) stated in Proposition C.12.

A direct computation by using the definition of v in (3.5.95) yields that

$$|D_{\mathbf{p}}g_{\mathrm{h}}^{\mathrm{w}}(Dv(\mathbf{y}), v(\mathbf{y}), \mathbf{y})| = |\frac{1}{v_{y_1}^2}(v_{y_2}, -v_{y_1})| \geq \frac{1}{|v_{y_1}|} = |\bar{\phi}_S| \qquad \text{for all } \mathbf{y} \in \overline{\mathcal{D}_{s^*}^\beta}.$$

Thus, (3.5.92) implies that

$$|D_{\mathbf{p}} g^{\text{w}}_{\text{h}}(Dv(\mathbf{y}), v(\mathbf{y}), \mathbf{y})| \geq \frac{d_1}{8} \qquad \text{for all } \mathbf{y} \in \overline{\mathcal{D}^{\beta}_{s^*}}.$$

This shows that $b^{(1)} = g^{\text{w}}_{\text{h}}$ satisfies condition (ii) of Proposition C.12. By (3.4.13), (A.18), Lemma 3.5, Remark 3.14, and Proposition 3.15, there exists a constant $\lambda_1 > 0$ depending only on $(v_{\infty}, \gamma, \sigma_{\text{s}})$ such that any admissible solution φ for $\beta \in [\beta^{(v_\infty)}_{\text{s}} + \sigma_{\text{s}}, \beta^{(v_\infty)}_{\text{d}})$ satisfies

$$D_{\mathbf{p}} g^{\text{sh}}_{\text{mod}}(D\varphi(\boldsymbol{\xi}), \varphi(\boldsymbol{\xi}), \boldsymbol{\xi}) \cdot \boldsymbol{\nu}_{\text{s}}(\boldsymbol{\xi}) \geq \lambda_1 \qquad \text{for all } \boldsymbol{\xi} \in \overline{\Gamma_{\text{shock}}} \cap B_{s^*}(P_\beta),$$

where $\boldsymbol{\nu}_{\text{s}}$ is the unit normal vector to Γ_{shock} towards the interior of Ω. Then a direct computation by using (3.5.92) and (3.5.94)–(3.5.95) shows that

$$\partial_{p_1} \hat{g}^{\text{sh}}(Dv(\mathbf{y}), v(\mathbf{y}), \mathbf{y}) = |D\bar{\phi}| \bar{\phi}_S D_{\mathbf{p}} g^{\text{sh}}_{\text{mod}}(D\varphi(\boldsymbol{\xi}), \varphi(\boldsymbol{\xi}), \boldsymbol{\xi}) \cdot \boldsymbol{\nu}_s(\boldsymbol{\xi}) \geq \lambda_1 \left(\frac{d_1}{8}\right)^2 \qquad \text{on } \Gamma^{(\text{h})}_{\text{shock}}.$$

This implies that $b^{(2)} = g^{\text{sh}}_{\text{h}}$ satisfies condition (iii) of Proposition C.12. In order to apply Proposition C.12, we also need to show that $(b^{(1)}, b^{(2)}) = (g^{\text{w}}_{\text{h}}, g^{\text{sh}}_{\text{h}})$ satisfies condition (iv). A direct computation by using Lemma 3.40, (3.5.92), and (3.5.94)–(3.5.95) yields that
(3.5.102)
$$\left| \det \begin{pmatrix} D_{\mathbf{p}} g^{\text{sh}}_{\text{h}}(Dv(\mathbf{y}), v(\mathbf{y}), \mathbf{y}) \\ D_{\mathbf{p}} g^{\text{w}}_{\text{h}}(Dv(\mathbf{y}), v(\mathbf{y}), \mathbf{y}) \end{pmatrix} \right| = \bar{\phi}^3_S |\partial_{p_1} g^{\text{sh}}(D\varphi(\boldsymbol{\xi}), \varphi(\boldsymbol{\xi}), \boldsymbol{\xi})| \geq \frac{1}{M} \left(\frac{d_1}{3}\right)^3 \qquad \text{for } \mathbf{y} \in \Gamma^{(\text{h})}_{\text{shock}}$$

for constant M from Lemma 3.40. We have shown that condition (iv) of Proposition C.12 holds.

Then we apply Proposition C.12 to conclude that there exist constants $\alpha_1 \in (0, 1), C > 0$, and $l^* > 0$ depending only on $(v_{\infty}, \gamma, \sigma_{\text{s}}, \sigma_{\text{d}})$ such that (3.5.101) holds.

3. We know from (3.5.98) that v satisfies that $|g^{\text{sh}}_{\text{h}}(Dv(\mathbf{y}), v(\mathbf{y}), \mathbf{y}) - g^{\text{sh}}_{\text{h}}(Dv(\mathbf{0}), v(\mathbf{0}), \mathbf{0})| \equiv 0$ on $\Gamma^{(\text{h})}_{\text{shock}}$. This, combined with (3.5.101), implies that condition (C.4.12) stated in Proposition C.13 is satisfied with $\alpha = \alpha_1$. It follows from (3.5.100) that condition (C.4.9) holds. Also, (3.5.102) implies that v satisfies condition (C.4.10) with $\mathbf{y}_0 = \mathbf{0}$. Moreover, condition (C.4.11) holds for the line segment $\Gamma^{(\text{h})}_{\text{shock}}$. Therefore, we obtain from Proposition C.13 that

$$(3.5.103) \qquad |Dv(\mathbf{y}) - Dv(\mathbf{0})| \leq C|\mathbf{y}|^{\alpha_1} \qquad \text{for } \mathbf{y} \in \Gamma^{(\text{h})}_{\text{shock}} \cap B_{l^*}(\mathbf{0})$$

for a constant $C > 0$ depending only on $(v_{\infty}, \gamma, \sigma_{\text{s}}, \sigma_{\text{d}})$.

Since $\bar{\phi}(\mathbf{0}) = 0$ in the (S, T)–coordinates, then $|\mathbf{y}| \leq |\bar{\phi}(S, T) - \bar{\phi}(\mathbf{0})| + |T|$ for each $\mathbf{y} = (\bar{\phi}(S, T), T) \in \mathcal{D}^{\beta}_{s^*}$. We apply Lemma 3.5 to obtain

$$(3.5.104) \qquad |\mathbf{y}| \leq C|(S, T)| = C|\boldsymbol{\xi} - P_\beta|$$

for a constant $C > 0$ depending only on (γ, v_{∞}).

By (3.5.95), $|\boldsymbol{\xi} - P_\beta| = |(S, T)| \leq |v(\mathbf{y}) - v(\mathbf{0})| + |y_2|$ for each $(S, T) \in B_{s^*}(P_\beta) \cap \Omega$. Then we apply (3.5.97) to obtain

$$(3.5.105) \qquad |\boldsymbol{\xi} - P_\beta| = |(S, T)| \leq (2K + 1)|\mathbf{y}|$$

for constant K from (3.5.97).

We write (3.5.101) and (3.5.103) back in the $\boldsymbol{\xi}$–coordinates, and apply (3.5.104)–(3.5.105) to obtain

(3.5.106)
$$|\varphi_{\xi_2}(\boldsymbol{\xi}) - \varphi_{\xi_2}(P_\beta)| \leq C|\boldsymbol{\xi} - P_\beta|^{\alpha_1} \qquad \text{in } \overline{\Omega \cap B_{s_1}(P_\beta)},$$
$$|D\varphi(\boldsymbol{\xi}) - D\varphi(P_\beta)| \leq C|\boldsymbol{\xi} - P_\beta|^{\alpha_1} \qquad \text{on } \Gamma_{\text{shock}} \cap B_{s_1}(P_\beta),$$

where $C > 0$ and $s_1 \in (0, s^*]$ depend only on $(v_\infty, \gamma, \sigma_s, \sigma_d)$.

For the rest of proof, each estimate constant is chosen depending only on $(v_\infty, \gamma, \sigma_s, \sigma_d)$, unless otherwise specified. For $\boldsymbol{\xi} \in \overline{\Omega}$, define $\mathfrak{f}(\boldsymbol{\xi}) := \boldsymbol{\tau}_w \cdot (D\bar{\phi}(\boldsymbol{\xi}) - D\bar{\phi}(P_\beta))$ for the unit tangent vector $\boldsymbol{\tau}_w = (1, 0)$ to Γ_{wedge}. Then (3.5.106) implies that

$$(3.5.107) \qquad |\mathfrak{f}(\boldsymbol{\xi}) - \mathfrak{f}(P_\beta)| \leq \hat{C} |\boldsymbol{\xi} - P_\beta|^{\alpha_1} \qquad \text{for } \boldsymbol{\xi} \in \Gamma_{\text{shock}} \cap B_{s_1}(P_\beta).$$

Denote $g_*^{\text{sh}}(\mathbf{p}) := \boldsymbol{\tau}_w \cdot (\mathbf{p} - D\bar{\phi}(P_\beta))$, and regard $g_*^{\text{sh}}(D\bar{\phi}) = \mathfrak{f}$ as a boundary condition for φ on Γ_{shock}. Since Γ_{wedge} is flat in the $\boldsymbol{\xi}$–coordinates, we can apply Proposition C.12 by setting $(\Gamma^1, \Gamma^2) := (\Gamma_{\text{shock}}, \Gamma_{\text{wedge}})$ and $(b^{(1)}, b^{(2)}) := (g_*^{\text{sh}}, \hat{g}^w)$ for $\Gamma^j, b^{(j)}, j = 1, 2$, from Proposition C.12. In particular, condition (C.4.8) holds with $\beta = \alpha_1$, owing to (3.5.107). Then we obtain constants $\alpha \in (0, \alpha_1], C > 0$, and $s_2 \in (0, s_1]$ such that

$$|g_*^{\text{sh}}(D\varphi(\boldsymbol{\xi})) - g_*^{\text{sh}}(D\varphi(P_\beta))| \leq C |\boldsymbol{\xi} - P_\beta|^\alpha \qquad \text{for } \boldsymbol{\xi} \in \overline{\Omega \cap B_{s_2}(P_\beta)}.$$

Combining this with (3.5.106), and noting that both boundary conditions $\hat{g}_w$ and g_*^{sh} are linear with constant coefficients and are linearly independent of each other, we finally have

$$(3.5.108) \qquad |D\varphi(\boldsymbol{\xi}) - D\varphi(P_\beta)| \leq C^* |\boldsymbol{\xi} - P_\beta|^\alpha \qquad \text{for } \boldsymbol{\xi} \in \overline{\Omega \cap B_{s_2}(P_\beta)}.$$

4. For each $\boldsymbol{\xi} \in \Gamma_{\text{shock}}$, define $d(\boldsymbol{\xi}) := |\boldsymbol{\xi} - P_\beta|$.

Claim: There exist constants $\omega_0 > 0$ and $s_3 \in (0, s_2]$ such that, for all $\boldsymbol{\xi} \in \Gamma_{\text{shock}} \cap B_{s_3}(P_\beta)$,

$$\text{dist}(\boldsymbol{\xi}, \Gamma_{\text{wedge}}) \geq \omega_0 \, d(\boldsymbol{\xi}).$$

If this claim holds, then $\Omega_{s_3} = \Omega \cap B_{s_3}(P_\beta)$ satisfies condition (ii) of Proposition C.14 so that Proposition 3.42 follows from (3.5.108) and Proposition C.14, where we use (3.5.108) to satisfy condition (C.4.13) stated in Proposition C.14.

Now we show the claim. For a fixed point $P \in \Gamma_{\text{shock}}$, let P' be the point on $S_{\mathcal{O}}$ so that $PP' \perp \Gamma_{\text{wedge}}$. Then

$$(3.5.109) \qquad \text{dist}(P, \Gamma_{\text{wedge}}) = d(P') \sin \beta - |P' - P| \geq d(P) \sin \beta - |P' - P|.$$

Denote $P = (\xi_1^P, \xi_2^P)$ and $P' = (\xi_1^{P'}, \xi_2^{P'})$ in the $\boldsymbol{\xi}$–coordinates. Then we see that $P' - P = (0, \xi_2^{P'} - \xi_2^P)$. Since $P' \in S_{\mathcal{O}}$ and $P \in \Gamma_{\text{shock}}$, $(\varphi_\infty - \varphi_{\mathcal{O}})(P') = (\varphi_\infty - \varphi)(P) = 0$ so that

$$v_\infty |\xi_2^{P'} - \xi_2^P| = |(\varphi_\infty - \varphi_{\mathcal{O}})(P') - (\varphi_\infty - \varphi_{\mathcal{O}})(P)| = |(\varphi_{\mathcal{O}} - \varphi)(P)|.$$

Since $(\varphi_{\mathcal{O}} - \varphi)(P_\beta) = 0$ by (2.5.12), the equation above gives

$$|P' - P| = \frac{1}{v_\infty} |(\varphi_{\mathcal{O}} - \varphi)(P) - (\varphi_{\mathcal{O}} - \varphi)(P_\beta)|.$$

Then we apply (3.5.108) to obtain

$$|P' - P| = \frac{1}{v_\infty} |(\varphi_{\mathcal{O}} - \varphi)(P)| \leq Cd(P)^{1+\alpha} \qquad \text{for } P \in \overline{\Omega \cap B_{s_2}(P_\beta)}$$

for some constant $C > 0$. Combining this estimate with (3.5.109), we can choose constants $\omega_0 > 0$ and $s_3 \in (0, s_2]$ so that the claim holds.

Then Proposition C.14, combined with (3.5.68) and the results from Steps 3–4, leads to Proposition 3.42. $\qquad\square$

3.6. Compactness of the Set of Admissible Solutions

Fix $\gamma \geq 1$, $v_\infty > 0$, and $\bar{\beta} \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$. According to all the *a priori* estimates obtained in Lemma 3.18, Corollary 3.19, Propositions 3.26, 3.30, 3.39, and 3.42, there exists $\bar{\alpha} \in (0, 1)$ depending only on $(v_\infty, \gamma, \bar{\beta})$ such that the set:

$$\left\{ \|\varphi\|_{C^{1,\bar{\alpha}}(\overline{\Omega})} + \|\Gamma_{\mathrm{shock}}\|_{C^{1,\bar{\alpha}}} \ : \ \begin{array}{l} \varphi \text{ is an admissible solution corresponding} \\ \text{to } (v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{0 \leq \beta \leq \bar{\beta}\} \end{array} \right\}$$

is bounded. For each admissible solution, its pseudo-subsonic region Ω is a bounded domain enclosed by $\Gamma_{\mathrm{sonic}}^{\mathcal{O}}$, $\Gamma_{\mathrm{sonic}}^{\mathcal{N}}$, Γ_{shock}, and Γ_{wedge}. These four curves intersect only at P_j for $j = 1, 2, 3, 4$. According to Definition 2.23, $\Gamma_{\mathrm{sonic}}^{\mathcal{N}}$, $O_{\mathcal{N}}$, P_2, and P_3 are fixed so as to be the same for all admissible solutions. Moreover, $\Gamma_{\mathrm{sonic}}^{\mathcal{O}}$, $O_{\mathcal{O}}$, P_1, and P_4 depend continuously on $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]$. From this observation, the following lemma is obtained:

LEMMA 3.43. *Fix $\gamma \geq 1$, $v_\infty > 0$, and $\bar{\beta} \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$. For each $\beta \in [0, \bar{\beta}]$, let Λ_β be defined by Definition 2.23. Let $\{\varphi^{(j)}\}$ be a sequence of admissible solutions corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{0 \leq \beta \leq \bar{\beta}\}$, and let $\lim_{j \to \infty} \beta_j = \beta_\infty$ for some $\beta_\infty \in [0, \bar{\beta}]$. For each j, let $\Omega^{(j)}$ and $\Gamma_{\mathrm{shock}}^{(j)}$ be the pseudo-subsonic region and the curved pseudo-transonic shock of $\varphi^{(j)}$, respectively. Then there exists a subsequence $\{\varphi^{(j_k)}\} \subset \{\varphi^{(j)}\}$ such that the following properties hold:*

(a) *$\{\varphi^{(j_k)}\}$ converges uniformly on any compact subset of $\overline{\Lambda_{\beta_\infty}}$ to a function $\varphi^{(\infty)} \in C_{\mathrm{loc}}^{0,1}(\overline{\Lambda_{\beta_\infty}})$, and $\varphi^{(\infty)}$ is an admissible solution corresponding to (v_∞, β_∞);*

(b) *$\Omega^{(j_k)} \to \Omega^{(\infty)}$ in the Hausdorff metric;*

(c) *If $\boldsymbol{\xi}^{(j_k)} \in \overline{\Omega^{(j_k)}}$, and $\boldsymbol{\xi}^{(j_k)}$ converges to $\boldsymbol{\xi}^{(\infty)} \in \overline{\Omega^{(\infty)}}$, then*

$$\varphi^{(j_k)}(\boldsymbol{\xi}^{(j_k)}) \to \varphi^{(\infty)}(\boldsymbol{\xi}^{(\infty)}), \qquad D\varphi^{(j_k)}(\boldsymbol{\xi}^{(j_k)}) \to D\varphi^{(\infty)}(\boldsymbol{\xi}^{(\infty)}),$$

where, in the case of $\boldsymbol{\xi}^{(j_k)} \in \Gamma_{\mathrm{shock}}^{(j_k)}$, $D\varphi^{(j_k)}(\boldsymbol{\xi}^{(j_k)}) := \lim_{\boldsymbol{\xi} \in \Omega^{(j_k)}, \boldsymbol{\xi} \to \boldsymbol{\xi}^{(j_k)}} D\varphi^{(j_k)}(\boldsymbol{\xi})$, and $D\varphi^{(\infty)}(\boldsymbol{\xi})$ for $\boldsymbol{\xi} \in \Gamma_{\mathrm{shock}}^{(\infty)}$ is defined similarly.

Iteration Set

In order to prove the existence of admissible solutions in the sense of Definition 2.24 for all $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ by employing the Leray-Schauder degree for a fixed point, we first introduce the iteration set.

4.1. Mapping the Admissible Solutions to the Functions Defined in $\mathcal{Q}^{\text{iter}}$

Fix $\gamma \geq 1$ and $v_\infty > 0$. We continue to follow Definition 2.23 for the notations: O_∞, $O_\mathcal{O}$, $O_\mathcal{N}$, $\Gamma^\mathcal{N}_{\text{sonic}}$, $\Gamma^\mathcal{O}_{\text{sonic}}$, and P_j for $j = 1, 2, 3, 4$, *etc.*. Denote $\mathcal{Q}^{\text{iter}} = (-1, 1) \times (0, 1)$.

DEFINITION 4.1. Let $(\varphi_\infty, \varphi_\mathcal{N}, \varphi_\mathcal{O})$ be defined by (2.5.1).
(i) *Definition of $\hat{c}_\mathcal{O}$.* For each $\beta \in [0, \beta_{\text{d}}^{(v_\infty)}]$, define $\hat{c}_\mathcal{O}$ by

$$\hat{c}_\mathcal{O} := \text{dist}(\Gamma^\mathcal{O}_{\text{sonic}}, O_\mathcal{O}) = \begin{cases} c_\mathcal{O} & \text{for } \beta < \beta_{\text{s}}^{(v_\infty)}, \\ |O_\mathcal{O} P_\beta| & \text{for } \beta \geq \beta_{\text{s}}^{(v_\infty)}. \end{cases}$$

Note that $\hat{c}_\mathcal{O} < c_\mathcal{O}$ if $\beta > \beta_{\text{s}}^{(v_\infty)}$.

(ii) *Extended sonic arcs.* Since $\hat{c}_\mathcal{O}$ depends continuously on $\beta \in [0, \frac{\pi}{2})$, a constant $\delta_0 > 0$ can be chosen depending only on (v_∞, γ) such that $S_\mathcal{N}^{\delta_0} = \{\boldsymbol{\xi} \in \mathbb{R}^2 : (\varphi_\infty - \varphi_\mathcal{N})(\boldsymbol{\xi}) = -\delta_0\}$ and $\partial B_{c_\mathcal{N}}(O_\mathcal{N})$ intersect at two distinct points, and $S_\mathcal{O}^{\delta_0} = \{\boldsymbol{\xi} \in \mathbb{R}^2 : (\varphi_\infty - \varphi_\mathcal{O})(\boldsymbol{\xi}) = -\delta_0\}$ and $\partial B_{\hat{c}_\mathcal{O}}(O_\mathcal{O})$ intersect at two distinct points for each $\beta \in [0, \beta_{\text{d}}^{(v_\infty)}]$. Let $\Gamma^{\mathcal{O}, \delta_0}_{\text{sonic}}$ be the smaller arc lying on $\partial B_{\hat{c}_\mathcal{O}}(O_\mathcal{O})$ with endpoints P_4 and P_1', where P_1' is the intersection point of $S_\mathcal{O}^{\delta_0}$ and $\partial B_{\hat{c}_\mathcal{O}}(O_\mathcal{O})$ closer to P_1. Similarly, let $\Gamma^{\mathcal{N}, \delta_0}_{\text{sonic}}$ be the smaller arc lying on $\partial B_{c_\mathcal{N}}(O_\mathcal{N})$ between $S_\mathcal{N}^{\delta_0}$ and $\xi_2 = 0$ with endpoints P_2' and P_3, where P_2' is the intersection point of $S_\mathcal{N}^{\delta_0}$ and $\partial B_{c_\mathcal{N}}(O_\mathcal{N})$ closer to P_2.

(iii) *Definition of Q^β.* Define Q^β as the bounded region enclosed by $\Gamma^{\mathcal{O}, \delta_0}_{\text{sonic}}$, $\Gamma^{\mathcal{N}, \delta_0}_{\text{sonic}}$, $S_\mathcal{O}^{\delta_0}$, $S_\mathcal{N}^{\delta_0}$, and Γ_{wedge}.

For each $\beta \in [0, \beta_{\text{d}}^{(v_\infty)}]$, we first define a mapping $G_1 : Q^\beta \to \mathbb{R}^2$ such that

$$(4.1.1) \qquad G_1(\boldsymbol{\xi}) = \begin{cases} (x + u_\mathcal{O} - c_\mathcal{O}, y) & \text{for } \boldsymbol{\xi} \text{ near } \Gamma^{\mathcal{O}, \delta_0}_{\text{sonic}}, \\ (c_\mathcal{N} - x, y) & \text{for } \boldsymbol{\xi} \text{ near } \Gamma^{\mathcal{N}, \delta_0}_{\text{sonic}}, \end{cases}$$

for the (x, y)–coordinates defined by (3.5.2) near $\Gamma^{\mathcal{O}, \delta_0}_{\text{sonic}}$ and by (3.4.18) near $\Gamma^{\mathcal{N}, \delta_0}_{\text{sonic}}$. We take several steps to construct G_1. The definition of G_1 is given in (4.1.28). First, we define a mapping $F_1 : Q^\beta \to \mathbb{R}^2$ such that $F_1(\boldsymbol{\xi}) \cdot (1, 0) = x + u_\mathcal{O} - c_\mathcal{O}$ for $\boldsymbol{\xi}$ near $\Gamma^{\mathcal{O}, \delta_0}_{\text{sonic}}$ and $F_1(\boldsymbol{\xi}) \cdot (1, 0) = c_\mathcal{N} - x$ for $\boldsymbol{\xi}$ near $\Gamma^{\mathcal{N}, \delta_0}_{\text{sonic}}$. Then we define a mapping $F_2 : F_1(Q^\beta) \to \mathbb{R}^2$ so that $(F_2 \circ F_1)(\boldsymbol{\xi}) \cdot (1, 0) = F_1(\boldsymbol{\xi})$, and $(F_2 \circ F_1)(\boldsymbol{\xi}) \cdot (0, 1) = y$ for $\boldsymbol{\xi}$ near $\Gamma^{\mathcal{O}, \delta_0}_{\text{sonic}} \cup \Gamma^{\mathcal{N}, \delta_0}_{\text{sonic}}$. Finally, G_1 is defined by $G_1 = F_2 \circ F_1$ as in (4.1.28).

For $\varepsilon > 0$, define two sets $\mathcal{D}_\varepsilon^{\mathcal{O}}$ and $\mathcal{D}_\varepsilon^{\mathcal{N}}$ by

$$
\begin{aligned}
\mathcal{D}_\varepsilon^{\mathcal{O}} &:= (Q^\beta \cap \{\xi_1 < u_{\mathcal{O}}\}) \setminus \overline{B_{\hat{c}_{\mathcal{O}} - \varepsilon}(O_{\mathcal{O}})}, \\
\mathcal{D}_\varepsilon^{\mathcal{N}} &:= (Q^\beta \cap \{\xi_1 > 0\}) \setminus \overline{B_{c_{\mathcal{N}} - \varepsilon}(O_{\mathcal{N}})}.
\end{aligned}
\tag{4.1.2}
$$

Since $\hat{c}_{\mathcal{O}}, S_{\mathcal{O}}$, and $O_{\mathcal{O}}$ depend continuously on $\beta \in [0, \frac{\pi}{2})$, there exist constants $k > 4$ and $\delta_1 \in (0, \frac{\pi}{2})$ depending only on (v_∞, γ) such that, for each $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]$, we have

$$
\begin{aligned}
\mathcal{D}_{\frac{4}{k}\hat{c}_{\mathcal{O}}}^{\mathcal{O}} &\subset \{x_{P_1} < x < x_{P_1} + \tfrac{4}{k}\hat{c}_{\mathcal{O}},\ \beta < y + \beta < \tfrac{\pi}{2} - \delta_1\}, \\
\mathcal{D}_{\frac{4}{k}c_{\mathcal{N}}}^{\mathcal{N}} &\subset \{0 < x < \tfrac{4}{k}c_{\mathcal{N}},\ 0 < y < \tfrac{\pi}{2} - \delta_1\}.
\end{aligned}
\tag{4.1.3}
$$

Define cut-off functions $\zeta_{\mathcal{O}}, \zeta_{\mathcal{N}}, \chi_{\mathcal{O}}$, and $\chi_{\mathcal{N}}$ as follows:

(i) $\zeta_{\mathcal{O}}, \zeta_{\mathcal{N}} \in C^4(\mathbb{R})$ satisfy

$$
\zeta_{\mathcal{O}}(r) = \begin{cases} 1 & \text{for } r \geq \hat{c}_{\mathcal{O}}(1 - \frac{2}{k}), \\ 0 & \text{for } r < \hat{c}_{\mathcal{O}}(1 - \frac{3}{k}), \end{cases} \qquad 0 \leq \zeta_{\mathcal{O}}'(r) \leq \frac{2k}{\hat{c}_{\mathcal{O}}} \quad \text{on } \mathbb{R};
\tag{4.1.4}
$$

$$
\zeta_{\mathcal{N}}(r) = \begin{cases} 1 & \text{for } r \geq c_{\mathcal{N}}(1 - \frac{2}{k}), \\ 0 & \text{for } r < c_{\mathcal{N}}(1 - \frac{3}{k}), \end{cases} \qquad 0 \leq \zeta_{\mathcal{N}}'(r) \leq \frac{2k}{c_{\mathcal{N}}} \quad \text{on } \mathbb{R};
\tag{4.1.5}
$$

(ii) Let $q_{\mathcal{O}}^{\delta_0}$ be the distance between $O_{\mathcal{O}} = (u_{\mathcal{O}}, 0)$ and $S_{\mathcal{O}}^{\delta_0}$, and denote

$$
u_{\mathcal{O}}^{\delta_0} := u_{\mathcal{O}} - q_{\mathcal{O}}^{\delta_0} \sin\beta.
\tag{4.1.6}
$$

Since $u_{\mathcal{O}} = -v_\infty \tan\beta < 0$, $u_{\mathcal{O}}^{\delta_0} < 0$. Then $\chi_{\mathcal{O}}, \chi_{\mathcal{N}} \in C^4(\mathbb{R})$ satisfy

$$
\chi_{\mathcal{O}}(\xi_1) = \begin{cases} 1 & \text{for } \xi_1 \leq u_{\mathcal{O}}^{\delta_0} - \frac{2\hat{c}_{\mathcal{O}}}{k}, \\ 0 & \text{for } \xi_1 \geq u_{\mathcal{O}}^{\delta_0}, \end{cases} \qquad -\frac{2k}{\hat{c}_{\mathcal{O}}} \leq \chi_{\mathcal{O}}'(\xi_1) \leq 0 \quad \text{on } \mathbb{R};
\tag{4.1.7}
$$

$$
\chi_{\mathcal{N}}(\xi_1) = \begin{cases} 0 & \text{for } \xi_1 \leq \frac{c_{\mathcal{N}}}{k}, \\ 1 & \text{for } \xi_1 \geq \frac{2c_{\mathcal{N}}}{k}, \end{cases} \qquad 0 \leq \chi_{\mathcal{N}}'(\xi_1) \leq \frac{2k}{c_{\mathcal{N}}} \quad \text{on } \mathbb{R}.
\tag{4.1.8}
$$

Choose constant $k > 4$ sufficiently large, depending only on (v_∞, γ), such that

$$
\mathcal{D}_{\frac{3}{k}\hat{c}_{\mathcal{O}}}^{\mathcal{O}} \cap \{\xi_1 < u_{\mathcal{O}}^{\delta_0}\} \subset \{\xi_1 < u_{\mathcal{O}}^{\delta_0} - \frac{3\hat{c}_{\mathcal{O}}}{k}\}, \quad \mathcal{D}_{\frac{3}{k}c_{\mathcal{N}}}^{\mathcal{N}} \subset \{\xi_1 > \frac{3c_{\mathcal{N}}}{k}\}.
\tag{4.1.9}
$$

Next, define a variable r by

$$
r = \begin{cases} \sqrt{(\xi_1 - u_{\mathcal{O}})^2 + \xi_2^2} & \text{for } \xi_1 \leq u_{\mathcal{O}}^{\delta_0}, \\ \sqrt{\xi_1^2 + \xi_2^2} & \text{for } \xi_1 \geq 0. \end{cases}
\tag{4.1.10}
$$

Since $u_{\mathcal{O}}^{\delta_0} < 0$, r is well defined by (4.1.10).

For the cut-off functions $(\zeta_{\mathcal{O}}, \zeta_{\mathcal{N}}, \chi_{\mathcal{O}}, \chi_{\mathcal{N}})$ given by (4.1.4)–(4.1.8) under the choice of k to satisfy (4.1.9), we define a function $h_1 : Q^\beta \to \mathbb{R}$ as

$$
\begin{aligned}
h_1(\xi_1, \xi_2) :=\ & \big((u_{\mathcal{O}} - r)\zeta_{\mathcal{O}}(r) + (1 - \zeta_{\mathcal{O}}(r))\xi_1\big)\chi_{\mathcal{O}} \\
& + \big(1 - \chi_{\mathcal{O}}\big)\big(\xi_1(1 - \chi_{\mathcal{N}}) + (r\zeta_{\mathcal{N}}(r) + (1 - \zeta_{\mathcal{N}}(r))\xi_1)\chi_{\mathcal{N}}\big).
\end{aligned}
\tag{4.1.11}
$$

In (4.1.11), $\chi_{\mathcal{O}}$ and $\chi_{\mathcal{N}}$ are evaluated at ξ_1.

Define a mapping $F_1 : Q^\beta \to \mathbb{R}^2$ by

$$(4.1.12) \qquad F_1(\xi_1, \xi_2) := \big(h_1(\xi_1, \xi_2), \xi_2\big).$$

LEMMA 4.2. *There exist constants $C > 0$ and $\delta_{F_1} > 0$ depending only on (v_∞, γ) such that, for each $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]$, F_1 defined by (4.1.12) satisfies the following properties:*

(a) $\|F_1\|_{C^4(\overline{Q^\beta})} + \|F_1^{-1}\|_{C^4(\overline{F_1(Q^\beta)})} \le C$, *and* $\det(DF_1) \ge \delta_{F_1}$ *in* Q^β;

(b) *Denoting* $F_1(\boldsymbol{\xi}) := (s, t)$, *then*

$$(4.1.13) \qquad F_1(\Gamma_{\text{wedge}}) = \{(s, 0) \,:\, s \in (u_\mathcal{O} - \hat{c}_\mathcal{O}, c_\mathcal{N})\};$$

(c) *For* $\phi_\infty := \varphi_\infty + \frac{1}{2}|\boldsymbol{\xi}|^2$,

$$\partial_t \phi_\infty\big(F_1^{-1}(s, t)\big) = -v_\infty \qquad \text{for all } (s, t) \in \overline{F_1(Q^\beta)};$$

(d) *For each* $j = 1, \cdots, 4$, *denote* $P_j = (\xi_1^{P_j}, \xi_2^{P_j})$ *in the* $\boldsymbol{\xi}$*–coordinates. Then*

$$F_1(P_1) = (u_\mathcal{O} - \hat{c}_\mathcal{O}, \xi_2^{P_1}), \quad F_1(P_2) = (c_\mathcal{N}, \xi_2^{P_2}),$$
$$F_1(P_3) = (c_\mathcal{N}, 0), \qquad\qquad F_1(P_4) = (u_\mathcal{O} - \hat{c}_\mathcal{O}, 0);$$

(e) *For* h_1 *defined by (4.1.11),*

$$h_1(\boldsymbol{\xi}) = \begin{cases} u_\mathcal{O} - c_\mathcal{O} + x & \text{if } \operatorname{dist}(\boldsymbol{\xi}, \Gamma_{\text{sonic}}^{\mathcal{O}, \delta_0}) < \frac{\hat{c}_\mathcal{O}}{k}, \\ c_\mathcal{N} - x & \text{if } \operatorname{dist}(\boldsymbol{\xi}, \Gamma_{\text{sonic}}^{\mathcal{N}, \delta_0}) < \frac{c_\mathcal{N}}{k} \end{cases}$$

for the (x, y)–coordinates defined by (3.4.18) and (3.5.2).

PROOF. By the definition of F_1 in (4.1.12), we have

$$(4.1.14) \qquad \det(DF_1) = \partial_{\xi_1} h_1.$$

Choose constant k large to satisfy that $\chi_\mathcal{N} \chi_\mathcal{O}' = 0$ and $\zeta_\mathcal{N} \chi_\mathcal{N}' = \zeta_\mathcal{O} \chi_\mathcal{O}' = 0$. Then, from definition (4.1.11) of h_1 and (4.1.4)–(4.1.8),

$$(4.1.15) \qquad \partial_{\xi_1} h_1(\boldsymbol{\xi}) = \sum_{j=1}^{3} a_j,$$

where

$$a_1 = \Big(\frac{u_\mathcal{O} - \xi_1}{r} \zeta_\mathcal{O} + (1 - \zeta_\mathcal{O}) + \frac{u_\mathcal{O} - \xi_1}{r}(r - (u_\mathcal{O} - \xi_1))\zeta_\mathcal{O}'\Big)\chi_\mathcal{O},$$
$$a_2 = \Big(\frac{\xi_1}{r} \zeta_\mathcal{N} + (1 - \zeta_\mathcal{N}) + \frac{\xi_1}{r}(r - \xi_1)\zeta_\mathcal{N}'\Big)\chi_\mathcal{N}(1 - \chi_\mathcal{O}),$$
$$a_3 = (1 - \chi_\mathcal{N})(1 - \chi_\mathcal{O}).$$

Then (4.1.3) implies that

$$(4.1.16) \quad \begin{aligned} \partial_{\xi_1} h_1 &\ge \Big(\frac{u_\mathcal{O} - \xi_1}{r}\zeta_\mathcal{O} + (1 - \zeta_\mathcal{O})\Big)\chi_\mathcal{O} + \Big(\big(\frac{\xi_1}{r}\zeta_\mathcal{N} + (1 - \zeta_\mathcal{N})\big)\chi_\mathcal{N} + (1 - \chi_\mathcal{N})\Big)(1 - \chi_\mathcal{O}) \\ &\ge \cos(\frac{\pi}{2} - \delta_1) \end{aligned}$$

for δ_1 from (4.1.3).

Moreover, it follows from (4.1.15) that

$$(4.1.17) \qquad \sup_{\boldsymbol{\xi} \in Q^\beta} \partial_{\xi_1} h_1(\boldsymbol{\xi}) \le C$$

for a constant $C > 0$ depending only on (γ, v_∞).

For a constant a, if $Q_\beta \cap \{\xi_2 = a\}$ is nonempty, then (4.1.16) implies that the one-dimensional mapping $(\xi_1, a) \in Q_\beta \cap \{\xi_2 = a\} \mapsto h_1(\xi_1, a)$ is invertible. Then it follows directly from the definition of F_1 given in (4.1.12) that F_1 is invertible. Also, we can directly check that F_1 and F_1^{-1} are C^4 from (4.1.11), which yields (a). Finally, (b), (d), and (e) follow from (4.1.11)–(4.1.12).

By (2.4.1) and (4.1.12), $\phi_\infty\big(F_1^{-1}(s,t)\big) = -v_\infty t$, which gives

$$\partial_t \phi_\infty\big(F_1^{-1}(s,t)\big) = -v_\infty \qquad \text{for all } (s,t) \in F_1(Q).$$

This proves (c). $\hfill\square$

By the definition of h_1 in (4.1.11), we have

$$\overline{F_1(Q^\beta)} \subset [u_\mathcal{O} - \hat{c}_\mathcal{O}, c_\mathcal{N}] \times [0, \infty).$$

LEMMA 4.3. *Fix $\gamma \geq 1$, $v_\infty > 0$, and $\bar{\beta} \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$. Then there exists a constant $m_0 > 0$ depending only on $(v_\infty, \gamma, \bar{\beta})$ such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{0 \leq \beta \leq \bar{\beta}\}$ satisfies*

(4.1.18) $$\partial_t(\varphi_\infty - \varphi)(F_1^{-1}(s,t)) \leq -m_0 < 0 \qquad in \ \overline{F_1(\Omega)}.$$

Therefore, there exists a unique function $\tilde{\mathfrak{g}}_{\mathrm{sh}} : [u_\mathcal{O} - \hat{c}_\mathcal{O}, c_\mathcal{N}] \to \mathbb{R}_+$ such that

$$F_1(\Gamma_{\mathrm{shock}}) = \{(s, \tilde{\mathfrak{g}}_{\mathrm{sh}}(s)) \ : \ u_\mathcal{O} - \hat{c}_\mathcal{O} < s < c_\mathcal{N}\}.$$

PROOF. For each $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]$, we represent F_1^{-1} as

$$F_1^{-1}(s,t) = (\tilde{h}_1(s,t), t) \qquad \text{in } \overline{F_1(Q^\beta)}.$$

This expression yields that

(4.1.19) $$\partial_t(\varphi_\infty - \varphi)(F_1^{-1}(s,t)) = D(\varphi_\infty - \varphi)|_{F_1^{-1}(s,t)} \cdot (\partial_t \tilde{h}_1(s,t), 1).$$

It follows from $(F_1 \circ F_1^{-1})(s,t) = \big(h_1(\tilde{h}_1(s,t)), t\big) = (s,t)$ that $\partial_t \tilde{h}_1(s,t) = -\frac{\partial_{\xi_2} h_1}{\partial_{\xi_1} h_1}$. This implies that

$$(\partial_t \tilde{h}_1(s,t), 1) = -\frac{1}{\partial_{\xi_1} h_1}(\partial_{\xi_2} h_1, -\partial_{\xi_1} h_1),$$

where $D_{(\xi_1, \xi_2)} h_1$ is evaluated at $\boldsymbol{\xi} = F_1^{-1}(s,t)$.

Next, we compute $\boldsymbol{v} := \frac{1}{\partial_{\xi_1} h_1}(-\partial_{\xi_2} h_1, \partial_{\xi_1} h_1)$.

Case 1. If $\chi_\mathcal{O} \neq 0$ so that $\chi_\mathcal{N} = \chi_\mathcal{N}' = 0$, we use $\zeta_\mathcal{O}(r)\chi_\mathcal{O}'(\xi_1) \equiv 0$ to obtain

(4.1.20) $$\partial_{\xi_1} h_1 \boldsymbol{v} = k_1 \boldsymbol{a}_1 + k_2 \boldsymbol{a}_2,$$

where

$$\boldsymbol{a}_1 = (\sin y, \cos y), \quad \boldsymbol{a}_2 = (0,1), \quad k_1 = \big(\zeta_\mathcal{O} + r(1 - \cos y)\zeta_\mathcal{O}'\big)\chi_\mathcal{O}, \quad k_2 = 1 - \zeta_\mathcal{O}\chi_\mathcal{O}$$

for the (x,y)–coordinates defined by (3.5.2).

Case 2. If $\chi_\mathcal{O} = 0$ so that $\chi_\mathcal{O} = \chi_\mathcal{O}' = 0$, we use $\zeta_\mathcal{N}(r)\chi_\mathcal{N}'(\xi_1) \equiv 0$ to obtain

(4.1.21) $$\partial_{\xi_1} h_1 \boldsymbol{v} = l_1 \boldsymbol{b}_1 + l_2 \boldsymbol{b}_2,$$

where

$$\boldsymbol{b}_1 = (-\sin y, \cos y), \quad \boldsymbol{b}_2 = (0,1), \quad l_1 = \big(\zeta_\mathcal{N} + r(1 - \cos y)\zeta_\mathcal{N}'\big)\chi_\mathcal{N}, \quad l_2 = 1 - \zeta_\mathcal{N}\chi_\mathcal{N}$$

for the (x,y)–coordinates defined by (3.4.18).

Claim: *There exists a constant* $\tilde{m} > 0$ *depending only on* $(v_\infty, \gamma, \bar{\beta})$ *such that any admissible solution* φ *corresponding to* $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{0 \le \beta \le \bar{\beta}\}$ *satisfies*

$$\sup_{P \in \overline{\Omega}} \big(D(\varphi_\infty - \varphi) \cdot \boldsymbol{v}\big)(P) \le -\tilde{m}.$$

Fix an admissible solution φ for $\beta \in [0, \bar{\beta}]$. Let the unit vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{b}_1$, and $\boldsymbol{b}_2$ be from (4.1.20)–(4.1.21). Then $\boldsymbol{a}_1, \boldsymbol{a}_2 \in \text{Cone}^0(\boldsymbol{e}_{S_\mathcal{O}}, \boldsymbol{e}_{S_\mathcal{N}})$ for all $y \in [0, \frac{\pi}{2} - \beta - \delta_1]$ for $\delta_1 > 0$ from (4.1.3), and $\boldsymbol{b}_1, \boldsymbol{b}_2 \in \text{Cone}^0(\boldsymbol{e}_{S_\mathcal{O}}, \boldsymbol{e}_{S_\mathcal{N}})$ for all $y \in [0, \frac{\pi}{2} - \delta_1]$. Moreover, k_j and l_j, $j = 1, 2$, are nonnegative and satisfy that $k_1 + k_2 \ge 1$ and $l_1 + l_2 \ge 1$ for all $P \in \overline{\Omega}$. Then (3.1.17) yields

$$\sup_{P \in \overline{\Omega}} \big(\partial_{\xi_1} h_1 \, D(\varphi_\infty - \varphi) \cdot \boldsymbol{v}\big)(P) \le -m_\varphi < 0$$

for a constant $m_\varphi > 0$. Furthermore, Lemma 3.43 implies that there exists a constant $m_1 > 0$ depending only on $(v_\infty, \gamma, \bar{\beta})$ such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{0 \le \beta \le \bar{\beta}\}$ satisfies

(4.1.22) $$\sup_{P \in \overline{\Omega}} \big(\partial_{\xi_1} h_1 \, D(\varphi_\infty - \varphi) \cdot \boldsymbol{v}\big)(P) \le -m_1.$$

Combining (4.1.22) with (4.1.16)–(4.1.17), we conclude that there exists a constant $m_0 > 0$ depending only on $(v_\infty, \gamma, \bar{\beta})$ such that any admissible solution φ for $\beta \in [0, \bar{\beta}]$ satisfies

(4.1.23) $$\partial_t(\varphi_\infty - \varphi)(F_1^{-1}(s,t)) = (D(\varphi_\infty - \varphi) \cdot \boldsymbol{v})(F_1^{-1}(s,t)) \le -m_0 < 0$$

for all $(s, t) \in F_1(\overline{\Omega})$. □

Next, we define a mapping $F_2 : F_1(\overline{Q^\beta}) \to \mathbb{R}^2$ so that mapping $G_1 := F_2 \circ F_1$ satisfies property (4.1.1) in $\overline{Q^\beta}$.

For each $\beta \in [0, \beta_{\text{d}}^{(v_\infty)}]$, we define $F_2 : F_1(\overline{Q^\beta}) \to \mathbb{R}^2$ by

(4.1.24) $$F_2(s, t) := \big(s, h_2(s, t)\big),$$

and define a function $h_2 : F_1(\overline{Q^\beta}) \to [0, \infty)$ by

(4.1.25) $$h_2(s, t) := \tilde{\chi}_\mathcal{O} \sin^{-1}\big(\frac{t}{u_\mathcal{O} - s}\big) + (1 - \tilde{\chi}_\mathcal{O})\big(t(1 - \tilde{\chi}_\mathcal{N}) + \tilde{\chi}_\mathcal{N} \sin^{-1}(\frac{t}{s})\big)$$

for the cut-off functions $\tilde{\chi}_\mathcal{O}, \tilde{\chi}_\mathcal{N} \in C^4(\mathbb{R})$ satisfying the following conditions:

$$\tilde{\chi}_\mathcal{O}(s) = \begin{cases} 1 & \text{for } s < u_\mathcal{O} - \hat{c}_\mathcal{O}(1 - \frac{1}{2k}), \\ 0 & \text{for } s > u_\mathcal{O} - \hat{c}_\mathcal{O}(1 - \frac{1}{k}), \end{cases}$$

$$\tilde{\chi}_\mathcal{N}(s) = \begin{cases} 0 & \text{for } s < c_\mathcal{N}(1 - \frac{1}{k}), \\ 1 & \text{for } s > c_\mathcal{N}(1 - \frac{1}{2k}), \end{cases}$$

$$0 \le \tilde{\chi}_\mathcal{O}, \tilde{\chi}_\mathcal{N} \le 1, \qquad -\frac{4k}{\hat{c}_\mathcal{O}} \le \tilde{\chi}'_\mathcal{O} \le 0 \le \tilde{\chi}'_\mathcal{N} \le \frac{4k}{c_\mathcal{N}}, \qquad \tilde{\chi}'_\mathcal{O}\tilde{\chi}'_\mathcal{N} = 0,$$

where $k > 4$ is the constant chosen to satisfy (4.1.9) and all the properties used in the proof of Lemma 4.2.

Then h_2 satisfies

(4.1.26) $$h_2(s, t) = y \qquad \text{for } (s, t) \text{ near } F_1(\Gamma_{\text{sonic}}^{\mathcal{O}, \delta_0} \cup \Gamma_{\text{sonic}}^{\mathcal{N}, \delta_0}).$$

LEMMA 4.4. *There exist constants $C > 0$ and $\kappa_1 > 0$ depending only on (v_∞, γ) such that, for each $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]$, F_2 defined by (4.1.24) satisfies the following properties*:

(a) $\|F_2\|_{C^4 l(F_1(Q^\beta))} + \|F_2^{-1}\|_{C^4(F_2 \circ F_1(Q^\beta))} \le C$, *and* $\det(DF_2) = \partial_t h_2 \ge \kappa_1$ *in* $\overline{F_1(Q^\beta)}$;

(b) *For* $F_2(s, t) := (\tilde{s}, \tilde{t})$, $(F_2 \circ F_1)(\Gamma_{\mathrm{wedge}}) = \{(\tilde{s}, 0) \,:\, \tilde{s} \in (u_\mathcal{O} - \hat{c}_\mathcal{O}, c_\mathcal{N})\}$.

PROOF. A direct computation by using (4.1.24) shows that

$$\det(DF_2) = \partial_t h_2(s, t) = \frac{\tilde{\chi}_\mathcal{O}}{\sqrt{(u_\mathcal{O} - s)^2 - t^2}} + (1 - \tilde{\chi}_\mathcal{O})\Big((1 - \tilde{\chi}_\mathcal{N}) + \frac{\tilde{\chi}_\mathcal{N}}{\sqrt{s^2 - t^2}}\Big).$$

For $s < u_\mathcal{O} - \hat{c}_\mathcal{O}(1 - \frac{1}{2k})$, we can write

$$\sqrt{(u_\mathcal{O} - s)^2 - t^2} = r \cos y,$$

by (4.1.4) and (4.1.11), where r and y are given by (4.1.10) and (3.5.2) for $\boldsymbol{\xi} = F_1^{-1}(s, t)$. Similarly, for $s > c_\mathcal{N}(1 - \frac{1}{2k})$, we can write as $\sqrt{s^2 - t^2} = r \cos y$, where r and y are given by (4.1.10) and (3.4.18) for $\boldsymbol{\xi} = F_1^{-1}(s, t)$. Then there exists a constant $\kappa_1 > 0$ depending only on (v_∞, γ) such that

(4.1.27) $$\det(DF_2) = \partial_t h_2 \ge \kappa_1 \qquad \text{in } \overline{F_1(Q^\beta)}.$$

For a constant a, if $F_1(Q_\beta) \cap \{s = a\}$ is nonempty, then (4.1.27) implies that the one-dimensional mapping $(a, t) \in F_1(Q_\beta) \cap \{s = a\} \mapsto h_2(a, t)$ is invertible. Then mapping F_2 given by (4.1.24) is also invertible.

The C^4–estimates of F_2 and F_2^{-1} and (b) are obtained directly from (4.1.13) and (4.1.25). □

By (4.1.25) and the invertibility of F_2, there exists a function $\tilde{h}_2 : [u_\mathcal{O} - \hat{c}_\mathcal{O}, c_\mathcal{N}] \to \mathbb{R}_+$ such that

$$F_2^{-1}(s', t') = (s', \tilde{h}_2(s', t')) \qquad \text{for all } (s', t') \in \overline{(F_2 \circ F_1)(Q^\beta)}.$$

For F_1 and F_2 given by (4.1.12) and (4.1.24) respectively, define a mapping $G_1 : \overline{Q^\beta} \to [u_\mathcal{O} - \hat{c}_\mathcal{O}, c_\mathcal{N}] \times \mathbb{R}_+$ by

(4.1.28) $$G_1 := F_2 \circ F_1,$$

and denote $G_1(\boldsymbol{\xi}) = (s', t')$. Mapping G_1 satisfies property (4.1.1).

For each $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]$, define

(4.1.29) $$s_\beta := u_\mathcal{O} - \hat{c}_\mathcal{O}.$$

Note that s_β varies continuously on (γ, v_∞) and $\beta \in [0, \frac{\pi}{2})$. Define a linear function $L_\beta(s')$ by

(4.1.30) $$L_\beta(s') := \frac{2}{c_\mathcal{N} - s_\beta}(s' - s_\beta) - 1.$$

Then L_β maps $[s_\beta, c_\mathcal{N}]$ onto $[-1, 1]$. We define a mapping $\mathcal{G}_1^\beta : \overline{Q^\beta} \to [-1, 1] \times \mathbb{R}_+$ by

(4.1.31) $$\mathcal{G}_1^\beta(\boldsymbol{\xi}) = (L_\beta(s'), t') \qquad \text{for } (s', t') = G_1(\boldsymbol{\xi}).$$

LEMMA 4.5. *There exist constants $C > 0$ and $\kappa > 0$ depending only on (v_∞, γ) such that, for any $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]$, $\mathcal{G}_1^\beta$ defined by (4.1.31) satisfies the following properties*:

(a) $\|\mathcal{G}_1^\beta\|_{C^4(\overline{Q^\beta})} + \|(\mathcal{G}_1^\beta)^{-1}\|_{C^4(\mathcal{G}_1^\beta(\overline{Q^\beta}))} \le C$;

(b) $|\det(D\mathcal{G}_1^\beta)| \ge \kappa \qquad \text{in } \overline{Q^\beta}$;

(c) $\mathcal{G}_1^\beta(\Gamma_{\text{wedge}}) = \{(s,0) \,:\, s \in (-1,1)\}$;

(d) For $\phi_\infty := \varphi_\infty + \frac{1}{2}|\boldsymbol{\xi}|^2$, $\partial_{t'}\phi_\infty\big((\mathcal{G}_1^\beta)^{-1}(s,t')\big) \le -\kappa < 0$ for all $(s,t') \in \overline{\mathcal{G}_1^\beta(Q^\beta)}$.

In addition, for any $\bar{\beta} \in (0, \beta_{\text{d}}^{(v_\infty)})$, there exists $m_2 > 0$ depending only on $(v_\infty, \gamma, \bar{\beta})$ such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{0 \le \beta \le \bar{\beta}\}$ satisfies

$$(4.1.32) \qquad \partial_{t'}(\varphi_\infty - \varphi)\big((\mathcal{G}_1^\beta)^{-1}(s,t')\big) \le -m_2 < 0 \qquad in\ \overline{\mathcal{G}_1^\beta(\Omega)}.$$

PROOF. Fix $\bar{\beta} \in (0, \beta_{\text{d}}^{(v_\infty)})$. It follows from (4.1.11), (4.1.25), (4.1.28), and Lemmas 4.2 and 4.4 that there exist constants $C, \kappa_2 > 0$ depending only on (v_∞, γ) such that, for any $\beta \in [0, \beta_{\text{d}}^{(v_\infty)}]$, mapping G_1 defined by (4.1.28) satisfies the following properties:

(a') $\|G_1\|_{C^4(Q^\beta)} + \|G_1^{-1}\|_{C^4(G_1(Q^\beta))} \le C$;

(b') $|\det(DG_1)| \ge \kappa_2$ in $\overline{Q^\beta}$;

(c') $G_1(\Gamma_{\text{wedge}}) = \{(s',0) \,:\, s' \in (u_{\mathcal{O}} - \hat{c}_{\mathcal{O}}, c_{\mathcal{N}})\}$.

These properties, combined with (4.1.31), yield (a)–(c) for some $\kappa < \kappa_2$.

By (4.1.12) and (4.1.24)–(4.1.28), we find that, at $\boldsymbol{\xi} = G_1^{-1}(s',t')$,

$$\partial_{t'}(\varphi_\infty - \varphi)(G_1^{-1}(s',t')) = D_{\boldsymbol{\xi}}(\varphi_\infty - \varphi) \cdot (\partial_t \tilde{h}_1, 1)\partial_{t'}\tilde{h}_2 = \frac{D_{(\xi_1,\xi_2)}(\varphi_\infty - \varphi) \cdot \boldsymbol{v}}{\partial_t h_2}$$

for $\boldsymbol{v}$ given by (4.1.20)–(4.1.21). Then (4.1.32) follows by combining (4.1.16) and (4.1.23) with Lemma 4.4(a) and (4.1.31). Assertion (d) can be verified similarly. $\qquad\square$

By using (2.4.3) and the definitions of $(\varphi_\infty, \varphi_{\mathcal{O}}, \varphi_{\mathcal{N}})$ given in (2.5.1), it can be directly checked that $S_{\mathcal{O}} = \{\boldsymbol{\xi} \,:\, (\varphi_\infty - \varphi_{\mathcal{O}})(\boldsymbol{\xi}) = 0\}$ and $S_{\mathcal{N}} = \{\boldsymbol{\xi} \,:\, (\varphi_\infty - \varphi_{\mathcal{N}})(\boldsymbol{\xi}) = 0\}$ intersect at a unique point:

$$(4.1.33) \qquad P_I = (\xi_1^I, \xi_2^{\mathcal{N}}) \qquad for\ \xi_1^I = -\frac{\xi_2^{(\beta)} - \xi_2^{\mathcal{N}}}{\tan\beta},$$

where $\xi_2^{(\beta)}$ is the ξ_2–intercept of $S_{\mathcal{O}}$. Then $S_{\mathcal{O}}^{\delta_0}$ and $S_{\mathcal{N}}^{\delta_0}$ intersect at $(\xi_1^I, \xi_2^{\mathcal{N}} + \frac{\delta_0}{v_\infty})$. It follows from (2.4.14) and (2.4.42) that $\frac{\mathrm{d}\xi_2^{(\beta)}}{\mathrm{d}\beta} > 0$ for $\beta \in (0, \frac{\pi}{2})$ so that

$$(4.1.34) \qquad \xi_1^I < 0.$$

Since point P_I lies on $S_{\mathcal{O}}$, and its ξ_2–coordinate is greater than the ξ_2–coordinate of P_1, we have

$$(4.1.35) \qquad \xi_1^I > \xi_1^{P_1}.$$

By (2.4.3), (4.1.3), and (4.1.20)–(4.1.21), there exists a constant $m_3 > 0$ depending only on (v_∞, γ) such that, for each $\beta \in [0, \beta_{\text{d}}^{(v_\infty)}]$,

$$(4.1.36) \qquad \begin{aligned} \partial_{t'}\big((\varphi_\infty - \varphi_{\mathcal{O}}) \circ (\mathcal{G}_1^\beta)^{-1}(s,t')\big) &\le -m_3, \\ \partial_{t'}\big((\varphi_\infty - \varphi_{\mathcal{N}}) \circ (\mathcal{G}_1^\beta)^{-1}(s,t')\big) &\le -m_3 \end{aligned}$$

for all $(s,t') \in \overline{\mathcal{G}_1^\beta(Q^\beta)}$. By the implicit function theorem, there exists a unique function $f_\beta \in C^{0,1}([-1,1])$ such that

$$(4.1.37) \qquad \mathcal{G}_1^\beta(Q^\beta) = \{(s,t') \,:\, -1 < s < 1,\ 0 < t' < f_\beta(s)\}, \quad \|f_\beta\|_{C^{0,1}([-1,1])} \le C$$

for a constant C depending only on (v_∞, γ).

PROPOSITION 4.6. Fix $\gamma \geq 1$ and $v_\infty > 0$. For each admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$, there exists a unique function

$$\mathfrak{g}_{\text{sh}} : [-1, 1] \to \mathbb{R}_+$$

satisfying the following properties:

(a) $\mathcal{G}_1^\beta(\Omega) = \{(s, t') : -1 < s < 1, 0 < t' < \mathfrak{g}_{\text{sh}}(s)\}$,

$\mathcal{G}_1^\beta(\Gamma_{\text{shock}}) = \{(s, \mathfrak{g}_{\text{sh}}(s)) : -1 < s < 1\}$.

(b) For any constant $\hat{\varepsilon} \in (0, \frac{1}{10}]$, there exists a constant $C_{\hat{\varepsilon}} > 0$ depending only on (v_∞, γ) such that

$$\|\mathfrak{g}_{\text{sh}}\|_{C^3([-1+\hat{\varepsilon}, 1-\hat{\varepsilon}])} \leq C_{\hat{\varepsilon}}.$$

(c) Let $\varepsilon_0^* > 0$ be the minimum of ε_0 from Lemmas 3.20 and 3.34. For each $\varepsilon \in (0, \varepsilon_0^*]$, denote

(4.1.38) $$\hat{\varepsilon} := \frac{2}{c_{\mathcal{N}} - s_\beta} \varepsilon.$$

Let Q_0^β be the bounded region enclosed by $\Gamma_{\text{sonic}}^\mathcal{O}$, $\Gamma_{\text{sonic}}^\mathcal{N}$, $S_\mathcal{O}$, $S_\mathcal{N}$, and Γ_{wedge}. Then

$$\Omega \subset Q_0^\beta \subset Q^\beta$$

for Q^β given by Definition 4.1(iii). For $\mathcal{D}_\varepsilon^\mathcal{N}$ and $\mathcal{D}_\varepsilon^\mathcal{O}$ defined by (4.1.2), there exist unique functions $\mathfrak{g}_\mathcal{N}$ and $\mathfrak{g}_\mathcal{O}$ so that

(4.1.39)
$$\mathcal{G}_1^\beta(Q_0^\beta \cap \mathcal{D}_\varepsilon^\mathcal{N}) = \{(s, t') : 1 - \hat{\varepsilon} < s < 1, \ 0 < t' < \mathfrak{g}_\mathcal{N}(s)\},$$
$$\mathcal{G}_1^\beta(Q_0^\beta \cap \mathcal{D}_\varepsilon^\mathcal{O}) = \{(s, t') : -1 < s < -1 + \hat{\varepsilon}, \ 0 < t' < \mathfrak{g}_\mathcal{O}(s)\},$$

for $\hat{\varepsilon}$ defined by (4.1.38). Moreover, there exists a constant $C > 0$ depending only on (v_∞, γ) such that

(4.1.40) $$\|\mathfrak{g}_\mathcal{N}\|_{C^3([1-\hat{\varepsilon}_0^*, 1])} + \|\mathfrak{g}_\mathcal{O}\|_{C^3([-1, -1+\hat{\varepsilon}_0^*])} \leq C.$$

For any $\alpha \in (0, 1)$, there exists $C_{\text{par}} > 0$ depending only on $(v_\infty, \gamma, \alpha)$ such that, for any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$,

$$\|\mathfrak{g}_\mathcal{N} - \mathfrak{g}_{\text{sh}}\|_{2, \alpha, (1-\hat{\varepsilon}_0^*, 1)}^{(\text{par})} \leq C_{\text{par}},$$

where the norm, $\|\cdot\|_{2, \alpha, (1-\hat{\varepsilon}_0^*, 1)}^{(\text{par})}$, is defined by Definition 3.25(iii) with the replacement of x by $1 - |s|$ for the weight of the norm.

(d) For each $\bar{\beta} \in (0, \beta_{\text{d}}^{(v_\infty)})$, there exist $\bar{\alpha} \in (0, 1)$ and $C_{\bar{\beta}} > 0$ depending only on $(v_\infty, \gamma, \bar{\beta})$ such that, for any admissible solution corresponding to $\beta \in [0, \bar{\beta}]$,

(4.1.41) $$\|\mathfrak{g}_{\text{sh}}\|_{2, \bar{\alpha}, (-1, -1+\hat{\varepsilon}_0^*)}^{(-1-\bar{\alpha}), \{-1\}} \leq C_{\bar{\beta}}, \quad (\mathfrak{g}_{\text{sh}} - \mathfrak{g}_\mathcal{O})(-1) = 0, \quad (\mathfrak{g}_{\text{sh}} - \mathfrak{g}_\mathcal{O})'(-1) = 0.$$

We note that (4.1.41) is equivalent to

$$\|\mathfrak{g}_{\text{sh}} - \mathfrak{g}_\mathcal{O}\|_{2, \hat{\alpha}, (-1, -1+\hat{\varepsilon}_0^*)}^{(1+\hat{\alpha}), (\text{par})} \leq C_{\bar{\beta}}'$$

for a constant $C_{\bar{\beta}}' > 0$ depending only on $(v_\infty, \gamma, \bar{\beta})$, where the norm, $\|\cdot\|_{2, \hat{\alpha}, (-1, -1+\hat{\varepsilon}_0^*)}^{(1+\hat{\alpha}), (\text{par})}$, is defined by Definition 3.25(iv) with the replacement of x by $1 - |s|$.

(e) For each $\bar{\beta} \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$, there exists a constant $\hat{k} > 1$ depending only on $(v_\infty, \gamma, \bar{\beta})$ such that, for any admissible solution φ for $\beta \in [0, \bar{\beta}]$,

$$\min\{\mathfrak{g}_{\mathrm{sh}}(-1) + \frac{s+1}{\hat{k}}, \frac{1}{\hat{k}}\} \leq \mathfrak{g}_{\mathrm{sh}}(s) \leq \min\{f_\beta(s) - \frac{1}{\hat{k}}, \mathfrak{g}_{\mathrm{sh}}(-1) + \hat{k}(s+1)\}$$

for all $-1 \leq s \leq 1$.

PROOF. By (4.1.32) and the implicit function theorem, property (a) is obtained. For an admissible solution φ, we differentiate the equation: $(\varphi_\infty - \varphi) \circ (\mathcal{G}_1^\beta)^{-1}(s, \mathfrak{g}_{\mathrm{sh}}(s)) = 0$ with respect to s to obtain

$$\mathfrak{g}_{\mathrm{sh}}'(s) = \frac{\partial_s\big((\varphi_\infty - \varphi) \circ (\mathcal{G}_1^\beta)^{-1}\big)}{\partial_{t'}\big((\varphi_\infty - \varphi) \circ (\mathcal{G}_1^\beta)^{-1}\big)},$$

where the right-hand side is evaluated at $(s, \mathfrak{g}_{\mathrm{sh}}(s))$. Then property (b) is obtained from Lemma 3.18, Corollary 3.19, and Lemma 4.5. Similarly, properties (c) and (d) are obtained from (2.5.8), (2.5.12), and Propositions 3.26, 3.32, 3.39, and 3.42.

By Lemma 3.34 and (4.1.1), there exist constants $\hat{\varepsilon}_1 \in (0, \hat{\varepsilon}_0^*]$ and $m > 1$ depending only on (v_∞, γ) such that, for each $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]$, $\mathfrak{g}_\mathcal{O}$ satisfies

$$\frac{1}{m} \leq \mathfrak{g}_\mathcal{O}'(s) \leq m \qquad \text{for all } -1 \leq s \leq -1 + \hat{\varepsilon}_1.$$

For each $\bar{\beta} \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$, by (4.1.41), we can choose $\hat{\varepsilon}_2 \in (0, \hat{\varepsilon}_1]$ depending only on $(v_\infty, \gamma, \bar{\beta})$ such that, for any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{0 \leq \beta \leq \bar{\beta}\}$,

$$\frac{1}{2m} \leq \mathfrak{g}_{\mathrm{sh}}'(s) \leq 2m \qquad \text{for } -1 \leq s \leq -1 + \hat{\varepsilon}_2.$$

By combining this estimate with Proposition 3.11, property (e) is obtained as a result. $\qquad\square$

REMARK 4.7. By Propositions 3.30 and 3.32, for each $\alpha \in (0, 1)$, there exist constants $\hat{\varepsilon}_3 > 0$ and $C_\alpha > 0$ depending only on $(v_\infty, \gamma, \alpha)$ such that, for any admissible solution corresponding to (v_∞, β) with $0 \leq \beta < \beta_{\mathrm{s}}^{(v_\infty)}$,

$$\|\mathfrak{g}_{\mathrm{sh}} - \mathfrak{g}_\mathcal{O}\|_{2,\alpha,(-1,-1+\hat{\varepsilon}_3)}^{(\mathrm{par})} \leq C_\alpha,$$

where the norm, $\|\cdot\|_{2,\alpha,(-1,-1+\hat{\varepsilon}_2)}^{(\mathrm{par})}$, is defined by Definition 3.25(iii) with the replacement of x by $1 - |s|$ for the weight of the norm.

By Proposition 3.39, for each $\alpha \in (0, 1)$, there exist constants $\hat{\varepsilon}_4 > 0$ and $C_\alpha' > 0$ depending only on $(v_\infty, \gamma, \alpha)$ such that, for any admissible solution corresponding to (v_∞, β) for $\beta_{\mathrm{s}}^{(v_\infty)} \leq \beta \leq \beta_{\mathrm{s}}^{(v_\infty)} + \sigma_3$,

$$\|\mathfrak{g}_{\mathrm{sh}} - \mathfrak{g}_\mathcal{O}\|_{C^{2,\alpha}([-1,-1+\hat{\varepsilon}_4])} \leq C_\alpha', \qquad \frac{\mathrm{d}^m}{\mathrm{d}s^m}(\mathfrak{g}_{\mathrm{sh}} - \mathfrak{g}_\mathcal{O})(-1) = 0 \quad \text{for } m = 0, 1, 2.$$

By (4.1.34)–(4.1.35), ξ_1^I given by (4.1.33) satisfies that $\xi_1^{P_1} < \xi_1^I < 0$ for any $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]$.

DEFINITION 4.8. Fix $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]$. For ξ_1^I given by (4.1.33), fix a smooth function χ_β^* such that

$$\chi_\beta^*(\xi_1) = \begin{cases} 1 & \text{for } \xi_1 \leq \xi_1^I - \frac{\xi_1^I - \xi_1^{P_1}}{10}, \\ 0 & \text{for } \xi_1 \geq \xi_1^I, \end{cases} \qquad -\frac{10C}{\xi_1^I - \xi_1^{P_1}} \leq (\chi_\beta^*)' \leq 0, \qquad \|\chi_\beta^*\|_{C^3(\mathbb{R})} \leq C$$

for some constant $C > 0$ depending only on (v_∞, γ). For such a smooth cut-off function, define

$$(4.1.42) \qquad \varphi_\beta^*(\boldsymbol{\xi}) := \varphi_\mathcal{O}(\boldsymbol{\xi})\chi_\beta^*(\xi_1) + \varphi_\mathcal{N}(\boldsymbol{\xi})(1 - \chi_\beta^*(\xi_1)).$$

For later use, we list the following useful properties of φ_β^* for $\beta \in [0, \beta_\mathrm{d}^{(v_\infty)}]$:

(i) Define

$$(4.1.43) \qquad \varphi_\beta := \max\{\varphi_\mathcal{O}, \varphi_\mathcal{N}\}.$$

By (2.5.1) and the definition of ξ_1^I given in (4.1.33), we have

$$\varphi_\beta(\xi_1, \xi_2) = \begin{cases} \varphi_\mathcal{O}(\xi_1, \xi_2) & \text{if } \xi_1 < \xi_1^I, \\ \varphi_\mathcal{O}(\xi_1, \xi_2) = \varphi_\mathcal{N}(\xi_1, \xi_2) & \text{if } \xi_1 = \xi_1^I, \\ \varphi_\mathcal{N}(\xi_1, \xi_2) & \text{if } \xi_1 > \xi_1^I, \end{cases}$$

so that

$$(4.1.44) \qquad \varphi_\beta^* \leq \varphi_\beta \qquad \text{in } \mathbb{R}^2.$$

(ii) Let $\mathcal{D}_r^\mathcal{O}$ and $\hat{c}_\mathcal{O}$ be given by (4.1.2) and Definition 4.1, respectively. Then there exists a sufficiently large constant $\bar{k} > 1$ depending only on (v_∞, γ) such that, for any $\beta \in [0, \beta_\mathrm{d}^{(v_\infty)}]$, φ_β^* satisfies

$$(4.1.45) \qquad \varphi_\beta^* = \varphi_\beta = \begin{cases} \varphi_\mathcal{O} & \text{in } \mathcal{D}_{\frac{\hat{c}_\mathcal{O}}{\bar{k}}}^\mathcal{O}, \\ \varphi_\mathcal{N} & \text{in } \{\boldsymbol{\xi} \in \mathbb{R}^2 \,:\, \xi_1 \geq 0\}. \end{cases}$$

(iii) The set, $\{\boldsymbol{\xi} \,:\, \xi_1^{P_1} \leq \xi_1 \leq \xi_1^{P_2}, \;\; (\varphi_\infty - \varphi_\beta^*)(\boldsymbol{\xi}) = 0\}$, is contained in Q^β and

$$(4.1.46) \qquad \sup_{Q^\beta}(\varphi_\infty - \varphi_\beta^*) - \inf_{Q^\beta}(\varphi_\infty - \varphi_\beta^*) \geq \bar{\delta} > 0$$

for some constant $\bar{\delta}$ depending only on (v_∞, γ).

LEMMA 4.9. *There exists a constant $m > 0$ depending only on (v_∞, γ) such that each φ_β^* for $\beta \in [0, \beta_\mathrm{d}^{(v_\infty)}]$ satisfies*

$$\partial_{t'}(\varphi_\infty - \varphi_\beta^*)((\mathcal{G}_1^\beta)^{-1}(s, t')) \leq -m \qquad \text{for all } (s, t') \in \overline{\mathcal{G}_1^\beta(Q^\beta)}.$$

PROOF. We have seen in the proof of Lemma 4.5 that

$$\partial_{t'}(\varphi_\infty - \varphi_\beta^*)((\mathcal{G}_1^\beta)^{-1}(s, t')) = \frac{1}{\partial_t h_2} D_{\boldsymbol{\xi}}(\varphi_\infty - \varphi_\beta^*) \cdot \boldsymbol{v}$$

for $\boldsymbol{v}$ given by (4.1.20)–(4.1.21), where $D_{\boldsymbol{\xi}}(\varphi_\infty - \varphi_\beta^*)$ is evaluated at $(\mathcal{G}_1^\beta)^{-1}(s, t')$. By using (2.5.1) and (4.1.42), a direct computation yields that

$$D_{\boldsymbol{\xi}}(\varphi_\infty - \varphi_\beta^*) = v_\infty \sec\beta(\sin\beta, -\cos\beta)\chi_\beta^* + (0, -v_\infty)(1 - \chi_\beta^*) + (\varphi_\mathcal{N} - \varphi_\mathcal{O})(\chi_\beta^*)'(1, 0).$$

From (4.1.3) and (4.1.20)–(4.1.21), there exists a constant $m_* > 0$ depending only on (v_∞, γ) such that

$$(4.1.47) \qquad D_{\boldsymbol{\xi}}(\varphi_\infty - \varphi_\mathcal{O}) \cdot \boldsymbol{v} \leq -m_*, \quad D_{\boldsymbol{\xi}}(\varphi_\infty - \varphi_\mathcal{N}) \cdot \boldsymbol{v} \leq -m_* \qquad \text{for all } (s, t') \in \overline{\mathcal{G}_1^\beta(Q^\beta)}.$$

By (4.1.7)–(4.1.8) and the definition of χ_β^*, we see that $\chi_{\mathcal{O}}(\chi_\beta^*)' = \chi_{\mathcal{N}}(\chi_\beta^*)' = 0$ on $\mathbb{R}$. This, combined with (4.1.20)–(4.1.21), yields that $(\varphi_{\mathcal{N}} - \varphi_{\mathcal{O}})(\chi_\beta^*)'(1,0) \cdot \boldsymbol{v} = 0$. Then (4.1.47) implies that

$$(4.1.48) \qquad\qquad D_{\boldsymbol{\xi}}(\varphi_\infty - \varphi_\beta^*) \cdot \boldsymbol{v} \le -m_* \qquad \text{for all } (s, t') \in \overline{\mathcal{G}_1^\beta(Q^\beta)}.$$

The proof is completed by (4.1.48) and Lemma 4.4. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Each admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ has a unique function $\mathfrak{g}_{\text{sh}} : (-1, 1) \to \mathbb{R}_+$ satisfying all the properties stated in Proposition 4.6. For such a function $\mathfrak{g}_{\text{sh}}$, define a mapping $G_{2,\mathfrak{g}_{\text{sh}}} : \mathcal{G}_1^\beta(Q^\beta) \to \mathbb{R}^2$ by

$$(4.1.49) \qquad\qquad G_{2,\mathfrak{g}_{\text{sh}}} : (s, t') \mapsto \Big(s, \frac{t'}{\mathfrak{g}_{\text{sh}}(s)}\Big) =: (s, t).$$

By Proposition 4.6(e), $G_{2,\mathfrak{g}_{\text{sh}}}$ is well defined and invertible with

$$G_{2,\mathfrak{g}_{\text{sh}}}^{-1}(s, t) = \big(s, t\mathfrak{g}_{\text{sh}}(s)\big).$$

More importantly, we have

$$G_{2,\mathfrak{g}_{\text{sh}}} \circ \mathcal{G}_1^\beta(\Omega) = (-1, 1) \times (0, 1) =: \mathcal{Q}^{\text{iter}}.$$

Therefore, a function u given by

$$(4.1.50) \qquad\qquad u(s, t) := (\varphi - \varphi_\beta^*) \circ (\mathcal{G}_1^\beta)^{-1} \circ G_{2,\mathfrak{g}_{\text{sh}}}^{-1}(s, t) \qquad \text{for } (s, t) \in \mathcal{Q}^{\text{iter}}$$

is well defined. To establish a uniform estimate of u given by (4.1.50) for admissible solutions corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$, we introduce a new weighted $C^{2,\alpha}$–norm in $\mathcal{Q}^{\text{iter}}$.

DEFINITION 4.10. Fix constants $\sigma > 0$, $\alpha \in (0, 1)$, and $m \in \mathbb{Z}_+$.

(i) For $\mathbf{s} = (s, t), \tilde{\mathbf{s}} = (\tilde{s}, \tilde{t}) \in \mathcal{Q}^{\text{iter}}$, define

$$\delta_\alpha^{(\text{subs})}(\mathbf{s}, \tilde{\mathbf{s}}) := \big((s - \tilde{s})^2 + (\max\{1 - |s|, 1 - |\tilde{s}|\})^2(t - \tilde{t})^2\big)^{\frac{\alpha}{2}}.$$

For an open set $U \subset \mathcal{Q}^{\text{iter}}$, define

$$\|u\|_{m,0,U}^{(\sigma),(\text{subs})} := \sum_{0 \le k+l \le m} \sup_{\mathbf{s} \in U} \big((1 - |s|)^{k-\sigma}|\partial_s^k \partial_t^l u(\mathbf{s})|\big),$$

$$[u]_{m,\alpha,U}^{(\sigma),(\text{subs})} := \sup_{\mathbf{s} \ne \tilde{\mathbf{s}} \in U} \Big(\min\big\{(1 - |s|)^{\alpha+k-\sigma}, (1 - |\tilde{s}|)^{\alpha+k-\sigma}\big\} \frac{|\partial_s^k \partial_t^l u(\mathbf{s}) - \partial_s^k \partial_t^l u(\tilde{\mathbf{s}})|}{\delta_\alpha^{(\text{subs})}(\mathbf{s}, \tilde{\mathbf{s}})} \Big),$$

$$\|u\|_{m,\alpha,U}^{(\sigma),(\text{subs})} := \|u\|_{m,0,U}^{(\sigma),(\text{subs})} + [u]_{m,\alpha,U}^{(\sigma),(\text{subs})}.$$

(ii) (Hölder norms with parabolic scaling). For $\mathbf{s} = (s, t), \tilde{\mathbf{s}} = (\tilde{s}, \tilde{t}) \in \mathcal{Q}^{\text{iter}}$, define

$$\delta_\alpha^{(\text{par})}(\mathbf{s}, \tilde{\mathbf{s}}) := \big((s - \tilde{s})^2 + \max\{1 - |s|, 1 - |\tilde{s}|\}(t - \tilde{t})^2\big)^{\frac{\alpha}{2}}.$$

For an open set $U \subset \mathcal{Q}^{\text{iter}}$, define

$$\|u\|_{m,0,U}^{(\sigma),(\text{par})} := \sum_{0 \le k+l \le m} \sup_{\mathbf{s} \in U} \big((1 - |s|)^{k+\frac{l}{2}-\sigma}|\partial_s^k \partial_t^l u(\mathbf{s})|\big),$$

$$[u]_{m,\alpha,U}^{(\sigma),(\text{par})} := \sum_{k+l=m} \sup_{\mathbf{s} \ne \tilde{\mathbf{s}} \in U} \Big(\min\big\{(1 - |s|)^{\alpha+k+\frac{l}{2}-\sigma}, (1 - |\tilde{s}|)^{\alpha+k+\frac{l}{2}-\sigma}\big\} \frac{|\partial_s^k \partial_t^l u(\mathbf{s}) - \partial_s^k \partial_t^l u(\tilde{\mathbf{s}})|}{\delta_\alpha^{(\text{par})}(\mathbf{s}, \tilde{\mathbf{s}})} \Big),$$

$$\|u\|_{m,\alpha,U}^{(\sigma),(\text{par})} := \|u\|_{m,0,U}^{(\sigma),(\text{par})} + [u]_{m,\alpha,U}^{(\sigma),(\text{par})}.$$

For a constant $r \in (0,1)$, denote
(4.1.51)
$$\mathcal{Q}_r^{\mathcal{O}} := \mathcal{Q}^{\mathrm{iter}} \cap \{-1 < s < -1+r\}, \ \ \mathcal{Q}_r^{\mathcal{N}} := \mathcal{Q}^{\mathrm{iter}} \cap \{1-r < s < 1\}, \ \ \mathcal{Q}_r^{\mathrm{int}} := \mathcal{Q}^{\mathrm{iter}} \cap \{|s| < 1-r\}.$$

REMARK 4.11 (Compact embedding properties of the norms in Definition 4.10). For $m \in \mathbb{Z}_+$, $\alpha \in [0,1)$, $\sigma > 0$, and an open bounded set U in $\mathbb{R}^2$, let $C_{(\sigma),\mathrm{par}}^{m,\alpha}(U)$ be the completion under the norm, $\|\cdot\|_{m,\alpha,U}^{(\sigma),(\mathrm{par})}$, of the set of all smooth functions whose $\|\cdot\|_{m,\alpha,U}^{(\sigma),(\mathrm{par})}$–norms are finite. Moreover, let $C_{(\sigma),(\mathrm{subs})}^{m,\alpha}(U)$ be the completion in norm $\|\cdot\|_{m,\alpha,U}^{(\sigma),(\mathrm{subs})}$ of the set of all smooth functions whose $\|\cdot\|_{m,\alpha,U}^{(\sigma),(\mathrm{subs})}$–norms are finite. Then the following compact embedding properties hold:

(i) Let $r \in (0,1)$, $\alpha, \hat{\alpha} \in [0,1)$ with $\alpha < \hat{\alpha}$, and $m \in \{1,2\}$. Then $C_{(1+\hat{\alpha}),(\mathrm{sub})}^{m,\hat{\alpha}}(\mathcal{Q}_r^{\mathcal{O}})$ is compactly embedded into $C_{(1+\alpha),(\mathrm{sub})}^{m,\alpha}(\mathcal{Q}_r^{\mathcal{O}})$; see [**11**, Corollary 17.2.7].

(ii) Let m_1 and m_2 be nonnegative integers, $\alpha_1, \alpha_2 \in [0,1)$, and $m_1 + \alpha_1 > m_2 + \alpha_2$, and let $\sigma_1 > \sigma_2 > 0$. Then $C_{(\sigma_1),(\mathrm{par})}^{m_1,\alpha_1}(U)$ is compactly embedded into $C_{(\sigma_2),(\mathrm{par})}^{m_2,\alpha_2}(U)$; see [**11**, Lemma 4.6.3].

For $\varepsilon_0 > 0$ from Proposition 3.36 and s_β from (4.1.29), define
(4.1.52)
$$\varepsilon_0' := \frac{\varepsilon_0}{\max_{\beta \in [0,\beta_{\mathrm{d}}^{(v_\infty)}]} s_\beta}.$$

PROPOSITION 4.12. For each $\bar{\beta} \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$, there exist constants $M > 0$ and $\bar{\alpha} \in (0, \frac{1}{3}]$ depending only on $(v_\infty, \gamma, \bar{\beta})$ such that, for any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{0 \le \beta \le \bar{\beta}\}$, $u : \mathcal{Q}^{\mathrm{iter}} \to \mathbb{R}$ defined by (4.1.50) satisfies
(4.1.53)
$$\|u\|_{C^{2,\bar{\alpha}}(\mathcal{Q}_{\varepsilon_0'/4}^{\mathrm{int}})} + \|u\|_{2,\bar{\alpha},\mathcal{Q}_{\varepsilon_0'}^{\mathcal{N}}}^{(2),(\mathrm{par})} + \|u\|_{2,\bar{\alpha},\mathcal{Q}_{\varepsilon_0'}^{\mathcal{O}}}^{(1+\bar{\alpha}),(\mathrm{par})} + \|u\|_{1,\bar{\alpha},\mathcal{Q}_{\varepsilon_0'}^{\mathcal{O}}}^{(1+\bar{\alpha}),(\mathrm{subs})} \le M.$$

PROOF. We divide the proof into six steps.

1. *Estimate of u away from $s = -1$*: A direct computation by using Corollary 3.19, Proposition 3.26, Lemma 4.2, Proposition 4.6, (4.1.45), and (4.1.50) shows that, for any $\alpha \in (0,1)$, there exists a constant $M_1 > 0$ depending only on $(v_\infty, \gamma, \alpha)$ such that
(4.1.54)
$$\|u\|_{C^{2,\alpha}(\mathcal{Q}_{\varepsilon_0'/4}^{\mathrm{int}})} + \|u\|_{2,\alpha,\mathcal{Q}_{\varepsilon_0'}^{\mathcal{N}}}^{(2),(\mathrm{par})} \le M_1$$

for any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$.

2. To obtain the *a priori* estimates of u near $s = -1$, the following two embedding inequalities from [**11**] are applied in the next two steps:

LEMMA 4.13 (Lemma 17.2.10 in [**11**]). *For a nonnegative integer m, $\alpha \in (0,1)$, and $\sigma > 0$, let both norms $\|\cdot\|_{m,\alpha,U}^{(\sigma),(\mathrm{subs})}$ and $\|\cdot\|_{m,\alpha,U}^{(\sigma),(\mathrm{par})}$ be defined in Definition 4.10. For $r \in (0,1]$, there exists a constant $C > 0$ independent of (r,α) such that*
$$\|u\|_{m,\alpha,\mathcal{Q}_r^{\mathcal{O}}}^{(\sigma),(\mathrm{par})} \le \|u\|_{m,\alpha,\mathcal{Q}_r^{\mathcal{O}}}^{(\sigma),(\mathrm{subs})}.$$

LEMMA 4.14 (Lemma 17.2.11 in [**11**]). *For a nonnegative integer m, $\alpha \in (0, \frac{1}{3}]$, $\sigma > 0$, and $r \in (0,1)$, there exists a constant $C > 0$ independent of (r,α) such that*
$$\|u\|_{1,\alpha,\mathcal{Q}_r^{\mathcal{O}}}^{(1+\alpha),(\mathrm{subs})} \le C\|u\|_{2,0,\mathcal{Q}_r^{\mathcal{O}}}^{(2),(\mathrm{par})}.$$

The estimates of u near $s = -1$ for the admissible solution are given for two cases separately: (i) $\beta \in [0, \beta_{\text{s}}^{(v_\infty)})$ and (ii) $\beta \in [\beta_{\text{s}}^{(v_\infty)}, \bar{\beta}]$.

3. *Estimate of u near $s = -1$ for $\beta \in [0, \beta_{\text{s}}^{(v_\infty)})$:* For each $\beta \in [0, \beta_{\text{d}}^{(v_\infty)}]$, by (4.1.1), (4.1.31), and Definition 4.15, we have

$$(4.1.55) \qquad u(s,t) = (\varphi - \varphi_{\mathcal{O}})(x, y) \qquad \text{for } (s,t) \in \mathcal{Q}^{\text{iter}} \cap \{-1 < s < -1 + \varepsilon_0'\}$$

with

$$(s,t) = \Big(L_\beta(x + u_{\mathcal{O}} - c_{\mathcal{O}}), \frac{y}{(\mathfrak{g}_{\text{sh}} \circ L_\beta)(x + u_{\mathcal{O}} - c_{\mathcal{O}})} \Big)$$

for the (x, y)–coordinates defined by (3.5.2). Differentiating (4.1.55), we have

$$(4.1.56) \quad \begin{aligned} u_s &= \frac{c_{\mathcal{N}} - s_\beta}{2} \psi_x + t \mathfrak{g}_{\text{sh}}' \psi_y, \qquad u_t = \mathfrak{g}_{\text{sh}} \psi_y, \\ u_{ss} &= \Big(\frac{c_{\mathcal{N}} - s_\beta}{2} \Big)^2 \psi_{xx} + 2 t \mathfrak{g}_{\text{sh}}' \frac{c_{\mathcal{N}} - s_\beta}{2} \psi_{xy} + t \mathfrak{g}_{\text{sh}}'' \psi_y + (t \mathfrak{g}_{\text{sh}}')^2 \psi_{yy}, \\ u_{st} &= \mathfrak{g}_{\text{sh}}' \psi_y + \frac{c_{\mathcal{N}} - s_\beta}{2} \mathfrak{g}_{\text{sh}} \psi_{xy} + t \mathfrak{g}_{\text{sh}}' \mathfrak{g}_{\text{sh}} \psi_{yy}, \\ u_{tt} &= \mathfrak{g}_{\text{sh}}^2 \psi_{yy}. \end{aligned}$$

A direct computation by using (4.1.50) and Propositions 3.30 and 3.32 shows that, for $\beta \in [0, \beta_{\text{s}}^{(v_\infty)})$ and $\alpha \in (0, 1)$, there exists a constant $C > 0$ depending only on $(v_\infty, \gamma, \alpha)$ such that

$$(4.1.57) \qquad \qquad \qquad \|u\|_{2,\alpha, \mathcal{Q}_{\varepsilon_0'}^{\mathcal{O}}}^{(2),(\text{par})} \leq C.$$

Furthermore, (4.1.57), combined with Lemma 4.14, implies that there exists a constant $M_2' > 0$ depending only on (v_∞, γ) such that

$$(4.1.58) \qquad \qquad \qquad \|u\|_{1,\frac{1}{3}, \mathcal{Q}_{\varepsilon_0'}^{\mathcal{O}}}^{(1+\frac{1}{3}),(\text{subs})} \leq M_2'$$

for any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{0 \leq \beta < \beta_{\text{s}}^{(v_\infty)}\}$. Combining the two estimates (4.1.57)–(4.1.58) together, we have

$$(4.1.59) \qquad \qquad \|u\|_{2,\frac{1}{3}, \mathcal{Q}_{\varepsilon_0'}^{\mathcal{O}}}^{(1+\frac{1}{3}),(\text{par})} + \|u\|_{1,\frac{1}{3}, \mathcal{Q}_{\varepsilon_0'}^{\mathcal{O}}}^{(1+\frac{1}{3}),(\text{subs})} \leq M_2$$

for a constant $M_2 > 0$ depending only on (v_∞, γ).

4. *Estimate of u near $s = -1$ for $\beta \in [\beta_{\text{s}}^{(v_\infty)}, \beta_{\text{s}}^{(v_\infty)} + \sigma_3]$:* Denote $\psi := \varphi - \varphi_{\mathcal{O}}$. By Proposition 3.39, any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta_{\text{s}}^{(v_\infty)} \leq \beta \leq \beta_{\text{s}}^{(v_\infty)} + \sigma_3\}$ satisfies

$$(4.1.60) \qquad \qquad \qquad \psi(P_\beta) = |D\psi(P_\beta)| = 0.$$

Regarding ψ as a function of (x,y) in $\mathcal{D}_{\varepsilon_0}^{\mathcal{O}}$ for $\varepsilon_0 > 0$ from Proposition 3.36, one can directly check by using (4.1.60) that ψ satisfies the following estimate: For $\mathbf{x} = (x,y)$, $\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}) \in \mathcal{D}_{\varepsilon_0}^{\mathcal{O}}$,

(4.1.61)
$$
\|\psi\|_{2,\alpha,\mathcal{D}_{\varepsilon_0}^{\mathcal{O}}}^{\prime(-1-\alpha)} := \sum_{0 \leq k+l \leq 2} \sup_{\mathbf{x} \in \mathcal{D}_{\varepsilon_0}^{\mathcal{O}}} \left(|x - x_{P_\beta}|^{k+l-(1+\alpha)} |\partial_x^k \partial_y^l \psi(\mathbf{x})| \right)
$$
$$
+ \sum_{k=0}^{2} \sup_{\mathbf{x},\tilde{\mathbf{x}} \in \mathcal{D}_{\varepsilon_0}^{\mathcal{O}}, \mathbf{x} \neq \tilde{\mathbf{x}}} \left(\min\{|x - x_{P_\beta}|, |\tilde{x} - x_{P_\beta}|\} \frac{|\partial_x^k \partial_y^{2-k} \psi(\mathbf{x}) - \partial_x^k \partial_y^{2-k} \psi(\tilde{\mathbf{x}})|}{|\mathbf{x} - \tilde{\mathbf{x}}|^\alpha} \right)
$$
$$
\leq \kappa_1 \|\psi\|_{2,\alpha,\Omega \cap \mathcal{D}_{\varepsilon_0}^{\mathcal{O}}}^{(-1-\alpha),\{P_\beta\}}
$$

for some constant $\kappa_1 > 0$ depending only on $(v_\infty, \gamma, \alpha)$.

Since $\mathfrak{g}_{\mathrm{sh}}(-1) = 0$ for $\beta \geq \beta_{\mathrm{s}}^{(v_\infty)}$, Proposition 4.6(e) implies that

$$
\frac{1 - |s|}{\hat{k}} \leq \mathfrak{g}_{\mathrm{sh}}(s) \leq \hat{k}(1 - |s|) \qquad \text{for } s \in [-1, -1 + \varepsilon_0'].
$$

Then, following the calculations in the proof of [**11**, Lemma 17.2.5], we obtain from (4.1.56) and Remark 4.7 that

$$
\|u\|_{2,\alpha,\mathcal{Q}_{\varepsilon_0'}^{\mathcal{O}}}^{(1+\alpha),(\mathrm{subs})} \leq \kappa_2 \|\psi\|_{2,\alpha,\mathcal{D}_{\varepsilon_0}^{\mathcal{O}}}^{\prime(-1-\alpha)}
$$

for some constant $\kappa_2 > 0$ depending only on $(v_\infty, \gamma, \alpha)$.

By Corollary 3.19 and Proposition 3.39, for each $\alpha \in (0,1)$, there exists a constant $C > 0$ depending only on $(v_\infty, \gamma, \alpha)$ such that any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta_{\mathrm{s}}^{(v_\infty)} \leq \beta \leq \beta_{\mathrm{s}}^{(v_\infty)} + \sigma_3\}$ satisfies

(4.1.62)
$$
\|\psi\|_{2,\alpha,\Omega \cap \mathcal{D}_{\varepsilon_0}^{\mathcal{O}}}^{(-1-\alpha),\{P_\beta\}} \leq C
$$

for $\varepsilon_0 > 0$ from Proposition 3.36. Therefore, there exists a constant $M_3 > 0$ depending only on $(v_\infty, \gamma, \alpha)$ such that u given by (4.1.50) associated with φ satisfies

(4.1.63)
$$
\|u\|_{2,\alpha,\mathcal{Q}_{\varepsilon_0'}^{\mathcal{O}}}^{(1+\alpha),(\mathrm{par})} \leq \|u\|_{2,\alpha,\mathcal{Q}_{\varepsilon_0'}^{\mathcal{O}}}^{(1+\alpha),(\mathrm{subs})} \leq M_3.
$$

5. *Estimate of u near $s = -1$ for $\beta \in [\beta_{\mathrm{s}}^{(v_\infty)} + \frac{\sigma_3}{2}, \bar{\beta}]$:* By Propositions 3.42 and 4.6, there exists $\hat{\alpha} \in (0,1)$ depending on $(v_\infty, \gamma, \bar{\beta})$ so that $\psi = \varphi - \varphi_{\mathcal{O}}$ still satisfies estimate (4.1.62) for all $\beta \in [\beta_{\mathrm{s}}^{(v_\infty)} + \frac{\sigma_3}{2}, \bar{\beta}]$ and $\alpha \in (0, \hat{\alpha}]$. Then there exists $M_4 > 0$ depending only on $(v_\infty, \gamma, \bar{\beta})$ such that any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta_{\mathrm{s}}^{(v_\infty)} + \frac{\sigma_2}{2} \leq \beta \leq \bar{\beta}\}$ satisfies estimate (4.1.63) with $\alpha = \hat{\alpha}$ and $M_3 = M_4$.

6. Finally, (4.1.53) is proved by choosing $\bar{\alpha} = \min\{\hat{\alpha}, \frac{1}{3}\}$ and $M = 4\max\{M_1, M_2, M_3, M_4\}$. $\square$

4.2. Mapping the Functions in $\mathcal{Q}^{\mathrm{iter}}$ to Approximate Admissible Solutions

Fix $\gamma \geq 1$ and $v_\infty > 0$. For each $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]$, let Q^β be defined by Definition 4.1(iii). For each $s^* \in (-1, 1)$, define

(4.2.1)
$$
Q^\beta(s^*) := Q^\beta \cap (\mathcal{G}_1^\beta)^{-1}(\{s = s^*\}).
$$

For each $\beta \in [0, \frac{\pi}{2})$, let φ_β^* be defined by (4.1.42). Then

$$
\inf_{Q^\beta(-1)} (\varphi_\infty - \varphi_\beta^*) < 0 \leq \sup_{Q^\beta(-1)} (\varphi_\infty - \varphi_\beta^*).
$$

In particular, the nonstrict inequality on the right above becomes strict when $\beta < \beta_{\text{s}}^{(v_\infty)}$ and becomes an equality when $\beta \geq \beta_{\text{s}}^{(v_\infty)}$.

DEFINITION 4.15. Fix $\alpha \in (0,1)$, $\bar{\beta} \in (0, \beta_{\text{d}}^{(v_\infty)})$, and $\beta \in (0, \bar{\beta}]$. Let $u \in C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}})$ be a function satisfying that, for any $s \in (-1,1)$,

$$
(4.2.2) \qquad \inf_{Q^\beta(s)} (\varphi_\infty - \varphi_\beta^*) < u(s,1) < \sup_{Q^\beta(s)} (\varphi_\infty - \varphi_\beta^*).
$$

We define functions $\mathfrak{g}_{\text{sh}}^{(u,\beta)}$, $\mathfrak{F}^{(u,\beta)}$, and $\varphi^{(u,\beta)}$ as follows:

(i) By Lemma 4.9, for each $s \in (-1,1)$, there exists a unique $\bar{t}' > 0$ such that

$$
(\varphi_\infty - \varphi_\beta^*) \circ (\mathcal{G}_1^\beta)^{-1}(s, \bar{t}') = u(s,1).
$$

Define a function $\mathfrak{g}_{\text{sh}}^{(u,\beta)} : (-1,1) \to \mathbb{R}^+$ by

$$
(4.2.3) \qquad \mathfrak{g}_{\text{sh}}^{(u,\beta)}(s) = \bar{t}'.
$$

(ii) For $\mathfrak{g}_{\text{sh}}^{(u,\beta)}$ from (i), define $G_{2,\mathfrak{g}_{\text{sh}}^{(u,\beta)}}$ by (4.1.49). For $\mathcal{G}_1^\beta$ given by (4.1.31), define a mapping $\mathfrak{F}_{(u,\beta)} : \mathcal{Q}^{\text{iter}} \to Q^\beta$ by

$$
\mathfrak{F}_{(u,\beta)} = (\mathcal{G}_1^\beta)^{-1} \circ G_{2,\mathfrak{g}_{\text{sh}}^{(u,\beta)}}^{-1}.
$$

(iii) For $\mathfrak{F}_{(u,\beta)}$ from (ii), define the sets:

$$
\Gamma_{\text{shock}}(u,\beta) := \mathfrak{F}_{(u,\beta)}((-1,1) \times \{1\}), \qquad \Omega(u,\beta) := \mathfrak{F}_{(u,\beta)}(\mathcal{Q}^{\text{iter}}).
$$

Moreover, define a function $\varphi^{(u,\beta)}$ in $\Omega(u,\beta)$ by

$$
(4.2.4) \qquad \varphi^{(u,\beta)}(\boldsymbol{\xi}) = (u \circ \mathfrak{F}_{(u,\beta)}^{-1})(\boldsymbol{\xi}) + \varphi_\beta^*(\boldsymbol{\xi}) \qquad \text{for all } \boldsymbol{\xi} \in \Omega(u,\beta).
$$

For $\alpha \in (0,1)$ and $\bar{\beta} \in (0, \beta_{\text{d}}^{(v_\infty)})$, define

$$
(4.2.5) \qquad \mathfrak{S}_\alpha^{\bar{\beta}} := \left\{ (u,\beta) \in C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}}) \times [0, \bar{\beta}] : \begin{array}{l} (u,\beta) \text{ satisfy } (4.2.2) \text{ for each } s \in (-1,1) \\ \text{and } (u, Du)(\pm 1, \cdot) = (0, \mathbf{0}) \end{array} \right\}.
$$

The next lemma follows from Definition 4.15. For details of the proof, we refer to [**11**, Lemmas 12.2.7 and 17.2.13].

LEMMA 4.16. *Fix* $\alpha \in (0,1)$ *and* $\bar{\beta} \in (0, \beta_{\text{d}}^{(v_\infty)})$. *For each* $(u,\beta) \in \mathfrak{S}_\alpha^{\bar{\beta}}$, *the following properties hold*:

(a) $\mathfrak{g}_{\text{sh}}^{(u,\beta)} \in C^{1,\alpha}([-1,1])$.

(b) *For domain* Λ_β *defined by Definition* 2.23,

$$
\Omega(u,\beta) \cup \Gamma_{\text{shock}}(u,\beta) \subset Q^\beta \subset \Lambda_\beta.
$$

Denote $P_1 = \mathfrak{F}_{(u,\beta)}(-1,1)$, $P_2 = \mathfrak{F}_{(u,\beta)}(1,1)$, $P_3 = \mathfrak{F}_{(u,\beta)}(1,0)$, *and* $P_4 = \mathfrak{F}_{(u,\beta)}(-1,0)$. *Then* $\Gamma_{\text{shock}}(u,\beta)$ *is a* $C^{1,\alpha}$–*curve up to its endpoints* P_1 *and* P_2, *and is tangential to* $S_{\mathcal{O}}$ *at* P_1 *and to* $S_{\mathcal{N}}$ *at* P_2. *For* $\hat{f}_{\mathcal{O},0}$ *and* $\hat{f}_{\mathcal{N},0}$ *defined in Lemmas* 3.20 *and* 3.27,

$$
(4.2.6) \qquad \begin{array}{ll} \mathfrak{g}_{\text{sh}}^{(u,\beta)}(-1) = \hat{f}_{\mathcal{O},0}(x_\beta), & \mathfrak{g}_{\text{sh}}^{(u,\beta)}(1) = \hat{f}_{\mathcal{N},0}(0), \\[2mm] \dfrac{\mathrm{d}}{\mathrm{d}s} \mathfrak{g}_{\text{sh}}^{(u,\beta)}(-1) = \dfrac{c_{\mathcal{N}} - s_\beta}{2} \hat{f}_{\mathcal{O},0}'(x_\beta), & \dfrac{\mathrm{d}}{\mathrm{d}s} \mathfrak{g}_{\text{sh}}^{(u,\beta)}(1) = -\dfrac{c_{\mathcal{N}} - s_\beta}{2} \hat{f}_{\mathcal{N},0}'(0), \end{array}
$$

where s_β is defined by (4.1.29) and x_β is given by

$$x_\beta = \begin{cases} 0 & \text{if } \beta < \beta_{\mathrm{s}}^{(v_\infty)}, \\ x_{P_\beta} & \text{if } \beta \geq \beta_{\mathrm{s}}^{(v_\infty)}. \end{cases}$$

In the above, P_β is the ξ_1–intercept of $S_\mathcal{O}$, and x_{P_β} represents the x–coordinate of P_β in the (x, y)–coordinates defined by (3.5.2). Note that $\frac{\mathrm{d}^k}{\mathrm{d}s^k}\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}(\pm 1)$, $k = 0, 1$, are uniquely determined depending only on (v_∞, β), but independent of $u \in \mathfrak{S}_\alpha^{\bar\beta}$. Boundary $\partial\Omega(u, \beta)$ consists of $\Gamma_{\mathrm{wedge}} = \mathfrak{F}_{(u,\beta)}\big((-1, 1) \times \{0\}\big)$, $\Gamma_{\mathrm{sonic}}^\mathcal{N} = \mathfrak{F}_{(u,\beta)}\big(\{1\} \times (0, 1)\big)$, $\Gamma_{\mathrm{sonic}}^\mathcal{O} = \mathfrak{F}_{(u,\beta)}\big(\{-1\} \times (0, 1)\big)$, and $\Gamma_{\mathrm{shock}}(u, \beta) = \mathfrak{F}_{(u,\beta)}\big((-1, 1) \times \{1\}\big)$ which do not intersect at the points of their relative interiors.

(c) *Let $\delta_0 > 0$ be from Definition 4.1. Let the (x, y)–coordinates be defined by (3.5.2) near $\Gamma_{\mathrm{sonic}}^\mathcal{O}$, and by (3.4.18) near $\Gamma_{\mathrm{sonic}}^\mathcal{N}$. For a constant $\varepsilon > 0$, define the two sets $\Omega_\varepsilon^\mathcal{O}$ and $\Omega_\varepsilon^\mathcal{N}$ by*

$$\Omega_\varepsilon^\mathcal{O} := \mathcal{N}_{\varepsilon_0}(\Gamma_{\mathrm{sonic}}^{\mathcal{O},\delta_0}) \cap \{x_{P_1} < x < x_{P_1} + \varepsilon\} \cap \Omega(u, \beta),$$

$$\Omega_\varepsilon^\mathcal{N} := \mathcal{N}_{\varepsilon_0}(\Gamma_{\mathrm{sonic}}^{\mathcal{N},\delta_0}) \cap \{0 < x < \varepsilon\} \cap \Omega(u, \beta)$$

for $\varepsilon_0 > 0$ to be fixed, where $\mathcal{N}_r(\Gamma)$ denotes an open r–neighborhood of Γ. Then there exists a constant $\varepsilon_0 > 0$ depending only on (v_∞, γ) such that the following holds: for L_β defined by (4.1.30), define the two functions $\hat f_{\mathcal{O},\mathrm{sh}}$ and $\hat f_{\mathcal{N},\mathrm{sh}}$ by

$$\hat f_{\mathcal{O},\mathrm{sh}}(x) = \mathfrak{g}_{\mathrm{sh}}^{(u,\beta)} \circ L_\beta(x + u_\mathcal{O} - c_\mathcal{O}), \qquad \hat f_{\mathcal{N},\mathrm{sh}}(x) = \mathfrak{g}_{\mathrm{sh}}^{(u,\beta)} \circ L_\beta(c_\mathcal{N} - x).$$

Then

$$\Omega_\varepsilon^\mathcal{O} = \{(x, y) \,:\, x \in (x_{P_1}, x_{P_1} + \varepsilon),\, 0 < y < \hat f_{\mathcal{O},\mathrm{sh}}(x)\},$$

$$\Gamma_{\mathrm{shock}}(u, \beta) \cap \partial\Omega_\varepsilon^\mathcal{O} = \{(x, \hat f_{\mathcal{O},\mathrm{sh}}(x)) \,:\, x \in (x_{P_1}, x_{P_1} + \varepsilon)\},$$

$$\Gamma_{\mathrm{wedge}} \cap \partial\Omega_\varepsilon^\mathcal{O} = \{(x, 0) \,:\, x \in (x_{P_1}, x_{P_1} + \varepsilon)\},$$

$$\Gamma_{\mathrm{sonic}}^\mathcal{O} = \Gamma_{\mathrm{sonic}}^\mathcal{O} \cap \partial\Omega_\varepsilon^\mathcal{O} = \{(0, y) \,:\, 0 < y < \hat f_{\mathcal{O},\mathrm{sh}}(0)\},$$

and

$$\Omega_\varepsilon^\mathcal{N} = \{(x, y) \,:\, x \in (0, \varepsilon),\, 0 < y < \hat f_{\mathcal{N},\mathrm{sh}}(x)\},$$

$$\Gamma_{\mathrm{shock}}(u, \beta) \cap \partial\Omega_\varepsilon^\mathcal{N} = \{(x, \hat f_{\mathcal{N},\mathrm{sh}}(x)) \,:\, x \in (0, \varepsilon)\},$$

$$\Gamma_{\mathrm{wedge}} \cap \partial\Omega_\varepsilon^\mathcal{N} = \{(x, 0) \,:\, x \in (0, \varepsilon)\},$$

$$\Gamma_{\mathrm{sonic}}^\mathcal{N} = \Gamma_{\mathrm{sonic}}^\mathcal{N} \cap \partial\Omega_\varepsilon^\mathcal{N} = \{(0, y) \,:\, 0 < y < \hat f_{\mathcal{N},\mathrm{sh}}(0)\}.$$

(d) *Suppose that $(u, \beta), (\tilde u, \tilde\beta) \in \mathfrak{S}_\alpha^{\bar\beta}$ satisfy that $\|(u, \tilde u)\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\mathrm{iter}}})} < M$ for some constant $M > 0$. Then there exists a constant $C > 0$, depending only on $(v_\infty, \gamma, M, \alpha)$, satisfying*

the following estimates:

$$\|\mathfrak{g}_{\text{sh}}^{(u,\beta)}\|_{C^{1,\alpha}([-1,1])} + \|\mathfrak{F}_{(u,\beta)}\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}})} \leq C,$$

$$\|\mathfrak{g}_{\text{sh}}^{(u,\beta)} - \mathfrak{g}_{\text{sh}}^{(\tilde{u},\tilde{\beta})}\|_{C^{1,\alpha}([-1,1])} \leq C\big(\|u - \tilde{u}\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}})} + |\beta - \tilde{\beta}|\big),$$

$$\|\mathfrak{F}_{(u,\beta)} - \mathfrak{F}_{(\tilde{u},\tilde{\beta})}\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}})} \leq C\big(\|u - \tilde{u}\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}})} + |\beta - \tilde{\beta}|\big),$$

$$\|\varphi^{(u,\beta)} \circ \mathfrak{F}_{(u,\beta)} - \varphi^{(\tilde{u},\tilde{\beta})} \circ \mathfrak{F}_{(\tilde{u},\tilde{\beta})}\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}})} \leq C\big(\|u - \tilde{u}\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}})} + |\beta - \tilde{\beta}|\big),$$

$$\|(\varphi^{(u,\beta)} - \varphi_\beta^*) \circ \mathfrak{F}_{(u,\beta)} - (\varphi^{(\tilde{u},\tilde{\beta})} - \varphi_{\tilde{\beta}}^*) \circ \mathfrak{F}_{(\tilde{u},\tilde{\beta})}\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}})}$$
$$\leq C\big(\|u - \tilde{u}\|_{C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}})} + |\beta - \tilde{\beta}|\big).$$

(e) $\psi^{(u,\beta)} := \varphi^{(u,\beta)} - \max\{\varphi_\mathcal{O}, \varphi_\mathcal{N}\} = 0$ *holds on* $\Gamma_{\text{sonic}}^\mathcal{O} \cup \Gamma_{\text{sonic}}^\mathcal{N}$.

(f) *For* $\varepsilon > 0$, *let* $\hat{\varepsilon}$ *be defined by* (4.1.38). *Let* $\varepsilon_0 > 0$ *be the constant from* (c). *Assume that, for constants* $\alpha \in (0,1)$, $\sigma \in (1,2]$, *and* $M > 0$,

$$(4.2.7) \qquad \|u\|_{2,\alpha,\mathcal{Q}^{\text{iter}} \cap \{|s| < 1 - \frac{\hat{\varepsilon}_0}{10}\}} + \|u\|_{2,\alpha,\mathcal{Q}^{\text{iter}} \cap \{|s| > 1 - \hat{\varepsilon}_0\}}^{(\sigma),(\text{par})} \leq M.$$

Then there exist $C > 0$ *depending only on* $(v_\infty, \gamma, \alpha, \sigma)$ *and* $C_0 > 0$ *depending only on* (v_∞, γ) *such that*

$$(4.2.8) \qquad \|\mathfrak{g}_{\text{sh}}^{(u,\beta)}\|_{2,\alpha,[-1+\frac{\hat{\varepsilon}_0}{10}, 1-\frac{\hat{\varepsilon}_0}{10}]} + \|\mathfrak{g}_{\text{sh}}^{(u,\beta)} - \mathfrak{g}_\mathcal{O}\|_{2,\alpha,(-1,-1+\hat{\varepsilon}_0)}^{(\sigma),(\text{par})} + \|\mathfrak{g}_{\text{sh}}^{(u,\beta)} - \mathfrak{g}_\mathcal{N}\|_{2,\alpha,(1-\hat{\varepsilon}_0,1)}^{(\sigma),(\text{par})} \leq CM,$$

$\mathfrak{F}_{(0,\beta)}$ *in* $\{1 - |s| < \varepsilon_0\} \times (0,\infty)$ *defined by*

$$\mathfrak{F}_{(0,\beta)}(s,t') = \begin{cases} (G_{2,\mathfrak{g}_\mathcal{O}} \circ \mathcal{G}_1^\beta)^{-1}(s,t') & \text{for } s \in (-1, -1 + \hat{\varepsilon}_0), \\ (G_{2,\mathfrak{g}_\mathcal{N}} \circ \mathcal{G}_1^\beta)^{-1}(s,t') & \text{for } s \in (1 - \hat{\varepsilon}_0, 1) \end{cases}$$

satisfies

$$\|\mathfrak{F}_{(0,\beta)}\|_{C^3(\overline{\mathcal{Q}^{\text{iter}}} \cap \{|s| \geq 1 - \hat{\varepsilon}_0\})} \leq C_0,$$

$$\|\mathfrak{F}_{(u,\beta)}\|_{2,\alpha,\mathcal{Q}^{\text{iter}} \cap \{|s| < 1 - \frac{\hat{\varepsilon}_0}{10}\}} + \|\mathfrak{F}_{(u,\beta)} - \mathfrak{F}_{(0,\beta)}\|_{2,\alpha,\mathcal{Q}^{\text{iter}} \cap \{|s| > 1 - \hat{\varepsilon}_0\}}^{(\sigma),(\text{par})} \leq C.$$

(g) *Let* f_β *be from* (4.1.37). *For constants* $M > 0$ *and* $\delta_{\text{sh}} > 0$, *assume that* $(u,\beta) \in \mathfrak{G}_\alpha^{\bar{\beta}}$ *satisfies* (4.2.7) *and*

$$\min\big\{\mathfrak{g}_{\text{sh}}^{(u,\beta)}(-1) + \frac{s+1}{M}, \delta_{\text{sh}}\big\} \leq \mathfrak{g}_{\text{sh}}^{(u,\beta)}(s) \leq \min\big\{\mathfrak{g}_{\text{sh}}^{(u,\beta)}(-1) + M(s+1), f_\beta(s) - \frac{1}{M}\big\}$$

for all $-1 \leq s \leq 1$ *and* $\delta_{\text{sh}} \geq \mathfrak{g}_{\text{sh}}^{(u,\beta)}(-1)$. *Then, for any* $\varepsilon \in (0, \frac{1}{4}\min\{s_\beta, c_\mathcal{N}\})$, *there exists a constant* $C_\varepsilon > 0$ *depending only on* $(v_\infty, \gamma, \alpha, \delta_{\text{sh}}, \varepsilon, M)$ *such that*

$$\|\mathfrak{F}_{(u,\beta)}^{-1}\|_{2,\alpha,\Omega(u,\beta) \setminus (\overline{\Omega_\varepsilon^\mathcal{O}} \cup \overline{\Omega_\varepsilon^\mathcal{N}})} + \|\mathfrak{F}_{(u,\beta)}^{-1} - \mathfrak{F}_{(0,\beta)}^{(-1)}\|_{2,\alpha,\Omega_{\varepsilon_0}^\mathcal{N}}^{(\sigma),(\text{par})} \leq C_\varepsilon,$$

$$\|\varphi - \varphi_\beta^*\|_{2,\alpha,\Omega(u,\beta) \setminus (\overline{\Omega_\varepsilon^\mathcal{O}} \cup \overline{\Omega_\varepsilon^\mathcal{N}})} + \|\varphi - \varphi_\beta^*\|_{2,\alpha,\Omega_{\varepsilon_0}^\mathcal{N}}^{(\sigma),(\text{par})} \leq C_\varepsilon.$$

(h) *Let* (u,β) *and* $(\tilde{u},\tilde{\beta})$ *be as in* (d). *For any open set* $K \Subset \mathcal{Q}^{\text{iter}}$ *so that* $K \subset (-1 + \delta, 1 - \delta) \times (0,1)$ *for some* $\delta > 0$, *there exists a constant* $C_\delta > 0$ *depending only on* $(v_\infty, \gamma, \alpha, \sigma, \delta)$

such that

$$\|\mathfrak{F}_{(u,\beta)} - \mathfrak{F}_{(\tilde{u},\tilde{\beta})}\|_{C^{2,\alpha}(\overline{K})} \le C_\delta \big(\|(u-\tilde{u})(\cdot,1)\|_{C^{2,\alpha}([-1+\delta,1-\delta])} + |\beta - \tilde{\beta}|\big),$$

$$\|\varphi^{(u,\beta)} \circ \mathfrak{F}_{(u,\beta)} - \varphi^{(\tilde{u},\tilde{\beta})} \circ \mathfrak{F}_{(\tilde{u},\tilde{\beta})}\|_{C^{2,\alpha}(\overline{K})} \le C_\delta \big(\|u-\tilde{u}\|_{C^{2,\alpha}(\overline{K})} + |\beta - \tilde{\beta}|\big),$$

$$\|\psi^{(u,\beta)} \circ \mathfrak{F}_{(u,\beta)} - \psi^{(\tilde{u},\tilde{\beta})} \circ \mathfrak{F}_{(\tilde{u},\tilde{\beta})}\|_{C^{2,\alpha}(\overline{K})} \le C_\delta \big(\|u-\tilde{u}\|_{C^{2,\alpha}(\overline{K})} + |\beta - \tilde{\beta}|\big),$$

where $\psi^{(u,\beta)}$ is given by $\psi^{(u,\beta)} := \varphi^{(u,\beta)} - \varphi_\beta^*$ for each $(u,\beta) \in \mathfrak{G}_\alpha^{\bar{\beta}}$.

REMARK 4.17. By (4.1.1) and (4.2.6), for any $(u,\beta) \in \mathfrak{G}_\alpha^{\bar{\beta}}$, we have

$$\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}(1) = \sin^{-1}\big(\frac{\xi_2^{\mathcal{N}}}{c_{\mathcal{N}}}\big) > 0$$

Fix $\delta \in (0, \beta_{\mathrm{s}}^{(v_\infty)})$, and suppose that $(u,\beta) \in \mathfrak{G}_\alpha^{\bar{\beta}}$ and $\beta \in [0, \beta_{\mathrm{s}}^{(v_\infty)} - \delta]$. Then it follows from (3.5.22), (4.1.1), and (4.2.6) that there exists a constant $l_{\mathrm{so}} > 0$ depending only on $(v_\infty, \gamma, \delta)$ such that

$$\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}(-1) \ge l_{\mathrm{so}}.$$

Therefore, there exists $b \in (0,1)$ depending only on $(v_\infty, \gamma, \sigma, \delta, M)$ such that, for any $(u,\beta) \in \mathfrak{G}_\alpha^{\bar{\beta}}$ with $\beta \in [0, \beta_{\mathrm{s}}^{(v_\infty)} - \delta]$, $\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}$ satisfies

(4.2.9) $$b \le \mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}(s) \le b^{-1} \qquad \text{for all } s \in [-1,1].$$

Then there exist $\hat{C} > 0$ depending on $(v_\infty, \gamma, \alpha, \sigma, \delta)$ and $\hat{C}_0 > 0$ depending only on $(v_\infty, \gamma, \delta)$ such that

(4.2.10)
$$\|\mathfrak{F}_{(0,\beta)}^{-1}\|_{C^3(\overline{Q^\beta \cap \mathcal{D}_{\varepsilon_0}})} \le \hat{C}_0 \qquad \text{for } \mathcal{D}_{\varepsilon_0} = \mathcal{N}_{\varepsilon_0}(\Gamma_{\mathrm{sonic}}^{\mathcal{O},\delta_0}) \cup \mathcal{N}_{\varepsilon_0}(\Gamma_{\mathrm{sonic}}^{\mathcal{N},\delta_0}),$$

$$\|\mathfrak{F}_{(u,\beta)}^{-1}\|_{C^{2,\alpha}(\overline{\Omega(u,\beta)\setminus\mathcal{D}_{\varepsilon_0/10}})} + \|\mathfrak{F}_{(u,\beta)}^{-1} - \mathfrak{F}_{(0,\beta)}^{-1}\|_{2,\alpha,\Omega(u,\beta)\cap\mathcal{D}_{\varepsilon_0}}^{(\sigma),(\mathrm{par})} \le \hat{C}M.$$

Furthermore, $\varphi = \varphi^{(u,\beta)}$ defined by (4.2.4) corresponding to (u,β) satisfies

(4.2.11) $$\|\varphi - \varphi_\beta^*\|_{C^{2,\alpha}(\overline{\Omega(u,\beta)\setminus\mathcal{D}_{\varepsilon_0/10}})} + \|\varphi - \varphi_\beta^*\|_{2,\alpha,\Omega(u,\beta)\cap\mathcal{D}_{\varepsilon_0}}^{(\sigma),(\mathrm{par})} \le \hat{C}M.$$

4.3. Definition of the Iteration Set

DEFINITION 4.18. For $\varepsilon_0 > 0$ from *Lemma* 4.16(c), let $\hat{\varepsilon}_0$ be given by (4.1.38).

(i) Define $u^{(\mathrm{norm})} \in C^3(\overline{\mathcal{Q}^{\mathrm{iter}}})$ by (4.1.50) with $\beta = 0$ and $\varphi = \varphi_\mathcal{N}$. Note that $\varphi_\beta^* \equiv \varphi_\mathcal{N}$ in Q^β by (4.1.42) because $\varphi_\mathcal{O} = \varphi_\mathcal{N}$ when $\beta = 0$, which yields that

$$u^{(\mathrm{norm})} \equiv 0 \qquad \text{in } \mathcal{Q}^{\mathrm{iter}}.$$

(ii) For $\alpha \in (0,1)$ and $\alpha' \in (0,1]$, we introduce the norm:

$$\|u\|_{2,\alpha,\mathcal{Q}^{\mathrm{iter}}}^{(*,\alpha')} := \|u\|_{C^{2,\alpha}(\overline{\mathcal{Q}_{\hat{\varepsilon}_0/4}^{int}})} + \|u\|_{2,\alpha,\mathcal{Q}_{\hat{\varepsilon}_0}^{\mathcal{N}}}^{(1+\alpha'),(\mathrm{par})} + \|u\|_{2,\alpha,\mathcal{Q}_{\hat{\varepsilon}_0}^{\mathcal{O}}}^{(1+\alpha),(\mathrm{par})} + \|u\|_{1,\alpha,\mathcal{Q}_{\hat{\varepsilon}_0}^{\mathcal{O}}}^{(1+\alpha),(\mathrm{subs})},$$

where $\mathcal{Q}_{\hat{\varepsilon}_0/4}^{int}$, $\mathcal{Q}_{\hat{\varepsilon}_0}^{\mathcal{N}}$, and $\mathcal{Q}_{\hat{\varepsilon}_0}^{\mathcal{O}}$ are defined in (4.1.51). Denote by $C_{(*,\alpha')}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}})$ the set of all $C^2(\mathcal{Q}^{\mathrm{iter}})$–functions whose $\|\cdot\|_{2,\alpha,\mathcal{Q}^{\mathrm{iter}}}^{(*,\alpha')}$–norms are finite. Note that $C_{(*,\alpha')}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}})$ is compactly embedded into $C_{(*,\tilde{\alpha}')}^{2,\tilde{\alpha}}(\mathcal{Q}^{\mathrm{iter}})$ whenever $0 \le \tilde{\alpha} < \alpha < 1$ and $0 \le \tilde{\alpha}' < \alpha' \le 1$.

For fixed $\gamma \geq 1, v_\infty > 0$, and $\beta_* \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$, we define the iteration set $\mathcal{K} \subset C^{1,\alpha}(\overline{\mathcal{Q}^{\mathrm{iter}}}) \times [0, \beta_*]$ for some appropriate $\alpha \in (0,1)$. For each $\beta \in [0, \beta_*]$, $\mathcal{K}_\beta := \{u \in C^{1,\alpha}(\overline{\mathcal{Q}^{\mathrm{iter}}}) : (u, \beta) \in \mathcal{K}\}$. In the definition to come, the iteration set $\mathcal{K}$ is given such that

- $\mathcal{K}_0$ contains $u^{(\mathrm{norm})}$;
- If β is sufficiently close to 0, then $u \in \mathcal{K}_\beta$ is also close to $u^{(\mathrm{norm})}$ in an appropriate norm;
- If β is away from 0, then any $\varphi^{(u,\beta)}$ given by (4.2.4) for $u \in \mathcal{K}_\beta$ satisfies the strict directional monotonicity properties (3.1.6)–(3.1.7);
- $\mathcal{K}_\beta$ varies continuously on $\beta \in [0, \beta_*]$.

For $\gamma \geq 1$ and $v_\infty > 0$, fix $\beta_* \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$. For $\bar{\alpha} \in (0, \frac{1}{3}]$ from Proposition 4.12, define

$$(4.3.1) \qquad\qquad\qquad\qquad \alpha_* := \frac{\bar{\alpha}}{2}.$$

Let $\varepsilon_0 > 0$ be from Lemma 4.16. For constants $\alpha \in (0, \alpha_*]$, $\alpha_1 \in (0,1)$, δ_1, δ_2, δ_3, $\varepsilon \in (0, \frac{\varepsilon_0}{2})$, and $N_1 > 1$ to be specified later, we now define the iteration set $\mathcal{K} \subset C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$.

DEFINITION 4.19. For fixed $\beta_* \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$, the iteration set $\mathcal{K} \subset C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$ is the set of all (u, β) satisfying the following properties:

(i) Fix $\alpha_1 = \frac{7}{8}$. Then (u, β) satisfies

$$\|u - u^{(\mathrm{norm})}\|_{2,\alpha,\mathcal{Q}^{\mathrm{iter}}}^{(*,\alpha_1)} < \mathscr{K}_1(\beta)$$

for $\mathscr{K}_1 \in C^{0,1}(\mathbb{R})$ given by

$$\mathscr{K}_1(\beta) = \begin{cases} \delta_1 & \text{if } \beta \leq \frac{\delta_1}{N_1}, \\ N_0 & \text{if } \beta \geq \frac{2\delta_1}{N_1}, \\ \text{linear} & \text{if } \beta \in (\frac{\delta_1}{N_1}, \frac{2\delta_1}{N_1}), \end{cases}$$

with $N_0 = \max\{10M, 1\}$ for constant M from Proposition 4.12.

(ii) For set $\mathfrak{S}_\alpha^{\beta_*}$ defined by (4.2.5), (u, β) is contained in $\mathfrak{S}_\alpha^{\beta_*}$. Moreover, let $\mathfrak{g}_{\mathrm{sh}} = \mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}$, $\Gamma_{\mathrm{shock}} = \Gamma_{\mathrm{shock}}(u, \beta)$, $\Omega = \Omega(u, \beta)$, and $\varphi = \varphi^{(u,\beta)}$ be defined by Definition 4.15.

(iii) Γ_{shock} and $\mathfrak{g}_{\mathrm{sh}}$ satisfy

$$\mathrm{dist}(\Gamma_{\mathrm{shock}}, B_1(O_\infty)) > N_2^{-1},$$

$$(4.3.2) \qquad \min\{\mathfrak{g}_{\mathrm{sh}}(-1) + N_3^{-1}(s+1), N_3^{-1}\} < \mathfrak{g}_{\mathrm{sh}}(s) < \min\{\mathfrak{g}_{\mathrm{sh}}(-1) + N_3(s+1), f_\beta(s) - N_3^{-1}\}$$

for all $-1 < s < 1$ with $N_2 = 2C$ for C from Proposition 3.7, and $N_3 = 2\hat{k}$ for $\hat{k}$ from Proposition 4.6(e) with $\mathfrak{g}_{\mathrm{sh}}(-1) \geq 0$, where f_β is defined by (4.1.37).

(iv) Let the (x, y)–coordinates be defined by (3.4.18) near $\Gamma_{\mathrm{sonic}}^{\mathcal{N}}$ and by (3.5.2) near $\Gamma_{\mathrm{sonic}}^{\mathcal{O}}$. For $\varphi_\beta = \max\{\varphi_{\mathcal{O}}, \varphi_{\mathcal{N}}\}$, denote $\psi := \varphi - \varphi_\beta$. For $r > 0$, let $\mathcal{D}_r^{\mathcal{O}}$ and $\mathcal{D}_r^{\mathcal{N}}$ be defined by

(4.1.2). Let φ and ψ satisfy the following:

(4.3.3) $\psi > \mathscr{K}_2(\beta) \quad$ in $\overline{\Omega} \setminus (\mathcal{D}^{\mathcal{O}}_{\frac{\varepsilon}{10}} \cup \mathcal{D}^{\mathcal{N}}_{\frac{\varepsilon}{10}})$,

(4.3.4) $\partial_{\boldsymbol{e}_{S_{\mathcal{O}}}}(\varphi_\infty - \varphi) < -\mathscr{K}_2(\beta) \qquad$ in $\overline{\Omega} \setminus \mathcal{D}^{\mathcal{O}}_{\frac{\varepsilon}{10}}$,

(4.3.5) $-\partial_{\xi_1}(\varphi_\infty - \varphi) < -\mathscr{K}_2(\beta) \qquad$ in $\overline{\Omega} \setminus \mathcal{D}^{\mathcal{N}}_{\frac{\varepsilon}{10}}$,

(4.3.6) $|\partial_x \psi(x,y)| < \dfrac{2-\mu_0}{1+\gamma} x \qquad$ in $\overline{\Omega} \cap (\mathcal{D}^{\mathcal{N}}_{\varepsilon_0} \setminus \mathcal{D}^{\mathcal{N}}_{\varepsilon/10})$,

(4.3.7) $|\partial_x \psi(x,y)| < \mathcal{K}_3(\beta) x \qquad$ in $\overline{\Omega} \cap (\mathcal{D}^{\mathcal{O}}_{\varepsilon_0} \setminus \mathcal{D}^{\mathcal{O}}_{\varepsilon/10})$,

(4.3.8) $|\partial_y \psi(x,y)| < N_4 x \qquad$ in $\overline{\Omega} \cap \big((\mathcal{D}^{\mathcal{O}}_{\varepsilon_0} \setminus \mathcal{D}^{\mathcal{O}}_{\varepsilon/10}) \cup (\mathcal{D}^{\mathcal{N}}_{\varepsilon_0} \setminus \mathcal{D}^{\mathcal{N}}_{\varepsilon/10})\big)$,

(4.3.9) $|(\partial_x \psi, \partial_y \psi)| < N_4 \varepsilon \qquad$ in $\overline{\Omega} \cap (\overline{\mathcal{D}^{\mathcal{O}}_{\varepsilon}} \cup \overline{\mathcal{D}^{\mathcal{N}}_{\varepsilon}})$,

(4.3.10) $\|\varphi - \varphi_{\mathcal{N}}\|_{C^{0,1}(\overline{\Omega})} + \|\varphi - \varphi_{\mathcal{O}}\|_{C^{0,1}(\overline{\Omega})} < N_5$,

(4.3.11) $\partial_{\boldsymbol{\nu}}(\varphi_\infty - \varphi) > \mu_1, \quad \partial_{\boldsymbol{\nu}}\varphi > \mu_1 \qquad$ on $\overline{\Gamma_{\text{shock}}}$,

for the unit normal vector $\boldsymbol{\nu}$ to Γ_{shock} towards the interior of Ω. In the above conditions, functions $\mathcal{K}_2, \mathcal{K}_3 \in C(\mathbb{R})$ are defined by

(4.3.12) $$\mathscr{K}_2(\beta) = \delta_2 \min\Big\{\beta - \frac{\delta_1}{N_1^2}, \frac{\delta_1}{N_1^2}\Big\},$$

$$\mathcal{K}_3(\beta) = \begin{cases} \frac{2-\mu_0}{1+\gamma} & \text{if } 0 \le \beta \le \beta_{\text{s}}^{(v_\infty)} + \frac{\sigma_2}{2}, \\ \text{linear} & \text{if } \beta_{\text{s}}^{(v_\infty)} + \frac{\sigma_2}{2} < \beta < \beta_{\text{s}}^{(v_\infty)} + \sigma_2, \\ N_4 & \text{if } \beta_{\text{s}}^{(v_\infty)} + \sigma_2 \le \beta, \end{cases}$$

for constants $\varepsilon_0, \sigma_2, \mu_0, \mu_1, N_4,$ and N_5 chosen as follows:

(iv-1) ε_0 is from Lemma 4.16.

(iv-2) $\sigma_2 > 0$ is from Lemma 3.36, and $\mu_0 = \frac{\delta}{2}$ for $\delta > 0$ from Lemmas 3.28 and 3.36.

(iv-3) $\mu_1 = \frac{\delta_1}{2}$ for $\delta_1 > 0$ from Corollary 3.17.

(iv-4) Choice of N_4: By (3.5.55)–(3.5.56), for each $\sigma \in (0, \beta_{\text{d}}^{(v_\infty)} - \beta_{\text{s}}^{(v_\infty)})$,

(4.3.13) $$\inf_{\beta_{\text{s}}^{(v_\infty)} + \sigma \le \beta < \beta_{\text{d}}^{(v_\infty)}} x_{P_\beta} = x_{P_\beta}\big|_{\beta = \beta_{\text{s}}^{(v_\infty)} + \sigma} =: x_\sigma > 0.$$

By Propositions 3.30, 3.32, and 3.39, there exists $C_1 > 0$ depending only on (v_∞, γ) such that any admissible solution $\varphi = \psi + \varphi_\beta$ for $\beta \in (0, \beta_{\text{s}}^{(v_\infty)} + \sigma_3]$ satisfies that $|(\partial_x, \partial_y)\psi(x,y)| \le C_1 x$ in $\overline{\Omega} \cap \mathcal{D}^{\mathcal{O}}_{\varepsilon_0}$.

Let $\bar{\alpha} \in (0,1)$ be from Proposition 4.12. By Proposition 3.42 and (4.3.13), any admissible solution $\varphi = \psi + \varphi_\beta$ for $\beta \ge \beta_{\text{s}}^{(v_\infty)} + \frac{\sigma_3}{2}$ satisfies

$$|(\partial_x, \partial_y)\psi(x,y)| \le C_2 x^{\bar{\alpha}} \le C_2\big(x_{P_\beta}\big|_{\beta = \beta_{\text{s}}^{(v_\infty)} + \frac{\sigma_3}{2}}\big)^{\bar{\alpha}-1} x \qquad \text{in } \overline{\Omega} \cap \mathcal{D}^{\mathcal{O}}_{\varepsilon_0}$$

for a constant $C_2 > 0$ depending only on $(v_\infty, \gamma, \beta_*)$. Then there exists a constant $C_1^* > 0$ depending only on $(v_\infty, \gamma, \beta_*, \sigma)$ such that any admissible solution $\varphi = \psi + \varphi_\beta$ for $\beta \in (0, \beta_*]$ satisfies

$$|(\partial_x, \partial_y)\psi(x,y)| \le C_1^* x \qquad \text{in } \overline{\Omega} \cap \mathcal{D}^{\mathcal{O}}_{\varepsilon_0}.$$

By combining this inequality with Proposition 3.26, there exists a constant $C^* > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ such that any admissible solution φ for $\beta \in [0, \beta_*]$ satisfies

$$|(\partial_x, \partial_y)\psi(x,y)| \leq C^* x \qquad \text{in } \overline{\Omega} \cap (\mathcal{D}^{\mathcal{O}}_{\varepsilon_0} \cup \mathcal{D}^{\mathcal{N}}_{\varepsilon_0}).$$

We choose $N_4 := 10 C^*$.

(iv-5) By Lemma 3.5 and the continuous dependence of $u_{\mathcal{O}}$ and $c_{\mathcal{O}}$ on $\beta \in [0, \frac{\pi}{2})$, there exists a constant $\hat{C} > 0$ depending only on (v_∞, γ) such that any admissible solution φ for $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)})$ satisfies

$$\|\varphi - \varphi_{\mathcal{N}}\|_{C^{0,1}(\overline{\Omega})} + \|\varphi - \varphi_{\mathcal{O}}\|_{C^{0,1}(\overline{\Omega})} \leq \hat{C}.$$

For such $\hat{C} > 0$, we choose $N_5 := 10 \hat{C}$.

(v) Let $c(|D\varphi|^2, \varphi)$ be defined by

(4.3.14) $$c(|D\varphi|^2, \varphi) := \rho^{\frac{\gamma-1}{2}}(|D\varphi|^2, \varphi)$$

for $\rho(|\mathbf{p}|^2, z)$ given by (2.4.2). Then φ satisfies

(4.3.15) $$\frac{|D\varphi(\boldsymbol{\xi})|^2}{c^2(|D\varphi(\boldsymbol{\xi})|^2, \varphi(\boldsymbol{\xi}))} < 1 - \tilde{\mu} \operatorname{dist}^\flat(\boldsymbol{\xi}, \Gamma^{\mathcal{O}}_{\mathrm{sonic}} \cup \Gamma^{\mathcal{N}}_{\mathrm{sonic}}) \qquad \text{for } \boldsymbol{\xi} \in \overline{\Omega} \setminus (\mathcal{D}^{\mathcal{N}}_{\varepsilon/10} \cup \mathcal{D}^{\mathcal{O}}_{\varepsilon/10}).$$

In (4.3.15), $\tilde{\mu} = \frac{\mu_{\mathrm{el}}}{2}$ for $\mu_{\mathrm{el}} > 0$ from Remark 3.16.

(vi) $\rho(|D\varphi|^2, \varphi)$ given by (2.4.2) satisfies

$$\frac{a_*}{2} < \rho(|D\varphi|^2, \varphi) < 2C \qquad \text{in } \overline{\Omega} \setminus (\mathcal{D}^{\mathcal{N}}_{\varepsilon/10} \cup \mathcal{D}^{\mathcal{O}}_{\varepsilon/10}),$$

for $a_* = (\frac{2}{\gamma+1})^{\frac{1}{\gamma-1}}$ and C from (3.1.26) in Lemma 3.5. For such constants, denote

$$\rho_{\min} := \frac{a_*}{2}, \qquad \rho_{\max} = 2C.$$

(vii) The boundary value problem

(4.3.16) $$\begin{cases} \mathcal{N}_{(u,\beta)}(\hat{\phi}) = A_{11}\hat{\phi}_{\xi_1\xi_1} + 2A_{12}\hat{\phi}_{\xi_1\xi_2} + A_{22}\hat{\phi}_{\xi_2\xi_2} = 0 & \text{in } \Omega, \\ \mathcal{M}_{(u,\beta)}(D\hat{\phi}, \hat{\phi}, \boldsymbol{\xi}) = 0 & \text{on } \Gamma_{\mathrm{shock}}, \\ \hat{\phi} = \max\{\varphi_{\mathcal{N}}, \varphi_{\mathcal{O}}\} - \varphi_{\mathcal{N}} & \text{on } \Gamma^{\mathcal{O}}_{\mathrm{sonic}} \cup \Gamma^{\mathcal{N}}_{\mathrm{sonic}}, \\ \hat{\phi}_{\xi_2} = 0 & \text{on } \Gamma_{\mathrm{wedge}} \end{cases}$$

has a unique solution $\hat{\phi} \in C^2(\Omega) \cap C^1(\overline{\Omega})$, where $\mathcal{N}_{(u,\beta)}$ and $\mathcal{M}_{(u,\beta)}$ are determined by (u, β) in §4.4. Moreover, this solution satisfies that $\hat{u}(s,t)$, defined by

(4.3.17) $$\hat{u}(s,t) := (\hat{\phi} + \varphi_{\mathcal{N}} - \varphi_\beta^*) \circ \mathfrak{F}_{(u,\beta)}(s,t) \qquad \text{in } \mathcal{Q}^{\mathrm{iter}},$$

satisfies

(4.3.18) $$\|\hat{u} - u\|_{2, \frac{\alpha}{2}, \mathcal{Q}^{\mathrm{iter}}}^{(*, \alpha_1)} < \delta_3.$$

REMARK 4.20. By (4.1.45), the boundary condition $\hat{\phi} = \max\{\varphi_{\mathcal{N}}, \varphi_{\mathcal{O}}\} - \varphi_{\mathcal{N}}$ on $\Gamma^{\mathcal{O}}_{\mathrm{sonic}} \cup \Gamma^{\mathcal{N}}_{\mathrm{sonic}}$ given in (4.3.16) is equivalent to

$$\hat{\phi} = \begin{cases} \varphi_{\mathcal{O}} - \varphi_{\mathcal{N}} & \text{on } \Gamma^{\mathcal{O}}_{\mathrm{sonic}}, \\ 0 & \text{on } \Gamma^{\mathcal{N}}_{\mathrm{sonic}}. \end{cases}$$

REMARK 4.21. For a fixed $\beta_* \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$, let the iteration set $\mathcal{K}$ be defined by Definition 4.19. For each $(u, \beta) \in \mathcal{K}$, let $\mathfrak{g}_{\mathrm{sh}} = \mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}$, $\Omega = \Omega(u, \beta)$, $\Gamma_{\mathrm{shock}} = \Gamma_{\mathrm{shock}}(u, \beta)$, and $\varphi = \varphi^{(u,\beta)}$ be defined by Definition 4.15. Then there exist constants $M_{\mathrm{dom}} > 0$ depending only on (v_∞, γ), $C > 0$ depending only on $(v_\infty, \gamma, \alpha)$, and $C_{\beta_*} > 0$ depending only on $(v_\infty, \gamma, \beta_*, \alpha)$ such that the following properties hold:

(i) Let $\mathfrak{g}_{\mathcal{O}}$ and $\mathfrak{g}_{\mathcal{N}}$ be from (4.1.39). For N_0 from Definition 4.19 (i), $\mathfrak{g}_{\mathrm{sh}}$ satisfies

(4.3.19)
$$\|\mathfrak{g}_{\mathrm{sh}}\|_{2,\alpha,(-1,1)}^{(-1-\alpha),\{\pm 1\}} \le C N_0,$$
$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}(\mathfrak{g}_{\mathrm{sh}} - \mathfrak{g}_{\mathcal{O}})(-1) = \frac{\mathrm{d}^k}{\mathrm{d}s^k}(\mathfrak{g}_{\mathrm{sh}} - \mathfrak{g}_{\mathcal{N}})(1) = 0 \qquad \text{for } k = 0, 1.$$

(ii) Γ_{shock} is a $C^{1,\alpha}$–curve up to its endpoints. Furthermore, $\Gamma_{\mathrm{shock}} \cap \mathcal{D}_{\varepsilon_0}^{\mathcal{O}}$ and $\Gamma_{\mathrm{shock}} \cap \mathcal{D}_{\varepsilon_0}^{\mathcal{N}}$ are graphs $y = \hat{f}_{\mathcal{O},\mathrm{sh}}(x)$ and $y = \hat{f}_{\mathcal{N},\mathrm{sh}}(x)$ for

(4.3.20) $\hat{f}_{\mathcal{O},\mathrm{sh}}(x) = (\mathfrak{g}_{\mathrm{sh}} \circ L_\beta^{-1})(s_\beta + x), \qquad \hat{f}_{\mathcal{N},\mathrm{sh}}(x) = (\mathfrak{g}_{\mathrm{sh}} \circ L_\beta^{-1})(c_\mathcal{N} - x),$

with $\hat{f}_{\mathcal{N},\mathrm{sh}}$ and $\hat{f}_{\mathcal{O},\mathrm{sh}}$ satisfying that

$$\|\hat{f}_{\mathcal{N},\mathrm{sh}} - \hat{f}_{\mathcal{N},0}\|_{2,\alpha,(0,\varepsilon_0)}^{(1+\alpha_1),(\mathrm{par})} + \|\hat{f}_{\mathcal{O},\mathrm{sh}} - \hat{f}_{\mathcal{O},0}\|_{2,\alpha,(0,\varepsilon_0)}^{(1+\alpha),(\mathrm{par})} < C\mathscr{K}_1(\beta)$$

for $\hat{f}_{\mathcal{N},0}$ and $\hat{f}_{\mathcal{O},0}$ from Lemmas 3.20(e) and 3.27(e), respectively.

(iii) $\Omega \subset B_{M_{\mathrm{dom}}}(\mathbf{0})$.

(iv) $\psi = \varphi - \varphi_\beta^*$ satisfies

$$D^k \psi = 0 \qquad \text{on } \overline{\Gamma_{\mathrm{sonic}}^{\mathcal{O}}} \cup \overline{\Gamma_{\mathrm{sonic}}^{\mathcal{N}}} \qquad \text{for } k = 0, 1,$$
$$\|\psi\|_{C^{1,\alpha}(\overline{\Omega})} < C\mathscr{K}_1(\beta).$$

By Lemma 3.27(e) and (4.3.19), we can adjust ε_0 depending on (v_∞, γ) to satisfy

$$0 < \frac{1}{2}\mathfrak{g}_{\mathcal{O}}'(-1) \le \mathfrak{g}_{\mathrm{sh}}'(s) \le 4\mathfrak{g}_{\mathcal{O}}'(-1) \qquad \text{for all } s \in [-1, -1 + \hat{\varepsilon}_0].$$

Then, for each $\beta < \beta_{\mathrm{s}}^{(v_\infty)}$,

$$|\partial_y \psi(x, y)| = \frac{|u_t(s, t)|}{\mathfrak{g}_{\mathrm{sh}}(s)} \le \|u\|_{1,\alpha,\mathcal{Q}_{\varepsilon_0}^{\mathcal{O}}}^{(1+\alpha),(\mathrm{subs})} \frac{(1 - |s|)^{1+\alpha}}{\mathfrak{g}_{\mathrm{sh}}(-1)} \le C x^{\frac{1}{2}+\alpha} \qquad \text{for } (x, y) \in \Omega \cap \mathcal{D}_{\hat{r}}^{\mathcal{O}},$$

where $\hat{r} = \min\{\mathfrak{g}_{\mathrm{sh}}^2(-1), \varepsilon_0\}$ (note that $\mathfrak{g}_{\mathrm{sh}}(-1) > 0$ for each $(u, \beta) \in \mathcal{K} \cap \{\beta < \beta_{\mathrm{s}}^{(v_\infty)}\}$). For each $\sigma \in (0, \beta_{\mathrm{s}}^{(v_\infty)})$, there exists a constant $N_0^*(\sigma)$ depending only on $(v_\infty, \gamma, \beta_*, \sigma)$ such that, if $(u, \beta) \in \mathcal{K} \cap \{\beta < \beta_{\mathrm{s}}^{(v_\infty)} - \sigma\}$, then

$$\|\psi\|_{2,\alpha,\mathcal{D}_{\varepsilon_0}^{\mathcal{O}}}^{(1+\alpha),(\mathrm{par})} < N_0^*(\sigma).$$

(v) For each $r \in (0, \varepsilon_0)$, there exists a constant $C_{\beta_*, r} > 0$ depending only on $(v_\infty, \gamma, \beta_*, r, \alpha)$ such that

$$\|\varphi\|_{C^{2,\alpha}(\overline{\Omega \setminus (\mathcal{D}_r^{\mathcal{O}} \cup \mathcal{D}_r^{\mathcal{N}})})} < C_{\beta_*, r}.$$

DEFINITION 4.22. Define the following sets:

(i) Denote $\mathcal{K}^{\mathrm{ext}}$ as

(4.3.21) $\mathcal{K}^{\mathrm{ext}} := \{(u, \beta) \in C_{(*, \alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}}) : (u, \beta) \text{ satisfy Definition 4.19(i)–(vi)}\};$

(ii) $\overline{\mathcal{K}}$ and $\overline{\mathcal{K}^{\text{ext}}}$ are the closures of $\mathcal{K}$ and $\mathcal{K}^{\text{ext}}$ in $C^{2,\alpha}_{(*,\alpha_1)}(\mathcal{Q}^{\text{iter}}) \times [0, \beta_*]$, respectively;

(iii) For each $\mathcal{C} \in \{\mathcal{K}, \mathcal{K}^{\text{ext}}, \overline{\mathcal{K}}, \overline{\mathcal{K}^{\text{ext}}}\}$ and each $\beta \in [0, \beta_*]$, denote

$$\mathcal{C}_\beta := \{u \,:\, (u, \beta) \in \mathcal{C}\}.$$

Note that $\mathcal{C}_\beta \subset C^{2,\alpha}_{(*,\alpha_1)}(\mathcal{Q}^{\text{iter}})$.

REMARK 4.23. Each $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$ satisfies property (ii) of Definition 4.19, as well as properties (i) and (iii)–(vi) of Definition 4.19, and all the properties stated in Remark 4.21 with nonstrict inequalities in the estimates.

4.4. Boundary Value Problem (4.3.16)

In order to complete Definition 4.19, it remains to define the nonlinear differential operators $\mathcal{N}_{(u,\beta)}$ and $\mathcal{M}_{(u,\beta)}$ in (4.3.16) for each $(u, \beta) \in \mathcal{K}$.

For each $(u, \beta) \in \mathcal{K}^{\text{ext}}$, let $\mathfrak{g}_{\text{sh}} = \mathfrak{g}^{(u,\beta)}_{\text{sh}}$, $\mathfrak{F} = \mathfrak{F}_{(u,\beta)}$, $\Omega = \Omega(u, \beta)$, and $\Gamma_{\text{shock}} = \Gamma_{\text{shock}}(u, \beta)$, and let $\varphi = \varphi^{(u,\beta)}$ be defined by (4.2.4).

4.4.1. Definition of $\mathcal{N}_{(u,\beta)}$ in (4.3.16). For $\varphi_\mathcal{N}$ defined by (2.5.1), denote

$$\phi := \varphi - \varphi_\mathcal{N}.$$

For a C^2–function $\hat{\phi}$ in Ω, we define $\mathcal{N}_{(u,\beta)}(\hat{\phi})$ by

$$(4.4.1) \qquad \mathcal{N}_{(u,\beta)}(\hat{\phi}) = \sum_{i,j=1}^{2} A_{ij}(D\hat{\phi}, \boldsymbol{\xi}) \partial_{\xi_i \xi_j} \hat{\phi}$$

so that the following properties hold:

- Equation $\mathcal{N}_{(u,\beta)}(\hat{\phi}) = 0$ is strictly elliptic in $\overline{\Omega} \setminus (\overline{\Gamma^{\mathcal{O}}_{\text{sonic}}} \cup \overline{\Gamma^{\mathcal{N}}_{\text{sonic}}})$;
- If ϕ is a solution of (4.3.16), then equation $\mathcal{N}_{(u,\beta)}(\phi) = 0$ coincides with (3.1.2).

The coefficient functions $A_{ij}(\mathbf{p}, \boldsymbol{\xi}), i, j = 1, 2$, of the nonlinear operator $\mathcal{N}_{(u,\beta)}$ are defined in the following six steps:

1. For a constant $r > 0$, let $\mathcal{D}^{\mathcal{O}}_r$ and $\mathcal{D}^{\mathcal{N}}_r$ be defined by (4.1.2), and let $\mathcal{D}_r := \mathcal{D}^{\mathcal{O}}_r \cup \mathcal{D}^{\mathcal{N}}_r$. Let $\varepsilon_0 > 0$ be from Lemma 4.16. For a constant $\varepsilon_{\text{eq}} \in (0, \frac{\varepsilon_0}{2})$ to be chosen later, we define $A^{(1)}_{ij}(\boldsymbol{\xi})$ for $\boldsymbol{\xi} \in \Omega \setminus \mathcal{D}_{\varepsilon_{\text{eq}}/10}$ by

$$(4.4.2) \qquad A^{(1)}_{ij}(\boldsymbol{\xi}) := A^{\text{potn}}_{ij}(D\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \boldsymbol{\xi}),$$

where

$$
\begin{aligned}
& A^{\text{potn}}_{11}(\mathbf{p}, z, \boldsymbol{\xi}) = c^2 - (p_1 + \partial_{\xi_1} \varphi_\mathcal{N})^2, \\
(4.4.3) \qquad & A^{\text{potn}}_{12}(\mathbf{p}, z, \boldsymbol{\xi}) = A^{\text{potn}}_{21}(\mathbf{p}, z, \boldsymbol{\xi}) = -(p_1 + \partial_{\xi_1} \varphi_\mathcal{N}(\boldsymbol{\xi}))(p_2 + \partial_{\xi_2} \varphi_\mathcal{N}(\boldsymbol{\xi})), \\
& A^{\text{potn}}_{22}(\mathbf{p}, z, \boldsymbol{\xi}) = c^2 - (p_2 + \partial_{\xi_2} \varphi_\mathcal{N}(\boldsymbol{\xi}))^2
\end{aligned}
$$

for $c^2 = c^2(|\mathbf{p} + D\varphi_\mathcal{N}|^2, z + \varphi_\mathcal{N})$ given by (4.3.14).

2. For $\mu_0 > 0$ from Definition 4.19(iv-1), fix a function $\zeta_1 \in C^3(\mathbb{R})$ such that

$$(4.4.4) \qquad \zeta_1(s) := \begin{cases} s & \text{if } |s| \leq \frac{2 - \frac{\mu_0}{5}}{1+\gamma}, \\ \frac{(2 - \frac{\mu_0}{10})\mathrm{sgn}(s)}{1+\gamma} & \text{if } |s| > \frac{2}{1+\gamma}, \end{cases}$$

$$(4.4.5) \qquad 0 \leq \zeta_1'(s) \leq 10, \quad \zeta_1(-s) = -\zeta_1(s) \qquad \text{for all } s \in \mathbb{R},$$

$$(4.4.6) \qquad -\frac{20(1+\gamma)}{\mu_0} \leq \zeta_1''(s) \leq 0 \qquad \text{for all } s \geq 0.$$

Define c_β, u_β, r, and ϕ_β by

$$(4.4.7) \qquad (c_\beta, u_\beta) := \begin{cases} (c_{\mathcal{O}}, u_{\mathcal{O}}) & \text{in } \mathcal{D}_{2\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}, \\ (c_{\mathcal{N}}, 0) & \text{in } \mathcal{D}_{2\varepsilon_{\mathrm{eq}}}^{\mathcal{N}}, \end{cases}$$

$$(4.4.8) \qquad r = \sqrt{(\xi_1 - u_\beta)^2 + \xi_2^2},$$

$$(4.4.9) \qquad \phi_\beta := \varphi_\beta^* - \varphi_{\mathcal{N}}$$

for φ_β^* given by (4.1.42).

Denote $\psi := \phi - \phi_\beta = \varphi - \varphi_\beta^*$. Suppose that $\hat{\phi}$ is a solution of (4.3.16). We denote

$$(4.4.10) \qquad \hat{\psi} := \hat{\phi} - \phi_\beta.$$

Let the (x, y)–coordinates be defined by (3.4.18) and (3.5.2) in $\mathcal{D}_{2\varepsilon_{\mathrm{eq}}}^{\mathcal{N}}$ and $\mathcal{D}_{2\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}$, respectively. For $\mathbf{p} \in \mathbb{R}^2$, denote

$$\mathbf{p}' := \mathbf{p} - D_{(x,y)}\phi_\beta.$$

Note that $\mathbf{p}' = \mathbf{p}$ in $\mathcal{D}_{2\varepsilon_{\mathrm{eq}}}^{\mathcal{N}}$ and $\mathbf{p}' = \mathbf{p} - D_{(x,y)}(\varphi_{\mathcal{O}} - \varphi_{\mathcal{N}})$ in $\mathcal{D}_{2\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}$. Let N_4 be the constant from Definition 4.19(iv-4). In $\mathcal{D}_{2\varepsilon_{\mathrm{eq}}} = \mathcal{D}_{2\varepsilon_{\mathrm{eq}}}^{\mathcal{N}} \cup \mathcal{D}_{2\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}$, define $O_j^{\mathrm{mod}}(\mathbf{p}, x, y)$ by

$$(4.4.11) \qquad O_j^{\mathrm{mod}}(p_1, p_2, x, y) = O_j\left(x^{3/4}\zeta_1\left(\frac{p_1'}{x^{3/4}}\right), (\gamma+1)N_4 x \zeta_1\left(\frac{p_2'}{(\gamma+1)N_4 x}\right), \psi(x,y), x, c_\beta\right)$$

for $j = 1, \cdots, 5$, where each $O_j(\mathbf{p}, z, x)$ is given by (3.2.29). With $O_j^{\mathrm{mod}} = O_j^{\mathrm{mod}}(\hat{\phi}_x, \hat{\phi}_y, x, y)$ for $j = 1, \cdots, 5$, define a nonlinear differential operator $\mathcal{N}_{(u,\beta)}^{\mathrm{polar}}$ by

$$(4.4.12) \qquad \begin{aligned} \mathcal{N}_{(u,\beta)}^{\mathrm{polar}}(\hat{\phi}) &:= \left(2x - (\gamma+1)x\zeta_1\left(\frac{\hat{\psi}_x}{x}\right) + O_1^{\mathrm{mod}}\right)\hat{\psi}_{xx} + O_2^{\mathrm{mod}}\hat{\psi}_{xy} + \left(\frac{1}{c_\beta} + O_3^{\mathrm{mod}}\right)\hat{\psi}_{yy} \\ &\quad - (1 + O_4^{\mathrm{mod}})\hat{\psi}_x + O_5^{\mathrm{mod}}\hat{\psi}_y \\ &=: a_{11}(D_{(x,y)}\hat{\phi}, x, y)\hat{\psi}_{xx} + 2a_{12}(D_{(x,y)}\hat{\phi}, x, y)\hat{\psi}_{xy} + a_{22}(D_{(x,y)}\hat{\phi}, x, y)\hat{\psi}_{yy} \\ &\quad + a_1(D_{(x,y)}\hat{\phi}, x, y)\hat{\psi}_x + a_2(D_{(x,y)}\hat{\phi}, x, y)\hat{\psi}_y. \end{aligned}$$

3. For a C^2–function $\hat{\phi} = \hat{\psi} + \phi_\beta$, the expression of $c_\beta \mathcal{N}_{(u,\beta)}^{\mathrm{polar}}(\hat{\phi})$ in the $\boldsymbol{\xi}$–coordinates is given in the form:

$$(4.4.13) \qquad c_\beta \mathcal{N}_{(u,\beta)}^{\mathrm{polar}}(\hat{\phi}) = \sum_{i,j=1}^{2} A_{ij}^{(2)}(D_{\boldsymbol{\xi}}\hat{\phi}, \boldsymbol{\xi})\partial_{\xi_i \xi_j}\hat{\phi} + \sum_{i=1}^{2} A_i^{(2)}(D_{\boldsymbol{\xi}}\hat{\phi}, \boldsymbol{\xi})\partial_{\xi_i}\hat{\phi} \qquad \text{in } \Omega \cap \mathcal{D}_{2\varepsilon_{\mathrm{eq}}},$$

where we have used that $D_{\boldsymbol{\xi}}^2 \hat{\psi} \equiv D_{\boldsymbol{\xi}}^2 \hat{\phi}$ holds in $\Omega \cap \mathcal{D}_{2\varepsilon_{eq}}$. In the expression above, c_β is multiplied to $\mathcal{N}_{(u,\beta)}^{polar}$ because the expression of $c_\beta \mathcal{N}_{(u,\beta)}^{polar}$ without cutoffs in the $\boldsymbol{\xi}$–coordinates coincides with the left-hand side of Eq. (3.1.2).

In $\Omega \cap \mathcal{D}_{2\varepsilon_{eq}}^{\mathcal{O}}$, a direct computation shows that

$$A_1^{(2)} = \left((c_{\mathcal{O}} - x)O_5^{mod} - O_2^{mod}\right)\sin y + \left((c_{\mathcal{O}} - x)\left(\frac{1}{c_{\mathcal{O}}} + O_3^{mod}\right) - (1 + O_4^{mod})\right)\cos y,$$

$$A_2^{(2)} = \left((c_{\mathcal{O}} - x)O_5^{mod} - O_2^{mod}\right)\cos y - \left((c_{\mathcal{O}} - x)\left(\frac{1}{c_{\mathcal{O}}} + O_3^{mod}\right) - (1 + O_4^{mod})\right)\sin y.$$

From this, combined with (3.2.29) and (4.4.11), we see that $A_1^{(2)} = A_2^{(2)} = 0$ in $\Omega \cap \mathcal{D}_{2\varepsilon_{eq}}^{\mathcal{O}}$. Similarly, it can be checked that $A_1^{(2)} = A_2^{(2)} = 0$ in $\Omega \cap \mathcal{D}_{2\varepsilon_{eq}}^{\mathcal{N}}$. Therefore, we have

$$A_1^{(2)} = A_2^{(2)} = 0 \qquad \text{in } \Omega \cap \mathcal{D}_{2\varepsilon_{eq}}.$$

For $\boldsymbol{\xi} \in \Omega \cap \mathcal{D}_{2\varepsilon_{eq}}^{\mathcal{N}}$, define $A_{ij}^{\mathcal{N}}$ as

(4.4.14) $$A_{ij}^{\mathcal{N}}(\mathbf{p}, \boldsymbol{\xi}) := A_{ij}^{(2)}(\mathbf{p}, \boldsymbol{\xi}).$$

For $\boldsymbol{\xi} \in \Omega \cap \mathcal{D}_{2\varepsilon_{eq}}^{\mathcal{O}}$, define $A_{ij}^{\mathcal{O}}$ as

(4.4.15) $$A_{ij}^{\mathcal{O}}(\mathbf{p}, \boldsymbol{\xi}) := A_{ij}^{(2)}(\mathbf{p}, \boldsymbol{\xi}).$$

By using Definition 4.19, the next two lemmas can be directly derived. We first discuss the properties of coefficients (a_{ij}, a_i) near $\Gamma_{sonic}^{\mathcal{N}}$.

LEMMA 4.24 (Coefficients $(a_{ij}, a_i)(\mathbf{p}, x, y)$ in $\Omega \cap \mathcal{D}_{2\varepsilon_{eq}}^{\mathcal{N}}$). *There exist constants $\lambda_1 \in (0, 1), \varepsilon_{eq} \in (0, \frac{\varepsilon_0}{2})$, and $N_{eq} \geq 1$ depending only on $(v_\infty, \gamma, \beta_*)$ such that, for any $(u, \beta) \in \overline{\mathcal{K}^{ext}} \cap \{0 \leq \beta < \beta_d^{(v_\infty)}\}$, coefficients $(a_{ij}, a_i)(\mathbf{p}, x, y)$ defined by (4.4.12) satisfy the following properties:*

(a) *For any $(x, y) \in \Omega \cap \mathcal{D}_{2\varepsilon_{eq}}^{\mathcal{N}}$ and $\mathbf{p}, \boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2$,*

$$\lambda_1 |\boldsymbol{\kappa}|^2 \leq \sum_{i,j=1}^{2} a_{ij}(\boldsymbol{p}, x, y)\frac{\kappa_i \kappa_j}{x^{2-\frac{i+j}{2}}} \leq \lambda_1^{-1}|\boldsymbol{\kappa}|^2.$$

(b) *$a_{ij}, a_i \in C^{1,\alpha}(\mathbb{R}^2 \times (\overline{\Omega \cap \mathcal{D}_{\varepsilon_{eq}}^{\mathcal{N}}} \setminus \overline{\Gamma_{sonic}^{\mathcal{N}}}))$ for $j = 1, 2$, and*

$$\|(a_{11}, a_{12}, a_2)\|_{C^{0,1}(\mathbb{R}^2 \times \overline{\Omega \cap \mathcal{D}_{\varepsilon_{eq}}^{\mathcal{N}}})} \leq N_{eq},$$

$$\|(a_{22}, a_1)\|_{L^\infty(\mathbb{R}^2 \times \overline{\Omega \cap \mathcal{D}_{\varepsilon_{eq}}^{\mathcal{N}}})} + \|D_{(\mathbf{p},y)}(a_{22}, a_1)\|_{L^\infty(\mathbb{R}^2 \times \overline{\Omega \cap \mathcal{D}_{\varepsilon_{eq}}^{\mathcal{N}}})} \leq N_{eq},$$

$$\sup_{(\mathbf{p},x,y) \in \mathbb{R}^2 \times \overline{\Omega \cap \mathcal{D}_{\varepsilon_{eq}}^{\mathcal{N}}}} |x^{1/4} D_x(a_{22}, a_2)(\mathbf{p}, x, y)| \leq N_{eq},$$

$$\sup_{\mathbf{p} \in \mathbb{R}^2} \|(a_{ij}, a_i)(\mathbf{p}, \cdot, \cdot)\|_{C^{3/4}(\overline{\Omega \cap \mathcal{D}_{\varepsilon_{eq}}^{\mathcal{N}}})} \leq N_{eq} \qquad \text{for } i, j = 1, 2.$$

(c) *For each $k = 1, 2$, $D_{\mathbf{p}}^k(a_{ij}, a_i) \in C^{1,\alpha}(\mathbb{R}^2 \times (\overline{\Omega \cap \mathcal{D}_{\varepsilon_{eq}}^{\mathcal{N}}} \setminus \overline{\Gamma_{sonic}^{\mathcal{N}}}))$ and*

$$\sup_{\mathbf{p} \in \mathbb{R}^2} \|D_{\mathbf{p}}^k(a_{ij}, a_i)(\mathbf{p}, \cdot, \cdot)\|_{C^{1,\alpha}(\mathbb{R}^2 \times (\overline{\Omega \cap \mathcal{D}_{\varepsilon_{eq}}^{\mathcal{N}}} \setminus \mathcal{N}_r(\Gamma_{sonic}^{\mathcal{N}})))} \leq N_{eq} r^{-5} \quad \text{for each } r \in (0, \frac{\varepsilon_{eq}}{2}).$$

(d) *There exists a constant $\hat{C} > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ such that*

$$|\partial_y(a_{11}, a_{12})(\mathbf{p}, x, y)| \le \hat{C} x^{1/2} \qquad \text{for all } \mathbf{p} \in \mathbb{R}^2 \text{ and } (x, y) \in \Omega \cap \mathcal{D}_{\varepsilon_{\text{eq}}}^{\mathcal{N}}.$$

(e) *For every $(\mathbf{p}, x, y) \in \mathbb{R}^2 \times \overline{\Omega \cap \mathcal{D}_{\varepsilon_{\text{eq}}}^{\mathcal{N}}}$,*

$$(a_{11}, a_{22}, a_2)((p_1, -p_2), x, y) = (a_{11}, a_{22}, a_2)((p_1, p_2), x, y),$$
$$|a_{ii}(\mathbf{p}, x, y) - a_{ii}(\mathbf{0}, 0, y)| \le N_{\text{eq}} x^{3/4} \qquad \text{for } i = 1, 2,$$
$$|a_{12}(\mathbf{p}, x, y)| \le N_{\text{eq}} x,$$
$$a_1(\mathbf{p}, x, y) \le -\frac{1}{2}.$$

(f) *For any $\mathbf{p} \in \mathbb{R}^2$, the values of $(a_{ij}, a_i)(\mathbf{p}, \cdot, \cdot)$ are given on $\Gamma_{\text{sonic}}^{\mathcal{N}} = \{x = 0\} \cap \partial(\Omega \cap \mathcal{D}_{\varepsilon_{\text{eq}}}^{\mathcal{N}})$ by fixing $\mathbf{p}$ and taking a limit in (x, y) from $\Omega \cap \mathcal{D}_{\varepsilon_{\text{eq}}} \subset \{x > 0\}$. More explicitly, for any $\mathbf{p} \in \mathbb{R}^2$ and $(0, y) \in \Gamma_{\text{sonic}}^{\mathcal{N}}$,*

$$a_{ij}(\mathbf{p}, 0, y) = 0 \qquad \text{for all } (i, j) \ne (2, 2),$$
$$a_{22}(\mathbf{p}, 0, y) = c_{\mathcal{N}}^{-1}, \quad a_1(\mathbf{p}, 0, y) = -1, \quad a_2(\mathbf{p}, 0, y) = 0.$$

(g) $\phi = \psi + \phi_\beta$ *satisfies*

$$O_j^{\text{mod}}(\phi_x, \phi_y, x, y) = O_j(\psi_x, \psi_y, \psi, x, y, c_\beta) \qquad \text{in } \Omega \cap \mathcal{D}_{\varepsilon_{\text{eq}}}^{\mathcal{N}} \text{ for } j = 1, \cdots, 5.$$

In addition, if ψ satisfies

$$|\psi_x| \le \frac{2 - \frac{\mu_0}{5}}{1 + \gamma} x \qquad \text{in } \Omega \cap \mathcal{D}_{\varepsilon/10}^{\mathcal{N}}$$

for $\varepsilon \in (0, \frac{\varepsilon_{\text{eq}}}{2}]$ from Definition 4.19(iv), then, in $\Omega \cap \mathcal{D}_{\varepsilon_{\text{eq}}}^{\mathcal{N}}$,

$$\mathcal{N}_{(u,\beta)}^{\text{polar}}(\phi) = (2x - (\gamma + 1)\psi_x + O_1)\psi_{xx} + O_2\psi_{xy} + (\frac{1}{c_\beta} + O_3)\psi_{yy} - (1 + O_4)\psi_x + O_5\psi_y$$

for $O_j = O_j(\psi_x, \psi_y, \psi, x, y, c_{\mathcal{N}})$. Therefore, equation $\mathcal{N}_{(u,\beta)}^{\text{polar}}(\phi) = 0$ coincides with Eq. (3.1.2) in $\Omega \cap \mathcal{D}_{\varepsilon_{\text{eq}}}^{\mathcal{N}}$.

Let σ_3 be from Proposition 3.39. Coefficients $A_{ij}^{\mathcal{O}}, i, j = 1, 2$, are used only for $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}} \cap \{\beta : \beta \in [0, \beta_s^{(v_\infty)} + \sigma_3]\}$ to define $\mathcal{N}_{(u,\beta)}$.

In the next lemma, we discuss the properties of coefficients (a_{ij}, a_i) near $\Gamma_{\text{sonic}}^{\mathcal{O}}$ for $\beta \le \beta_s^{(v_\infty)} + \sigma_3$. While $\Gamma_{\text{sonic}}^{\mathcal{N}}$ is fixed to be the same for all $\beta \in [0, \frac{\pi}{2})$, $\Gamma_{\text{sonic}}^{\mathcal{O}}$ changes as β varies. As $\beta \in [0, \beta_s^{(v_\infty)})$ tends to $\beta_s^{(v_\infty)}$, $\Gamma_{\text{sonic}}^{\mathcal{O}}$ shrinks to a point set $\{P_1\}$ for P_1 given in Definition 2.23, and it remains to be the point set $\{P_1\}$ for $\beta > \beta_s^{(v_\infty)}$. For that reason, the properties of (a_{ij}, a_i) near $\Gamma_{\text{sonic}}^{\mathcal{O}}$ are different from Lemma 4.24.

LEMMA 4.25 (Coefficients $(a_{ij}, a_i)(\mathbf{p}, x, y)$ in $\Omega \cap \mathcal{D}_{2\varepsilon_{\text{eq}}}^{\mathcal{O}}$). *For each $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}} \cap \{\beta : \beta \in [0, \beta_s^{(v_\infty)} + \sigma_3]\}$, let (a_{ij}, a_i) be defined by (4.4.12). Then there exists a constant $\varepsilon_{\text{eq}} \in (0, \frac{\varepsilon_0}{2})$ depending only on $(v_\infty, \gamma, \beta_*)$ satisfying the following properties:*

(a) *There exist constants $\lambda_1 \in (0,1)$ and $N_{\mathrm{eq}} \geq 1$ depending only on $(v_\infty, \gamma, \beta_*)$ such that, for each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$ with $\beta \in [0, \beta_{\mathrm{s}}^{(v_\infty)} + \sigma_3]$, coefficients (a_{ij}, a_i) satisfy all the assertions of Lemma 4.24 except for assertions (d) and (g) of Lemma 4.24 by replacing $(\mathcal{D}_{\varepsilon_{\mathrm{eq}}}^{\mathcal{N}}, \Gamma_{\mathrm{sonic}}^{\mathcal{N}})$ with $(\mathcal{D}_{\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}, \Gamma_{\mathrm{sonic}}^{\mathcal{O}})$.*

(b) *Assertion (d) of Lemma 4.24 now takes the following form:*

(b-1) *There exists a constant $\hat{C} > 0$ depending only on $(v_\infty, \gamma, \beta_*, \alpha)$ such that, for each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$ with $\beta \in [0, \beta_{\mathrm{s}}^{(v_\infty)})$,*

$$|D_y(a_{11}, a_{12})(\mathbf{p}, x, y)| \leq \hat{C} x^{1/2} \qquad \text{for } (\mathbf{p}, x, y) \in \mathbb{R}^2 \times (\Omega \cap \mathcal{D}_r^{\mathcal{O}}),$$

where $r = \min\{\mathfrak{g}_{\mathrm{sh}}^2(-1), \varepsilon_{\mathrm{eq}}\}$;

(b-2) *Let $\sigma_1 > 0$ be from Proposition 3.32. For any $\delta \in (0, \frac{\sigma_1}{2})$, there exists a constant $\hat{C}_\delta > 0$ depending on $(v_\infty, \gamma, \beta_*, \delta)$ such that, for each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}} \cap \{\beta \in (0, \beta_{\mathrm{s}}^{(v_\infty)} - \delta]\}$,*

$$|D_y(a_{11}, a_{12})(\mathbf{p}, x, y)| \leq \hat{C}_\delta x^{1/2} \qquad \text{for } (\mathbf{p}, x, y) \in \mathbb{R}^2 \times (\Omega \cap \mathcal{D}_{\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}).$$

(c) *Assertion (g) of Lemma 4.24 now takes the following form: suppose that ψ satisfies*

(4.4.16)
$$|\psi_x| \leq C' x, \quad |\psi_y| \leq C' x^{3/2} \qquad \text{in } \Omega \cap \mathcal{D}_{\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}$$

for some constant $C' > 0$; then there exists a small constant $\varepsilon^{(1)} \in (0, \frac{\varepsilon_{\mathrm{eq}}}{2})$ depending on (v_∞, γ, C') such that, whenever ε from Definition 4.19(iv) with $\varepsilon \leq \varepsilon^{(1)}$, $\phi = \psi + \phi_\beta$ satisfies

$$O_j^{\mathrm{mod}}(\phi_x, \phi_y, x, y) = O_j(\psi_x, \psi_y, \psi, x, y, c_\beta) \qquad \text{in } \Omega \cap \mathcal{D}_{\varepsilon_{\mathrm{eq}}}^{\mathcal{O}} \text{ for } j = 1, \cdots, 5.$$

(c-1) *For P_β given by (2.5.3), suppose that*

$$x_{P_\beta} < \frac{\varepsilon}{10}, \quad \text{i.e., } \Omega \cap \mathcal{D}_{\varepsilon/10}^{\mathcal{O}} \neq \emptyset.$$

If ψ satisfies

$$|\psi_x| \leq \frac{2 - \frac{\mu_0}{5}}{1 + \gamma} x \qquad \text{in } \Omega \cap \mathcal{D}_{\varepsilon/10}^{\mathcal{O}},$$

then, in $\Omega \cap \mathcal{D}_{\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}$,

$$\mathcal{N}_{(u,\beta)}^{\mathrm{polar}}(\phi) = (2x - (\gamma+1)\psi_x + O_1)\psi_{xx} + O_2 \psi_{xy} + (\frac{1}{c_\beta} + O_3)\psi_{yy} - (1 + O_4)\psi_x + O_5 \psi_y$$

for $O_j = O_j(\psi_x, \psi_y, \psi, x, y, c_\beta)$. Therefore, if $\mathcal{N}_{(u,\beta)}^{\mathrm{polar}}(\phi) = 0$ holds in $\Omega \cap \mathcal{D}_{\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}$, then φ satisfies Eq. (3.1.2) in $\Omega \cap \mathcal{D}_{\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}$.

(c-2) *For $\beta \in (\beta_{\mathrm{s}}^{(v_\infty)}, \beta_{\mathrm{s}}^{(v_\infty)} + \sigma_3]$, suppose that*

$$x_{P_\beta} \geq \frac{\varepsilon}{10},$$

which is equivalent to the case that $\Omega \cap \mathcal{D}_{\varepsilon/10}^{\mathcal{O}} = \emptyset$. Then equation $\mathcal{N}_{(u,\beta)}^{\mathrm{polar}}(\phi) = 0$ coincides with Eq. (3.1.2) in $\Omega \cap \mathcal{D}_{\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}$.

(d) *For all $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$ with $\beta > \beta_{\text{s}}^{(v_\infty)}$, $(a_{ij}, a_i)(\mathbf{p}, \cdot, \cdot)$ and $D_{\mathbf{p}}^k(a_{ij}, a_i)(\mathbf{p}, \cdot, \cdot)$, $k = 1, 2$, are in $C^{1,\alpha}(\overline{\Omega \cap \mathcal{D}_{\varepsilon_{\text{eq}}}^{\mathcal{O}}})$. In particular, for each $\delta \in (0, \frac{\sigma_3}{2})$, there exists a constant $C_\delta > 0$ depending only on $(v_\infty, \gamma, \beta_*, \delta)$ such that, if $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$ with $\beta \in [\beta_{\text{s}}^{(v_\infty)} + \delta, \beta_{\text{s}}^{(v_\infty)} + \frac{\sigma_3}{2})$, then*

$$\sup_{\mathbf{p} \in \mathbb{R}^2} \|(a_{ij}, a_i)(\mathbf{p}, \cdot, \cdot)\|_{C^{1,\alpha}(\overline{\Omega \cap \mathcal{D}_{\varepsilon_{\text{eq}}}^{\mathcal{O}}})} \leq C_\delta,$$

$$\sup_{\mathbf{p} \in \mathbb{R}^2} \|D_{\mathbf{p}}^k(a_{ij}, a_i)(\mathbf{p}, \cdot, \cdot)\|_{C^{1,\alpha}(\overline{\Omega \cap \mathcal{D}_{\varepsilon_{\text{eq}}}^{\mathcal{O}}})} \leq C_\delta \qquad \text{for } k = 1, 2.$$

4. In this step, we define $\mathcal{N}_{(u, \beta)}$ near $\Gamma_{\text{sonic}}^{\mathcal{O}}$ for $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$ with $\beta \geq \beta_{\text{s}}^{(v_\infty)} + \frac{\sigma_3}{4}$.

LEMMA 4.26. *For each $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$, let $\mathfrak{g}_{\text{sh}} = \mathfrak{g}_{\text{sh}}^{(u, \beta)}$, $\mathfrak{F} = \mathfrak{F}_{(u, \beta)}$, $\varphi = \varphi^{(u, \beta)}$, and $\Omega = \Omega(u, \beta)$ be defined by Definition 4.15, and let*

(4.4.17) $$\phi := \varphi^{(u, \beta)} - \varphi_{\mathcal{N}}$$

for $\varphi_{\mathcal{N}}$ given by (2.5.1). For any given $\sigma \in (0, 1)$, there exists a constant $C_\sigma > 0$ depending only on $(v_\infty, \gamma, \beta_*, \sigma)$ such that, for each $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$, there exists a function $v_\sigma^{(u, \beta)} \in C^4(\overline{\Omega})$ satisfying the following two properties:

(a) $\|v_\sigma^{(u, \beta)} - \phi\|_{C^1(\overline{\Omega})} \leq \sigma^2$ *and* $\|v_\sigma^{(u, \beta)}\|_{C^4(\overline{\Omega})} \leq C_\sigma$;

(b) $v_\sigma^{(u, \beta)}$ *depends continuously on $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$ in the sense that, if $\{(u_k, \beta_k)\} \subset \overline{\mathcal{K}^{\text{ext}}}$ converges to (u, β) in $C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}}) \times [0, \beta_*]$ for some $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$, then*

$$v_\sigma^{(u_k, \beta_k)} \circ \mathfrak{F}_{(u_k, \beta_k)} \to v_\sigma^{(u, \beta)} \circ \mathfrak{F}_{(u, \beta)} \qquad \text{in } C^{1,\alpha}(\overline{\mathcal{Q}^{\text{iter}}}).$$

PROOF. For $\mathcal{G}_1^\beta$ defined by (4.1.31), denote

$$w(s, t') := \phi \circ (\mathcal{G}_1^\beta)^{-1}(s, t')$$

for $(s, t') \in \mathcal{G}_1^\beta(\Omega) = \{(s, t') : -1 < s < 1, 0 < t' < \mathfrak{g}_{\text{sh}}^{(u, \beta)}(s)\}$. For each small constant $\varepsilon > 0$, define a function $\tilde{w}_\varepsilon(s, t')$ by

$$\tilde{w}_\varepsilon(s, t') := w\Big(\frac{s}{1 + \frac{\varepsilon}{M_1}}, \frac{t' + \frac{\varepsilon}{2M_2}}{1 + \varepsilon}\Big)$$

for constants $M_1 > 1$ and $M_2 > 1$ to be determined later. Then $\tilde{w}_\varepsilon$ is well defined in the set:

$$\mathcal{A}_\varepsilon := \Big\{(s, t') : |s| < 1 + \frac{\varepsilon}{M_1}, \ -\frac{\varepsilon}{2M_2} < t' < (1 + \varepsilon)\mathfrak{g}_{\text{sh}}\Big(\frac{s}{1 + \varepsilon/M_1}\Big) - \frac{\varepsilon}{2M_2}\Big\}.$$

Using (i) and (iii) of Definition 4.19, and Remark 4.21(i), we choose constants $M_1, M_2, M_3 > 1$ depending only on $(v_\infty, \gamma, \beta_*)$ such that the $\frac{\varepsilon}{M_3}$–neighborhood $\mathcal{N}_{\frac{\varepsilon}{M_3}}(\mathcal{G}_1^\beta(\Omega))$ of $\mathcal{G}_1^\beta(\Omega)$ is contained in $\mathcal{A}_\varepsilon$.

Define

$$w_\varepsilon(s, t') := (\tilde{w}_\varepsilon * \chi_{\frac{\varepsilon}{2M_3}})(s, t') \qquad \text{in } \mathcal{G}_1^\beta(\Omega)$$

with $\chi_\delta(\boldsymbol{\xi}) := \frac{1}{\delta^2}\chi(\frac{\boldsymbol{\xi}}{\delta})$, where $\chi(\cdot)$ is a standard mollifier: $\chi \in C_0^\infty(\mathbb{R}^2)$ is a nonnegative function with $\text{supp}(\chi) \subset B_1(\mathbf{0})$ and $\int_{\mathbb{R}^2} \chi(\boldsymbol{\xi}) \, d\boldsymbol{\xi} = 1$. Then we define

$$V_\varepsilon^{(u, \beta)}(\boldsymbol{\xi}) := w_\varepsilon \circ \mathcal{G}_1^\beta(\boldsymbol{\xi}) \qquad \text{in } \Omega.$$

For each $\sigma \in (0, 1)$, there exists a small constant $\varepsilon_*(\sigma) > 0$ depending on $(v_\infty, \gamma, \beta_*, \sigma)$ such that $v_\sigma^{(u, \beta)} := V_{\varepsilon_*(\sigma)}^{(u, \beta)}$ satisfies properties (a)–(b). $\qquad \square$

Let $\varsigma \in C^\infty(\mathbb{R})$ be a cut-off function satisfying that

$$\varsigma(t) = \begin{cases} 1 & \text{for } t < 1, \\ 0 & \text{for } t > 2, \end{cases} \qquad 0 \le \varsigma \le 1 \quad \text{on } \mathbb{R}.$$

For a constant $\sigma > 0$, denote

$$(4.4.18) \qquad\qquad\qquad\qquad \varsigma_\sigma(t) := \varsigma\left(\frac{t}{\sigma}\right).$$

Let $\sigma_{\mathrm{cf}} \in (0,1)$ be a constant to be specified later. For each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, let $v_{\sigma_{\mathrm{cf}}}^{(u,\beta)}$ be the function given by Lemma 4.26. For each $i, j = 1, 2$, we define

$$
\begin{aligned}
A_{ij}^{\mathcal{O},\mathrm{subs}}(\mathbf{p}, \boldsymbol{\xi}) = {} & \varsigma_{\sigma_{\mathrm{cf}}}(|\mathbf{p} - Dv_{\sigma_{\mathrm{cf}}}^{(u,\beta)}(\boldsymbol{\xi})|) A_{ij}^{\mathrm{potn}}(\mathbf{p}, \phi(\boldsymbol{\xi}), \boldsymbol{\xi}) \\
(4.4.19) \qquad & + \left(1 - \varsigma_{\sigma_{\mathrm{cf}}}(|\mathbf{p} - Dv_{\sigma_{\mathrm{cf}}}^{(u,\beta)}(\boldsymbol{\xi})|)\right) A_{ij}^{\mathrm{potn}}(Dv_{\sigma_{\mathrm{cf}}}^{(u,\beta)}(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \boldsymbol{\xi})
\end{aligned}
$$

for $A_{ij}^{\mathrm{potn}}(\mathbf{p}, z, \boldsymbol{\xi})$ defined by (4.4.3).

LEMMA 4.27. *There exist two small constants $\varepsilon^{(2)} > 0$ and $\delta_1^{(1)} > 0$ depending only on (v_∞, γ) such that, whenever ε and δ_1 from Definition 4.19 satisfy*

$$\varepsilon \le \varepsilon^{(2)}, \quad \delta_1 \le \delta_1^{(1)},$$

there exist $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ and $\lambda \in (0,1)$ depending only on (v_∞, γ) so that, for each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}} \cap \{\beta \ge \beta_{\mathrm{s}}^{(v_\infty)} + \frac{\sigma_3}{4}\}$, the associated coefficients $A_{ij}^{\mathcal{O},\mathrm{subs}}$ defined by (4.4.19) with $\sigma_{\mathrm{cf}} = \sqrt{\delta_1}$ satisfy the following properties:

(a) *For all $(\mathbf{p}, \boldsymbol{\xi}) \in \mathbb{R}^2 \times \overline{\Omega \cap \mathcal{D}_{\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}}$ satisfying that $|\mathbf{p} - D\phi(\boldsymbol{\xi})| < \frac{\sqrt{\delta_1}}{2}$,*

$$A_{ij}^{\mathcal{O},\mathrm{subs}}(\mathbf{p}, \boldsymbol{\xi}) = A_{ij}^{\mathrm{potn}}(\mathbf{p}, \phi(\boldsymbol{\xi}), \boldsymbol{\xi}),$$

   so that

$$A_{ij}^{\mathcal{O},\mathrm{subs}}(D\phi(\boldsymbol{\xi}), \boldsymbol{\xi}) = A_{ij}^{\mathrm{potn}}(D\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \boldsymbol{\xi}) \qquad \text{in } \Omega;$$

(b) *For all $(\mathbf{p}, \boldsymbol{\xi}) \in \mathbb{R}^2 \times \overline{\Omega \cap \mathcal{D}_{\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}}$,*

$$|A_{ij}^{\mathcal{O},\mathrm{subs}}(\mathbf{p}, \boldsymbol{\xi}) - A_{ij}^{\mathcal{O},\mathrm{subs}}(D\phi(\boldsymbol{\xi}), \boldsymbol{\xi})| \le C\sqrt{\delta_1};$$

(c) *For each $\mathbf{p} \in \mathbb{R}^2$, $D_{\mathbf{p}}^k A_{ij}^{\mathcal{O},\mathrm{subs}}(\mathbf{p}, \cdot)$ are in $C^{1,\alpha}(\overline{\Omega \cap \mathcal{D}_{\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}})$ for $k = 0, 1, 2$, with*

$$\sum_{k=0}^{2} \|D_{\mathbf{p}}^k A_{ij}^{\mathcal{O},\mathrm{subs}}(\mathbf{p}, \cdot)\|_{C^{1,\alpha}(\overline{\Omega \cap \mathcal{D}_{\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}})} \le C;$$

(d) *For all $\boldsymbol{\xi} \in \Omega \cap \mathcal{D}_{\varepsilon_{\mathrm{eq}}}^{\mathcal{O}}$ and $\mathbf{p}$, $\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2$,*

$$\lambda|\boldsymbol{\kappa}|^2 \le \sum_{i,j=1}^{2} A_{ij}^{\mathcal{O},\mathrm{subs}}(\mathbf{p}, \boldsymbol{\xi})\kappa_i \kappa_j \le \lambda^{-1}|\boldsymbol{\kappa}|^2.$$

5. Let $\chi_{\mathrm{eq}} \in C^\infty(\mathbb{R})$ be a function satisfying that

$$\chi_{\mathrm{eq}}(\beta) = \begin{cases} 1 & \text{if } \beta \le \beta_{\mathrm{s}}^{(v_\infty)} + \frac{\sigma_3}{4}, \\ 0 & \text{if } \beta \ge \beta_{\mathrm{s}}^{(v_\infty)} + \frac{\sigma_3}{2}, \end{cases} \qquad \chi_{\mathrm{eq}}'(\beta) \le 0 \ \text{ on } \mathbb{R}.$$

For such a cut-off function χ_{eq}, we define

$$(4.4.20) \quad A_{ij}^{(3)}(\mathbf{p}, \boldsymbol{\xi}) = \begin{cases} \chi_{\mathrm{eq}}(\beta) A_{ij}^{\mathcal{O}}(\mathbf{p}, \boldsymbol{\xi}) + (1 - \chi_{\mathrm{eq}}(\beta)) A_{ij}^{\mathcal{O},\mathrm{subs}}(\mathbf{p}, \boldsymbol{\xi}) =: A_{ij}^{(3,\mathcal{O})}(\mathbf{p}, \boldsymbol{\xi}) & \text{for } \xi_1 < 0, \\ A_{ij}^{\mathcal{N}}(\mathbf{p}, \boldsymbol{\xi}) & \text{for } \xi_1 > 0 \end{cases}$$

for $A_{ij}^{\mathcal{N}}$ and $A_{ij}^{\mathcal{O}}$ given by (4.4.14) and (4.4.15), respectively.

6. Finally, we combine (4.4.2) with (4.4.20) to complete the definition of $\mathcal{N}_{(u,\beta)}(\hat{\phi})$ in (4.4.1).

DEFINITION 4.28. We define the following:

(i) For a parameter $\tau \in (0, \frac{1}{2}]$, introduce a family of functions $\bar{\zeta}_2(s, t; \tau)$ so that

- $\bar{\zeta}_2(\cdot, \cdot; \tau) \in C^4(\mathbb{R}^2)$ for each $\tau \in (0, \frac{1}{2}]$;
- $\partial_t \bar{\zeta}_2(s, t; \tau) = 0$ for each $\tau \in (0, \frac{1}{2}]$ and $(s, t) \in \mathbb{R}^2$;
- For each $\tau \in (0, \frac{1}{2}]$, $\bar{\zeta}_2(s, t; \tau) = \begin{cases} 1 & \text{for } |s| < 1 - \tau, \\ 0 & \text{for } |s| \geq 1 - \frac{\tau}{2}; \end{cases}$
- $\bar{\zeta}_2(-s, t; \tau) = \bar{\zeta}(s, t; \tau)$ for all $s \in \mathbb{R}$ and $\tau \in (0, \frac{1}{2}]$;
- $-\frac{10}{\tau} \leq \partial_s \bar{\zeta}_2(s, t; \tau) \leq 0$ for all $s \geq 0$ and $\tau \in (0, \frac{1}{2}]$;
- $\|\bar{\zeta}_2(\cdot, \cdot; \tau)\|_{C^4(\mathbb{R}^2)}$ is a continuous function of $\tau \in (0, \frac{1}{2}]$.

(ii) For $\beta_* \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$, define a set $Q_{\beta_*}^{\cup} \subset \mathbb{R}_+^2 \times [0, \frac{\pi}{2})$ as

$$Q_{\beta_*}^{\cup} := \cup_{\beta \in [0, \beta_*]} Q^\beta \times \{\beta\}$$

for Q^β defined by Definition 4.1(iii).

For $\varepsilon > 0$ and $\beta \in [0, \beta_*]$, let $\hat{\varepsilon}$ be given by (4.1.38). For $(\boldsymbol{\xi}, \beta) \in Q_{\beta_*}^{\cup}$, define a function $\zeta_2^{(\varepsilon, \beta)} : Q_{\beta_*}^{\cup} \to \mathbb{R}$ by

$$(4.4.21) \qquad\qquad \zeta_2^{(\varepsilon, \beta)}(\boldsymbol{\xi}) := \bar{\zeta}_2(\mathcal{G}_1^\beta(\boldsymbol{\xi}); \hat{\varepsilon}).$$

The C^1–dependence of $(s_\beta, c_\beta, u_{\mathcal{O}})$ on $\beta \in [0, \frac{\pi}{2})$ yields the following lemma:

LEMMA 4.29. *Let $\varepsilon_0 > 0$ be from Lemma 4.16(c). For each $\varepsilon \in (0, \frac{\varepsilon_0}{2})$, $\zeta_2^{(\varepsilon, \beta)}$ satisfies the following properties:*

(a) *$\zeta_2^{(\varepsilon, \beta)} : Q_{\beta_*}^{\cup} \to \mathbb{R}$ is C^4 with respect to $\boldsymbol{\xi} \in Q^\beta$ for $\beta \in [0, \beta_*]$, and is continuous with respect to $\beta \in [0, \beta_*]$;*

(b) *There exists a constant $C_\varepsilon > 0$ depending only on $(v_\infty, \gamma, \varepsilon)$ such that*

$$\|\zeta_2^{(\varepsilon, \beta)}\|_{C^4(\overline{Q^\beta})} \leq C_\varepsilon;$$

(c) *$\zeta_2^{(\varepsilon, \beta)} = \begin{cases} 1 & \text{in } \Omega(u, \beta) \setminus \mathcal{D}_\varepsilon, \\ 0 & \text{in } \Omega(u, \beta) \cap \mathcal{D}_{\varepsilon/2}. \end{cases}$*

Finally, we define coefficients $A_{ij}(\mathbf{p}, \boldsymbol{\xi})$ for the nonlinear differential operator $\mathcal{N}_{(u,\beta)}$ given by (4.4.1) as follows:

$$(4.4.22) \qquad A_{ij}(\mathbf{p}, \boldsymbol{\xi}) := \zeta_2^{(\varepsilon_{\mathrm{eq}}, \beta)}(\boldsymbol{\xi}) A_{ij}^{(1)}(\boldsymbol{\xi}) + \left(1 - \zeta_2^{(\varepsilon_{\mathrm{eq}}, \beta)}(\boldsymbol{\xi})\right) A_{ij}^{(3)}(\mathbf{p}, \boldsymbol{\xi}), \quad i, j = 1, 2.$$

Hereafter, we continue to adjust $\varepsilon_{\mathrm{eq}} > 0$ depending only on (v_∞, γ).

LEMMA 4.30. *For each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, let coefficients $A_{ij}(\mathbf{p}, \boldsymbol{\xi}), i, j = 1, 2$, of $\mathcal{N}_{(u,\beta)}$ in (4.4.1) be given by (4.4.22). Then there exist constants $\varepsilon_{\mathrm{eq}} \in (0, \frac{\varepsilon_0}{2})$, $\lambda_0 \in (0, 1)$, $N_{\mathrm{eq}} \geq 1$, and $C > 0$ with λ_0 depending only on (v_∞, γ), $(N_{\mathrm{eq}}, \varepsilon_{\mathrm{eq}})$ depending on $(v_\infty, \gamma, \beta_*)$, and $C > 0$ depending only on $(v_\infty, \gamma, \beta_*, \alpha)$ such that the following properties hold:*

(a) *For all $\boldsymbol{\xi} \in \Omega$ with $\Omega = \Omega(u, \beta)$ and all $\mathbf{p}, \boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2$,*

$$\lambda_0 \operatorname{dist}(\boldsymbol{\xi}, \Gamma^{\mathcal{O}}_{\mathrm{sonic}} \cup \Gamma^{\mathcal{N}}_{\mathrm{sonic}}) |\boldsymbol{\kappa}|^2 \leq \sum_{i,j=1}^2 A_{ij}(\boldsymbol{p}, \boldsymbol{\xi}) \kappa_i \kappa_j \leq \lambda_0^{-1} |\boldsymbol{\kappa}|^2;$$

(b) *$A_{12}(\mathbf{p}, \boldsymbol{\xi}) = A_{12}(\mathbf{p}, \boldsymbol{\xi})$ holds in $\mathbb{R}^2 \times \Omega$, and each A_{ij} satisfies*

$$\|A_{ij}\|_{L^\infty(\mathbb{R}^2 \times \Omega)} \leq N_{\mathrm{eq}};$$

(c) *For $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \Omega \setminus \mathcal{D}_{\varepsilon_{\mathrm{eq}}}$, $A_{ij}(\mathbf{p}, \boldsymbol{\xi}) = A_{ij}^{(1)}(\boldsymbol{\xi})$ and*

$$\|A_{ij}\|_{C^{1,\alpha}(\overline{\Omega \setminus \mathcal{D}_{\varepsilon_{\mathrm{eq}}}})} \leq C;$$

(d) *For each $\mathbf{p} \in \mathbb{R}^2$,*

$$\|A_{ij}(\mathbf{p}, \cdot, \cdot)\|_{C^{3/4}(\overline{\Omega})} + \|D_{\mathbf{p}} A_{ij}(\mathbf{p}, \cdot, \cdot)\|_{L^\infty(\Omega)} \leq N_{\mathrm{eq}};$$

(e) *For each $k = 0, 1, 2$, $D_{\mathbf{p}}^k A_{ij} \in C^{1,\alpha}(\mathbb{R}^2 \times (\overline{\Omega} \setminus \overline{\Gamma^{\mathcal{O}}_{\mathrm{sonic}} \cup \Gamma^{\mathcal{N}}_{\mathrm{sonic}}})$. Furthermore, for each $s \in (0, \frac{\varepsilon_0}{2})$, $D_{\mathbf{p}}^k A_{ij}$ satisfies*

$$\|D_{\mathbf{p}}^k A_{ij}\|_{C^{1,\alpha}(\mathbb{R}^2 \times (\overline{\Omega} \setminus \mathcal{N}_s(\overline{\Gamma^{\mathcal{O}}_{\mathrm{sonic}} \cup \Gamma^{\mathcal{N}}_{\mathrm{sonic}})))} \leq Cs^{-5};$$

(f) *For each $i, j = 1, 2$, $A_{ij}(\mathbf{p}, \boldsymbol{\xi}) = A_{ij}^{\mathcal{N}}(\mathbf{p}, \boldsymbol{\xi})$ holds for all $(\mathbf{p}, \boldsymbol{\xi}) \in \mathbb{R}^2 \times (\Omega \cap \mathcal{D}^{\mathcal{N}}_{\varepsilon_{\mathrm{eq}}/2})$;*

(g) *If $\beta \leq \beta_{\mathrm{s}}^{(v_\infty)} + \frac{\sigma_3}{4}$, then $A_{ij}(\mathbf{p}, \boldsymbol{\xi}) = A_{ij}^{\mathcal{O}}(\mathbf{p}, \boldsymbol{\xi})$ holds for all $(\mathbf{p}, \boldsymbol{\xi}) \in \mathbb{R}^2 \times \mathcal{D}^{\mathcal{O}}_{\varepsilon_{\mathrm{eq}}/2}$;*

(h) *If $\beta \in [\beta_{\mathrm{s}}^{(v_\infty)} + \delta, \beta_*]$ for $\delta \in (0, \frac{\sigma_3}{2})$, then $A_{ij}(\mathbf{p}, \boldsymbol{\xi}) = A_{ij}^{(3)}(\mathbf{p}, \boldsymbol{\xi})$ holds for all $(\mathbf{p}, \boldsymbol{\xi}) \in \mathbb{R}^2 \times (\Omega \cap \mathcal{D}^{\mathcal{O}}_{\varepsilon_{\mathrm{eq}}/2})$, and*

$$\lambda_0 \big(\operatorname{dist}(\boldsymbol{\xi}, \overline{\Gamma^{\mathcal{O}}_{\mathrm{sonic}}}) + \delta\big) |\boldsymbol{\kappa}|^2 \leq \sum_{i,j=1}^2 A_{ij}(\mathbf{p}, \boldsymbol{\xi}) \kappa_i \kappa_j \leq \lambda_0^{-1} |\boldsymbol{\kappa}|^2 \qquad \text{for all } \boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2,$$

$$\sup_{\mathbf{p} \in \mathbb{R}^2} \|D_{\mathbf{p}}^k A_{ij}(\mathbf{p}, \cdot, \cdot)\|_{C^{1,\alpha}(\overline{\Omega \cap \mathcal{D}^{\mathcal{O}}_{\varepsilon_{\mathrm{eq}}/2}})} \leq C \qquad \text{for } k = 0, 1, 2;$$

(i) *For each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, let $\phi = \phi^{(u,\beta)}$ be defined by (4.4.17). Suppose that ε from Definition 4.19 satisfies that $0 < \varepsilon < \frac{\varepsilon_{\mathrm{eq}}}{2}$. Then equation $\mathcal{N}_{(u,\beta)}(\phi) = 0$ coincides with (3.1.2) in $\Omega \setminus (\mathcal{D}^{\mathcal{O}}_{\varepsilon/10} \cup \mathcal{D}^{\mathcal{N}}_{\varepsilon/10})$. In addition, if $x_{P_\beta} \geq \frac{\varepsilon}{10}$ or $\beta \geq \beta_{\mathrm{s}}^{(v_\infty)} + \frac{\sigma_3}{2}$ holds, then equation $\mathcal{N}_{(u,\beta)}(\phi) = 0$ coincides with (3.1.2) in $\Omega \setminus \mathcal{D}^{\mathcal{N}}_{\varepsilon/10}$.*

4.4.2. *Definition of $\mathcal{M}_{(u,\beta)}(\mathbf{p}, z, \boldsymbol{\xi})$ in (4.3.16).* The definition of $\mathcal{M}_{(u,\beta)}(\mathbf{p}, z, \boldsymbol{\xi})$ in (4.3.16) is given in the following five steps:

1. For $\varphi_{\mathcal{N}}$ and g^{sh} given by (2.5.1) and (3.4.13), respectively, define

$$(4.4.23) \qquad \mathcal{M}_0(\mathbf{p}, z, \boldsymbol{\xi}) := g^{\mathrm{sh}}(\mathbf{p} + D\varphi_{\mathcal{N}}(\boldsymbol{\xi}), z + \varphi_{\mathcal{N}}(\boldsymbol{\xi}), \boldsymbol{\xi}) \qquad \text{for } \mathbf{p}, \boldsymbol{\xi} \in \mathbb{R}^2 \text{ and } z \in \mathbb{R}.$$

The nonlinear function $\mathcal{M}_0(\boldsymbol{p}, z, \boldsymbol{\xi})$ is well defined on the set:

$$\mathcal{A}_{\mathcal{M}_0} := \left\{ \begin{array}{l} (\boldsymbol{p}, z, \boldsymbol{\xi}) \in B_{4N_5}(\mathbf{0}) \times (-4N_5, 4N_5) \times B_{4M_{\text{dom}}}(\mathbf{0}) \\ : \quad 2\rho_{\max}^{\gamma-1} > \rho_{\mathcal{N}}^{\gamma-1} + (\gamma-1)\big(\boldsymbol{\xi} \cdot \mathbf{p} - \frac{|\boldsymbol{p}|^2}{2} - z\big) > \frac{\rho_{\min}^{\gamma-1}}{2}, \\ \quad |\mathbf{p} - (0, -v_\infty)| > \frac{\mu_1}{2} \end{array} \right\}$$

for constants $(\mu_1, N_5, \rho_{\min}, \rho_{\max})$ from properties (iv) and (vi) of Definition 4.19, and M_{dom} from Remark 4.21. Since these constants are chosen depending only on (v_∞, γ), for each $k = 1, 2, \cdots$, there exists a constant $C_k > 0$ depending only on (v_∞, γ, k) to satisfy

(4.4.24) $\|\mathcal{M}_0\|_{C^k(\overline{\mathcal{A}_{\mathcal{M}_0}})} \le C_k.$

2. Similarly to (3.4.22), we define a function $\mathcal{M}_1(\mathbf{p}, z, \xi_1)$ by

(4.4.25) $\mathcal{M}_1(\mathbf{p}, z, \xi_1) = \mathcal{M}_0(\mathbf{p}, z, \xi_1, \xi_2^{\mathcal{N}} - \frac{z}{v_\infty}).$

$\mathcal{M}_1$ is well defined in the set:

$$\mathcal{A}_{\mathcal{M}_1} := \left\{ \begin{array}{l} (\mathbf{p}, z, \boldsymbol{\xi}) \in B_{3N_5}(\mathbf{0}) \times (-3N_5, 3N_5) \times B_{3M_{\text{dom}}}(\mathbf{0}) \\ : \quad 2\rho_{\max}^{\gamma-1} > \rho_{\mathcal{N}}^{\gamma-1} + (\gamma-1)\big(p_1\xi_2^{\mathcal{N}} + p_2(\xi_1 - \frac{z}{v_\infty}) - \frac{|\mathbf{p}|^2}{2} - z\big) > \frac{\rho_{\min}^{\gamma-1}}{2}, \\ \quad |\mathbf{p} - (0, -v_\infty)| > \frac{\mu_1}{2} \end{array} \right\}.$$

For each $k = 1, 2, \cdots$, there exists a constant $C_k > 0$ depending only on (v_∞, γ, k) such that

(4.4.26) $\|\mathcal{M}_1\|_{C^k(\overline{\mathcal{A}_{\mathcal{M}_1}})} \le C_k.$

In particular, $\mathcal{M}_1$ is homogeneous in the sense of

(4.4.27) $\mathcal{M}_1(\mathbf{0}, 0, \xi_1) = 0, \quad \mathcal{M}_1(D(\varphi_{\mathcal{O}} - \varphi_{\mathcal{N}}), \varphi_{\mathcal{O}} - \varphi_{\mathcal{N}}, \xi_1) = 0 \qquad$ for all $\xi_1 \in \mathbb{R}.$

3. For $(\varphi_{\mathcal{O}}, \varphi_{\mathcal{N}})$ given by (2.5.1), denote

(4.4.28) $\phi_{\mathcal{O}} := \varphi_{\mathcal{O}} - \varphi_{\mathcal{N}}.$

For a constant $\sigma > 0$, let function ς_σ be given by (4.4.18). For a constant $\sigma_{\text{bc}} > 0$ to be determined later, we define

$$\mathcal{M}(\boldsymbol{p}, z, \boldsymbol{\xi}) = \varsigma_{\sigma_{\text{bc}}}(|(\mathbf{p}, z)|)\mathcal{M}_1(\mathbf{p}, z, \xi_1)$$

(4.4.29)
$$+ \big(1 - \varsigma_{\sigma_{\text{bc}}}(|(\mathbf{p}, z)|)\big)\Big(\varsigma_{\sigma_{\text{bc}}}(|(\mathbf{p}, z) - (D\phi_{\mathcal{O}}, \phi_{\mathcal{O}}(\boldsymbol{\xi}))|)\mathcal{M}_1(\mathbf{p}, z, \xi_1)$$

$$+ \big(1 - \varsigma_{\sigma_{\text{bc}}}(|(\mathbf{p}, z) - (D\phi_{\mathcal{O}}, \phi_{\mathcal{O}}(\boldsymbol{\xi}))|)\big)\mathcal{M}_0(\mathbf{p}, z, \boldsymbol{\xi})\Big)$$

for $(\mathbf{p}, z, \boldsymbol{\xi}) \in \mathcal{A}_{\mathcal{M}} := \mathcal{A}_{\mathcal{M}_0} \cap \mathcal{A}_{\mathcal{M}_1}$.

For each $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$, let $\mathfrak{g}_{\text{sh}} = \mathfrak{g}_{\text{sh}}^{(u,\beta)}$, $\mathfrak{F} = \mathfrak{F}_{(u,\beta)}$, $\Omega = \Omega(u, \beta)$, $\Gamma_{\text{shock}} = \Gamma_{\text{shock}}(u, \beta)$, and $\varphi = \varphi^{(u,\beta)}$ be defined by Definition 4.15. Denote $\phi := \varphi - \varphi_{\mathcal{N}}$.

For a constant $\sigma > 0$, we define

$$\mathcal{E}(\phi, \Gamma_{\text{shock}}) = \{(\mathbf{p}, z, \boldsymbol{\xi}) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 : \boldsymbol{p} = D\phi(\boldsymbol{\xi}), \ z = \phi(\boldsymbol{\xi}), \ \boldsymbol{\xi} \in \Gamma_{\text{shock}}\}$$

and

$$\mathcal{E}_\sigma(\phi, \Gamma_{\text{shock}}) = \{(\mathbf{p}, z, \boldsymbol{\xi}) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 : \ \text{dist}(\boldsymbol{\xi}, \Gamma_{\text{shock}}) < \sigma, |\mathbf{p} - D\phi(\boldsymbol{\xi})| < \sigma, |z - \phi(\boldsymbol{\xi})| < \sigma\}.$$

LEMMA 4.31. *There exists a constant $\bar{\sigma}_{\mathrm{bc}} > 0$ depending only on (v_∞, γ) such that, whenever $\sigma_{\mathrm{bc}} \in (0, \bar{\sigma}_{\mathrm{bc}}]$, there exists a constant $C_{\sigma_{\mathrm{bc}}} > 0$ depending only on $(v_\infty, \gamma, \sigma_{\mathrm{bc}})$ such that*

$$\|\mathcal{M}\|_{C^4(\overline{\mathcal{A}_\mathcal{M}})} \leq C_{\sigma_{\mathrm{bc}}}.$$

Furthermore, it holds that, for each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$,

  (a) $\mathcal{E}_{\sigma_{\mathrm{bc}}}(\phi, \Gamma_{\mathrm{shock}}) \subset \mathcal{A}_M$;
  (b) *The mapping: $\beta \mapsto \mathcal{M}$ is in $C([0, \beta_*]; C^4(\overline{\mathcal{A}_\mathcal{M}}))$;*
  (c) *On Γ_{shock}, $\mathcal{M}(D\phi, \phi, \boldsymbol{\xi}) = \mathcal{M}_0(D\phi, \phi, \boldsymbol{\xi})$ and $\partial_{\mathbf{p}}\mathcal{M}(D\phi, \phi, \boldsymbol{\xi}) = \partial_{\mathbf{p}}\mathcal{M}_0(D\phi, \phi, \boldsymbol{\xi})$;*
  (d) *ϕ satisfies*

(4.4.30) $$\mathcal{M}(D\phi, \phi, \boldsymbol{\xi}) = 0 \qquad \text{on } \Gamma_{\mathrm{shock}}$$

   if and only if φ satisfies (3.4.12);

  (e) *$\mathcal{M}$ is homogeneous in the sense that*

(4.4.31) $$\mathcal{M}(\mathbf{0}, 0, \boldsymbol{\xi}) = 0, \quad \mathcal{M}(D(\varphi_\mathcal{O} - \varphi_\mathcal{N}), \varphi_\mathcal{O} - \varphi_\mathcal{N}, \boldsymbol{\xi}) = 0 \qquad \text{for all } \boldsymbol{\xi} \in B_{2M_{\mathrm{dom}}}(\mathbf{0}).$$

LEMMA 4.32. *For constant $\bar{\sigma}_{\mathrm{bc}}$ from Lemma 4.31, there exist constants $\sigma_{\mathrm{bc}} \in (0, \bar{\sigma}_{\mathrm{bc}}]$, $\bar{\varepsilon}_{\mathrm{bc}} > 0$, and $\delta_{bc} > 0$ depending only on (v_∞, γ) such that, if ε from Definition 4.19 satisfies that $0 < \varepsilon \leq \bar{\varepsilon}_{\mathrm{bc}}$, then, for each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, $\mathcal{M}(\mathbf{p}, z, \boldsymbol{\xi})$ satisfies that, for all $\boldsymbol{\xi} \in \Gamma_{\mathrm{shock}}$,*

(4.4.32) $$\delta_{\mathrm{bc}} \leq D_{\mathbf{p}}\mathcal{M}(D\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \boldsymbol{\xi}) \cdot \boldsymbol{\nu}_{\mathrm{sh}}(\boldsymbol{\xi}) \leq \delta_{\mathrm{bc}}^{-1},$$

(4.4.33) $$D_z\mathcal{M}(D\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \boldsymbol{\xi}) \leq -\delta_{\mathrm{bc}},$$

where $\boldsymbol{\nu}_{\mathrm{sh}}$ is the unit normal vector to Γ_{shock} towards the interior of Ω.

PROOF. By Lemma 4.31(c), it suffices to estimate $D_{\mathbf{p}}\mathcal{M}_0(D\phi, \phi, \boldsymbol{\xi}) \cdot \boldsymbol{\nu}_{\mathrm{sh}}$ to prove (4.4.32). Following Definition 2.23, let $\boldsymbol{\xi}^{P_1}$ and $\boldsymbol{\xi}^{P_2}$ be the $\boldsymbol{\xi}$–coordinates of points P_1 and P_2, respectively. By Definition 4.19(i), $Du(\pm 1, 1) = 0$, which implies that $D\phi = D\phi_\beta - D\varphi_\mathcal{N}$ at $\boldsymbol{\xi}^{P_1}$ and $\boldsymbol{\xi}^{P_2}$, for ϕ_β given by (4.4.9). By (4.1.45), we have

$$D_{\boldsymbol{p}}\mathcal{M}_0(D\phi(\boldsymbol{\xi}^{P_j}), \phi(\boldsymbol{\xi}^{P_j}), \boldsymbol{\xi}^{P_j}) \cdot \boldsymbol{\nu}_{\mathrm{sh}}(\boldsymbol{\xi}^{P_j}) = \begin{cases} \rho_\mathcal{O}(1 - M_\mathcal{O}^2) & \text{for } j = 1, \\ \rho_\mathcal{N}\left(1 - (\frac{\xi_2^\mathcal{N}}{c_\mathcal{N}})^2\right) & \text{for } j = 2, \end{cases}$$

for $M_\mathcal{O}$ given by (2.4.6). For each $\beta \in [0, \frac{\pi}{2})$, $M_\mathcal{O} < 1 \leq \rho_\mathcal{O}$. Furthermore, it is shown in (2.4.40)–(2.4.43) that $\frac{d\rho_\mathcal{O}}{d\beta} > 0$ and $\frac{dM_\mathcal{O}}{d\beta} < 0$ for all $\beta \in (0, \frac{\pi}{2})$. Then there exists a constant $\delta_{\mathrm{bc}}^{(1)} \in (0, 1)$ depending only on (v_∞, γ) such that

$$\delta_{\mathrm{bc}}^{(1)} \leq \inf_{\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]} D_{\boldsymbol{p}}\mathcal{M}_0(D\phi(\boldsymbol{\xi}^{P_j}), \phi(\boldsymbol{\xi}^{P_j}), \boldsymbol{\xi}^{P_j}) \cdot \boldsymbol{\nu}_{\mathrm{sh}}(\boldsymbol{\xi}^{P_j}) \leq \frac{1}{\delta_{\mathrm{bc}}^{(1)}} \qquad \text{for } j = 1, 2.$$

By (4.4.24), there exists a constant $\bar{\varepsilon}_{\mathrm{bc}} \in (0, \varepsilon_0)$ depending only on (v_∞, γ) such that, for each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$,

$$\frac{\delta_{\mathrm{bc}}^{(1)}}{2} \leq D_{\boldsymbol{p}}\mathcal{M}_0(D\phi, \phi, \boldsymbol{\xi}) \cdot \boldsymbol{\nu}_{\mathrm{sh}}(\boldsymbol{\xi}) \leq \frac{2}{\delta_{\mathrm{bc}}^{(1)}} \qquad \text{for all } \boldsymbol{\xi} \in \Gamma_{\mathrm{shock}} \cap \mathcal{D}_{\bar{\varepsilon}_{\mathrm{bc}}}.$$

By Definition 4.19(v)–(vi), if ε from Definition 4.19 satisfies that $0 < \varepsilon < \bar{\varepsilon}_{\mathrm{bc}}$, then there exists a constant $\delta_{\mathrm{bc}}^{(2)} > 0$ depending only on (v_∞, γ) such that

$$D_{\boldsymbol{p}}\mathcal{M}_0(D\phi, \phi, \boldsymbol{\xi}) \cdot \boldsymbol{\nu}_{\mathrm{sh}}(\boldsymbol{\xi}) = \rho\Big(1 - \frac{|D\varphi(\boldsymbol{\xi})|^2}{c^2(|D\varphi(\boldsymbol{\xi})|^2, \varphi(\boldsymbol{\xi}))}\Big) \geq \delta_{\mathrm{bc}}^{(2)} \qquad \text{for all } \boldsymbol{\xi} \in \Gamma_{\mathrm{shock}} \setminus \mathcal{D}_{\bar{\varepsilon}_{\mathrm{bc}}/4}.$$

Then (4.4.32) is obtained from the previous two inequalities.

A direct computation by using (4.4.25) yields that, for all $\boldsymbol{\xi} = (\xi_1, \xi_2) \in B_{M_{\mathrm{dom}}}(\mathbf{0})$,

$$D_z\mathcal{M}_1(D\phi_\mathcal{O}(\boldsymbol{\xi}), \phi_\mathcal{O}(\boldsymbol{\xi}), \xi_1) = -\rho_\mathcal{O} M_\mathcal{O} - (\rho_\mathcal{O} - 1)\frac{\cos\beta}{v_\infty},$$

$$D_z\mathcal{M}_1(\mathbf{0}, 0, \xi_1) = -\rho_\mathcal{N}^{2-\gamma}\xi_2^\mathcal{N} - \frac{\rho_\mathcal{N} - 1}{v_\infty}.$$

Then there exists a constant $\delta_{\mathrm{bc}}^{(3)} > 0$ depending only on (v_∞, γ) such that

$$\max_{\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]} \{D_z\mathcal{M}_1(D\phi_\mathcal{O}(\boldsymbol{\xi}), \phi_\mathcal{O}(\boldsymbol{\xi}), \xi_1), D_z\mathcal{M}_1(\mathbf{0}, 0, \xi_1)\} \leq -\delta_{\mathrm{bc}}^{(3)} \qquad \text{for all } \boldsymbol{\xi} \in \Gamma_{\mathrm{shock}}.$$

By (4.4.26), there exists a constant $\sigma_{\mathrm{bc}} \in (0, \bar{\sigma}_{\mathrm{bc}}]$ depending on (v_∞, γ) such that

$$(4.4.34) \qquad\qquad D_z\mathcal{M}_1(\mathbf{p}, z, \xi_1) \leq -\frac{\delta_{\mathrm{bc}}^{(3)}}{2}$$

for all $\boldsymbol{\xi} \in B_{M_{\mathrm{dom}}}(\mathbf{0})$ and for all $(\mathbf{p}, z)$ satisfying that either $|(\mathbf{p}, z)| \leq \sigma_{\mathrm{bc}}$ or $|(\mathbf{p}, z) - (D\phi_\mathcal{O}, \phi_\mathcal{O}(\boldsymbol{\xi}))| \leq \sigma_{\mathrm{bc}}$. By (4.3.11), (4.4.23), and Definition 4.19(vi), there exists a constant $\delta_{\mathrm{bc}}^{(4)} > 0$ depending on (v_∞, γ) such that

$$D_z\mathcal{M}_0(D\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \boldsymbol{\xi}) = -\frac{1}{\rho^{\gamma-2}}D\varphi \cdot \boldsymbol{\nu}_{\mathrm{sh}}(\boldsymbol{\xi}) \leq -\delta_{\mathrm{bc}}^{(4)} \qquad \text{on } \Gamma_{\mathrm{shock}} \setminus (\mathcal{D}_{\varepsilon/10}^\mathcal{N} \cup \mathcal{D}_{\varepsilon/10}^\mathcal{O}).$$

By Definition 4.19(i), $\rho(|D\varphi|^2, \varphi) = \rho_\mathcal{O}$ on $\Gamma_{\mathrm{sonic}}^\mathcal{O}$ and $\rho(|D\varphi|^2, \varphi) = \rho_\mathcal{N}$ on $\Gamma_{\mathrm{sonic}}^\mathcal{N}$. Using Definition 4.19(i), we can further reduce $\bar{\varepsilon}_{\mathrm{bc}} > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ so that $\rho(|D\varphi|^2, \varphi) \geq \frac{1}{10}\min\{\rho_\mathcal{O}, \rho_\mathcal{N}\} > 0$ on $\Gamma_{\mathrm{shock}} \cap (\mathcal{D}_{\bar{\varepsilon}_{\mathrm{bc}}}^\mathcal{N} \cup \mathcal{D}_{\bar{\varepsilon}_{\mathrm{bc}}}^\mathcal{O})$. Therefore, if $\varepsilon \in (0, \bar{\varepsilon}_{\mathrm{bc}})$, then we obtain

$$(4.4.35) \qquad D_z\mathcal{M}_0(D\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \boldsymbol{\xi}) = -\frac{1}{\rho^{\gamma-2}}D\varphi \cdot \boldsymbol{\nu}_{\mathrm{sh}}(\boldsymbol{\xi}) \leq -\delta_{\mathrm{bc}}^{(5)} \qquad \text{on } \Gamma_{\mathrm{shock}}$$

for a constant $\delta_{\mathrm{bc}}^{(5)} > 0$ depending on (v_∞, γ).

Then (4.4.33) is obtained by combining inequalities (4.4.34)–(4.4.35). $\qquad\square$

Hereafter, let $\sigma_{\mathrm{bc}} > 0$ in (4.4.29) be fixed as in Lemma 4.32. This completes the definition of $\mathcal{M}$ in (4.4.29).

4. For ϕ_β given by (4.4.9), denote $\psi := \phi - \phi_\beta = \varphi - \varphi_\beta^*$.

Let the (x, y)–coordinates be defined by (3.4.18) and (3.5.2) near $\Gamma_{\mathrm{sonic}}^\mathcal{N}$ and $\Gamma_{\mathrm{sonic}}^\mathcal{O}$, respectively. For $\mathcal{M}$ given by (4.4.29), and for $\boldsymbol{\xi} = ((c_\mathcal{N} - x)\cos y, (c_\mathcal{N} - x)\sin y)$ near $\Gamma_{\mathrm{sonic}}^\mathcal{N}$, we use (3.4.25) to define $\hat{\mathcal{M}}^\mathcal{N}$ by

$$(4.4.36) \qquad \begin{aligned} &\hat{\mathcal{M}}^\mathcal{N}(q_1, q_2, z, x, y) \\ &:= \mathcal{M}\Big(-q_1\cos y - \frac{q_2\sin y}{c_\mathcal{N} - x}, -q_1\sin y + \frac{q_2\cos y}{c_\mathcal{N} - x}, z, (c_\mathcal{N} - x)\cos y, (c_\mathcal{N} - x)\sin y\Big). \end{aligned}$$

For $\boldsymbol{\xi} = (u_\mathcal{O} - (c_\mathcal{O} - x)\cos(\pi - y), (c_\mathcal{O} - x)\sin(\pi - y))$ near $\Gamma_{\mathrm{sonic}}^\mathcal{O}$, we first denote

$$\mathcal{M}^\mathcal{O}(\mathbf{q}, z, \boldsymbol{\xi}) := \mathcal{M}(\mathbf{q} + D\phi_\mathcal{O}, z + \phi_\mathcal{O}, \boldsymbol{\xi}),$$

and then define $\hat{\mathcal{M}}^{\mathcal{O}}$ by

$$(4.4.37) \qquad \hat{\mathcal{M}}^{\mathcal{O}}(q_1, q_2, z, x, y)$$
$$:= \mathcal{M}^{\mathcal{O}}\Big(-q_1 \cos(\pi - y) + \frac{q_2 \sin(\pi - y)}{c_{\mathcal{O}} - x}, -q_1 \sin(\pi - y) - \frac{q_2 \cos(\pi - y)}{c_{\mathcal{O}} - x}, z,$$
$$u_{\mathcal{O}} - (c_{\mathcal{O}} - x) \cos(\pi - y), (c_{\mathcal{O}} - x) \sin(\pi - y)\Big).$$

LEMMA 4.33. *Let constant $\sigma_2 > 0$ be from Lemma 3.36. Following Definition 2.23, let (x_{P_j}, y_{P_j}) be the (x, y)–coordinates of P_j for $j = 1, 2$. Let $\bar{\varepsilon}_{\mathrm{bc}}$ be from Lemma 4.32. Then there exist $\varepsilon_{\mathrm{bc}} \in (0, \bar{\varepsilon}_{\mathrm{bc}})$, $\hat{\sigma}_{\mathrm{bc}} > 0$, and $C > 0$ depending only on (v_∞, γ) such that, for any $\beta \in [0, \beta_{\mathrm{s}}^{(v_\infty)} + \frac{\sigma_2}{4}]$ and all $(\mathbf{q}, z)$ satisfying that*

$$(4.4.38) \qquad\qquad\qquad |(\mathbf{q}, z)| \le \hat{\sigma}_{\mathrm{bc}},$$

the following properties hold:

(a) *If $0 < x - x_{P_1} \le \varepsilon_{\mathrm{bc}}$, then*
$$D_{q_i} \hat{\mathcal{M}}^{\mathcal{O}}(\mathbf{q}, z, x, y) \le -C^{-1} \ \ for \ i = 1, 2, \qquad D_z \hat{\mathcal{M}}^{\mathcal{O}}(\mathbf{q}, z, x, y) \le -C^{-1};$$

(b) *If $0 < x - x_{P_2} \le \varepsilon_{\mathrm{bc}}$, then*
$$D_{q_i} \hat{\mathcal{M}}^{\mathcal{N}}(\mathbf{q}, z, x, y) \le -C^{-1} \ \ for \ i = 1, 2, \qquad D_z \hat{\mathcal{M}}^{\mathcal{N}}(q_1, q_2, z, x, y) \le -C^{-1}.$$

PROOF. By (3.4.25) and (3.5.10), there exists a constant $\hat{\sigma}_{\mathrm{bc}}^*$ depending only on (v_∞, γ) such that, for each $\beta \in [0, \beta_{\mathrm{s}}^{(v_\infty)} + \frac{\sigma_2}{4}]$, if $|(\mathbf{q}, z)| \le \hat{\sigma}_{\mathrm{bc}}^*$, then $\mathcal{M}$ on the right-hand side of (4.4.36) and (4.4.37) is the same as $\mathcal{M}_1$ given by (4.4.25). A direct computation shows that there exists a constant $\tilde{C} > 0$ depending only on (v_∞, γ) such that, for each $\beta \in [0, \beta_{\mathrm{s}}^{(v_\infty)} + \frac{\sigma_2}{4}]$,

$$D_{q_i} \hat{\mathcal{M}}^{\mathcal{O}}(\mathbf{0}, 0, x_{P_1}, y_{P_1}) \le -\tilde{C}^{-1}, \qquad D_z \hat{\mathcal{M}}^{\mathcal{O}}(\mathbf{0}, 0, x_{P_1}, y_{P_1}) \le -\tilde{C}^{-1},$$
$$D_{q_i} \hat{\mathcal{M}}^{\mathcal{N}}(\mathbf{0}, 0, x_{P_2}, y_{P_2}) \le -\tilde{C}^{-1}, \qquad D_z \hat{\mathcal{M}}^{\mathcal{N}}(\mathbf{0}, 0, x_{P_2}, y_{P_2}) \le -\tilde{C}^{-1}$$

for $i = 1, 2$. Then, by Lemma 4.31, there exist constants $\hat{\sigma}_{\mathrm{bc}} \in (0, \hat{\sigma}_{\mathrm{bc}}^*]$ and $C > 0$ depending only on (v_∞, γ) such that properties (a) and (b) hold. $\qquad\square$

5. The next step is to extend the definition of $\mathcal{M}$ in (4.4.29) to all $(\mathbf{p}, z) \in \mathbb{R}^2 \times \mathbb{R}$.

For each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$ and a constant $\sigma > 0$, let $v_\sigma^{(u, \beta)} \in C^4(\overline{\Omega})$ (from Lemma 4.26) be given. For a constant $\sigma > 0$ to be fixed later, we define a linear operator:

$$(4.4.39) \quad \mathcal{L}_\sigma^{(u, \beta)}(\mathbf{p}, z, \boldsymbol{\xi}) := \mathcal{M}(Dv_\sigma^{(u, \beta)}(\boldsymbol{\xi}), v_\sigma^{(u, \beta)}(\boldsymbol{\xi}), \boldsymbol{\xi})$$
$$+ D_{\mathbf{p}} \mathcal{M}(Dv_\sigma^{(u, \beta)}(\boldsymbol{\xi}), v_\sigma^{(u, \beta)}(\boldsymbol{\xi}), \boldsymbol{\xi}) \cdot \mathbf{p} + D_z \mathcal{M}(Dv_\sigma^{(u, \beta)}(\boldsymbol{\xi}), v_\sigma^{(u, \beta)}(\boldsymbol{\xi}), \boldsymbol{\xi}) z.$$

Let $\sigma_{\mathrm{bc}} > 0$ be from Lemma 4.32. By Lemma 4.26(a), if $\sigma^2 < \sigma_{\mathrm{bc}}$, then $\mathcal{L}_\sigma^{(u, \beta)}$ is well defined for all $(\mathbf{p}, z, \boldsymbol{\xi}) \in \mathbb{R}^2 \times \mathbb{R} \times \overline{\Omega}$. For a constant $\sigma \in (0, \sigma_{\mathrm{bc}})$ to be determined later, depending only on $(v_\infty, \gamma, \beta_*)$, we finally define $\mathcal{M}_{(u, \beta)}(\mathbf{p}, z, \boldsymbol{\xi})$ by

$$(4.4.40) \qquad \mathcal{M}_{(u, \beta)}(\mathbf{p}, z, \boldsymbol{\xi}) := \varsigma_\sigma \mathcal{M}(\mathbf{p}, z, \boldsymbol{\xi}) + (1 - \varsigma_\sigma) \mathcal{L}_\sigma^{(u, \beta)}(\mathbf{p} - Dv_\sigma^{(u, \beta)}(\boldsymbol{\xi}), z - v_\sigma^{(u, \beta)}(\boldsymbol{\xi}), \boldsymbol{\xi})$$

for $\varsigma_\sigma = \varsigma_\sigma(|(\mathbf{p}, z) - (Dv_\sigma^{(u, \beta)}(\boldsymbol{\xi}), v_\sigma^{(u, \beta)}(\boldsymbol{\xi}))|)$, where ς_σ is defined by (4.4.18).

The following lemma is obtained by adjusting the proofs of [**11**, Lemmas 12.5.7 and 17.3.23] via use of Definition 4.19, Lemmas 4.31–4.33, and (4.4.39)–(4.4.40):

LEMMA 4.34. *Let constants $\bar{\varepsilon}_{\mathrm{bc}}$ and $\varepsilon_{\mathrm{bc}}$ be from Lemmas 4.32 and 4.33, respectively. Then there exist positive constants $\delta_1^{(1)}$, $N_1^{(1)}$, δ_{bc}, C, C_{β_*}, and $\varepsilon_{\mathcal{M}} \in (0, \varepsilon_{\mathrm{bc}}]$ with $(\delta_1^{(1)}, N_1^{(1)}, \delta_{\mathrm{bc}}, C)$ depending on (v_∞, γ), $\varepsilon_{\mathcal{M}}$ depending on $(v_\infty, \gamma, \beta_*)$, and C_{β_*} depending on $(v_\infty, \gamma, \beta_*, \alpha)$ such that, if parameters $(\varepsilon, \delta_1, N_1)$ from Definition 4.19 satisfy that $\varepsilon \in (0, \bar{\varepsilon}_{\mathrm{bc}}]$, $\delta_1 \in (0, \delta_1^{(1)}]$, and $N_1 \geq N_1^{(1)}$, then, for each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, $\mathcal{M}_{(u,\beta)} : \mathbb{R}^2 \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ given by (4.4.40) with $\sigma = \sqrt{\delta_1}$ satisfies the following properties:*

(a) *$\mathcal{M}_{(u,\beta)} : \mathbb{R}^2 \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$ is in C^3 and, for all $(\mathbf{p}, z) \in \mathbb{R}^2 \times \mathbb{R}$,*

$$\|(\mathcal{M}_{(u,\beta)}(\mathbf{0}, 0, \cdot), D_{(\mathbf{p},z)}^k \mathcal{M}_{(u,\beta)}(\mathbf{p}, z, \cdot))\|_{C^3(\overline{\Omega})} \leq C_{\beta_*} \qquad \text{for } k = 1, 2, 3;$$

(b) *For $|\mathbf{p} - D\phi(\boldsymbol{\xi})| + |z - \phi(\boldsymbol{\xi})| \leq \frac{\sqrt{\delta_1}}{2}$,*

$$\mathcal{M}_{(u,\beta)}(\mathbf{p}, z, \boldsymbol{\xi}) = \mathcal{M}(\mathbf{p}, z, \boldsymbol{\xi})$$

for $\mathcal{M}$ defined by (4.4.29);

(c) *For all $(\mathbf{p}, z, \boldsymbol{\xi}) \in \mathbb{R}^2 \times \mathbb{R} \times \overline{\Omega}$,*

$$|D_{(\mathbf{p},z)} \mathcal{M}_{(u,\beta)}(\mathbf{p}, z, \boldsymbol{\xi}) - D_{(\mathbf{p},z)} \mathcal{M}(D\phi(\boldsymbol{\xi}), \phi(\boldsymbol{\xi}), \boldsymbol{\xi})| \leq C\sqrt{\delta_1};$$

(d) *For all $(\mathbf{p}, z, \boldsymbol{\xi}) \in \mathbb{R}^2 \times \mathbb{R} \times \overline{\Gamma_{\mathrm{shock}}}$,*

$$\delta_{\mathrm{bc}} \leq D_{\mathbf{p}} \mathcal{M}_{(u,\beta)}(\mathbf{p}, z, \boldsymbol{\xi}) \cdot \boldsymbol{\nu}_{\mathrm{sh}} \leq \frac{1}{\delta_{\mathrm{bc}}}, \quad D_z \mathcal{M}_{(u,\beta)}(\mathbf{p}, z, \boldsymbol{\xi}) \leq -\delta_{\mathrm{bc}},$$

where $\boldsymbol{\nu}_{\mathrm{sh}}$ is the unit normal vector to Γ_{shock} towards the interior of Ω;

(e) *Representing as $\mathcal{L}_\sigma^{(u,\beta)}(\mathbf{p} - Dv_{\sqrt{\delta_1}}^{(u,\beta)}(\boldsymbol{\xi}), z - v_{\sqrt{\delta_1}}^{(u,\beta)}(\boldsymbol{\xi}), \boldsymbol{\xi}) = \mathcal{B}_{\sigma, \Gamma_{\mathrm{shock}}}^{(u,\beta)}(\mathbf{p}, z, \boldsymbol{\xi})$, define*

$$\mathcal{B}_{\sigma, \Gamma_{\mathrm{shock}}}^{(u,\beta)}(\mathbf{p}, z, \boldsymbol{\xi}) = b_1^{(\mathrm{sh})}(\boldsymbol{\xi}) p_1 + b_2^{(\mathrm{sh})}(\boldsymbol{\xi}) p_2 + b_0^{(\mathrm{sh})}(\boldsymbol{\xi}) z + h^{(\mathrm{sh})}(\boldsymbol{\xi}).$$

Then

$$\|(b_i^{(\mathrm{sh})}, h^{(\mathrm{sh})})\|_{C^3(\overline{\Gamma_{\mathrm{shock}}})} \leq C_{\beta_*} \qquad \text{for } i = 0, 1, 2,$$

and, for all $(\mathbf{p}, z, \boldsymbol{\xi}) \in \mathbb{R}^2 \times \mathbb{R} \times \overline{\Omega}$,

$$|\mathcal{M}_{(u,\beta)}(\mathbf{p}, z, \boldsymbol{\xi}) - \mathcal{B}_{\sqrt{\delta_1}, \Gamma_{\mathrm{shock}}}^{(u,\beta)}(\mathbf{p}, z, \boldsymbol{\xi})| \leq C\sqrt{\delta_1}(|\mathbf{p} - Dv_{\sqrt{\delta_1}}^{(u,\beta)}(\boldsymbol{\xi})| + |z - v_{\sqrt{\delta_1}}^{(u,\beta)}(\boldsymbol{\xi})|),$$
$$|D_{(\mathbf{p},z)} \mathcal{M}_{(u,\beta)}(\mathbf{p}, z, \boldsymbol{\xi}) - D_{(\mathbf{p},z)} \mathcal{B}_{\sqrt{\delta_1}, \Gamma_{\mathrm{shock}}}^{(u,\beta)}(\boldsymbol{\xi})| \leq C\sqrt{\delta_1};$$

(f) *$\mathcal{M}_{(u,\beta)}$ is homogeneous in the sense that*

$$\begin{cases} \mathcal{M}_{(u,\beta)}(\mathbf{0}, 0, \boldsymbol{\xi}) = 0, \\ \mathcal{M}_{(u,\beta)}(D\phi_{\mathcal{O}}(\boldsymbol{\xi}), \phi_{\mathcal{O}}(\boldsymbol{\xi}), \boldsymbol{\xi}) = 0 \end{cases}$$

for all $\boldsymbol{\xi} \in \Gamma_{\mathrm{shock}}$ when $\beta \in [0, \frac{\delta_1}{N_1}]$, and for all $\boldsymbol{\xi} \in \Gamma_{\mathrm{shock}} \cap \mathcal{D}_{\varepsilon_{\mathcal{M}}}$ when $\beta \in (\frac{\delta_1}{N_1}, \beta_*]$.

(g) *Let the (x, y)–coordinates be defined by (3.4.18) and (3.5.2) near $\Gamma_{\mathrm{sonic}}^{\mathcal{N}}$ and $\Gamma_{\mathrm{sonic}}^{\mathcal{O}}$, respectively. For $\boldsymbol{\xi} \in \overline{\Gamma_{\mathrm{shock}} \cap \mathcal{D}_{\varepsilon_{\mathrm{bc}}}^{\mathcal{N}}}$, define*

(4.4.41)

$$\hat{\mathcal{M}}_{(u,\beta)}^{\mathcal{N}}(q_1, q_2, z, x, y)$$
$$:= \mathcal{M}_{(u,\beta)}\Big(-q_1 \cos y - \frac{q_2 \sin y}{c_{\mathcal{N}} - x}, -q_1 \sin y + \frac{q_2 \cos y}{c_{\mathcal{N}} - x}, z, (c_{\mathcal{N}} - x) \cos y, (c_{\mathcal{N}} - x) \sin y\Big).$$

For $\boldsymbol{\xi} \in \overline{\Gamma_{\text{shock}} \cap \mathcal{D}_{\varepsilon_{\text{bc}}}^{\mathcal{O}}}$, *define* $\mathcal{M}_{(u,\beta)}^{\mathcal{O}}(\mathbf{p}, z, \boldsymbol{\xi}) := \mathcal{M}_{(u,\beta)}(\mathbf{p} + D\phi_{\mathcal{O}}, z + \phi_{\mathcal{O}}, \boldsymbol{\xi})$, *and*

$$\hat{\mathcal{M}}_{(u,\beta)}^{\mathcal{O}}(q_1, q_2, z, x, y)$$

(4.4.42)
$$:= \mathcal{M}_{(u,\beta)}^{\mathcal{O}}\left(-q_1 \cos(\pi - y) + \frac{q_2 \sin(\pi - y)}{c_{\mathcal{O}} - x}, -q_1 \sin(\pi - y) - \frac{q_2 \cos(\pi - y)}{c_{\mathcal{O}} - x},\right.$$
$$\left. u_{\mathcal{O}} - (c_{\mathcal{O}} - x) \cos(\pi - y), (c_{\mathcal{O}} - x) \sin(\pi - y)\right).$$

Then $\hat{\mathcal{M}}_{(u,\beta)}^{\mathcal{N}}$ *and* $\hat{\mathcal{M}}_{(u,\beta)}^{\mathcal{O}}$ *satisfy the following properties, provided that* $\Gamma_{\text{shock}} \cap \mathcal{D}_{\varepsilon_{\text{bc}}}^{\mathcal{O}}$ *is nonempty:*

(g-1) $\|\hat{\mathcal{M}}_{(u,\beta)}^{\mathcal{N}}\|_{C^3(\mathbb{R}^2 \times \mathbb{R} \times \overline{\Gamma_{\text{shock}} \cap \mathcal{D}_{\varepsilon_{\text{bc}}}^{\mathcal{N}}})} + \|\hat{\mathcal{M}}_{(u,\beta)}^{\mathcal{O}}\|_{C^3(\mathbb{R}^2 \times \mathbb{R} \times \overline{\Gamma_{\text{shock}} \cap \mathcal{D}_{\varepsilon_{\text{bc}}}^{\mathcal{O}}})} \leq C_{\beta_*};$

(g-2) *For all* $|(\mathbf{q}, z)| \leq \frac{\delta_{\text{bc}}}{C}$,

$$\hat{\mathcal{M}}_{(u,\beta)}^{\mathcal{N}}(\mathbf{q}, z, x, y) = \hat{\mathcal{M}}^{\mathcal{N}}(\mathbf{q}, z, x, y) \qquad in\ \Gamma_{\text{shock}} \cap \mathcal{D}_{\varepsilon_{\text{bc}}}^{\mathcal{N}},$$
$$\hat{\mathcal{M}}_{(u,\beta)}^{\mathcal{O}}(\mathbf{q}, z, x, y) = \hat{\mathcal{M}}^{\mathcal{O}}(\mathbf{q}, z, x, y) \qquad in\ \Gamma_{\text{shock}} \cap \mathcal{D}_{\varepsilon_{\text{bc}}}^{\mathcal{O}}$$

for $\hat{\mathcal{M}}^{\mathcal{N}}$ *and* $\hat{\mathcal{M}}^{\mathcal{O}}$ *defined by* (4.4.36) *and* (4.4.37)*, respectively;*

(g-3) *For each* $(\mathbf{q}, z) \in \mathbb{R}^2 \times \mathbb{R}$ *and* $i = 1, 2$,

$$D_{q_i} \hat{\mathcal{M}}_{(u,\beta)}^{\mathcal{N}}(\mathbf{q}, z, x, y) \leq -\delta_{\text{bc}}, \quad D_z \hat{\mathcal{M}}_{(u,\beta)}^{\mathcal{N}}(\mathbf{q}, z, x, y) \leq -\delta_{\text{bc}} \qquad in\ \Gamma_{\text{shock}} \cap \mathcal{D}_{\varepsilon_{\mathcal{M}}}^{\mathcal{N}},$$

$$D_{q_i} \hat{\mathcal{M}}_{(u,\beta)}^{\mathcal{O}}(\mathbf{q}, z, x, y) \leq -\delta_{\text{bc}}, \quad D_z \hat{\mathcal{M}}_{(u,\beta)}^{\mathcal{O}}(\mathbf{q}, z, x, y) \leq -\delta_{\text{bc}} \qquad in\ \Gamma_{\text{shock}} \cap D_{\varepsilon_{\mathcal{M}}}^{\mathcal{O}},$$

provided that $\Gamma_{\text{shock}} \cap \mathcal{D}_{\varepsilon_{\mathcal{M}}}^{\mathcal{O}}$ *is nonempty;*

(h) $\mathcal{M}_{(u,\beta)}(D\phi, \phi, \boldsymbol{\xi}) = 0$ *on* Γ_{shock} *if and only if* $\varphi = \phi + \varphi_{\mathcal{N}}$ *satisfies the Rankine-Hugoniot jump condition* (3.4.12) *on* $\Gamma_{\text{shock}} = \{\varphi = \varphi_{\infty}\}$.

By (4.4.22) and (4.4.40), the definition of the nonlinear boundary value problem (4.3.16) is completed.

4.4.3. Well-posedness of the boundary value problem (4.3.16).

LEMMA 4.35. *Fix* $\gamma \geq 1$, $v_{\infty} > 0$, *and* $\beta_* \in (0, \beta_{\text{d}}^{(v_{\infty})})$. *Let* $\varepsilon_0 > 0$ *be from Lemma* 4.16(c) *with* $\bar{\beta}$ *replaced by* β_*. *Let constant* $\sigma_2 > 0$ *be from Lemma* 3.36. *Moreover, let* $\bar{\alpha} \in (0, 1)$ *be from Proposition* 4.12 *with* $\bar{\beta}$ *replaced by* β_*. *Then there exist constants* $\varepsilon^{(\text{w})} \in (0, \varepsilon_0]$, $\delta_1^{(\text{w})} \in (0, 1)$, $N_1^{(\text{w})} \geq 1$, *and* $\alpha_1^* \in (0, \bar{\alpha}]$ *depending only on* $(v_{\infty}, \gamma, \beta_*)$ *such that, whenever parameters* $(\varepsilon, \delta_1, N_1)$ *from Definition* 4.19 *satisfy that* $\varepsilon \in (0, \varepsilon^{(\text{w})}]$, $\delta_1 \in (0, \delta_1^{(\text{w})}]$, *and* $N_1 \geq N_1^{(\text{w})}$, *the following properties hold:*

Case 1. *If* $\beta \leq \beta_{\text{s}}^{(v_{\infty})} + \sigma_2$, *then the boundary value problem* (4.3.16) *associated with* $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}} \cap \{\beta \leq \beta_{\text{s}}^{(v_{\infty})} + \sigma_2\}$ *has a unique solution* $\hat{\phi} \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus (\overline{\Gamma_{\text{sonic}}^{\mathcal{O}} \cup \Gamma_{\text{sonic}}^{\mathcal{N}}})) \cap C^0(\overline{\Omega})$ *for* $\Omega = \Omega(u, \beta)$. *Moreover, there exists a constant* $C > 0$ *depending only on* $(v_{\infty}, \gamma, \beta_*, \alpha)$ *such that solution* $\hat{\phi}$ *satisfies*

(4.4.43) $$\|\hat{\phi}\|_{L^{\infty}(\Omega)} \leq C, \qquad |\hat{\phi}(\boldsymbol{\xi}) - \phi_{\beta}^*(\boldsymbol{\xi})| \leq C \operatorname{dist}(\boldsymbol{\xi}, \Gamma_{\text{sonic}}^{\mathcal{O}} \cup \Gamma_{\text{sonic}}^{\mathcal{N}}) \quad in\ \Omega$$

for $\phi_{\beta}^* = \max\{\varphi_{\mathcal{O}}, \varphi_{\mathcal{N}}\} - \varphi_{\mathcal{N}}$. *Furthermore, for each* $d \in (0, \varepsilon_0)$, *there exists a constant* $C_d > 0$ *depending only on* $(v_{\infty}, \gamma, \beta_*, d, \alpha)$ *such that*

(4.4.44) $$\|\hat{\phi}\|_{2, \alpha_1^*, \Omega \setminus \mathcal{D}_d} \leq C_d.$$

Case 2. *For each $\delta \in (0, \frac{\sigma_2}{2})$, if $\beta_{\mathrm{s}}^{(v_\infty)} + \delta \leq \beta \leq \beta_*$, then the boundary value problem* (4.3.16) *associated with $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$ has a unique solution $\hat{\phi} \in C^2(\Omega) \cap C^1(\overline{\Omega} \setminus (\overline{\Gamma_{\mathrm{sonic}}^{\mathcal{O}} \cup \Gamma_{\mathrm{sonic}}^{\mathcal{N}}})) \cap C^0(\overline{\Omega})$ for $\Omega = \Omega(u, \beta)$, and the solution satisfies* (4.4.43)–(4.4.44) *for constants $C > 0$ depending only on $(v_\infty, \gamma, \beta_*, \delta)$ and $C_d > 0$ depending only on $(v_\infty, \gamma, \beta_*, \delta, d, \alpha)$.*

PROOF. Fix $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}} \cap \{\beta \leq \beta_{\mathrm{s}}^{(v_\infty)} + \sigma_2\}$. Using $\mathcal{G}_1^\beta$ defined by (4.1.31), we rewrite the boundary value problem (4.3.16) associated with fixed (u, β) in domain $\mathcal{R} = \mathcal{G}_1^\beta(\Omega(u, \beta))$. Then we follow the argument of Step 1 in the proof of [**11**, Proposition 17.4.2], by using Lemmas 4.2, 4.5, 4.24–4.26, and 4.34, to choose constants $\varepsilon^{(\mathrm{w})} \in (0, \varepsilon_0]$, $\delta_1^{(\mathrm{w})} \in (0, 1)$, and $N_1^{(\mathrm{w})} \geq 1$ such that, whenever parameters $(\varepsilon, \delta_1, N_1)$ from Definition 4.19 satisfy that $\varepsilon \in (0, \varepsilon^{(\mathrm{w})}]$, $\delta_1 \in (0, \delta_1^{(\mathrm{w})}]$, and $N_1 \geq N_1^{(\mathrm{w})}$, the newly written boundary value problem in $\mathcal{R}$ satisfies all the conditions of Proposition C.15. Then the existence and uniqueness of solution $\hat{\phi}$ of problem (4.3.16) satisfying (4.4.43)–(4.4.44) directly follows from Proposition C.15.

In the case of $\beta_{\mathrm{s}}^{(v_\infty)} + \delta \leq \beta \leq \beta_*$ for $\delta \in (0, \frac{\sigma_2}{2})$, we follow the argument of Step 2 in the proof of [**11**, Proposition 17.4.2] by using Lemma 4.27 and Proposition C.16 to prove that the boundary value problem (4.3.16) associated with $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$ has a unique solution $\hat{\phi}$ that satisfies (4.4.43)–(4.4.44). □

For each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, the corresponding pseudo-subsonic region $\Omega = \Omega(u, \beta)$ depends continuously on (u, β). For later discussions, it is useful to rewrite (4.3.16) as a boundary value problem for

$$(4.4.45) \qquad\qquad \hat{u}(s, t) = (\hat{\phi} + \varphi_{\mathcal{N}} - \varphi_\beta^*) \circ \mathfrak{F}_{(u,\beta)}(s, t) \qquad \text{in } \mathcal{Q}^{\mathrm{iter}}$$

for mapping $\mathfrak{F} = \mathfrak{F}_{(u,\beta)}$ defined by Definition 4.15(ii), where φ_β^* is given by (4.1.42).

Substitute expression $\hat{\phi} = \hat{u} \circ (\mathfrak{F}_{(u,\beta)})^{-1} - (\varphi_{\mathcal{N}} - \varphi_\beta^*)$ into (4.3.16) and then rewrite (4.3.16) in terms of $\hat{u}$ to obtain

(4.4.46)

$$\sum_{i,j=1}^{2} \mathcal{A}_{ij}^{(u,\beta)}(D\hat{u}, s, t)\partial_{ij}\hat{u} + \sum_{i=1}^{2} \mathcal{A}_i^{(u,\beta)}(D\hat{u}, s, t)\partial_i \hat{u} = f^{(u,\beta)} \qquad \text{in } \mathcal{Q}^{\mathrm{iter}} = (-1, 1) \times (0, 1),$$

$$\mathscr{M}_{(u,\beta)}(D\hat{u}, \hat{u}, s) = 0 \qquad \text{on } \partial_{\mathrm{sh}}\mathcal{Q}^{\mathrm{iter}} := (-1, 1) \times \{1\},$$

$$\hat{u} = 0 \qquad \text{on } \partial_{\mathrm{so}}\mathcal{Q}^{\mathrm{iter}} := \{-1, 1\} \times (0, 1),$$

$$\mathscr{B}_{(u,\beta)}^{(\mathrm{w})}(D\hat{u}, s) := b_1^{(\mathrm{w})}(s)\partial_1 \hat{u} + b_2^{(\mathrm{w})}(s)\partial_2 \hat{u} = 0 \qquad \text{on } \partial_{\mathrm{w}}\mathcal{Q}^{\mathrm{iter}} := (-1, 1) \times \{0\},$$

where $(\partial_1, \partial_2) = (\partial_s, \partial_t)$.

Since $\varphi_{\mathcal{N}} - \varphi_\beta^* = 0$ when $\beta = 0$, we have

$$(4.4.47) \qquad\qquad f^{(u,\beta)} \equiv 0 \qquad\qquad \text{if } \beta = 0,$$

$$(4.4.48) \qquad\qquad \mathscr{M}_{(u,0)}(\mathbf{0}, 0, s) = 0 \qquad \text{on } \partial_{\mathrm{sh}}\mathcal{Q}^{\mathrm{iter}},$$

where (4.4.48) follows from Lemma 4.34(f).

From Lemmas 4.16, 4.30, and 4.34–4.35, the following lemma is obtained:

LEMMA 4.36. *For each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, let $\mathcal{A}_{ij}^{(u,\beta)}$, $\mathcal{A}_i^{(u,\beta)}$, $f^{(u,\beta)}$, $\mathscr{M}_{(u,\beta)}$, $\mathscr{B}_{(u,\beta)}^{(\mathrm{w})}$, and $b_{j,(u,\beta)}^{(\mathrm{w})}$ be as those in* (4.4.46). *Then the following properties hold:*

(a) $\mathcal{A}_{ij}^{(u,\beta)}, \mathcal{A}_i^{(u,\beta)} \in C(\mathbb{R}^2 \times \mathcal{Q}^{\text{iter}})$, $f^{(u,\beta)} \in C(\mathcal{Q}^{\text{iter}})$, $\mathscr{M}_{(u,\beta)} \in C(\mathbb{R}^2 \times \mathbb{R} \times \partial_{\text{sh}}\mathcal{Q}^{\text{iter}})$, and
$\mathscr{B}_{(u,\beta)}^{(\text{w})} \in C(\mathbb{R}^2 \times \mathbb{R} \times \partial_{\text{w}}\mathcal{Q}^{\text{iter}})$;

(b) *Suppose that a sequence* $\{(u_k, \beta_k)\}_{k=1}^\infty \subset \overline{\mathcal{K}^{\text{ext}}}$ *converges to* $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$ *in* $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\text{iter}}) \times [0, \beta_*]$ *as* $k \to \infty$. *Then the following properties hold:*

  – $(\mathcal{A}_{ij}^{(u_k,\beta_k)}, \mathcal{A}_i^{(u_k,\beta_k)}) \to (\mathcal{A}_{ij}^{(u,\beta)}, \mathcal{A}_i^{(u,\beta)})$ *uniformly on compact subsets of* $\mathbb{R}^2 \times \mathcal{Q}^{\text{iter}}$;

  – $f^{(u_k,\beta_k)} \to f^{(u,\beta)}$ *uniformly on compacts subsets of* $\mathcal{Q}^{\text{iter}}$;

  – $\mathscr{M}_{(u_k,\beta_k)} \to \mathscr{M}_{(u,\beta)}$ *uniformly on compact subsets of* $\mathbb{R}^2 \times \mathbb{R} \times \partial_{\text{sh}}\mathcal{Q}^{\text{iter}}$;

  – $\mathscr{B}_{(u_k,\beta_k)}^{(\text{w})} \to \mathscr{B}_{(u,\beta)}^{(\text{w})}$ *uniformly on compact subsets of* $\mathbb{R}^2 \times \partial_{\text{w}}\mathcal{Q}^{\text{iter}}$.

From Lemmas 4.16 and 4.35–4.36, we obtain the following corollary:

COROLLARY 4.37. Let constants $\varepsilon^{(\text{w})}$, $\delta_1^{(\text{w})}$, and $N_1^{(\text{w})}$ be from Lemma 4.35. Let parameters ε, δ_1, and N_1 from Definition 4.19 satisfy that $\varepsilon \in (0, \varepsilon^{(\text{w})}]$, $\delta_1 \in (0, \delta_1^{(\text{w})}]$, and $N_1 \geq N_1^{(\text{w})}$.

(a) For each $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$, $\hat{\phi}$ solves the boundary value problem (4.3.16) if and only if $\hat{u}$ given by (4.4.45) solves the boundary value problem (4.4.46). Thus, (4.4.46) has a unique solution $\hat{u} \in C^2(\mathcal{Q}^{\text{iter}}) \cap C^1(\overline{\mathcal{Q}^{\text{iter}}} \setminus \overline{\partial_{\text{so}}\mathcal{Q}^{\text{iter}}}) \cap C(\overline{\mathcal{Q}^{\text{iter}}})$.
    Furthermore, there exists a constant $C \geq 1$ depending on $(v_\infty, \gamma, \beta_*, \alpha)$ such that

$$|\hat{u}(s,t)| \leq C(1 - |s|) \qquad \text{in } \mathcal{Q}^{\text{iter}}.$$

For each $\hat{d} \in (0, \frac{1}{2})$, there exists $C_{\hat{d}}$ depending on $(v_\infty, \gamma, \beta_*, \hat{d}, \alpha)$ such that

$$\|\hat{u}\|_{2,\alpha_1^*, \mathcal{Q}^{\text{iter}} \cap \{1-|s|>\hat{d}\}} \leq C_{\hat{d}},$$

where constant $\alpha_1^* \in (0, \bar{\alpha}]$ is from Lemma 4.35.

(b) For each $(u_k, \beta_k) \in \overline{\mathcal{K}^{\text{ext}}}$, let $\hat{u}_k$ be the solution of the boundary value problem (4.4.46) associated with (u_k, β_k). Suppose that sequence $\{(u_k, \beta_k)\}$ converges to $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$ in $C^1(\overline{\mathcal{Q}^{\text{iter}}}) \times [0, \beta_*]$. Then there exists a unique solution $\hat{u} \in C^2(\mathcal{Q}^{\text{iter}}) \cap C^1(\overline{\mathcal{Q}^{\text{iter}}} \setminus \overline{\partial_{\text{so}}\mathcal{Q}^{\text{iter}}}) \cap C(\overline{\mathcal{Q}^{\text{iter}}})$ to the boundary value problem (4.4.46) associated with (u, β). Moreover, $\hat{u}_k$ converges to $\hat{u}$ in the following senses:

  – uniformly in $\overline{\mathcal{Q}^{\text{iter}}}$,

  – in $C^{1,\alpha'}(K)$ for any compact subset $K \subset \overline{\mathcal{Q}^{\text{iter}}} \setminus \overline{\partial_{\text{so}}\mathcal{Q}^{\text{iter}}}$ and any $\alpha' \in [0, \alpha_1^*)$,

  – in $C^{2,\alpha'}(K)$ for any compact subset $K \subset \mathcal{Q}^{\text{iter}}$ and any $\alpha' \in [0, \alpha_1^*)$.

(c) If $(u, \beta) \in \overline{\mathcal{K}}$, then (u, β) satisfies property (vii) of Definition 4.19 with nonstrict inequality in (4.3.18).

REMARK 4.38. For a constant $M > 0$, define a set $\mathcal{K}_M^E$ by

$$\mathcal{K}_M^E := \left\{ (u, \beta) \in C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\text{iter}}) \; : \; \|u\|_{2,\alpha,\mathcal{Q}^{\text{iter}}}^{(*,\alpha_1)} \leq M, \, (u, \beta) \text{ satisfy (ii)–(vi) of Definition 4.19} \right\}.$$

Let $\overline{\mathcal{K}_M^E}$ be the closure of $\mathcal{K}_M^E$ in $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\text{iter}}) \times [0, \beta_*]$. Then Lemma 4.36 and Corollary 4.37 still hold when $\overline{\mathcal{K}^{\text{ext}}}$ is replaced by $\overline{\mathcal{K}_M^E}$ for some constant $M > 0$.

4.5. Properties of the Iteration Set $\mathcal{K}$

4.5.1. Admissible solutions. As stated in Definition 4.19, parameter α for the iteration set $\mathcal{K}$ will be chosen in $(0, \frac{\bar{\alpha}}{2}]$, where $\bar{\alpha} \in (0, 1)$ is the constant in Proposition 4.12.

LEMMA 4.39. *Given $\gamma \geq 1$ and $v_\infty > 0$, fix $\beta_* \in (0, \beta_{\mathrm{d}}^{(v_\infty)}]$. Take a sequence $\{\beta_j\}_{j=1}^\infty \subset (0, \beta_*]$ such that β_j converges to 0 as $j \to \infty$. For each $j \in \mathbb{N}$, let $\varphi^{(j)}$ be an admissible solution corresponding to (v_∞, β_j). Let $u^{(j)}$ be defined by (4.1.50) corresponding to (φ_j, β_j). Then there exists a subsequence of $\{u^{(j)}\}$ converging in $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}})$ to $u^{(\mathrm{norm})} \equiv 0$.*

PROOF. By Proposition 4.12 and (4.3.1), sequence $\{u^{(j)}\}$ is uniformly bounded in $C_{(*,1)}^{2,2\alpha}(\mathcal{Q}^{\mathrm{iter}})$. Since $C_{(*,1)}^{2,2\alpha}(\mathcal{Q}^{\mathrm{iter}})$ is compactly embedded into $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}})$, there exists a subsequence (still denoted as) $\{u^{(j)}\}$ such that the subsequence converges in $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}})$ to a function $u^{(\infty)} \in C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}})$.

By (4.4.47), Lemma 4.36, Corollary 4.37, and Remark 4.38, we see that $u = u^{(\infty)}$ is the solution of the nonlinear boundary value problem:

$$(4.5.1) \quad \begin{aligned} \sum_{i,j=1}^2 \mathcal{A}_{ij}^{(u,0)}(Du,s,t)\partial_{ij}u + \sum_{i=1}^2 \mathcal{A}_i^{(u,0)}(Du,s,t)\partial_i u &= 0 && \text{in } \mathcal{Q}^{\mathrm{iter}}, \\ \mathcal{M}_{(u,0)}(Du,u,s) &= 0 && \text{on } \partial_{\mathrm{sh}}\mathcal{Q}^{\mathrm{iter}}, \\ u &= 0 && \text{on } \partial_{\mathrm{so}}\mathcal{Q}^{\mathrm{iter}}, \\ \mathcal{B}_{(u,0)}^{(\mathrm{w})}(Du,s) := b_1^{(\mathrm{w})}(s)\partial_1 u + b_2^{(\mathrm{w})}(s)\partial_2 u &= 0 && \text{on } \partial_{\mathrm{w}}\mathcal{Q}^{\mathrm{iter}}. \end{aligned}$$

Owing to (4.4.48), $u = 0$ is the solution of the boundary value problem (4.5.1). Then $u^{(\infty)} = 0$ in $\mathcal{Q}^{\mathrm{iter}}$ by the uniqueness of solutions. In other words, $u^{(\infty)} = u^{(\mathrm{norm})}$ in $\mathcal{Q}^{\mathrm{iter}}$. $\square$

COROLLARY 4.40. Let constants $\varepsilon^{(\mathrm{w})}$, $\delta_1^{(\mathrm{w})}$, and $N_1^{(\mathrm{w})}$ be from Lemma 4.35, and let parameters (ε, δ_1) in Definition 4.19 be fixed from $(0, \varepsilon^{(\mathrm{w})}] \times (0, \delta_1^{(\mathrm{w})}]$. For each admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{0 \leq \beta \leq \beta_*\}$ in the sense of Definition 2.24, let a function $u = u^{(\varphi,\beta)}$ be given by (4.1.50). Let N_1 be the parameter in Definition 4.19. For each $\delta_1 \in (0, \delta_1^{(\mathrm{w})}]$, there exists a constant $N_1^{(\mathrm{a})} \in [N_1^{(\mathrm{w})}, \infty)$ depending only on $(v_\infty, \gamma, \beta_*, \delta_1)$ such that, if $N_1 \geq N_1^{(\mathrm{a})}$, then $(u^{(\varphi,\beta)}, \beta) \in \mathcal{K}$ for each admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{0 \leq \beta \leq \beta_*\}$.

PROOF. For a fixed admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{0 \leq \beta \leq \beta_*\}$, let $u = u^{(\varphi,\beta)}$ be given by (4.1.50). For simplicity of notation, denote u as $u^{(\varphi,\beta)}$ in this proof.

By the choice of constants N_i ($i = 2, 3, 4, 5$), μ_j ($j = 0, 1$), $\tilde{\mu}$, σ_1, $\hat{\zeta}$, and C in Definition 4.19, (u, β) satisfy properties (ii)–(vi) of Definition 4.19.

By the choice of constant N_0 in Definition 4.19(i), u satisfies

$$\|u - u^{(\mathrm{norm})}\|_{2,\alpha,\mathcal{Q}^{\mathrm{iter}}}^{(*,\alpha_1)} < N_0$$

for any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{0 \leq \beta \leq \beta_*\}$. Lemma 4.39 implies that, for any given constant $\delta_1 \in (0, \delta_1^{(\mathrm{w})}]$, a constant $N_1^{(\mathrm{a})} \in [N_1^{(\mathrm{w})}, \infty)$ can be chosen depending

only on $(v_\infty, \gamma, \beta_*, \delta_1)$ such that, whenever $\beta \in [0, \frac{2\delta_1}{N_1^{(\mathrm{a})}}]$, u satisfies

$$\|u - u^{(\mathrm{norm})}\|_{2,\alpha,\mathcal{Q}^{\mathrm{iter}}}^{(*,\alpha_1)} < \frac{\delta_1}{2}.$$

Therefore, if $N_1 \geq N_1^{(\mathrm{a})}$, then any (u, β) given by (4.1.50) for an admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{0 \leq \beta \leq \beta_*\}$ satisfies property (i) of Definition 4.19. This implies that $(u, \beta) \in \mathcal{K}^{\mathrm{ext}}$. Therefore, Lemmas 4.24, 4.27, 4.30, and 4.34 apply to the nonlinear differential operators $(\mathcal{N}_{(u,\beta)}, \mathcal{M}_{(u,\beta)})$. Then, by Propositions 3.30, 3.32, and 3.39, and Corollary 4.37, we conclude that u is the unique solution of the boundary value problem (4.4.46) associated with (u, β). That is, $\hat{u} = u$ in $\mathcal{Q}^{\mathrm{iter}}$, for $\hat{u}$ is given by (4.3.17). Thus, (u, β) satisfies property (vii) of Definition 4.19.

Therefore, we conclude that $(u^{(\varphi,\beta)}, \beta) \in \mathcal{K}$ for any admissible solution φ corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{0 \leq \beta \leq \beta_*\}$ in the sense of Definition 2.24. $\square$

4.5.2. Openness of $\mathcal{K}$. Let ε, δ_1, δ_2, δ_3, and N_1 be the parameters from Definition 4.19. In this chapter, we further adjust parameters (ε, δ_1), then choose $\delta_3 > 0$ small, depending only on (ε, δ_1) such that Definition 4.19 determines a relatively open subset of $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$.

LEMMA 4.41. *For each $\beta_* \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$, the function set $\mathcal{K}^{\mathrm{ext}}$ given by Definition 4.22 is relatively open in $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$.*

PROOF. For each $j = 1, 2, 3$, function $\mathscr{K}_j(\beta)$ of β in Definition 4.19 is continuous for $\beta \in [0, \beta_*]$. Since $\varphi_\mathcal{O}$ defined in (2.4.1) depends continuously on $\beta \in [0, \frac{\pi}{2})$, $\varphi_\beta = \max\{\varphi_\mathcal{O}, \varphi_\mathcal{N}\}$ and φ_β^* defined in (4.1.42) also depend continuously on $\beta \in [0, \frac{\pi}{2})$. Moreover, s_β and L_β defined in (4.1.29) and (4.1.30), respectively, depend continuously on $\beta \in [0, \frac{\pi}{2})$. Furthermore, for each $\beta \in [0, \beta_*]$,

$$\sup_{Q^\beta(s^*)} (\varphi_\infty - \varphi_\beta^*) - \inf_{Q^\beta(s^*)} (\varphi_\infty - \varphi_\beta^*) > 0 \qquad \text{for all } s^* \in [s_\beta, c_\mathcal{N}],$$

where $Q^\beta(s^*)$ is defined in (4.2.1).

By Lemma 4.16 and the observations stated above, the set determined by conditions (i)–(vi) of Definition 4.19 is relatively open in $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$, because $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}})$ is compactly embedded in $C^1(\overline{\mathcal{Q}^{\mathrm{iter}}})$; for further details, we refer to the proofs of [**11**, Lemmas 12.8.1 and 17.5.1]. $\square$

LEMMA 4.42. *Let $\varepsilon^{(\mathrm{w})}, \delta_1^{(\mathrm{w})}, N_1^{(\mathrm{w})}$, and $\alpha_1 \in (0, \bar{\alpha}]$ be from Lemma 4.35. Let $\varepsilon_0 > 0$ be from Lemma 4.16(c). Then there exists $\varepsilon^{(\mathrm{lb})} \in (0, \varepsilon^{(\mathrm{w})}]$ depending only on $(v_\infty, \gamma, \beta_*)$ such that, whenever parameters $(\varepsilon, \delta_1, N_1)$ in Definition 4.19 are from $(0, \varepsilon^{(\mathrm{lb})}] \times (0, \delta_1^{(\mathrm{w})}] \times [N_1^{(\mathrm{w})}, \infty)$, there is $\bar{\delta}_3 > 0$ depending only on $(v_\infty, \gamma, \beta_*, \delta_1, \delta_2, N_1)$ for δ_2 from Definition 4.19(iv) so that, if parameter δ_3 in Definition 4.19(vii) satisfies that $\delta_3 \in (0, \bar{\delta}_3]$, then the following properties hold: For each $(u^\sharp, \beta^\sharp) \in \overline{\mathcal{K}}$, a constant $\delta^\sharp > 0$ can be chosen depending only on $(v_\infty, \gamma, \beta_*, u^\sharp, \beta^\sharp)$ such that solution $\hat{\phi}$ of the boundary value problem (4.3.16) associated with (u, β) satisfies*

$$(4.5.2) \qquad \hat{\phi} - (\varphi_\beta^* - \varphi_\mathcal{N}) > 0 \qquad in \ \Omega$$

for $\Omega = \Omega(u, \beta)$, provided that $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$ satisfies

$$(4.5.3) \qquad \|u^\sharp - u\|_{C^1(\overline{\mathcal{Q}^{\mathrm{iter}}})} + |\beta^\sharp - \beta| \leq \delta^\sharp.$$

PROOF. We consider two cases separately: (i) $\beta^\sharp \in [\frac{2\delta_1}{N_1^2}, \beta_*]$ and (ii) $\beta^\sharp \in [0, \frac{2\delta_1}{N_1^2}]$.

1. Suppose that $\beta^\sharp \in [\frac{2\delta_1}{N_1^2}, \beta_*]$. By (4.3.3) in Definition 4.19(iv), $u^\sharp$ satisfies

$$u^\sharp > \frac{\delta_1 \delta_2}{N_1^2} \qquad \text{in } \overline{\mathcal{Q}^{\text{iter}}} \cap \{1 - |s| \geq \frac{\tilde{\varepsilon}}{10}\}$$

for $\tilde{\varepsilon} = \frac{2\varepsilon}{c_\mathcal{N} - s_\beta}$. If $\delta_3 > 0$ satisfies

$$(4.5.4) \qquad\qquad \delta_3 \leq \frac{\delta_1}{2N_1^2} \delta_2,$$

then it follows from (4.3.18) that $\hat{u}^\sharp := (\hat{\phi}^\sharp + \varphi_\mathcal{N} - \varphi_{\beta^\sharp}^*) \circ \mathfrak{F}_{(u^\sharp, \beta^\sharp)}$ satisfies

$$(4.5.5) \qquad\qquad \hat{u}^\sharp > \frac{\delta_1}{2N_1^2} \delta_2 \qquad \text{in } \overline{\mathcal{Q}^{\text{iter}}} \cap \{1 - |s| \geq \frac{\tilde{\varepsilon}}{10}\}$$

for $\tilde{\varepsilon} = \frac{2\varepsilon}{c_\mathcal{N} - s_\beta}$, provided that $\hat{\phi}^\sharp$ is the solution of the boundary value problem (4.3.16) associated with $(u^\sharp, \beta^\sharp)$.

Note that $\hat{u}^\sharp$ is the solution of (4.4.46) determined by $(u^\sharp, \beta^\sharp)$. Then, by Corollary 4.37, there exists a constant $\delta^\sharp > 0$ small, depending on $(v_\infty, \gamma, \beta_*, \delta_3, u^\sharp, \beta^\sharp)$, such that, if $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$ satisfies (4.5.3), then (4.5.5) implies that $\hat{u}$ given by (4.3.17) satisfies

$$(4.5.6) \qquad\qquad \hat{u} > \frac{\delta_1}{4N_1^2} \delta_2 \qquad \text{in } \overline{\mathcal{Q}^{\text{iter}}} \cap \{1 - |s| \geq \frac{\tilde{\varepsilon}}{10}\}.$$

For a constant $r > 0$, denote $\mathcal{D}_r := \mathcal{D}_r^\mathcal{N} \cup \mathcal{D}_r^\mathcal{O}$ for $\mathcal{D}_r^\mathcal{N}$ and $\mathcal{D}_r^\mathcal{O}$ defined by (4.1.2). By Proposition 4.16(c), $\mathfrak{F}_{(u,\beta)}^{-1}(\mathcal{D}_{\varepsilon/10}) = \mathcal{Q}^{\text{iter}} \cap \{1 - |s| < \frac{\tilde{\varepsilon}}{10}\}$. Thus, (4.5.6) implies

$$(4.5.7) \qquad\qquad \hat{\phi} - (\varphi_\beta^* - \varphi_\mathcal{N}) = \hat{u} \circ \mathfrak{F}_{(u,\beta)}^{-1} > 0 \qquad \text{in } \Omega \setminus \mathcal{D}_{\varepsilon/10}.$$

Define

$$(4.5.8) \qquad\qquad \hat{\psi} := \hat{\phi} - (\varphi_\beta^* - \varphi_\mathcal{N}) \qquad \text{in } \Omega \cap \mathcal{D}_{\varepsilon/2}.$$

By (4.1.45), we have

$$(4.5.9) \qquad\qquad \hat{\psi} = \begin{cases} \hat{\phi} - (\varphi_\mathcal{O} - \varphi_\mathcal{N}) & \text{in } \Omega \cap \mathcal{D}_{\varepsilon/2}^\mathcal{O}, \\ \hat{\phi} & \text{in } \Omega \cap \mathcal{D}_{\varepsilon/2}^\mathcal{N}, \end{cases}$$

provided that the condition:

$$(4.5.10) \qquad\qquad \varepsilon < \frac{2\hat{c}_\mathcal{O}}{\bar{k}}$$

holds for $\bar{k} > 1$ from (4.1.45).

By (2.5.1), $\varphi_\mathcal{O} - \varphi_\mathcal{N}$ is a linear function depending only on ξ_1. Since $\hat{\phi}$ is a solution of the boundary value problem (4.3.16) associated with (u, β), $\hat{\psi}$ satisfies

$$\mathcal{L}_{(u,\beta)}(\hat{\psi}) := \sum_{i,j=1}^{2} A_{ij}(D\hat{\phi}, \boldsymbol{\xi}) \partial_{\xi_i \xi_j} \hat{\psi} = 0 \qquad \text{in } \Omega \cap \mathcal{D}_{\varepsilon/2}^\mathcal{O},$$

$$\hat{\psi} = 0 \qquad \text{on } \Gamma_{\text{sonic}}^\mathcal{O},$$

$$\partial_{\xi_2} \hat{\psi} = 0 \qquad \text{on } \Gamma_{\text{wedge}} \cap \partial \mathcal{D}_{\varepsilon/2}^\mathcal{O},$$

where $\{A_{ij}(D\hat{\phi}, \boldsymbol{\xi})\}_{i,j=1}^2$ is given by (4.4.22). By Lemma 4.30(g)–(h), $\mathcal{L}_{(u,\beta)}(\hat{\psi}) = 0$ is strictly elliptic in $\mathcal{D}_{\varepsilon/2}^{\mathcal{O}}$. By Lemma 4.34(f), the boundary condition $\mathcal{M}_{(u,\beta)}(D\hat{\phi}, \hat{\phi}, \boldsymbol{\xi}) = 0$ on $\Gamma_{\text{shock}} \cap \partial \mathcal{D}_{\varepsilon/2}^{\mathcal{O}}$ is equivalent to

$$\mathcal{M}_{(u,\beta)}(D\hat{\phi}, \hat{\phi}, \boldsymbol{\xi}) - \mathcal{M}_{(u,\beta)}(D(\varphi_{\mathcal{O}} - \varphi_{\mathcal{N}}), \varphi_{\mathcal{O}} - \varphi_{\mathcal{N}}, \boldsymbol{\xi}) = 0 \qquad \text{on } \Gamma_{\text{shock}} \cap \partial \mathcal{D}_{\varepsilon/2}^{\mathcal{O}}.$$

By Lemma 4.34(d), the boundary condition stated immediately above can be rewritten as

$$\boldsymbol{\beta} \cdot \nabla \hat{\psi} - \mu \hat{\psi} = 0 \qquad \text{on } \Gamma_{\text{shock}} \cap \partial \mathcal{D}_{\varepsilon/2}^{\mathcal{O}},$$

where $\boldsymbol{\beta}$ and μ satisfy

$$\delta_{\text{bc}} \le \boldsymbol{\beta} \cdot \boldsymbol{\nu}_{\text{sh}} \le \delta_{\text{bc}}^{-1}, \quad \mu \ge \delta_{\text{bc}} \qquad \text{on } \Gamma_{\text{shock}} \cap \partial \mathcal{D}_{\varepsilon/2}^{\mathcal{O}}$$

for constant $\delta_{\text{bc}} > 0$ from Lemma 4.34(d) and the unit normal vector $\boldsymbol{\nu}_{\text{sh}}$ to Γ_{shock} towards the interior of Ω.

By (4.5.7), the strong maximum principle, and Hopf's lemma, we obtain that $\hat{\psi} > 0$ in $\mathcal{D}_{\varepsilon/2}^{\mathcal{O}}$, which implies that

$$\hat{u} > 0 \qquad \text{in } \mathcal{Q}^{\text{iter}} \cap \{-1 < s < -1 + \frac{\tilde{\varepsilon}}{2}\}, \tag{4.5.11}$$

provided that condition (4.5.10) holds.

By using (4.5.9), Lemma 4.30(a), and properties (d) and (f) of Lemma 4.34, it can be similarly checked that

$$\hat{u} > 0 \qquad \text{in } \mathcal{Q}^{\text{iter}} \cap \{1 - \frac{\tilde{\varepsilon}}{2} < s < 1\}. \tag{4.5.12}$$

From (4.5.6) and (4.5.11)–(4.5.12), we obtain that $\hat{u} > 0$ in $\mathcal{Q}^{\text{iter}}$, provided that $\delta^\sharp > 0$ is chosen sufficiently small and ε satisfies (4.5.10). This proves (4.5.2) for $\beta^\sharp \in [\frac{2\delta_1}{N_1^2}, \beta_*]$.

2. Suppose that $\beta^\sharp \in [0, \frac{2\delta_1}{N_1^2}]$. Choose $\delta^\sharp \in (0, \frac{2\delta_1}{N_1^2})$ so that (4.5.3) implies that $\beta \in [0, \frac{\delta_1}{N_1})$. By Lemma 4.34(d), the maximum principle applies to solution $\hat{\phi}$ of the boundary value problem (4.3.16) associated with $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$ satisfying (4.5.3) so that

$$\hat{\phi} > 0 \qquad \text{in } \Omega. \tag{4.5.13}$$

For $(\varphi_{\mathcal{O}}, \varphi_{\mathcal{N}})$ given by (2.5.1), denote $\phi_\beta := \varphi_{\mathcal{O}} - \varphi_{\mathcal{N}}$. Since ϕ_β is a linear function of $\boldsymbol{\xi}$, $\hat{\phi} - \phi_\beta$ satisfies

$$\mathcal{N}_{(u,\beta)}(\hat{\phi} - \phi_\beta) = \mathcal{N}_{(u,\beta)}(\hat{\phi}) = 0 \qquad \text{in } \Omega$$

for the second-order differential operator (4.4.1). From properties (d) and (f) of Lemma 4.34, it follows that $\mathcal{M}_{(u,\beta)}(D\hat{\phi}, \hat{\phi}, \boldsymbol{\xi}) - \mathcal{M}_{(u,\beta)}(D\phi_\beta, \phi_\beta, \boldsymbol{\xi}) = 0$ for all $\boldsymbol{\xi} \in \Gamma_{\text{shock}}$. This condition can be written as

$$\mathbf{b} \cdot D_{\boldsymbol{\xi}}(\hat{\phi} - \phi_\beta) + b_0(\hat{\phi} - \phi_\beta) = 0 \qquad \text{on } \Gamma_{\text{shock}},$$

where $\mathbf{b}$ and b_0 satisfy that $\mathbf{b} \cdot \boldsymbol{\nu}_{\text{sh}} > 0$ and $b_0 < 0$ on Γ_{shock} for the unit normal vector $\boldsymbol{\nu}_{\text{sh}}$ to Γ_{shock} towards the interior of Ω. Then the comparison principle implies that $\hat{\phi} \ge \phi_\beta$ in Ω. Furthermore, $\hat{\phi} = 0 > \phi_\beta$ on $\Gamma_{\text{sonic}}^{\mathcal{N}}$. By the strong maximum principle, we conclude that

$$\hat{\phi} > \phi_\beta \qquad \text{in } \Omega. \tag{4.5.14}$$

Then (4.5.2) is obtained from (4.5.13)–(4.5.14), because $\max\{0, \phi_\beta\} \ge \varphi_\beta^* - \varphi_{\mathcal{N}}$ holds in Ω. $\qquad \square$

LEMMA 4.43 (*Estimate of $\hat{\phi}$ away from $\Gamma^{\mathcal{O}}_{\mathrm{sonic}}$*). *Let $\varepsilon_0 > 0$ be from Lemma 4.16(c). Let $\varepsilon^{(\mathrm{w})}$, $\delta_1^{(\mathrm{w})}$, $N_1^{(\mathrm{w})}$, and $\alpha_1^* \in (0, \bar{\alpha}]$ be from Lemma 4.35. Let $\varepsilon^{(\mathrm{lb})}$ and $\bar{\delta}_3$ be from Lemma 4.42. For a constant $r > 0$, let $\mathcal{D}_r^{\mathcal{O}}$ be defined by (4.1.2). Then there exist $\varepsilon^{(\mathrm{par})} \in (0, \varepsilon^{(\mathrm{lb})}]$ depending only on $(v_\infty, \gamma, \beta_*)$ and $C > 0$ depending only on $(v_\infty, \gamma, \beta_*, \alpha)$ such that, whenever parameters $(\varepsilon, \delta_1, N_1)$ in Definition 4.19 are from $(0, \varepsilon^{(\mathrm{lb})}] \times (0, \delta_1^{(\mathrm{w})}] \times [N_1^{(\mathrm{w})}, \infty)$, and $\delta_3 \in (0, \bar{\delta}_3]$, then the following properties hold: For each $(u^\sharp, \beta^\sharp) \in \overline{\mathcal{K}}$, a constant $\delta^\sharp > 0$ can be chosen depending only on $(v_\infty, \gamma, \beta_*, u^\sharp, \beta^\sharp)$ so that, if $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$ satisfies (4.5.3), solution $\hat{\phi}$ of the boundary value problem (4.3.16) associated with (u, β) satisfies the estimate*:

$$(4.5.15) \qquad\qquad \|\hat{\phi}\|^{(2),(\mathrm{par})}_{2,\alpha_1^*,\Omega \setminus \mathcal{D}^{\mathcal{O}}_{\varepsilon_0/10}} \leq C$$

for $\Omega = \Omega(u, \beta)$, where norm $\|\cdot\|^{(2),(\mathrm{par})}_{2,\alpha_1^*,\Omega \setminus \mathcal{D}^{\mathcal{O}}_{\varepsilon_0/10}}$ is defined by Definition 3.25.

PROOF. The proof is divided into two steps.

1. *Claim: There exists a constant $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ such that, for each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, $\hat{\phi}$ satisfies*

$$(4.5.16) \qquad\qquad \hat{\phi}(x, y) \leq C x^2 \qquad in \ \Omega \cap \mathcal{D}^{\mathcal{N}}_{\varepsilon_0}$$

in the (x, y)–coordinates defined by (3.4.18).

For the (x, y)–coordinates defined by (3.4.18), denote

$$v(x, y) := \frac{A}{2} x^2$$

for a constant $A \geq \frac{2 - \frac{\mu_0}{10}}{\gamma + 1}$ to be determined later, where μ_0 is from Definition 4.19(iv-1). For the elliptic cut-off ζ_1 defined by (4.4.4), $\zeta_1(\frac{v_x}{x}) = \frac{2 - \frac{\mu_0}{10}}{\gamma + 1}$. By Lemma 4.24 and (4.4.22), equation $\mathcal{N}_{(u,\beta)}(\hat{\phi}) = 0$ is rewritten in the (x, y)–coordinates as

$$\mathcal{N}^{\mathrm{polar}}_{(u,\beta)}(\hat{\phi}) = 0 \qquad in \ \Omega \cap \mathcal{D}^{\mathcal{N}}_{\varepsilon_{\mathrm{eq}}/2}$$

for the nonlinear differential operator $\mathcal{N}^{\mathrm{polar}}_{(u,\beta)}$ given by (4.4.12), where $\varepsilon_{\mathrm{eq}} \in (0, \frac{\varepsilon_0}{2})$ is from Lemma 4.24.

By $\zeta_1(\frac{v_x}{x}) = \frac{2 - \frac{\mu_0}{10}}{\gamma + 1}$ and (4.4.12), we have

$$\mathcal{N}^{\mathrm{polar}}_{(u,\beta)}(v) = Ax \Big(-(1 - \frac{\mu_0}{10}) + \frac{O_1^{\mathrm{mod}}}{x} + O_4^{\mathrm{mod}} \Big) \qquad in \ \Omega \cap \mathcal{D}^{\mathcal{N}}_{\varepsilon_{\mathrm{eq}}/2},$$

with $O_j^{\mathrm{mod}} = O_j^{\mathrm{mod}}(v_x, 0, x, y)$ for $j = 1, 4$. It follows from (4.4.11) that $|\frac{O_1^{\mathrm{mod}}}{x}| + |O_4^{\mathrm{mod}}| \leq C\sqrt{x}$ for $C > 0$ depending only on (v_∞, γ). Therefore, there exists $\bar{\varepsilon} \in (0, \frac{1}{2}\min\{\varepsilon_0, \varepsilon_{\mathrm{eq}}, \bar{\varepsilon}_{\mathrm{bc}}\})$ depending only on (v_∞, γ) such that

$$\mathcal{N}^{\mathrm{polar}}_{(u,\beta)}(v) \leq Ax \big(-(1 - \frac{\mu_0}{10}) + C\sqrt{\bar{\varepsilon}} \big) < -\frac{Ax}{2} \big(1 - \frac{\mu_0}{10} \big) < 0 = \mathcal{N}^{\mathrm{polar}}_{(u,\beta)}(\hat{\phi}) \qquad in \ \Omega \cap \mathcal{D}^{\mathcal{N}}_{\bar{\varepsilon}}.$$

Note that $0 < \mu_0 < 1$ by Definition 4.19(iv-1) and Lemma 3.28.

On $\Gamma_{\mathrm{shock}}(u, \beta) \cap \mathcal{D}^{\mathcal{N}}_{\bar{\varepsilon}}$, properties (f)–(g) of Lemma 4.34 imply that

$$\mathcal{M}_{(u,\beta)}(Dv, v, \boldsymbol{\xi}) = \mathcal{M}_{(u,\beta)}(Dv, v, \boldsymbol{\xi}) - \mathcal{M}_{(u,\beta)}(\mathbf{0}, 0, \boldsymbol{\xi})$$

$$\leq -\delta_{\mathrm{bc}}(Ax + \frac{A}{2}x^2) < 0 = \mathcal{M}_{(u,\beta)}(D\hat{\phi}, \hat{\phi}, \boldsymbol{\xi})$$

for constant $\delta_{\mathrm{bc}} > 0$ from Lemma 4.34(g). On $\Gamma_{\mathrm{wedge}} \cap \overline{\mathcal{D}_{\bar{\varepsilon}}^{\mathcal{N}}}$, $\partial_{\xi_2} v = \partial_y v = 0 = \partial_{\mathbf{n}_\mathrm{w}} \hat{\phi}$. On $\Gamma_{\mathrm{sonic}}^{\mathcal{N}}$, $v = 0 = \hat{\phi}$ holds.

By (4.4.43) and Remark 4.21(ii), there exists a constant $\hat{C} > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ such that $\hat{\phi}$ satisfies

$$(4.5.17) \qquad \hat{\phi}(x, y) \leq \hat{C} x \qquad \text{on } \Omega \cap \mathcal{D}_{\varepsilon_0}^{\mathcal{N}}.$$

Choose $A = \max\{\frac{2\hat{C}}{\bar{\varepsilon}}, \frac{2 - \frac{\mu_0}{10}}{1+\gamma}\}$ so that v satisfies

$$\hat{\phi} \leq v \qquad \text{on } \Omega \cap \{x = \bar{\varepsilon}\}.$$

By Lemmas 4.30 and 4.34, and the comparison principle, we have

$$(4.5.18) \qquad \hat{\phi} \leq v \qquad \text{in } \Omega \cap \mathcal{D}_{\bar{\varepsilon}}^{\mathcal{N}}.$$

In order to extend this result onto $\Omega \cap \mathcal{D}_{\varepsilon_0}^{\mathcal{N}}$, we adjust the choice of A as

$$A = \max\left\{ \frac{2\hat{C}}{\bar{\varepsilon}}, \frac{2 - \frac{\mu_0}{10}}{1 + \gamma}, \frac{2\hat{C}\varepsilon_0}{\bar{\varepsilon}^2} \right\},$$

so that, from (4.5.17),

$$(4.5.19) \qquad \hat{\phi}(x, y) \leq \hat{C}\varepsilon_0 \leq \frac{A}{2}\bar{\varepsilon}^2 \leq v(x, y) \qquad \text{in } \Omega \cap (\mathcal{D}_{\varepsilon_0}^{\mathcal{N}} \setminus \mathcal{D}_{\bar{\varepsilon}}^{\mathcal{N}}).$$

Combining (4.5.18) with (4.5.19), we obtain (4.5.16) with $C = A$ for A given above before (4.5.19).

2. By Definition 4.19(iii) and Remark 4.21(ii), there exists a constant $l > 0$ depending only on (γ, v_∞) such that

$$(4.5.20) \qquad \hat{f}_{\mathcal{N}, \mathrm{sh}}(x) \geq l \qquad \text{on } [0, \varepsilon_0].$$

By Remark 4.21(ii), $\hat{f}_{\mathcal{N}, \mathrm{sh}}$ satisfies the estimate:

$$(4.5.21) \qquad \|\hat{f}_{\mathcal{N}, \mathrm{sh}}\|_{2, \alpha, (0, \varepsilon_0)}^{(-1-\alpha), \{0\}} \leq \|\hat{f}_{\mathcal{N}, 0}\|_{C^3([0, \varepsilon_0])} + C N_0.$$

By (4.5.16), (4.5.20)–(4.5.21), Lemmas 4.24 and 4.33–4.34, the boundary value problem (4.3.16) associated with $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$ satisfying (4.5.3) satisfies all the conditions of Theorem C.11. Therefore, we conclude from Theorem C.11 that, for each $\alpha' \in (0, 1)$, there exists a constant $C_{\alpha'} > 0$ depending only on $(v_\infty, \gamma, \beta_*, \alpha')$ such that $\hat{\phi}$ satisfies

$$(4.5.22) \qquad \|\hat{\phi}\|_{2, \alpha', \Omega \cap \mathcal{D}_{\varepsilon_0}^{\mathcal{N}}}^{(2), (\mathrm{par})} \leq C_{\alpha'}.$$

Finally, (4.5.15) is obtained by combining estimate (4.5.22) with Lemma 4.35. □

As pointed out earlier, $\Gamma_{\mathrm{sonic}}^{\mathcal{O}}$ defined in Definition 2.23 depends continuously on $\beta \in [0, \frac{\pi}{2})$. Therefore, the pseudo-subsonic region $\Omega(u, \beta)$ associated with $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$ depends continuously on (u, β). In particular, $\Omega(u, \beta) \cap \mathcal{D}_{\varepsilon_0}^{\mathcal{O}}$ changes from a rectangular domain to a triangular domain as β increases from $\beta < \beta_{\mathrm{s}}^{(v_\infty)}$ to $\beta > \beta_{\mathrm{s}}^{(v_\infty)}$. Furthermore, the ellipticity of equation $\mathcal{N}_{(u, \beta)}(\hat{\phi}) = 0$ near $\Gamma_{\mathrm{sonic}}^{\mathcal{O}}$ changes as β varies. For that reason, the *a priori* estimate of a solution $\hat{\phi}$ of the boundary value problem (4.3.16) is given for the three cases separately:
(i) $\beta < \beta_{\mathrm{s}}^{(v_\infty)}$;
(ii) $\beta \geq \beta_{\mathrm{s}}^{(v_\infty)}$ close to $\beta_{\mathrm{s}}^{(v_\infty)}$;
(iii) $\beta > \beta_{\mathrm{s}}^{(v_\infty)}$ away from $\beta_{\mathrm{s}}^{(v_\infty)}$.

LEMMA 4.44 (Estimates of $\hat{\phi}$ near $\Gamma^{\mathcal{O}}_{\text{sonic}}$). *Let* $\varepsilon^{(\text{par})}$ *be from Lemma 4.43. There exist* $\varepsilon^{\mathcal{O}} \in$ $(0, \varepsilon^{(\text{par})}]$ *and* $\delta_1^{(\text{E})}$ *depending only on* $(v_\infty, \gamma, \beta_*)$ *such that, whenever parameters* $(\varepsilon, \delta_1, \delta_3, N_1)$ *in Definition 4.19 are chosen as in Lemma 4.43, and* (ε, δ_1) *further satisfy*

$$0 < \varepsilon < \varepsilon^{\mathcal{O}}, \quad 0 < \delta_1 \le \delta_1^{(E)},$$

then, for each $(u^\sharp, \beta^\sharp) \in \overline{\mathcal{K}}$, *there is a constant* $\delta^\sharp$ *depending on* $(v_\infty, \gamma, \beta_*, \delta_2, \delta_3, u^\sharp, \beta^\sharp)$ *so that, if* $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$ *satisfies* (4.5.3), *then the following properties hold:*

(i) *If* $\beta \in [0, \beta_s^{(v_\infty)})$, *for each* $\alpha' \in (0, 1)$, *there exist constants* $\hat{\varepsilon}_p \in (0, \varepsilon_0]$ *and* $C_{\alpha'} > 0$ *depending only on* $(v_\infty, \gamma, \beta_*, \alpha')$ *such that solution* $\hat{\phi} \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *of the boundary value problem* (4.3.16) *associated with* (u, β) *satisfies*

$$\|\hat{\phi} - (\varphi_{\mathcal{O}} - \varphi_{\mathcal{N}})\|^{(2), (\text{par})}_{2, \alpha', \Omega \cap \mathcal{D}^{\mathcal{O}}_{\hat{\varepsilon}_p}} \le C_{\alpha'};$$

(ii) *There exists a constant* $\hat{\delta} \in (0, \beta_* - \beta_s^{(v_\infty)})$ *depending only on* $(v_\infty, \gamma, \beta_*)$ *such that, if* $\beta \in [\beta_s^{(v_\infty)}, \beta_s^{(v_\infty)} + \hat{\delta}]$, *then, for each* $\alpha' \in (0, 1)$, *there exist constants* $\hat{\varepsilon}_p \in (0, \varepsilon_0]$ *depending on* $(v_\infty, \gamma, \beta_*)$ *and* $C_{\alpha'} > 0$ *depending only on* $(v_\infty, \gamma, \beta_*, \alpha')$ *so that* $\hat{\phi}$ *satisfies*

$$\|\hat{\phi} - (\varphi_{\mathcal{O}} - \varphi_{\mathcal{N}})\|_{C^{2,\alpha'}(\Omega \cap \mathcal{D}^{\mathcal{O}}_{\hat{\varepsilon}_p})} \le C_{\alpha'},$$
$$D^m(\hat{\phi} - \varphi_{\mathcal{O}} + \varphi_{\mathcal{N}})(P_\beta) = 0 \qquad \text{for } m = 0, 1, 2,$$

where P_β *is defined in Definition 2.23;*

(iii) *There exist constants* $\hat{\alpha} \in (0, \frac{1}{3})$ *depending only on* $(v_\infty, \gamma, \beta_*)$ *and* $C > 0$ *depending only on* $(v_\infty, \gamma, \beta_*)$ *so that, if* $\beta \in [\beta_s^{(v_\infty)} + \frac{\hat{\delta}}{2}, \beta_*]$, *then* $\hat{\phi}$ *satisfies*

$$(4.5.23) \qquad \|\hat{\phi} - (\varphi_{\mathcal{O}} - \varphi_{\mathcal{N}})\|^{(-1-\hat{\alpha}), \{P_\beta\}}_{2, \hat{\alpha}, \Omega \cap \mathcal{D}^{\mathcal{O}}_{\varepsilon_0}} \le C,$$

$$(4.5.24) \qquad D^m(\hat{\phi} - \varphi_{\mathcal{O}} + \varphi_{\mathcal{N}})(P_\beta) = 0 \qquad \text{for } m = 0, 1.$$

PROOF. We divide the proof into two steps.

1. *Assertion* (i): Owing to Remark 3.31, we need to consider two cases separately: (i) $\beta < \beta_s^{(v_\infty)}$ away from $\beta_s^{(v_\infty)}$ and (ii) $\beta < \beta_s^{(v_\infty)}$ close to $\beta_s^{(v_\infty)}$.

By Lemma 4.2(e), (4.1.26), (4.1.31), Proposition 4.6, and Definition 4.19(iii), there exist $\hat{\varepsilon} \in (0, \varepsilon^{(\text{par})}]$ and $\hat{\sigma}_1 \in (0, \frac{\beta_s^{(v_\infty)}}{10})$ such that, for any $(u, \beta) \in \overline{\mathcal{K}^{\text{ext}}}$, it holds that, if $\sigma \in (0, \hat{\sigma}_1]$, then we can fix $\hat{m} > 1$ depending only on $(v_\infty, \gamma, \sigma)$ and $\hat{k} > 1$ depending only on (v_∞, γ) such that

(a) if $0 \le \beta \le \beta_s^{(v_\infty)} - \frac{\sigma}{2}$, then

$$(4.5.25) \qquad \{0 < x < 2\hat{\varepsilon}, 0 < y < \frac{1}{2\hat{m}}\} \subset \Omega \cap \mathcal{D}^{\mathcal{O}}_{2\hat{\varepsilon}} \subset \{0 < x < 2\hat{\varepsilon}, 0 < y < 2\hat{m}\};$$

(b) if $\beta_s^{(v_\infty)} - \sigma \le \beta < \beta_s^{(v_\infty)}$, then

$$(4.5.26) \qquad \{0 < x < 2\hat{\varepsilon}, 0 < y < y_{P_1} + \frac{x}{2\hat{k}}\} \subset \Omega \cap \mathcal{D}^{\mathcal{O}}_{2\hat{\varepsilon}} \subset \{0 < x < 2\hat{\varepsilon}, 0 < y < y_{P_1} + 2\hat{k}x\}.$$

For a fixed $\sigma \in (0, \hat{\sigma}_1]$, suppose that $0 \le \beta \le \beta_s^{(v_\infty)} - \frac{\sigma}{2}$. Let $\hat{\psi}$ be given by (4.5.8). By Lemma 4.42, we have

$$(4.5.27) \qquad \hat{\psi} > 0 \qquad \text{in } \Omega \cap \mathcal{D}^{\mathcal{O}}_{\varepsilon/2},$$

provided that $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$ satisfies (4.5.2) for $\delta^{\sharp} > 0$ from Lemma 4.42.

Owing to (4.1.45), if condition (4.5.10) holds, then we can repeat Step 1 in the proof of Lemma 4.43 to obtain

$$(4.5.28) \qquad \hat{\psi}(x, y) \leq C x^2 \qquad \text{in } \Omega \cap \mathcal{D}^{\mathcal{O}}_{\hat{\varepsilon}_0} \text{ for } \hat{\varepsilon}_0 := \min\{\varepsilon_0, \frac{\hat{c}_{\mathcal{O}}}{\bar{k}}\}$$

for $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$, where the (x, y)–coordinates are given by (3.5.2), and $\hat{c}_{\mathcal{O}}$ and $\bar{k}$ are given by Definition 4.1 and (4.1.45), respectively. Repeating Step 2 in the proof of Lemma 4.43 with (4.5.27)–(4.5.28) and $\hat{f}_{\mathcal{O},\mathrm{sh}}$ given by (4.3.20), and using (4.5.25), we can show that, for each $\alpha' \in (0, 1)$, there exists a constant $C_{\alpha'} > 0$ depending only on $(v_\infty, \gamma, \beta_*, \alpha')$ such that

$$\|\hat{\phi} - (\varphi_{\mathcal{O}} - \varphi_{\mathcal{N}})\|^{(2),(\mathrm{par})}_{2,\alpha',\Omega \cap \mathcal{D}^{\mathcal{O}}_{\hat{\varepsilon}_0}} = \|\hat{\psi}\|^{(2),(\mathrm{par})}_{2,\alpha',\Omega \cap \mathcal{D}^{\mathcal{O}}_{\hat{\varepsilon}_0}} \leq C_{\alpha'}.$$

Next, suppose that $\beta_{\mathrm{s}}^{(v_\infty)} - \sigma \leq \beta < \beta_{\mathrm{s}}^{(v_\infty)}$. In this case, we need to combine two estimates: (i) in $\Omega \cap \{x < y_{P_1}^2\}$ and (ii) in $\Omega \cap \{x > \frac{y_{P_1}^2}{10}\}$ near $\Gamma^{\mathcal{O}}_{\mathrm{sonic}}$.

In $\Omega \cap \{x < y_{P_1}^2\}$, we repeat the argument of Step 2 in the proof of Lemma 4.43 to obtain

$$\|\hat{\phi} - (\varphi_{\mathcal{O}} - \varphi_{\mathcal{N}})\|^{(2),(\mathrm{par})}_{2,\alpha',\Omega \cap \mathcal{D}^{\mathcal{O}}_{y_{P_1}^2}} = \|\hat{\psi}\|^{(2),(\mathrm{par})}_{2,\alpha',\Omega \cap \mathcal{D}^{\mathcal{O}}_{y_{P_1}^2}} \leq C_{\alpha'}$$

for each $\alpha' \in (0, 1)$, where $C^{\alpha'} > 0$ is given, depending only on $(v_\infty, \gamma, \beta_*, \alpha')$.

In $\Omega \cap \{x > y_{P_1}^2\}$ near $\Gamma^{\mathcal{O}}_{\mathrm{sonic}}$, we adjust the argument in Step 2 in the proof of Proposition 3.32 to show that there exist sufficiently small constants $\bar{\sigma} \in (0, \sigma_1]$ and $\varepsilon^* \in (0, \hat{\varepsilon}_0] \cap (0, \varepsilon^{(\mathrm{par})}]$ depending only on $(v_\infty, \gamma, \beta_*)$ so that $\hat{\psi}$ satisfies

$$\hat{\psi}(x, y) \leq C x^4 \qquad \text{in } \Omega \cap \mathcal{D}^{\mathcal{O}}_{\varepsilon^*} \cap \{x > \frac{y_{P_1}^2}{10}\}$$

for $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$. For $\hat{f}_{\mathcal{O},\mathrm{sh}}$ defined by (4.3.20) and $z_0 = (x_0, y_0) \in \Omega \cap \mathcal{D}^{\mathcal{O}}_{\varepsilon^*} \cap \{x > \frac{y_{P_1}^2}{5}\}$, we define $F^{(z_0)}(S)$ by (3.5.39) given in the proof of Proposition 3.32. By Remark 4.21(i)–(ii), $F^{(z_0)}$ satisfies

$$\|F^{(z_0)}\|_{C^2([-1,1])} \leq C N_0 \sqrt{x_0}$$

for $C > 0$ depending only on $(v_\infty, \gamma, \alpha)$. Then we apply Theorem C.6 and adjust the later part of Step 4 in the proof of Proposition 3.32 to conclude that

$$\|\hat{\phi} - (\varphi_{\mathcal{O}} - \varphi_{\mathcal{N}})\|^{(2),(\mathrm{par})}_{2,\alpha',\Omega \cap \mathcal{D}^{\mathcal{O}}_{\varepsilon^*}} = \|\hat{\psi}\|^{(2),(\mathrm{par})}_{2,\alpha',\Omega \cap \mathcal{D}^{\mathcal{O}}_{\varepsilon^*}} \leq C_{\alpha'}$$

for each $\alpha' \in (0, 1)$, where $C_{\alpha'} > 0$ is given, depending only on $(v_\infty, \gamma, \beta_*, \alpha')$, provided that $\sigma \in (0, \bar{\sigma}]$.

The proof of assertion (i) is completed.

2. *Assertions* (ii) *and* (iii): Assertion (ii) can be proved in a way similar to Proposition 3.39. Estimate (4.5.23) in assertion (iii) directly follows from Proposition C.16.

For $\beta \geq \beta_{\mathrm{s}} + \frac{\hat{\delta}}{2}$, (4.4.43) implies that

$$(4.5.29) \qquad\qquad\qquad (\hat{\phi} - \phi_{\mathcal{O}})(P_\beta) = 0$$

for $\phi_{\mathcal{O}} = \varphi_{\mathcal{O}} - \varphi_{\mathcal{N}}$. By Lemma 4.34(f) and (4.5.29), $\hat{\phi}$ satisfies

$$\int_0^1 \mathcal{M}_{(u,\beta)}(tD\hat{\phi} + (1-t)D\phi_0, t\hat{\phi} + (1-t)\phi_0, \boldsymbol{\xi})\, dt \cdot D(\hat{\phi} - \phi_0) = 0 \qquad \text{at } \boldsymbol{\xi} = P_\beta.$$

By (4.4.23), (4.4.25), (4.4.29), and Lemma 4.34,

$$|\partial_{p_1}\mathcal{M}_{(u,\beta)}(\mathbf{p}, z, P_\beta) - \partial_{p_1}g^{\mathrm{sh}}(D\varphi_{\mathcal{O}}(P_\beta), \varphi_{\mathcal{O}}(P_\beta), P_\beta)| \leq C\sqrt{\delta_1}$$

for some $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$. This inequality, combined with Lemma 3.37, implies that, if $\delta_1 > 0$ is chosen small, depending only on $(v_\infty, \gamma, \beta_*)$, then the boundary conditions: $\mathcal{M}_{(u,\beta)}(D\hat{\phi}, \hat{\phi}, \boldsymbol{\xi}) = 0$ on Γ_{shock} and $\hat{\phi}_{\xi_2} = 0$ on Γ_{wedge} are functionally independent at P_β so that

$$D(\hat{\phi} - \phi_{\mathcal{O}})(P_\beta) = 0.$$

In proving assertions (i)–(iii), all the required properties of $\mathcal{N}_{(u,\beta)}$ and $\mathcal{M}_{(u,\beta)}$ are provided by Lemmas 4.25, 4.27, 4.30, and 4.32–4.34.   □

COROLLARY 4.45. In Definition 4.19, choose parameters $(\alpha, \varepsilon, \delta_1, \delta_3, N_1)$ as follows:

(i) For $\bar{\alpha}$, α_1, and $\hat{\alpha}$ from Lemmas 4.35, 4.43, and 4.44, respectively, choose

$$\alpha = \frac{1}{2}\min\{\bar{\alpha}, \alpha_1, \hat{\alpha}\};$$

(ii) Choose $(\varepsilon, \delta_1, N_1)$ to satisfy

$$(\varepsilon, \delta_1, N_1) \in (0, \varepsilon^{\mathcal{O}}] \times (0, \delta_1^{(\mathrm{w})}] \times [N_1^{(\mathrm{a})}, \infty)$$

for $N_1^{(\mathrm{a})} \in [N_1^{(\mathrm{w})}, \infty)$ from Corollary 4.40;

(iii) For $(\delta_1, N_1) \in (0, \delta_1^{(\mathrm{w})}] \times [N_1^{(\mathrm{a})}, \infty)$, denote $\bar{\delta} := \frac{\delta_1}{2N_1^2}\delta_2$, where $\delta_2 > 0$ is a parameter to be determined later. Choose δ_3 to satisfy

$$\delta_3 \in (0, \bar{\delta}_3].$$

Under the choices of parameters $(\alpha, \varepsilon, \delta_1, \delta_3, N_1)$ above, there exists a constant $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ such that, for each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, denoting the unique solution of the boundary value problem (4.3.16) associated with (u, β) by $\hat{\phi} \in C^2(\Omega(u, \beta)) \cap C^1(\overline{\Omega(u, \beta)})$ and defining $\hat{u} : \overline{\mathcal{Q}^{\mathrm{iter}}} \to \mathbb{R}$ by (4.3.17), we have

(4.5.30) $\|\hat{u}\|_{2,2\alpha,\mathcal{Q}^{\mathrm{iter}}}^{(*,1)} \leq C.$

PROOF. By the choice of parameters $\alpha \in (0, \frac{1}{6})$ and $(\varepsilon, \delta_1, \delta_3, N_1)$, estimate (4.5.30) follows from Lemmas 4.43–4.44 by repeating the argument in the proof of Proposition 4.12.   □

PROPOSITION 4.46. Under the choices of parameters $(\alpha, \varepsilon, \delta_1, \delta_3, N_1)$ as in Corollary 4.45, the iteration set $\mathcal{K}$ defined in Definition 4.19 is relatively open in $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$.

PROOF. We have shown in Lemma 4.41 that $\mathcal{K}^{\mathrm{ext}}$ is relatively open in $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$. Therefore, it remains to check that property (vii) of Definition 4.19 defines a relatively open subset of $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$ under the choice of δ_3 given by (iii) in the statement of Corollary 4.45.

Suppose that this is not true. Then there exist $(u^\sharp, \beta^\sharp) \in \mathcal{K}$ and a sequence $\{(u_n, \beta_n)\}_{n=1}^\infty \subset \mathcal{K}^{\mathrm{ext}}$ such that

$$\lim_{n \to \infty} \|u_n - u^\sharp\|_{2,\alpha/2,\mathcal{Q}^{\mathrm{iter}}}^{(*,\alpha_1)} + |\beta_n - \beta^\sharp| = 0, \qquad \|\hat{u}_n - u_n\|_{2,\alpha/2,\mathcal{Q}^{\mathrm{iter}}}^{(*,\alpha_1)} \geq \delta_3 \qquad \text{for all } n \in \mathbb{N},$$

where each $\hat{u}_n$ for $n \in \mathbb{N}$ is given by (4.3.18) for $(u, \beta) = (u_n, \beta_n)$.

Let $\hat{u}^\sharp$ be given by (4.3.17) with $(u, \beta) = (u^\sharp, \beta^\sharp)$, and denote

$$\delta^\sharp := \frac{\delta_3 - \|\hat{u}^\sharp - u^\sharp\|_{2,\alpha/2,\mathcal{Q}^{\mathrm{iter}}}^{(*,\alpha_1)}}{10}.$$

By (4.3.18), it holds that $\delta^\sharp > 0$. Therefore, we can choose $n^\sharp \in \mathbb{N}$ sufficiently large such that $\|u_n - u^\sharp\|_{2,\alpha/2,\mathcal{Q}^{\mathrm{iter}}}^{(*,\alpha_1)} + |\beta_n - \beta^\sharp| \leq \delta^\sharp$ for all $n \geq n^\sharp$. Then we have

$$\|\hat{u}_n - \hat{u}^\sharp\|_{2,\alpha/2,\mathcal{Q}^{\mathrm{iter}}}^{(*,\alpha_1)} \geq 9\delta^\sharp \qquad \text{for all } n \geq n^\sharp.$$

By Corollary 4.45, $\{\hat{u}_n\}$ is bounded in $C_{(*,1)}^{2,2\alpha}(\mathcal{Q}^{\mathrm{iter}})$. It is noted in Definition 4.18 that $C_{(*,1)}^{2,2\alpha}(\mathcal{Q}^{\mathrm{iter}})$ is compactly embedded into $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}})$. Therefore, $\{\hat{u}_n\}$ has a subsequence $\{\hat{u}_{n_j}\}$ that converges in $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}})$ to a function $\hat{u}^* \in C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}})$ so that

(4.5.31)
$$\|\hat{u}^* - \hat{u}^\sharp\|_{2,\alpha/2,\mathcal{Q}^{\mathrm{iter}}}^{(*,\alpha_1)} \geq 9\delta^\sharp.$$

Define

$$\hat{\phi}^* := \hat{u}^* \circ \mathfrak{F}_{(u^\sharp,\beta^\sharp)}^{-1} - \varphi_{\mathcal{N}} + \varphi_{\beta^\sharp}^* \qquad \text{in } \mathfrak{F}_{(u^\sharp,\beta^\sharp)}^{-1}(\overline{\mathcal{Q}^{\mathrm{iter}}}) = \overline{\Omega(u^\sharp,\beta^\sharp)}.$$

By Lemma 4.36, $\hat{\phi}^*$ solves the nonlinear boundary value problem (4.3.16) associated with $(u^\sharp, \beta^\sharp)$. Then the uniqueness of solutions of (4.3.16) stated in Lemma 4.35 implies that $\hat{u}^* = u^\sharp$, which is in contradiction to (4.5.31). Therefore, we conclude that property (vii) of Definition 4.19 defines a relatively open subset of $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$ under the choice of δ_3 given by (iii) in the statement of Corollary 4.45. $\qquad \square$

REMARK 4.47. In Proposition 4.46, the choice of $(\alpha, \varepsilon, \delta_1, N_1)$ depends only on $(v_\infty, \gamma, \beta_*)$, and the choice of δ_3 depends only on $(v_\infty, \gamma, \beta_*, \delta_1, \delta_2, N_1)$, where parameter δ_2 is to be determined later.

Existence of Admissible Solutions up to $\beta_{\mathrm{d}}^{(v_\infty)}$ − Proof of Theorem 2.31

Fix $\gamma \geq 1$, $v_\infty > 0$, and $\beta_* \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$. For the iteration set $\mathcal{K}$ defined in Definition 4.19, define

$$\mathcal{K}(\beta) := \{u \in C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}}) \, : \, (u, \beta) \in \mathcal{K}\} \qquad \text{for each } \beta \in [0, \beta_*].$$

In this chapter, we define an iteration mapping $\mathcal{I} : \overline{\mathcal{K}} \to C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}})$ with the following properties:

(i) For each $\beta \in [0, \beta_*]$, there exists $u \in \mathcal{K}(\beta)$ such that $\mathcal{I}(u, \beta) = u$;

(ii) If $\mathcal{I}(u, \beta) = u$, then φ given by (4.1.50) yields an admissible solution corresponding to (v_∞, β).

5.1. Definition of the Iteration Mapping

Let parameters $(\alpha, \varepsilon, \delta_1, \delta_3, N_1)$ in Definition 4.19 be fixed as in Proposition 4.46.

In order to define an iteration map satisfying (i)–(ii) stated above, and to employ the Leray-Schauder degree argument for proving the existence of a fixed point of $\mathcal{I}(\cdot, \beta)$ in $\mathcal{K}(\beta)$ for all $\beta \in (0, \beta_{\mathrm{d}}^{(v_\infty)})$, we require the compactness of $\mathcal{I}$.

For each $(u, \beta) \in \overline{\mathcal{K}}$, let $(\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}, \Gamma_{\mathrm{shock}}(u, \beta), \Omega(u, \beta), \varphi^{(u,\beta)})$ be defined by Definition 4.15, and denote them as $(\mathfrak{g}_{\mathrm{sh}}, \Gamma_{\mathrm{shock}}, \Omega, \varphi)$. For such a function $\mathfrak{g}_{\mathrm{sh}}$, we define $(\mathcal{G}_1^\beta, G_{2,\mathfrak{g}_{\mathrm{sh}}})$ by (4.1.31) and (4.1.49), respectively. Let $\hat{\phi} \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be the unique solution of the boundary value problem (4.3.16) associated with (u, β). Then function $\hat{u} : \overline{\mathcal{Q}^{\mathrm{iter}}} \to \mathbb{R}$ is given by (4.3.17), and function $\hat{\varphi} = \hat{\varphi}^{(u,\beta)}$ is given by

$$\hat{\varphi}^{(u,\beta)} = \varphi_\beta^* + \hat{u} \circ \mathfrak{F}_{(u,\beta)}^{-1} \tag{5.1.1}$$

for φ_β^* given by (4.1.42).

Next, we define functions w, w_∞, and $\hat{w}$ by

$$
\begin{aligned}
w(s, t') &:= (\varphi - \varphi_\beta^*) \circ (\mathcal{G}_1^\beta)^{-1}(s, t'), \\
w_\infty(s, t') &:= (\varphi_\infty - \varphi_\beta^*) \circ (\mathcal{G}_1^\beta)^{-1}(s, t'), \\
\hat{w}(s, t') &:= (\hat{\varphi} - \varphi_\beta^*) \circ (\mathcal{G}_1^\beta)^{-1}(s, t').
\end{aligned}
\tag{5.1.2}
$$

LEMMA 5.1. *For each $\beta \in [0, \beta_{\mathrm{d}}^{(v_\infty)}]$, there exists a unique function $\mathfrak{g}_\beta : [-1, 1] \to \mathbb{R}_+$ such that*

(a) $w_\infty(s, \mathfrak{g}_\beta(s)) = 0$ *for all $s \in [-1, 1]$;*

(b) $\{(s, \mathfrak{g}_\beta(s)) \, : \, s \in (-1, 1)\} \subset \mathcal{G}_1^\beta(Q^\beta)$ *for Q^β defined in Definition 4.1(iii);*

(c) $\|\mathfrak{g}_\beta\|_{C^3([-1,1])} \leq C$ *for $C > 0$ depending only on (γ, v_∞).*

PROOF. By property (iii) stated right after Definition 4.8, the set:

$$\{(s,t') \,:\, w_\infty(s,t') = 0\}$$

is contained in $\mathcal{G}_1^\beta(Q^\beta)$. Then the existence and uniqueness of $\mathfrak{g}_\beta$ satisfying statements (a)–(b) follow from Lemma 4.9, combined with the implicit function theorem. Statement (c) is obtained from Lemma 4.9 and the smoothness of $\varphi_\infty - \varphi_\beta^*$, owing to (4.1.42). □

For each $(u,\beta) \in \overline{\mathcal{K}}$, $\mathfrak{g}_{\mathrm{sh}} : [-1,1] \to \mathbb{R}_+$ is in $C^{0,1}([-1,1])$ and satisfies $\mathfrak{g}_{\mathrm{sh}} > 0$ on $(-1,1)$. Define

$$\begin{aligned}
R_{\mathfrak{g}_{\mathrm{sh}}} &:= \{(s,t') \in \mathbb{R}^2 \,:\, -1 < s < 1, 0 < t' < \mathfrak{g}_{\mathrm{sh}}(s)\}, \\
\Sigma_{\mathfrak{g}_{\mathrm{sh}}} &:= \{(s, \mathfrak{g}_{\mathrm{sh}}(s)) \,:\, -1 < s < 1\}.
\end{aligned}$$
(5.1.3)

Note that w and $\hat{w}$ are defined in $R_{\mathfrak{g}_{\mathrm{sh}}}$, and w_∞ is defined in $R_\infty := (-1,1) \times \mathbb{R}_+$.

In order to define an iteration mapping $\mathcal{I}$, the first step is to introduce an extension of $\hat{w}$ onto $R_{(1+\kappa)\mathfrak{g}_{\mathrm{sh}}}$ for some $\kappa \in (0, \frac{1}{3}]$.

LEMMA 5.2 (Regularized distance). *Let* $R_\infty := (-1,1) \times \mathbb{R}^+$. *For each* $g \in C^{0,1}([-1,1])$ *satisfying*

$$g > 0 \qquad on \ (-1,1),$$
(5.1.4)

define

(5.1.5) $\quad R_g := \{(s,t') \in \mathbb{R}^2 \,:\, -1 < s < 1, 0 < t' < g(s)\}, \quad \Sigma_g := \{(s,g(s)) \,:\, -1 < s < 1\}.$

Then there exists a function $\delta_g \in C^\infty(\overline{R_\infty} \setminus \overline{R_g})$, *the regularized distance, such that*

(i) *For all* $\mathbf{x} = (s,t') \in \overline{R_\infty} \setminus \Sigma_g$,

$$\frac{1}{2}\mathrm{dist}(\mathbf{x}, \Sigma_g) \le \delta_g(\mathbf{x}) \le \frac{3}{2}\mathrm{dist}(\mathbf{x}, \Sigma_g).$$

(ii) *For all* $\mathbf{x} = (s,t') \in \overline{R_\infty} \setminus \Sigma_g$,

$$|D^m \delta_g(\mathbf{x})| \le C(m)\big(\mathrm{dist}(\mathbf{x}, \Sigma_g)\big)^{1-m} \qquad for \ m = 1, 2, 3, \cdots,$$

where $C(m)$ *depends only on* m.

(iii) *There exists* $C_* > 0$ *depending only on* $\mathrm{Lip}[g]$ *such that*

$$\delta_g(\mathbf{x}) \ge C_*(t' - g(s)) \qquad for \ all \ \mathbf{x} \in \overline{R_\infty} \setminus \overline{R_g}.$$

(iv) *Suppose that* $g_i \in C^{0,1}([-1,1])$ *and* $g \in C^{0,1}([-1,1])$ *satisfy* (5.1.4) *and*

$$\|g_i\|_{C^{0,1}([-1,1])} \le L \qquad for \ all \ i \in \mathbb{N}$$

for some constant $L > 0$. *If* $\{g_i(s)\}_{i\in\mathbb{N}}$ *converges to* $g(s)$ *uniformly on* $[-1,1]$, *then* $\{\delta_{g_i}(\mathbf{x})\}_{i\in\mathbb{N}}$ *converges to* $\delta_g(\mathbf{x})$ *in* $C^m(K)$ *for any* $m = 0, 1, 2, \cdots$, *and any compact set* $K \subset \overline{R_\infty} \setminus \overline{R_g}$.

(v) *For* C_* *from* (iii), *define*

(5.1.6) $$\delta_g^*(\mathbf{x}) := \frac{2}{C_*}\delta_g(\mathbf{x}).$$

Then there exists $\kappa \in (0, \frac{1}{3}]$ depending only on $\mathrm{Lip}[g]$ such that, for each $\mathbf{x} = (s, t') \in R_{(1+\kappa)g} \setminus \overline{R}_g$,

$$(s, t' - \lambda \delta_g^*(\mathbf{x})) \in \{s\} \times [\frac{g(s)}{3}, g(s) - (t' - g(s))] \Subset R_g \qquad \text{for all } \lambda \in [1, 2].$$

(vi) *There exist constants $C_* > 0$ and $\kappa \in (0, \frac{1}{3}]$ depending only on $(\gamma, v_\infty, \beta_*)$ such that, for each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, the regularized distance $\delta_{\mathfrak{g}_{\mathrm{sh}}}^{(u,\beta)}$ can be given so that properties* (i)–(iii) *and* (v) *stated above are satisfied.*

(vii) *If $\{(u_j, \beta_j)\}_{j=1}^n \subset \overline{\mathcal{K}^{\mathrm{ext}}}$ converges to (u, β) in $C_{(*)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$, then $\delta_{\mathfrak{g}_{\mathrm{sh}}}^{(u_j, \beta_j)}$ converges to $\delta_{\mathfrak{g}_{\mathrm{sh}}}^{(u,\beta)}$ in $C^m(K)$ for any $m = 0, 1, 2, \cdots$, and any compact $K \subset \overline{R_\infty} \setminus \overline{R_{\mathfrak{g}_{\mathrm{sh}}}^{(u,\beta)}}$.*

PROOF. Statements (i)–(iv) of this lemma follow directly from [**11**, Lemma 13.9.1]. Statement (v) can be verified by using statement (iii). We refer to [**11**, Lemma 13.9.4] for a proof of statement (v). Finally, statements (i)–(v), combined with (d) and (g)–(h) of Lemma 4.16 and (i) of Remark 4.21, lead to statements (vi) and (vii). $\qquad\square$

By [**11**, Lemma 13.9.2], there exists a function $\Psi \in C_c^\infty(\mathbb{R})$ satisfying that

$$\mathrm{supp}\Psi \subset [1, 2],$$

(5.1.7)
$$\int_{-\infty}^\infty \Psi(y) \, d\lambda = 1, \qquad \int_{-\infty}^\infty \lambda^m \Psi(\lambda) \, d\lambda = 0 \quad \text{for } m = 1, 2.$$

For a function $g \in C^{0,1}([-1, 1])$ satisfying (5.1.4), let R_g and δ_g^* be given by (5.1.5) and (5.1.6), respectively. Let $\kappa \in (0, \frac{1}{3}]$ be fixed depending on $\mathrm{Lip}[g]$ to satisfy Lemma 5.2(v). For a function $v \in C^0(\overline{R}_g) \cap C^2(R_g \cup \Sigma_g)$, we define its extension $\mathcal{E}_g(v)$ onto $R_{(1+\kappa)g}$ by

(5.1.8)
$$\mathcal{E}_g(v)(\mathbf{x}) = \begin{cases} v(\mathbf{x}) & \text{for } \mathbf{x} = (s, t') \in \overline{R}_g, \\ \int_{-\infty}^\infty v\left(s, t' - \lambda \delta_g^*(\mathbf{x})\right) \Psi(\lambda) \, d\lambda & \text{for } \mathbf{x} \in R_{(1+\kappa)g} \setminus \overline{R}_g. \end{cases}$$

DEFINITION 5.3 (Extension mapping). For each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, let g denote $\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}$, and let δ_g be the regularized distance given in Lemma 5.2. For constant $C_* > 0$ from Lemma 5.2(vi), let δ_g^* be given by (5.1.6). Let $\kappa \in (0, \frac{1}{3}]$ be from Lemma 5.2(vi). Then, for each $v \in C^0(\overline{R}_g) \cap C^2(R_g \cup \Sigma_g)$, define its extension $\mathcal{E}_g(v)$ onto $R_{(1+\kappa)g}$ by (5.1.8) for Ψ given by (5.1.7).

PROPOSITION 5.4 (Properties of extension operator $\mathcal{E}$). For each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, the extension operator $\mathcal{E}_g$ given by Definition 5.3 maps $C^2(R_g \cup \Sigma_g)$ into $C^2(R_{(1+\kappa)g})$ with the following properties: Fix $\alpha \in (0, 1)$. Then

(a) Fix b_1, b_2 with $-1 < b_1 < b_2 < 1$.

(a-1) There exists $C > 0$ depending only on $(v_\infty, \gamma, \beta_*, \alpha)$ such that

$$\|\mathcal{E}_g(v)\|_{2,\alpha, R_{(1+\kappa)g} \cap \{b_1 < s < b_1\}} \le C\|v\|_{2,\alpha, R_g \cap \{b_1 < s < b_1\}}.$$

More precisely,

$$\|\mathcal{E}_g(v)\|_{m,0, R_{(1+\kappa)g} \cap \{b_1 < s < b_2\}} \le C\|v\|_{m,0, R_g \cap \{b_1 < s < b_2\}} \qquad \text{for } m = 0, 1, 2,$$

$$[D^2 \mathcal{E}_g(v)]_{\alpha, R_{(1+\kappa)g} \cap \{b_1 < s < b_2\}} \le C[D^2 v]_{\alpha, R_g \cap \{b_1 < s < b_2\}}.$$

(a-2) $\mathcal{E}_g : C^{2,\alpha}(\overline{R_g \cap \{b_1 < s < b_2\}}) \longrightarrow C^{2,\alpha}(\overline{R_{(1+\kappa)g} \cap \{b_1 < s < b_2\}})$ is linear and continuous.

(a-3) Suppose that $\{(u_j, \beta_j)\} \subset \overline{\mathcal{K}^{\mathrm{ext}}}$ converges to (u, β) in $C^{2,\tilde{\alpha}}_{(*,\alpha_1)}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$ for some $\tilde{\alpha} \in (0, 1)$. If $\{v_j\}$ satisfies

$$v_j \in C^{2,\alpha}(\overline{R^{(u_j,\beta_j)}_{\mathfrak{g}_{\mathrm{sh}}} \cap \{b_1 < s < b_2\}}), \ \|v_j\|_{2,\alpha, R^{(u_j,\beta_j)}_{\mathfrak{g}_{\mathrm{sh}}} \cap \{b_1 < s < b_2\}} \leq M \quad \text{for all } j \in \mathbb{N}$$

for some constant $M > 0$ and converges uniformly to v on compact subsets of $R_{\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}}$
for some $v \in C^{2,\alpha}(\overline{R^{(u,\beta)}_{\mathfrak{g}_{\mathrm{sh}}} \cap \{b_1 < s < b_2\}})$, then $\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}^{(u_j,\beta_j)}}(v_j)$ converges to $\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}}(v)$ in
$C^{2,\alpha'}(\overline{R_{(1+\frac{\kappa}{2})g} \cap \{b_1 < s < b_2\}})$ for all $\alpha' \in (0, \alpha)$, where $\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}^{(u_j,\beta_j)}}(v_j)$ is well defined
on $\overline{R_{(1+\frac{\kappa}{2})\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}} \cap \{b_1 < s < b_2\}}$ for large j.

(b) Fix $\sigma > 0$ and $\varepsilon \in (0, \frac{1}{4}]$.

(b-1) There exists $C_{\mathrm{par}} > 0$ depending only on $(v_\infty, \gamma, \beta_*, \alpha, \sigma)$ such that

$$\|\mathcal{E}_g(v)\|^{(\sigma),(\mathrm{par})}_{2,\alpha, R_{(1+\kappa)g} \cap \{-1 < s < -1+\varepsilon\}} \leq C_{\mathrm{par}} \|v\|^{(\sigma),(\mathrm{par})}_{2,\alpha, R_g \cap \{-1 < s < -1+\varepsilon\}},$$

$$\|\mathcal{E}_g(v)\|^{(\sigma),(\mathrm{par})}_{2,\alpha, R_{(1+\kappa)g} \cap \{1-\varepsilon < s < 1\}} \leq C_{\mathrm{par}} \|v\|^{(\sigma),(\mathrm{par})}_{2,\alpha, R_g \cap \{1-\varepsilon < s < 1\}}.$$

(b-2) The mapping

$$\mathcal{E}_g : C^{2,\alpha}_{(\sigma),(\mathrm{par})}(R_g \cap \{-1 < s < -1+\varepsilon\}) \to C^{2,\alpha}_{(\sigma),(\mathrm{par})}(R_{(1+\kappa)g} \cap \{-1 < s < -1+\varepsilon\})$$

is linear and continuous. The same is true when we replace $\{-1 < s < -1+\varepsilon\}$ by
$\{1 - \varepsilon < s < 1\}$.

(b-3) If $\{(u_j, \beta_j)\} \subset \overline{\mathcal{K}^{\mathrm{ext}}}$ converges to (u, β) in $C^{2,\tilde{\alpha}}_{(*,\alpha_1)}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$ for some $\tilde{\alpha} \in (0, 1)$,
and if

$$\{v_j\} \subset C^{2,\alpha}_{(\sigma),(\mathrm{par})}(\overline{R^{(u_j,\beta_j)}_{\mathfrak{g}_{\mathrm{sh}}} \cap \{-1 < s < -1+\varepsilon\}}),$$

$$v \in C^{2,\alpha}_{(\sigma),(\mathrm{par})}(\overline{R^{(u,\beta)}_{\mathfrak{g}_{\mathrm{sh}}} \cap \{-1 < s < -1+\varepsilon\}}),$$

and v_j converges uniformly to v on compact subsets of $R_{\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}}$, then $\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}^{(u_j,\beta_j)}}(v_j)$
converges to $\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}}(v)$ in $C^{2,\alpha'}_{(\sigma'),(\mathrm{par})}(R_{(1+\frac{\kappa}{2})g} \cap \{-1 < s < -1+\varepsilon\})$ for all $\alpha' \in (0, \alpha)$
and all $\sigma' \in (0, \sigma)$. The same is true when we replace $\{-1 < s < -1+\varepsilon\}$ by
$\{1 - \varepsilon < s < 1\}$.

(c) Consider the case that $s \in (-1, \frac{1}{2})$.

(c-1) There exists $C_{\mathrm{sub}} > 0$ depending only on $(v_\infty, \gamma, \beta_*, \alpha)$ such that

$$\|\mathcal{E}_g(v)\|^{(-1-\alpha),\{s=-1\}}_{2,\alpha, R_{(1+\kappa)g} \cap \{-1 < s < -\frac{1}{2}\}} \leq C_{\mathrm{sub}} \|v\|^{(-1-\alpha),\{s=-1\}}_{2,\alpha, R_g \cap \{-1 < s < -\frac{1}{2}\}}.$$

Furthermore, if $(v, Dv) = (0, \mathbf{0})$ on $\overline{R_g} \cap \{s = -1\}$, then

$$(\mathcal{E}_g(v), D\mathcal{E}_g(v)) = (0, \mathbf{0}) \qquad \text{on } \overline{R_{(1+\kappa)g}} \cap \{s = -1\}.$$

(c-2) $\mathcal{E}_g : C^{2,\alpha}_{(-1-\alpha),\{s=-1\}}(R_g \cap \{-1 < s < -\frac{1}{2}\})$
$\longrightarrow C^{2,\alpha}_{(-1-\alpha),\{s=-1\}}(R_{(1+\kappa)g} \cap \{-1 < s < -\frac{1}{2}\})$ is linear and continuous.

(c-3) If $\{(u_j, \beta_j)\} \subset \overline{\mathcal{K}^{\mathrm{ext}}}$ converges to (u, β) in $C^{2,\tilde{\alpha}}_{(*,\alpha_1)}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$ for some $\tilde{\alpha} \in (0, 1)$, and if

$$\{v_j\} \subset C^{2,\alpha}_{(-1-\alpha),\{s=-1\}}(R_{\mathfrak{g}_{\mathrm{sh}}^{(u_j,\beta_j)}} \cap \{-1 < s < -\frac{1}{2}\}),$$

$$v \in C^{2,\alpha}_{(-1-\alpha),\{s=-1\}}(R_{\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}} \cap \{-1 < s < -\frac{1}{2}\}),$$

and v_j converges uniformly to v on compact subsets of $R_{\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}}$, then $\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}^{(u_j,\beta_j)}}(v_j)$ converges to $\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}}(v)$ in $C^{2,\alpha'}_{(-1-\alpha'),\{s=-1\}}(R_{(1+\frac{\kappa}{2})g} \cap \{-1 < s < -\frac{1}{2}\})$ for all $\alpha' \in (0, \alpha)$.

PROOF. We divide the proof into three steps.

1. By Remark 4.21, $\mathrm{Lip}[\mathfrak{g}_{\mathrm{sh}}]$ is uniformly bounded by a constant $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ for all $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$. Then statements (a-1)–(a-2) follow from [**11**, Lemma 13.9.6(i)–(ii)]. By Lemma 4.16(d), if $\{(u_j, \beta_j)\} \subset \overline{\mathcal{K}^{\mathrm{ext}}}$ converges to (u, β) in $C^{2,\tilde{\alpha}}_{(*,\alpha_1)}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$ for some $\tilde{\alpha} \in (0, 1)$, then $\mathfrak{g}_{\mathrm{sh}}^{(u_j,\beta_j)}$ converges to $\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}$ in $C^1([-1, 1])$. Thus, we apply [**11**, Lemma 13.9.6 (iii)] to obtain statement (a-3).

2. Statements (b-1)–(b-2) can be proved by following Steps 2–3 in the proof of [**11**, Theorem 13.9.5]. Since $\mathrm{Lip}[\mathfrak{g}_{\mathrm{sh}}]$ is uniformly bounded by a constant $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ for all $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, the estimate constant C_{par} in (b-1) can be given uniformly, depending only on $(v_\infty, \gamma, \beta_*, \alpha, \sigma)$, for all $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$. Moreover, statement (b-3) can be proved by following Step 4 in the proof of [**11**, Theorem 13.9.5] and using the uniform convergence of $\mathfrak{g}_{\mathrm{sh}}^{(u_j,\beta_j)}$ to $\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}$ on $[-1, 1]$ when $\{(u_j, \beta_j)\} \subset \overline{\mathcal{K}^{\mathrm{ext}}}$ converges to (u, β) in $C^{2,\tilde{\alpha}}_{(*,\alpha_1)}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$ for some $\tilde{\alpha} \in (0, 1)$.

3. Finally, we follow the proof of [**11**, Theorem 13.9.8] to obtain statements (c-1)–(c-3). Similarly to Steps 1–2, the uniform boundedness of $\mathrm{Lip}[\mathfrak{g}_{\mathrm{sh}}]$ for all $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$ implies that the estimate constant C_{sub} depends only on $(v_\infty, \gamma, \beta_*, \alpha)$ for all $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$. To prove (c-3), we use the uniform convergence of $\mathfrak{g}_{\mathrm{sh}}^{(u_j,\beta_j)}$ to $\mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}$ on $[-1, 1]$ when $\{(u_j, \beta_j)\} \subset \overline{\mathcal{K}^{\mathrm{ext}}}$ converges to (u, β) in $C^{2,\tilde{\alpha}}_{(*,\alpha_1)}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$ for some $\tilde{\alpha} \in (0, 1)$. $\square$

LEMMA 5.5. *Let parameters* $(\alpha, \varepsilon, \delta_1, \delta_3, N_1)$ *in Definition 4.19 be fixed as in Proposition 4.46. Then there exists a constant* $\delta_3^{(\mathrm{imp})} > 0$ *depending only on* $(v_\infty, \gamma, \beta_*, \delta_2)$ *(where parameter* δ_2 *in Definition 4.19 is determined later) such that, if* δ_3 *further satisfies* $0 < \delta_3 \leq \delta_3^{(\mathrm{imp})}$, *for each* $(u, \beta) \in \overline{\mathcal{K}}$, *there exists a unique function* $\hat{\mathfrak{g}}_{\mathrm{sh}} : [-1, 1] \to \mathbb{R}_+$ *such that*

$$(5.1.9) \qquad (w_\infty - \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(\hat{w}))(s, \hat{\mathfrak{g}}_{\mathrm{sh}}(s)) = 0 \qquad \text{for all } s \in [-1, 1].$$

Furthermore, there exists a constant $C > 0$ *depending only on* $(v_\infty, \gamma, \beta_*)$ *such that* $\hat{\mathfrak{g}}_{\mathrm{sh}}$ *satisfies*

$$(5.1.10) \qquad \begin{cases} \|\hat{\mathfrak{g}}_{\mathrm{sh}} - \mathfrak{g}_\beta\|^{(2),(\mathrm{par})}_{2,2\alpha,(-\frac{1}{2},1)} + \|\hat{\mathfrak{g}}_{\mathrm{sh}} - \mathfrak{g}_\beta\|^{(-1-2\alpha),\{-1\}}_{2,2\alpha,(-1,0)} \leq C, \\ \frac{\mathrm{d}^k}{\mathrm{d}x^k}(\hat{\mathfrak{g}}_{\mathrm{sh}} - \mathfrak{g}_\beta)(-1) = 0 \qquad \text{for } k = 0, 1, \end{cases}$$

$$(5.1.11) \qquad \|\hat{\mathfrak{g}}_{\mathrm{sh}} - \mathfrak{g}_{\mathrm{sh}}\|_{1,\frac{\alpha}{2},(-1,1)} \leq C\delta_3,$$

$$(5.1.12) \qquad (\hat{\mathfrak{g}}_{\mathrm{sh}} - \mathfrak{g}_{\mathrm{sh}})(\pm 1) = (\hat{\mathfrak{g}}_{\mathrm{sh}} - \hat{\mathfrak{g}}_{\mathrm{sh}})'(\pm 1) = 0,$$

where $\mathfrak{g}_\beta$ *is from Lemma 5.1.*

PROOF. We divide the proof into three steps.

1. By Definition 4.15(i), w given by (5.1.2) satisfies

$$(5.1.13) \qquad\qquad w_\infty - w = 0 \qquad \text{on } \Sigma_{\mathfrak{g}_{\mathrm{sh}}}.$$

By (4.3.11) in Definition 4.19(iv), Lemma 4.5(a), and (5.1.2), there exists a constant $C' > 0$ depending on (v_∞, γ) such that

$$(5.1.14) \qquad\qquad |D(w_\infty - w)| \geq C' \mu_1 > 0 \qquad \text{on } \Sigma_{\mathfrak{g}_{\mathrm{sh}}}.$$

Therefore, we have

$$\frac{D(w_\infty - w)}{|D(w_\infty - w)|} = -\frac{(-\mathfrak{g}_{\mathrm{sh}}', 1)}{\sqrt{1 + (\mathfrak{g}_{\mathrm{sh}}')^2}} \qquad \text{on } \Sigma_{\mathfrak{g}_{\mathrm{sh}}}.$$

Since $\mathrm{Lip}[\mathfrak{g}_{\mathrm{sh}}]$ is uniformly bounded by a constant $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ for all $(u, \beta) \in \overline{\mathcal{K}}$, there exists a constant $\bar{\mu} > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ to satisfy

$$(5.1.15) \qquad\qquad \partial_{t'}(w_\infty - w) = -\frac{|D(w_\infty - w)|}{\sqrt{1 + (\mathfrak{g}_{\mathrm{sh}}')^2}} \leq -\bar{\mu} \qquad \text{on } \Sigma_{\mathfrak{g}_{\mathrm{sh}}}.$$

For each $(u, \beta) \in \overline{\mathcal{K}^{\mathrm{ext}}}$, the corresponding function $\mathfrak{g}_{\mathrm{sh}} = \mathfrak{g}_{\mathrm{sh}}^{(u,\beta)}$ satisfies that $\mathfrak{g}_{\mathrm{sh}}(-1) \geq 0$. Therefore, Definition 4.19(iii) implies that

$$(5.1.16) \qquad\qquad \frac{1}{N_3}(1 + s) \leq \mathfrak{g}_{\mathrm{sh}}(s) \leq \mathfrak{g}_{\mathrm{sh}}(-1) + N_3(1 + s) \qquad \text{for } -1 \leq s \leq -1 + \hat{\varepsilon}_0$$

for $\hat{\varepsilon}_0 = \frac{1}{5}$, where $N_3 > 1$ is the constant from Definition 4.19(iii). The lower bound of $\mathfrak{g}_{\mathrm{sh}}(s)$ in (5.1.16) is obtained from Definition 4.19(iii), and $\mathfrak{g}_{\mathrm{sh}}(-1) \geq 0$ which follows from (4.2.6).

Let $\kappa \in (0, \frac{1}{3}]$ be fixed as in Definition 5.3. In other words, let κ be from Lemma 5.2(vi). By Definition 4.19(i), Remark 4.21, (5.1.15), and Proposition 5.4, there exists a small constant $\sigma \in (0, \frac{1}{4}\min\{1, \kappa\}]$ depending only on $(v_\infty, \gamma, \beta_*)$ such that, for each $(u, \beta) \in \overline{\mathcal{K}}$, $\mathfrak{g}_{\mathrm{sh}}$ satisfies

$$0 < \mathfrak{g}_{\mathrm{sh}}(s) - \sigma < \mathfrak{g}_{\mathrm{sh}}(s) + \sigma < (1 + \kappa)\mathfrak{g}_{\mathrm{sh}}(s) \qquad \text{for } -1 + \frac{\hat{\varepsilon}_0}{2} \leq s \leq 1,$$

and the corresponding function w given by (5.1.2) satisfies

$$(5.1.17) \qquad \begin{aligned} \partial_{t'}(w_\infty - \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(w))(s, t') &\leq -\frac{\bar{\mu}}{2} \qquad \text{for } -1 \leq s \leq -1 + \hat{\varepsilon}_0 \text{ and } 1 - \sigma \leq \frac{t'}{\mathfrak{g}_{\mathrm{sh}}(s)} \leq 1 + \sigma, \\ \partial_{t'}(w_\infty - \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(w))(s, t') &\leq -\frac{\bar{\mu}}{2} \qquad \text{for } -1 + \frac{\hat{\varepsilon}_0}{2} \leq s \leq 1 \text{ and } |t' - \mathfrak{g}_{\mathrm{sh}}(s)| \leq \sigma. \end{aligned}$$

2. By (5.1.13) and the linearity of the extension operator $\mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}$, we have

$$(w_\infty - \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(\hat{w}))(s, (1 + \sigma)\mathfrak{g}_{\mathrm{sh}}(s)) = \mathcal{A}_1 + \mathcal{A}_2,$$

where

$$\begin{aligned} \mathcal{A}_1 &= (w_\infty - \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(w))(s, (1 + \sigma)\mathfrak{g}_{\mathrm{sh}}(s)) - (w_\infty - \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(w))(s, \mathfrak{g}_{\mathrm{sh}}(s)), \\ \mathcal{A}_2 &= \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(w - \hat{w})(s, (1 + \sigma)\mathfrak{g}_{\mathrm{sh}}(s)). \end{aligned}$$

By (5.1.16)–(5.1.17), we have

$$(5.1.18) \qquad\qquad \mathcal{A}_1 \leq -\frac{\bar{\mu}\sigma}{2N_3}(1 - |s|) \qquad \text{for } -1 \leq s \leq -1 + \hat{\varepsilon}_0.$$

By (4.3.18), (5.1.2), and properties (b-1) and (c-1) of Proposition 5.4, there exists a constant $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ such that

$$(5.1.19) \qquad |\mathcal{A}_2| \le C\delta_3(1 - |s|) \qquad \text{for } -1 \le s \le 1,$$

where $\delta_3 > 0$ is the constant in (4.3.18). From (5.1.18)–(5.1.19), we obtain

$$(w_\infty - \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(\hat{w}))(s, (1 + \sigma)\mathfrak{g}_{\mathrm{sh}}(s)) \le (1 - |s|)\Big(C\delta_3 - \frac{\bar{\mu}\sigma}{2N_3}\Big) \qquad \text{for } -1 \le s \le -1 + \hat{\varepsilon}_0.$$

Therefore, a constant $\delta_3^{(\mathrm{imp})} \in (0, \bar{\delta}_3]$ can be chosen depending only on $(v_\infty, \gamma, \beta_*)$ such that, whenever $\delta_3 \in (0, \delta_3^{(\mathrm{imp})}]$, the inequality above implies that, for any $(u, \beta) \in \overline{\mathcal{K}}$,

$$(5.1.20) \qquad (w_\infty - \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(\hat{w}))\,(s, (1 + \sigma)\mathfrak{g}_{\mathrm{sh}}(s)) < 0 \qquad \text{for } -1 < s \le -1 + \hat{\varepsilon}_0.$$

Under the same choice of δ_3, we also have

$$(5.1.21) \qquad (w_\infty - \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(\hat{w}))\,(s, (1 - \sigma)\mathfrak{g}_{\mathrm{sh}}(s)) > 0 \qquad \text{for } -1 < s \le -1 + \hat{\varepsilon}_0.$$

Adjusting the argument above, we can further reduce $\delta_3^{(\mathrm{imp})} > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ so that, whenever $\delta_3 \in (0, \delta_3^{(\mathrm{imp})}]$,

$$(5.1.22) \quad (w_\infty - \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(\hat{w}))(s, \mathfrak{g}_{\mathrm{sh}}(s) + \sigma) < 0 < (w_\infty - \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(\hat{w}))(s, \mathfrak{g}_{\mathrm{sh}}(s) - \sigma) \quad \text{for } -1 + \frac{\hat{\varepsilon}_0}{2} \le s \le 1.$$

3. Finally, by (4.3.18), (5.1.17), and Proposition 5.4, we can reduce $\delta_3^{(\mathrm{imp})} > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ so that, whenever $\delta_3 \in (0, \delta_3^{(\mathrm{imp})}]$, $\hat{w}$ satisfies

$$(5.1.23) \qquad \begin{aligned} &\partial_{t'}(w_\infty - \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(\hat{w}))(s, t') \le -\frac{\bar{\mu}}{4} \quad \text{for } -1 \le s \le -1 + \hat{\varepsilon}_0 \text{ and } 1 - \sigma \le \frac{t'}{\mathfrak{g}_{\mathrm{sh}}(s)} \le 1 + \sigma, \\ &\partial_{t'}(w_\infty - \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(\hat{w}))(s, t') \le -\frac{\bar{\mu}}{4} \quad \text{for } -1 + \frac{\hat{\varepsilon}_0}{2} \le s \le 1 \text{ and } |t' - \mathfrak{g}_{\mathrm{sh}}(s)| \le \sigma. \end{aligned}$$

Then (5.1.9) follows from the implicit function theorem. By (5.1.16) and (5.1.20)–(5.1.22), there exists a constant $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ such that

$$\|\hat{\mathfrak{g}}_{\mathrm{sh}} - \mathfrak{g}_{\mathrm{sh}}\|_{C^0([-1,1])} < C\sigma.$$

By Lemmas 4.5 and 4.43, and definition (5.1.2), for any $\varepsilon \in (0, 1)$, we have

$$\|\hat{w}\|_{2,2\alpha, R_{\mathfrak{g}_{\mathrm{sh}}} \cap \{s > -1 + \varepsilon\}}^{(2),(\mathrm{par})} \le C_\varepsilon,$$

where constant $C_\varepsilon > 0$ depends only on $(v_\infty, \gamma, \beta_*)$ and ε. Furthermore, by Lemmas 4.5 and 4.44, we obtain

$$\|\hat{w}\|_{2,2\alpha, R_{\mathfrak{g}_{\mathrm{sh}}} \cap \{-1 < s < 0\}}^{(-1-2\alpha), \{s = -1\}} \le C, \qquad \hat{w}(-1, t') = D\hat{w}(-1, t') = 0 \quad \text{for } 0 < t' < \mathfrak{g}_{\mathrm{sh}}(-1)$$

for a constant $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$. Combining these two estimates of $\hat{w}$ with (5.1.9), (5.1.23), and Proposition 5.4, we obtain (5.1.10).

Next, we use (5.1.1)–(5.1.2), Lemma 4.5, Definition 4.15(ii), Lemma 4.16(d), and estimate (4.3.18) given in Definition 4.19(vii) to obtain

$$\|\hat{w} - w\|_{1, \frac{\alpha}{2}, \mathcal{G}_1^\beta(\Omega)} = \|(\hat{u} - u) \circ \mathfrak{F}_{(u,\beta)}^{-1} \circ (\mathcal{G}_1^\beta)^{-1}\|_{1, \alpha/2, \mathcal{G}_1^\beta(\Omega)} \le C\delta_3$$

for a constant $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$. Using this estimate and (5.1.17), we obtain (5.1.11). Finally, (5.1.12) follows directly from (5.1.10) and the fact that $\frac{\mathrm{d}^k}{\mathrm{d}x^k}(\mathfrak{g}_{\mathrm{sh}} - \mathfrak{g}_\beta)(\pm 1) = 0$ for $k = 0, 1$. $\qquad \square$

Let parameters $(\alpha, \varepsilon, \delta_1, \delta_3, N_1)$ in Definition 4.19 be chosen as in Lemma 5.5. For each $(u, \beta) \in \overline{\mathcal{K}}$, let $\hat{\mathfrak{g}}_{\mathrm{sh}} : [-1, 1] \to \mathbb{R}_+$ be given by (5.1.9). From (5.1.11)–(5.1.12), further reducing δ_3, we obtain that $\hat{\mathfrak{g}}_{\mathrm{sh}}$ satisfies estimate (4.3.2) in Definition 4.19(iii) with N_3 replaced by $2N_3$. We define a function $\tilde{u} : \overline{\mathcal{Q}^{\mathrm{iter}}} \to \mathbb{R}$ by

$$(5.1.24) \qquad \tilde{u} = \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(\hat{w}) \circ (G_{2, \hat{\mathfrak{g}}_{\mathrm{sh}}})^{-1}$$

for $G_{2, \hat{\mathfrak{g}}_{\mathrm{sh}}}$ defined by (4.1.49). By Corollary 4.45, Proposition 5.4, and Lemma 5.5, there exists a constant $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ such that $\tilde{u}$ satisfies

$$(5.1.25) \qquad \|\tilde{u}\|_{2, 2\alpha, \mathcal{Q}^{\mathrm{iter}}}^{(*, 1)} \leq C.$$

Now we define the iteration mapping $\mathcal{I} : \overline{\mathcal{K}} \to C_{(*, \alpha_1)}^{2, \alpha}(\mathcal{Q}^{\mathrm{iter}})$.

DEFINITION 5.6. Let parameters $(\alpha, \varepsilon, \delta_1, \delta_3, N_1)$ in Definition 4.19 be fixed as in Proposition 4.46. Then we adjust $\delta_3 \in (0, \delta_3^{\mathrm{imp}}]$ for δ_3^{imp} from Lemma 5.5 so that Lemma 5.5 holds for all $(u, \beta) \in \overline{\mathcal{K}}$. For each $(u, \beta) \in \overline{\mathcal{K}}$, let $\tilde{u}$ be given by (5.1.24). Then define an iteration mapping $\mathcal{I} : \overline{\mathcal{K}} \to C_{(*, \alpha_1)}^{2, \alpha}(\mathcal{Q}^{\mathrm{iter}})$ by

$$\mathcal{I}(u, \beta) = \tilde{u}.$$

LEMMA 5.7. *The iteration mapping $\mathcal{I}$ defined in Definition* 5.6 *satisfies the following properties:*

(a) *For any $\beta \in [0, \beta_*]$, define*

$$\overline{\mathcal{K}}(\beta) := \{u \in C_{(*, \alpha_1)}^{2, \alpha}(\mathcal{Q}^{\mathrm{iter}}) : (u, \beta) \in \overline{\mathcal{K}}\}.$$

For each $(u, \beta) \in \overline{\mathcal{K}}$, define

$$\mathcal{I}_1^{(\beta)}(u) = \hat{u},$$

where $\hat{u}$ is given by (4.3.17). Then $u \in \overline{\mathcal{K}}(\beta)$ satisfies $\mathcal{I}(u, \beta) = u$ if and only if $\mathcal{I}_1^{(\beta)}(u) = u$.

(b) *For $\tilde{\alpha} = \frac{\alpha}{2}$, there exists a constant $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ such that, for each $(u, \beta) \in \overline{\mathcal{K}}$,*

$$\|\mathcal{I}(u, \beta)\|_{2, \alpha + \tilde{\alpha}, \mathcal{Q}^{\mathrm{iter}}}^{(*, 1)} \leq C.$$

PROOF. For a fixed $\beta \in [0, \beta_*]$, suppose that $\mathcal{I}(u, \beta) = u$ for some $u \in \overline{\mathcal{K}}(\beta)$; that is, $\tilde{u} = u$ for $\tilde{u}$ given by (5.1.24). Then, by Definition 4.15 and (5.1.24), we see that, for all $s \in [-1, 1]$,

$$w_\infty(s, \mathfrak{g}_{\mathrm{sh}}(s)) = u(s, 1) = \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(\hat{w})(s, \hat{\mathfrak{g}}_{\mathrm{sh}}(s)) = w_\infty(s, \hat{\mathfrak{g}}_{\mathrm{sh}}(s)).$$

This, combined with Lemma 4.9 and (5.1.2), implies that $\mathfrak{g}_{\mathrm{sh}} = \hat{\mathfrak{g}}_{\mathrm{sh}}$ on $[-1, 1]$. Then it follows from (5.1.24) that $\tilde{u} = \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(\hat{w}) \circ (G_{2, \mathfrak{g}_{\mathrm{sh}}})^{-1} = \hat{u}$, which implies that $u = \hat{u} = \mathcal{I}_1^{(\beta)}(u)$ in $\mathcal{Q}^{\mathrm{iter}}$.

Next, suppose that $\mathcal{I}_1^{(\beta)}(u) = u$ for some $u \in \overline{\mathcal{K}}(\beta)$. Then $\mathfrak{g}_{\mathrm{sh}} = \hat{\mathfrak{g}}_{\mathrm{sh}}$ on $[-1, 1]$. This, combined with (5.1.24), implies that $\tilde{u} = \mathcal{I}(u, \beta) = \mathcal{E}_{\mathfrak{g}_{\mathrm{sh}}}(\hat{w}) \circ (G_{2, \mathfrak{g}_{\mathrm{sh}}})^{-1} = \hat{u}$. Therefore, we obtain that $\tilde{u} = u$ in $\mathcal{Q}^{\mathrm{iter}}$.

Finally, statement (b) directly follows from (5.1.25). $\qquad\square$

5.2. Fixed Points of $\mathcal{I}(\cdot, \beta)$ and Admissible Solutions

For the iteration map $\mathcal{I}$ defined in Definition 5.6, we show that, if $u \in \overline{\mathcal{K}}(\beta)$ is a fixed point of $\mathcal{I}(\cdot, \beta)$ for some $\beta \in (0, \beta_*]$, then φ defined by (4.2.4) in Definition 4.15 is an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ in the sense of Definition 2.24.

PROPOSITION 5.8. Let parameters $(\alpha, \varepsilon, \delta_1, \delta_3, N_1)$ in Definition 4.19 be fixed as in Definition 5.6. Then parameters (ε, δ_1) can be further reduced depending only on $(v_\infty, \gamma, \beta_*)$ so that, for each $\beta \in (0, \beta_*]$, $u \in \mathcal{K}(\beta)$ is a fixed point of $\mathcal{I}(\cdot, \beta) : \overline{\mathcal{K}}(\beta) \to C^{2,\alpha}_{(*,\alpha_1)}(\mathcal{Q}^{\text{iter}})$ if and only if φ, defined by (4.2.4) in Definition 4.15, yields an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ in the sense of Definition 2.24 by extending φ into Λ_β via (2.5.8) if $\beta < \beta_s^{(v_\infty)}$, and via (2.5.12) if $\beta \geq \beta_s^{(v_\infty)}$.

PROOF. By Corollary 4.40, it suffices to prove that, if $u \in \mathcal{K}(\beta)$ is a fixed point of $\mathcal{I}(\cdot, \beta) : \overline{\mathcal{K}}(\beta) \to C^{2,\alpha}_{(*,\alpha_1)}(\mathcal{Q}^{\text{iter}})$, then φ, defined by (4.2.4) in Definition 4.15, yields an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ in the sense of Definition 2.24. We divide the proof into six steps.

1. For $(u, \beta) \in \overline{\mathcal{K}}$, let $(\Omega, \Gamma_{\text{shock}}, \varphi) = (\Omega(u, \beta), \Gamma_{\text{shock}}(u, \beta), \varphi^{(u,\beta)})$ be defined by Definition 4.15, and denote $\phi := \varphi - \varphi_{\mathcal{N}}$. Let $\hat{\phi} \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be the unique solution of the boundary value problem (4.3.16) determined by (u, β).

Suppose that

$$\mathcal{I}(u, \beta) = u \qquad \text{for some } u \in \overline{\mathcal{K}}(\beta).$$

By Lemma 5.7, we have

(5.2.1) $$\hat{\phi} = \phi \qquad \text{in } \Omega.$$

Let φ be extended onto Λ_β by (2.5.8) for $\beta < \beta_s^{(v_\infty)}$, and by (2.5.12) for $\beta \geq \beta_s^{(v_\infty)}$. Moreover, let $\Gamma^{\mathcal{O}}_{\text{sonic}}$, $\Gamma^{\mathcal{N}}_{\text{sonic}}$, $\boldsymbol{e}_{S_\mathcal{O}}$, $\boldsymbol{e}_{S_\mathcal{N}}$, φ_∞, $\varphi_\mathcal{O}$, and $\varphi_\mathcal{N}$ be defined by Definition 2.23.

2. *Verification of properties* (i-2)–(i-4) *and* (ii-1)–(ii-3) *of Definition* 2.24.

Properties (i-2)–(i-3) follows from Remark 4.21(i). By using Lemma 4.16(b), it can be directly checked that property (i-4) holds.

By Definition 4.19(i) (or Corollary 4.45) and the extension of φ onto Λ_β described in Step 1, φ satisfies properties (ii-1) and (ii-3).

We define

$$A_{ij}(\boldsymbol{\xi}) := A_{ij}(D\hat{\phi}, \boldsymbol{\xi}), \qquad i, j = 1, 2,$$

for $A_{ij}(D\hat{\phi}, \boldsymbol{\xi})$ given by (4.4.22). By Definition 4.19(i) (or Corollary 4.45), coefficients $A_{ij}(\boldsymbol{\xi}), i, j = 1, 2$, of equation $\mathcal{N}_{(u,\beta)}(\hat{\phi}) = 0$ in (4.3.16) are in $C^{1,\alpha}(\overline{\Omega} \setminus (\overline{\Gamma^{\mathcal{O}}_{\text{sonic}}} \cup \overline{\Gamma^{\mathcal{N}}_{\text{sonic}}}))$. Furthermore, Lemma 4.30(a) implies that $\mathcal{N}_{(u,\beta)}(\hat{\phi}) = 0$ is strictly elliptic in Ω. Then the standard interior Schauder estimates for linear elliptic equations imply that $\varphi \in C^{3,\alpha}(\Omega)$. This, combined with Definition 4.19(i) (or Corollary 4.45), implies that φ satisfies property (ii-2).

3. *Verification of property* (iv) *of Definition* 2.24.

For $A_{ij}(\boldsymbol{\xi})$ defined in Step 2, we define a linear operator $\mathcal{L}_{(u,\beta)}$ by

$$\mathcal{L}_{(u,\beta)}(v) := \sum_{i,j=1}^{2} A_{ij} \partial_{\xi_i \xi_j} v.$$

Since $\varphi_\infty - \varphi_\mathcal{N}$ is a linear function of $\boldsymbol{\xi}$, and $\varphi - \varphi_\infty = \hat{\phi} - (\varphi_\infty - \varphi_\mathcal{N})$, we have

$$(5.2.2) \qquad \mathcal{L}_{(u,\beta)}(\varphi - \varphi_\infty) = \mathcal{L}_{(u,\beta)}(\hat{\phi}) = 0 \qquad \text{in } \Omega.$$

By Lemma 4.30(a), the equation stated above is strictly elliptic in Ω so that the maximum principle applies to $\varphi - \varphi_\infty$ in Ω. From (5.1.13) and (5.2.1), we obtain that $\varphi - \varphi_\infty = 0$ on Γ_{shock}. By Definition 4.8(ii), it follows directly from the boundary condition $\hat{\phi} = \max\{\varphi_\mathcal{O}, \varphi_\mathcal{N}\} - \varphi_\mathcal{N}$ on $\Gamma^\mathcal{O}_{\mathrm{sonic}} \cup \Gamma^\mathcal{N}_{\mathrm{sonic}}$ given in (4.3.16) that $\varphi - \varphi_\infty = \varphi_\mathcal{O} - \varphi_\infty \leq 0$ on $\Gamma^\mathcal{O}_{\mathrm{sonic}}$, and $\varphi - \varphi_\infty = \varphi_\mathcal{N} - \varphi_\infty \leq 0$ on $\Gamma^\mathcal{N}_{\mathrm{sonic}}$. Furthermore, the boundary condition for $\hat{\phi}_{\xi_2} = 0$ on Γ_{wedge} given in (4.3.16) implies that

$$(5.2.3) \qquad \partial_{\xi_2}(\varphi_\infty - \varphi) = -v_\infty < 0 \qquad \text{on } \Gamma_{\mathrm{wedge}}.$$

Therefore, by the maximum principle and Hopf's lemma, we obtain

$$(5.2.4) \qquad \varphi \leq \varphi_\infty \quad \text{in } \Omega.$$

When $\beta < \frac{2\delta_1}{N_1^2}$, we have shown in Step 2 in the proof of Lemma 4.42 that

$$(5.2.5) \qquad \max\{\varphi_\mathcal{O}, \varphi_\mathcal{N}\} \leq \varphi \qquad \text{in } \Omega.$$

When $\beta \geq \frac{2\delta_1}{N_1^2}$, (4.3.3) in Definition 4.19(iv) implies that $\max\{\varphi_\mathcal{O}, \varphi_\mathcal{N}\} \leq \varphi$ holds in $\Omega \backslash (\mathcal{D}^\mathcal{O}_{\varepsilon/10} \cup \mathcal{D}^\mathcal{N}_{\varepsilon/10})$. Note that parameter ε in Definition 4.19 has been chosen so that $\varepsilon < \frac{\hat{c}_\mathcal{O}}{k}$ for $\frac{\hat{c}_\mathcal{O}}{k}$ from (4.1.45) in Definition 4.8. Therefore, $\varphi^*_\beta = \max\{\varphi_\mathcal{O}, \varphi_\mathcal{N}\}$ in $\Omega \cap (\mathcal{D}^\mathcal{O}_\varepsilon \cup \mathcal{D}^\mathcal{N}_\varepsilon)$ for φ^*_β given by (4.1.42). Then we obtain from (4.5.2) in Lemma 4.43 that $\max\{\varphi_\mathcal{O}, \varphi_\mathcal{N}\} \leq \varphi$ holds in $\Omega \cap (\mathcal{D}^\mathcal{O}_{\varepsilon/10} \cup \mathcal{D}^\mathcal{N}_{\varepsilon/10})$.

Therefore, we conclude that inequality (5.2.5) holds for any $\beta \in (0, \beta_*]$. Combining this inequality with (5.2.4), we conclude that φ satisfies property (iv) of Definition 2.24.

4. *Verification of property* (v) *of Definition* 2.24. In order to show that φ satisfies property (v) of Definition 2.24, it suffices to verify the following claim:

Claim. There exist small constants $\varepsilon_{\mathrm{fp}} > 0$ and $\delta_{\mathrm{fp}} > 0$ depending only on $(\gamma, v_\infty, \beta_*)$ so that, if parameters (ε, δ_1) in Definition 4.19 satisfy $\varepsilon \in (0, \varepsilon_{\mathrm{fp}}]$ and $\delta_1 \in (0, \delta_{\mathrm{fp}}]$, then φ satisfies

$$(5.2.6) \qquad \partial_{\boldsymbol{e}_{S_\mathcal{O}}}(\varphi_\infty - \varphi) \leq 0, \quad \partial_{\boldsymbol{e}_{S_\mathcal{N}}}(\varphi_\infty - \varphi) \leq 0 \qquad \text{in } \Omega.$$

Similarly to the previous step, we consider two cases: $\beta \in [\frac{\delta_1}{N_1^2}, \beta_*]$ and $\beta \in (0, \frac{\delta_1}{N_1^2})$, separately.

4-1. Suppose that $\beta \in [\frac{\delta_1}{N_1^2}, \beta_*]$. Define

$$W := \varphi_\infty - \varphi \qquad \text{in } \Omega.$$

Let (X, Y) be the rectangular coordinates such that $(\boldsymbol{e}_{S_\mathcal{O}}, \boldsymbol{e}^\perp_{S_\mathcal{O}}) = (\boldsymbol{e}_X, \boldsymbol{e}_Y)$. By (5.2.2), W satisfies that $\mathcal{L}_{(u,\beta)}(W) = 0$ in Ω. Since the (X, Y)–coordinates are obtained from rotating the (ξ_1, ξ_2)–plane by β counter-clockwise, equation $\mathcal{L}_{(u,\beta)}(W) = 0$ can be rewritten in the (X, Y)–coordinates as follows:

$$(5.2.7) \qquad \hat{A}_{11}W_{XX} + 2\hat{A}_{12}W_{XY} + \hat{A}_{22}W_{YY} = 0 \qquad \text{in } \Omega,$$

with $\hat{A}_{ij} \in C^\alpha(\overline{\Omega}) \cap C^{1,\alpha}(\overline{\Omega} \setminus (\overline{\Gamma^\mathcal{O}_{\mathrm{sonic}}} \cup \overline{\Gamma^\mathcal{N}_{\mathrm{sonic}}})), i, j = 1, 2$.

Define

$$w := W_X = \partial_{\boldsymbol{e}_{S_\mathcal{O}}}(\varphi_\infty - \varphi).$$

By (4.3.4) in Definition 4.19(iv), w satisfies

$$(5.2.8) \qquad w < 0 \qquad \text{in } \overline{\Omega} \setminus \mathcal{D}^\mathcal{O}_{\varepsilon/10}.$$

Next, we prove that $w \leq 0$ in $\Omega \cap \mathcal{D}_{\varepsilon/10}^{\mathcal{O}}$.

Differentiating (5.2.7) with respect to X, we have

$$\hat{A}_{11} w_{XX} + 2\hat{A}_{12} w_{XY} + \hat{A}_{22} w_{YY} + \partial_X \hat{A}_{11} w_X + 2\partial_X \hat{A}_{12} w_Y + \partial_X \hat{A}_{22} W_{YY} = 0 \qquad \text{in } \Omega.$$

Using the strict ellipticity of operator $\mathcal{L}_{(u,\beta)}$ following from Lemma 4.30(a), we obtain that $\hat{A}_{22} > 0$ in Ω such that W_{YY} can be expressed as

$$W_{YY} = -\frac{\hat{A}_{11} w_X + 2\hat{A}_{12} w_Y}{\hat{A}_{22}} \qquad \text{in } \Omega.$$

Substituting this expression into the equation immediately above, we obtain a strictly elliptic equation for w in the following form:

$$(5.2.9) \qquad \hat{A}_{11} w_{XX} + 2\hat{A}_{12} w_{XY} + \hat{A}_{22} w_{YY} + \hat{A}_1 w_X + \hat{A}_2 w_Y = 0 \qquad \text{in } \Omega.$$

Since $\hat{A}_{ij} \in C^\alpha(\overline{\Omega}) \cap C^{1,\alpha}(\overline{\Omega} \backslash (\overline{\Gamma_{\text{sonic}}^{\mathcal{O}}} \cup \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}})), i,j = 1, 2$, we see that $\hat{A}_i \in C^\alpha(\overline{\Omega} \backslash (\overline{\Gamma_{\text{sonic}}^{\mathcal{O}}} \cup \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}})), i = 1, 2$.

By a direct computation, applying Lemma 4.44 and the definitions of $(\boldsymbol{e}_{S_\mathcal{O}}, \varphi_\infty, \varphi_\mathcal{O})$ given in Definition 2.23, we have

$$(5.2.10) \qquad w = \partial_{\boldsymbol{e}_{S_\mathcal{O}}}(\varphi_\infty - \varphi_\mathcal{O}) = 0 \qquad \text{on } \Gamma_{\text{sonic}}^{\mathcal{O}}.$$

On Γ_{wedge}, w satisfies the homogeneous oblique boundary condition:

$$(5.2.11) \qquad \mathbf{b}_w \cdot \nabla w = 0 \qquad \text{with } \mathbf{b}_w \cdot \mathbf{n}_w > 0 \text{ on } \Gamma_{\text{wedge}}$$

for the inward unit normal vector $\mathbf{n}_w$ to Γ_{wedge}. This can be verified as follows: Differentiating the boundary condition (5.2.3) along $\Gamma_{\text{wedge}} \subset \{\xi_2 = 0\}$, we find that $W_{\xi_1 \xi_2 = 0}$ on Γ_{wedge}. Equation (5.2.2), combined with $\partial_{\xi_1 \xi_2} W = 0$ on Γ_{wedge}, leads to

$$A_{11} W_{\xi_1 \xi_1} + A_{22} W_{\xi_2 \xi_2} = 0 \qquad \text{on } \Gamma_{\text{wedge}}.$$

Note that $A_{11} > 0$ and $A_{22} > 0$ hold on Γ_{wedge} by Lemma 4.30(a). Then a direct computation by using the definition of $\boldsymbol{e}_{S_\mathcal{O}}$ shows that

$$\frac{A_{11}}{\cos \beta} w_{\xi_1} + \frac{A_{22}}{\sin \beta} w_{\xi_2} = 0 \qquad \text{with } \frac{A_{22}}{\sin \beta} > 0 \text{ on } \Gamma_{\text{wedge}}.$$

This implies the strict obliqueness of the boundary condition for w on Γ_{wedge}.

In order to obtain a boundary condition for w on Γ_{shock}, we apply [**11**, Lemma 13.4.5]. For this purpose, we need to check that all the conditions to apply [**11**, Lemma 13.4.5] are satisfied.

Let $M_\mathcal{O}$ and $c_\mathcal{O}$ be given by (2.4.6), and let $S_\mathcal{O}$ and $O_\mathcal{O}$ be given by Definition 2.23. Then $c_\mathcal{O} - \text{dist}(S_\mathcal{O}, O_\mathcal{O}) > 0$ if and only if $M_\mathcal{O} < 1$. By Lemma 2.13, $M_\mathcal{O} < 1$ for $\beta = 0$. Then (2.4.43) given in the proof of Lemma 2.22 implies that $M_\mathcal{O} < 1$ for $\beta \in (0, \beta_*]$. Therefore, there exists a constant $\mu_0 > 0$ depending only on (v_∞, γ) such that

$$(5.2.12) \qquad c_\mathcal{O} - \text{dist}(S_\mathcal{O}, O_\mathcal{O}) \geq \mu_0 \qquad \text{for all } \beta \in (0, \beta_*].$$

By Lemma 4.34(h) and (5.2.1), φ satisfies the Rankine-Hugoniot condition (2.5.37) on Γ_{shock}.

Let $\boldsymbol{\nu}$ be the unit normal vector to Γ_{shock} towards the interior of Ω, and let $\boldsymbol{\tau}$ be obtained from rotating $\boldsymbol{\nu}$ by $\frac{\pi}{2}$ counter-clockwise ($\boldsymbol{\tau}$ is a unit tangent vector to Γ_{shock}). By Definition 4.19(i) (or by Corollary 4.45), we have

$$(5.2.13) \qquad \|\varphi - \varphi_\mathcal{O}\|_{C^1(\overline{\Omega \cap \mathcal{D}_\varepsilon^\mathcal{O}})} + \|\boldsymbol{\tau} - \mathbf{e}_X\|_{C^0(\overline{\Gamma_{\text{shock}} \cap \mathcal{D}_\varepsilon^\mathcal{O}})} + \|\boldsymbol{\nu} - (-\mathbf{e}_Y)\|_{C^0(\overline{\Gamma_{\text{shock}} \cap \mathcal{D}_\varepsilon^\mathcal{O}})} \leq C\varepsilon^\alpha$$

for a constant $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$, where point P_1 is defined in Definition 2.23. Note that point P_1 lies on $\Gamma_{\mathrm{sonic}}^{\mathcal{O}}$. At P_1, $\boldsymbol{\tau} = \boldsymbol{e}_{S_\mathcal{O}} = \mathbf{e}_X$ and $\boldsymbol{\nu} = -\mathbf{e}_Y$.

By the definition of A_{ij} given in (4.4.22), Corollary 4.45, and (5.2.1), we have

$$(5.2.14) \qquad A_{ij} = A_{ij}^{\mathrm{potn}}(D(\varphi_\mathcal{O} - \varphi_\mathcal{N}), (\varphi_\mathcal{O} - \varphi_\mathcal{N})(P_1), P_1) \qquad \text{at } P_1$$

for $A_{ij}^{\mathrm{potn}}, i, j = 1, 2$, defined by (4.4.3). By (2.5.1), this yields

$$A_{11} = c_\mathcal{O}^2 - (\partial_{\xi_1} \varphi_\mathcal{O})^2, \;\; A_{12} = A_{21} = -\partial_{\xi_1} \varphi_\mathcal{O} \partial_{\xi_2} \varphi_\mathcal{O} = 0, \;\; A_{22} = c_\mathcal{O}^2 - (\partial_{\xi_2} \varphi_\mathcal{O})^2 \qquad \text{at } P_1.$$

Then we have

$$(5.2.15) \qquad \sum_{i,j=1}^{2} A_{ij} \nu_i \nu_j = c_\mathcal{O}^2 - (\partial_{\boldsymbol{\nu}} \varphi_\mathcal{O})^2 = c_\mathcal{O}^2 - (\mathrm{dist}(S_\mathcal{O}, O_\mathcal{O}))^2 = c_\mathcal{O}^2(1 - M_\mathcal{O}^2) > \lambda_0 \qquad \text{at } P_1$$

for some constant $\lambda_0 > 0$. By (5.2.12), constant $\lambda_0 > 0$ in (5.2.15) can be fixed, depending only on (v_∞, γ). By (5.2.13) and (5.2.15), there exists a small constant $\varepsilon_{\mathrm{fp}}^{(1)} > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ such that

$$(5.2.16) \qquad \sum_{i,j=1}^{2} A_{ij} \nu_i \nu_j \geq \frac{\lambda_0}{2} \qquad \text{in } \Gamma_{\mathrm{shock}} \cap \mathcal{D}_{\varepsilon_{\mathrm{fp}}^{(1)}}^{\mathcal{O}}.$$

By Lemma 4.30(a), there exists a constant $\lambda_1 > 0$ depending only on $(\gamma, v_\infty, \beta_*)$ such that

$$(5.2.17) \qquad \sum_{i,j=1}^{2} A_{ij} \nu_i \nu_j \geq \lambda_1 \qquad \text{in } (\Gamma_{\mathrm{shock}} \cap \mathcal{D}_{\hat{c}_\mathcal{O}/10}^{\mathcal{O}}) \setminus \mathcal{D}_{\varepsilon_{\mathrm{fp}}^{(1)}/2}^{\mathcal{O}}$$

for $\hat{c}_\mathcal{O}$ defined in Definition 4.1.

Since φ satisfies the Rankine-Hugoniot condition (2.5.37) on Γ_{shock}, it follows from (5.2.13) and (5.2.16)–(5.2.17) that φ satisfies all the conditions required to apply [**11**, Lemma 13.4.5]. Then, by [**11**, Lemma 13.4.5], we obtain a boundary condition for w in the form:

$$(5.2.18) \qquad \mathbf{b}_{\mathrm{sh}} \cdot \nabla w = 0 \qquad \text{on } \Gamma_{\mathrm{shock}} \cap \mathcal{D}_{\varepsilon_{\mathrm{fp}}^{(2)}}^{\mathcal{O}}$$

for some small constant $\varepsilon_{\mathrm{fp}}^{(2)} > 0$ depending on $(\gamma, v_\infty, \beta_*)$, where $\mathbf{b}_{\mathrm{sh}}$ satisfies

$$\mathbf{b}_{\mathrm{sh}} \cdot \boldsymbol{\nu} > 0 \qquad \text{on } \Gamma_{\mathrm{shock}} \cap \mathcal{D}_{\varepsilon_{\mathrm{fp}}^{(2)}}^{\mathcal{O}}.$$

In conclusion, w satisfies the strictly elliptic equation (5.2.9) in $\Omega \cap \mathcal{D}_\varepsilon^{\mathcal{O}}$ for $\varepsilon > 0$ to be specified later, the boundary condition $w = 0$ on $\Gamma_{\mathrm{sonic}}^{\mathcal{O}}$, and the oblique boundary conditions (5.2.11) on Γ_{wedge} and (5.2.18) on $\Gamma_{\mathrm{shock}} \cap \mathcal{D}_{\varepsilon_{\mathrm{fp}}^{(2)}}^{\mathcal{O}}$. Therefore, if parameter $\varepsilon > 0$ in Definition 4.19 satisfies

$$(5.2.19) \qquad 0 < \varepsilon \leq \varepsilon_{\mathrm{fp}}^{(2)},$$

then it follows from the maximum principle, Hopf's lemma, and (5.2.8) that

$$w \leq 0 \qquad \text{in } \Omega \cap \mathcal{D}_{\varepsilon_{\mathrm{fp}}^{(2)}}^{\mathcal{O}}.$$

Finally, we combine this result with (5.2.8) to conclude that

$$\partial_{\boldsymbol{e}_{S_\mathcal{O}}}(\varphi_\infty - \varphi) \leq 0 \qquad \text{in } \Omega \text{ for } \beta \in [\frac{\delta_1}{N_1^2}, \beta_*],$$

provided that ε satisfies condition (5.2.19).

4-2. Suppose that $\beta \in (0, \frac{\delta_1}{N_1^2})$. Note that w satisfies (5.2.9)–(5.2.11). By the definitions of $(\boldsymbol{e}_{S_\mathcal{O}}, \varphi_\infty, \varphi_\mathcal{N})$ given in Definition 2.23 and Corollary 4.45, w satisfies

$$w = \partial_{\boldsymbol{e}_{S_\mathcal{O}}}(\varphi_\infty - \varphi_\mathcal{N}) = -v_\infty \sin \beta < 0 \qquad \text{on } \Gamma_{\text{sonic}}^\mathcal{N}.$$

By (2.4.3) and (2.5.1), $\varphi_\mathcal{O} - \varphi_\mathcal{N} = v_\infty(\xi_1 \tan \beta - \xi_2^{(\beta)} + \xi_2^\mathcal{N})$. Note that $\xi_2^\mathcal{N} = \xi_2^{(\beta)}|_{\beta=0}$. Then, by (2.4.14) and the continuous differentiability of M_∞ with respect to $\beta \in [0, \beta_{\text{d}}^{(v_\infty)}]$, there exists a constant $C > 0$ depending only on (γ, v_∞) such that

(5.2.20) $$\|\varphi_\mathcal{O} - \varphi_\mathcal{N}\|_{C^{1,\alpha}(\overline{\Omega})} \leq C\beta \qquad \text{for all } \beta \in [0, \beta_{\text{d}}^{(v_\infty)}].$$

By Definition 4.19(i) and (5.2.20), we see that, for any $\beta \in (0, \frac{\delta_1}{N_1^2})$,

(5.2.21) $$\|\varphi - \varphi_\mathcal{O}\|_{C^{1,\alpha}(\overline{\Omega})} \leq \|\varphi - \varphi_\mathcal{N}\|_{C^{1,\alpha}(\overline{\Omega})} + \|\varphi_\mathcal{O} - \varphi_\mathcal{N}\|_{C^{1,\alpha}(\overline{\Omega})} \leq C\delta_1$$

for some constant $C > 0$ depending only on $(\gamma, v_\infty, \beta_*)$ so that

$$[A_{ij}]_{\alpha,\Omega} + [\boldsymbol{\nu}]_{\alpha,\Gamma_{\text{shock}}} + [\boldsymbol{\tau}]_{\alpha,\Gamma_{\text{shock}}} \leq C\delta_1$$

for $C > 0$ depending only on $(\gamma, v_\infty, \beta_*)$. By (5.2.15) and the estimate immediately above, there exists a small constant $\delta_{\text{fp}} > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ so that, if

(5.2.22) $$\delta_1 \in (0, \delta_{\text{fp}}],$$

then

$$\sum_{i,j=1}^2 A_{ij}\nu_i\nu_j \geq \frac{\lambda_0}{2} \qquad \text{on } \Gamma_{\text{shock}}$$

for $\lambda_0 > 0$ from (5.2.15). Then [**11**, Lemma 13.4.5] implies that w satisfies a boundary condition in the form:

(5.2.23) $$\mathbf{b}_{\text{sh}} \cdot \nabla w = 0 \qquad \text{on } \Gamma_{\text{shock}},$$

with $\mathbf{b}_{\text{sh}}$ satisfying $\mathbf{b}_{\text{sh}} \cdot \boldsymbol{\nu} > 0$ on Γ_{shock}.

Since w satisfies the strictly elliptic equation (5.2.9) in Ω, $w \leq 0$ on $\Gamma_{\text{sonic}}^\mathcal{O} \cup \Gamma_{\text{sonic}}^\mathcal{N}$, and the strictly oblique boundary conditions (5.2.11) on Γ_{wedge} and (5.2.23) on Γ_{shock}, it follows from the maximum principle and Hopf's lemma that

$$w \leq 0 \qquad \text{in } \Omega,$$

provided that parameter $\delta_1 > 0$ in Definition 4.19 satisfies (5.2.22).

4-3. By repeating the argument in Steps 4-1 and 4-2 with $w = \partial_{\boldsymbol{e}_{S_\mathcal{O}}}(\varphi_\infty - \varphi)$ replaced by $w = \partial_{\boldsymbol{e}_{S_\mathcal{N}}}(\varphi_\infty - \varphi)$, we can also show that

$$\partial_{\boldsymbol{e}_{S_\mathcal{N}}}(\varphi_\infty - \varphi) \leq 0 \qquad \text{in } \Omega,$$

provided that constants $(\varepsilon_{\text{fp}}^{(2)}, \delta_{\text{fp}})$ from (5.2.19) and (5.2.22) are adjusted, depending only on $(v_\infty, \gamma, \beta_*)$.

For the rest of the proof, parameters (ε, δ_1) in Definition 4.19 satisfy

$$0 < \delta_1 < \delta_{\text{fp}}, \qquad 0 < \varepsilon < \min\{\varepsilon_{\text{fp}}^{(1)}, \varepsilon_{\text{fp}}^{(2)}\}.$$

5. *Verification of property* (ii-4) *of Definition* 2.24. Since Eq. (2.1.19) is equivalent to (3.1.2), it suffices to check that equation $\mathcal{N}_{(u,\beta)}(\phi) = 0$ coincides with Eq. (3.1.2).

5-1. *Equation $\mathcal{N}_{(u,\beta)}(\phi) = 0$ away from* $\Gamma_{\mathrm{sonic}}^{\mathcal{O}} \cup \Gamma_{\mathrm{sonic}}^{\mathcal{N}}$. In order to show that φ satisfies property (ii-4), it suffices to show that equation $\mathcal{N}_{(u,\beta)}(\phi) = 0$ from (4.3.16) coincides with Eq. (3.1.2) in Ω. By Lemma 4.30(i), equation $\mathcal{N}_{(u,\beta)}(\phi) = 0$ coincides with Eq. (3.1.2) in $\Omega \setminus (\mathcal{D}_{\varepsilon/10}^{\mathcal{O}} \cup \mathcal{D}_{\varepsilon/10}^{\mathcal{N}})$ for parameter $\varepsilon > 0$ in Definition 4.19 fixed as in Definition 5.6.

5-2. *Equation $\mathcal{N}_{(u,\beta)}(\phi) = 0$ near* $\Gamma_{\mathrm{sonic}}^{\mathcal{N}}$. In $\Omega_\varepsilon^{\mathcal{N}} := \Omega \cap \mathcal{D}_\varepsilon^{\mathcal{N}}$, let the (x,y)–coordinates be defined by (3.4.18). Define $\psi := \varphi - \varphi_{\mathcal{N}} = \hat{\varphi} - \varphi_{\mathcal{N}}$ in $\Omega_\varepsilon^{\mathcal{N}}$. By Lemma 4.24(g), if it can be shown that

$$(5.2.24) \qquad \left| \psi_x(x,y) \right| < \frac{2 - \frac{\mu_0}{5}}{1 + \gamma} x \qquad \text{in } \Omega \cap \mathcal{D}_{\frac{\varepsilon}{2}}^{\mathcal{N}}$$

for $\mu_0 \in (0,1)$ from Definition 4.19(iv-1), then equation $\mathcal{N}_{(u,\beta)}(\phi) = 0$ coincides with Eq. (3.1.2) in $\Omega_{\varepsilon/10}^{\mathcal{N}}$.

Define

$$v(x,y) := Ax - \psi_x(x,y) \qquad \text{for } A = \frac{2 - \frac{\mu_0}{5}}{1 + \gamma}.$$

Then v satisfies

$$(5.2.25) \qquad v = 0 \ \text{ on } \Gamma_{\mathrm{sonic}}^{\mathcal{N}} = \{x = 0\}, \qquad v_y = 0 \ \text{ on } \Gamma_{\mathrm{wedge}} \cap \partial \Omega_\varepsilon^{\mathcal{N}},$$

because $\partial_{\xi_2} \varphi = \partial_{\xi_2} \varphi_{\mathcal{N}} = 0$ on Γ_{wedge}.

By (5.2.1) and properties (a), (f), and (g-3) of Lemma 4.34, the boundary condition on Γ_{shock} in (4.3.16) can be written as

$$b_1 \psi_x + b_2 \psi_y + b_0 \psi = 0 \qquad \text{on } \Gamma_{\mathrm{shock}} \cap \mathcal{D}_\varepsilon^{\mathcal{N}}$$

for (b_0, b_1, b_2) satisfying that

$$-\delta^{-1} \le b_j \le -\delta \qquad \text{on } \Gamma_{\mathrm{shock}} \cap \mathcal{D}_\varepsilon^{\mathcal{N}}$$

for a constant $\delta \in (0,1)$ depending only on $(v_\infty, \gamma, \beta_*)$. Then $|\psi_x| \le C(|\psi_y| + |\psi|)$ on $\Gamma_{\mathrm{shock}} \cap \mathcal{D}_\varepsilon^{\mathcal{N}}$ for $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$. By combining this inequality with estimate (4.5.15) given in Lemma 4.43, we have

$$|\psi_x| \le C x^{3/2} \qquad \text{on } \Gamma_{\mathrm{shock}} \cap \mathcal{D}_\varepsilon^{\mathcal{N}}$$

for $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$. Then we can fix a small constant $\varepsilon_{\mathrm{fp}}^{(3)}$ depending only on $(v_\infty, \gamma, \beta_*)$ so that, if

$$(5.2.26) \qquad 0 \le \varepsilon \le \varepsilon_{\mathrm{fp}}^{(3)},$$

we have

$$(5.2.27) \qquad v \ge 0 \qquad \text{on } \Gamma_{\mathrm{shock}} \cap \partial \mathcal{D}_\varepsilon^{\mathcal{N}}.$$

By (4.3.6) in Definition 4.19(iv), we obtain

$$(5.2.28) \qquad v \ge \frac{4\mu_0 \varepsilon}{5(1 + \gamma)} > 0 \qquad \text{on } \partial \Omega_\varepsilon^{\mathcal{N}} \cap \{x = \varepsilon\}.$$

By Lemma 4.43, $\varepsilon_{\mathrm{fp}}^{(3)}$ can be further reduced, depending only on $(v_\infty, \gamma, \beta_*)$, so that, if (5.2.26) holds, then

$$\zeta_1 \left(\frac{\psi_x}{x^{3/4}} \right) = \frac{\psi_x}{x^{3/4}}, \quad \zeta_1 \left(\frac{\psi_y}{(\gamma+1) N_4 x} \right) = \frac{\psi_y}{(\gamma+1) N_4 x} \qquad \text{in } \Omega_{\varepsilon_{\mathrm{fp}}^{(3)}}^{\mathcal{N}}$$

for ζ_1 given by (4.4.4). This implies that

$$O_j^{\mathrm{mod}}(\psi_x, \psi_y, x, y) = O_j(\psi_x, \psi_y, \psi, x, y) \qquad \text{in } \Omega^{\mathcal{N}}_{\varepsilon^{(3)}_{\mathrm{fp}}} \text{ for all } j = 1, \cdots, 5,$$

for O_j^{mod} and O_j defined by (4.4.11) and (3.2.29), respectively.

By (4.4.22) and (5.2.1), equation $\mathcal{N}_{(u,\beta)}(\hat{\phi}) = 0$ in $\Omega^{\mathcal{N}}_{\varepsilon/2}$ becomes $\mathcal{N}^{\mathrm{polar}}_{(u,\beta)}(\psi) = 0$ in the (x, y)–coordinates given by (3.4.18) for $\mathcal{N}^{\mathrm{polar}}_{(u,\beta)}$ defined by (4.4.12). We differentiate $\mathcal{N}^{\mathrm{polar}}_{(u,\beta)}(\psi) = 0$ with respect to x in $\Omega^{\mathcal{N}}_{\varepsilon/2}$ and then rewrite the resulting equation as an equation for $v(x, y)$ in the following form:

$$(5.2.29) \qquad a_{11}v_{xx} + a_{12}v_{xy} + a_{22}v_{yy} + a_1 v_x + a_0 v = -A\big((\gamma+1)A - 1\big) + E(x, y) \qquad \text{in } \Omega^{\mathcal{N}}_{\varepsilon/2},$$

where

$$a_{ij} = a_{ij}(D_{(x,y)}\psi, x, y) \qquad \text{for } a_{ij}(D_{(x,y)}\psi, x, y) \text{ given by (4.4.12)},$$

$$a_1 = 1 - (\gamma+1)\left(\zeta_1(A - \frac{v}{x}) + \zeta_1'(A - \frac{v}{x})(\frac{v}{x} - v_x + A)\right),$$

$$a_0 = (\gamma+1)\frac{A}{x}\left(\zeta_1'(A - \frac{v}{x}) - \int_0^1 \zeta_1'(A - s\frac{v}{x})\, \mathrm{d}s\right),$$

$$E(x, y) = \psi_{xx}\partial_x\hat{O}_1 + \psi_{xy}\partial_x\hat{O}_2 + \psi_{yy}\partial_x\hat{O}_3 - \psi_{xx}\hat{O}_4 - \psi_x\partial_x\hat{O}_4 + \psi_{xy}\hat{O}_5 + \psi_y\partial_x\hat{O}_5,$$

$$\hat{O}_j(x, y) = O_j(\psi_x(x, y), \psi_y(x, y), \psi(x, y), x, y) \quad \text{for } j = 1, \cdots, 5.$$

By Lemma 4.24(a), Eq. (5.2.29) is strictly elliptic in $\Omega^{\mathcal{N}}_{\varepsilon/2}$. Estimate (4.5.15) given in Lemma 4.43 implies that $a_{ij}, a_1, a_0 \in C(\overline{\Omega} \setminus \{x = 0\})$. Since $\zeta_1'' \le 0$ by (4.4.6), $a_0 v \ge 0$ in $\Omega^{\mathcal{N}}_{\varepsilon/2}$. By (3.2.29) and (4.5.15), there exists a constant $C > 0$ depending on $(v_\infty, \gamma, \beta_*)$ such that $|E(x, y)| \le Cx$ in $\Omega^{\mathcal{N}}_{\varepsilon/2}$. Therefore, we can fix a small constant $\varepsilon^{(4)}_{\mathrm{fp}}$ depending only on $(v_\infty, \gamma, \beta_*)$ so that, if

$$(5.2.30) \qquad\qquad\qquad 0 \le \varepsilon \le \varepsilon^{(4)}_{\mathrm{fp}},$$

then $-A\big((\gamma+1)A - 1\big) + E(x, y) < 0$ in $\Omega^{\mathcal{N}}_{\varepsilon/2}$. Thus, for such ε, we have

$$(5.2.31) \qquad\qquad a_{11}v_{xx} + a_{12}v_{xy} + a_{22}v_{yy} + a_1 v_x + a_0 v < 0 \qquad \text{in } \Omega^{\mathcal{N}}_{\varepsilon/2}.$$

By properties (5.2.25), (5.2.27)–(5.2.28), and (5.2.31), we can apply the maximum principle and Hopf's lemma to conclude that, if

$$(5.2.32) \qquad\qquad\qquad 0 < \varepsilon < \min\{\varepsilon^{(3)}_{\mathrm{fp}}, \varepsilon^{(4)}_{\mathrm{fp}}\},$$

then $v \ge 0$ in $\Omega^{\mathcal{N}}_{\varepsilon/2}$, which is equivalent to stating that

$$\psi_x(x, y) \le \frac{2 - \frac{\mu_0}{5}}{1 + \gamma} \qquad \text{in } \Omega^{\mathcal{N}}_{\varepsilon/2}.$$

Next, we show that $\psi_x \ge -\frac{2 - \frac{\mu_0}{5}}{1 + \gamma}x$ in $\Omega^{\mathcal{N}}_{\varepsilon/2}$. Since $\partial_{\boldsymbol{e}_{S_\mathcal{N}}}(\varphi_\infty - \varphi_\mathcal{N}) = 0$, we obtain from (5.2.6) that

$$(5.2.33) \qquad\qquad \partial_{\boldsymbol{e}_{S_\mathcal{N}}}\psi = \partial_{\boldsymbol{e}_{S_\mathcal{N}}}(\varphi - \varphi_\infty) \ge 0 \qquad \text{in } \Omega.$$

By (3.4.25), $\partial_{\boldsymbol{e}_{S_\mathcal{N}}}\psi$ is represented as

$$(5.2.34) \qquad\qquad \partial_{\boldsymbol{e}_{S_\mathcal{N}}}\psi = \psi_x \cos y + \frac{\sin y}{c_\mathcal{N} - x}\psi_y \qquad \text{in } \Omega^{\mathcal{N}}_{c_\mathcal{N}/2}.$$

By Remark 4.21(i)–(ii), we can fix a small constant $\varepsilon_{\mathrm{fp}}^{(5)} > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ such that $\Omega_{\varepsilon_{\mathrm{fp}}^{(5)}}^{\mathcal{N}} \subset \{(x,y) : x \in (0, \varepsilon_{\mathrm{fp}}^{(5)}), 0 < y < \frac{\pi}{2} - \sigma_0\}$ for some constant $\sigma_0 > 0$ that is chosen depending only on (v_∞, γ). Then it follows from estimate (4.5.15) given in Lemma 4.43 and (5.2.33)–(5.2.34) that there exists a constant $C > 0$ depending only on $(v_\infty, \gamma, \beta_*)$ such that

$$\psi_x \geq -\tan(\frac{\pi}{2} - \sigma_0)\psi_y \geq -Cx^{3/2} \qquad \text{in } \Omega_{\varepsilon_{\mathrm{fp}}^{(5)}}^{\mathcal{N}}.$$

Therefore, $\varepsilon_{\mathrm{fp}}^{(5)}$ can be further reduced, depending only on $(v_\infty, \gamma, \beta_*)$, so that the inequality above implies

$$\psi_x \geq -\frac{2 - \frac{\mu_0}{5}}{1 + \gamma}x \qquad \text{in } \Omega_{\varepsilon_{\mathrm{fp}}^{(5)}}^{\mathcal{N}}.$$

We finally conclude that φ satisfies (5.2.24), provided that parameter ε in Definition 4.19 satisfies

(5.2.35)
$$0 < \varepsilon \leq \min\{\varepsilon_{\mathrm{fp}}^{(3)}, \varepsilon_{\mathrm{fp}}^{(4)}, \varepsilon_{\mathrm{fp}}^{(5)}\}.$$

Therefore, equation $\mathcal{N}_{(u,\beta)}(\phi) = 0$ coincides with Eq. (3.1.2) in $\Omega_{\varepsilon/10}^{\mathcal{N}}$, provided that condition (5.2.35) holds.

5-3. *Equation $\mathcal{N}_{(u,\beta)}(\phi) = 0$ near $\Gamma_{\mathrm{sonic}}^{\mathcal{O}}$.* In $\Omega_\varepsilon^{\mathcal{O}} := \Omega \cap \mathcal{D}_\varepsilon^{\mathcal{O}}$, let the (x,y)–coordinates be defined by (3.5.2).

By (3.5.54)–(3.5.56), there exists a small constant $\varepsilon_{\mathrm{fp}}^{(6)} > 0$ depending only on (v_∞, γ) so that, if $x_{P_\beta} < \frac{\varepsilon_{\mathrm{fp}}^{(6)}}{10}$, then $\beta < \beta_{\mathrm{s}}^{(v_\infty)} + \frac{1}{2}\min\{\sigma_3, \hat{\delta}\}$ for $\hat{\delta} > 0$ from Lemma 4.44(ii) and σ_3 from Proposition 3.39.

Assume that parameter ε in Definition 4.19 satisfies

(5.2.36)
$$0 < \varepsilon < \varepsilon_{\mathrm{fp}}^{(6)},$$

and suppose that $x_{P_\beta} < \frac{\varepsilon}{10}$. By (4.4.20) and (4.4.22), if we can show that

(5.2.37)
$$\left|\psi_x(x,y)\right| < \frac{2 - \frac{\mu_0}{5}}{1 + \gamma}x \qquad \text{in } \Omega \cap \mathcal{D}_{\varepsilon/2}^{\mathcal{O}},$$

then it follows from Lemma 4.25(c-1) that equation $\mathcal{N}_{(u,\beta)}(\phi) = 0$ coincides with Eq. (3.1.2) in $\Omega_{\varepsilon/10}^{\mathcal{O}}$. To prove (5.2.37), we can mostly repeat the argument in Step 5-2 by using Lemma 4.44(i)–(ii) and the positivity of $\partial_{\boldsymbol{e}_{S_\mathcal{O}}}(\varphi - \varphi_\infty)$ in Ω given in (5.2.6), instead of Lemma 4.43 and the positivity of $\partial_{\boldsymbol{e}_{S_\mathcal{N}}}(\varphi - \varphi_\infty)$ in Ω. Then there exists a small constant $\varepsilon_{\mathrm{fp}}^{(6)} > 0$ depending only on (v_∞, γ) such that, if ε satisfies condition (5.2.36), then equation $\mathcal{N}_{(u,\beta)}(\phi) = 0$ coincides with Eq. (3.1.2) in $\Omega_{\varepsilon/10}^{\mathcal{O}}$.

If parameter ε in Definition 4.19 satisfies condition (5.2.36), and if $x_{P_\beta} \geq \frac{\varepsilon}{10}$, then it follows from Lemma 4.30(i) that equation $\mathcal{N}_{(u,\beta)}(\phi) = 0$ coincides with Eq. (3.1.2) in $\Omega_{\varepsilon/10}^{\mathcal{O}}$.

For the rest of the proof, parameters (ε, δ_1) in Definition 4.19 satisfy

(5.2.38)
$$0 < \delta_1 < \delta_{\mathrm{fp}}, \qquad 0 < \varepsilon < \min\{\varepsilon_{\mathrm{fp}}^{(j)} : j = 1, \cdots, 6\},$$

where δ_{fp} is from (5.2.22).

6. It remains to check that properties (i-1) and (iii) of Definition 2.24 hold.

Verification of property (iii) *of Definition* 2.24. In Step 5, we have shown that Eq. (3.1.2) coincides with equation $\mathcal{N}_{(u,\beta)}(\phi) = 0$ in Ω. Therefore, it directly follows from Lemma 4.30(a)

and Lemmas 4.43–4.44 that Eq. (3.1.2) is strictly elliptic in $\overline{\Omega} \setminus (\overline{\Gamma^{\mathcal{O}}_{\text{sonic}}} \cup \overline{\Gamma^{\mathcal{N}}_{\text{sonic}}})$. This proves that property (iii) of Definition 2.24 holds, because Eq. (2.1.19) is equivalent to (3.1.2) in Ω.

Verification of property (i-1) *of Definition* 2.24. The strict ellipticity of Eq. (3.1.2) in $\overline{\Omega} \setminus (\overline{\Gamma^{\mathcal{O}}_{\text{sonic}}} \cup \overline{\Gamma^{\mathcal{N}}_{\text{sonic}}})$ implies

$$\frac{|\partial_{\boldsymbol{\nu}}\varphi(\boldsymbol{\xi})|^2}{c^2(|\nabla\varphi(\boldsymbol{\xi})|^2, \varphi(\boldsymbol{\xi}), \boldsymbol{\xi})} \leq \frac{|\nabla\varphi(\boldsymbol{\xi})|^2}{c^2(|\nabla\varphi(\boldsymbol{\xi})|^2, \varphi(\boldsymbol{\xi}), \boldsymbol{\xi})} < 1 \qquad \text{on } \Gamma_{\text{shock}} \setminus (\overline{\Gamma^{\mathcal{O}}_{\text{sonic}}} \cup \overline{\Gamma^{\mathcal{N}}_{\text{sonic}}}).$$

for a unit normal vector $\boldsymbol{\nu}$ to Γ_{shock}. We have shown in Step 4-1 that φ satisfies the Rankine-Hugoniot condition (2.5.37) on Γ_{shock}. Define $M := \frac{|\partial_{\boldsymbol{\nu}}\varphi(\boldsymbol{\xi})|}{c^2(|\nabla\varphi(\boldsymbol{\xi})|^2, \varphi(\boldsymbol{\xi}), \boldsymbol{\xi})}$ and $M_\infty := |\partial_{\boldsymbol{\nu}}\varphi_\infty(\boldsymbol{\xi})|$. We substitute $M_{\mathcal{O}} = M$ into the left-hand side of (2.4.9) in the proof of Lemma 2.17. Then, by repeating the argument right after (2.4.9) in the proof of Lemma 2.17, we obtain that $M_\infty > 1$ on Γ_{shock}, which yields that

$$(5.2.39) \qquad\qquad |D\varphi_\infty(\boldsymbol{\xi})| > 1 \qquad \text{on } \Gamma_{\text{shock}}.$$

By the definition of φ_∞ given in (2.5.1), (5.2.39) implies that $\boldsymbol{\xi} \notin \overline{B_1(O_\infty)}$ for all $\boldsymbol{\xi} \in \Gamma_{\text{shock}}$. Furthermore, $\{P_1, P_2\} \not\subset \overline{B_1(O_\infty)}$, because P_1 and P_2 lie on $S_{\mathcal{O}}$ and $S_{\mathcal{N}}$, respectively.

Now it remains to show that $\xi_1^{P_1} \leq \xi_1 \leq \xi_1^{P_2}$ for all $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \Gamma_{\text{shock}}$. Since we have shown that φ satisfies properties (i-2), (i-4), and (ii)–(v) of Definition 2.24 in the previous steps, we can repeat the proof of Lemma 3.2 to show that φ satisfies the directional monotonicity properties (3.1.6)–(3.1.7). Then, by repeating the proof of Proposition 3.4, we obtain a function f_{sh} satisfying

$$\Gamma_{\text{shock}} = \{\boldsymbol{\xi} = (\xi_1, \xi_2) : \xi_2 = f_{\text{sh}}(\xi_1), \ \xi_1^{P_1} < \xi_2 < \xi_1^{P_2}\}.$$

Therefore, property (i-1) holds.

With these, we complete the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

5.3. Existence of Admissible Solutions for All $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$

In order to prove the existence of admissible solutions for all $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$, we employ the Leray-Schauder fixed point index and its generalized homotopy invariance property.

5.3.1. Leray-Schauder degree theorem.

DEFINITION 5.9 (Compact mapping). Let X and Y be two Banach spaces. For an open subset G in X, a mapping $\mathbf{f} : \overline{G} \to Y$ is called *compact* if

 (i) $\mathbf{f}$ is continuous;
 (ii) $\mathbf{f}(U)$ is precompact in Y for any bounded subset U of $\overline{G}$.

DEFINITION 5.10. Let G be an open bounded set in a Banach space X. Denote by $V(G, X)$ the set of all mappings $f : \overline{G} \to X$ satisfying the following:

 (i) f is compact in the sense of Definition 5.9;
 (ii) f has no fixed points on the boundary ∂G.

DEFINITION 5.11. Two mappings $\mathbf{f}, \mathbf{g} \in V(G, X)$ are called *compactly homotopic* on ∂G if there exists a mapping $\mathbf{H}$ with the following properties:

 (i) $\mathbf{H} : \overline{G} \times [0, 1] \to X$ is compact in the sense of Definition 5.9;
 (ii) $\mathbf{H}(\mathbf{x}, \tau) \neq \mathbf{x}$ for all $(\mathbf{x}, \tau) \in \partial G \times [0, 1]$;
 (iii) $\mathbf{H}(\mathbf{x}, 0) = \mathbf{f}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x}, 1) = \mathbf{g}(\mathbf{x})$ in $\overline{G}$.

We write $\partial G : \mathbf{f} \cong \mathbf{g}$ if $\mathbf{f}$ and $\mathbf{g}$ are compactly homotopic on ∂G, and call $\mathbf{H}$ a compact homotopy.

THEOREM 5.12 (Leray-Schauder degree theorem). *Let G be an open bounded set in a Banach space* X. *Then, to each mapping $\mathbf{f} \in V(G, X)$, a unique integer $\mathbf{Ind}(\mathbf{f}, G)$ can be assigned with the following properties*:

(i) *If $\mathbf{f}(\mathbf{x}) \equiv \mathbf{x}_0$ for all $\mathbf{x} \in \overline{G}$ and some fixed $\mathbf{x}_0 \in G$, then $\mathbf{Ind}(\mathbf{f}, G) = 1$*;

(ii) *If $\mathbf{Ind}(\mathbf{f}, G) \neq 0$, then there exists $\mathbf{x} \in G$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{x}$*;

(iii) $\mathbf{Ind}(\mathbf{f}, G) = \sum_{j=1}^n \mathbf{Ind}(\mathbf{f}, G_j)$, *whenever $f \in V(G, X) \cap (\cap_{j=1}^n V(G_j, X))$, where $G_i \cap G_j = \emptyset$ for $i \neq j$ and $\overline{G} = \cup_{j=1}^n \overline{G_j}$*;

(iv) *If $\partial G : \mathbf{f} \cong \mathbf{g}$, then $\mathbf{Ind}(\mathbf{f}, G) = \mathbf{Ind}(\mathbf{g}, G)$*.

Such a number $\mathbf{Ind}(\mathbf{f}, G)$ is called the fixed point index *of $\mathbf{f}$ over G.*

A generalized homotopy invariance of the fixed point index is given in the following theorem:

THEOREM 5.13 ([**47**], §13.6, A4*). *Let* X *be a Banach space, and let $t_2 > t_1$. Let* $\mathrm{U} \subset \mathrm{X} \times [t_1, t_2]$, *and let $U_t = \{\mathbf{x} : (\mathbf{x}, t) \in \mathrm{U}\}$. Then*

$$\mathbf{Ind}(\mathbf{h}(\cdot, t), U_t) = const. \qquad \text{for all } t \in [t_1, t_2],$$

provided that U *is bounded and open in* $\mathrm{X} \times [t_1, t_2]$, *and mapping $\mathbf{h} : \overline{\mathrm{U}} \to \mathrm{X}$ is compact in the sense of Definition 5.9 with $\mathbf{h}(\mathbf{x}, t) \neq \mathbf{x}$ on $\partial \mathrm{U}$.*

5.3.2. Proof of Theorem 2.31.

In this subsection, we complete the proof of Theorem 2.31.

Parameters $(\alpha, \varepsilon, \delta_1, \delta_3, N_1)$ ***in Definition*** 4.19: Let parameters $(\alpha, \varepsilon, \delta_1, \delta_3, N_1)$ in Definition 4.19 be fixed as in Definition 5.6. We further reduce (ε, δ_1) depending only on $(v_\infty, \gamma, \beta_*)$ so that Proposition 5.8 implies that, for each $\beta \in (0, \beta_*]$, $u \in \mathcal{K}(\beta)$ is a fixed point of $\mathcal{I}(\cdot, \beta) : \overline{\mathcal{K}}(\beta) \to C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}})$ if and only if φ, defined by (4.2.4) in Definition 4.15, yields an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$ in the sense of Definition 2.24.

In the proof of Theorem 2.31, we adjust N_1 and choose δ_2 so that $\mathcal{I}(\cdot, \beta)$ has a fixed point in $\mathcal{K}(\beta)$ for each $\beta \in (0, \beta_*]$. Then the existence of an admissible solution for each $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta \leq \beta_*\}$ follows from Proposition 5.8. This proves Theorem 2.31, since β_* is arbitrarily chosen in $(0, \beta_{\mathrm{d}}^{(v\infty)})$.

Further adjustment of δ_3 ***in Definition*** 4.19: Note that, if parameter N_1 in Definition 4.19 is adjusted such that the new choice of N_1 is greater than the previous one, all the properties stated previously hold. Then we choose N_1 greater than the previous choice in the proof of Theorem 2.31. Also, once parameters (N_1, δ_2) are fixed, δ_3 can be adjusted to satisfy the conditions of δ_3 in Lemmas 4.42–4.43. As long as the new choice of δ_3 is less than the previous choice, all the properties stated previously hold. Since N_1 is adjusted to be greater than the previous one, the new choice of δ_3 is less than the previous one. Since the previous choice of $(\alpha, \varepsilon, \delta_1, \delta_2, N_1)$ was independent of δ_3, we can reduce δ_3 as described above.

PROOF OF THEOREM 2.31. The proof is divided into three steps.

1. *Claim* 1: *The iteration mapping $\mathcal{I} : \overline{\mathcal{K}} \to C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}})$ defined by Definition 5.6 is continuous. Moreover, $\mathcal{I} : \overline{\mathcal{K}} \to C_{(*,\alpha_1)}^{2,\alpha}$ is compact in the sense of Definition 5.9.*

1-1. *Continuity of $\mathcal{I} : \overline{\mathcal{K}} \to C_{(*,\alpha_1)}^{2,\alpha}$.* Suppose that $\{(u_j, \beta_j)\}_{j=1}^\infty \subset \mathcal{K}$ converges to (u, β) in $C_{(*,\alpha_1)}^{2,\alpha}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$. For each $j \in \mathbb{N}$, define $(\Omega_j, \mathbf{g}_{\mathrm{sh}}^{(j)}) := (\Omega(u_j, \beta_j), \mathbf{g}_{\mathrm{sh}}^{(u_j, \beta_j)})$ for $\Omega(u_j, \beta_j)$ and

$\mathfrak{g}_{\text{sh}}^{(u_j, \beta_j)}$ given by Definition 4.15. By Lemma 4.35, the nonlinear boundary value problem (4.3.16) associated with (u_j, β_j) has a unique solution $\hat{\phi}^{(j)} \in C^2(\Omega_j) \cap C^1(\overline{\Omega_j} \setminus (\overline{\Gamma_{\text{sonic}}^{\mathcal{O},j}} \cup \overline{\Gamma_{\text{sonic}}^{\mathcal{N}}})) \cap C^0(\overline{\Omega_j})$, where $\Gamma_{\text{sonic}}^{\mathcal{O},j}$ is $\Gamma_{\text{sonic}}^{\mathcal{O}}$ corresponding to (v_∞, β_j). For such $\hat{\phi}^{(j)}$, define

$$(5.3.1) \qquad \hat{w}^{(j)} := (\hat{\phi}^{(j)} + \varphi_\mathcal{N} - \varphi_{\beta_j}^*) \circ (\mathcal{G}_1^{\beta_j})^{-1}$$

for $\mathcal{G}_1^{\beta_j}$ and $\varphi_{\beta_j}^*$ defined by (4.1.31) and (4.1.42), respectively.

Let $\hat{u}_j$ be given by (4.3.17) associated with $(u_j, \beta_j, \hat{\phi}^{(j)})$. Then Definition 4.15(ii) implies that

$$(5.3.2) \qquad \hat{w}^{(j)} = \hat{u}_j \circ G_{2, \mathfrak{g}_{\text{sh}}^{(j)}}$$

for $G_{2, \mathfrak{g}_{\text{sh}}^{(j)}}$ defined by (4.1.49).

For each $\hat{w}^{(j)}$, let $\hat{\mathfrak{g}}_{\text{sh}}^{(j)}$ be given from (5.1.9) with $\hat{w} = \hat{w}^{(j)}$. We also define Ω, $\mathfrak{g}_{\text{sh}}$, $\hat{\phi}$, $\hat{w}$, $\hat{u}$, and $\hat{\mathfrak{g}}_{\text{sh}}$, similarly associated with $(u, \beta) \in \overline{\mathcal{K}}$.

By Lemma 4.16(d), we have

$$(5.3.3) \qquad \mathfrak{g}_{\text{sh}}^{(j)} \to \mathfrak{g}_{\text{sh}} \qquad \text{in } C^{1,\alpha}([-1,1]).$$

Fix a compact set $K \subset \mathcal{G}_1^\beta(\Omega) = \{(s, t') : -1 < s < 1, 0 < t' < \mathfrak{g}_{\text{sh}}(s)\}$. Then there exists a constant $\sigma_K \in (0,1)$ depending only on K such that $K \subset \{s \geq -1 + \sigma_K\}$. Thus, by Lemma 4.16(g), there exists a constant $C_K > 1$ depending only on $(v_\infty, \gamma, \beta_*)$ and K such that, for any $(u^\sharp, \beta^\sharp) \in \overline{\mathcal{K}}$,

$$(5.3.4) \qquad C_K^{-1} < \mathfrak{g}_{\text{sh}}^\sharp(s) < C_K \qquad \text{for all } (s, t') \in K.$$

By (4.1.49) and (5.3.3)–(5.3.4), we have

$$(5.3.5) \qquad G_{2, \mathfrak{g}_{\text{sh}}^{(j)}} \to G_{2, \mathfrak{g}_{\text{sh}}} \qquad \text{in } C^{1,\alpha}(K).$$

This implies that there exists a compact set $\mathcal{Q}_K \subset \mathcal{Q}^{\text{iter}}$ such that $G_{2, \mathfrak{g}_{\text{sh}}^{(j)}}(K) \subset \mathcal{Q}_K$ for all j, and $G_{2, \mathfrak{g}_{\text{sh}}}(K) \subset \mathcal{Q}_k$. By Corollary 4.37(b), $\hat{u}_j$ converges to $\hat{u}$ in $C^2(\mathcal{Q}_K)$. Therefore, it follows from (5.3.2) and (5.3.5) that

$$(5.3.6) \qquad \hat{w}^{(j)} \to \hat{w} \qquad \text{in } C^{1,\alpha}(K).$$

Since K is an arbitrary compact subset of $\mathcal{G}_1^\beta(\Omega)$, we conclude that $\hat{w}_j$ converges to $\hat{w}$ in $C^{1,\alpha}$ for any compact subset of $\mathcal{G}_1^\beta(\Omega)$.

By (5.3.1), (5.3.6), and Lemmas 4.5 and 4.43–4.44, we can apply Proposition 5.4(a-3) to obtain the convergence of sequence $\{\mathcal{E}_{\mathfrak{g}_{\text{sh}}^{(j)}}(\hat{w}^{(j)})\}$ to $\mathcal{E}_{\mathfrak{g}_{\text{sh}}}(\hat{w}^{(\infty)})$ in $C^{2,\alpha}(R_{(1+\frac{\kappa}{2})\mathfrak{g}_{\text{sh}}} \cap \{b_1 < s < b_2\})$ for any b_1 and b_2 with $-1 < b_1 < b_2 < 1$, where $\kappa \in (0, \frac{1}{3}]$ is from Definition 5.3. Note that, for any $\sigma \in (0,1)$, $\{(s, \hat{\mathfrak{g}}_{\text{sh}}^{(j)}(s)) : -1 + \sigma < s < 1 - \sigma\} \subset R_{(1+\frac{\kappa}{2})\mathfrak{g}_{\text{sh}}}$ holds for all j sufficiently large depending on σ. Therefore, by using the C^2–estimate of $\hat{\mathfrak{g}}_{\text{sh}}$ given in Lemma 5.5 and (5.1.17), it can be directly checked that $\{\hat{\mathfrak{g}}_{\text{sh}}^{(j)}\}$ converges to $\hat{\mathfrak{g}}_{\text{sh}}$ in $C^2([-1 + \sigma, 1 - \sigma])$ for any $\sigma \in (0,1)$. Then we obtain from (5.1.10) that

$$(5.3.7) \qquad \hat{\mathfrak{g}}_{\text{sh}}^{(j)} \to \hat{\mathfrak{g}}_{\text{sh}} \qquad \text{in } C_{(*, \alpha_1)}^{2,\alpha}((-1,1)).$$

By (5.1.24), (5.3.7), and properties (a-3), (b-3), and (c-3) of Proposition 5.4, we conclude that $\tilde{u}_j := \mathcal{I}(u_j, \beta_j)$ converges to $\tilde{u} = \mathcal{I}(u, \beta)$ in $C_{(*, \alpha_1)}^{2,\alpha}(\mathcal{Q}^{\text{iter}})$. This implies that $\mathcal{I} : \overline{\mathcal{K}} \to C_{(*, \alpha_1)}^{2,\alpha}$ is continuous.

1-2. *Compactness of* $\mathcal{I} : \overline{\mathcal{K}} \to C^{2,\alpha}_{(*,\alpha_1)}$. Let U be a subset of $\overline{\mathcal{K}} \subset C^{2,\alpha}_{(*,\alpha_1)}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$. Then U is bounded in $C^{2,\alpha}_{(*,\alpha_1)}(\mathcal{Q}^{\mathrm{iter}}) \times [0, \beta_*]$. Since $C^{2,2\alpha}_{(*,1)}(\mathcal{Q}^{\mathrm{iter}})$ is compactly embedded into $C^{2,\alpha}_{(*,\alpha_1)}(\mathcal{Q}^{\mathrm{iter}})$, Lemma 5.7(b) implies that $\mathcal{I}(U)$ is pre-compact in $C^{2,\alpha}_{(*,\alpha_1)}(\mathcal{Q}^{\mathrm{iter}})$. From this property, combined with the continuity of $\mathcal{I}$ proved in the previous step, we conclude that $\mathcal{I} : \overline{\mathcal{K}} \to C^{2,\alpha}_{(*,\alpha_1)}(\mathcal{Q}^{\mathrm{iter}})$ is compact in the sense of Definition 5.9. This verifies *Claim* 1.

2. *Claim* 2: *In Definition* 4.19, N_1 *can be increased, and* $\delta_2 > 0$ *can be fixed such that, for any* $\beta \in (0, \beta_*]$, *no fixed point of* $\mathcal{I}(\cdot, \beta)$ *lies on boundary* $\partial\mathcal{K}(\beta)$ *of* $\mathcal{K}(\beta)$, *where* $\partial\mathcal{K}(\beta)$ *is considered relative to space* $C^{2,\alpha}_{(*,\alpha_1)}(\mathcal{Q}^{\mathrm{iter}})$. *Furthermore, the choices of* (N_1, δ_2) *depend only on* $(v_\infty, \gamma, \beta_*)$.

2-1. Let $\mathcal{I}(u, \beta) = u$ for some $(u, \beta) \in \overline{\mathcal{K}}$, and let $\varphi = \varphi^{(u,\beta)}$ be given by (4.2.4). We extend φ onto Λ_β by (2.5.8) if $\beta < \beta_{\mathrm{s}}^{(v_\infty)}$, and by (2.5.12) if $\beta \geq \beta_{\mathrm{s}}^{(v_\infty)}$. By Proposition 5.8, φ is an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}}$ in the sense of Definition 2.24.

In order to verify Claim 2, we need to show the following:

- u satisfies the strict inequality given in condition (i) of Definition 4.19;
- φ satisfies all the strict inequalities given in conditions (iii)–(vi) given in Definition 4.19.

2-2. *The strict inequalities in condition* (i) *of Definition* 4.19: Note that N_1 satisfies that $N_1 \geq N_1^{(\mathrm{a})}$ for $N_1^{(\mathrm{a})}$ from Corollary 4.40. Therefore, u satisfies the strict inequality given in condition (i) of Definition 4.19.

2-3. *The strict inequalities in conditions* (iii) *and* (v)–(vi) *of Definition* 4.19. In conditions (iii) and (v)–(vi) of Definition 4.19, constants $(N_2, \tilde{\zeta}, \tilde{\mu}, a_*, C)$ are fixed so that any admissible solution satisfies the strict inequalities in conditions (iii) and (v)–(vi) of Definition 4.19 by Propositions 3.7 and 4.6, Remark 3.16, and Lemma 3.5.

2-4. *The strict inequalities in condition* (iv) *of Definition* 4.19. Suppose that $0 < \beta < \frac{\delta_1}{N_1^2}$. Then $\mathscr{K}_2(\beta)$ defined by (4.3.12) satisfies that $\mathscr{K}_2(\beta) < 0$ for any $\delta_2 > 0$. Moreover, φ satisfies (4.3.3) in the whole domain Ω by Definition 2.24(iv), the strong maximum principle, and Hopf's lemma. The strict inequalities in (4.3.4)–(4.3.5) are satisfied by Lemma 3.2.

Next, suppose that $\beta \geq \frac{\delta_1}{N_1^2}$. Then it follows directly from (2.5.1) that $\varphi_\mathcal{N} - \varphi_\mathcal{O}$ is a nontrivial linear function. By Definition 2.24(iv), $\psi = \varphi - \max\{\varphi_\mathcal{O}, \varphi_\mathcal{N}\} \geq 0$ in Ω. Since $\varphi = \varphi_\mathcal{O}$ on $\Gamma^\mathcal{O}_{\mathrm{sonic}}$, $\varphi_\mathcal{N}$ on $\Gamma^\mathcal{N}_{\mathrm{sonic}}$, and $\varphi_\mathcal{O} - \varphi_\mathcal{N}$ is a nonzero function, the strong maximum principle and Hopf's lemma apply to φ, so that $\varphi - \varphi_\mathcal{O} > 0$ and $\varphi - \varphi_\mathcal{N} > 0$ in Ω hold, which yields that

$$(5.3.8) \qquad \psi = \varphi - \max\{\varphi_\mathcal{O}, \varphi_\mathcal{N}\} > 0 \qquad \text{in } \overline{\Omega} \setminus (\mathcal{D}^\mathcal{O}_{\varepsilon/10} \cup \mathcal{D}^\mathcal{N}_{\varepsilon/10})$$

for fixed $\varepsilon > 0$ in Definition 4.19. By (5.3.8), Lemmas 3.2 and 3.43, and the continuous dependence of $(\Gamma^\mathcal{O}_{\mathrm{sonic}}, \varphi_\mathcal{O})$ on β, there exists a constant $\sigma > 0$ depending only on $(\gamma, v_\infty, \beta_*)$ such that

$$\psi = \varphi - \max\{\varphi_\mathcal{O}, \varphi_\mathcal{N}\} > \sigma \qquad \text{in } \overline{\Omega} \setminus (\mathcal{D}^\mathcal{O}_{\varepsilon/10} \cup \mathcal{D}^\mathcal{N}_{\varepsilon/10}).$$

By Lemma 3.2, we also have

$$\partial_{\boldsymbol{e}_{S_\mathcal{O}}}(\varphi_\infty - \varphi) < 0 \quad \text{in } \overline{\Omega} \setminus \mathcal{D}^\mathcal{O}_{\varepsilon/10}, \qquad -\partial_{\xi_1}(\varphi_\infty - \varphi) < 0 \quad \text{in } \overline{\Omega} \setminus \mathcal{D}^\mathcal{N}_{\varepsilon/10}.$$

By Corollary 3.19, and Propositions 3.26, 3.30, 3.32, 3.39, and 3.42, the set of admissible solutions corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\mathrm{weak}} \cap \{\beta \leq \beta_*\}$ is uniformly bounded in $C^{1,\alpha}$. Therefore, there exists a constant $\hat{\sigma} > 0$ depending only on $(\gamma, v_\infty, \beta_*)$ such that

$$\partial_{\boldsymbol{e}_{S_\mathcal{O}}}(\varphi_\infty - \varphi) < -\hat{\sigma} \quad \text{in } \overline{\Omega} \setminus \mathcal{D}^\mathcal{O}_{\varepsilon/10}, \qquad -\partial_{\xi_1}(\varphi_\infty - \varphi) < -\hat{\sigma} \quad \text{in } \overline{\Omega} \setminus \mathcal{D}^\mathcal{N}_{\varepsilon/10}.$$

Since $\delta_1 > 0$ is fixed, depending on $(v_\infty, \gamma, \beta_*)$, we can choose N_1 sufficiently large and $\delta_2 > 0$ sufficiently small, depending only on $(v_\infty, \gamma, \beta_*, \delta_1, N_1)$, such that

$$\mathscr{K}_2(\beta) \leq \frac{\delta_1 \delta_2}{N_1^2} < \min\{\sigma, \hat{\sigma}\} \qquad \text{for all } \beta \in [0, \beta_*].$$

With the choices of (N_1, δ_2), φ satisfies (4.3.3)–(4.3.5) in Definition 4.19(iv).

In inequalities (4.3.6)–(4.3.11), parameters μ_0, $\mathcal{K}_3(\beta)$, N_4, N_5, and μ_1 are fixed so that any admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{\beta < \beta_*\}$ satisfies all the strict inequalities.

2-5. With the choices of (N_1, δ_2) determined in Step 2-4, we conclude that any fixed point of $\mathcal{I}(\cdot, \beta)$ for $\beta \in (0, \beta_*]$ lies in $\mathcal{K}(\beta)$. In the next step, we also show that no fixed point of $\mathcal{I}(\cdot, 0)$ lies on $\partial \mathcal{K}(0)$.

3. Let parameters $(\alpha, \varepsilon, \delta_1, \delta_3, N_1)$ in Definition 4.19 be fixed as described at the beginning of §5.3.2. Let N_1 be further adjusted, and let δ_2 be fixed as in Step 2 so that *Claim* 2 holds. Finally, let δ_3 be further adjusted to satisfy the conditions in Lemmas 4.42 and 4.43 as described at the beginning of §5.3.2. In particular, let δ_3 be adjusted to satisfy (4.5.4) given in the proof of Lemma 4.42. With these choices of parameters $(\alpha, \varepsilon, \delta_1, \delta_2, \delta_3, N_1)$, the definition for the iteration set $\mathcal{K}$ given in Definition 4.19 is now complete.

3-1. *Claim* 3: *The iteration map* $\mathcal{I}(\cdot, 0)$ *has a unique fixed point* 0 *with*

$$\mathbf{Ind}(\mathcal{I}(\cdot, 0), \mathcal{K}(0)) = 1.$$

At $\beta = 0$, it follows from (2.5.1) that $\varphi_{\mathcal{O}} - \varphi_{\mathcal{N}} \equiv 0$, so that the boundary condition on $\Gamma_{\text{sonic}}^{\mathcal{O}} \cup \Gamma_{\text{sonic}}^{\mathcal{N}}$ of the boundary value problem (4.3.16) associated with any $u \in \overline{\mathcal{K}}(0)$ becomes homogeneous. Then it follows from Lemmas 4.34(f) and 4.35 that, for any $u \in \overline{\mathcal{K}}(0)$, the associated boundary value problem (4.3.16) has a unique solution $\hat{\phi} = 0$ in $\Omega(u, 0)$. From this, we have

$$\mathcal{I}(u, 0) = 0 \qquad \text{for all } u \in \overline{\mathcal{K}}(0).$$

It can be directly checked from Definition 4.19 that the fixed point $u = 0$ of $\mathcal{I}(\cdot, 0)$ lies in $\mathcal{K}(0)$. Also, we have shown in Step 1 that $\mathcal{I} : \overline{\mathcal{K}} \to C_{(*, \alpha_1)}^{2, \alpha}$ is compact in the sense of Definition 5.9. Therefore, the fixed point index $\mathbf{Ind}(\mathcal{I}(\cdot, \beta), \mathcal{K}(\beta))$ satisfying properties (i)–(iv) stated in Theorem 5.12 is well defined. Then Theorem 5.12(i) implies that

$$(5.3.9) \qquad \qquad \mathbf{Ind}(\mathcal{I}(\cdot, 0), \mathcal{K}(0)) = 1.$$

3-2. Combining Claim 2 in Step 2 with Claim 3 in Step 3-1, we see that no fixed point of $\mathcal{I}(\cdot, \beta)$ lies on the boundary $\partial \mathcal{K}(\beta)$ of $\mathcal{K}(\beta)$ for all $\beta \in [0, \beta_*]$. Then, using (5.3.9) and properties (a) and (d) of Theorem 5.13, we have

$$(5.3.10) \qquad \mathbf{Ind}(\mathcal{I}(\cdot, \beta), \mathcal{K}(\beta)) = \mathbf{Ind}(\mathcal{I}(\cdot, 0), \mathcal{K}(0)) \qquad \text{for all } \beta \in [0, \beta_*].$$

By Theorem 5.12(ii), (5.3.10) implies that $\mathcal{I}(\cdot, \beta)$ has a fixed point in $\mathcal{K}(\beta)$ for all $\beta \in [0, \beta_*]$. Then Proposition 5.8 implies that, for each $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}} \cap \{0 \leq \beta \leq \beta_*\}$, an admissible solution corresponding to (v_∞, β) exists. Since $v_\infty > 0$ is arbitrary, and β_* is also arbitrary in $(0, \beta_{\text{d}}^{(v_\infty)})$, we finally conclude that there exists an admissible solution for any $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$. This completes the proof of Theorem 2.31. $\qquad \square$

Optimal Regularity of Admissible Solutions
– Proof of Theorem 2.33

This chapter is devoted to the complete proof of Theorem 2.33.

Let φ be an admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{weak}}$ in the sense of Definition 2.24. We now prove statements (a)–(e) of Theorem 2.33, respectively.

1. *Proof of statement* (a) *of Theorem* 2.33. It follows from Lemmas 3.9 and 3.18 that Γ_{shock} is C^∞ in its relative interior, and $\varphi \in C^\infty(\overline{\Omega} \setminus \overline{\Gamma^{\mathcal{O}}_{\text{sonic}}} \cup \overline{\Gamma^{\mathcal{N}}_{\text{sonic}}})$. By Definition 2.23, $\overline{\Gamma^{\mathcal{O}}_{\text{sonic}}}$ is a closed portion of a circle when $\beta < \beta_s^{(v_\infty)}$ and becomes a point P_β when $\beta \geq \beta_s^{(v_\infty)}$. Near $\Gamma^{\mathcal{N}}_{\text{sonic}}$, we combine Proposition 3.26 with the smoothness of φ away from $\overline{\Gamma^{\mathcal{O}}_{\text{sonic}}} \cup \overline{\Gamma^{\mathcal{N}}_{\text{sonic}}}$ to obtain $\varphi \in C^{1,1}(\overline{\Omega} \setminus \overline{\Gamma^{\mathcal{O}}_{\text{sonic}}})$.

Near $\Gamma^{\mathcal{O}}_{\text{sonic}}$, we consider two cases separately: (i) $\beta < \beta_s^{(v_\infty)}$ and (ii) $\beta \geq \beta_s^{(v_\infty)}$. If $\beta < \beta_s^{(v_\infty)}$, it follows from Propositions 3.30 and 3.32 that φ is $C^{1,1}$ up to $\overline{\Gamma^{\mathcal{O}}_{\text{sonic}}}$. If $\beta \geq \beta_s^{(v_\infty)}$, then Propositions 3.39 and 3.42 imply that φ is $C^{1,\alpha}$ up to $\overline{\Gamma^{\mathcal{O}}_{\text{sonic}}} = \{P_\beta\}$ for some $\alpha \in (0, 1)$. This completes the proof of statement (a).

2. *Proof of statements* (b)–(c) *of Theorem* 2.33. Let the (x, y)–coordinates be defined by (3.4.18) and (3.5.2) near $\Gamma^{\mathcal{N}}_{\text{sonic}}$ and $\Gamma^{\mathcal{O}}_{\text{sonic}}$, respectively. Define

$$\psi := \varphi - \max\{\varphi_\mathcal{N}, \varphi_\mathcal{O}\}$$

for $\varphi_\mathcal{O}$ and $\varphi_\mathcal{N}$ given by (2.5.1). Note that $\psi = \varphi - \varphi_\mathcal{N}$ near $\Gamma^{\mathcal{N}}_{\text{sonic}}$ and $\psi = \varphi - \varphi_\mathcal{O}$ near $\Gamma^{\mathcal{O}}_{\text{sonic}}$.

By (3.2.29), (3.4.21), (3.4.26), Lemma 3.21, and Proposition 3.26, we can apply the following theorem to ψ near $\Gamma^{\mathcal{N}}_{\text{sonic}}$:

THEOREM 6.1 (Theorem 3.1 in [**1**]). *For constants $r, R > 0$, define $Q^+_{r,R}$ by*

$$Q^+_{r,R} := \{(x, y) : x \in (0, r), \ |y| < R\}.$$

For positive constants a, b, M, N, and $\kappa \in (0, \frac{1}{4})$, suppose that $\psi \in C(\overline{Q^+_{r,R}}) \cap C^2(Q^+_{r,R})$ satisfies

$$(2x - a\psi_x + O_1)\psi_{xx} + O_2\psi_{xy} + (b + O_3)\psi_{xy} - (1 + O_4)\psi_x + O_5\psi_y = 0 \qquad in\ Q^+_{r,R},$$

$$\psi > 0 \qquad in\ Q^+_{r,R},$$

$$\psi = 0 \qquad on\ \partial Q^+_{r,R} \cap \{x = 0\},$$

$$-Mx \leq \psi_x \leq \frac{2 - \kappa}{a}x \qquad in\ Q^+_{r,R},$$

where terms $O_i(x, y)$, $i = 1, \cdots, 5$, are continuously differentiable and

$$(6.1) \qquad \frac{|O_1(x, y)|}{x^2} + \frac{|DO_1(x, y)|}{x^2} + \sum_{k=2}^{5}\left(\frac{|O_k(x, y)|}{x} + |DO_k(x, y)|\right) \leq N \qquad in\ Q^+_{r,R}.$$

Then

$$\psi \in C^{2,\alpha}(\overline{Q^+_{r/2,R/2}}) \qquad \text{for any } \alpha \in (0,1),$$

with

$$\psi_{xx}(0,y) = \frac{1}{a}, \quad \psi_{xy}(0,y) = \psi_{yy}(0,y) = 0 \qquad \text{for all } |y| < \frac{R}{2}.$$

For $\beta \in [0, \beta_s^{(v_\infty)})$, it can be directly checked from the results in §3.5.1 that Theorem 6.1 applies to ψ near $\Gamma_{\text{sonic}}^{\mathcal{O}}$. Then the admissible solution φ satisfies statements (b)–(c) of Theorem 2.33.

3. *Proof of statement* (d) *in Theorem* 2.33. By Lemma 3.21(d), $\Gamma_{\text{shock}} \cap \mathcal{D}^{\mathcal{N}}_{\bar{\varepsilon}}$ is represented as the graph of $y = \hat{f}_{\mathcal{N},\text{sh}}(x)$ for $0 \leq x \leq \bar{\varepsilon}$, where $\mathcal{D}^{\mathcal{N}}_{\bar{\varepsilon}}$ is defined by (4.1.2).

Let $\{y_m^{(1)}\}$ be a sequence satisfying $0 < y_m^{(1)} < \hat{f}_{\mathcal{N},\text{sh}}(0)$ for each $m \in \mathbb{N}$, and $\lim_{m\to\infty} y_m^{(1)} = \hat{f}_{\mathcal{N},\text{sh}}(0)$. By (2.5.30), (2.5.32), and Theorem 2.33(c), we can choose a sequence $\{x_m^{(1)}\}$ such that $\{(x_m^{(1)}, y_m^{(1)})\} \subset \Omega$, $x_m^{(1)} \in (0, \frac{1}{m})$, and

$$\left|\psi_{xx}(x_m^{(1)}, y_m^{(1)}) - \frac{1}{\gamma+1}\right| < \frac{1}{m} \qquad \text{for each } m \in \mathbb{N}.$$

By Lemma 3.21(d), $0 < y_m^{(1)} < \hat{f}_{\mathcal{N},\text{sh}}(0) < \hat{f}_{\mathcal{N},\text{sh}}(x_m^{(1)})$ for each $m \in \mathbb{N}$. Therefore, we have

$$(6.2) \qquad \lim_{m\to\infty} (x_m^{(1)}, y_m^{(1)}) = (0, \hat{f}_{\mathcal{N},\text{sh}}(0)), \qquad \lim_{m\to\infty} \psi_{xx}(x_m^{(1)}, y_m^{(1)}) = \frac{1}{\gamma+1}.$$

By properties (a) and (c) of Lemma 3.23, and Proposition 3.26, there exists $\varepsilon \in (0, \bar{\varepsilon}]$ such that, on $\Gamma_{\text{shock}} \cap \mathcal{D}^{\mathcal{N}}_\varepsilon$, the boundary condition (3.4.23) can be rewritten as

$$(6.3) \qquad \psi_x + b_1 \psi_y + b_0 \psi = 0 \qquad \text{on } \Gamma_{\text{shock}} \cap \mathcal{D}^{\mathcal{N}}_\varepsilon$$

for $(b_0, b_1) = (b_0, b_1)(\psi_x, \psi_y, \psi, x, \hat{f}_{\mathcal{N},\text{sh}}(x))$. Let $\omega > 0$ be from Lemma 3.21(d). Then

$$\left\{(x, \hat{f}_{\mathcal{N},\text{sh}}(x) - \frac{\omega}{10}x) \, : \, 0 < x < \varepsilon\right\} \subset \Omega.$$

Denote $\mathcal{F}(x) := \psi_x(x, \hat{f}_{\mathcal{N},\text{sh}}(x) - \frac{\omega}{10}x)$. By (6.3), we have

$$\mathcal{F}(x) = \psi_x(x, \hat{f}_{\mathcal{N},\text{sh}}(x)) - \frac{\omega}{10}x \int_0^1 \psi_{xy}(x, \hat{f}_{\mathcal{N},\text{sh}}(x) - \frac{t\omega}{10}x) \, dt$$

$$= -(b_1\psi_y + b_0\psi)(x, \hat{f}_{\mathcal{N},\text{sh}}(x)) - \frac{\omega}{10}x \int_0^1 \psi_{xy}(x, \hat{f}_{\mathcal{N},\text{sh}}(x) - \frac{t\omega}{10}x) \, dt \qquad \text{for } 0 < x < \varepsilon.$$

From the last equality and Proposition 3.26, we obtain that $\mathcal{F}(0) = 0$, $\mathcal{F} \in C([0,\varepsilon]) \cap C^1((0,\varepsilon))$, and $\lim_{x\to 0+} \frac{\mathcal{F}(x)}{x} = 0$. Then, by the mean value theorem, there exists a sequence $\{x_m^{(2)}\} \subset (0, \varepsilon)$ such that

$$(6.4) \qquad \lim_{m\to\infty} x_m^{(2)} = 0, \qquad \mathcal{F}'(x_m^{(2)}) = 0.$$

For each $m \in \mathbb{N}$, define $y_m^{(2)} := \hat{f}_{\mathcal{N},\text{sh}}(x_m^{(2)}) - \frac{\omega}{10}x_m^{(2)}$ so that $\{(x_m^{(2)}, y_m^{(2)})\} \subset \Omega$. By the definition of $\mathcal{F}$ and (6.4), we have

$$(6.5) \qquad \begin{aligned} \lim_{m\to\infty} \psi_{xx}(x_m^{(2)}, y_m^{(2)}) &= \lim_{m\to\infty} \mathcal{F}'(x_m^{(2)}) - \lim_{m\to\infty} (\hat{f}'_{\mathcal{N},\text{sh}}(x_m^{(2)}) - \frac{\omega}{10})\psi_{xy}(x_m^{(2)}, y_m^{(2)}) \\ &= -\lim_{m\to\infty} (\hat{f}'_{\mathcal{N},\text{sh}}(x_m^{(2)}) - \frac{\omega}{10})\psi_{xy}(x_m^{(2)}, y_m^{(2)}). \end{aligned}$$

Since $\lim_{m\to\infty}(x_m^{(2)}, y_m^{(2)}) = (0, \hat{f}_{\mathcal{N},\mathrm{sh}}(0))$, we combine (6.5) with Proposition 3.26 to obtain

$$(6.6) \qquad\qquad \lim_{m\to\infty} \psi_{xx}(x_m^{(2)}, y_m^{(2)}) = 0.$$

In (6.2) and (6.6), we have shown that there are two sequences, $\{(x_m^{(1)}, y_m^{(1)})\}$ and $\{(x_m^{(2)}, y_m^{(2)})\}$, in Ω such that the limits of both sequences are $(0, \hat{f}_{\mathcal{N},\mathrm{sh}}(0))$. On the other hand,

$$\lim_{m\to\infty} \psi_{xx}(x_m^{(1)}, y_m^{(2)}) \neq \lim_{m\to\infty} \psi_{xx}(x_m^{(1)}, y_m^{(2)}).$$

For $\beta \in (0, \beta_{\mathrm{s}}^{(v_\infty)})$, we can repeat the argument above by using Lemma 3.28(d) and Propositions 3.30 and 3.32 to show that there are two sequences, $\{(\tilde{x}_m^{(1)}, \tilde{y}_m^{(1)})\}$ and $\{(\tilde{x}_m^{(2)}, \tilde{y}_m^{(2)})\}$, in Ω such that the limits of both sequences are $(0, \hat{f}_{\mathcal{O},\mathrm{sh}}(0))$, but it can similarly be shown that

$$\lim_{m\to\infty} \psi_{xx}(\tilde{x}_m^{(1)}, \tilde{y}_m^{(1)}) = \frac{1}{\gamma+1} \neq 0 = \lim_{m\to\infty} \psi_{xx}(\tilde{x}_m^{(2)}, \tilde{y}_m^{(2)}),$$

where $\hat{f}_{\mathcal{O},\mathrm{sh}}$ is from Lemma 3.28. This proves statement (d) of Theorem 2.33.

3. *Proof of statement* (e) *of Theorem* 2.33. By Lemma 3.20(e), $S_{\mathcal{N}}$ is represented as the graph of $y = \hat{f}_{\mathcal{N},0}(x)$ near point P_2 in the (x, y)–coordinates given by (3.4.18). We extend the definition of $\hat{f}_{\mathcal{N},\mathrm{sh}}$ into $(-\bar{\varepsilon}, \bar{\varepsilon})$ by

$$(6.7) \qquad\qquad \hat{f}_{\mathcal{N},\mathrm{sh}}(x) = \hat{f}_{\mathcal{N},0}(x) \qquad \text{for } x \in (-\bar{\varepsilon}, 0].$$

By Proposition 3.26, $\hat{f}_{\mathcal{N},\mathrm{sh}}$ satisfies

$$(6.8) \qquad\qquad (\hat{f}_{\mathcal{N},\mathrm{sh}} - \hat{f}_{\mathcal{N},0})(0) = (\hat{f}_{\mathcal{N},\mathrm{sh}} - \hat{f}_{\mathcal{N},0})'(0) = 0,$$

so that curve $\overline{\Gamma_{\mathrm{shock}} \cup S_{\mathcal{N},\mathrm{seg}}}$ is $C^{1,1}$, including at point P_2.

Define

$$\phi_\infty^{\mathcal{N}} := \varphi_\infty - \varphi_{\mathcal{N}}.$$

Since $\phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},0}(x)) = 0$ and $(\varphi_\infty - \varphi)(x, \hat{f}_{\mathrm{sh}}(x)) = 0$, ψ satisfies

$$(6.9) \qquad \phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},0}(x)) - \phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},\mathrm{sh}}(x)) = \psi(x, \hat{f}_{\mathrm{sh}}(x)) \qquad \text{for } 0 < x < \bar{\varepsilon}.$$

A direct computation yields that

(6.10)

$$\frac{\mathrm{d}^2\phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},0}(x))}{\mathrm{d}x^2} = \hat{f}_{\mathcal{N},0}''(x)\partial_y\phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},0}(x)) + \sum_{k=0}^{2} a_k(\hat{f}_{\mathcal{N},0}'(x))^k \partial_x^{2-k}\partial_y^k \phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},0}(x)),$$

$$\frac{\mathrm{d}^2\phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},\mathrm{sh}}(x))}{\mathrm{d}x^2} = \hat{f}_{\mathcal{N},\mathrm{sh}}''(x)\partial_y\phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},\mathrm{sh}}(x)) + \sum_{k=0}^{2} a_k(\hat{f}_{\mathcal{N},\mathrm{sh}}'(x))^k \partial_x^{2-k}\partial_y^k \phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},\mathrm{sh}}(x)),$$

with $(a_0, a_1, a_2) = (1, 2, 1)$.

We differentiate (6.9) with respect to x twice and use (6.10) to obtain the following expression:

$$(6.11) \qquad\qquad (\hat{f}_{\mathcal{N},\mathrm{sh}} - \hat{f}_{\mathcal{N},0})''(x) = \frac{A_1(x) + A_2(x) + A_3(x)}{\partial_y\phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},\mathrm{sh}}(x))},$$

where

$$A_1(x) = \sum_{k=0}^{2} a_k \left((\hat{f}'_{\mathcal{N},0}(x))^k \partial_x^{2-k} \partial_y^k \phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},0}(x)) - (\hat{f}'_{\mathcal{N},\mathrm{sh}}(x))^k \partial_x^{2-k} \partial_y^k \phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},\mathrm{sh}}(x)) \right),$$

$$A_2(x) = \left(\partial_y \phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},0}(x)) - \partial_y \phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},\mathrm{sh}}(x)) \right) \hat{f}''_{\mathcal{N},0}(x),$$

$$A_3(x) = -\left(\hat{f}''_{\mathcal{N},\mathrm{sh}}(x)\psi_y(x, \hat{f}_{\mathcal{N},\mathrm{sh}}(x)) + \sum_{k=0}^{2} a_k(\hat{f}'_{\mathcal{N},\mathrm{sh}}(x))^k \partial_x^{2-k} \partial_y^k \psi(x, \hat{f}_{\mathcal{N},\mathrm{sh}}(x)) \right).$$

By (6.8), we have

(6.12)
$$A_1(0) = A_2(0) = 0.$$

We differentiate the boundary condition (3.4.23) in the tangential direction along Γ_{shock}, and apply Lemma 3.23(a)–(c) and Proposition 3.26 to obtain that there exists a constant $C > 0$ such that

$$|\psi_{xx}(x, \hat{f}_{\mathcal{N},\mathrm{sh}}(x))|$$
$$\leq C\left(|\psi(x, \hat{f}_{\mathcal{N},\mathrm{sh}}(x))| + |D_{(x,y)}\psi(x, \hat{f}_{\mathcal{N},\mathrm{sh}}(x))| + |D_{(x,y)}\psi_y(x, \hat{f}_{\mathcal{N},\mathrm{sh}}(x))| \right) \qquad \text{on } \Gamma_{\mathrm{shock}} \cap \mathcal{D}_{\bar{\varepsilon}}^{\mathcal{N}}.$$

From this estimate and Proposition 3.26, we see that $\lim_{x \to 0+} \psi_{xx}(x, \hat{f}_{\mathcal{N},\mathrm{sh}}(x)) = 0$, which implies that

(6.13)
$$\lim_{x \to 0+} A_3(x) = 0.$$

By Lemma 3.20(c), $\partial_y \phi_\infty^{\mathcal{N}}(x, \hat{f}_{\mathcal{N},\mathrm{sh}}(x)) \neq 0$ on $\Gamma_{\mathrm{shock}} \cap \mathcal{D}_{\bar{\varepsilon}}^{\mathcal{N}}$. Then we conclude from (6.11)–(6.13) that

$$(\hat{f}_{\mathcal{N},\mathrm{sh}} - \hat{f}_{\mathcal{N},0})''(0) = 0.$$

This implies that the extension of $\hat{f}_{\mathcal{N},\mathrm{sh}}$ given by (6.7) is in $C^2([-\bar{\varepsilon}, \bar{\varepsilon}])$. Furthermore, we conclude from (6.11) and Proposition 3.26 that the extension of $\hat{f}_{\mathcal{N},\mathrm{sh}}$ given by (6.7) is in $C^{2,\alpha}((-\bar{\varepsilon}, \bar{\varepsilon}))$ for any $\alpha \in (0,1)$. This implies that $\overline{\Gamma_{\mathrm{shock}} \cup S_{\mathcal{N},\mathrm{seg}}}$ is $C^{2,\alpha}$ for any $\alpha \in (0,1)$, including at point $P_2 = (0, \hat{f}_{\mathcal{N},\mathrm{sh}}(0))$. For $\beta \in (0, \beta_{\mathrm{s}}^{(v_\infty)})$, it can similarly be checked that $\overline{S_{\mathcal{O},\mathrm{seg}} \cup \Gamma_{\mathrm{shock}}}$ is $C^{2,\alpha}$ for any $\alpha \in (0,1)$, including at point $P_1 = (0, \hat{f}_{\mathcal{O},\mathrm{sh}}(0))$ for $\hat{f}_{\mathcal{O},\mathrm{sh}}$ from Lemma 3.28. Therefore, statement (e) of Theorem 2.33 is proved.

The Shock Polar for Steady Potential Flow

According to [22], for any given uniform supersonic state, a shock polar curve for the two-dimensional steady full Euler system should exist and be convex. In this appendix, we show the same for the potential flow. The convexity of the shock polar curve leads to Lemma A.4, which is the key ingredient for proving the existence of admissible solutions in the sense of Definition 2.14 for $(u_\infty, u_0) \in \mathfrak{P}_{\text{weak}}$ with $u_0 \leq u_s^{(\rho_\infty, u_\infty)}$, and the non-existence of admissible solutions for $(u_\infty, u_0) \in \mathfrak{P}_{\text{strong}}$. The existence of convex shock polar curves for potential flow is proved by combining the results from [24, 33].

The two-dimensional steady potential flow for an ideal polytropic gas is governed by the equations:

$$\begin{cases} (\rho u)_{x_1} + (\rho v)_{x_2} = 0, \\ u_{x_2} - v_{x_1} = 0, \\ \frac{1}{2}(u^2 + v^2) + i(\rho) = B_0 \qquad \text{(Bernoulli's law)} \end{cases}$$

for a constant $B_0 > 0$, where $i(\rho)$ is given by

$$i(\rho) = \begin{cases} \frac{\rho^{\gamma-1}-1}{\gamma-1} & \text{for } \gamma > 1, \\ \ln \rho & \text{for } \gamma = 1. \end{cases}$$

LEMMA A.1. *Fix $\gamma \geq 1$ and the incoming constant state $(\rho_\infty, \mathbf{u}_\infty) = (\rho_\infty, (u_\infty, 0))$, with $u_\infty > \rho_\infty^{(\gamma-1)/2} > 0$. Denote $M_\infty := \frac{u_\infty}{\rho_\infty^{(\gamma-1)/2}} > 1$ as the Mach number of the incoming supersonic flow. For each $\beta \in [0, \cos^{-1}(\frac{1}{M_\infty}))$, there exists a unique $\mathbf{u} = (u_\mathcal{O}, v_\mathcal{O}) \in (\mathbb{R}_+)^2 \setminus \{\mathbf{u}_\infty\}$ such that*

(A.1) $$\rho_\mathcal{O} \mathbf{u} \cdot \mathbf{n} = \rho_\infty \mathbf{u}_\infty \cdot \mathbf{n},$$

(A.2) $$(\mathbf{u}_\infty - \mathbf{u}) \cdot \mathbf{t} = 0,$$

(A.3) $$\frac{1}{2}(\mathbf{u} \cdot \mathbf{n})^2 + i(\rho_\mathcal{O}) = \frac{1}{2}(\mathbf{u}_\infty \cdot \mathbf{n})^2 + i(\rho_\infty)$$

for $\mathbf{n} = (\cos \beta, -\sin \beta)$ and $\mathbf{t} = (\sin \beta, \cos \beta)$, where $\rho_\mathcal{O}$ is given by

(A.4) $$\rho_\mathcal{O} = i^{-1}(i(\rho_\infty) + \frac{1}{2}(u_\infty^2 - |\mathbf{u}|^2)).$$

In other words, $\mathbf{u}$ becomes the downstream velocity behind a straight oblique shock $S_\mathcal{O}$ of angle $\frac{\pi}{2} - \beta$ from the horizontal axis. Moreover, the collection of such $\mathbf{u} = (u_\mathcal{O}, v_\mathcal{O})$ for $\beta \in [0, \cos^{-1}(\frac{1}{M_\infty}))$ forms a concave curve on the (u, v)–plane.

PROOF. The existence of the curve for $(u_\mathcal{O}, v_\mathcal{O})$ is verified by following the proof of [33, Proposition 2.1], and the convexity of this curve can be checked by adjusting the proof of [24, Theorem 1]. We prove the lemma for the case that $\gamma > 1$. The case that $\gamma = 1$ can be treated in the same way. The proof is divided into two steps.

1. *Existence of shock polar.* Fix constants $\gamma > 1$, $\rho_\infty > 0$, and u_∞ with $u_\infty > \rho_\infty^{(\gamma-1)/2}$. Let $S_{\mathcal{O}}$ be a straight oblique shock with angle $\frac{\pi}{2} - \beta$ from the horizontal ground, and let $\rho_{\mathcal{O}}$ and $\mathbf{u} = (u_{\mathcal{O}}, v_{\mathcal{O}})$ be the density and the velocity behind shock $S_{\mathcal{O}}$. By (A.2), the angle between vector $\mathbf{u} - \mathbf{u}_\infty$ and the horizontal axis in Fig. A.1 is β. By the expression of $\{\mathbf{n}, \mathbf{t}\}$, we have
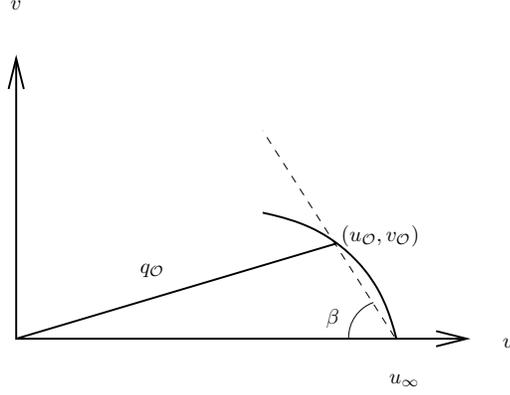
FIGURE A.1. The shock polar for potential flow

$$
\text{(A.5)} \qquad
\begin{aligned}
\mathbf{u}_\infty \cdot \mathbf{n} &= u_\infty \cos\beta, & \mathbf{u}_\infty \cdot \mathbf{t} &= u_\infty \sin\beta, \\
\mathbf{u} \cdot \mathbf{n} &= u_{\mathcal{O}} \cos\beta - v_{\mathcal{O}} \sin\beta, & \mathbf{u} \cdot \mathbf{t} &= u_{\mathcal{O}} \sin\beta + v_{\mathcal{O}} \cos\beta.
\end{aligned}
$$

Denote $M_{\infty,n} = \frac{\mathbf{u}_\infty \cdot \mathbf{n}}{\rho_\infty^{(\gamma-1)/2}}$. For each $\beta \in [0, \frac{\pi}{2})$, $M_{\infty,n}$ is fixed and $M_{\infty,n} > 0$ holds. It has been shown in the proof of Lemma 2.17 that there exists a unique M_n with $M_n \neq M_{\infty,n}$ as a solution of the equation:

$$
\text{(A.6)} \qquad \mathfrak{g}(M_n) = \mathfrak{g}(M_{\infty,n})
$$

for $\mathfrak{g}(M) = (M^2 + \frac{2}{\gamma-1})M^{-\frac{2(\gamma-1)}{\gamma+1}}$, unless $M_{\infty,n} = 1$. Substitute $\mathbf{u} \cdot \mathbf{n} = M_n \rho_{\mathcal{O}}^{\frac{\gamma-1}{2}}$ into (A.3) and solve the resultant equation for $\rho_{\mathcal{O}}$ to obtain

$$
\rho_{\mathcal{O}}^{\gamma-1} = \frac{(\mathbf{u}_\infty \cdot \mathbf{n})^2 + 2i(\rho_\infty) + \frac{2}{\gamma-1}}{M_n^2 + \frac{2}{\gamma-1}}.
$$

By the entropy condition, shock $S_{\mathcal{O}}$ is admissible only if $\rho_\infty < \rho_{\mathcal{O}}$, which is equivalent to saying that $0 < M_n < 1 < M_{\infty,n}$. Since $M_{\infty,n} = M_\infty \cos\beta$ for $M_\infty = \frac{u_\infty}{\rho_\infty^{(\gamma-1)/2}}$, we restrict our consideration only to the case that $\beta \in [0, \cos^{-1}(\frac{1}{M_\infty}))$. Then (A.2) and (A.5) yield that

$$
\begin{pmatrix} u_{\mathcal{O}} \\ v_{\mathcal{O}} \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} M_n \rho_{\mathcal{O}}^{\frac{\gamma-1}{2}} \\ u_\infty \sin\beta \end{pmatrix}.
$$

Therefore, curve $(u_{\mathcal{O}}, v_{\mathcal{O}})(\beta)$ is given for $\beta \in [0, \cos^{-1}(\frac{1}{M_\infty}))$ in the (u,v)–plane; see Fig. A.1.

Since $\lim_{\beta \to \cos^{-1}(\frac{1}{M_\infty})} M_n = 1 = \lim_{\beta \to \cos^{-1}(\frac{1}{M_\infty})} M_{\infty,n}$, the shock polar curve is extended up to $\beta = \cos^{-1}(\frac{1}{M_\infty})$ by $(u_{\mathcal{O}}, v_{\mathcal{O}}) = (u_\infty, 0)$.

This curve $(u,v) = (u_{\mathcal{O}}, v_{\mathcal{O}})(\beta)$ for $\beta \in [0, \cos^{-1}(\frac{1}{M_\infty})]$ is called a *shock polar for potential flow*.

2. *Convexity of shock polar.* Let $\mathbf{u} = (u, v)$ denote each point on the shock polar curve. By (A.1)–(A.2), each point $\mathbf{u}$ on the shock polar satisfies the equation:

$$(A.7) \qquad g(\mathbf{u}) = \left(\rho(|\mathbf{u}|^2)\mathbf{u} - \rho_\infty \mathbf{u}_\infty\right) \cdot \frac{\mathbf{u}_\infty - \mathbf{u}}{|\mathbf{u}_\infty - \mathbf{u}|} = 0$$

for $\mathbf{u}_\infty = (u_\infty, 0)$, where $\rho(|\mathbf{u}|^2)$ is given by (A.4) so that $D_{\mathbf{u}}\rho = -\frac{\mathbf{u}}{c^2}\rho$ for $c^2(|\mathbf{u}|^2) = \rho^{\gamma-1}(|\mathbf{u}|^2)$. Combining this with (A.7) gives that

$$(A.8) \qquad g_{\mathbf{u}} \cdot \mathbf{n} = \rho\left(1 - \left(\frac{\mathbf{u} \cdot \mathbf{n}}{c}\right)^2\right), \qquad g_{\mathbf{u}} \cdot \mathbf{t} = -(\mathbf{u}_\infty \cdot \mathbf{t})\left(\frac{\rho \mathbf{u} \cdot \mathbf{n}}{c^2} + \frac{\rho - \rho_\infty}{|\mathbf{u}_\infty - \mathbf{u}|}\right).$$

By the entropy condition, we have

$$(A.9) \qquad g_{\mathbf{u}} \cdot \mathbf{n} > 0.$$

Define

$$\mathbf{q} := \frac{g_{\mathbf{u}}}{g_{\mathbf{u}} \cdot \mathbf{n}},$$

and express $\mathbf{q}$ as $\mathbf{q} = \mathbf{n} + \frac{g_{\mathbf{u}} \cdot \mathbf{t}}{g_{\mathbf{u}} \cdot \mathbf{n}}\mathbf{t}$.

Claim: $\mathbf{q} \times \frac{d\mathbf{q}}{d\beta} < 0$ for all $\beta \in (0, \cos^{-1}(\frac{1}{M_\infty}))$.

Denote $A := -\frac{g_{\mathbf{u}} \cdot \mathbf{t}}{g_{\mathbf{u}} \cdot \mathbf{n}}$. Then $\frac{d\mathbf{q}}{d\beta} = -(1 + \frac{dA}{d\beta})\mathbf{t} - A\mathbf{n}$, which implies that

$$(A.10) \qquad \mathbf{q} \times \frac{d\mathbf{q}}{d\beta} = -\left(1 + A^2 + \frac{dA}{d\beta}\right).$$

By (A.1), (A.5), and (A.8), we can rewrite A as $A = \frac{u_\infty \sin\beta}{1 - M_n^2}\left(\frac{M_n}{c} + \frac{1}{u_\infty \cos\beta}\right)$ for $M_n := \frac{\mathbf{u} \cdot \mathbf{n}}{c}$. Differentiate (A.6) with respect to β to obtain

$$\frac{dM_n}{d\beta} = -\frac{\mathbf{g}'(M_{\infty,n})}{\mathbf{g}'(M_n)}\frac{u_\infty \sin\beta}{\rho_\infty^{(\gamma-1)/2}} > 0 \qquad \text{for } \beta \in (0, \cos^{-1}(\frac{1}{M_\infty})).$$

From $\rho^{\frac{\gamma+1}{2}}M_n = \rho_\infty^{\frac{\gamma+1}{2}}M_{\infty,n} = \rho_\infty u_\infty \cos\beta$ and $\frac{dM_n}{d\beta} > 0$, we see that $\frac{d\rho}{d\beta} < 0$ so that $\frac{dA}{d\beta} \geq 0$ holds for all $\beta \in (0, \cos^{-1}(\frac{1}{M_\infty}))$. Combining this with (A.10), we have

$$\mathbf{q} \times \frac{d\mathbf{q}}{d\beta} \leq -1 \qquad \text{for } \beta \in (0, \cos^{-1}(\frac{1}{M_\infty})).$$

The claim is verified.

The inequality above gives the useful property:

$$(A.11) \qquad \frac{\mathbf{q}}{|\mathbf{q}|} \times \frac{d}{d\beta}\left(\frac{\mathbf{q}}{|\mathbf{q}|}\right) = \frac{\mathbf{q} \times \frac{d\mathbf{q}}{d\beta}}{|\mathbf{q}|^2} \leq -\frac{1}{|\mathbf{q}|^2} < 0$$

at each point on the shock polar curve.

Fix a point $\mathbf{u}_0 = (u_0, v_0)$ on the shock polar $\{\mathbf{u} = (u, v) : g(\mathbf{u}) = 0\}$, and define $\mathbf{n}_0 = \frac{\mathbf{u}_0 - \mathbf{u}_\infty}{|\mathbf{u}_0 - \mathbf{u}_\infty|}$. We introduce a new coordinate system (s, t) so that the following properties hold in the new (s, t)–coordinates:

(i) $\mathbf{u}_0 = (0, 0)$, $\mathbf{n}_0 = (0, 1)$;
(ii) If $\boldsymbol{\tau}_0$ is the unit vector perpendicular to $\mathbf{n}_0$ and oriented to satisfy $\mathbf{u}_\infty \cdot \boldsymbol{\tau}_0 > 0$, then $\boldsymbol{\tau}_0 = (1, 0)$.

Define a function $\mathcal{G}(s, t)$ by

$$\mathcal{G}(s^{(\mathbf{u})}, t^{(\mathbf{u})}) = g(\mathbf{u}),$$

where $(s^{(\mathbf{u})}, t^{(\mathbf{u})})$ is the (s, t)–coordinates of $\mathbf{u}$ on the shock polar. Since the value of $g_{\mathbf{u}} \cdot \mathbf{n}$ for $\mathbf{n} = \frac{\mathbf{u}_\infty - \mathbf{u}}{|\mathbf{u}_\infty - \mathbf{u}|}$ is invariant under the rotation, $\mathcal{G}_t(0, 0) = -(g_{\mathbf{u}} \cdot \mathbf{n})(\mathbf{u}_0) < 0$. By the implicit function theorem, there exists a function $f_{\mathbf{u}_0} : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ for some small constant $\varepsilon_0 > 0$ so that the shock polar curve is represented by $t = f_{\mathbf{u}_0}(s)$ near $\mathbf{u}_0$ in the (s, t)–coordinates. Such a function $f_{\mathbf{u}_0}$ satisfies the relation:

$$\frac{f_{\mathbf{u}_0}''(0)}{\sqrt{1 + (f_{\mathbf{u}_0}'(0))^2}} = \frac{\mathbf{q}}{|\mathbf{q}|} \times \frac{d}{d\beta}\left(\frac{\mathbf{q}}{|\mathbf{q}|}\right)\Big|_{\mathbf{u}=\mathbf{u}_0} \leq -\frac{1}{|\mathbf{q}(\mathbf{u}_0)|^2} < 0.$$

Therefore, we conclude that the shock polar for potential flow is concave.   □

REMARK A.2. Fix $\gamma \geq 1$ and (ρ_∞, u_∞) with $u_\infty > \rho_\infty^{(\gamma-1)/2} > 0$. Let $\Upsilon^{(\rho_\infty, u_\infty)}$ be the shock polar curve lying in the first quadrant in the (u, v)–plane for the steady potential flow with the incoming supersonic state (ρ_∞, u_∞). Owing to the concavity of the shock polar, there exists a unique $\theta_d^{(\rho_\infty, u_\infty)} \in (0, \frac{\pi}{2})$ such that the following properties hold:

(i) If $0 \leq \theta_w < \theta_d^{(\rho_\infty, u_\infty)}$, then line $\frac{v}{u} = \tan\theta_w$ intersects with $\Upsilon^{(\rho_\infty, u_\infty)}$ at two distinct points;

(ii) Line $\frac{v}{u} = \tan\theta_d^{(\rho_\infty, u_\infty)}$ and $\Upsilon^{(\rho_\infty, u_\infty)}$ have a unique intersection point so that $\frac{v}{u} = \tan\theta_d^{(\rho_\infty, u_\infty)}$ is tangential to $\Upsilon^{(\rho_\infty, u_\infty)}$ at the intersection point;

(iii) If $\theta_d^{(\rho_\infty, u_\infty)} < \theta_w < \frac{\pi}{2}$, then line $\frac{v}{u} = \tan\theta_w$ never intersects with $\Upsilon^{(\rho_\infty, u_\infty)}$.

LEMMA A.3. Fix $\gamma \geq 1$. For each (ρ_∞, u_∞) with $u_\infty > \rho_\infty^{(\gamma-1)/2} > 0$, there exist a unique constant $\hat{u}_0^{(\rho_\infty, u_\infty)} =: \hat{u}_0 \in (0, u_\infty)$ and a unique smooth function $f_{\text{polar}} \in C^0([\hat{u}_0, u_\infty]) \cap C^\infty((\hat{u}_0, u_\infty))$ such that

(A.12) $\Upsilon^{(\rho_\infty, u_\infty)} = \{(u, f_{\text{polar}}(u)) : u \in [\hat{u}_0, u_\infty]\}.$

Furthermore, the following properties hold:

(a) Let $\theta_s^{(\rho_\infty, u_\infty)}$ be from Lemma 2.4(c). Then there exist unique $u_d, u_s \in (\hat{u}_0, u_\infty)$ such that

(A.13) $\frac{f_{\text{polar}}(u_s)}{u_s} = \tan\theta_s^{(\rho_\infty, u_\infty)}, \qquad \frac{f_{\text{polar}}(u_d)}{u_d} = \tan\theta_d^{(\rho_\infty, u_\infty)}.$

Moreover, $u_d < u_s$ holds, and (u_d, u_s) vary continuously on (ρ_∞, u_∞).

(b) Denote by $f_{\text{polar}}(\cdot, \rho_\infty, u_\infty)$ the shock polar function $f_{\text{polar}}(\cdot)$ for the incoming flow (ρ_∞, u_∞). Then f_{polar} as a function of $(u, \rho_\infty, u_\infty)$ is C^∞ on the domain:

$$\{(u, \rho_\infty, u_\infty) : \rho_\infty > 0, \ u_\infty > \rho_\infty^{(\gamma-1)/2}, \ u \in (\hat{u}^{(\rho_\infty, u_\infty)}, u_\infty)\}.$$

PROOF. The proof is divided into four steps.

1. For each $\beta \in [0, \cos^{-1}(\frac{1}{M_\infty})]$, let $(\rho_\mathcal{O}, u_\mathcal{O}, v_\mathcal{O})$ be from Lemma A.1, and let $q_\mathcal{O} := \sqrt{u_\mathcal{O}^2 + v_\mathcal{O}^2}$. Since $(\rho_\mathcal{O}, u_\mathcal{O}, v_\mathcal{O})$ is uniquely determined for $\beta \in [0, \cos^{-1}(\frac{1}{M_\infty})]$, $q_\mathcal{O}$ is considered as a function of β. Substituting (A.5) into (A.1)–(A.2), we obtain

$$(u_\mathcal{O}, v_\mathcal{O}) = u_\infty(1 - (1 - \frac{\rho_\infty}{\rho_\mathcal{O}})\cos^2\beta, \ (1 - \frac{\rho_\infty}{\rho_\mathcal{O}})\cos\beta\sin\beta),$$

so that

$$(A.14) \qquad \cos^2 \beta = \frac{1 - (\frac{q_{\mathcal{O}}}{u_\infty})^2}{1 - (\frac{\rho_\infty}{\rho_{\mathcal{O}}})^2} =: h(q_{\mathcal{O}}).$$

It follows from (A.4) and (A.14) that

$$h'(q_{\mathcal{O}}) =: \frac{2q_{\mathcal{O}}}{(1 - \frac{\rho_\infty^2}{\rho_{\mathcal{O}}^2})^2 \rho_{\mathcal{O}}^2 c_{\mathcal{O}}^2 u_\infty^2} I(q_{\mathcal{O}})$$

for $I(q_{\mathcal{O}})$ satisfying $I(u_\infty) = 0$ and $I'(q_{\mathcal{O}}) = (\gamma+1)q_{\mathcal{O}}(\rho_{\mathcal{O}}^2 - \rho_\infty^2)$. Inequality $\rho_{\mathcal{O}} > \rho_\infty$ holds, owing to the entropy condition for the admissible shock so that $I'(q_{\mathcal{O}}) > 0$ and $I(q_{\mathcal{O}}) < I(u_\infty) = 0$ for $0 < q_{\mathcal{O}} < u_\infty$, which implies that $h'(q_{\mathcal{O}}) < 0$ for $0 < q_{\mathcal{O}} < u_\infty$. Then (A.14) yields that

$$(A.15) \qquad \frac{\mathrm{d}q_{\mathcal{O}}}{\mathrm{d}\beta} = -\frac{2 \cos \beta \sin \beta}{h'(q_{\mathcal{O}})} > 0 \qquad \text{for all } \beta \in (0, \cos^{-1} \frac{1}{M_\infty}).$$

2. Let $g(\mathbf{u}), \mathbf{n}$, and $\mathbf{t}$ be given by (A.7). Then (A.8) implies

$$\partial_v g(\mathbf{u}) = -(g_{\mathbf{u}} \cdot \mathbf{n}) \sin \beta + (g_{\mathbf{u}} \cdot \mathbf{t}) \cos \beta < 0$$

for any interior point $\mathbf{u} = (u, v)$ in $\Upsilon^{(\rho_\infty, u_\infty)}$. By the implicit function theorem, there exists a unique function $f_{\mathrm{polar}} : [\hat{u}_0, u_\infty] \to [0, \infty)$ so that (A.12) holds, where $\hat{u}_0 = q_{\mathcal{O}}(\beta)|_{\beta=0}$ for $q_{\mathcal{O}}$ defined through (A.14). The smoothness of mapping $(u, \rho_\infty, u_\infty) \mapsto f_{\mathrm{polar}}(u, \rho_\infty, u_\infty)$ follows from the implicit function theorem and the smooth dependence of $g(\mathbf{u})$ on (ρ_∞, u_∞).

3. The existence and uniqueness of $u_{\mathrm{d}} \in (\hat{u}_0, u_\infty)$ result directly from the concavity of the shock polar curve $\Upsilon^{(\rho_\infty, u_\infty)}$. Since point $(\hat{u}_0, 0)$ on the shock polar $\Upsilon^{(\rho_\infty, u_\infty)}$ corresponds to a normal shock, $(\hat{u}_0, 0)$ is subsonic; that is, $\rho_{\mathcal{O}}^{\gamma-1} - q_{\mathcal{O}}^2 > 0$ holds at $\beta = 0$. At $\beta = \cos^{-1}(\frac{1}{M_\infty})$, $\rho_{\mathcal{O}}^{\gamma-1} - q_{\mathcal{O}}^2 < 0$, because $(\rho_{\mathcal{O}}, q_{\mathcal{O}})|_{\beta=\cos^{-1}(\frac{1}{M_\infty})} = (\rho_\infty, u_\infty)$. From (A.15) and Bernoulli's law that $\frac{1}{2}q_{\mathcal{O}}^2 + \rho_\infty(\rho_{\mathcal{O}}) = B_0$, we have

$$\frac{\mathrm{d}(\rho_{\mathcal{O}}^{\gamma-1} - q_{\mathcal{O}}^2)}{\mathrm{d}\beta} < 0 \qquad \text{for all } \beta \in (0, \cos^{-1}(\frac{1}{M_\infty})).$$

Therefore, there exists a unique $u_{\mathrm{s}} \in (\hat{u}_0, u_\infty)$ such that $\frac{f_{\mathrm{polar}}(u_{\mathrm{s}})}{u_{\mathrm{s}}} = \tan \theta_{\mathrm{s}}^{(\rho_\infty, u_\infty)}$ holds. Furthermore, Lemma 2.4(c) and the concavity of $\Upsilon^{(\rho_\infty, u_{\mathcal{O}})}$ imply that $u_{\mathrm{d}} < u_{\mathrm{s}}$.

4. By Bernoulli's law and the concavity of $\Upsilon^{(\rho_\infty, u_\infty)}$, (A.13) is equivalent to

$$(A.16) \qquad \begin{aligned} u_{\mathrm{s}}^2 + f_{\mathrm{polar}}^2(u_{\mathrm{s}}, \rho_\infty, u_\infty) &= \frac{2(\gamma-1)}{\gamma+1}\left(\frac{1}{2}u_\infty^2 + \frac{\rho_\infty^{\gamma-1}}{\gamma-1}\right), \\ f_{\mathrm{polar}}(u_{\mathrm{d}}, \rho_\infty, u_\infty) - u_{\mathrm{d}} f_{\mathrm{polar}}'(u_{\mathrm{d}}, \rho_\infty, u_\infty) &= 0 \end{aligned}$$

for each (ρ_∞, u_∞) with $u_\infty > \rho_\infty^{(\gamma-1)/2} > 0$.

For each $k \in \mathbb{N}$, let a sequence $\{(\rho_\infty^{(k)}, u_\infty^{(k)})\}$ satisfy $u_\infty^{(k)} > (\rho_\infty^{(k)})^{(\gamma-1)/2} > 0$. Also, suppose that $\{(\rho_\infty^{(k)}, u_\infty^{(k)})\}$ converges to $(\rho_\infty^*, u_\infty^*)$ with $u_\infty^* > (\rho_\infty^*)^{(\gamma-1)/2} > 0$. Let $(u_{\mathrm{d}}^{(k)}, u_{\mathrm{s}}^{(k)})$ and $(u_{\mathrm{d}}^*, u_{\mathrm{s}}^*)$ be the values of $(u_{\mathrm{d}}, u_{\mathrm{s}})$ corresponding to $(\rho_\infty^{(k)}, u_\infty^{(k)})$ and $(\rho_\infty^*, u_\infty^*)$, respectively. Note that

$$(u_{\mathrm{d}}^{(k)}, \rho_\infty^{(k)}, u_\infty^{(k)}), (u_{\mathrm{s}}^{(k)}, \rho_\infty^{(k)}, u_\infty^{(k)}) \in \{(u, \rho_\infty, u_\infty) : \rho_\infty > 0, u_\infty > \rho_\infty^{(\gamma-1)/2}, u \in (\hat{u}^{(\rho_\infty, u_\infty)}, u_\infty)\}$$

for each $k \in \mathbb{N}$ and that $\hat{u}_0$ varies continuously on (ρ_∞, u_∞) so that $\{(u_{\mathrm{d}}^{(k)}, u_{\mathrm{s}}^{(k)})\}$ is bounded in $(\mathbb{R}^+)^2$. Therefore, there exist a convergent subsequence $\{(u_{\mathrm{d}}^{(k_j)}, u_{\mathrm{s}}^{(k_j)})\}$ and states $(u_{\mathrm{d}}^\sharp, u_{\mathrm{s}}^\sharp)$ such that $\lim_{j\to\infty}(u_{\mathrm{d}}^{(k_j)}, u_{\mathrm{s}}^{(k_j)}) = (u_{\mathrm{d}}^\sharp, u_{\mathrm{s}}^\sharp)$. Then assertion (b) (proved in Step 2) and (A.16) yield

$$(u_{\mathrm{s}}^\sharp)^2 + f_{\mathrm{polar}}^2(u_{\mathrm{s}}^\sharp, \rho_\infty^*, u_\infty^*) = \frac{2(\gamma-1)}{\gamma+1}\left(\frac{1}{2}(u_\infty^*)^2 + \frac{(\rho_\infty^*)^{\gamma-1}}{\gamma-1}\right),$$

$$f_{\mathrm{polar}}(u_{\mathrm{d}}^\sharp, \rho_\infty^*, u_\infty^*) - u_{\mathrm{d}}^\sharp f_{\mathrm{polar}}'(u_{\mathrm{d}}^\sharp, \rho_\infty^*, u_\infty^*) = 0.$$

This implies that $(u_{\mathrm{d}}^\sharp, u_{\mathrm{s}}^\sharp) = (u_{\mathrm{d}}^*, u_{\mathrm{s}}^*)$, since it has been shown in Step 3 that $(u_{\mathrm{d}}, u_{\mathrm{s}})$ satisfying (A.13) for $(\rho_\infty^*, u_\infty^*)$ uniquely exists. Therefore, we conclude that $(u_{\mathrm{d}}, u_{\mathrm{s}})$ varies continuously on (ρ_∞, u_∞). $\qquad\square$

In Lemma 2.19, the one-to-one correspondence between parameter sets $\mathfrak{P}$ and $\mathfrak{R}$ is established. For each $(u_\infty, u_0) \in \mathfrak{P}$, there exists a unique $\theta_{\mathrm{w}} \in (0, \frac{\pi}{2})$ such that v_∞ is given by (2.4.23), where $(v_\infty, \beta) \in \mathfrak{R}$ corresponds to (u_∞, u_0). The convexity of the shock polar obtained in Lemma A.3 yields the following property:

LEMMA A.4. *Fix $\gamma \geq 1$ and $v_\infty > 0$. For each $\beta \in (0, \frac{\pi}{2})$, let $\varphi_\infty, \varphi_{\mathcal{O}}, \rho_{\mathcal{O}}$, and P_β be defined by (2.4.1), (2.4.4), (2.4.5), (2.5.3), respectively. Denote $G(\mathbf{p}, z, \boldsymbol{\xi}) = g^{\mathrm{sh}}(\mathbf{p}, z, \boldsymbol{\xi})$ for $g^{\mathrm{sh}}(\mathbf{p}, z, \boldsymbol{\xi})$ defined by (3.4.13). Then there exists $\beta_{\mathrm{d}}^{(v_\infty)} \in (0, \frac{\pi}{2})$ depending only on (v_∞, γ) such that $G(\mathbf{p}, z, \boldsymbol{\xi})$ satisfies*

$$(A.17) \qquad G_{p_1}(D\varphi_{\mathcal{O}}, \varphi_\infty, P_\beta) \begin{cases} < 0 & \text{for } \beta \in (0, \beta_{\mathrm{d}}^{(v_\infty)}), \\ = 0 & \text{for } \beta = \beta_{\mathrm{d}}^{(v_\infty)}, \\ > 0 & \text{for } \beta \in (\beta_{\mathrm{d}}^{(v_\infty)}, \frac{\pi}{2}). \end{cases}$$

PROOF. The following facts are useful to compute $G_{q_1}(D\varphi_{\mathcal{O}}, \varphi_\infty, P_\beta)$:

  (i) The unit normal vector $\boldsymbol{n}_{\mathcal{O}}$ to $S_{\mathcal{O}}$ towards the downstream is $\boldsymbol{n}_{\mathcal{O}} = \frac{D\varphi_\infty - D\varphi_{\mathcal{O}}}{|D\varphi_\infty - D\varphi_{\mathcal{O}}|} = (\sin\beta, -\cos\beta)$ so that $(\rho_{\mathcal{O}} D\varphi_{\mathcal{O}} - D\varphi_\infty) \cdot (1, 0) = (\rho_{\mathcal{O}} - 1)(u_{\mathcal{O}} - \xi_1)\cos^2\beta$, where $D\varphi_{\mathcal{O}}$ and $D\varphi_\infty$ are evaluated at $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$.

  (ii) It is shown from a direct computation that, if $G(\mathbf{p}, z, \boldsymbol{\xi}) = 0$, then

$$(A.18) \qquad G_{\mathbf{p}}(\mathbf{p}, z, \boldsymbol{\xi}) = \frac{1}{\rho^{\gamma-2}}\left(c^2 \frac{D\varphi_\infty - \mathbf{p}}{|D\varphi_\infty - \mathbf{p}|} - \left(\mathbf{p} \cdot \frac{D\varphi_\infty - \mathbf{p}}{|D\varphi_\infty - \mathbf{p}|}\right)\mathbf{p}\right) - \frac{\rho\mathbf{p} - D\varphi_\infty}{|D\varphi_\infty - \mathbf{p}|}$$

  for $\rho = \rho(\mathbf{p}, z)$.

It follows from (i)–(ii) that

$$(A.19) \qquad G_{p_1}(D\varphi_{\mathcal{O}}, \varphi_\infty, P_\beta) = \left(c_{\mathcal{O}}^2 - (u_{\mathcal{O}} - \xi_1^{P_\beta})^2\right)\frac{\sin\beta}{\rho_{\mathcal{O}}^{\gamma-1}} - \frac{(\rho_{\mathcal{O}} - 1)(u_{\mathcal{O}} - \xi_1^{P_\beta})\cos^2\beta}{\sqrt{u_{\mathcal{O}}^2 + v_\infty^2}}$$

for $c_{\mathcal{O}} = \rho_{\mathcal{O}}^{(\gamma-1)/2}$. Denote $q_{\mathcal{O}} := D\varphi_{\mathcal{O}}(P_\beta) \cdot \boldsymbol{n}_{\mathcal{O}}$. Then $u_{\mathcal{O}} - \xi_1^{P_\beta} = q_{\mathcal{O}}\csc\beta$, where P_β is denoted as $P_\beta = (\xi_1^{P_\beta}, 0)$. Also, ξ_2^m in the proof of Lemma 2.22 can be written as $\xi_2^m = q_{\mathcal{O}}\cos\beta$. Substituting these two expressions into (A.19) and using the relations that $u_{\mathcal{O}} = -v_\infty\tan\beta$ and $\frac{(\rho_{\mathcal{O}} - 1)q_{\mathcal{O}}}{v_\infty\sec\beta} = 1$ obtained from (2.4.1), (2.4.3), and (2.4.29), we have

$$G_{p_1}(D\varphi_{\mathcal{O}}, \varphi_\infty, P_\beta) = \rho_{\mathcal{O}}(1 - M_{\mathcal{O}}^2)\sin\beta - \frac{(\xi_2^m)^2}{\rho_{\mathcal{O}}^{\gamma-2}}\csc\beta - \frac{\cos\beta}{\tan\beta},$$

where $M_{\mathcal{O}}$ is defined by (2.4.6) with $c = c_{\mathcal{O}}$. Then it can be directly checked that $\frac{dG_{p_1}(D\varphi_{\mathcal{O}}, \varphi_\infty, P_\beta)}{d\beta} > 0$ for all $0 < \beta < \frac{\pi}{2}$.

It follows from $\lim\limits_{\beta \to 0+} (\rho_{\mathcal{O}}, \xi_2^m) = (\rho_{\mathcal{N}}, \xi_2^{\mathcal{N}})$ that $\lim_{\beta \to 0+} G_{p_1}(D\varphi_{\mathcal{O}}, \varphi_\infty, P_\beta) = -\infty$.

Relations (2.4.11) and (2.4.35) yield $\xi_2^m = q_{\mathcal{O}} \cos \beta$, which gives that

$$G_{p_1}(D\varphi_{\mathcal{O}}, \varphi_\infty, P_\beta) = \rho_{\mathcal{O}}\big((1 - M_{\mathcal{O}}^2)\sin\beta - M_{\mathcal{O}}^2 \cos\beta^2 \csc\beta\big) - \frac{\cos\beta}{\tan\beta}.$$

It is shown in the proof of Lemma 2.22 that $\lim_{\beta \to \frac{\pi}{2}-} c_{\mathcal{O}} = \infty$ and $\frac{dM_{\mathcal{O}}}{d\beta} < 0$ for all $0 < \beta < \frac{\pi}{2}$. This implies that $\lim_{\beta \to \frac{\pi}{2}-} G_{p_1}(D\varphi_{\mathcal{O}}, \varphi_\infty, P_\beta) = \infty$. Therefore, there exists a unique $\beta_{\mathrm{d}}^{(v_\infty)} \in (0, \frac{\pi}{2})$ satisfying (A.17). □

APPENDIX B

Non-Existence of Self-Similar Strong Shock Solutions

For the completeness of this monograph, we include the proof of the non-existence of admissible solutions corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{strong}}$ in the sense of Definition 2.24, or equivalently, the non-existence of admissible solutions corresponding to $(u_\infty, u_0) \in \mathfrak{P}_{\text{strong}}$ in the sense of Definition 2.14. The non-existence of self-similar strong shock solutions was first studied in Elling [**25**]. In this appendix, we combine the convexity of the shock polar shown in Lemma A.1 for steady potential flow with the result from [**25**] to show the non-existence of admissible solutions corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{strong}}$.

PROPOSITION (Non-existence of admissible solutions with a strong shock). For each $\gamma \geq 1$, there is no admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{strong}}$ in the sense of Definition 2.24. Equivalently, there is no admissible solution corresponding to $(u_\infty, u_0) \in \mathfrak{P}_{\text{strong}}$.

PROOF. The proof is divided into six steps.

1. On the contrary, suppose that there is an admissible solution φ for some $(v_\infty, \beta) \in \mathfrak{R}_{\text{strong}}$ in the sense of Definition 2.24. Then $\psi := \varphi - \varphi_\mathcal{O} \in C^3(\overline{\Omega} \setminus (\overline{\Gamma^\mathcal{O}_{\text{sonic}}} \cup \overline{\Gamma^\mathcal{N}_{\text{sonic}}}) \cap C^1(\overline{\Omega})$ satisfies

(B.1) $$(c^2 - \varphi^2_{\xi_1})\psi_{\xi_1\xi_1} - 2\varphi_{\xi_1}\varphi_{\xi_2}\psi_{\xi_1\xi_2} + (c^2 - \varphi^2_{\xi_2})\psi_{\xi_2\xi_2} = 0 \qquad \text{in } \Omega,$$

(B.2) $$\psi = \varphi_\infty - \varphi_\mathcal{O}, \quad \mathfrak{g}(D\psi, \psi, \boldsymbol{\xi}) = 0 \qquad \text{on } \Gamma_{\text{shock}},$$

(B.3) $$\psi = |D\psi| = 0 \quad \text{on } \Gamma^\mathcal{O}_{\text{sonic}}, \qquad \psi = \varphi_\mathcal{N} - \varphi_\mathcal{O} \qquad \text{on } \Gamma^\mathcal{N}_{\text{sonic}},$$

(B.4) $$\partial_{\xi_2}\psi = 0 \qquad \text{on } \Gamma_{\text{wedge}}$$

for $c^2 = c^2(|D\varphi|^2, \varphi)$ and $\Gamma^\mathcal{O}_{\text{sonic}} = \{P_\beta\}$ by (2.5.6), where

$$\mathfrak{g}(\mathbf{q}, z, \boldsymbol{\xi}) := G(D\varphi_\mathcal{O}(\boldsymbol{\xi}) + \mathbf{q}, \varphi_\mathcal{O}(\boldsymbol{\xi}) + z, \boldsymbol{\xi}),$$

$$G(\mathbf{q}, z, \boldsymbol{\xi}) := \big(\rho(\mathbf{q}, z)\mathbf{q} - D\varphi_\infty(\boldsymbol{\xi})\big) \cdot \frac{D\varphi_\infty(\boldsymbol{\xi}) - \mathbf{q}}{|D\varphi_\infty(\boldsymbol{\xi}) - \mathbf{q}|},$$

(B.5) $$\rho(\mathbf{q}, z) := \begin{cases} \big(1 + (\gamma - 1)(\frac{1}{2}v^2_\infty - \frac{1}{2}|\mathbf{q}|^2 - z)\big)^{\frac{1}{\gamma-1}} & \text{for } \gamma > 1, \\ \exp\big(\frac{v^2_\infty}{2} - \frac{1}{2}|\mathbf{q}|^2 - z\big) & \text{for } \gamma = 1, \end{cases}$$

$$c^2(|\mathbf{q}|^2, z) = \rho^{\gamma-1}(|\mathbf{q}|^2, z),$$

for $\mathbf{q} \in \mathbb{R}^2, z \in \mathbb{R}$, and $\boldsymbol{\xi} \in \overline{\Omega}$.

2. *Claim*: ψ attains its minimum at P_β.

Since (B.3), combined with Remark 2.35, implies that ψ is not a constant in $\overline{\Omega}$, then the minimum of ψ over $\overline{\Omega}$ is attained on $\partial\Omega$ by the strong maximum principle. Also, ψ cannot attain its minimum over $\overline{\Omega}$ on Γ_{wedge} by Hopf's lemma. The proof of Proposition 3.4 applies to φ such that Γ_{shock} lies strictly below $S_\mathcal{O}$, and $\psi > 0$ on Γ_{shock}. Therefore, we conclude that $\min_{\overline{\Omega}} \psi = \psi(P_\beta) = 0$.

3. Divide equation (B.1) by $c^2(|D\varphi|^2, \varphi)$ to rewrite (B.1) as

$$\mathcal{L}\psi := \left(1 - \frac{|D\varphi_\mathcal{O}(P_\beta)|^2}{c_\mathcal{O}^2} + \mathcal{O}_{11}(\boldsymbol{\xi})\right)\psi_{\xi_1\xi_1} + 2\mathcal{O}_{12}(\boldsymbol{\xi})\psi_{\xi_1\xi_2} + \left(1 + O_{22}(\boldsymbol{\xi})\right)\psi_{\xi_2\xi_2} = 0 \qquad \text{in } \Omega$$

for $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \Omega$, where each $\mathcal{O}_{ij} = \mathcal{O}_{ij}(D\varphi, \varphi)$ satisfies that $\lim_{\boldsymbol{\xi} \to P_\beta} |\mathcal{O}_{ij}(\boldsymbol{\xi})| = 0$ for $i, j = 1, 2$. Define $k := \frac{1}{\sqrt{1 - |D\varphi_\mathcal{O}(P_\beta)|^2/c_\mathcal{O}^2}}$ and $\tilde{\xi}_1 := k(\xi_1 - \xi_1^{P_\beta})$. Let (r, θ) be the polar coordinates of $(\tilde{\xi}_1, \xi_2)$ centered at P_β. Then $\Omega \subset \{r > 0, 0 < \theta < \tilde{\beta}\}$ for $\tan \tilde{\beta} = \frac{\tan \beta}{k}$.

Next, define

(B.6) $$\Psi(r, \theta) := \varepsilon r \cos(\omega_0 \theta)$$

for constants $\varepsilon, \omega_0 > 0$ to be determined later. As in [**25**], choose $\varepsilon > 0$ small and $\omega_0 \in (0, 1)$ close to 1. A direct computation by using the definition of (r, θ) shows that

(B.7) $$\mathcal{L}\Psi = \frac{\varepsilon}{r}(1 - \omega_0^2)\left(\cos(\omega_0\theta) + \mathcal{O}_1^{(\text{polar})}(r, \theta)\right) \qquad \text{in } \Omega,$$

with $\lim_{r \to 0+} |\mathcal{O}_1^{(\text{polar})}(r, \theta)| = 0$.

A direct computation by using (A.18) and Lemma A.4 gives that

$$\mathfrak{g}_{\mathbf{q}}(\mathbf{0}, 0, P_\beta) \cdot (\cos\beta, \sin\beta) < 0 < \mathfrak{g}_{\mathbf{q}}(\mathbf{0}, 0, P_\beta) \cdot (1, 0).$$

Therefore, there exists $\theta_0 \in (-\frac{\pi}{2}, -\frac{\pi}{2} + \beta)$ satisfying that $\frac{\mathfrak{g}_{\mathbf{q}}(\mathbf{0},0,P_\beta)}{|\mathfrak{g}_{\mathbf{q}}(\mathbf{0},0,P_\beta)|} = (\cos\theta_0, \sin\theta_0)$. Then it can be directly checked that

(B.8) $$\mathfrak{g}_{\mathbf{q}}(\mathbf{0}, 0, P_\beta) \cdot D_{\boldsymbol{\xi}}\Psi(r, \theta) = \varepsilon\left(k\cos\theta_0 \cos((1 - \omega_0)\theta) + \mathcal{O}_2^{(\text{polar})}(\theta)\right),$$

where $|\mathcal{O}_2^{(\text{polar})}(\theta)| \leq C^\sharp |1 - \omega_0|$ for all $\theta \in [0, \tilde{\beta}]$ with a constant $C^\sharp > 0$ chosen independently of ε and r.

4. *Claim: There exist $\omega_* \in (0, 1)$ and $\underline{R_2 > 0}$ such that, whenever $\omega_0 \in [\omega_*, 1)$ in* (B.6) *and $R \leq R_2$, the minimum of $\psi - \Psi$ over $\overline{\Omega \cap B_R(P_\beta)}$ cannot be attained on $\Gamma_{\text{shock}} \cap B_R(P_\beta)$. Furthermore, ω_* and R_2 can be chosen independently of ε.*

Suppose that $(\psi - \Psi)(P_*) = \min_{\overline{\Omega \cap \partial B_R(P_\beta)}} (\psi - \Psi)$ for $P_* \in \Gamma_{\text{shock}} \cap \partial\Omega_R(P_\beta)$ for some $R > 0$. Since $\psi - \Psi = 0$ at P_β, $\psi - \Psi \leq 0$ at P_*. Let $\boldsymbol{\nu}_{\text{sh}}$ be the unit normal vector to Γ_{shock} at P_* oriented towards the interior of Ω, and let $\boldsymbol{\tau}_{\text{sh}}$ be a unit tangent vector to Γ_{shock} at P_*. Then $\psi - \Psi$ satisfies

(B.9) $$\partial_{\boldsymbol{\tau}_{\text{sh}}}(\psi - \Psi)(P_*) = 0, \qquad \partial_{\boldsymbol{\nu}_{\text{sh}}}(\psi - \Psi)(P_*) \geq 0.$$

Let $P_\beta P_*'$ be the projection of $P_\beta P_*$ onto $S_\mathcal{O}$. Since $(\varphi_\infty - \varphi_\mathcal{O})(P_*') = 0$, it follows from (2.4.1) and (2.4.3)–(2.4.4) that

$$\varepsilon|P_* - P_\beta| \geq \Psi(P_*) - \Psi(P_\beta) \geq \psi(P_*) = [(\varphi_\infty - \varphi_\mathcal{O})(\boldsymbol{\xi})]_{\boldsymbol{\xi}=P_*'}^{P_*} \geq v_\infty \sec\beta \, |P_* - P_*'|,$$

which yields that

(B.10) $$|P_* - P_*'| \leq \frac{\varepsilon}{v_\infty \sec\beta} |P_* - P_\beta|.$$

From (B.9), we have

(B.11) $$D\psi(P_*) = D\Psi(P_*') + \left(D\Psi(P_*) - D\Psi(P_*')\right) + |D(\psi - \Psi)(P_*)|\boldsymbol{\nu}_{\text{sh}}.$$

Since $|D(\varphi_\infty - \varphi) \cdot \boldsymbol{\nu}_{\text{sh}}| > 0$ on $\overline{\Gamma_{\text{shock}}}$, there exist constants $\hat{\varepsilon}, \delta > 0$ such that φ satisfies $|D(\varphi_\infty - \varphi)| \geq \delta$ on the open $\hat{\varepsilon}$–neighborhood $\mathcal{N}_{\hat{\varepsilon}}(\Gamma_{\text{shock}})$ of Γ_{shock}. Since $\psi = \varphi_\infty - \varphi_\mathcal{O}$ on Γ_{shock},

$\mathfrak{g}(D\psi, \psi, \boldsymbol{\xi}) = \mathfrak{g}(D\psi, \varphi_\infty - \varphi_\mathcal{O}, \boldsymbol{\xi})$ on Γ_{shock}. Define $\mathfrak{g}_\sharp(\mathbf{q}, \boldsymbol{\xi}) := \mathfrak{g}(\mathbf{q}, (\varphi_\infty - \varphi_\mathcal{O})(\boldsymbol{\xi}), \boldsymbol{\xi})$. Choose constants $\sigma_0, R_1 > 0$ small so that

(i) $\mathfrak{g}_\sharp(\mathbf{q}, \boldsymbol{\xi})$ is well defined in $U_{\sigma_0, R_1} = \{(\mathbf{q}, \boldsymbol{\xi}) : |\mathbf{q}| \le 2\sigma_0, |\boldsymbol{\xi} - P_\beta| \le 2R_1\}$;

(ii) There is a constant $C_\mathfrak{g} > 0$ such that

$$\|\mathfrak{g}_\sharp\|_{C^1(\overline{U_{\sigma_0, R_1}})} \le C_\mathfrak{g},$$

(B.12)

$$\partial_\mathbf{q}\mathfrak{g}_\sharp(\mathbf{q}, \boldsymbol{\xi}) \cdot \frac{D\varphi_\infty(\boldsymbol{\xi}') - \mathbf{q}'}{|D\varphi_\infty(\boldsymbol{\xi}') - \mathbf{q}'|} \ge C_\mathfrak{g}^{-1} \qquad \text{for } (\mathbf{q}, \boldsymbol{\xi}), (\mathbf{q}', \boldsymbol{\xi}') \in U_{\sigma_0, R_1}.$$

Such a constant $C_\mathfrak{g}$ can be chosen independently of (ε, ω_0).

Owing to $|D\psi(P_\beta)| = 0$, there exists $R_1 > 0$ small, depending on σ_0, such that $(D\psi(\boldsymbol{\xi}), \boldsymbol{\xi}) \in \mathfrak{R}_{\sigma_0, R_1}$ for all $\boldsymbol{\xi} \in \overline{\Omega \cap B_{R_1}(P_\beta)}$.

If $P_* \in \overline{\Omega \cap B_{R_1/2}(P_\beta)}$ and $\frac{\varepsilon}{v_\infty \sec \beta} \le \frac{1}{4}$, then (B.10) implies that $P_*' \in B_{3R_1/4}(P_\beta)$. Choose $\varepsilon_1 \in (0, \frac{v_\infty \sec \beta}{4}]$ so that, whenever $\varepsilon \in (0, \varepsilon_1]$, $(\nabla\Psi(P_*'), P_*') \in U_{\sigma_0, R_1}$. Note that ε_1 can be chosen depending only on σ_0. Then

$$\begin{aligned} 0 &= \mathfrak{g}_\sharp(D\psi(P_*), P_*) - \mathfrak{g}_\sharp(\mathbf{0}, P_*') \\ &= \big(\mathfrak{g}_\sharp(D\psi(P_*), P_*) - \mathfrak{g}_\sharp(D\psi(P_*), P_*')\big) + \big(\mathfrak{g}_\sharp(D\psi(P_*), P_*') - \mathfrak{g}_\sharp(\mathbf{0}, P_*')\big) \\ &=: J_1 + J_2. \end{aligned}$$

By (B.10) and (B.12), J_1 is estimated as

(B.13)
$$|J_1| \le \frac{C_\mathfrak{g}\varepsilon}{v_\infty \sec \beta}|P_* - P_\beta|.$$

J_2 is estimated more carefully by using (B.8) and (B.10)–(B.12) as follows:

$$\begin{aligned} J_2 &= \big(D\Psi(P_*') + (D\Psi(P_*) - D\Psi(P_*')) + |D(\psi - \Psi)(P_*)|\boldsymbol{\nu}_{\text{sh}}\big) \cdot \int_0^1 \partial_\mathbf{q}\mathfrak{g}_\sharp(tD\psi(P_*), P_*') \, dt \\ &\ge \big(D\Psi(P_*') + (D\Psi(P_*) - D\Psi(P_*'))\big) \cdot \int_0^1 \partial_\mathbf{q}\mathfrak{g}_\sharp(tD\psi(P_*), P_*') \, dt. \end{aligned}$$

Let $C^\sharp$ be from Step 3. By (B.8) and (B.12),

$$\begin{aligned} D\Psi(P_*') &\cdot \int_0^1 \partial_\mathbf{q}\mathfrak{g}_\sharp(tD\psi(P_*), P_*') \, dt \\ &\ge \varepsilon\big(k\cos\theta_0 \cos((1 - \omega_0)\beta) - C^\sharp|1 - \omega_0| - C|P_* - P_\beta|^\alpha\big) \end{aligned}$$

for some $C > 0$ depending on $C_\mathfrak{g}$ and $\|\psi\|_{C^{1,\alpha}(\overline{\Omega})}$. By (B.6), (B.10), and (B.12),

$$(D\Psi(P_*) - D\Psi(P_*')) \cdot \int_0^1 \partial_\mathbf{q}\mathfrak{g}_\sharp(tD\psi(P_*), P_*')dt \ge C\varepsilon^2|P_* - P_\beta|$$

for some $C > 0$ depending on $C_\mathfrak{g}$. Therefore, J_2 is estimated as

$$J_2 \ge \varepsilon\big(k\cos\theta_0 \cos((1 - \omega_0)\beta) - C^\sharp|1 - \omega_0| - Ch(|P - P_\beta|)\big)$$

for a non-increasing continuous function $h(r)$ that tends to 0 as r tends to 0, where $C^\sharp$ and C are chosen, independent of P_* and ω_0. Combine this estimate with (B.13) to obtain

(B.14) $\varepsilon\big(k\cos\theta_0 \cos((1 - \omega_0)\beta) - C^\sharp|1 - \omega_0| - C(h(|P_* - P_\beta|) + |P_* - P_\beta|)\big) \le 0.$

Choose $\omega_* \in (0,1)$ close to 1 and $R_2 \in (0, R_1]$ small, so that

$$k \cos\theta_0 \cos((1 - \omega_*)\beta) - C^\sharp |1 - \omega_*| - C(h(R_2) + R_2) \geq \frac{\varepsilon}{2} k \cos\theta_0.$$

Under such choices of (ω_*, R_2), we arrive at a contradiction whenever $\omega_0 \in [\omega_*, 1)$ and $P_* \in \Gamma_{\text{shock}} \cap B_{R_2}(P_\beta)$. Thus, $\psi - \Psi$ cannot attain its minimum on $\Gamma_{\text{shock}} \cap B_R(P_\beta)$ whenever $\omega_0 \in [\omega_*, 1)$ and $R \leq R_2$.

5. *Claim: Let ω_* and R_2 be from Step 4. There exist $\varepsilon > 0$, $\omega_0 \in [\omega_*, 1)$, and $R \in (0, R_2]$ such that, for Ψ defined by* (B.6), *$\psi - \Psi$ attains its minimum over $\Omega_R(P_\beta) := \Omega \cap B_R(P_\beta)$ at P_β.*

By (B.7), there exists a small constant $R_3 \in (0, R_2]$ so that $\mathcal{L}$ is uniformly elliptic in $\Omega_{R_3}(P_\beta)$ and

$$\mathcal{L}(\psi - \Psi) \leq -\frac{\varepsilon}{2R_3}(1 - \omega_0^2)\cos(\omega_0 \tilde{\beta}) \qquad \text{in } \Omega_{R_3}(P_\beta).$$

By the strong maximum principle and Hopf's lemma, the minimum of $\psi - \Psi$ over $\overline{\Omega_R(P_\beta)}$ must be attained on $\partial\Omega_{R_3}(P_\beta) \setminus \Gamma_{\text{wedge}}$. It is shown in Step 4 that $\psi - \Psi$ cannot attain its minimum on $\Gamma_{\text{shock}} \cap B_{R_3}(P_\beta)$.

Denote $m := \inf_{\Omega \cap \partial B_{R_3}(P_\beta)} \psi$. The claim in Step 2 implies that $m > 0$. Choose $\varepsilon > 0$ small, depending only on R_3, so that $\psi - \Psi > 0$ on $\Omega \cap \partial B_{R_3}(P_\beta)$. For such a choice of ε, since $(\psi - \Psi)(P_\beta) = 0$, we conclude that

$$\min_{\Omega_{R_3}(P_\beta)} (\psi - \Psi) = (\psi - \Psi)(P_\beta) = 0.$$

6. In Steps 4–5, it is shown that we can choose (ε, ω_0) in (B.6) so that, if $R > 0$ is sufficiently small, the minimum of $\psi - \Psi$ over $\Omega_R(P_\beta)$ must be attained at P_β, provided that there is an admissible solution φ corresponding to some $(v_\infty, \beta) \in \mathfrak{R}_{\text{strong}}$, and that ψ is given by $\psi = \varphi - \varphi_\mathcal{O}$.

By the definition of Ψ with $\omega_0 \in (0,1)$ and (B.3), and by the C^1–regularity of φ up to P_β, there exists a small constant $\delta > 0$ so that $\partial_r(\psi - \Psi) < -\frac{\varepsilon}{2}$ in $\Omega_\delta(P_\beta)$. However, this contradicts the fact that

$$(\psi - \Psi)(P_\beta) = \min_{\Omega_R(P_\beta)} (\psi - \Psi).$$

Therefore, we conclude that there exists no admissible solution corresponding to $(v_\infty, \beta) \in \mathfrak{R}_{\text{strong}}$ in the sense of Definition 2.24. $\qquad\qquad\square$

Quasilinear Elliptic Equations in Two Variables

For the completeness of this work, this appendix includes several properties of quasilinear elliptic equations, which are used to prove Theorem 2.31. We refer the reader to [**11**] for the proofs of these properties as stated below.

C.1. Ellipticity Principle for Self-Similar Potential Flow

The following lemma is an extension of the ellipticity principle of Elling-Liu [**26**]:

LEMMA C.1 (Theorem 5.2.1, [**11**]). *Fix $\gamma \geq 1$ and $v_\infty > 0$. In a bounded domain $\Omega \subset \mathbb{R}^2$, let $\varphi \in C^3(\Omega)$ satisfy the equation:*

$$(\text{C.1.1}) \qquad \text{div}\left(\rho(|D\varphi|^2, \varphi)D\varphi\right) + 2\rho(|D\varphi|^2, \varphi) = 0$$

for $\rho(|D\varphi|^2, \varphi)$ given by (2.4.2). Denote the pseudo-Mach number as $M := \frac{|D\varphi|}{c(|D\varphi|^2, \varphi)}$ for $c(|D\varphi|^2, \varphi) = \rho^{\frac{(\gamma-1)}{2}}(|D\varphi|^2, \varphi)$. Let φ satisfy that $\rho > 0$ and $M \leq 1$ in Ω. Then the following properties hold:

(a) *Either $M \equiv 0$ holds in Ω or M does not attain its maximum in Ω;*

(b) *Suppose that $\text{diam}(\Omega) \leq d$ for some constant $d > 0$. Then there exists a constant $C_0 > 0$ depending only on (v_∞, γ, d) such that, for any given $\delta \geq 0$, $\hat{c} \geq 1$, and $b \in C^2(\Omega)$ with $|Db| + \hat{c}|D^2 b| \leq \frac{\delta}{\hat{c}}$ in Ω, if $c(|D\varphi|^2, \varphi) \leq \hat{c}$ holds in Ω, then either $M^2 \leq C_0 \delta$ holds in Ω or $M^2 + b$ does not attain its maximum in Ω.*

LEMMA C.2 (Theorem 5.3.1, [**11**]). *In a bounded domain $\Omega \subset \mathbb{R}^2$ with a relatively open flat segment $\Gamma \subset \partial\Omega$, let $\varphi \in C^3(\Omega \cup \Gamma)$ satisfy (C.1.1) in Ω and*

$$\partial_{\boldsymbol{\nu}}\varphi = 0 \qquad on\ \Gamma$$

for the unit normal vector $\boldsymbol{\nu}$ to Γ towards the interior of Ω. Assume that $\rho > 0$ and $M \leq 1$ in $\Omega \cup \Gamma$. Then the following properties hold:

(a) *Either $M \equiv 0$ holds in $\Omega \cup \Gamma$ or M does not attain its maximum in $\Omega \cup \Gamma$;*

(b) *Let $\text{diam}(\Omega) \leq d$ for some constant $d > 0$. Then there exists a constant $C_0 > 0$ depending only on (v_∞, γ, d) such that, for any given $\delta \geq 0$, $\hat{c} \geq 1$, and $b \in C^2(\Omega)$ with $|Db| + \hat{c}|D^2 b| \leq \frac{\delta}{\hat{c}}$ in Ω and $\partial_{\boldsymbol{\nu}}b = 0$ on Γ, if $c(|D\varphi|^2, \varphi) \leq \hat{c}$ holds in $\Omega \cup \Gamma$, then either $M^2 \leq C_0 \delta$ holds in $\Omega \cup \Gamma$ or $M^2 + b$ does not attain its maximum in $\Omega \cup \Gamma$.*

C.2. Uniformly Elliptic Equations Away From the Corners

Consider a quasilinear elliptic equation of the form:

$$(\text{C.2.1}) \qquad \mathcal{N}(u) = f(\mathbf{x}) \qquad in\ \Omega,$$

with

$$\mathcal{N}(u) := \sum_{i,j=1}^{2} A_{ij}(Du, u, \mathbf{x})D_{ij}u + A(Du, u, \mathbf{x}),$$

where

(C.2.2) $A_{ij}(\mathbf{p}, z, \mathbf{x}) = A_{ji}(\mathbf{p}, z, \mathbf{x}),\ \ A(\mathbf{0}, 0, \mathbf{x}) = 0$ for all $(\mathbf{p}, z, \mathbf{x}) \in \mathbb{R}^2 \times \mathbb{R} \times \Omega$ and $i, j = 1, 2$.

Suppose that there exist $\lambda > 0$ and $\alpha \in (0, 1)$ such that

(C.2.3) $\lambda|\boldsymbol{\mu}|^2 \le \sum_{i,j=1}^{2} A_{ij}(Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x})\mu_i \mu_j \le \lambda^{-1}|\boldsymbol{\mu}|^2$ for all $\mathbf{x} \in \Omega$ and $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathbb{R}^2$,

(C.2.4) $\|(A_{ij}, A)(\mathbf{p}, z, \cdot)\|_{0,\alpha,\overline{\Omega}} \le \lambda^{-1}$ for all $(\mathbf{p}, z) \in \mathbb{R}^2 \times \mathbb{R}$,

(C.2.5) $\|D_{(\mathbf{p},z)}(A_{ij}, A)\|_{0,\mathbb{R}^2 \times \mathbb{R} \times \overline{\Omega}} \le \lambda^{-1}$.

For $r > 0$, let B_r denote a ball of radius r in $\mathbb{R}^2$.

THEOREM C.3 (Theorem 4.2.1, [11]). *For $\Omega = B_2$, if $u \in C^{2,\alpha}(B_2)$ is a solution of (C.2.1) with*

$$\|u\|_{0,B_2} + \|f\|_{0,\alpha,B_2} \le M,$$

then there exists a constant $C > 0$ depending only on (λ, M, α) such that

$$\|u\|_{2,\alpha,B_1} \le C\big(\|u\|_{0,B_2} + \|f\|_{0,\alpha,B_2}\big).$$

Applying Theorem C.3 to $v(x) = \frac{1}{r}u(rx)$, we have the following corollary:

COROLLARY C.4. *If $u \in C^{2,\alpha}(B_{2r})$ is a solution of (C.2.1) for $r \in (0, 1]$ with*

$$\|u\|_{0,B_{2r}} + \|f\|_{0,\alpha,B_{2r}} \le M,$$

then there exists a constant $C > 0$ depending only on (λ, M, α) such that

$$\|u\|_{2,\alpha,B_r} \le \frac{C}{r^{2+\alpha}}\big(\|u\|_{0,B_{2r}} + r^2\|f\|_{0,\alpha,B_{2r}}\big).$$

THEOREM C.5 (Theorem 4.2.3, [11]). *For $\lambda \in (0, 1)$, let $\Phi \in C^1(\mathbb{R})$ satisfy*

$$\|\Phi\|_{1,\mathbb{R}} \le \lambda^{-1}, \qquad \Phi(0) = 0.$$

For $R > 0$, denote

$$\Omega_R := B_R(\mathbf{0}) \cap \{x_2 > \varepsilon\Phi(x_1)\}, \qquad \Gamma_R := B_R(\mathbf{0}) \cap \{x_2 = \varepsilon\Phi(x_1)\}.$$

In addition to assumptions (C.2.2)–(C.2.5) with $\Omega = \Omega_{2r}$, let $W(p_2, z, x)$ satisfy

$$W(0, 0, \mathbf{x}) = 0 \qquad on\ \Gamma_{2r},$$

$$|\partial_{p_2}W(p_2, z, \mathbf{x})| \le \varepsilon \qquad for\ all\ (p_2, z, \mathbf{x}) \in \mathbb{R} \times \mathbb{R} \times \Gamma_{2r},$$

$$\|D_{(p_2,z)}W(p_2, z, \cdot)\|_{1,\Gamma_{2r}} \le \lambda^{-1} \qquad for\ all\ (p_2, z) \in \mathbb{R} \times \mathbb{R}.$$

Then there exist constants $\varepsilon, \beta \in (0, 1)$ and $C > 0$ depending only on λ such that, for $u \in C^2(\Omega_{2r}) \cap C^{1,\beta}(\Omega_{2r} \cup \Gamma_{2r})$ satisfying (C.2.1) with $f = 0$ in Ω_{2r} and

(C.2.6) $u_{x_1} = W(u_{x_2}, u, \mathbf{x})$ *on Γ_{2r},*

we have

$$\|u\|_{1,\beta,\Omega_{9r/5}} \le \frac{C}{r^{1+\beta}}\|u\|_{0,\Omega_{2r}}.$$

THEOREM C.6 (Theorem 4.2.8, [**11**]). *In addition to the assumptions of Theorem C.5, for $\alpha \in (0,1)$, assume that*

$$\|\Phi\|_{1,\alpha,\mathbb{R}} \leq \lambda^{-1},$$

$$\|D_{(p_2,z)}W(p_2,z,\cdot)\|_{1,\alpha,\Gamma_{2r}} \leq \lambda^{-1} \qquad \text{for all } (p_2,z) \in \mathbb{R} \times \mathbb{R},$$

$$\|D^2_{(p_2,z)}W\|_{1,0,\mathbb{R}\times\mathbb{R}\times\Gamma_{2r}} \leq \lambda^{-1}.$$

Then there exist $\varepsilon \in (0,1)$ and $C > 0$ depending only on $(\lambda, \alpha, \|u\|_{0,\Omega_{2r}})$ such that, for $u \in C^{2,\alpha}(\Omega_{2r} \cup \Gamma_{2r})$ satisfying (C.2.1) with $f = 0$ in Ω_{2r} and (C.2.6) on Γ_{2r},

$$\|u\|_{2,\alpha,\Omega_{9r/5}} \leq \frac{C}{r^{2+\alpha}} \|u\|_{0,\Omega_{2r}}.$$

THEOREM C.7 (Theorem 4.2.10, [**11**]). *For $\lambda \in (0,1)$ and $\alpha \in (0,1)$, let $\Phi \in C^{2,\alpha}(\mathbb{R})$ satisfy*

$$\|\Phi\|_{2,\alpha,\mathbb{R}} \leq \lambda^{-1}, \qquad \Phi(0) = \Phi'(0) = 0,$$

and denote

$$\Omega_R := B_R(0) \cap \{x_2 > \Phi(x_1)\}, \quad \Gamma_R := \partial\Omega_R \cap \{x_2 = \Phi(x_1)\} \qquad \text{for } R \in (0,2).$$

Let $u \in C^{2,\alpha}(\Omega_R \cup \Gamma_R)$ satisfy (C.2.1) in Ω_R and

$$\boldsymbol{\omega} \cdot Du + b_0 u = h \qquad \text{on } \Gamma_R.$$

Assume that $\boldsymbol{\omega} = (\omega_1, \omega_2)(\mathbf{x})$ and $b_0 = b_0(\mathbf{x})$ satisfy the following conditions:

$$\boldsymbol{\omega} \cdot \boldsymbol{\nu} \geq \lambda \text{ on } \Gamma_R, \qquad \|(\boldsymbol{\omega}, b_0)\|_{1,\alpha,\Gamma_R} \leq \lambda^{-1},$$

where $\boldsymbol{\nu}$ represents the unit normal vector to Γ_R towards the interior of Ω_R. If u satisfies

$$\|u\|_{0,\Omega_R} + \|f\|_{0,\alpha,\Omega_R} + \|h\|_{1,\alpha,\Gamma_R} \leq M,$$

then there exists a constant $C > 0$ depending only on (λ, α) such that

$$\|u\|_{2,\alpha,\Omega_{R/2}} \leq \frac{C}{R^{2+\alpha}}\Big(\|u\|_{0,\Omega_R} + R^2 \|f\|_{0,\alpha,\Omega_R} + R\|h\|_{1,\alpha,\Gamma_R}\Big).$$

In addition, there exist $\beta \in (0,1)$ and $\hat{C} > 0$ depending only on λ such that

$$\|u\|_{1,\beta,\Omega_{R/2}} \leq \frac{\hat{C}}{R^{1+\beta}}\Big(\|u\|_{0,\Omega_R} + R^2 \|f\|_{0,\alpha,\Omega_R} + R\|h\|_{0,\beta,\Gamma_R}\Big).$$

Note that β is independent of α.

THEOREM C.8 (Theorem 4.3.2, [**11**]). *Let $R > 0$, $\lambda \in (0,1)$, $\gamma \in (0,1)$, and $K > 0$. Let $\Phi \in C^1(\mathbb{R})$ satisfy*

$$\|\Phi\|_{0,1,\mathbb{R}} \leq \lambda^{-1}, \qquad \Phi(0) = 0.$$

Let Ω_R and Γ_R be as in Theorem C.7 for $R > 0$. Define

$$d(\mathbf{x}) := \text{dist}(\mathbf{x}, \Gamma_R) \qquad \text{for } \mathbf{x} \in \Omega_R.$$

Assume that $u \in C^3(\Omega_R) \cap C^1(\overline{\Omega_R})$ is a solution of (C.2.1) with $f = 0$ in Ω_R and the boundary condition:

$$B(Du, u, \mathbf{x}) = 0 \qquad \text{on } \Gamma_R.$$

Assume that $A_{ij}(\mathbf{p}, z, \mathbf{x}), i, j = 1, 2,$ and $A(\mathbf{p}, z, \mathbf{x})$ satisfy (C.2.3)–(C.2.5) and the additional property:

$$d(\mathbf{x})^\gamma |D_{\mathbf{x}}(A_{ij}, A)(\mathbf{p}, z, \mathbf{x})| \leq \lambda^{-1} \qquad \text{for all } \mathbf{x} \in \Omega_R \text{ and } |\mathbf{p}| + |z| \leq 2K,$$

and that $B(\mathbf{p}, z, \mathbf{x})$ satisfies

(C.2.7) $\qquad |D_{\mathbf{p}}B(Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x})| \geq \lambda \ \text{ for all } \mathbf{x} \in \overline{\Omega_R}, \qquad \|B\|_{2,\{|\mathbf{p}|+|z| \leq 2K, \mathbf{x} \in \overline{\Omega_R}\}} \leq \lambda^{-1}.$

Assume that u satisfies

$$|u| + |Du| \leq K \qquad \text{on } \Omega_R \cup \Gamma_R.$$

Then there exist both $\beta \in (0, 1]$ depending only on (λ, K, γ) and $C > 0$ depending only on (R, λ, K, γ) such that

$$\|u\|_{1,\beta,\Omega_{R/2}} \leq C, \qquad \|u\|_{2,\beta,\Omega_{R/2}}^{(-1-\beta),\Gamma_{R/2}} \leq C.$$

THEOREM C.9 (Theorem 4.3.4, [**11**]). *Let the assumptions of Theorem C.8 be satisfied with $\gamma = 0$. In addition, for $\alpha, \sigma \in (0, 1)$, assume that*

$$\|\Phi\|_{C^{1,\sigma}(\mathbb{R})} \leq \lambda^{-1}, \qquad \Phi(0) = 0,$$

$$\|(A_{ij}, A)\|_{C^{1,\alpha}(\{|\mathbf{p}|+|z| \leq 2K, \mathbf{x} \in \overline{\Omega_R}\})} + \|B\|_{C^{2,\alpha}(\{|\mathbf{p}|+|z| \leq 2K, \mathbf{x} \in \overline{\Omega_R}\})} \leq \lambda^{-1} \qquad \text{for } j = 1, 2.$$

Then

$$\|u\|_{2,\sigma,\Omega_{R/4}} \leq C,$$

where C depends only on $(\lambda, K, \alpha, \sigma, R)$.

COROLLARY C.10 (Corollary 4.3.5, [**11**]). *Let the assumptions of Theorem C.8 be satisfied with $\gamma = 0$. In addition, for $\alpha \in (0, 1)$ and $k \in \mathbb{N}$, assume that*

$$\|\Phi\|_{k,\alpha,\mathbb{R}} \leq \lambda^{-1}, \qquad \Phi(0) = 0,$$

$$\|(A_{ij}, A)\|_{C^{k,\alpha}(\{|\mathbf{p}|+|z| \leq 2K, \mathbf{x} \in \overline{\Omega_R}\})} + \|B\|_{C^{k+1,\alpha}(\{|\mathbf{p}|+|z| \leq 2K, \mathbf{x} \in \overline{\Omega_R}\})} \leq \lambda^{-1} \qquad \text{for } j = 1, 2.$$

Then

$$\|u\|_{k+1,\alpha,\Omega_{R/2}} \leq C,$$

where C depends only on $(\lambda, K, k, \alpha, R)$.

C.3. Quasilinear Degenerate Elliptic Equations

Consider a domain $U \subset \mathbb{R}^2$ of the form:

$$U = \{\mathbf{x} = (x_1, x_2) \ : \ x_1 > 0, x_2 \in (0, f(x_1))\},$$

where $f \in C^1(\mathbb{R}_+)$ and $f > 0$ on $\mathbb{R}_+$. For a constant $r > 0$, denote

$$U_r = U \cap \{x_1 < r\},$$
$$\Gamma_{n,r} = \partial U \cap \{(x_1, 0) \ : \ 0 < x_1 < r\},$$
$$\Gamma_{f,r} = \partial U \cap \{(x_1, f(x_2)) \ : \ 0 < x_1 < r\}.$$

Consider a boundary value problem of the form:

$$\sum_{i,j=1}^{2} A_{ij}(Du, u, \mathbf{x})\partial_{x_i x_j} u + \sum_{i=1}^{2} A_i(Du, u, \mathbf{x})\partial_{x_i} u = 0 \qquad \text{in } U_r,$$

(C.3.1) $\qquad\qquad\qquad\qquad\qquad\qquad B(Du, u, \mathbf{x}) = 0 \qquad \text{on } \Gamma_{f,r},$

$$\partial_{x_2} u = 0 \qquad \text{on } \Gamma_{n,r},$$

$$u = 0 \qquad \text{on } \Gamma_0 = \partial U \cap \{x_1 = 0\}.$$

THEOREM C.11 (Theorem 4.7.4, [**11**]). *Given constants $r > 0$, $M \geq 1$, and $l, \lambda \in (0, 1)$, assume that the following conditions are satisfied:*

(i) Conditions for $\Gamma_{f,r}$: f is in $C^{1,\beta}([0,r])$ for some $\beta \in (0,1)$ and satisfies

$$\|f\|_{2,\beta,(0,r)}^{(-1-\beta),\{0\}} \leq M, \qquad f \geq l \quad on\ \mathbb{R}_+.$$

(ii) Conditions for (A_{ij}, A_i): For any $(\mathbf{p}, z, \mathbf{x}) \in \mathbb{R}^2 \times \mathbb{R} \times U_r$ and $\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2$,

$$\lambda|\boldsymbol{\kappa}|^2 \leq \sum_{i,j=1}^{2} A_{ij}(\mathbf{p}, z, \mathbf{x}) \frac{\kappa_i \kappa_j}{x_1^{2-\frac{i+j}{2}}} \leq \lambda^{-1}|\boldsymbol{\kappa}|^2.$$

In addition, (A_{ij}, A_i) satisfy the following estimates:

$$\|(A_{11}, A_{12})\|_{0,1,\mathbb{R}^2 \times \mathbb{R} \times U_r} \leq M,$$

$$|\partial_{x_2} A_{11}(\mathbf{p}, z, \mathbf{x})| \leq M x_1^{1/2} \quad in\ \mathbb{R}^2 \times \mathbb{R} \times U_r,$$

$$\|(A_{22}, A_1, A_2)\|_{0,\mathbb{R}^2 \times \mathbb{R} \times U_r} + \|D_{(\mathbf{p},z)}(A_{22}, A_1, A_2)\|_{0,\mathbb{R}^2 \times \mathbb{R} \times U_r} \leq M,$$

$$\sup_{(\mathbf{p},z) \in \mathbb{R}^2 \times \mathbb{R}, \mathbf{x} \in U_r} |(x_1 \partial_{x_1}, x_1^{1/2} \partial_{x_2})(A_{22}, A_1, A_2)(\mathbf{p}, z, \mathbf{x})| \leq M.$$

(iii) Conditions for B: For any $(\mathbf{p}, z, \mathbf{x}) \in \mathbb{R}^2 \times \mathbb{R} \times \Gamma_{f,r}$,

(C.3.2) $$\partial_{p_1} B(\mathbf{p}, z, \mathbf{x}) \leq -M^{-1}.$$

In addition, B satisfies the following estimates:

$$\|B\|_{3,\mathbb{R}^2 \times \mathbb{R} \times \Gamma_{f,r}} \leq M, \qquad B(\mathbf{0}, 0, \mathbf{x}) = 0 \quad on\ \Gamma_{f,r}.$$

Let $u \in C(\overline{U_r}) \cap C^2(\overline{U_r} \setminus \overline{\Gamma_0})$ be a solution of the boundary value problem (C.3.1) satisfying that

$$|u(\mathbf{x})| \leq M x_1^2 \quad in\ U_r.$$

Then, for any $\alpha \in (0,1)$, there exist constants $r_0 \in (0,1]$ and $C > 0$ depending only on (M, λ, α) such that, for $\varepsilon := \min\{\frac{r}{2}, r_0, l^2\}$,

$$\|u\|_{2,\alpha,U_\varepsilon}^{(2),(\mathrm{par})} \leq C.$$

C.4. Estimates at a Corner for the Oblique Derivative Boundary Value Problems

PROPOSITION C.12 (Proposition 4.3.7, [**11**]). Let $R > 0$, $\beta \in (0,1)$, $\gamma \in [0,1)$, $\lambda > 0$, and $K, M \geq 1$. Let $\Omega \subset \mathbb{R}^2$ be a domain with $\mathbf{x}_0 \in \partial\Omega$ and $\partial\Omega \cap B_R(\mathbf{x}_0) = \Gamma^1 \cup \Gamma^2$, where Γ^k, $k = 1, 2$, are two Lipschitz curves intersecting only at $\mathbf{x}_0$ and contained within $\mathbf{x}_0 + \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 > \tau|x_1|\}$ for some $\tau > 0$. Denote

$$\Omega_R := \Omega \cap B_R(\mathbf{x}_0).$$

Assume that Γ^2 is $C^{1,\sigma}$ up to the endpoints for some $\sigma \in (0,1)$ with $\|\overline{\Gamma^2}\|_{C^{1,\sigma}} \leq M$ in the sense that there exist $c^{(2)} > 0$ and $f^{(2)} \in C^{1,\sigma}([0, c^{(2)}])$ such that, in an appropriate basis in $\mathbb{R}^2$,

$$\Omega_R \subset \{\mathbf{x} : x_2 > f^{(2)}(x_1), 0 < x_1 < c^{(2)}\}, \quad \overline{\Gamma^2} = \{x_2 = f^{(2)}(x_1) : 0 < x_1 < c^{(2)}\}.$$

Let $u \in C^1(\overline{\Omega_R}) \cap C^2(\Omega_R \cup \Gamma^2) \cap C^3(\Omega_R)$ satisfy

(C.4.1) $$\|u\|_{C^{0,1}(\overline{\Omega_R})} \leq K.$$

Assume that u is a solution of

(C.4.2) $$\sum_{i,j=1}^{2} a_{ij}(Du, u, \mathbf{x})D_{ij}u + a(Du, u, \mathbf{x}) = 0 \qquad \text{in } \Omega_R,$$

(C.4.3) $$b^{(1)}(Du, u, \mathbf{x}) = h(\mathbf{x}) \qquad \text{on } \Gamma^1,$$

(C.4.4) $$b^{(2)}(Du, u, \mathbf{x}) = 0 \qquad \text{on } \Gamma^2,$$

where $(a_{ij}, a, b^{(k)})$ are defined in $V = \{(\mathbf{p}, z, \mathbf{x}) \in \mathbb{R}^2 \times \mathbb{R} \times \Omega : |\mathbf{p}| + |z| \leq 2K\}$. Assume that $(a_{ij}, a) \in C(\overline{V}) \cap C^1(\overline{V} \setminus \{\mathbf{x} = \mathbf{x}_0\})$, $b^{(1)} \in C^2(\overline{V})$, $b^{(2)} \in C^1(\overline{V})$, and $h \in C(\overline{\Gamma^1})$ with

(C.4.5) $$\|(a_{ij}, a)\|_{C^0(\overline{V})} + \|D_{(\mathbf{p}, z)}(a_{ij}, a)\|_{C^0(\overline{V})} \leq M,$$

(C.4.6) $$|D_{\mathbf{x}}(a_{ij}, a)(\mathbf{p}, z, \mathbf{x})| \leq M|\mathbf{x} - \mathbf{x}_0|^{-\gamma} \qquad \text{for all } (\mathbf{p}, z, \mathbf{x}) \in V,$$

(C.4.7) $$\|b^{(1)}\|_{C^2(\overline{V})} + \|b^{(2)}\|_{C^1(\overline{V})} \leq M,$$

(C.4.8) $$|h(\mathbf{x}) - h(\mathbf{x}_0)| \leq \frac{\lambda^{-1}}{R^\beta}|\mathbf{x} - \mathbf{x}_0|^\beta \qquad \text{for all } \mathbf{x} \in \Gamma^1.$$

In addition to the conditions stated above, assume that the following properties hold:

(i) For any $\mathbf{x} \in \Omega_R$ and $\boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2$,

$$\lambda|\boldsymbol{\kappa}|^2 \leq \sum_{i,j=1}^{2} a_{ij}(Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x})\kappa_i\kappa_j \leq \lambda^{-1}|\boldsymbol{\kappa}|^2;$$

(ii) For any $\mathbf{x} \in \Gamma^1$, $|D_{\mathbf{p}}b^{(1)}(Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x})| \geq \lambda$;

(iii) For any $\mathbf{x} \in \Gamma^2$, $D_{\mathbf{p}}b^{(2)}(Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x}) \cdot \boldsymbol{\nu} \geq \lambda$, where $\boldsymbol{\nu}$ is the inner unit normal vector to Γ^2;

(iv) $b^{(1)}$ and $b^{(2)}$ are independent for u on Γ^2 in the sense that, for any $\mathbf{x} \in \Gamma^2$,

$$\left|\det\begin{pmatrix} D_{\mathbf{p}}b^{(1)}(Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x}) \\ D_{\mathbf{p}}b^{(2)}(Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x}) \end{pmatrix}\right| \geq \lambda \qquad \text{for any } \mathbf{x} \in \Gamma^2.$$

Then there exist $\alpha \in (0, \beta]$ and C depending only on (λ, K, M), and $R' \in (0, R]$ depending only on $(\lambda, \gamma, K, M, \alpha)$ so that, for any $\mathbf{x} \in \overline{\Omega_{R'}}$,

$$|b^{(1)}(Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x}) - b^{(1)}(Du(\mathbf{x}_0), u(\mathbf{x}_0), \mathbf{x}_0)| \leq C|\mathbf{x} - \mathbf{x}_0|^\alpha.$$

PROPOSITION C.13 (Proposition 4.3.9, [**11**]). *In addition to the assumptions of Proposition C.12, assume that*

(C.4.9) $$|b^{(k)}(\mathbf{p}, z, \mathbf{x}) - b^{(k)}(\tilde{\mathbf{p}}, \tilde{z}, \tilde{\mathbf{x}})| \leq M|(\mathbf{p}, z, \mathbf{x}) - (\tilde{\mathbf{p}}, \tilde{z}, \tilde{\mathbf{x}})| \qquad \text{for } k = 1, 2,$$

for all $(\mathbf{p}, z, \mathbf{x})$, $(\tilde{\mathbf{p}}, \tilde{z}, \tilde{\mathbf{x}}) \in V$. *Moreover, denoting* $h^{(k)}(\mathbf{p}) = b^{(k)}(\mathbf{p}, u(\mathbf{x}_0), \mathbf{x}_0)$, $k = 1, 2$, *and noting that functions* $h^{(k)}$ *are defined in* $B_K(Du(\mathbf{x}_0))$, *assume that* $h^{(k)} \in C^{1,\alpha}(\overline{B_K(Du(\mathbf{x}_0))})$ *with* $\|h^{(k)}\|_{C^{1,\alpha}(\overline{B_{K/2}(Du(\mathbf{x}_0))})} \leq M$ *for some* $\alpha \in (0, 1)$, *and*

(C.4.10) $$\left|\det\begin{pmatrix} D_{\mathbf{p}}h^{(1)}(Du(\mathbf{x}_0)) \\ D_{\mathbf{p}}h^{(2)}(Du(\mathbf{x}_0)) \end{pmatrix}\right| \geq \lambda.$$

Let $W \subset \overline{\Omega_R}$ *satisfy*

(C.4.11) $$\mathbf{x}_0 \in W, \qquad \emptyset \neq W \cap \partial B_r(\mathbf{x}_0) \subset \overline{W \cap B_r(\mathbf{x}_0)} \quad \text{for all } r \in (0, R).$$

For each $k = 1, 2$, let

(C.4.12) $|b^{(k)}(Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x}) - b^{(k)}(Du(\mathbf{x}_0), u(\mathbf{x}_0), \mathbf{x}_0)| \leq M|\mathbf{x} - \mathbf{x}_0|^{\alpha}$ for all $\mathbf{x} \in W$.

Then there exists a constant $C > 0$ depending only on (K, M, R, α) such that, for all $\mathbf{x} \in W$,

$$|Du(\mathbf{x}) - Du(\mathbf{x}_0)| \leq C|\mathbf{x} - \mathbf{x}_0|^{\alpha}.$$

PROPOSITION C.14 (Proposition 4.3.11, [**11**]). Let $R, \lambda > 0$, $\alpha \in (0, 1]$, $\gamma \in [0, 1)$, and $M \geq 1$.

(a) Let Ω_R be as in Proposition C.12. Assume that Γ^1 and Γ^2 satisfy that, for each $k = 1, 2$,

   (i) $\Gamma^k \in C^1$ with $\|\Gamma^k\|_{C^{0,1}} \leq M$,

   (ii) $B_{\frac{d(\mathbf{x})}{M}}(\mathbf{x}) \cap \partial\Omega_R = B_{\frac{d(\mathbf{x})}{M}} \cap \Gamma^k$ for all $\mathbf{x} \in \Gamma^k \cap B_{\frac{3R}{4}}(\mathbf{x}_0)$, for $d(\mathbf{x}) := |\mathbf{x} - \mathbf{x}_0|$.

Let $u \in C^1(\overline{\Omega_R}) \cap C^3(\Omega_R)$ be a solution of (C.4.2)–(C.4.4) with $h \equiv 0$, where $(a_{ij}, a)(\mathbf{p}, z, \mathbf{x})$ satisfy all the conditions stated in Proposition C.12. In addition, assume that, for each $k = 1, 2$,

$$\|b^{(k)}\|_{C^2(\overline{V})} \leq M,$$
$$|D_{\mathbf{p}}b^{(k)}(Du(\mathbf{x}), u(\mathbf{x}), \mathbf{x})| \geq \lambda \qquad \text{for all } \mathbf{x} \in \Omega_R.$$

Moreover, assume that u satisfies

(C.4.13) $|Du(\mathbf{x}) - Du(\mathbf{x}_0)| \leq M|\mathbf{x} - \mathbf{x}_0|^{\alpha}$ for all Ω_R.

Then there exist $\beta \in (0, \alpha]$ depending only on (λ, K, M, α) and $C > 0$ depending on $(\lambda, K, M, R, \alpha)$ such that $u \in C^{1,\beta}(\overline{\Omega_{R/2}})$ with

$$\|u\|_{C^{1,\beta}(\overline{\Omega_{R/2}})} \leq C.$$

(b) In addition to the previous assumptions, if $\|\Gamma^k\|_{C^{1,\sigma}} \leq M$, $k = 1, 2$, for some $\sigma \in (0, 1)$, if (a_{ij}, a) satisfy

$$\|(a_{ij}, a)(\mathbf{0}, 0, \cdot), D^m_{(\mathbf{p},z)}(a_{ij}, a)(\mathbf{p}, z, \cdot)\|_{1,\delta,\Omega_R}^{(-\delta),\{\mathbf{x}_0\}} \leq M$$

for any $(\mathbf{p}, z)$ satisfying $|\mathbf{p}| + |z| \leq 2K$ and for $m = 1, 2$, and if each $b^{(k)}$ satisfies

$$\|b^{(k)}\|_{C^{2,\delta}(\overline{V})} \leq M \qquad \text{for } k = 1, 2,$$

for some $\delta \in (0, 1)$, then there exists a constant $C > 0$ depending only on $(\lambda, K, M, R, \alpha, \sigma, \delta)$ such that u satisfies

$$\|u\|_{2,\sigma,\Omega_{R/2}}^{(-1-\alpha),\{\mathbf{x}_0\}} \leq C.$$

C.5. Well-Posedness of a Nonlinear Boundary Value Problem

For a constant $h > 0$ and a function $f_{\mathrm{bd}} : [0, h] \to \mathbb{R}_+$, Denote a bounded domain $\Omega \subset \mathbb{R}^2$ as

(C.5.1) $\Omega := \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1 \in (0, h), \ x_2 \in (0, f_{\mathrm{bd}}(x_1))\},$

where f_{bd} satisfies that, for constants $t_0 \geq 0$, $t_1 > 0$, $t_2 > 0$, $t_h > 0$, $\alpha \in (0, 1)$, and $M > 0$,

(C.5.2)
$$f_{\mathrm{bd}} \in C^1([0, h]), \quad f_{\mathrm{bd}}(0) = t_0, \quad f_{\mathrm{bd}}(h) = t_h,$$
$$f_{\mathrm{bd}}(x_1) \geq \min\{t_1 x_1 + t_0, t_2\},$$
$$\|f_{\mathrm{bd}}\|_{2,\alpha,(0,h)}^{(-1-\alpha),\{0,h\}} \leq M.$$

We denote the boundary vertices and segments as follows:

$$P_1 = (0, t_0), \quad P_2 = (h, t_h), \quad P_3 = (h, 0), \quad P_4 = (0, 0),$$

(C.5.3)
$$\overline{\Gamma_{\mathrm{l}}} = \partial\Omega \cap \{x_1 = 0\}, \quad \overline{\Gamma_{\mathrm{r}}} = \partial\Omega \cap \{x_1 = h\},$$

$$\overline{\Gamma_{\mathrm{t}}} = \partial\Omega \cap \{x_2 = f_{\mathrm{bd}}(x_1)\}, \quad \overline{\Gamma_{\mathrm{b}}} = \partial\Omega \cap \{x_2 = 0\};$$

and Γ_{l}, Γ_{r}, Γ_{t}, and Γ_{b} are the relative interiors of the segments defined above.

Let $\phi_0(\mathbf{x})$ be a piecewise smooth function defined in $\mathbb{R}^2$ such that

- $\phi_0 \in C^\infty(\{x_1 \le \frac{h}{3}\}) \cap C^\infty(\{x_1 \ge \frac{2h}{3}\})$ with $\|\phi_0\|_{C^3(\overline{\Omega \setminus \{\frac{h}{3} < x_1 < \frac{2h}{3}\}})} \le C_{\phi_0}$,

- $\phi_0 \equiv 0$ in $\{x_1 \le \frac{h}{4}\}$,

- ϕ_0 is linear in $\{x_1 \ge \frac{3h}{4}\}$,

- $\partial_{x_2}\phi_0 = 0$ on Γ_{b}.

Consider a nonlinear boundary value problem:

$$\sum_{i,j=1}^{2} A_{ij}(Du, \mathbf{x})D_{ij}u + \sum_{i=1}^{2} A_i(Du, \mathbf{x})D_i u = 0 \qquad \text{in } \Omega,$$

(C.5.4)
$$u = \phi_0 \qquad \text{on } \Gamma_{\mathrm{l}} \cup \Gamma_{\mathrm{r}},$$

$$B(Du, u, \mathbf{x}) = 0 \qquad \text{on } \Gamma_{\mathrm{t}},$$

$$\partial_{x_2}u = 0 \qquad \text{on } \Gamma_{\mathrm{b}}.$$

Assume that (C.5.4) satisfies that, for constants $\lambda \in (0, 1)$, $M < \infty$, $\alpha \in (0, 1)$, $\beta \in [\frac{1}{2}, 1)$, $\sigma \in (0, 1)$, and $\varepsilon \in (0, \frac{h}{10})$, the following properties holds:

(i) For any $\mathbf{x} \in \overline{\Omega}$, and $\mathbf{p}, \boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2$,

$$\lambda \operatorname{dist}(\mathbf{x}, \Gamma_{\mathrm{l}} \cup \Gamma_{\mathrm{r}})|\boldsymbol{\kappa}|^2 \le \sum_{i,j=1}^{2} A_{ij}(\mathbf{p}, \mathbf{x})\kappa_i \kappa_j \le \lambda^{-1}|\boldsymbol{\kappa}|^2.$$

(ii) For any $\mathbf{x} \in \overline{\Omega} \setminus \{\frac{\varepsilon}{2} < x_1 < h - \frac{\varepsilon}{2}\}$ and $\mathbf{p}, \boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2$,

$$\lambda|\boldsymbol{\kappa}|^2 \le \sum_{i,j=1}^{2} \frac{A_{ij}(\mathbf{p}, \mathbf{x})\kappa_i \kappa_j}{(\min\{x_1, h - x_1, \delta\})^{2 - \frac{i+j}{2}}} \le \lambda^{-1}|\boldsymbol{\kappa}|^2.$$

(iii) $(A_{ij}, A_i)(\mathbf{p}, \mathbf{x})$ are independent of $\mathbf{p}$ on $\Omega \cap \{\varepsilon \le x_1 \le h - \varepsilon\}$ with

$$\|A_{ij}\|_{L^\infty(\Omega \cap \{\varepsilon \le x_1 \le h - \varepsilon\})} + \|(A_{ij}, A_i)\|_{C^{1,\alpha}(\overline{\Omega \cap \{\varepsilon \le x \le h - \varepsilon\}})} \le M.$$

(iv) For any $\mathbf{p} \in \mathbb{R}^2$,

$$\|(A_{ij}, A_i)(\mathbf{p}, \cdot)\|_{C^\beta(\overline{\Omega \setminus \{2\varepsilon < x_1 < h - 2\varepsilon\}})} + \|(D_{\mathbf{p}}A_{ij}, D_{\mathbf{p}}A_i)(\mathbf{p}, \cdot)\|_{L^\infty(\Omega \setminus \{2\varepsilon < x_1 < h - 2\varepsilon\})} \le M.$$

(v) $(A_{ij}, A_i) \in C^{1,\alpha}(\mathbb{R}^2 \times (\overline{\Omega} \setminus \overline{\Gamma_{\mathrm{l}} \cup \Gamma_{\mathrm{r}}}))$ and

$$\|(A_{ij}, A_i)\|_{C^{1,\alpha}(\mathbb{R}^2 \times (\overline{\Omega} \cap \{s \le x_1 \le h - s\}))} \le M\left(\frac{h}{s}\right)^M \qquad \text{for all } s \in (0, \frac{h}{4}).$$

(vi) For each $(\mathbf{p}, \mathbf{x}) \in \mathbb{R}^2 \times \overline{\Omega} \setminus \{\frac{h}{4} \le x_1 \le \frac{3h}{4}\}$, define

$$\hat{\mathbf{p}} = \mathbf{p} - D\phi_0(\mathbf{x}), \qquad (a_{ij}, a_i)(\hat{\mathbf{p}}, \mathbf{x}) = (A_{ij}, A_i)(\mathbf{p}, \mathbf{x}).$$

For each $(\mathbf{p}, (x_1, 0)) \in \mathbb{R}^2 \times (\Gamma_{\mathrm{b}} \setminus \{\varepsilon \leq x_1 \leq h - \varepsilon\})$,

$$(a_{11}, a_{22}, a_1)((\hat{p}_1, -\hat{p}_2), (x_1, 0)) = (a_{11}, a_{22}, a_1)((\hat{p}_1, \hat{p}_2), (x_1, 0)),$$

and, for all $(\mathbf{p}, \mathbf{x}) \in \mathbb{R}^2 \times (\Omega \setminus \{\varepsilon \leq x_1 \leq h - \varepsilon\})$, $i = 1, 2$,

$$|a_{ii}(\mathbf{p}, (x_1, x_2)) - a_{ii}(D\phi_0(0, x_2), (0, x_2))| \leq M|x_1|^\beta \qquad \text{when } x_1 < \varepsilon,$$
$$|a_{ii}(\mathbf{p}, (x_1, x_2)) - a_{ii}(D\phi_0(h, x_2), (0, x_2))| \leq M|x_1 - h|^\beta \qquad \text{when } x_1 > h - \varepsilon.$$

In $\Omega \setminus \{\varepsilon \leq x_1 < h - \varepsilon\}$, ϕ_0 satisfies

$$\sum_{i,j=1}^{2} A_{ij}(Du, \mathbf{x})D_{ij}\phi_0 + \sum_{i=1}^{2} A_i(Du, \mathbf{x})D_i\phi_0 = 0,$$

so that the equation for u in (C.5.4) is written as an equation for $\hat{u} = u - \phi_0$ in the form:

$$\sum_{i,j=1}^{2} a_{ij}(D\hat{u}, \mathbf{x})D_{ij}\hat{u} + \sum_{i=1}^{2} a_i(D\hat{u}, \mathbf{x})D_i\hat{u} = 0.$$

(vii) For any $\mathbf{p} \in \mathbb{R}^2$ and $\mathbf{x} \in \Gamma_l \cup \Gamma_r$, $(A_{12}, A_{21})(\mathbf{p}, \mathbf{x}) = 0$.

(viii) For any $\mathbf{p} \in \mathbb{R}^2$ and $\mathbf{x} \in \Omega \setminus \{\frac{\varepsilon}{2} \leq x_1 \leq h - \frac{\varepsilon}{2}\}$, $A_1(\mathbf{p}, \mathbf{x}) \leq -\lambda$.

(ix) For any $(\mathbf{p}, z, \mathbf{x}) \in \mathbb{R}^2 \times \mathbb{R} \times \Gamma_{\mathrm{t}}$, $D_{\mathbf{p}}B(\mathbf{p}, z, \mathbf{x}) \cdot \boldsymbol{\nu}^{(1)}(\mathbf{x}) \geq \lambda$, where $\boldsymbol{\nu}^{(1)}$ is the inner unit normal vector to Γ_{t} towards the interior of Ω;

(x) For any $(\mathbf{p}, z) \in \mathbb{R}^2 \times \mathbb{R}$,

$$\|(B(D\phi_0, \phi_0, \cdot)\|_{C^3(\overline{\Omega \setminus \{\frac{h}{3} < x_1 < \frac{2h}{3}\}})} + \|D^k_{(\mathbf{p}, z)}(\mathbf{p}, z, \cdot))\|_{C^3(\overline{\Omega})} \leq M \qquad \text{for } k = 1, 2, 3,$$
$$\|D_{\mathbf{p}}B(\mathbf{p}, z, \cdot)\|_{C^0(\overline{\Omega})} \leq \lambda^{-1},$$
$$D_z B(\mathbf{p}, z, \mathbf{x}) \leq -\lambda \qquad \text{for all } \mathbf{x} \in \Gamma_{\mathrm{t}},$$
$$D_{p_1}B(\mathbf{p}, z, \mathbf{x}) \leq -\lambda \qquad \text{for all } \Gamma_{\mathrm{t}} \setminus \{\varepsilon \leq x_1 \leq h - \varepsilon\}.$$

(xi) There exist $v \in C^3(\overline{\Gamma_{\mathrm{t}}})$ and a nonhomogeneous linear operator:

$$L(\mathbf{p}, z, \mathbf{x}) = \mathbf{b}^{(1)}(\mathbf{x}) \cdot \mathbf{p} + b_0^{(1)}(\mathbf{x})z + g_1(\mathbf{x}),$$

defined for $\mathbf{x} \in \Gamma_{\mathrm{t}}$ and $(\mathbf{p}, z) \in \mathbb{R}^2 \times \mathbb{R}$, satisfying

$$\|v\|_{C^3(\overline{\Omega})} + \|(\mathbf{b}^{(1)}, b_0^{(1)}, g_1)\|_{C^3(\overline{\Gamma_{\mathrm{t}}})} \leq M$$

such that, for all $(\mathbf{p}, z, \mathbf{x}) \in \mathbb{R}^2 \times \mathbb{R} \times \Gamma_{\mathrm{t}}$,

$$|B(\mathbf{p}, z, \mathbf{x}) - L(\mathbf{p}, z, \mathbf{x})| \leq \sigma(|\mathbf{p} - Dv(\mathbf{x})| + |z - v(\mathbf{x})|),$$
$$|D_{\mathbf{p}}B(\mathbf{p}, z, \mathbf{x}) - \mathbf{b}^{(1)}(\mathbf{x})| + |D_z B(\mathbf{p}, z, \mathbf{x}) - b_0^{(1)}(\mathbf{x})| \leq \sigma.$$

From [**11**, Propositions 4.7.2 and 4.8.7], the following two propositions are obtained:

PROPOSITION C.15. *For fixed constants $\lambda > 0$, $M < \infty$, $\alpha \in (0, 1)$, $\beta \in [\frac{1}{2}, 1)$, and $\varepsilon \in (0, \frac{h}{10})$, there exist constants $\alpha_1 \in (0, \frac{1}{2})$, $\sigma \in (0, 1)$, and $\delta_0 > 0$ with α_1 depending only on λ, and (σ, δ_0) depending only on $(\lambda, M, C_{\phi_0}, \alpha, \beta, \varepsilon)$ such that the following statement holds: let domain Ω be defined by (C.5.1), and let the nonlinear boundary value problem (C.5.4) satisfy all the conditions*

stated above with h, t_h, t_1, t_2, $t_0 \geq 0$, $\varepsilon \in (0, \frac{h}{10})$, and $\delta \in [0, \delta_0)$. Then the boundary value problem (C.5.4) has a unique solution $u \in C(\overline{\Omega}) \cap C^1(\overline{\Omega} \setminus (\overline{\Gamma_1} \cup \overline{\Gamma_r})) \cap C^2(\Omega)$. Moreover, u satisfies

(C.5.5) $$\|u\|_{C^0(\overline{\Omega})} \leq C, \qquad |u(\mathbf{x}) - \phi_0(\mathbf{x})| \leq C \min\{x_1, h - x_1\} \quad \text{in } \Omega$$

with a constant $C > 0$ depending only on $(\lambda, M, C_{\phi_0}, \varepsilon)$. Furthermore, u is in $C(\overline{\Omega}) \cap C^{2,\alpha_1}(\overline{\Omega} \setminus \overline{\Gamma_1} \cup \overline{\Gamma_r})$ and satisfies

(C.5.6) $$\|u\|_{C^{2,\alpha_1}(\overline{\Omega \cap \{s < x_1 < h-s\}})} \leq C_s$$

for each $s \in (0, \frac{h}{10})$ with a constant $C_s > 0$ depending only on $(\lambda, M, C_{\phi_0}, \alpha, \beta, \varepsilon, s)$.

PROPOSITION C.16. For fixed constants $\lambda > 0$, $\delta > 0$, $M < \infty$, $\alpha \in (0, 1)$, $\beta \in [\frac{1}{2}, 1)$, and $\varepsilon \in (0, \frac{h}{10})$, there exist constants $\alpha_1 \in (0, \frac{1}{2})$, $\sigma \in (0, 1)$ with α_1 depending only on (λ, δ), and $\sigma > 0$ depending only on $(\lambda, \delta, M, C_{\phi_0}, \alpha, \beta, \varepsilon)$ such that the following statement holds: let domain Ω be of the structure of (C.5.1)–(C.5.3) with $h > 0$, $t_h > 0$, $t_1 \geq 0$, $t_2 \geq 0$, and $t_0 = 0$, that is,

$$P_1 = P_4 = (0, 0), \qquad \overline{\Gamma_1} = \{(0, 0)\},$$

and let the nonlinear boundary value problem (C.5.4) satisfy conditions (iii), (v), and (ix)–(xi) above, and the following modified conditions:

(i*) For any $\mathbf{x} \in \overline{\Omega}$ and $\mathbf{p}, \boldsymbol{\kappa} = (\kappa_1, \kappa_2) \in \mathbb{R}^2$,

$$\min\{\lambda \operatorname{dist}(\mathbf{x}, \Gamma_1) + \delta, \lambda \operatorname{dist}(\mathbf{x}, \Gamma_r)\}|\boldsymbol{\kappa}|^2 \leq \sum_{i,j=1}^{2} A_{ij}(\mathbf{p}, \mathbf{x})\kappa_i \kappa_j \leq \lambda^{-1}|\boldsymbol{\kappa}|^2,$$

$$\|(A_{ij}, A_i)(D\phi_0, \cdot), D_{\mathbf{p}}^m (A_{ij}, A_i)(\mathbf{p}, \cdot)\|_{1,\alpha,\Omega \cap \{x_1 < 2\varepsilon\}}^{(-\alpha), \{P_1\}} \leq M \qquad \text{for } m = 1, 2.$$

(ii*) Condition (ii) holds for any $\mathbf{x} \in \overline{\Omega} \cap \{\operatorname{dist}(\mathbf{x}, \Gamma_r) < \frac{\varepsilon}{2}\}$ and $\mathbf{p}, \boldsymbol{\kappa} \in \mathbb{R}^2$.

(iv*) For any $\mathbf{p} \in \mathbb{R}^2$,

$$\|(A_{ij}, A_i)(\mathbf{p}, \cdot)\|_{C^\beta(\overline{\Omega \cap \{x_1 \geq h-2\varepsilon\}})} + \|(D_{\mathbf{p}} A_{ij}, D_{\mathbf{p}} A_i)(\mathbf{p}, \cdot)\|_{L^\infty(\Omega \cap \{x_1 > h-2\varepsilon\})} \leq M.$$

(vi*) For each $(\mathbf{p}, (x_1, 0)) \in \mathbb{R}^2 \times (\Gamma_b \cap \{x_1 > h - \varepsilon\})$,

$$(a_{11}, a_{22}, a_1)((\hat{p}_1, -\hat{p}_2), (x_1, 0)) = (a_{11}, a_{22}, a_1)((\hat{p}_1, \hat{p}_2), (x_1, 0)),$$

and, for all $(\mathbf{p}, \mathbf{x}) \in \mathbb{R}^2 \times (\Omega \cap \{x_1 > h - \varepsilon\})$,

$$|a_{ii}(\mathbf{p}, (x_1, x_2)) - a_{ii}(D\phi_0(h, x_2), (0, x_2))| \leq M|x_1 - h|^\beta, \qquad i = 1, 2.$$

(vii*) Condition (vii) holds for all $\mathbf{p} \in \mathbb{R}^2$ and $\mathbf{x} \in \Gamma_r$.

(viii*) Condition (viii) holds for all $\mathbf{p} \in \mathbb{R}^2$ and $\mathbf{x} \in \Omega \cap \{x_1 > h - \frac{\varepsilon}{2}\}$.

Then the boundary value problem (C.5.4) has a unique solution $u \in C(\overline{\Omega}) \cap C^1(\overline{\Omega} \setminus (\overline{\Gamma_l} \cup \overline{\Gamma_r})) \cap C^2(\Omega)$. Moreover, solution u is in $C(\overline{\Omega}) \cap C^{2,\alpha_1}(\overline{\Omega} \setminus (\overline{\Gamma_l} \cup \overline{\Gamma_r}))$ and satisfies (C.5.5)–(C.5.6) for $C > 0$ in (C.5.5) depending only on $(\lambda, \delta, M, C_{\phi_0}, \varepsilon)$, and $C_s > 0$ depending on $(\lambda, \delta, M, C_{\phi_0}, \varepsilon, s)$. Furthermore, u satisfies

$$\|u\|_{2,\alpha_1,\Omega \cap \{x_1 < \frac{h}{4}\}}^{(-1-\alpha_1), \{P_1\}} \leq \hat{C}$$

for constant $\hat{C} > 0$ depending only on $(\delta, \lambda, M, \alpha, \varepsilon)$.

Acknowledgments

Bibliography

[1] Bae, M., Chen, G.-Q. and Feldman, M., Regularity of solutions to regular shock reflection for potential flow, *Invent. Math.* **175** (2009), 505–543.

[2] Bae, M., Chen, G.-Q. and Feldman, M., Prandtl-Meyer reflection for supersonic flow past a solid ramp, *Quart. Appl. Math.* **71** (2013), 583–600.

[3] Busemann, A., *Gasdynamik*, Handbuch der Experimentalphysik, Vol. IV, Akademische Verlagsgesellschaft, Leipzig, 1931.

[4] Chang, T., Chen, G.-Q., and Yang, S., On the Riemann problem for two-dimensional Euler equations I: Interaction of shocks and rarefaction waves, *Discrete Contin. Dynam. Systems*, **1** (1995), 555–584.

[5] Chang, T., Chen, G.-Q., and Yang, S., On the Riemann problem for two-dimensional Euler equations II: Interaction of contact discontinuities, *Discrete Contin. Dynam. Systems*, **6** (2000), 419–430.

[6] Chang, T. and Hsiao, L., *The Riemann Problem and Interaction of Waves in Gas Dynamics*, Longman Scientific & Technical: Harlow; and John Wiley & Sons, Inc.: New York, 1989.

[7] Chen, G.-Q., Supersonic flow onto solid wedges, multidimensional shock waves and free boundary problems, *Sci. China Math.* **60** (2017), 1353–1370.

[8] Chen, G.-Q., Chen, J., and Feldman, M., Transonic flows with shocks past curved wedges for the full Euler equations, *Discrete Contn. Dyn. Syst.* **36** (2016), 4179–4211.

[9] Chen, G.-Q., Chen, J., and Feldman, M., Stability and asymptotic behavior of transonic flows past wedges for the full Euler equations, *Interfaces Free Bound.* **19** (2017), 591–626.

[10] Chen, G.-Q. and Feldman, M., Global solutions to shock reflection by large-angle wedges, *Ann. Math.* **171** (2010), 1019–1134.

[11] Chen, G.-Q. and Feldman, M., *Mathematics of Shock Reflection-Diffraction and von Neumann's Conjectures*, Research Monograph, Annals of Mathematics Studies, **197**, Princeton University Press, 2018.

[12] Chen, G.-Q., Feldman, M., and Xiang, W., Convexity of self-similar transonic shock waves for potential flow, *Arch. Ration. Mech. Anal.*, 2020 (to appear). [Preprint arXiv:1804.04769, 2018]

[13] Chen, G.-Q., Feldman, M., and Xiang, W., Uniqueness and stability of the shock reflection-diffraction problem for potential flow, In: *Proceedings of the 17th International Conference on Hyperbolic Problems – Theory, Numerics, Applications*, June 25-29, 2018 (University Park, Pennsylvania), AIMS: Springfield, MO, USA, 2019 [arXiv:1904.00114].

[14] Chen, G.-Q., Feldman, M., and Xiang, W., Uniqueness and stability of regular shock reflection-diffraction configurations by wedges for potential flow, Preprint, 2020.

[15] Chen, G.-Q. and Li, T.-H., Well-posedness for two-dimensional steady supersonic Euler flows past a Lipschitz wedge, *J. Diff. Eqs.* **244** (2008), 1521–1550.

[16] Chen, G.-Q., Shahgholian, H., and Vazquez, J. L., *Free Boundary Problems and Related Topics*, Theme Issue 2050, Philosophical Transactions of the Royal Society A: Mathematical, Physical & Engineering Sciences, **373**, 2015.

[17] Chen, G.-Q., Zhang, Y., and Zhu, D., Existence and stability of supersonic Euler flows past Lipschitz wedges, *Arch. Ration. Mech. Anal.* **181** (2006), 261–310.

[18] Chen, J., Christoforou, C., and Jegdié, K., Existence and uniqueness analysis of a detached shock problem for the potential flow, *Nonlinear Anal.* **74** (2011), 705–720.

[19] Chen, S.-X. and Fang, B.-X., Stability of transonic shocks in supersonic flow past a wedge, *J. Diff. Eqs.* **233** (2007), 105–135.

[20] Chiodaroli, E., De Lellis, C. and Kreml, O., Global ill-posedness of the isentropic system of gas dynamics, *Comm. Pure Appl. Math.* **68** (2015), 1157–1190.

[21] Chiodaroli, E. and Kreml, O., On the energy dissipation rate of solutions to the compressible isentropic Euler system, *Arch. Ration. Mech. Anal.* **214** (2014), 1019–1049.

[22] Courant, R. and Friedrichs, K. O., *Supersonic Flow and Shock Waves*, Springer-Verlag: New York, 1948.

[23] Dafermos, C. M., *Hyperbolic Conservation Laws in Continuum Physics*, Fourth Ed., Springer-Verlag: Berlin, 2016.

[24] Elling, V., Counter exammples to the sonic criterion, *Arch. Ration. Mech. Anal.* **194** (2009), 987–1010.

[25] Elling, V. Non-existence of strong regular reflections in self-similar potential flow, *J. Diff. Eqs.* **252** (2012), 2085–2103.

[26] Elling, V. and Liu, T.-P., The ellipticity principle for self-similar potential flows, *J. Hyperbolic Diff. Eqs.* **2** (2005), 909–917.

[27] Elling, V. and Liu, T.-P., Supersonic flow onto a solid wedge, *Comm. Pure Appl. Math.* **61** (2008), 1347–1448.

[28] Fang, B.-X., Stability of transonic shocks for the full Euler system in supersonic flow past a wedge, *Math. Methods Appl. Sci.* **29** (2006), 1–26.

[29] Feireisl, E., Klingenberg, C., Kreml, O. and Markfelder, S, On oscillatory solutions to the complete Euler system, J. Diff. Eqs. **269**(2020), 1521–1543.

[30] Gilbarg, D. and Trudinger, N., *Elliptic Partial Differential Equations of Second Order,* Second Ed., Springer-Verlag: Berlin, 1983.

[31] Glimm, J., Klingenberg, C., McBryan, O., Plohr, B., Sharp, D. and Yaniv, S., Front tracking and two-dimensional Riemann problems, *Adv. Appl. Math.* **6** (1985), 259–290.

[32] Glimm, J. and Majda, A., *Multidimensional Hyperbolic Problems and Computations*, Springer-Verlag: New York, 1991.

[33] Keyfitz, B. L. and Warnecke, G., The existence of viscous profiels and admissibility for transonic shocks, *Comm. Partial Diff. Eqs.* **16** (1991), 1197–1221.

[34] Klingenberg, C. and Markfelder S., Non-uniqueness of energy-conservative solutions to the isentropic compressible two-dimensional Euler equations, *J. Hyper. Diff. Eqs.* **15** (2018), 721–730.

[35] Kurganov, A. and Tadmor, E., Solution of two-dimensional Riemann problems for gas dynamics without Riemann problem solvers, *Numer. Methods Partial Diff. Eqs.* **18** (2002), 584–608.

[36] Lax, P. D. and Liu, X.-D., Solution of two-dimensional Riemann problems of gas dynamics by positive schemes, *SIAM J. Sci. Comput.* **19** (1998), 319–340.

[37] Li, J., Zhang, T. and Yang, S. (1998). *The Two-Dimensional Riemann Problem in Gas Dynamics*, Longman (Pitman Monographs 98): Essex.

[38] Lieberman, G. M., Hölder continuity of the gradient at a corner for the capillary problem and related result, *Pacific J. Math.* **133** (1988), 115–135.

[39] Liu, T.-P., Multi-dimensional gas flow: some historical perespectives, *Bull. Inst. Math. Acad. Sinica* (*N.S.*), **6** (2011), 269–291.

[40] Meyer, Th., Über zweidimensionale Bewegungsvorgänge in einem Gas, das mit Überschallgeschwindigkeit strömt. Dissertation, Göttingen, 1908. *Forschungsheft des Vereins deutscher Ingenieure*, Vol. **62**, pp. 31–67, Berlin, 1908,

[41] von Neumann, J., Discussion on the existence and uniqueness or multiplicity of solutions of the aerodynamical equations, *Bull. Amer. Math. Soc.* **47** (2010), 145–154.

[42] Prandtl, L., Allgemeine Überlegungen über die Strömung zusammendrückbarer Flüssigkeiten, *Zeitschrift für angewandte Mathematik und Mechanik*, **16** (1936), 129–142.

[43] Schulz-Rinne, C. W., Collins, J. P. and Glaz, H. M., Numerical solution of the Riemann problem for two-dimensional gas dynamics, *SIAM J. Sci. Comput.* **14** (1993), 1394–1414.

[44] Serre, D., von Neumann's comments about existence and uniqueness for the initial-boundary value problem in gas dynamics, *Bull. Amer. Math. Soc. (N.S.)*, **47** (2010), 139–144.

[45] Whitham, G. B., *Linear and Nonlinear Waves*, Wiley-Interscience [John Wiley & Sons]: New York-London-Sydney, 1974.

[46] Yuan, H., *On transonic shocks in two-dimensional variable-area ducts for steady Euler system*, *SIAM J. Math. Anal.* **38** (2006), 1343–1370.

[47] Zeidler, E., *Nonlinear Functional Analysis and Its Applications I. Fixed Point Theorems,* Translated by P. R. Wadsack, Springer-Verlag: New York, 1986.

[48] Zhang, Y., Steady supersonic flow past an almost straight wedge with large vertex angle. *J. Diff. Eqs.* **192** (2003), 1–46.

[49] Zheng, Y., *Systems of Conservation Laws*: *Two-Dimensional Riemann Problems*, Birkhäuser: Boston, 2001.