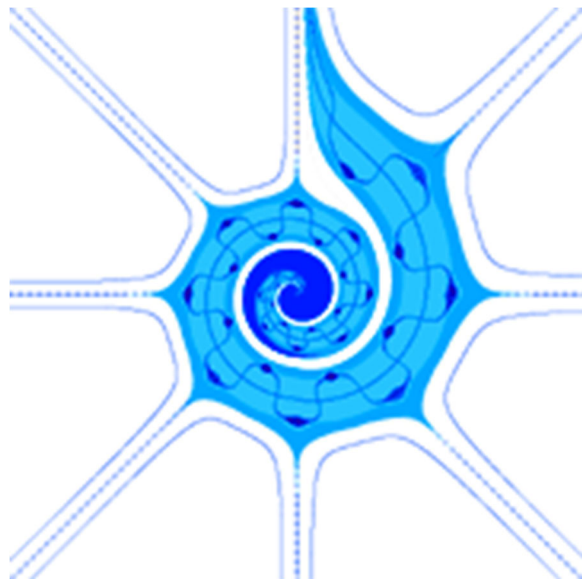


PRECALCULUS

Math 2412

Turnell

Updated Dec. 2019-page 10 #20 answer



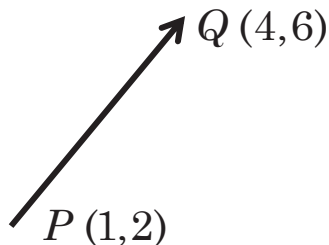
Vectors

Definition: A vector has _____ and _____.

Definition: A scalar has only _____.

Vectors	Scalars

A vector is represented geometrically as a directed line segment where the magnitude of the vector is taken to be the length of the line segment and the direction is made clear with the use of an arrow at one endpoint of the segment.



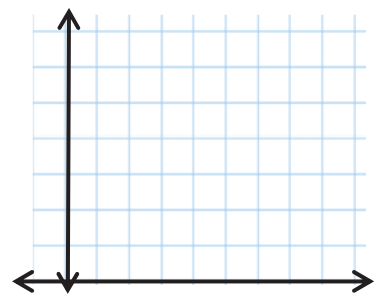
The point P is called the _____ point or _____ and the point Q is

called the _____ point or _____ of the vector \vec{v} . $\vec{v} = \overrightarrow{PQ}$

Component Form:

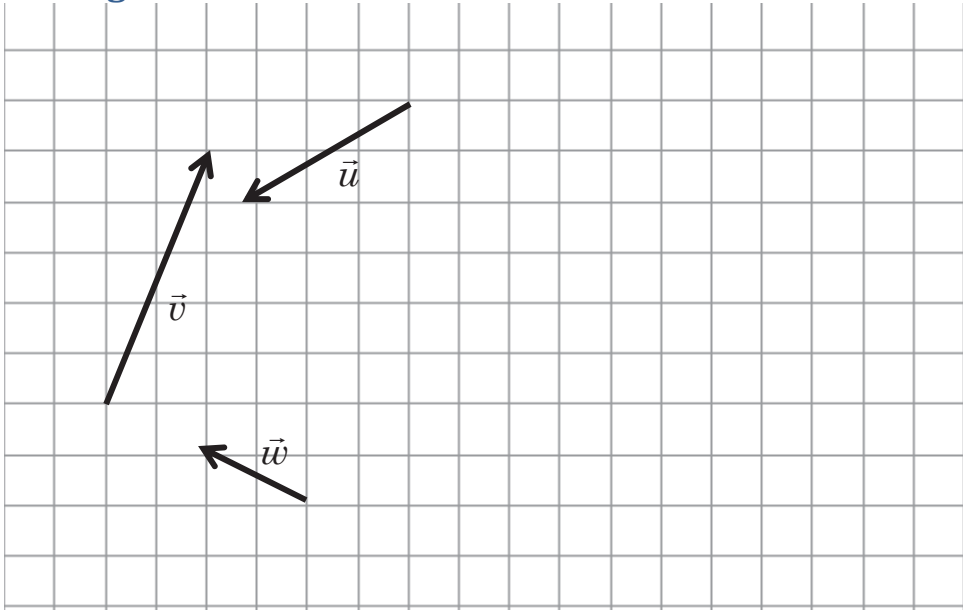
One way to write a vector is in component form:
 $\vec{v} = \langle a, b \rangle$, where a is the x -component and b is the y -component.

$$\vec{v} = \langle (x_2 - x_1), (y_2 - y_1) \rangle \quad (\text{Terminal} - \text{Initial})$$



Write \overrightarrow{PQ} in component form and graph \overrightarrow{PQ} . Some books call this a _____ vector. It is convenient because the initial point is $(0,0)$

Adding Vectors.



Find the following:

1. $\vec{u} + \vec{v}$
2. $\vec{w} + \vec{v}$
3. $\vec{u} - \vec{w}$
4. $\vec{u} + \vec{v} + \vec{w}$
5. $\vec{v} + 2\vec{w}$

Find the magnitude of a vector :

If $\vec{v} = \langle a, b \rangle$, the magnitude of a vector is the _____ or _____ of a directed line segment. The magnitude is defined to be:

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2}$$

If $P_1 = (-3, 2)$ and $P_2 = (6, 5)$, find the magnitude of $\overrightarrow{P_1P_2}$

Find a unit vector in the direction of a given vector:

A unit vector is a vector that has a length or magnitude of _____.

For any nonzero vector \mathbf{v} , the vector $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector that has the same direction as \mathbf{v} .

Example:

Find a unit vector in the direction of $\mathbf{v} = \langle 3, -4 \rangle$

Principal Unit Vectors

The vector \hat{i} is defined by $\hat{i} = \langle 1, 0 \rangle$ and The vector \hat{j} is defined by $\hat{j} = \langle 0, 1 \rangle$

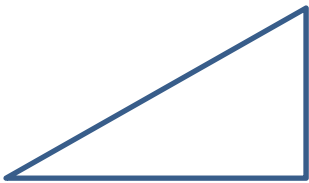
We can think of the vector \hat{i} as representing the positive x -direction, while \hat{j} represents the positive y -direction.

Example:

Find a unit vector in the direction of $\mathbf{v} = \langle 2, -1 \rangle$. Write your answer in the form of $\bar{v} = a\hat{i} + b\hat{j}$

Find the components of a vector.

Review of trigonometry:



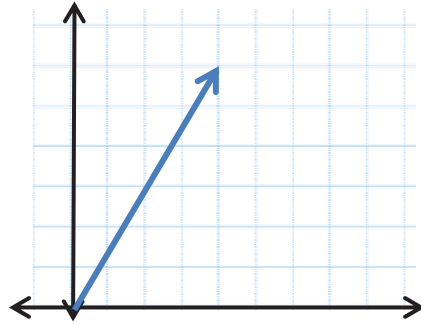
Find the vector with the given information. Write your answer in the form of $\bar{v} = a\hat{i} + b\hat{j}$

Given $\|\mathbf{v}\| = 8$ and the angle, θ , that \mathbf{v} makes with the positive x -axis is 45°

Given $\|\mathbf{v}\| = 10$ and the angle, θ , that \mathbf{v} makes with the positive x -axis is 240°

Direction angle of a vector:

The direction angle of a vector is the angle that the vector makes with the positive x -axis:



Find the direction angle of the given vectors.

$$\mathbf{v} = \langle 3, 4 \rangle$$

$$\mathbf{v} = \langle -5, 5 \rangle$$

$$\mathbf{v} = \langle -2, -7 \rangle$$

$$\mathbf{v} = \langle 3, -8 \rangle$$

Application: A car weighing 3000 lbs. is parked on a driveway that is inclined 15° to the horizontal. Find the magnitude of the force required to prevent the car from rolling down the driveway. Find the magnitude of the force exerted by the car on the driveway. (How does this relate to the "Normal" force that you learned in physics class?)

3. An airplane has an airspeed of 600 km/hr bearing 30° east of south. The wind velocity is 40 km/hr in the direction of 45° west of north. Find the resultant vector representing the path of the plane relative to the ground. What is the ground speed of the plane? What is its direction?

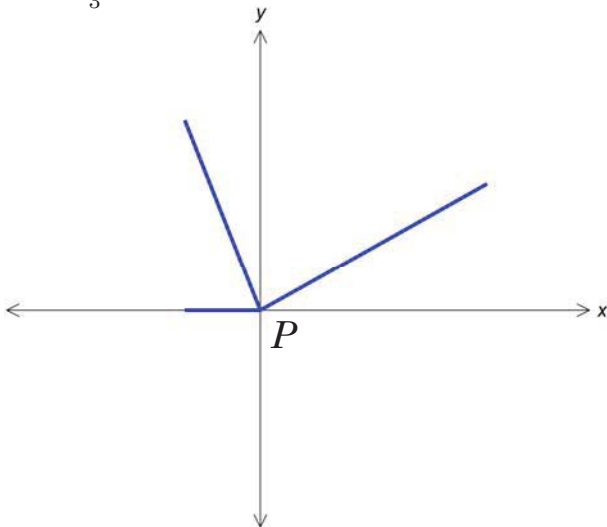
Definition: When all the forces that act upon an object are balanced, then the object is said to be in a state of **equilibrium**. The forces are considered to be balanced if the rightward forces are balanced by the leftward forces and the upward forces are balanced by the downward forces. In other words, their **resultant** equals 0. This is a very important concept in physics (engineering).

4. Find the direction and magnitude of the force that will keep object P in static equilibrium.

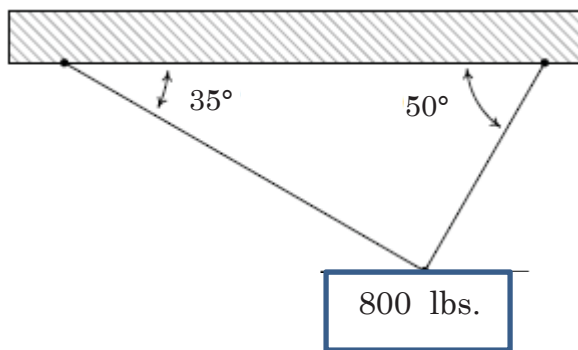
$$F_1 = 150 \text{ lbs. at } 110^\circ$$

$$F_2 = 75 \text{ lbs. at } 180^\circ$$

$$F_3 = 240 \text{ lbs. at } 35^\circ$$

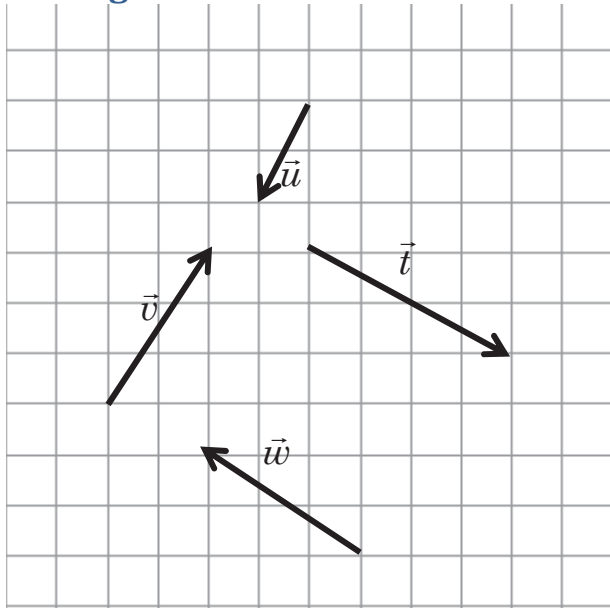


5. A weight of 800 pounds is suspended from two cables. What are the tensions in the 2 cables?



Vectors

Adding Vectors.



Graph the following vectors.

1. $\vec{u} + 2\vec{v}$

2. $\vec{w} - \vec{t}$

Write the following in component form.

3. $3\vec{u} - \vec{w}$

4. $\vec{u} + \vec{t} + \vec{w}$

5. $3\vec{t} - 2\vec{w}$

6. The vector \vec{v} has initial point $P = (9, 7)$ and terminal point $Q = (6, 8)$.

a) Write \vec{v} in component form, that is, find its position vector.

b) Find the magnitude of \vec{v} .

7. The vector \vec{v} has initial point $P = (3, 6)$ and terminal point $Q = (-3, 4)$.

a) Write \vec{v} in component form, that is, find its position vector.

b) Find the magnitude of \vec{v} .

Find a unit vector in the direction of \vec{v} .

Write your answer in the form of $\vec{v} = a\hat{i} + b\hat{j}$

8. $\mathbf{v} = \langle 6, -8 \rangle$.

9. $\mathbf{v} = \langle 2, -1 \rangle$.

Find the vector with the given information.

Write your answer in the form of $\vec{v} = a\hat{i} + b\hat{j}$ (exact values)

10. Given $\|\mathbf{v}\| = 9$ and the angle, θ , that \mathbf{v} makes with the positive x -axis is 30°

11. Given $\|\mathbf{v}\| = 12$ and the angle, θ , that \mathbf{v} makes with the positive x -axis is 300°

12-14 Find the direction angle of the given vector. Round approximations to two decimal places.

12. $\vec{v} = \langle -5, 5 \rangle$

13. $\vec{v} = \langle 2, -5 \rangle$

14. $\vec{v} = \langle -2, -4 \rangle$

15. A car weighing 2500 lbs. is parked on a driveway that is inclined 10° to the horizontal. Find the magnitude of the force required to prevent the car from rolling down the driveway. Find the magnitude of the force exerted by the car on the driveway.

16. Three forces act on an object: $\vec{F}_1 = \langle 2, 5 \rangle$, $\vec{F}_2 = \langle 8, 3 \rangle$, $\vec{F}_3 = \langle 0, -7 \rangle$. Find the direction and magnitude of the resultant force.

Find the magnitude and direction of the resultant vector. All angles are measured from the positive x-axis.

17. $\|\vec{F}\| = 37$ and $\alpha = 175^\circ$, $\|\vec{G}\| = 15$ and $\beta = 230^\circ$, $\|\vec{J}\| = 29$ and $\varphi = 310^\circ$.

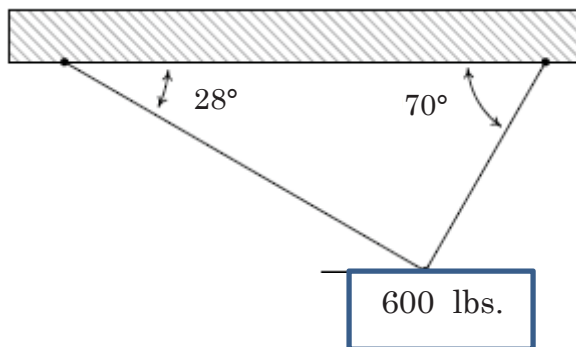
18. $\|\vec{F}\| = 25$ and $\alpha = 78^\circ$, $\|\vec{G}\| = 31$ and $\beta = 160^\circ$, $\|\vec{J}\| = 16$ and $\varphi = 240^\circ$.

19. Two forces of magnitude 20 newtons and 50 newtons act on an object at angles of 35° and 160° with the positive x-axis. Find the direction and magnitude of the resultant force.

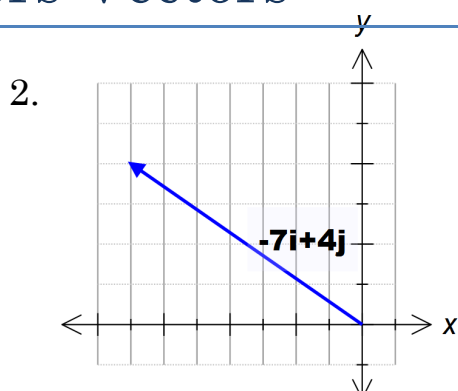
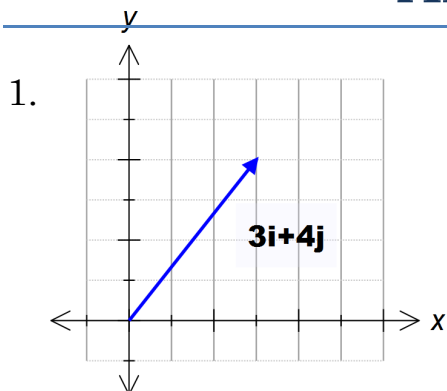
20. A jet maintains a constant airspeed of 750 mi/hr headed due west. The jet stream is 80 mi/hr in the southeasterly direction. Find the resultant vector representing the path of the plane relative to the ground. What is the ground speed of the plane? What is its direction?

21. An airplane has an airspeed of 650 km/hr bearing 15° north of west. The wind velocity is 19 km/hr in the direction of 30° east of south. Find the resultant vector representing the path of the plane relative to the ground. What is the ground speed of the plane? What is its direction?

22. A weight of 600 pounds is suspended from two cables. What are the tensions in the 2 cables?



Answers-Vectors



3. $\langle 0, -8 \rangle$

4. $\langle 0, -2 \rangle$

5. $\langle 18, -10 \rangle$

6. a) $\langle -3, 1 \rangle$

b) $\sqrt{10}$

7. a) $\langle -6, -2 \rangle$

b) $2\sqrt{10}$

8. $\frac{3}{5}i - \frac{4}{5}j$

9. $\frac{2}{\sqrt{5}}i - \frac{1}{\sqrt{5}}j$

10. $\frac{9\sqrt{3}}{2}i + \frac{9}{2}j$

11. $6i - 6\sqrt{3}j$

12. $\theta = 135^\circ$

13. $\theta = 291.8^\circ$

14. $\theta = 243.4^\circ$

15. 2462 lbs. & 434 lbs.

16. $\theta = 5.7^\circ, \sqrt{101}$

17. $R = -27.86i - 30.48j$

$\|R\| = 41.3, \theta = 227.6^\circ$

18. $R = -31.93i + 21.2j$

$\|R\| = 38.33, \theta = 146.4^\circ$

19. $R = -30.6i + 28.57j$

$\|R\| = 41.87, \theta = 137^\circ$

20. $R = -693.4i - 56.57j$

$\|R\| = 695.7$ (speed), $\theta = 184.7^\circ$

or $\theta = 4.7^\circ$ south of west

21. $R = -618.4i + 151.8j$

$\|R\| = 636.7$ (speed), $\theta = 166.2^\circ$

or $\theta = 13.8^\circ$ north of west

22. $T_1 = 535$ lbs. $T_2 = 207$ lbs.

Dot Product

Definition: If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j}$ are two vectors, the **dot** product is defined as

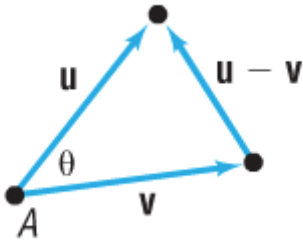
$$\mathbf{v} \cdot \mathbf{w} =$$

The **dot product** is a _____, NOT a _____

Example 1: Given two vectors: $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} - 7\mathbf{j}$. Find $\mathbf{v} \cdot \mathbf{w}$

Property of Dot Product is $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

The dot product can be used to find the angle between 2 vectors.



$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{u}\|}, \text{ where } 0 \leq \theta \leq 180^\circ$$

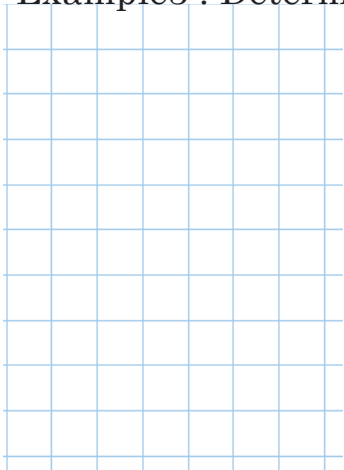


Example 2 : Find the angle between the given vectors: $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} - 7\mathbf{j}$.

How to determine if 2 vectors are **parallel**:

The angle between the two vectors are 0° or 180°

Example3 : Determine if the two vectors are parallel: $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = 6\mathbf{i} - 2\mathbf{j}$.



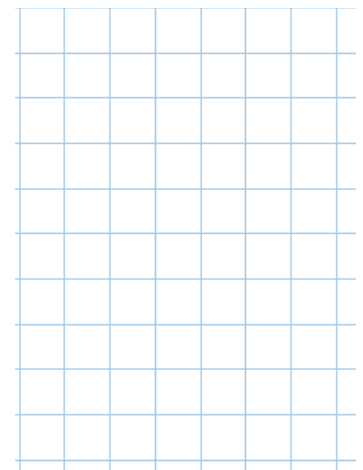
How to determine if 2 vectors are **orthogonal**:

Two vectors are **orthogonal** if the angle between the two vectors is 90°

Since the $\cos(90^\circ) = \underline{\hspace{2cm}}$ and $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{u}\|}$, then we can conclude:

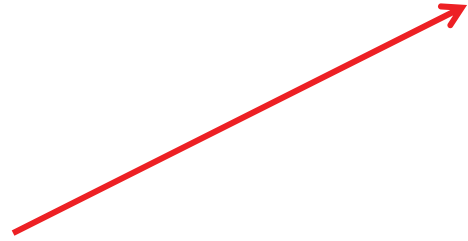
Example 4 : Determine if the two vectors are **orthogonal**:

$\mathbf{v} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = 3\mathbf{i} + 6\mathbf{j}$.



Decompose a Vector into Two Orthogonal Vectors

Decompose \mathbf{v} into 2 vectors \mathbf{v}_1 and \mathbf{v}_2
where \mathbf{v}_1 is parallel to \mathbf{w}
and \mathbf{v}_2 is orthogonal to \mathbf{w}



$$\mathbf{v}_1 = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{w}\|^2} \right) \mathbf{w} \text{ and } \mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1$$

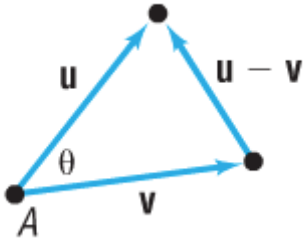
Example 5:

Find the vector projection of $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$ onto $\mathbf{w} = \mathbf{i} + \mathbf{j}$. Decompose \mathbf{v} into 2 vectors \mathbf{v}_1 and \mathbf{v}_2 where \mathbf{v}_1 is parallel to \mathbf{w} and \mathbf{v}_2 is orthogonal to \mathbf{w} .

Example 6:

Find the vector projection of $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$ onto $\mathbf{w} = \mathbf{i} - 2\mathbf{j}$. Decompose \mathbf{v} into 2 vectors \mathbf{v}_1 and \mathbf{v}_2 where \mathbf{v}_1 is parallel to \mathbf{w} and \mathbf{v}_2 is orthogonal to \mathbf{w} .

The dot product can be used to find the angle between 2 vectors.



$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{u}\|}, \text{ where } 0 \leq \theta \leq 180^\circ$$

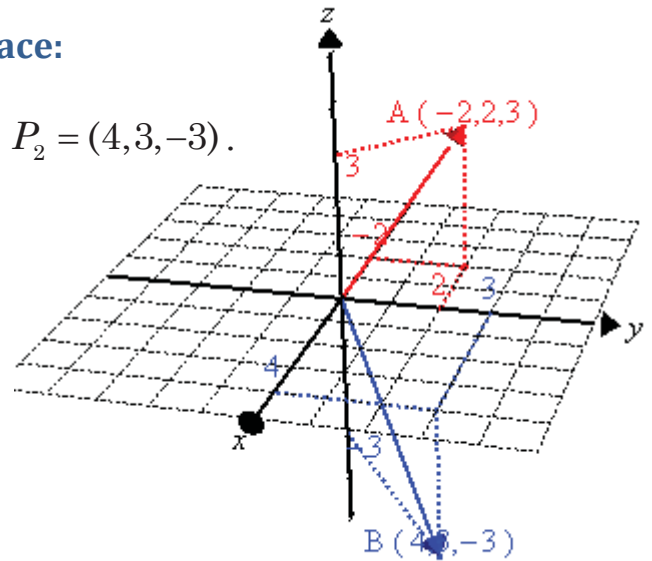
$$\text{Law of cosines: } c^2 = a^2 + b^2 - 2ab \cos \theta$$

Vectors in Space

Find the distance between two points in 3-space:

Find the distance between $P_1 = (-2, 2, 3)$ and $P_2 = (4, 3, -3)$.

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



Find a position vector.

Suppose that \mathbf{v} is a vector with initial point $P_1 = (x_1, y_1, z_1)$ and terminal point

$P_2 = (x_2, y_2, z_2)$. If $\mathbf{v} = \overrightarrow{P_1P_2}$, then \mathbf{v} is equal to the _____ vector.

$$\mathbf{v} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$$

(Terminal - Initial)

Find the position vector of the vector $\mathbf{v} = \overrightarrow{P_1P_2}$ if $P_1 = (-2, 2, 3)$ and $P_2 = (4, 3, -3)$.

Find the magnitude of a vector in space:

If $\mathbf{v} = \langle a, b, c \rangle$ then the magnitude of a vector is the _____ of a directed line segment. The magnitude is defined to be:

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}$$

Find the magnitude of $\mathbf{v} = \overrightarrow{P_1P_2}$ (see previous example)

Find the Dot Product of a vector in space:

Definition: If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{u} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ are two vectors, the **dot product** is defined as

$$\mathbf{v} \cdot \mathbf{u} =$$

The **dot product** is a _____, NOT a _____

Example: Given two vectors: $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{u} = \mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$. Find $\mathbf{v} \cdot \mathbf{u}$

Find the angle between 2 vectors in space:

Recall that the angle between 2 vectors can be found by using

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{u}\|}, \text{ where } 0 \leq \theta \leq 180^\circ$$

Find the angle between \mathbf{v} and \mathbf{u} (see given vectors in previous example)

Find a unit vector in the direction of a given vector:

Recall a unit vector is a vector that has a length or magnitude of _____.

For any nonzero vector \mathbf{v} , the vector $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ is a unit vector that

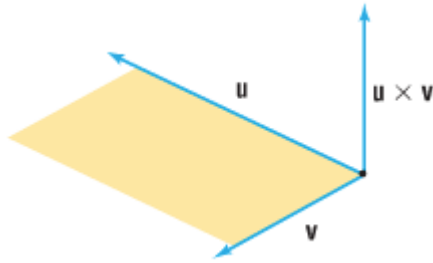
has the same direction as \mathbf{v} .

Example:

Find a unit vector in the direction of $\mathbf{v} = 9\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

Cross Product

The **cross product** is a _____ that is orthogonal to the plane of \mathbf{v} and \mathbf{u}



Investigate the "Right-Hand" Rule

Definition: If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{u} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ are two vectors, the **cross product** is defined as

$$\mathbf{v} \times \mathbf{u} = (b_1c_2 - c_1b_2)\mathbf{i} + (c_1a_2 - a_1c_2)\mathbf{j} + (a_1b_2 - b_1c_2)\mathbf{k}$$

Instead of memorizing such an intricate formula, we can use the following methods to help us find the cross product.

Example 1: Given two vectors: $\mathbf{v} = -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = 3\mathbf{i} - 2\mathbf{j} - \mathbf{k}$. Find $\mathbf{v} \times \mathbf{w}$

Option 1: Using determinants :

A 2×2 determinant: $\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix}$

$$\begin{vmatrix} i & j & k \\ -1 & 3 & 2 \\ 3 & -2 & -1 \end{vmatrix}$$

Option 2: Using diagonals:

$$\begin{vmatrix} i & j & k \\ -1 & 3 & 2 \\ 3 & -2 & -1 \end{vmatrix}$$

Example 2: Given 3 vectors: $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = -3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$.
Find $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

Dot and Cross Product

Find $\vec{u} \cdot \vec{v}$

1. $\vec{u} = -4\vec{i} - 2\vec{j}$, $\vec{v} = \vec{i} - 5\vec{j}$

2. $\vec{u} = -5\vec{i} + 6\vec{j}$, $\vec{v} = 4\vec{i} - 7\vec{j}$

3. $\vec{u} = -3\vec{i} - 2\vec{j} + 5\vec{k}$, $\vec{v} = 2\vec{i} - \vec{j} + 6\vec{k}$

4. $\vec{u} = \vec{i} + 4\vec{j} - 2\vec{k}$, $\vec{v} = 3\vec{i} + \vec{j} - 6\vec{k}$

Find the angle between the given vectors. Round to nearest degree.

5. $\vec{u} = \langle -2, -7 \rangle$, $\vec{v} = \langle 5, -9 \rangle$

6. $\vec{u} = \langle -8, 3 \rangle$, $\vec{v} = \langle 2, 6 \rangle$

7. $\vec{u} = \langle 1, -4, 2 \rangle$, $\vec{v} = \langle 2, 5, -3 \rangle$

8. $\vec{u} = \langle 3, -5, -1 \rangle$, $\vec{v} = \langle 4, 6, -10 \rangle$

Determine if the given vectors are parallel, orthogonal, or neither.

9. $\vec{u} = \langle 3, 4 \rangle$, $\vec{v} = \langle -6, -8 \rangle$

10. $\vec{u} = \langle -8, 4 \rangle$, $\vec{v} = \langle 2, 4 \rangle$

11. $\vec{u} = \langle -2, 7 \rangle$, $\vec{v} = \langle 3, -9 \rangle$

12. $\vec{u} = \langle -5, -6 \rangle$, $\vec{v} = \langle 12, -10 \rangle$

Decompose \vec{v} into two vectors \vec{v}_1 and \vec{v}_2 , where \vec{v}_1 is parallel to \vec{w} and \vec{v}_2 is orthogonal to \vec{w} . Leave answers in fractional form. NO DECIMALS

13. $\vec{v} = -8\vec{i} + 3\vec{j}$, $\vec{w} = 2\vec{i} + 6\vec{j}$

14. $\vec{v} = \vec{i} - 3\vec{j}$, $\vec{w} = 4\vec{i} - \vec{j}$

The vector \vec{v} has initial point P and terminal point Q . a) Write \vec{v} in the form $a\vec{i} + b\vec{j} + c\vec{k}$; that is, find its position vector. b) Find the magnitude of \vec{v}

15. $P = (-2, 5, 1)$; $Q = (-7, 1, -4)$

16. $P = (1, -2, 3)$; $Q = (5, -7, 4)$

Find a unit vector in the direction of \vec{v} .

17. $\vec{v} = \langle 1, 4, -3 \rangle$.

18. $\vec{v} = \langle -2, 6, 1 \rangle$.

Evaluate the given determinants.

19. $\begin{vmatrix} -5 & 6 \\ 4 & 2 \end{vmatrix}$

20. $\begin{vmatrix} 8 & -9 \\ -4 & 6 \end{vmatrix}$

Use the given vectors \vec{v} , \vec{u} , \vec{w} to find each expression.

$\vec{u} = \langle 2, -3, 1 \rangle$, $\vec{v} = \langle -3, 3, 2 \rangle$, $\vec{w} = \langle 1, 1, 3 \rangle$

21. $\vec{v} \times \vec{w}$

22. $\vec{u} \times \vec{w}$

23. $\vec{u} \cdot (\vec{w} \times \vec{v})$

24. $(\vec{v} \times \vec{u}) \cdot \vec{w}$

Answers—Dot and Cross Product

1. 6
2. -62
3. 26
4. 19
5. $\theta = 45^\circ$
6. $\theta = 87.8^\circ$
7. $\theta = 148.2^\circ$
8. $\theta = 96.4^\circ$
9. parallel

10. orthogonal

11. neither

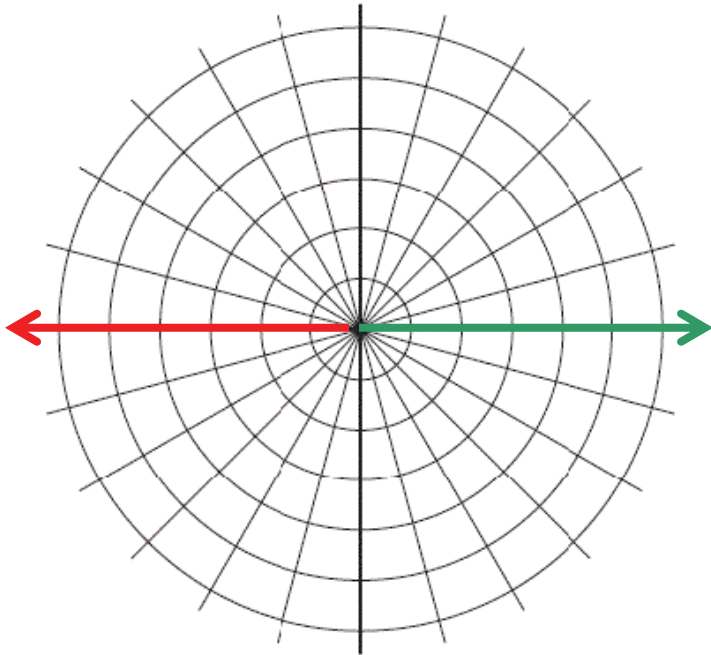
12. orthogonal
13. $v_1 = \frac{1}{10}i + \frac{3}{10}j$
 $v_2 = \frac{-81}{10}i + \frac{27}{10}j$

14. $v_1 = \frac{28}{17}i - \frac{7}{17}j$
 $v_2 = \frac{-11}{17}i - \frac{44}{17}j$
15. a) $PQ = -5i - 4j - 5k$
b) $\|PQ\| = \sqrt{66}$
16. a) $PQ = 4i - 5j + k$
b) $\|PQ\| = \sqrt{42}$
17. $\frac{1}{\sqrt{26}}i + \frac{4}{\sqrt{26}}j - \frac{3}{\sqrt{26}}k$
18. $\frac{-2}{\sqrt{41}}i + \frac{6}{\sqrt{41}}j + \frac{1}{\sqrt{41}}k$
19. -34
20. 12
21. $\langle 7, 11, -6 \rangle$
22. $\langle -10, -5, 5 \rangle$
23. 25
24. 25

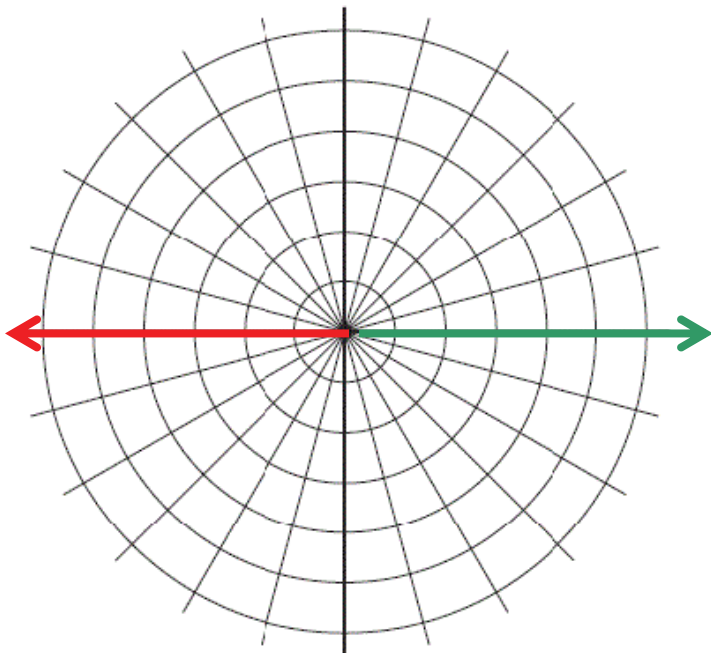
Polar Coordinates

How to plot polar coordinates: (r, θ)

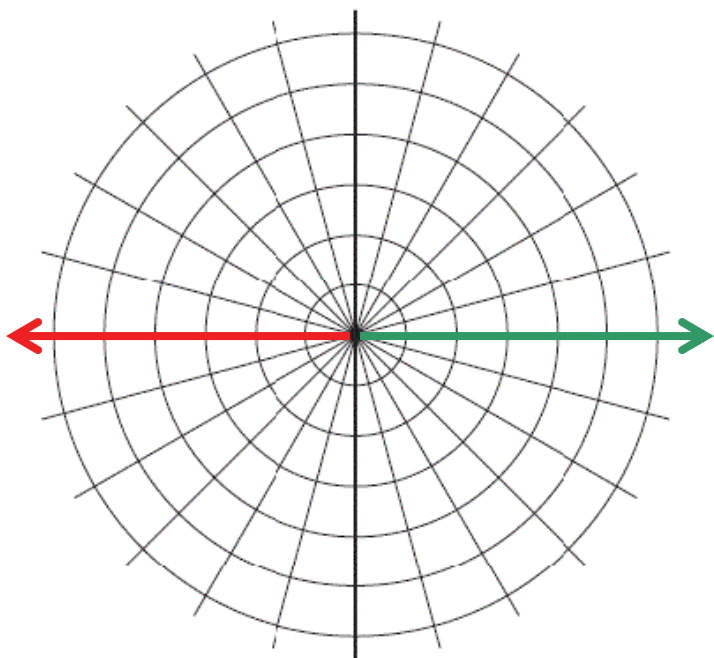
Example 1: Plot the point $(5, 120^\circ)$. Give three equivalent expressions for the point.



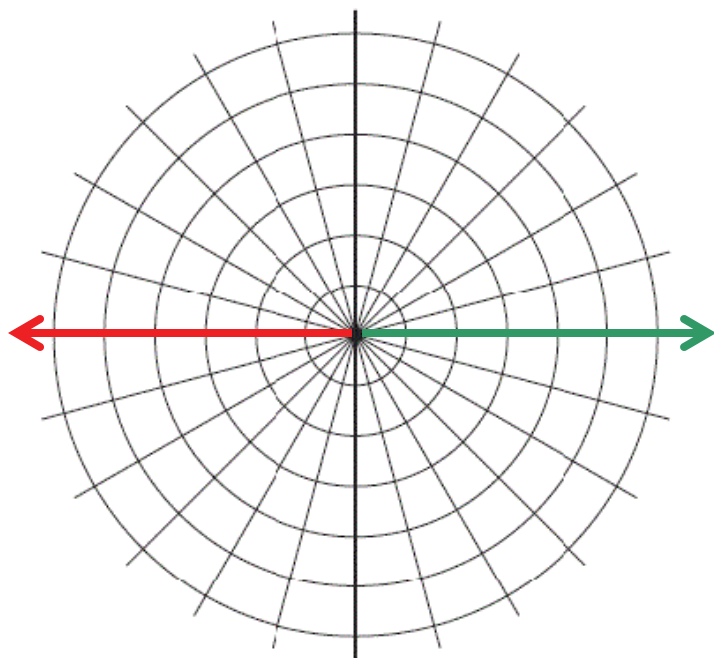
Example 2: Plot the point $(-4, 210^\circ)$. Give three equivalent expressions for the point.



Example 3: Plot the point $\left(5, -\frac{\pi}{3}\right)$. Give three equivalent expressions for the point.

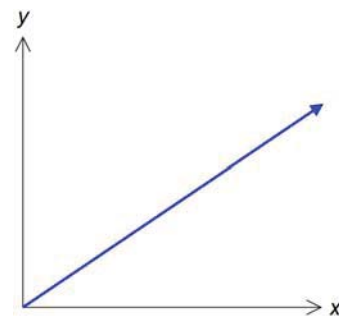


Example 4: Plot the point $\left(-6, -\frac{5\pi}{4}\right)$. Give three equivalent expressions for the point.



Convert Polar to Rectangular and Rectangular to Polar

Polar (r, θ)	Rectangular (x, y)



Convert the point from rectangular coordinates into polar coordinates where $r \geq 0$ and $0 \leq \theta < 2\pi$

Example 5: $(-2, -2\sqrt{3})$

Example 6: $(-3, 2)$

Convert the point from polar coordinates into rectangular coordinates.

Ex7: $(5, 300^\circ)$

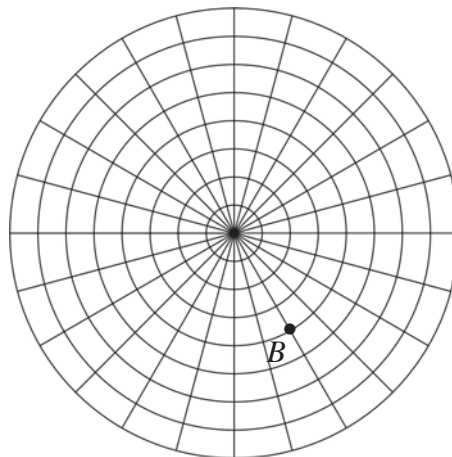
Ex 8: $(-4, \frac{5\pi}{6})$

Ex 9: $(8.1, 5.2)$

Polar Coordinates

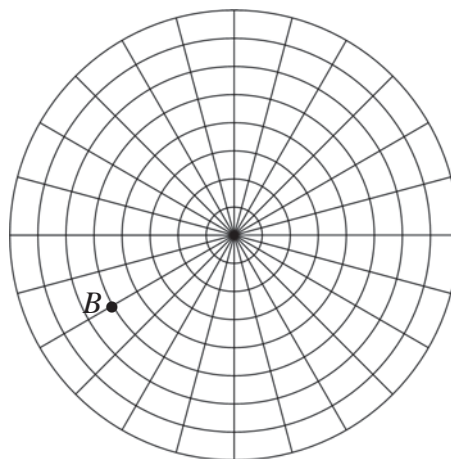
1. Given point B on the graph, give 4 different polar coordinates for point B with the following restrictions: RADIANS

- (a) $r > 0, 0 \leq \theta < 2\pi$
- (b) $r < 0, 0 \leq \theta < 2\pi$
- (c) $r > 0, -2\pi < \theta \leq 0$
- (d) $r < 0, -2\pi < \theta \leq 0$



2. Given point B on the graph, give 4 different polar coordinates for point B with the following restrictions: RADIANS

- (a) $r > 0, 0 \leq \theta < 2\pi$
- (b) $r < 0, 0 \leq \theta < 2\pi$
- (c) $r > 0, -2\pi < \theta \leq 0$
- (d) $r < 0, -2\pi < \theta \leq 0$



Convert the point from rectangular coordinates into polar coordinates where $r \geq 0$ and $0 \leq \theta < 2\pi$

- 3. (4,2)
- 4. (-4,6)
- 5. (3,-5)
- 6. (-4,-7)

Convert the point from polar coordinates into rectangular coordinates. Use exact values, if possible.

- 7. $\left(7, \frac{7\pi}{6}\right)$
- 8. $\left(6, -\frac{\pi}{4}\right)$
- 9. $\left(12, -\frac{\pi}{3}\right)$
- 10. $\left(3, \frac{\pi}{2}\right)$
- 11. (3,2)
- 12. (7,1)

Answers—Polar Coordinates

1. a) $\left(4, \frac{5}{3}\pi\right)$

b) $\left(-4, \frac{2}{3}\pi\right)$

c) $\left(4, \frac{-1}{3}\pi\right)$

d) $\left(-4, \frac{-4}{3}\pi\right)$

2. a) $\left(5, \frac{7}{6}\pi\right)$

b) $\left(-5, \frac{1}{6}\pi\right)$

c) $\left(5, \frac{-5}{6}\pi\right)$

d) $\left(-5, \frac{-11}{6}\pi\right)$

3. $(2\sqrt{5}, 0.46)$

4. $(2\sqrt{13}, 2.16)$

5. $(\sqrt{34}, 5.25)$

6. $(\sqrt{65}, 4.19)$

7. $\left(\frac{-7\sqrt{3}}{2}, \frac{-7}{2}\right)$

8. $(3\sqrt{2}, -3\sqrt{2})$

9. $(6, -6\sqrt{3})$

10. $(0, 3)$

11. $(-1.25, 2.73)$

12. $(3.78, 5.89)$

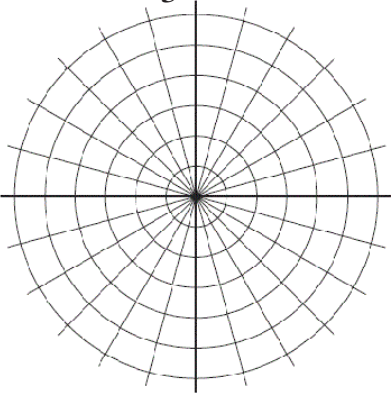
Polar Graphs

Lines: $\theta = a$

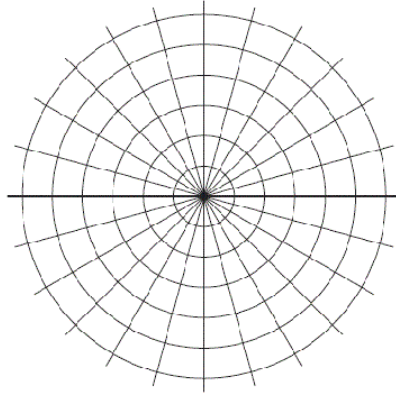
$$r \cos \theta = a$$

$$r \sin \theta = a$$

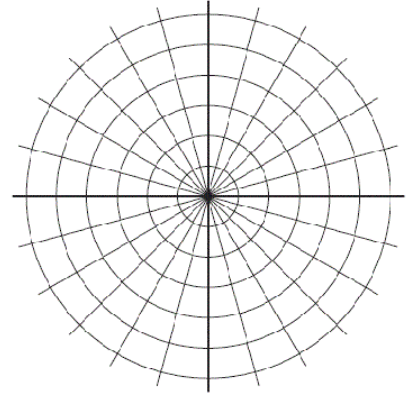
$$\theta = \frac{\pi}{3}$$



$$r \cos \theta = 4$$



$$r \sin \theta = -3$$



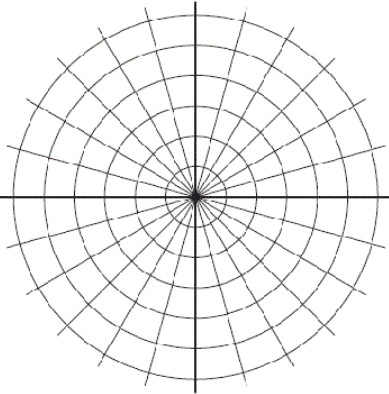
Circles: $r = a$

$$r = \pm 2a \cos \theta$$

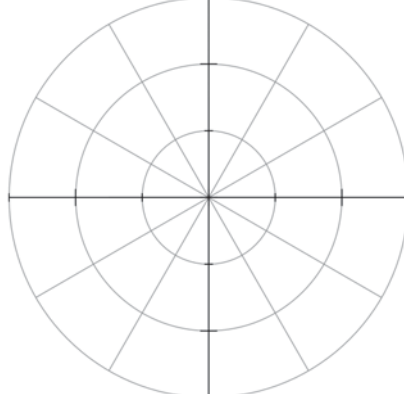
$$r = \pm 2a \sin \theta$$

The _____ of the circle is $2a$

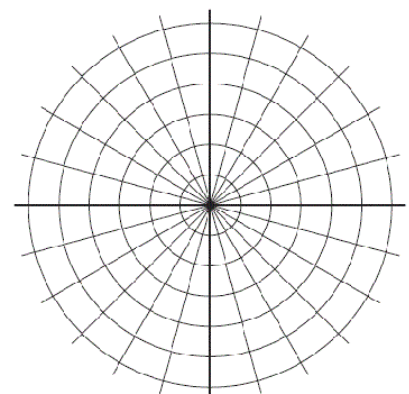
$$r = 3$$



$$r = -2 \cos \theta$$



$$r = 6 \sin \theta$$



Use Graphing Calculator to investigate symmetry:

For TI-83: MODE is Pol, Deg. or Rad,

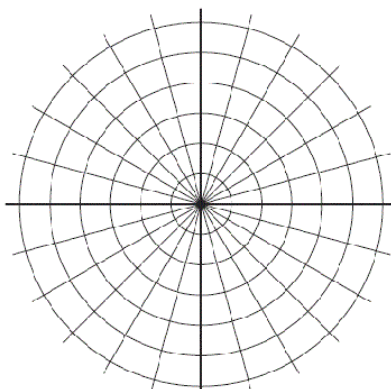
WINDOW must correspond with Deg. or Rad.

Graph the following with calculator: $r = 3 + 3 \cos \theta$, $r = 3 - 3 \cos \theta$, $r = 3 + 4 \sin \theta$,
 $r = 5 - 3 \sin \theta$

Cardioids (hearts): $r = a \pm a \cos \theta$ $r = a \pm a \sin \theta$

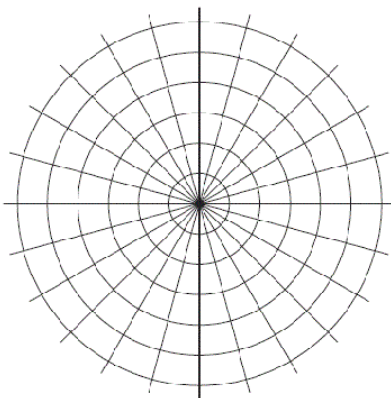
$$r = 3 + 3 \cos \theta$$

θ	r
0°	
30°	
45°	
60°	
90°	
180°	
270°	



$$r = 2 - 2 \sin \theta$$

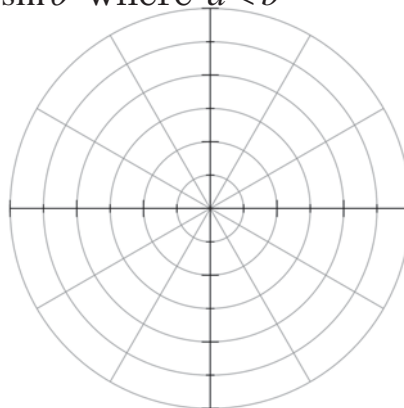
θ	r
0°	
90°	
180°	
270°	



Limaçons with a loop: $r = a \pm b \cos \theta$ $r = a \pm b \sin \theta$ where $a < b$

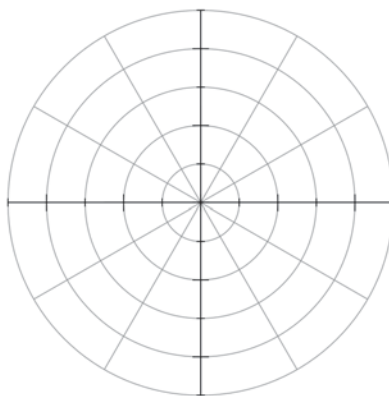
$$r = 2 - 4 \cos \theta$$

θ	r
0°	
60°	
90°	
180°	
270°	
300°	



$$r = 1 + 4 \sin \theta$$

θ	r
0°	
90°	
180°	
270°	
360°	

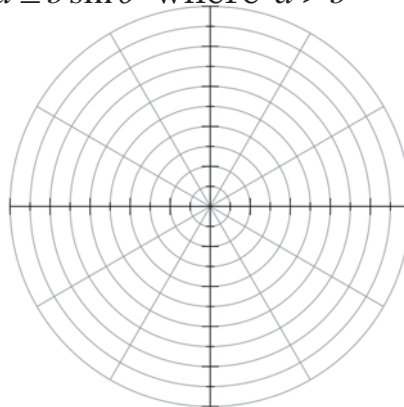


The graph passes through the pole when $\theta =$ _____ and _____.

Limaçons without a loop : $r = a \pm b \cos \theta$ $r = a \pm b \sin \theta$ where $a > b$

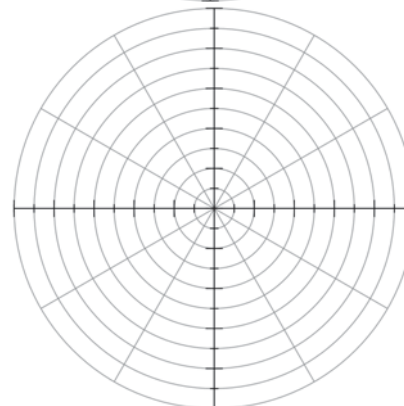
$$r = 6 + 4 \cos \theta$$

θ	r
0°	
90°	
180°	
270°	



$$r = 5 - 4 \sin \theta$$

θ	r
0°	
90°	
180°	
270°	

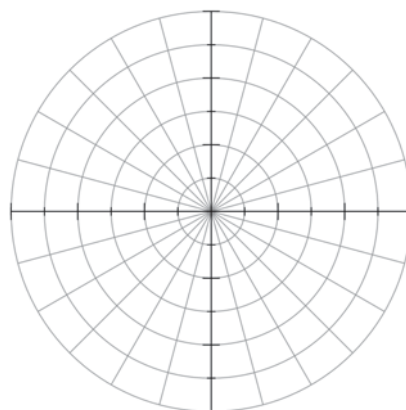
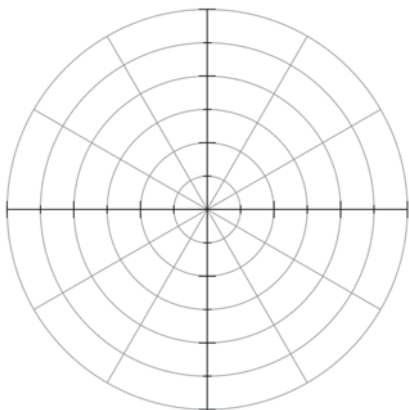


Lemniscate (a figure 8) : $r^2 = a^2 \cos(2\theta)$

$$r^2 = 25 \cos(2\theta)$$

$r^2 = a^2 \sin(2\theta)$

$$r^2 = 16 \sin(2\theta)$$



Rose Curves: $r = a \cos(n\theta)$ $r = a \sin(n\theta)$

The **length** (from pole to tip) of each "petal" is _____

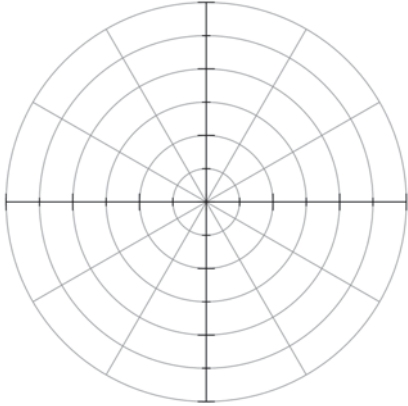
Number of petals: If n is even, then curve has _____ petals.

If n is odd, then curve has _____ petals.

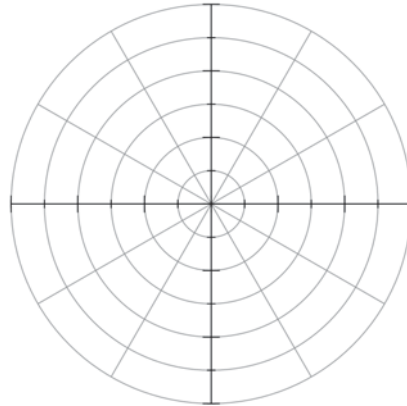
Where to place each petal:

1. For $r = a\cos(n\theta)$, the first petal will always be when $\theta = \underline{\hspace{2cm}}$. The spacing (how far apart) of each petal is found by dividing 360° by number of .

$$r = 4\cos(5\theta)$$

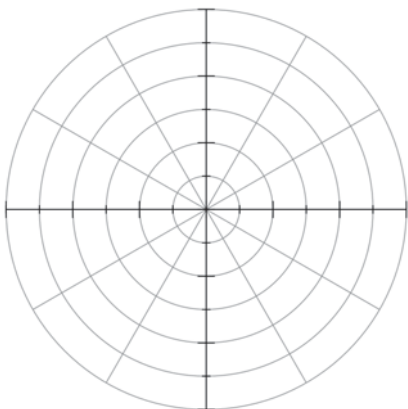


$$r = 5\cos(4\theta)$$

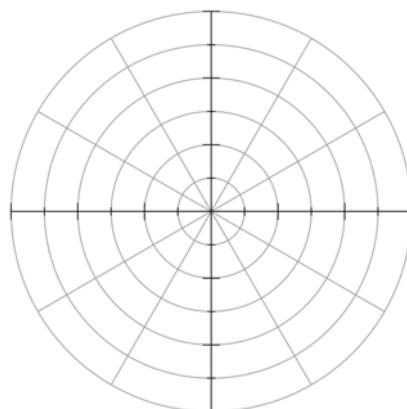


2. For $r = a\sin(n\theta)$, the first petal is found by dividing by . The spacing (how far apart) of each petal is found by dividing 360° by number of .

$$r = 4\sin(5\theta)$$



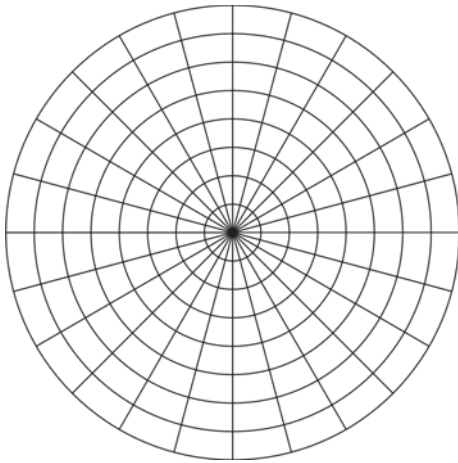
$$r = 5\sin(4\theta)$$



Polar Graphs

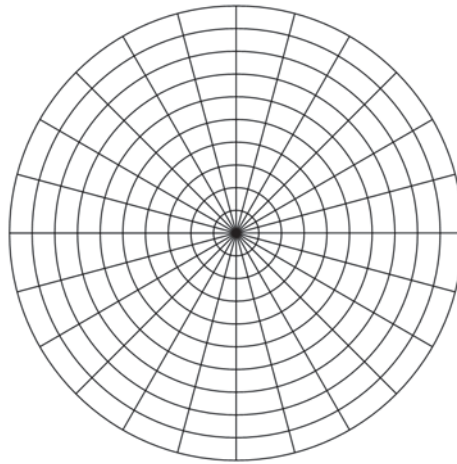
Graph and complete the table:

1. Graph $r = 2 + 2\cos\theta$



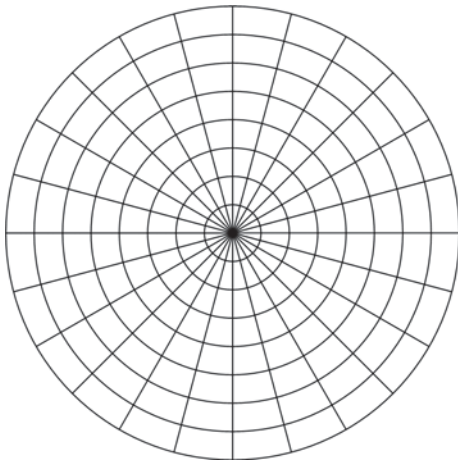
θ	r
0°	
90°	
180°	
270°	

2. Graph: $r = 6 + 4\sin\theta$

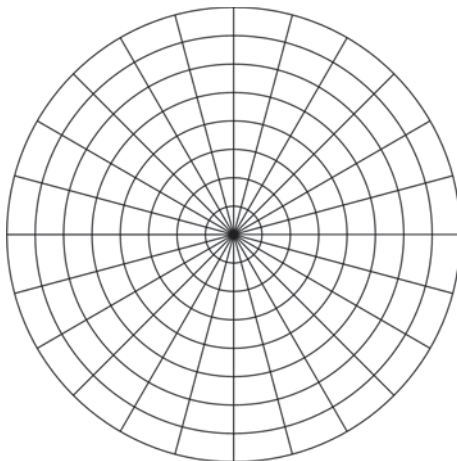


θ	r
0°	
90°	
180°	
270°	

3. Graph $r^2 = 25\sin(2\theta)$

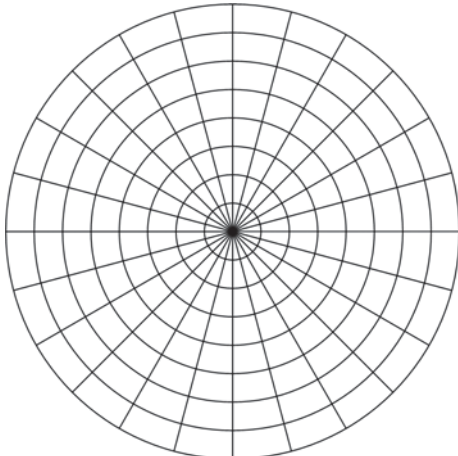


4. Graph: $r = 1 - 3\cos\theta$



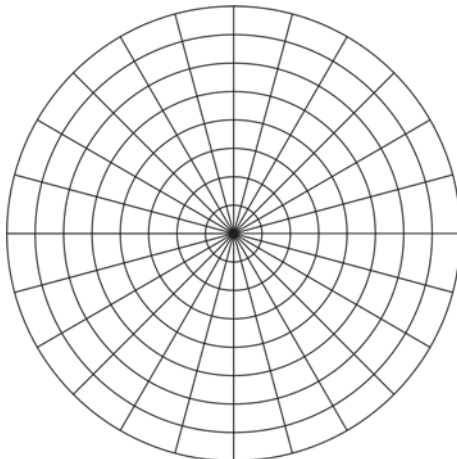
θ	r
0°	
90°	
180°	
270°	

5. Graph $r = 3 - 3\sin\theta$



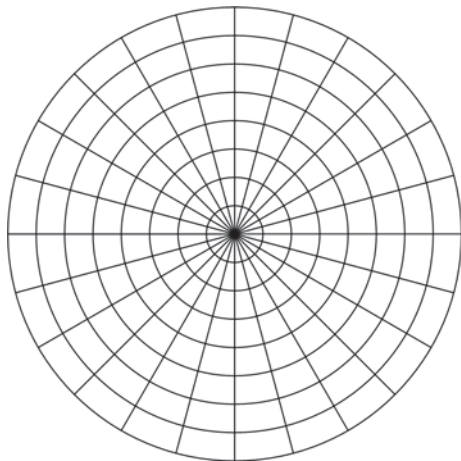
θ	r
0°	
90°	
180°	
270°	

6. Graph: $r = 5 - 3\cos\theta$

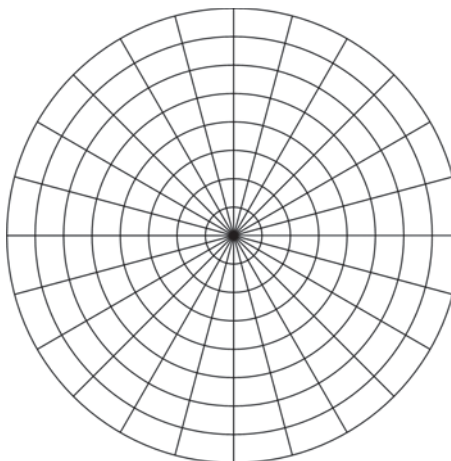


θ	r
0°	
90°	
180°	
270°	

7. Graph $r = 4\cos(3\theta)$

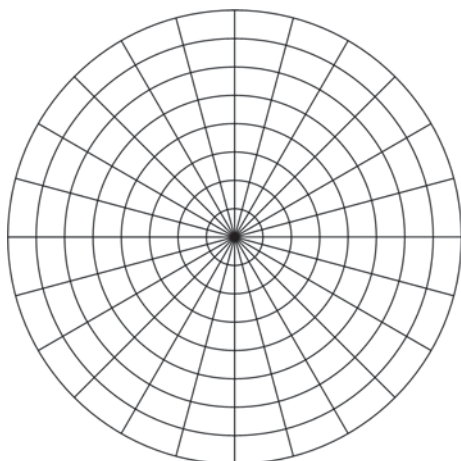


8. Graph: $r = 2 + 5\sin\theta$

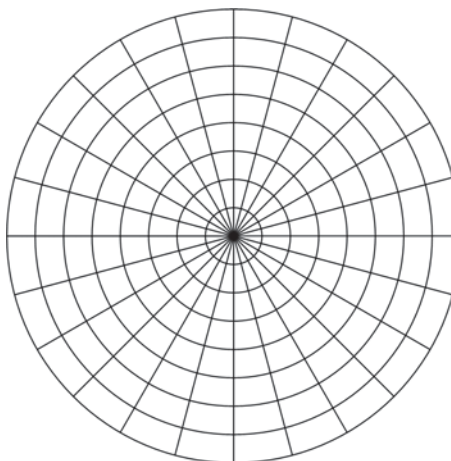


θ	r
0°	
90°	
180°	
270°	

9. Graph $r^2 = 16\cos(2\theta)$



10. Graph: $r = 5\sin(2\theta)$



11. Refer to problem #4: Show work!

The graph passes through the pole when $\theta =$ _____ and _____

12. Refer to problem #8: Show work!

The graph passes through the pole when $\theta =$ _____ and _____

14. The following is a rose curve: $r = 5\cos(10\theta)$

How many petals are there? _____ How far apart is each petal? _____

Where is the first petal located? (at what angle?) _____

15. The following is a rose curve: $r = 5\sin(12\theta)$

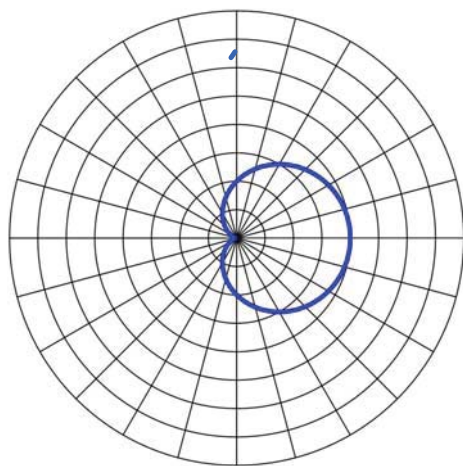
How many petals are there? _____ How far apart is each petal? _____

Where is the first petal located? (at what angle?) _____

Answers—Polar Graphs

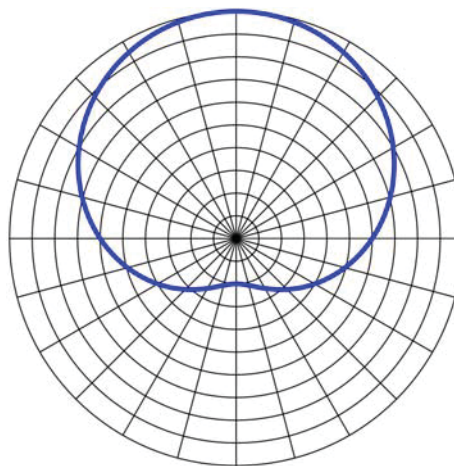
Graph and complete the table:

1. Graph $r = 2 + 2\cos\theta$



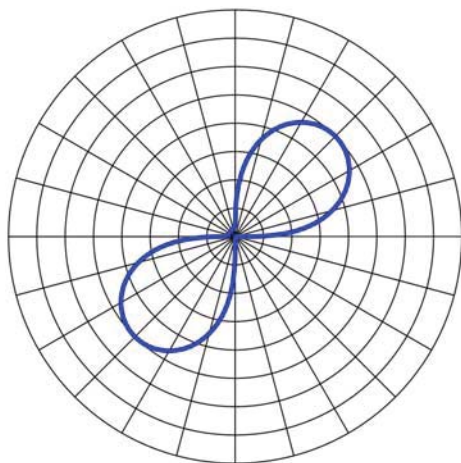
θ	r
0°	4
90°	2
180°	0
270°	2

2. Graph: $r = 6 + 4\sin\theta$

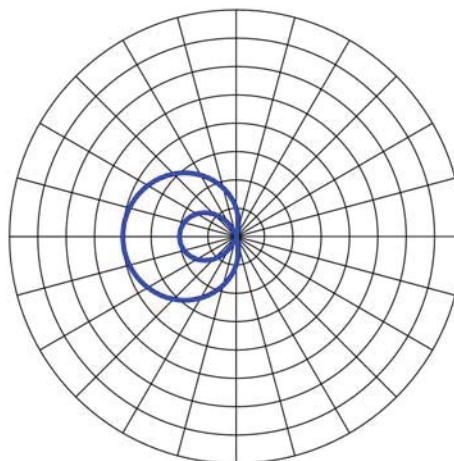


θ	r
0°	6
90°	10
180°	6
270°	2

3. Graph $r^2 = 25\sin(2\theta)$

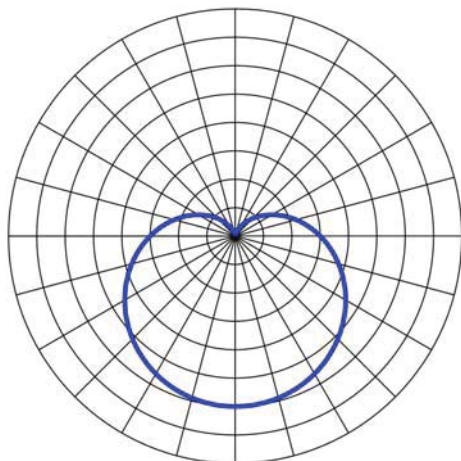


4. Graph: $r = 1 - 3\cos\theta$



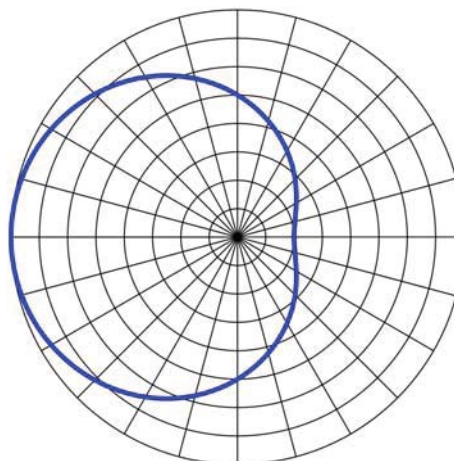
θ	r
0°	-2
90°	1
180°	4
270°	1

5. Graph $r = 3 - 3\sin\theta$



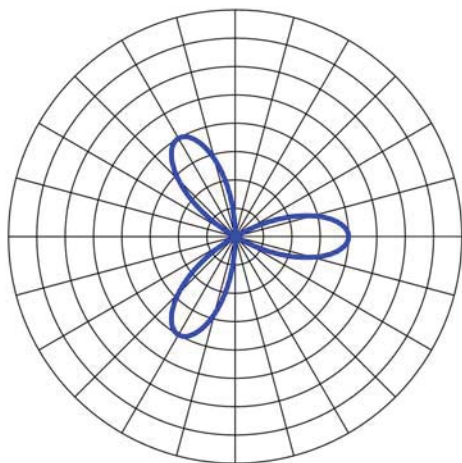
θ	r
0°	3
90°	0
180°	3
270°	6

6. Graph: $r = 5 - 3\cos\theta$

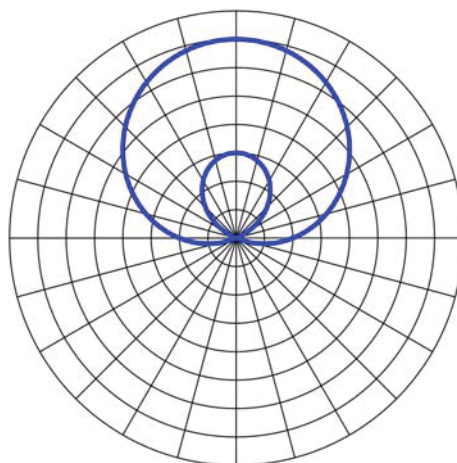


θ	r
0°	2
90°	5
180°	8
270°	5

7. Graph $r = 4\cos(3\theta)$

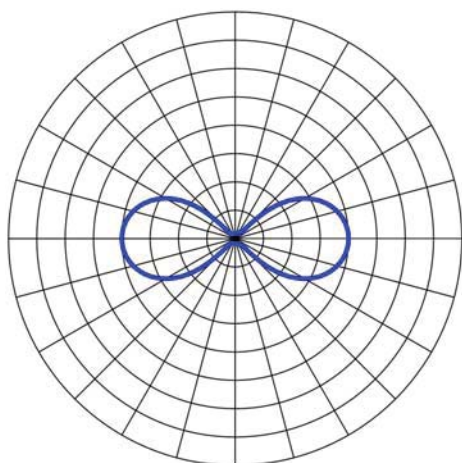


8. Graph: $r = 2 + 5\sin\theta$

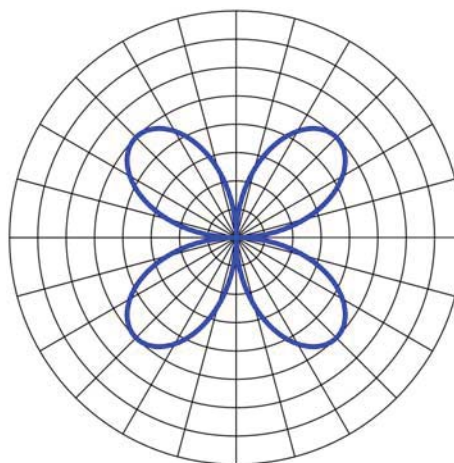


θ	r
0°	2
90°	7
180°	2
270°	-3

9. Graph $r^2 = 16\cos(2\theta)$



10. Graph: $r = 5\sin(2\theta)$



11. Refer to problem #4: Show work!

The graph passes through the pole when $\theta = \underline{70.5^\circ}$ and $\underline{289.5^\circ}$

12. Refer to problem #8: Show work!

The graph passes through the pole when $\theta = \underline{203.6^\circ}$ and $\underline{336.4^\circ}$

14. The following is a rose curve: $r = 5\cos(10\theta)$

How many petals are there? 20 How far apart is each petal? 18°

Where is the first petal located? (at what angle?) 0°

15. The following is a rose curve: $r = 5\sin(12\theta)$

How many petals are there? 24 How far apart is each petal? 15°

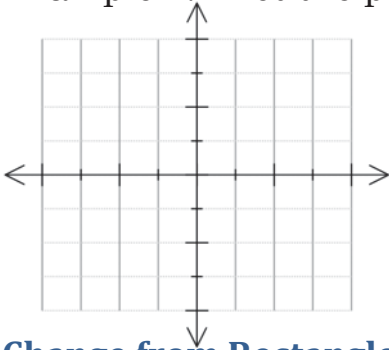
Where is the first petal located? (at what angle?) 7.5°

Complex Numbers

Definition: A complex number $z = a + bi$ can be interpreted geometrically as the point (a, b) in the xy -plane, where the x -axis is the **real** axis and the y -axis is the **imaginary** axis.

Definition: The magnitude or modulus of z is the distance from the point (a, b) to the origin. $|z| = \sqrt{a^2 + b^2}$

Example 1: Plot the point $z = 3 + 2i$. Find the magnitude (Modulus) of z .



Change from Rectangle Form to Polar Form:

Rectangular Form: $z = a + bi$

Polar Form: $z = r(\cos \theta + i \sin \theta)$ or $z = r(\text{cis } \theta)$, $r \geq 0$ and $0 \leq \theta < 2\pi$ or $0 \leq \theta < 360^\circ$

Example 2: Write in polar form: $z = 3 + 2i$. Express angle in degrees.

Example 3: Write in polar form: $z = -3 + 5i$. Express angle in degrees.

Example 4: Write in rectangular form: $z = 3\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$.

Multiplication and Division of Complex Numbers:

Given 2 complex numbers, $z = r_1(\cos\theta_1 + i\sin\theta_1)$ and $w = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$zw = (r_1r_2)(cis(\theta_1 + \theta_2)) \text{ and } \frac{z}{w} = \left(\frac{r_1}{r_2}\right)(cis(\theta_1 - \theta_2))$$

Example 5: Given $z = 2(cis(80^\circ))$ and $w = 6(cis(200^\circ))$, Find zw and $\frac{z}{w}$.

Write answer in rectangular form.

DeMoivre's Theorem:

This theorem allows us to raise complex numbers to powers.

If $z = r(\cos \theta + i \sin \theta)$ is a complex number, then $z^n = r^n(\cos(n\theta) + i \sin(n\theta))$, where n is a positive integer.

Example 6: If $z = \sqrt{2} \left(\text{cis} \frac{5\pi}{16} \right)$, find z^4 . Write answer with **exact** values in rectangular form.

Example 7: Find $(\sqrt{3} - i)^6$. Write answer in rectangular form.

We can also use DeMoivre's Theorem to find roots of complex numbers. Remember, if you are asked to find the cube root of a number, there are 3 complex roots. The fourth root yields 4 complex roots, and so on.

When you look in a text book, this is the formula that you will see:

$$z_k = \sqrt[n]{r} \left(\operatorname{cis} \left(\frac{\theta}{n} + \frac{2\pi}{n} k \right) \right), \text{ where } k = 0, 1, 2, \dots, n-1$$

Looks scary, but don't panic.

Example 8: Find the 4 fourth roots of $\sqrt{3} - i$. Leave answers in polar form.

Example 9: Find the 5 fifth roots of $-i$. Leave answers in polar form.

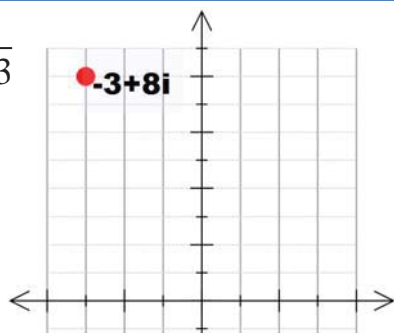
Complex Numbers

Round to 2 decimal places or use exact values, if possible:

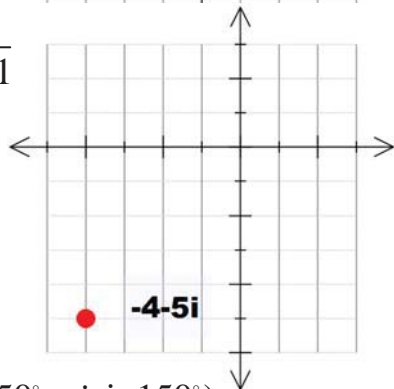
1. Plot the complex number in the complex plane and find the magnitude (Modulus) of z . $z = -3 + 8i$
2. Plot the complex number in the complex plane and find the magnitude (Modulus) of z . $z = -4 - 5i$
3. Write in polar form: $z = -\sqrt{3} + i$. Express the angle in degrees ($0^\circ \leq \theta < 360^\circ$)
4. Write in polar form: $z = -5 - 2i$. Express the angle in degrees ($0^\circ \leq \theta < 360^\circ$)
5. Write in rectangular form: $z = 3\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$.
6. Write in rectangular form: $z = 6\left(\cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5}\right)$.
7. If $z = 8(\cos 30^\circ + i \sin 30^\circ)$ and $w = 2(\cos 45^\circ + i \sin 45^\circ)$, find zw and $\frac{z}{w}$. Leave your answers in rectangular form.
8. If $z = 12(\cos 80^\circ + i \sin 80^\circ)$ and $w = 2(\cos 20^\circ + i \sin 20^\circ)$, find zw and $\frac{z}{w}$. Leave your answers in rectangular form.
9. Find: $\left[3\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)\right]^4$. Write the expression in the rectangular form.
10. Find: $\left[\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)\right]^{10}$. Write the expression in the rectangular form.
11. Find: $(4 + 4i)^6$. Write the expression in the rectangular form.
12. Find: $(-\sqrt{3} + i)^5$. Write the expression in the rectangular form.
13. Find the complex fifth roots of $5 + 3i$. Leave your answers in polar form with the argument in degrees. $0^\circ \leq \theta < 360^\circ$
14. Find the complex sixth roots of 2 . Leave your answers in polar form with the argument in degrees. $0^\circ \leq \theta < 360^\circ$

Answers—Complex Numbers

1. $|z| = \sqrt{73}$



2. $|z| = \sqrt{41}$



3. $2(\cos 150^\circ + i \sin 150^\circ)$

4. $\sqrt{29}(\cos 201.80^\circ + i \sin 201.80^\circ)$

5. $\frac{-3\sqrt{3}}{2} + \frac{3}{2}i$

6. $-1.85 + 5.71i$

7. $zw = 4.14 + 15.45i$

$$\frac{z}{w} = 3.86 - 1.04i$$

8. $zw = -4.17 + 23.64i$

$$\frac{z}{w} = 3 + 3\sqrt{3}i$$

9. $\frac{-81}{2} + \frac{81\sqrt{3}}{2}i$

10. $0 + 32i$

11. $0 - 32768i$

12. $16\sqrt{3} + 16i$

13.

$$\sqrt[5]{\sqrt{34}}(\cos 6.19^\circ + i \sin 6.19^\circ)$$

$$\sqrt[5]{\sqrt{34}}(\cos 78.19^\circ + i \sin 78.19^\circ)$$

$$\sqrt[5]{\sqrt{34}}(\cos 150.19^\circ + i \sin 150.19^\circ)$$

$$\sqrt[5]{\sqrt{34}}(\cos 222.19^\circ + i \sin 222.19^\circ)$$

$$\sqrt[5]{\sqrt{34}}(\cos 294.19^\circ + i \sin 294.19^\circ)$$

$$\sqrt[6]{2}(\cos 0^\circ + i \sin 0^\circ)$$

$$\sqrt[6]{2}(\cos 60^\circ + i \sin 60^\circ)$$

$$\sqrt[6]{2}(\cos 120^\circ + i \sin 120^\circ)$$

14. $\sqrt[6]{2}(\cos 180^\circ + i \sin 180^\circ)$

$$\sqrt[6]{2}(\cos 240^\circ + i \sin 240^\circ)$$

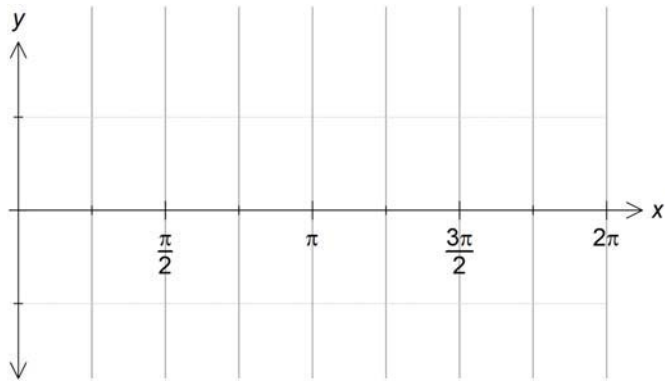
$$\sqrt[6]{2}(\cos 300^\circ + i \sin 300^\circ)$$

Graphing Trig Functions

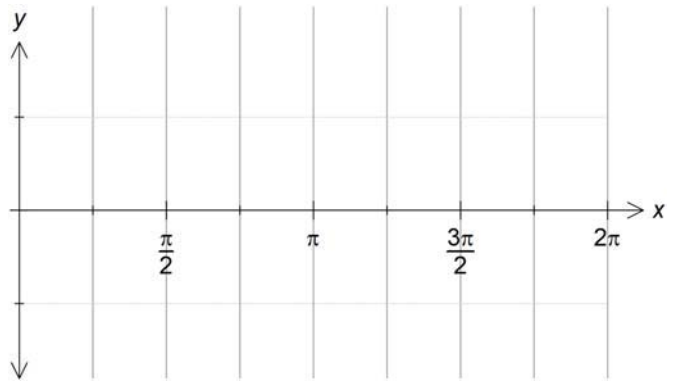
Review of Cosine and Sine

$$y = \sin(x)$$

1. Period = 2π 2. Range = $[-1,1]$



$$y = \cos(x)$$



$$y = A\sin(Bx - C) + D$$

_____ = Amplitude

_____ = Phase Shift

$$y = A\cos(Bx - C) + D$$

_____ = Vertical Shift

_____ = Period

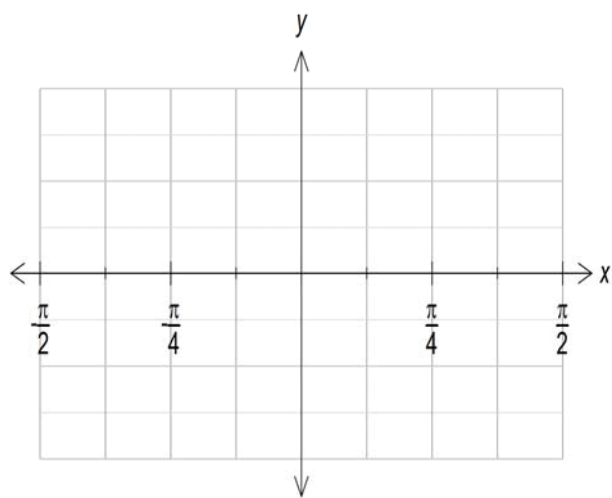
Ex. 1: Graph the following over a one period interval. $y = 3\sin\left(2x + \frac{\pi}{2}\right) - 1$

Ex. 2: Graph the following over a one period interval. $y = -2\cos\left(\frac{1}{2}x - \frac{\pi}{4}\right) + 1$

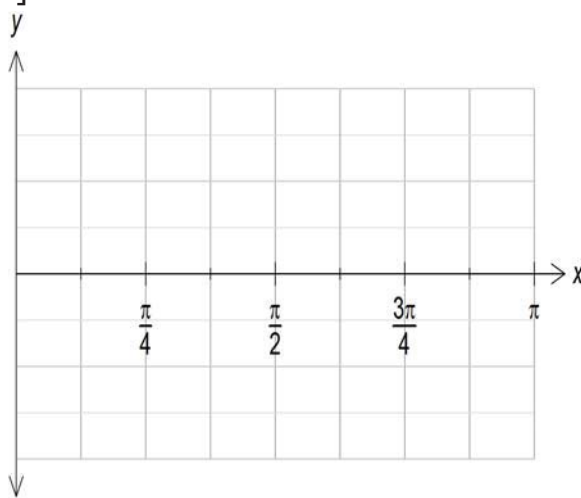
Review of Tangent and Cotangent

$$y = \tan(x)$$

Period = π Range = $[-\infty, \infty]$



$$y = \cot(x)$$



$$y = A \tan(Bx - C) + D$$

_____ = "Amplitude"

$$y = A \cot(Bx - C) + D$$

_____ = Vertical Shift

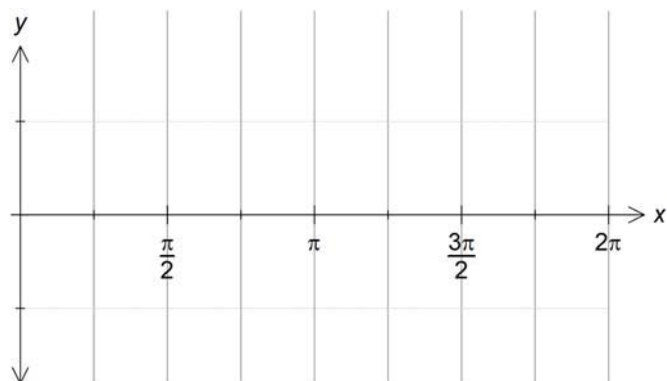
_____ = Period ★

Ex. 3: Graph the following over a one period interval. $y = \frac{1}{2} \tan\left(2x - \frac{\pi}{3}\right) + 2$

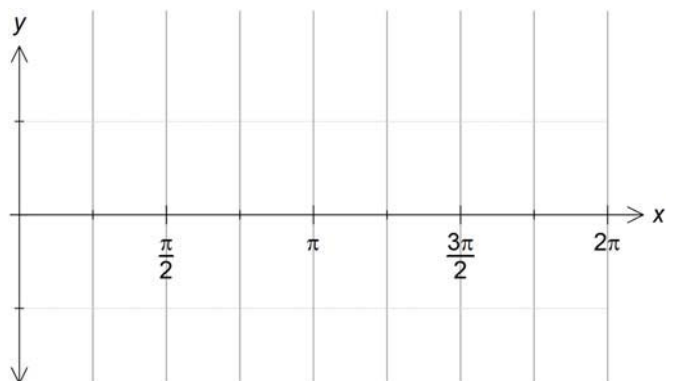
Ex. 4: Graph the following over a one period interval. $y = 2 \cot\left(\frac{1}{2}\pi x + \frac{\pi}{4}\right) - 1$

Review of Cosecant and Secant

$$y = \csc(x)$$

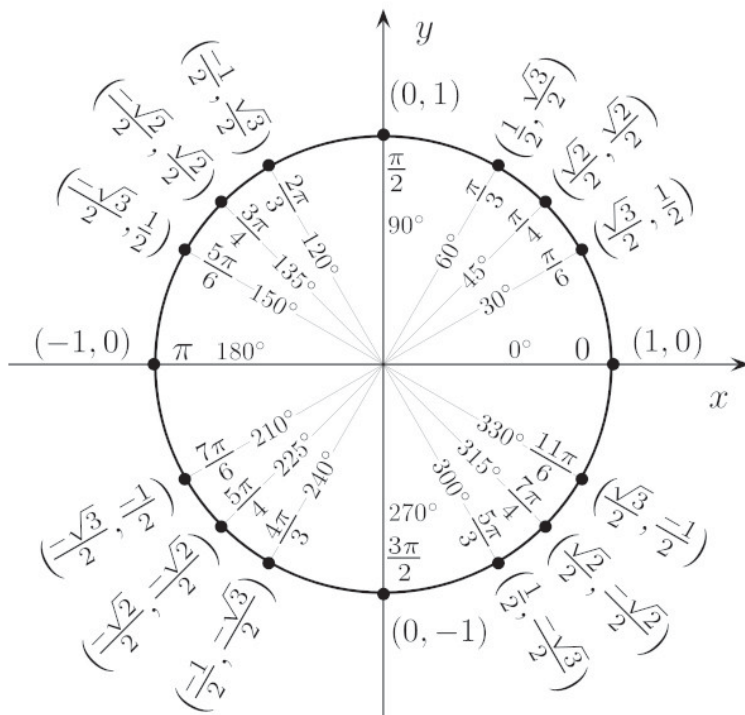


$$y = \sec(x)$$



Ex. 5: Graph the following over a one period interval. $y = 3\csc\left(2x - \frac{\pi}{4}\right) - 1$

Ex. 6: Graph the following over a one period interval. $y = -\frac{1}{3}\sec\left(\frac{1}{2}x + \frac{\pi}{3}\right)$



Deg.	Rad.	Sin θ	Cos θ	Tan θ	Csc θ	Sec θ	Cot θ
0°	0	0	1	0	Und.	0	Und.
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$	2	$2\sqrt{3}/3$	$\sqrt{3}$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$2\sqrt{3}/3$	2	$\sqrt{3}/3$
90°	$\pi/2$	1	0	Und.	1	Und.	0
120°	$2\pi/3$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$	$2\sqrt{3}/3$	-2	$-\sqrt{3}/3$
135°	$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1	$\sqrt{2}$	$-\sqrt{2}$	1
150°	$5\pi/6$	$1/2$	$-\sqrt{3}/2$	$-\sqrt{3}/3$	2	$-2\sqrt{3}/3$	$-\sqrt{3}$
180°	π	0	-1	0	Und.	-1	Und.
210°	$7\pi/6$	$-1/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$	-2	$-2\sqrt{3}/3$	$\sqrt{3}$
225°	$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
240°	$4\pi/3$	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$	$-2\sqrt{3}/3$	-2	$\sqrt{3}/3$
270°	$3\pi/2$	-1	0	Und.	-1	Und.	0
300°	$5\pi/3$	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$	$-2\sqrt{3}/3$	2	$-\sqrt{3}/3$
315°	$7\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
330°	$11\pi/6$	$-1/2$	$\sqrt{3}/2$	$-\sqrt{3}/3$	-2	$2\sqrt{3}/3$	$-\sqrt{3}$

Graphing Trig Functions

Graph over a one period interval. Following class procedure, label the vertical and horizontal axes accordingly.

1. $y = 2\sin\left(x + \frac{\pi}{2}\right) - 3$

2. $y = -\frac{1}{4}\cos\left(\frac{3}{4}x + \frac{\pi}{8}\right) - 2$

3. $y = \tan\left(x + \frac{\pi}{3}\right) - 3$

4. $y = \frac{1}{2}\sin(2x - 3\pi) - 1$

5. $y = 2\cot\left(x + \frac{\pi}{4}\right)$

6. $y = \frac{1}{2}\tan\left(\frac{\pi x}{4} + \frac{\pi}{4}\right)$

7. $y = \cot(3x - \pi) + 1$

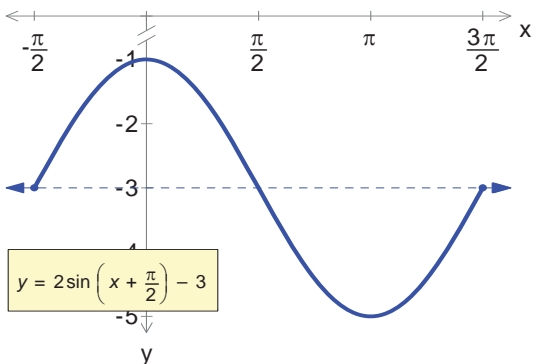
8. $y = \frac{2}{3}\cos\left(\frac{3}{4}x - \pi\right) - 2$

9. $y = -\frac{1}{2}\csc\left(x - \frac{3\pi}{4}\right) + 1$

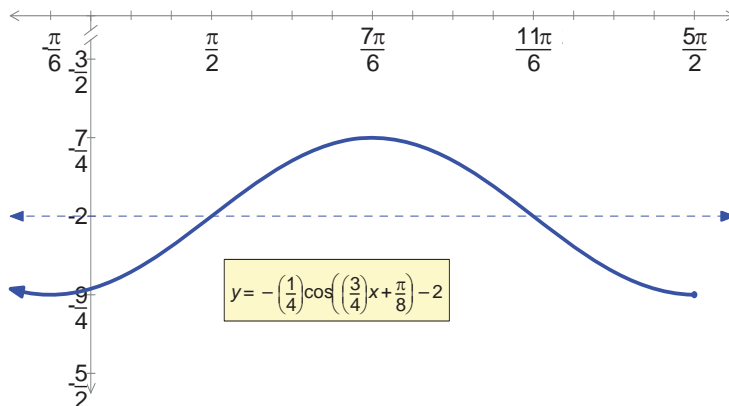
10. $y = \frac{1}{4}\sec\left(\frac{1}{2}x - \pi\right) + 2$

Answers—Graphing Trig Functions

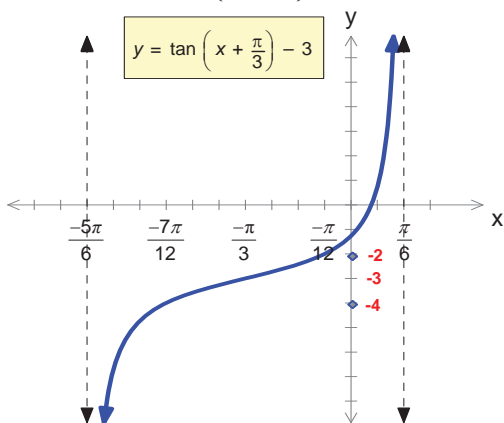
1. $y = 2 \sin\left(x + \frac{\pi}{2}\right) - 3$



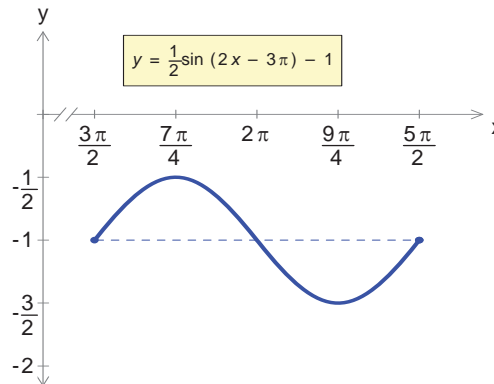
2. $y = -\frac{1}{4} \cos\left(\frac{3}{4}x + \frac{\pi}{8}\right) - 2$



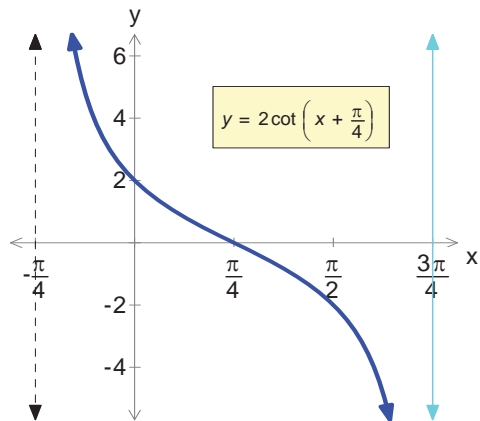
3. $y = \tan\left(x + \frac{\pi}{3}\right) - 3$



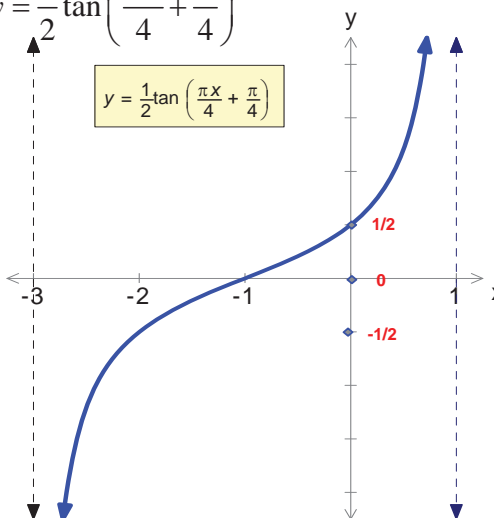
4. $y = \frac{1}{2} \sin(2x - 3\pi) - 1$



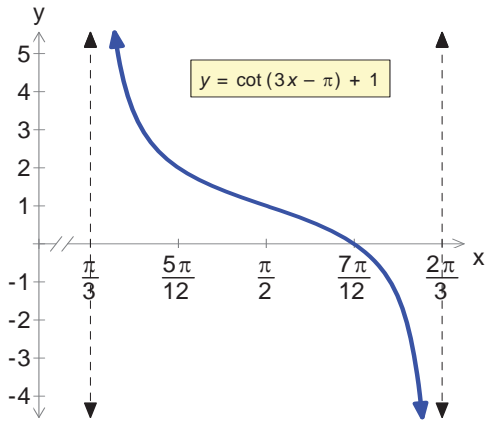
5. $y = 2 \cot\left(x + \frac{\pi}{4}\right)$



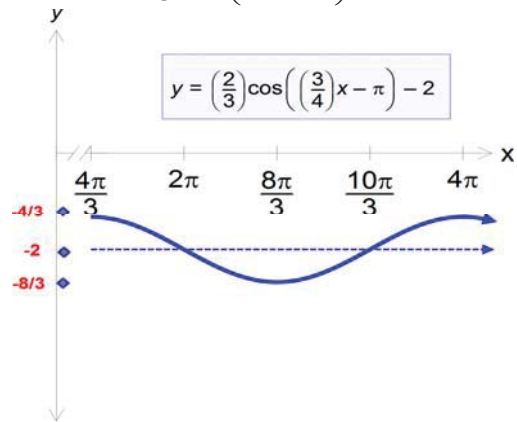
6. $y = \frac{1}{2} \tan\left(\frac{\pi x}{4} + \frac{\pi}{4}\right)$



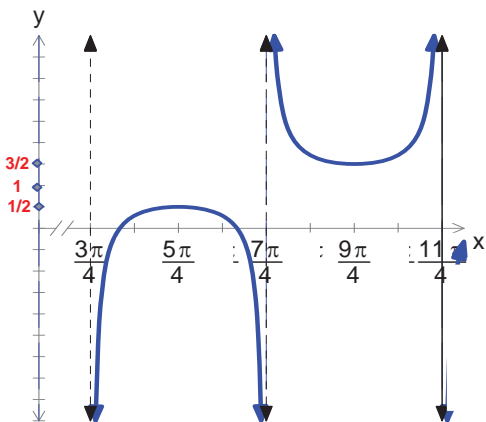
7. $y = \cot(3x - \pi) + 1$



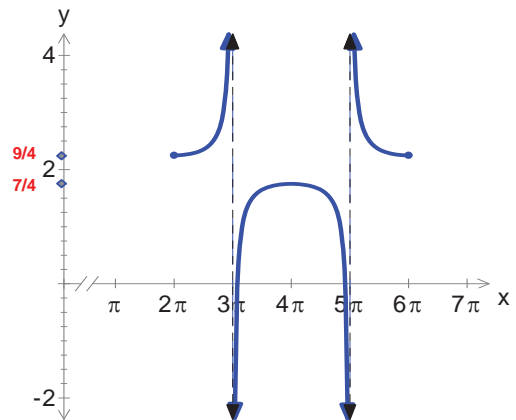
8. $y = \frac{2}{3} \cos\left(\frac{3}{4}x - \pi\right) - 2$



9. $y = -\frac{1}{2} \csc\left(x - \frac{3\pi}{4}\right) + 1$



10. $y = \frac{1}{4} \sec\left(\frac{1}{2}x - \pi\right) + 2$



Trig Equations—Part 1

Solve the following in the given interval. Give exact answers, if possible. Round answers to 2 decimal places when necessary.

1. $\sin x = \frac{1}{2}$ $[0, 2\pi)$

2. $2\cos x = -\sqrt{3}$ $[0, \pi)$

3. $\sec x = -4$ $(0, 2\pi)$

4. $\csc x = -3$ $(0, 2\pi)$

$$5. 2\sin^2 x - 3\sin x + 1 = 0 \quad [0, 4\pi)$$

$$6. 2\cos^2 x - 5\cos x + 2 = 0 \quad [0, 2\pi)$$

$$7. 2\cos^2 x = \frac{6}{\pi} \quad \left[-\frac{\pi}{4}, \frac{\pi}{4} \right]$$

$$8. 2\sin x \cos x + \sqrt{3} \cos x = 0 \quad [-\pi, \pi]$$

$$9. 2\tan^2 x - 6 = 0 \quad [0, 2\pi)$$

Trig Equations—Part 2

Pythagorean Identities:

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$\cot^2 \theta + 1 = \csc^2 \theta$$

Double Angle Identities:

$$\sin 2\theta = 2\cos \theta \sin \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\cos 2\theta = 2\cos^2 \theta - 1$$

$$\cos 2\theta = 1 - 2\sin^2 \theta$$

1. $3\tan^2 \theta - \sec^2 \theta - 5 = 0$ $[0, 2\pi)$

2. $\cos^2 \theta - \cos \theta - 1 = 0$ $[0, 2\pi)$

3. $2\sin^2 \theta - \cos 2\theta = 0$ $[0, 2\pi)$

4. $3\cos \theta + 3 = 2\sin^2 \theta$ $[0, 2\pi)$

5. $\cos 2\theta + 3 = 5\cos \theta$ $[0, 2\pi)$

6. $3\cos^2 x + \sin x = 2$ $[0, 2\pi)$

7. $\sin 2\theta = 1$ $[0, 2\pi)$

8. $\cos 2\theta = \frac{\sqrt{3}}{2}$ $[0, 2\pi)$

9. $2\sin^2 3x - \sin 3x - 1 = 0$ $[0, 2\pi)$

10. $\tan^2(3x) = 1$ $[0, 2\pi)$

11. $2\cos x - 1 = \sec x$ $[0, 2\pi)$

Trigonometric Equations

Find all solutions of the equation in the given interval. Give exact answers where possible. Otherwise, round answers to 2 decimal places.

#	Problem	Interval
1.	$1 + \cos x = 0$	$[0, 2\pi)$
2.	$\sqrt{3} - 2\sin x = 0$	$[0, 2\pi)$
3.	$\cos x = 0$	$[0, 2\pi)$
4.	$-\sin x = 0$	$[0, 2\pi)$
5.	$\frac{1}{2} - \sin x = 0$	$[0, 2\pi)$
6.	$-\sin x - 1 = 0$	$[0, 4\pi)$
7.	$1 - 2\cos x = 1$	$[-\pi, \pi]$
8.	$\cos x = \frac{3\sqrt{3}}{2\pi}$	$\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$
9.	$\sin x = \frac{2}{\pi}$	$[0, \pi]$
10.	$\cos x = \frac{2}{\pi}$	$\left[0, \frac{\pi}{2}\right]$
11.	$2\sin x \cos x - \sin x = 0$	$[0, 2\pi)$
12.	$2\sin x \cos x + \cos x = 0$	$[0, 2\pi)$
13.	$2 - \sec^2 x = 0$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
14.	$2 - \csc^2 x = 0$	$(0, \pi)$
15.	$2\sec^2 x = \frac{8}{\pi}$	$\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
16.	$\sec x \tan x = 0$	$\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
17.	$2\csc^2 x \cot x = 0$	$(0, \pi)$

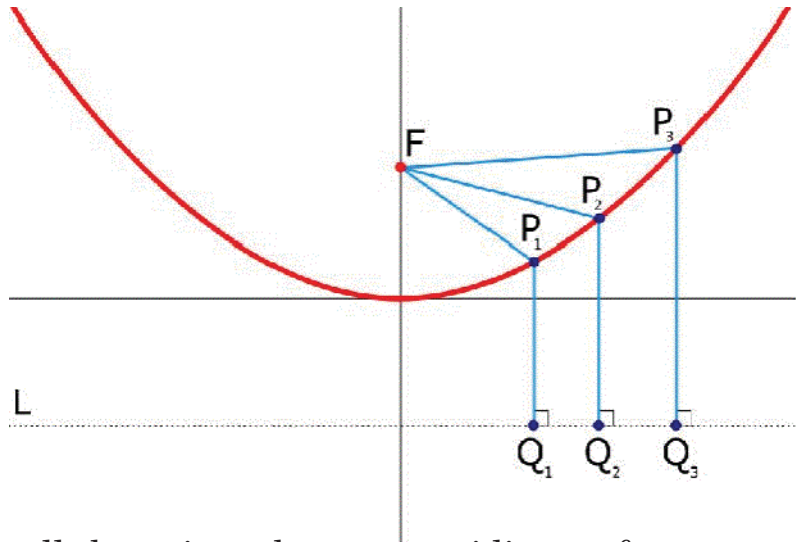
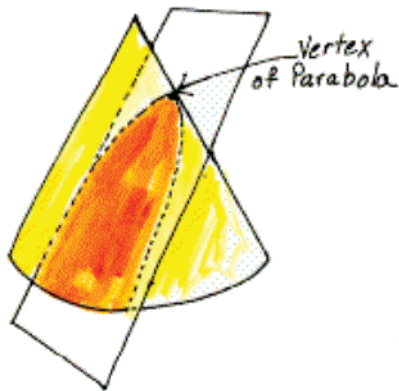
#	Problem	Interval
18.	$2\sec \theta \tan \theta + \sec^2 \theta = 0$	$[0, 2\pi)$
19.	$2\sin x = \tan x$	$\left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$
20.	$2\cos x + 2\cos 2x = 0$	$[0, \pi]$
21.	$2\cos x - 2\sin 2x = 0$	$[0, 2\pi)$
22.	$-\sin x + \sin 2x = 0$	$[0, 2\pi)$
23.	$-2\sin x - 4\sin 2x = 0$	$[0, 2\pi)$
24.	$\sin 2x = \cos x$	$\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$
25.	$2\sin x + \sin 2x = 0$	$[0, \pi]$
26.	$\cos^2 x - \sin^2 x = 0$	$[0, 2\pi)$
27.	$-\cos x + 2\cos 2x = 0$	$[0, 2\pi)$
28.	$-\pi \sin \pi x = 0$	$\left[0, \frac{1}{6}\right)$
29.	$2\cos 2x = 0$	$\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$
30.	$\frac{1}{2}\cos \frac{x}{2} = 0$	$[0, 4\pi)$
31.	$-\frac{1}{4}\sin \frac{x}{2} = 0$	$[0, 4\pi)$
32.	$\frac{1}{2} - \frac{\pi}{6}\cos \frac{\pi}{6}x = 0$	$[-1, 0]$
33.	$4 - \pi \sec^2 \pi x = 0$	$\left[-\frac{1}{4}, \frac{1}{4}\right]$
34.	$-3\csc \frac{3x}{2} \cot \frac{3x}{2} = 0$	$(0, 2\pi)$

Answers—Trig Equations

1.	$\{\pi\}$
2.	$\left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$
3.	$\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$
4.	$\{0, \pi\}$
5.	$\left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$
6.	$\left\{\frac{3\pi}{2}, \frac{7\pi}{2}\right\}$
7.	$\left\{\frac{\pi}{2}, -\frac{\pi}{2}\right\}$
8.	$\{\pm 0.5971\}$
9.	$\{0.6901, 2.4515\}$
10.	$\{0.8807\}$
11.	$\left\{0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$
12.	$\left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$
13.	$\left\{-\frac{\pi}{4}, \frac{\pi}{4}\right\}$
14.	$\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$
15.	$\{\pm 0.4817\}$
16.	$\{0\}$
17.	$\left\{\frac{\pi}{2}\right\}$

18.	$\left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$
19.	$\left\{\pm \frac{\pi}{3}, 0\right\}$
20.	$\left\{\frac{\pi}{3}, \pi\right\}$
21.	$\left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}\right\}$
22.	$\left\{0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}\right\}$
23.	$\{0, \pi, 1.8235, 4.4597\}$
24.	$\left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}\right\}$
25.	$\{0, \pi\}$
26.	$\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$
27.	$\{0.5678, 2.2057, 4.0775, 5.7154\}$
28.	$\{0\}$
29.	$\left\{\frac{\pi}{4}\right\}$
30.	$\{\pi, 3\pi\}$
31.	$\{0, 2\pi\}$
32.	$\{-0.5756\}$
33.	$\{\pm 0.1533\}$
34.	$\left\{\frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$

Parabolas—Conic Section

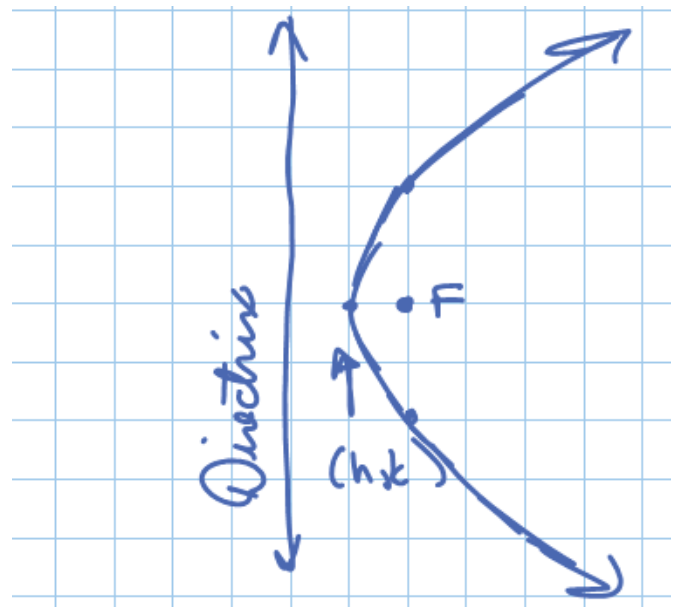
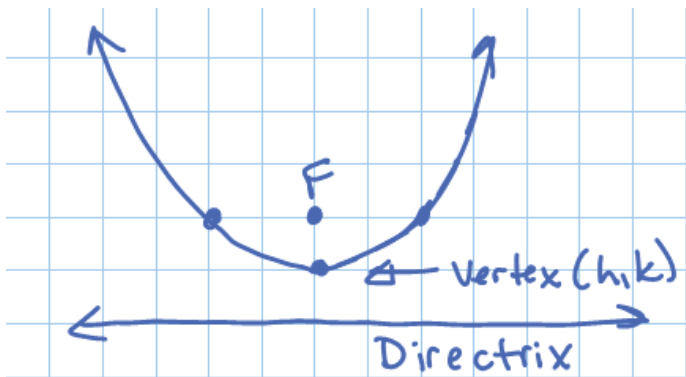


Definition: A parabola is defined to be all the points that are equidistant from a point (focus) and a line (directrix).

The standard form of the equations are:

$$(x - h)^2 = 4a(y - k)$$

$$(y - k)^2 = 4a(x - h)$$



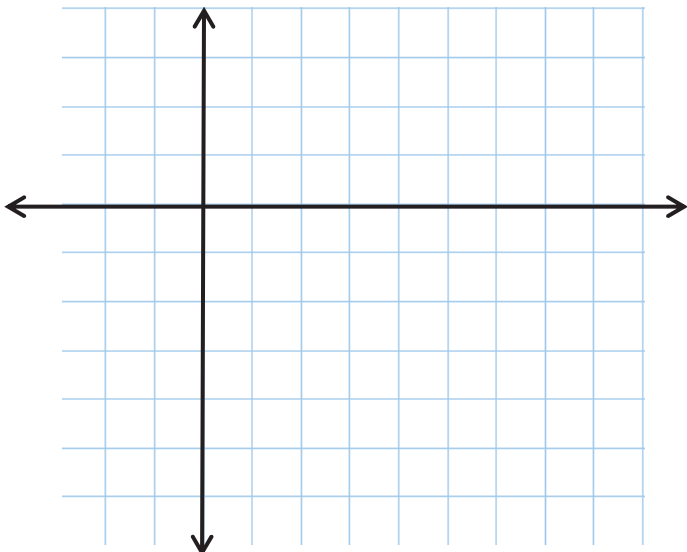
The parabola has a vertex: _____

_____ is the distance from the vertex to the focus.

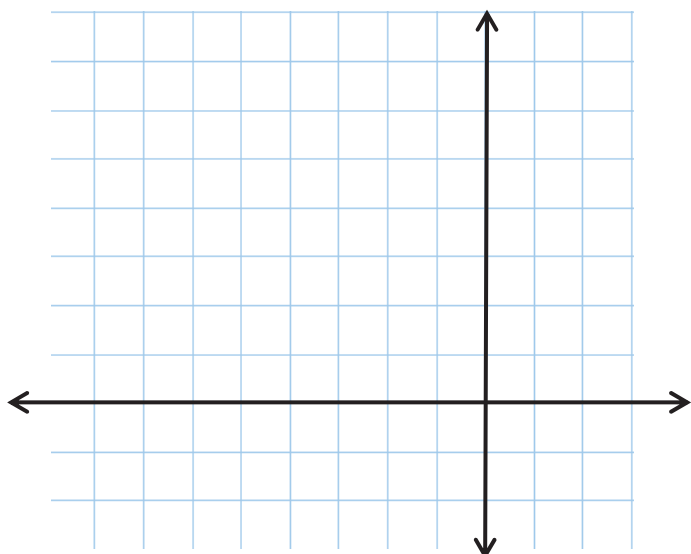
_____ is the distance from the vertex to the directrix.

_____ is the "focal width"

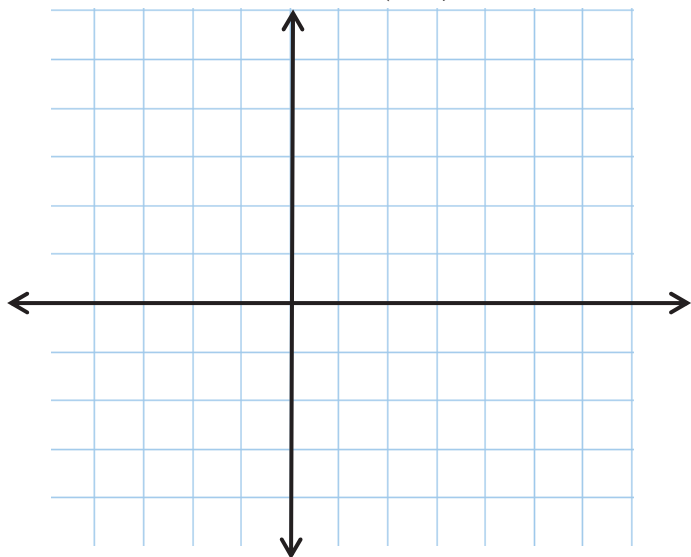
Find an equation of a parabola which fits the given criteria. Graph the equation.
Find the directrix. Example 1: Vertex $(4, -2)$ and Focus $(6, -2)$.



Example 2: Directrix: $y = 2$ and Focus $(-4, 4)$

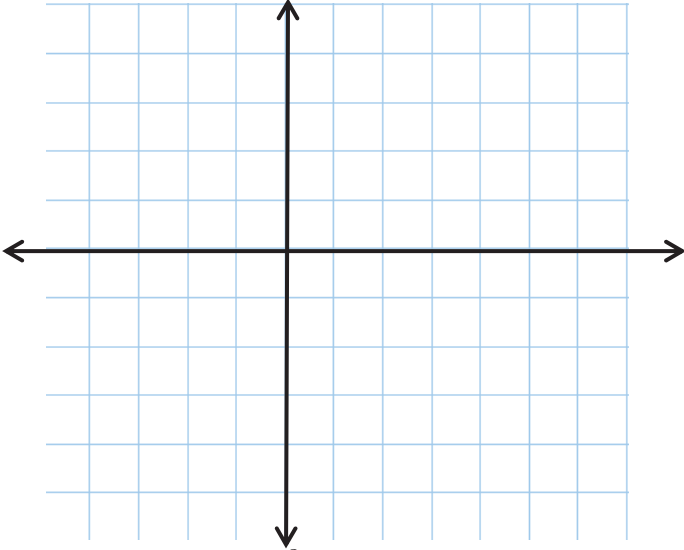


Example 3: Vertex $(1, 2)$, passes through the point $(2, 1)$ and has a vertical axis of symmetry.

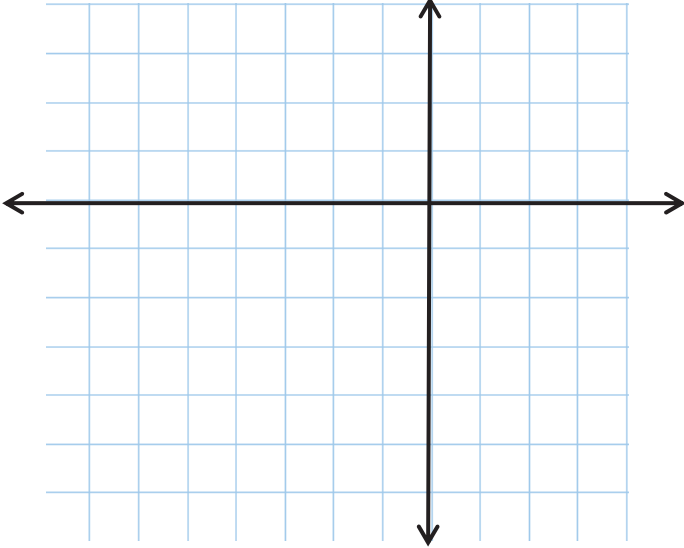


Find the vertex, focus, directrix and graph the following equations:

Example 4: $(y + 1)^2 = -4(x - 2)$

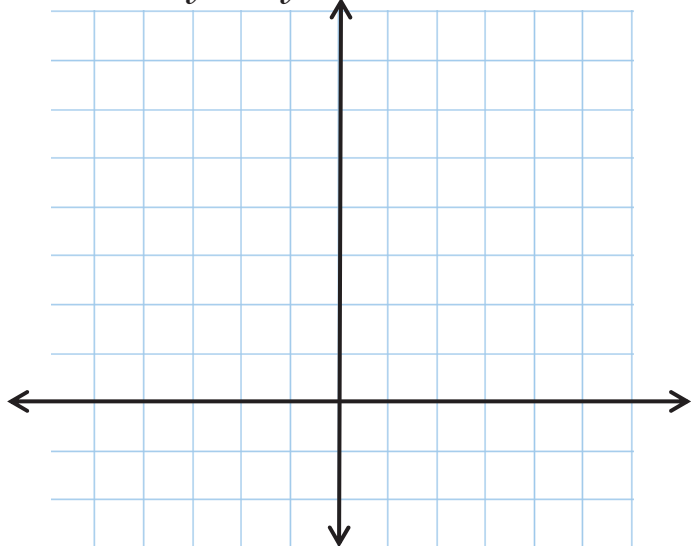


Example 5: $x^2 + 6x + 4y + 17 = 0$



Example 6: The mirror in Carl's flashlight is a paraboloid of revolution. If the mirror is 12 centimeters in diameter and 4 centimeters deep, where should the light bulb be placed so it is at the focus of the mirror?

Extra: $y^2 - 8y - 3x + 13 = 0$



Parabolas

Find an equation of a parabola which fits the given criteria. Graph the equation.

1. Vertex $(-3,1)$ and Focus $(0,1)$.
2. Vertex $(2,-3)$ and Focus $(2,-5)$.
3. Directrix: $y = 0$ and Focus $(-3,4)$.
4. Directrix: $x = 4$ and Focus $(-2,4)$.
5. Vertex $(1,-1)$, passes through the point $(0,1)$ and has a vertical axis of symmetry.
6. Vertex $(-2,0)$, passes through the point $(1,3)$ and has a horizontal axis of symmetry.

Find the vertex, focus, directrix and graph the following equations:

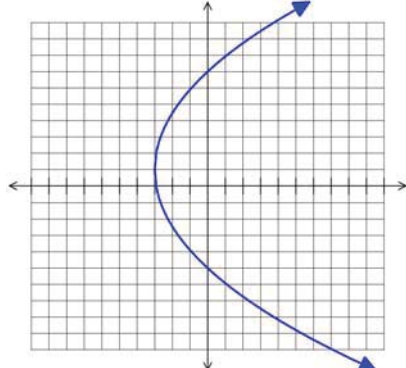
7. $(x - 1)^2 = 4(y + 3)$
8. $(y - 4)^2 = 8(x + 2)$
9. $x^2 + 4x + y + 1 = 0$
10. $y^2 + 2y + 2x - 5 = 0$
11. $x^2 + 6x - 4y + 1 = 0$
12. $y^2 + 12y + x + 28 = 0$

Solve the following:

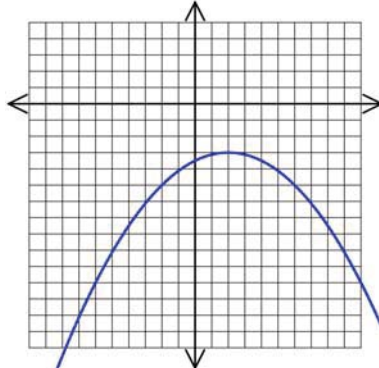
13. The mirror in Carl's flashlight is a paraboloid of revolution. If the mirror is 5 centimeters in diameter and 2.5 centimeters deep, where should the light bulb be placed so it is at the focus of the mirror?
14. A parabolic Wi-Fi antenna is constructed by taking a sheet of metal and bending it into a parabolic shape. If the cross section of the antenna is a parabola which is 45 centimeters wide and 25 centimeters deep, where should the receiver be placed to maximize reception?
15. A parabolic arch is constructed which is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch.

Answers—Parabolas

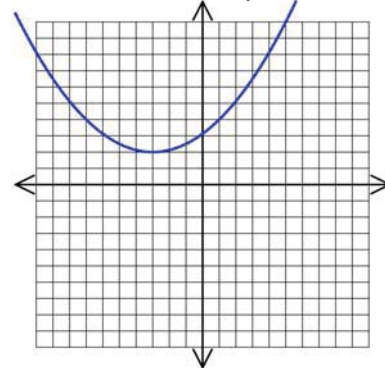
1. $(y-1)^2 = 12(x+3)$



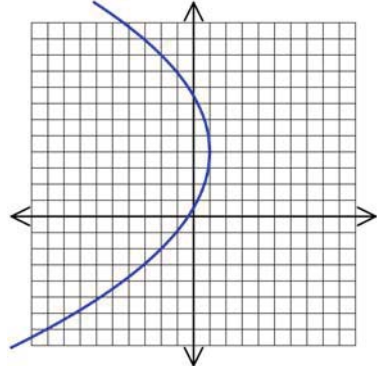
2. $(x-2)^2 = -8(y+3)$



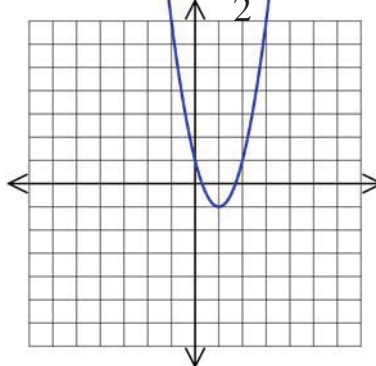
3. $(x+3)^2 = 8(y-2)$



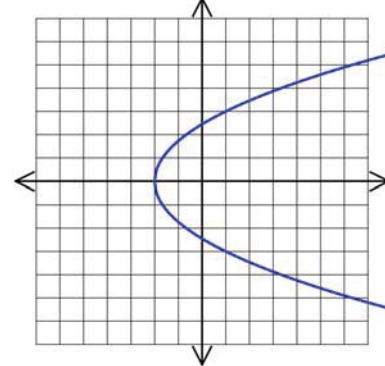
4. $(y-4)^2 = -12(x-1)$



5. $(x-1)^2 = \frac{1}{2}(y+1)$

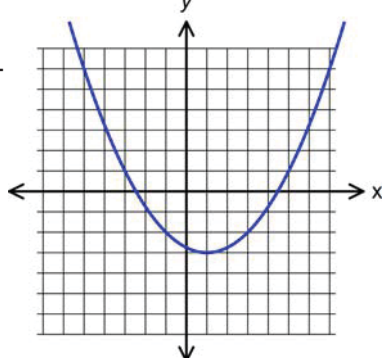


6. $y^2 = 3(x+2)$



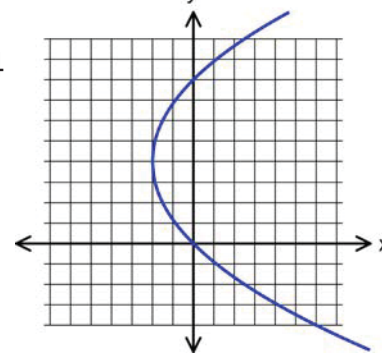
7. Vertex: $(1, -3)$ Focus: $(1, -2)$

Directrix: $y = -4$

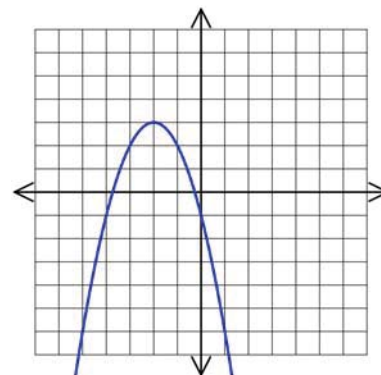


8. Vertex: $(-2, 4)$ Focus: $(0, 4)$

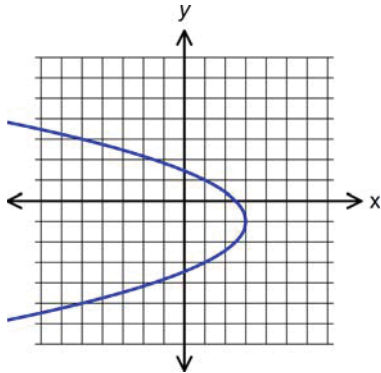
Directrix: $x = -4$



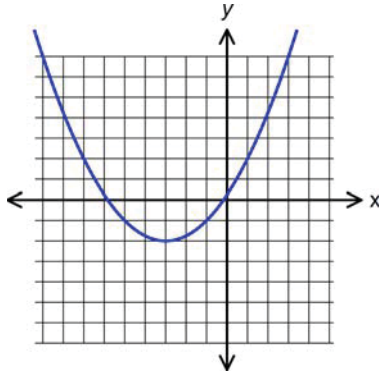
9. Vertex: $(-2, 3)$ Focus: $(-2, \frac{11}{4})$ Directrix: $y = \frac{13}{4}$



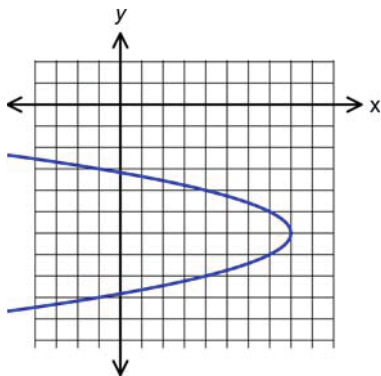
10. Vertex: $(3, -1)$ Focus: $\left(\frac{5}{2}, -1\right)$ Directrix: $x = \frac{7}{2}$



11. Vertex: $(-3, -2)$ Focus: $(-3, -1)$ Directrix: $y = -3$



12. Vertex: $(8, -6)$ Focus: $\left(\frac{31}{4}, -6\right)$ Directrix: $x = \frac{33}{4}$

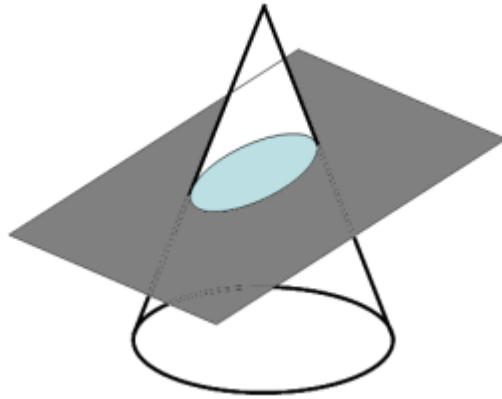


13. $\frac{5}{8} = .625$ cm. above the bottom of mirror.

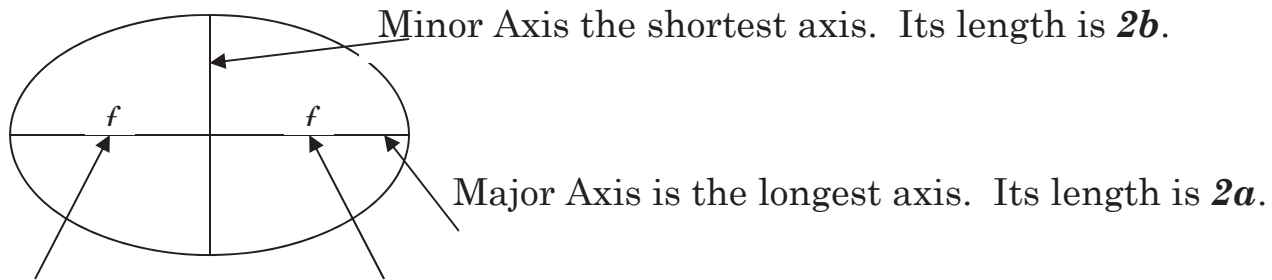
14. $\frac{81}{16} = 5.0625$ cm. above the bottom of antenna.

15. 5 feet

Ellipses—Conic Section



Definition: An ellipse is the set of all points, the sum of whose distances from 2 fixed points (Foci) is a constant.



The foci are always located on the _____ axis.
They are equidistant from the center.

_____ is the distance from the center to each focus.

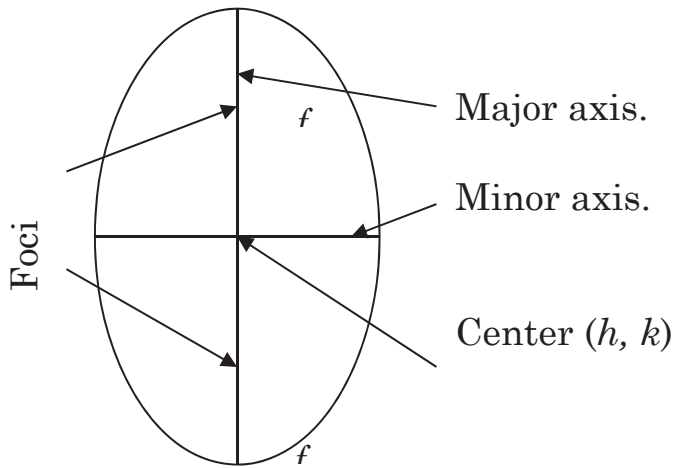
_____ are the endpoints of the major axis.

The standard form of an equation of an ellipse where the major axis is parallel to

the x -axis with center (h, k) is:
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The relationship between a , b , and c is: $c^2 = a^2 - b^2$, where $a > b$.

The ellipse can also be drawn where the major axis is parallel to the y -axis.

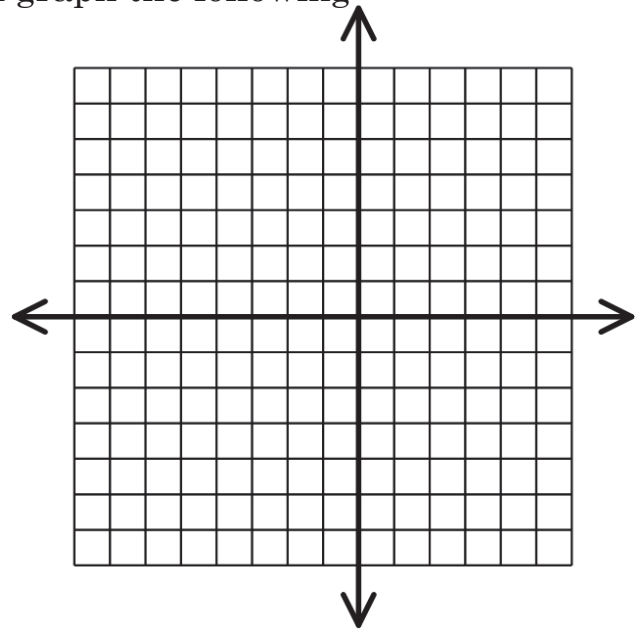


The standard form of an equation of an ellipse where the major axis is parallel to the y -axis with center (h, k) is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

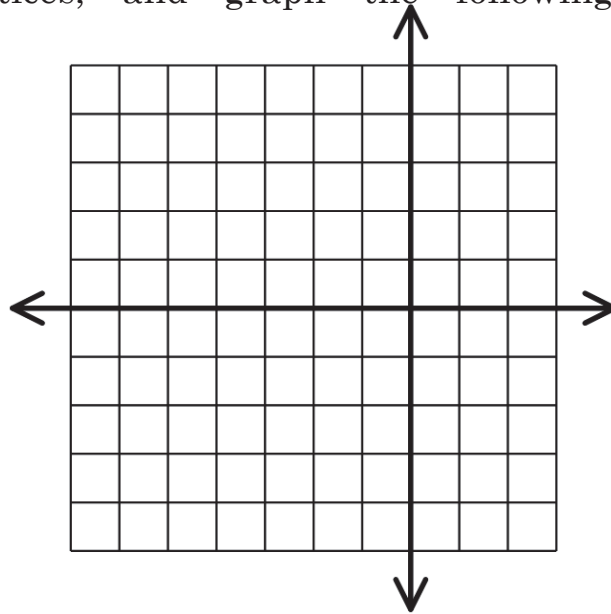
Example 1: Find the center, foci, vertices, and graph the following

$$\frac{(x+4)^2}{4} + \frac{(y-2)^2}{9} = 1$$



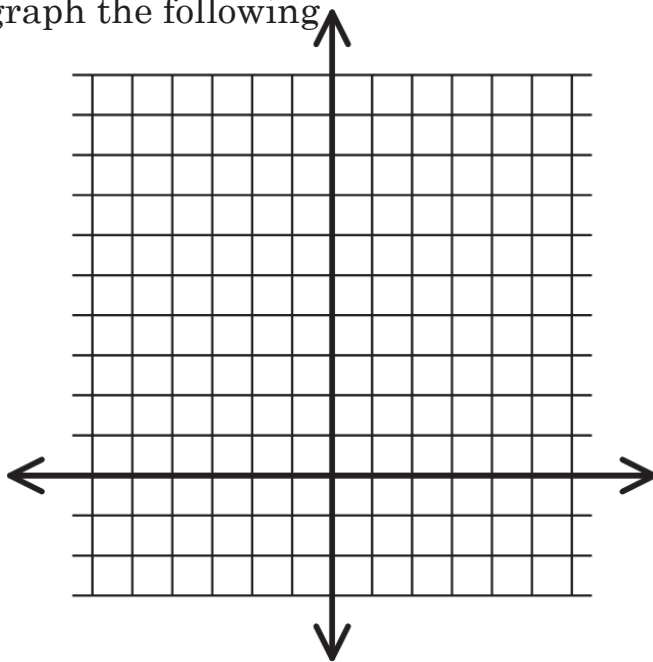
Example 2: Find the center, foci, vertices, and graph the following

$$x^2 + 9y^2 + 6x - 18y + 9 = 0$$



Example 3: Find the center, foci, vertices, and graph the following

$$25x^2 + 4y^2 + 100x - 32y + 64 = 0$$



Example 4:

Find the equation of the ellipse with foci $(2, 1)$ and $(4, 1)$ and vertex $(0, 1)$.

Example 5:

Find the equation of the ellipse with center $(-3,1)$, focus $(-3,0)$ and vertex $(-3,3)$.

Example 6:

Find the equation of the ellipse with center $(1,2)$, focus $(1,4)$ and passes through the point $(2,2)$.

Example 7: Jamie and Jason want to exchange secrets from across a crowded whispering gallery. Recall that a whispering gallery is a room which, in cross section, is half of an ellipse. If the room is 40 feet high at the center and 100 feet wide at the floor, how far from the outer wall should each of them stand so that they will be positioned at the foci of the ellipse?

Example 8: An elliptical arch is constructed which is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch.

Ellipses

Find the center, foci, vertices, and graph the following:

1. $\frac{(x+5)^2}{25} + \frac{(y-4)^2}{16} = 1$

2. $\frac{(x-1)^2}{4} + \frac{(y+3)^2}{9} = 1$

3. $9(x-3)^2 + (y+2)^2 = 36$

4. $4(x+4)^2 + 25(y+1)^2 = 100$

5. $x^2 + 9y^2 + 2x - 18y + 1 = 0$

6. $9x^2 + 4y^2 - 18x + 24y + 9 = 0$

7. $9x^2 + 16y^2 - 36x + 96y + 36 = 0$

Find an equation of an ellipse which fits the given criteria.

8. Center $(3,7)$, Focus $(3,3)$ and Vertex $(3,2)$.

9. Foci $(0, \pm 5)$ and Vertices $(0, \pm 8)$.

10. Vertices $(3,2)$ and $(13,2)$. Endpoints of minor axis: $(8,4)$ and $(8,0)$

11. Foci: $(-5, -2)$ and $(3, -2)$. Vertex $(7, -2)$

12. Foci: $(1,4)$ and $(1,-2)$. Length of minor axis is 10.

13. Center $(1,2)$, Focus $(4,2)$ and passes through the point $(1,3)$.

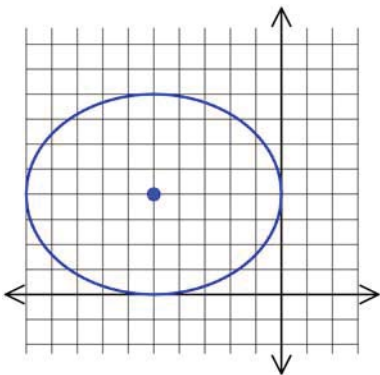
14. Center $(1,2)$, Vertex $(1,5)$ and passes through the point $(3,2)$.

Solve the following:

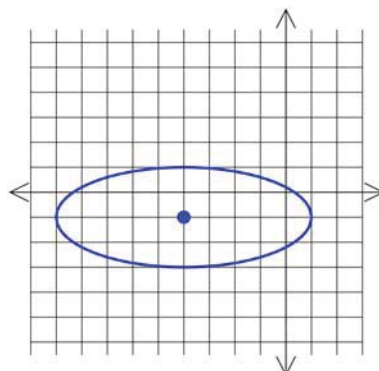
15. Denise and Donna want to exchange secrets from across a crowded whispering gallery. Recall that a whispering gallery is a room which, in cross section, is half of an ellipse. If the room is 50 feet high at the center and 180 feet wide at the floor, how far from the outer wall should each of them stand so that they will be positioned at the foci of the ellipse?
16. A hall 120 feet in length is to be designed as a whispering gallery. If the foci are located 40 feet from the center, how high will the ceiling be at the center?
17. Juan, standing at one focus of a whispering gallery, is 10 feet from the nearest wall. His friend is standing at the other focus, 90 feet away. What is the length of this gallery? How high is its elliptical ceiling at the center?
18. A semielliptical arch is constructed which is 160 feet wide at the base and 10 feet tall in the middle. Find the height of the arch exactly 50 feet in from the base of the arch.

Answers—Ellipses

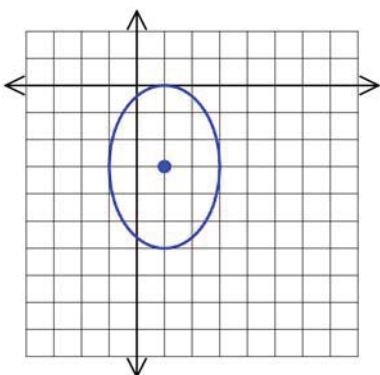
1. Center: $(-5, 4)$ Foci: $(-8, 4)$ $(-2, 4)$
 Vertices: $(-10, 4)$ $(0, 4)$



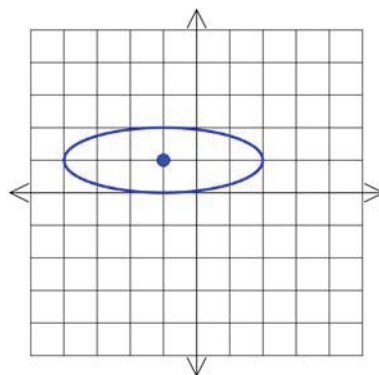
4. Center: $(-4, -1)$ Foci: $(-4 \pm \sqrt{21}, -1)$
 Vertices: $(1, -1)$ $(-9, -1)$



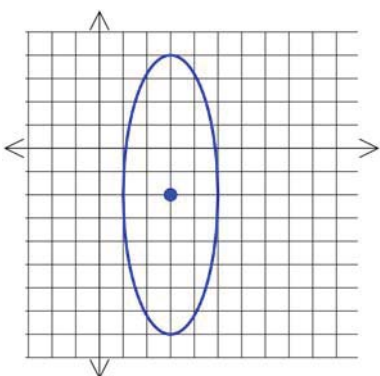
2. Center: $(1, -3)$ Foci: $(1, -3 \pm \sqrt{5})$
 Vertices: $(1, 0)$ $(1, -6)$



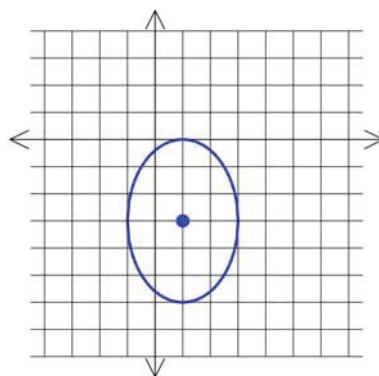
5. Center: $(-1, 1)$ Foci: $(-1 \pm 2\sqrt{2}, 1)$
 Vertices: $(2, 1)$ $(-4, 1)$



3. Center: $(3, -2)$ Foci: $(3, -2 \pm 4\sqrt{2})$
 Vertices: $(3, 4)$ $(3, -8)$

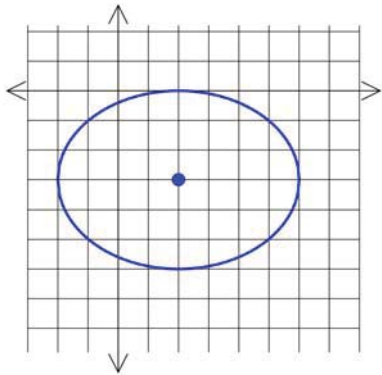


6. Center: $(1, -3)$ Foci: $(1, -3 \pm \sqrt{5})$
 Vertices: $(1, 0)$ $(1, -6)$



7. Center: $(2, -3)$ Foci: $(2 \pm \sqrt{7}, -3)$

Vertices: $(6, -3)$ $(-2, -3)$



8. $\frac{(x-3)^2}{9} + \frac{(y-7)^2}{25} = 1$

9. $\frac{x^2}{39} + \frac{y^2}{64} = 1$

10. $\frac{(x-8)^2}{25} + \frac{(y-2)^2}{4} = 1$

11. $\frac{(x+1)^2}{64} + \frac{(y+2)^2}{48} = 1$

12. $\frac{(x-1)^2}{25} + \frac{(y-1)^2}{34} = 1$

13. $\frac{(x-1)^2}{10} + \frac{(y-2)^2}{1} = 1$

14. $\frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} = 1$

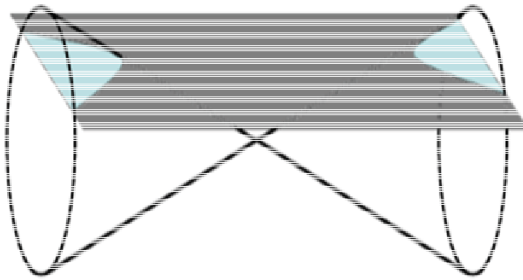
15. ≈ 15.2 feet from the outer wall.

16. $\sqrt{2000} \approx 44.7$ feet tall

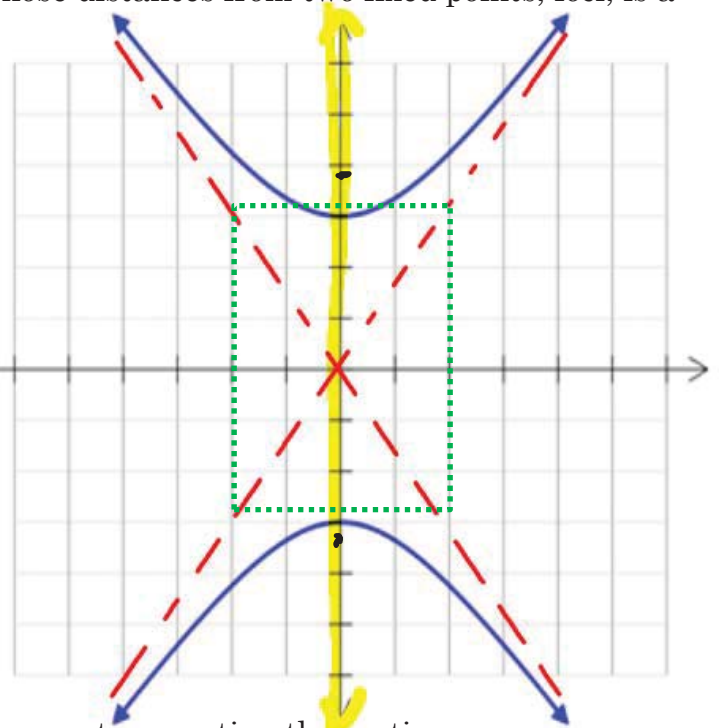
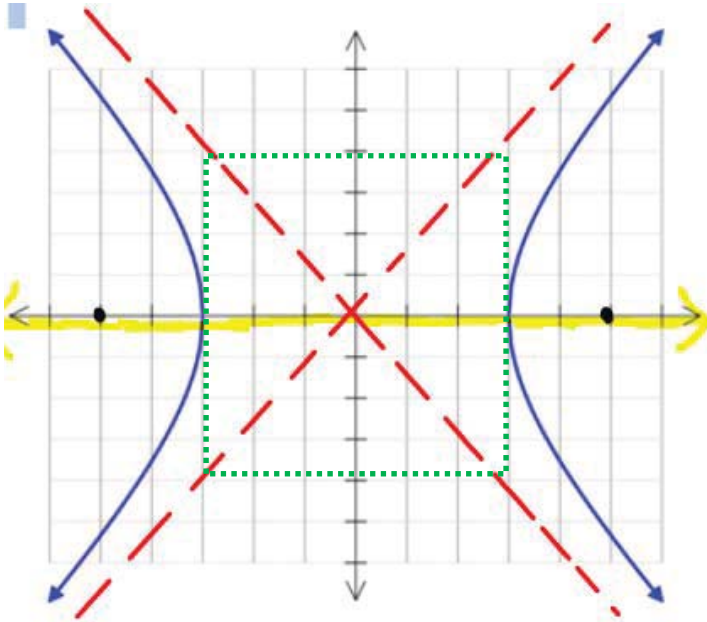
17. Length is 110 feet and
 $\sqrt{1000} \approx 31.6$ feet tall

18. ≈ 9.3 feet

Hyperbolas—Conic Section



Definition: The set of all points, the difference of whose distances from two fixed points, foci, is a constant.



The **transverse axis** of the hyperbola is the line segment connecting the vertices.
 The **conjugate axis** of a hyperbola is the line segment through the center which is perpendicular to the transverse axis.

The foci are always located on the _____ axis.

_____ is the distance from the center to each focus.

_____ is the distance from center to each vertex.

_____ are the endpoints of the transverse axis.

The standard form of an equation of a hyperbola with center (h, k) AND transverse axis parallel to the x -axis is:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

The standard form of an equation of a hyperbola with center (h, k) AND transverse axis parallel to the y -axis is:

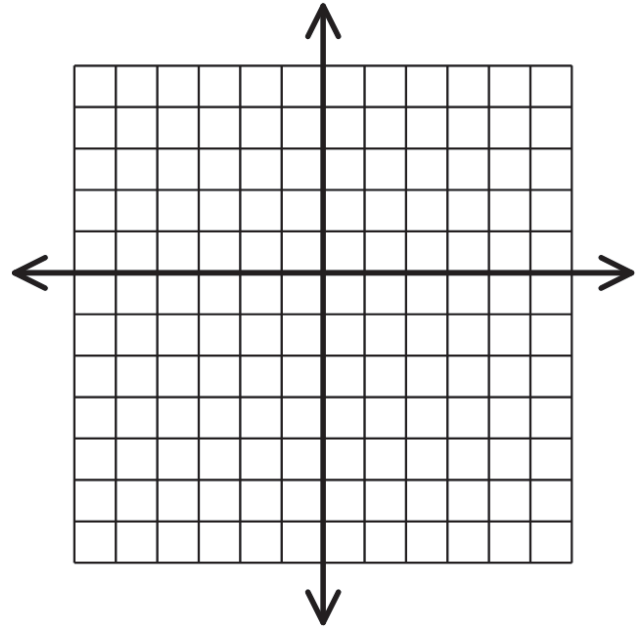
$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Equations of Asymptotes: $(y - k) = \pm \frac{\text{rise}}{\text{run}}(x - h)$

The relationship between a , b , and c is: $c^2 = a^2 + b^2$

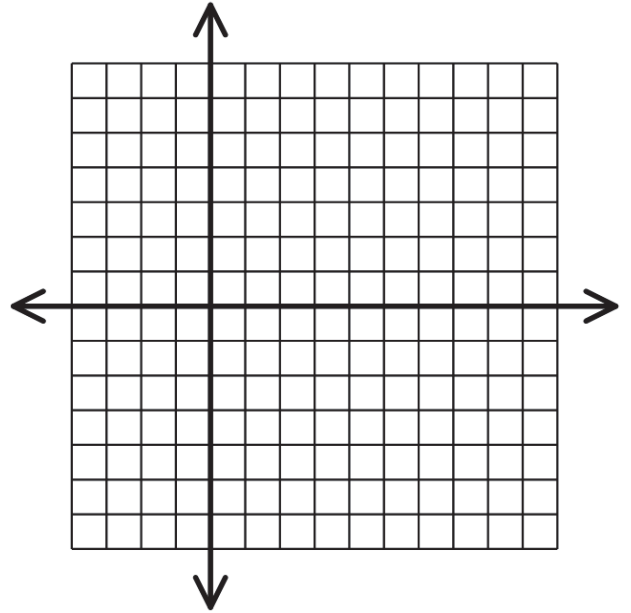
Example 1: Find the center, foci, vertices, asymptotes, and graph the following

$$\frac{(y+3)^2}{4} - \frac{(x-2)^2}{9} = 1$$



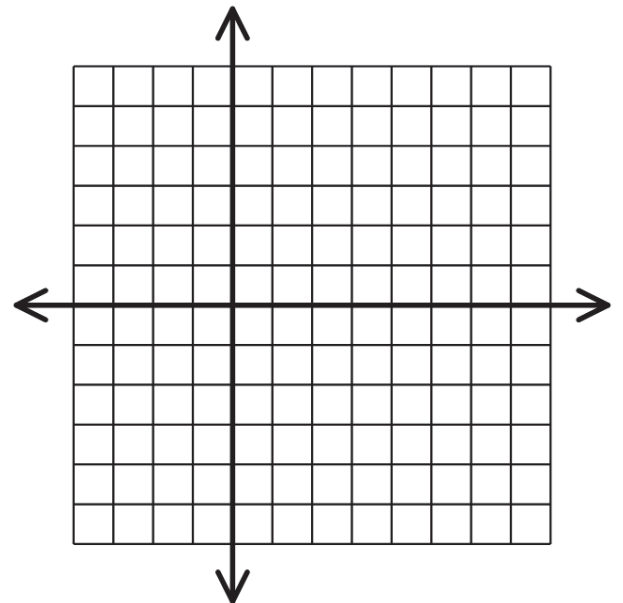
Example 2: Find the center, foci, vertices, asymptotes, and graph the following

$$9x^2 - 25y^2 - 54x - 50y - 169 = 0$$



Example 3: Find the center, foci, vertices, asymptotes, and graph the following

$$18y^2 - 5x^2 + 30x + 72y - 63 = 0$$



Example 4:

Find the equation of the hyperbola with focus $(-2,4)$; center $(1,4)$ and vertex $(0,4)$.

Example 5:

Find the equation of the hyperbola with vertices at $(-4,5)$ and $(-4,1)$, focus $(-4,-1)$

Example 6:

Find the equation of the hyperbola with vertices at $(1,-3)$ and $(1,1)$; asymptote is the line $y + 1 = \frac{3}{2}(x - 1)$

Hyperbolas

Find the center, foci, vertices, asymptotes, and graph the following:

1. $\frac{(x+3)^2}{4} - \frac{(y-2)^2}{9} = 1$

2. $\frac{(y+5)^2}{25} - \frac{(x-4)^2}{16} = 1$

3. $9(x-3)^2 - (y+2)^2 = 36$

4. $4(y+4)^2 - 25(x+1)^2 = 100$

5. $x^2 - y^2 - 2x - 2y - 1 = 0$

6. $4y^2 - 9x^2 - 32y - 18x + 19 = 0$

7. $49x^2 - 25y^2 - 294x - 100y - 884 = 0$

Find an equation of a hyperbola which fits the given criteria.

8. Center $(4, -1)$, Focus $(7, -1)$ and Vertex $(6, -1)$.

9. Foci $(0, \pm 8)$ and Vertices $(0, \pm 5)$.

10. Vertices $(3, 2)$ and $(11, 2)$. Endpoints of conjugate axis: $(7, 4)$ and $(7, 0)$

11. Foci: $(-4, 0)$ and $(-4, 4)$. Vertex $(-4, 3)$

12. Vertices: $(0, 1)$ and $(8, 1)$. Focus $(-3, 1)$.

13. Vertices: $(-2, -1)$ and $(4, -1)$; asymptote is the line $y + 1 = 2(x - 1)$.

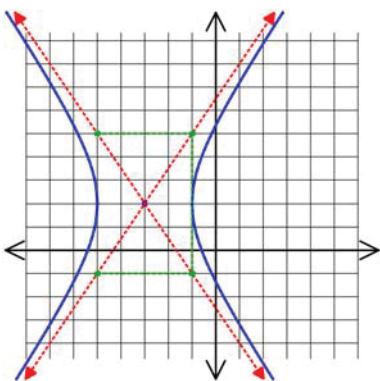
14. Vertices: $(2, 3)$ and $(2, -9)$; asymptote is the line $y + 3 = \frac{4}{3}(x - 2)$.

Answers—Hyperbolas

1. Center: $(-3, 2)$ Foci: $(-3 \pm \sqrt{13}, 2)$

Vertices: $(-1, 2)$ $(-5, 2)$

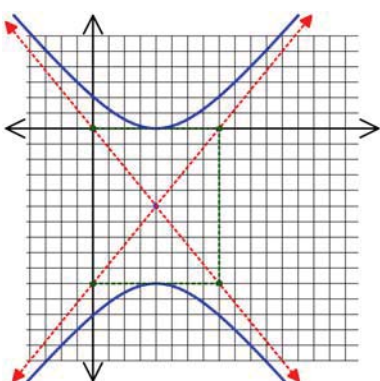
Asymp.: $(y - 2) = \pm \frac{3}{2}(x + 3)$



2. Center: $(4, -5)$ Foci: $(4, -5 \pm \sqrt{41})$

Vertices: $(4, 0)$ $(4, -10)$

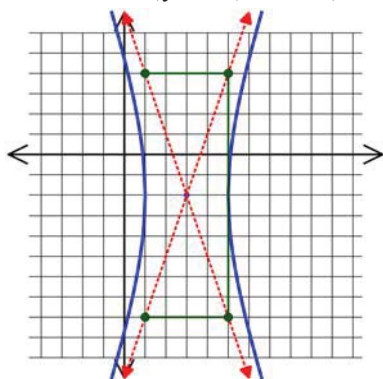
Asymp.: $(y + 5) = \pm \frac{5}{4}(x - 4)$



3. Center: $(3, -2)$ Foci: $(3 \pm 2\sqrt{10}, -2)$

Vertices: $(5, -2)$ $(1, -2)$

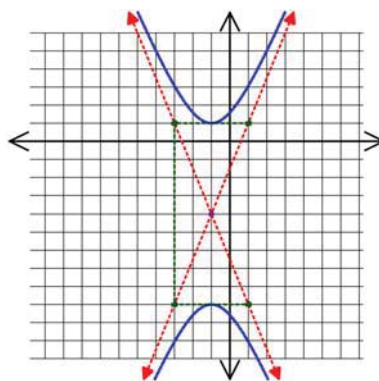
Asymp.: $(y + 2) = \pm 3(x - 3)$



4. Center: $(-1, -4)$ Foci: $(-1, -4 \pm \sqrt{29})$

Vertices: $(-1, -9)$ $(-1, 1)$

Asymp.: $(y + 4) = \pm \frac{5}{2}(x + 1)$

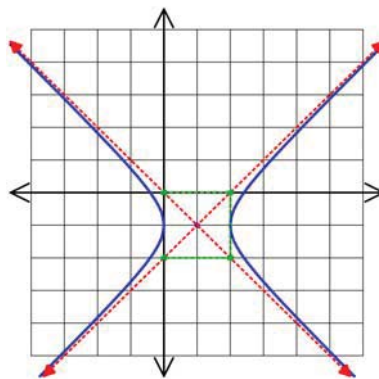


5. Center: $(1, -1)$ Foci: $(1 \pm \sqrt{2}, -1)$

Vertices: $(2, -1)$ $(0, -1)$

Asymp.: $(y + 1) = \pm(x - 1)$

Equation: $(x - 1)^2 - (y + 1)^2 = 1$

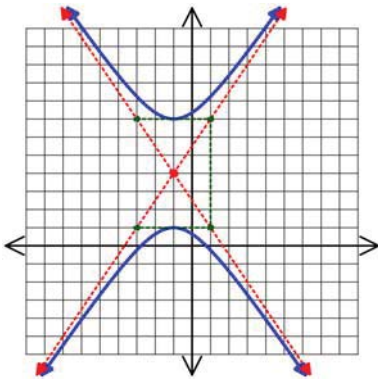


6. Center: $(-1, 4)$ Foci: $(-1, 4 \pm \sqrt{13})$

Vertices: $(-1, 7)$ $(-1, 1)$

Asymp.: $(y - 4) = \pm \frac{3}{2}(x + 1)$

Equation: $\frac{(y - 4)^2}{9} - \frac{(x + 1)^2}{4} = 1$

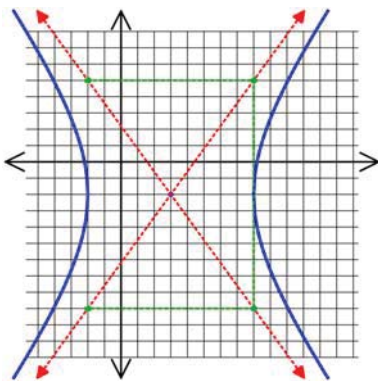


7. Center: $(3, -2)$ Foci: $(3 \pm \sqrt{74}, -2)$

Vertices: $(-2, -2)$ $(8, -2)$

Asymp.: $(y + 2) = \pm \frac{7}{5}(x - 3)$

Equation: $\frac{(x - 3)^2}{25} - \frac{(y + 2)^2}{49} = 1$



8. $\frac{(x - 4)^2}{4} - \frac{(y + 1)^2}{5} = 1$

9. $\frac{y^2}{25} - \frac{x^2}{39} = 1$

10. $\frac{(x - 7)^2}{16} - \frac{(y - 2)^2}{4} = 1$

11. $\frac{(y - 2)^2}{1} - \frac{(x + 4)^2}{3} = 1$

12. $\frac{(x - 4)^2}{16} - \frac{(y - 1)^2}{33} = 1$

13. $\frac{(x - 1)^2}{9} - \frac{(y + 1)^2}{36} = 1$

14. $\frac{(y + 3)^2}{36} - \frac{(x - 2)^2}{81/4} = 1$

Rotation of Conics

Identify a Conic:

Given the equation: $Ax^2 + Cy^2 + Dx + Ey + F = 0$

1) If parabola, then only one variable is squared.

$$AC=0$$

2) If ellipse, then both variables are squared and both coefficients are positive.

$$AC>0$$

3) If hyperbola, then both variables are squared and one coefficient is negative.

$$AC<0$$

These are conics that are parallel to either the x -axis or the y -axis.

However, there are conics that are NOT parallel to either axis.

The general equation is: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

To identify these conics, you must look at the quantity: $B^2 - 4AC$

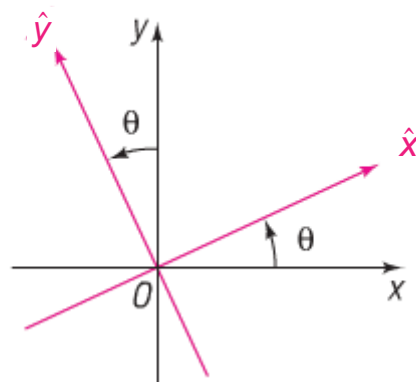
1) If parabola, then $B^2 - 4AC$ _____

2) If ellipse, then $B^2 - 4AC$ _____

3) If hyperbola, then $B^2 - 4AC$ _____

Identify the following conic: $4x^2 + 12xy + 9y^2 - x - y = 0$

Angle of Rotation:

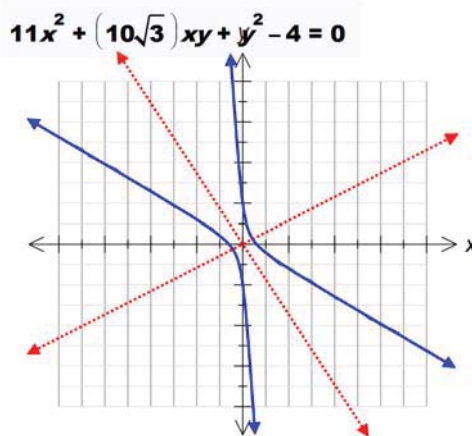
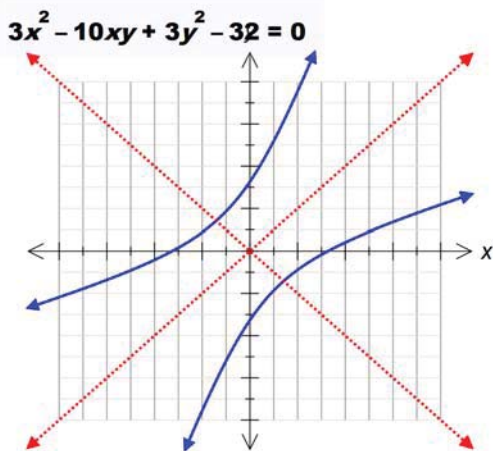


$$0^\circ < \theta \leq 90^\circ$$

To view the derivation of the following formulas, please watch the first 7 minutes of this video: [Conic Sections—Rotations](#)

Rotation Formulas: $x = \hat{x} \cos \theta - \hat{y} \sin \theta$
 $y = \hat{x} \sin \theta + \hat{y} \cos \theta$

Angle of Rotation: $\cot(2\theta) = \frac{A-C}{B}$

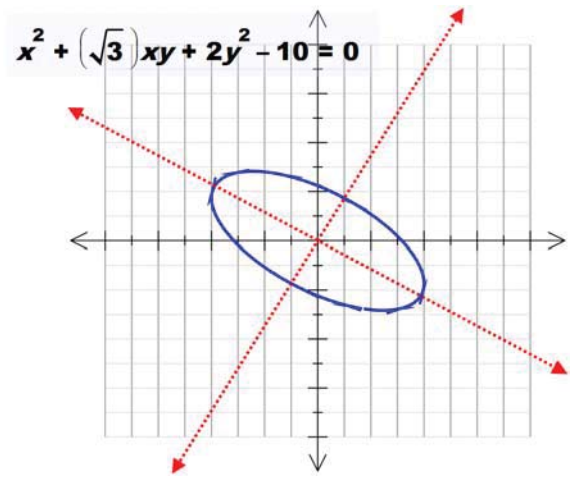


Find the rotation formulas. Use only EXACT values for $\cos \theta$ and $\sin \theta$.

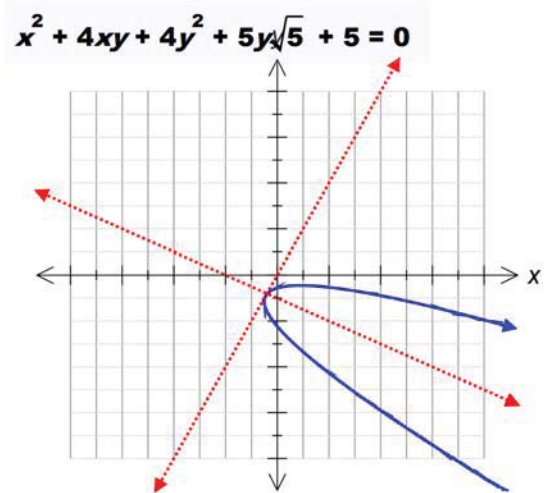
Ex. 1: $3x^2 - 10xy + 3y^2 - 32 = 0$

Ex. 2: $11x^2 + 10\sqrt{3}xy + y^2 - 4 = 0$

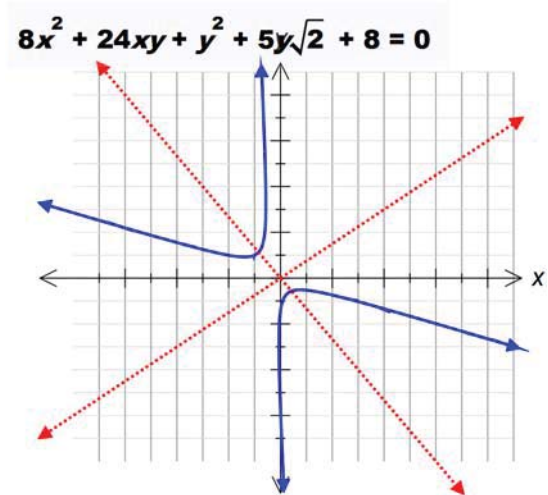
Ex. 3: $x^2 + \sqrt{3}xy + 2y^2 - 10 = 0$



Ex. 4: $x^2 + 4xy + 4y^2 + 5\sqrt{5}y + 5 = 0$



Ex. 5: $8x^2 + 24xy + y^2 + 3\sqrt{2}y + 8 = 0$



Eliminate the xy -term.

Rotate the axes so that the new equation has no xy -term.

Using results from Ex. 3: $x^2 + \sqrt{3}xy + 2y^2 - 10 = 0$ Substitute for x and y .

Rotation of Conics

Identify the following conics:

1. $x^2 + y + 4x + 9 = 0$
2. $4x^2 + 10y^2 - 5x - 2y = 0$
3. $9y^2 - 4x^2 + 4x - y = 0$
4. $10x^2 - 12xy + 4y^2 - x - y = 0$
5. $x^2 + 3xy - 2y^2 + 3x + 2y + 5 = 0$
6. $3x^2 - 2xy + y^2 - 5x + 3y - 6 = 0$

Find the rotation formulas. Use only EXACT values for $\cos\theta$ and $\sin\theta$.

7. $5x^2 + 26xy + 5y^2 - 16x\sqrt{2} + 16y\sqrt{2} - 104 = 0$
8. $21x^2 + 10\sqrt{3}xy + 31y^2 - 144 = 0$
9. $13x^2 + 6\sqrt{3}xy + 7y^2 - 16 = 0$
10. $9x^2 + 24xy + 16y^2 + 15x - 20y = 0$
11. $25x^2 - 36xy + 40y^2 - 12\sqrt{13}x - 8\sqrt{13}y = 0$

Rotate the axes so that the new equation has no xy -term. Graph the new equation.

12. $21x^2 + 10\sqrt{3}xy + 31y^2 - 144 = 0$ (Use results from #8)
13. $x^2 + 2\sqrt{3}xy - y^2 + 8 = 0$

Answers—Rotation of Conics

1. parabola
2. ellipse
3. hyperbola
4. ellipse
5. hyperbola
6. ellipse

$$7. \quad x = \frac{\sqrt{2}}{2}\hat{x} - \frac{\sqrt{2}}{2}\hat{y}$$

$$y = \frac{\sqrt{2}}{2}\hat{x} + \frac{\sqrt{2}}{2}\hat{y}$$

$$8. \quad x = \frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{y}$$

$$y = \frac{\sqrt{3}}{2}\hat{x} + \frac{1}{2}\hat{y}$$

$$9. \quad x = \frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{y}$$

$$y = \frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}$$

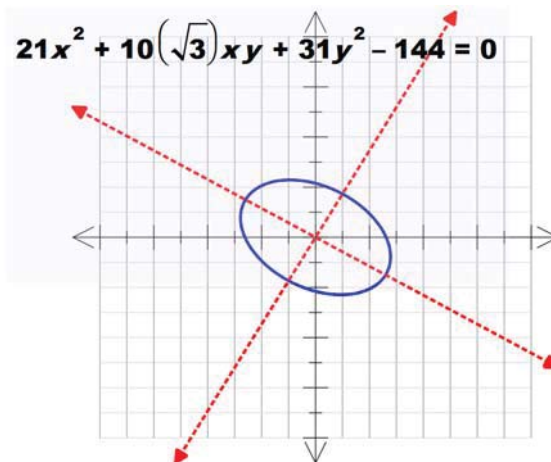
$$10. \quad x = \frac{3}{5}\hat{x} - \frac{4}{5}\hat{y}$$

$$y = \frac{4}{5}\hat{x} + \frac{3}{5}\hat{y}$$

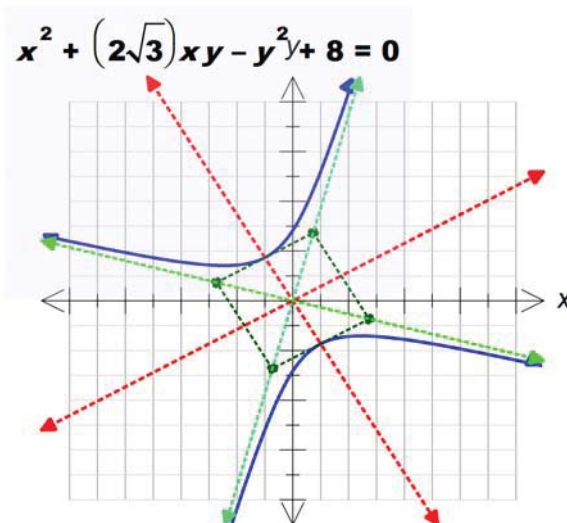
$$11. \quad x = \frac{3}{\sqrt{13}}\hat{x} - \frac{2}{\sqrt{13}}\hat{y}$$

$$y = \frac{2}{\sqrt{13}}\hat{x} + \frac{3}{\sqrt{13}}\hat{y}$$

$$12. \quad \frac{\hat{x}^2}{4} + \frac{\hat{y}^2}{9} = 1, \theta = 60^\circ$$

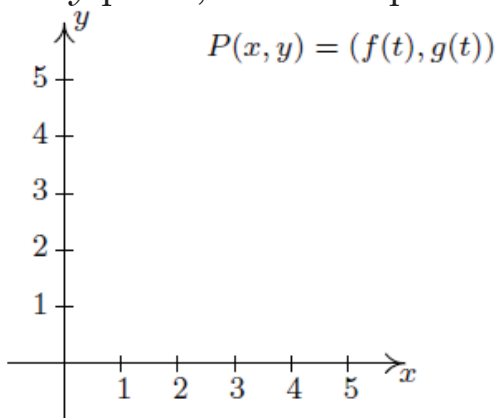


$$13. \quad \frac{\hat{y}^2}{4} - \frac{\hat{x}^2}{4} = 1, \theta = 30^\circ$$



Parametric Equations

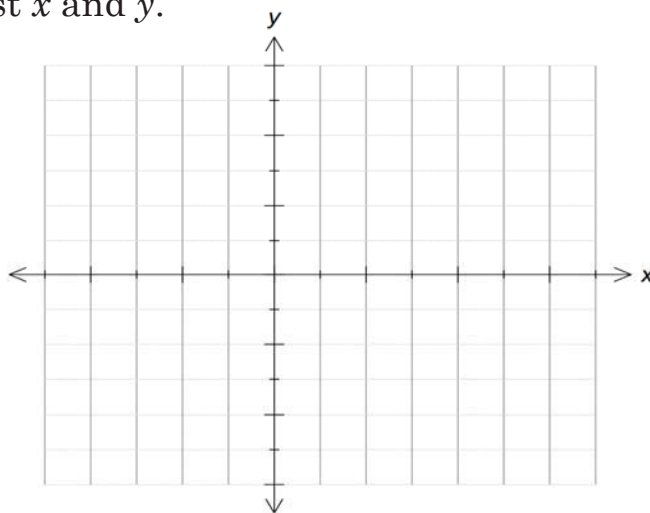
In this section, we present a new concept which allows us to use functions to study curves which, when plotted in the xy -plane, neither represent y as a function of x nor x as a function of y .



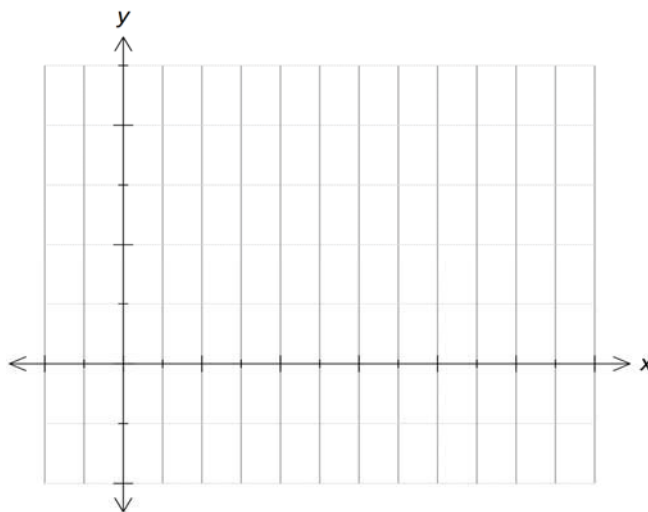
We can define the x -coordinate of P as a function of t and the y -coordinate of P as a (usually, but not necessarily) different function of t . (Traditionally, $f(t)$ is used for x and $g(t)$ is used for y .) The independent variable t in this case is called a parameter and the system of equations $\begin{cases} x = f(t) \\ y = g(t) \end{cases}$ is called a system of parametric equations.

For the following, sketch the curve, indicate its orientation, and eliminate the parameter to get an equation involving just x and y .

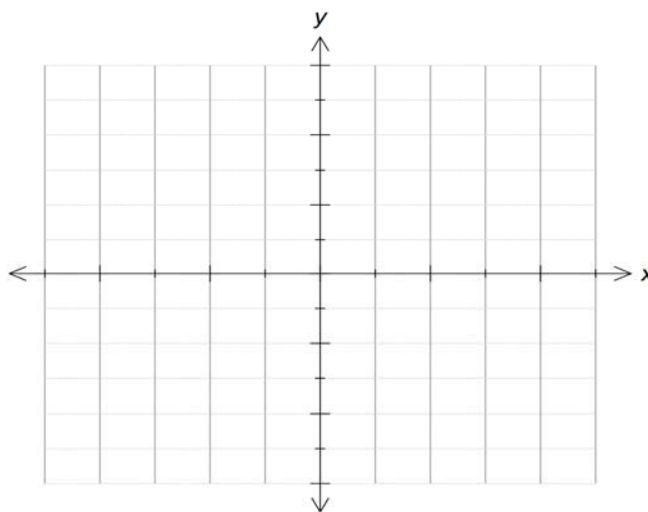
Ex. 1 $\begin{cases} x = t^2 - 3 \\ y = 2t - 1 \end{cases}$ for $-2 \leq t \leq 3$



Ex. 2 $\begin{cases} x = t^{3/2} + 1 \\ y = \sqrt{t} \end{cases}$ for $t \geq 0$.



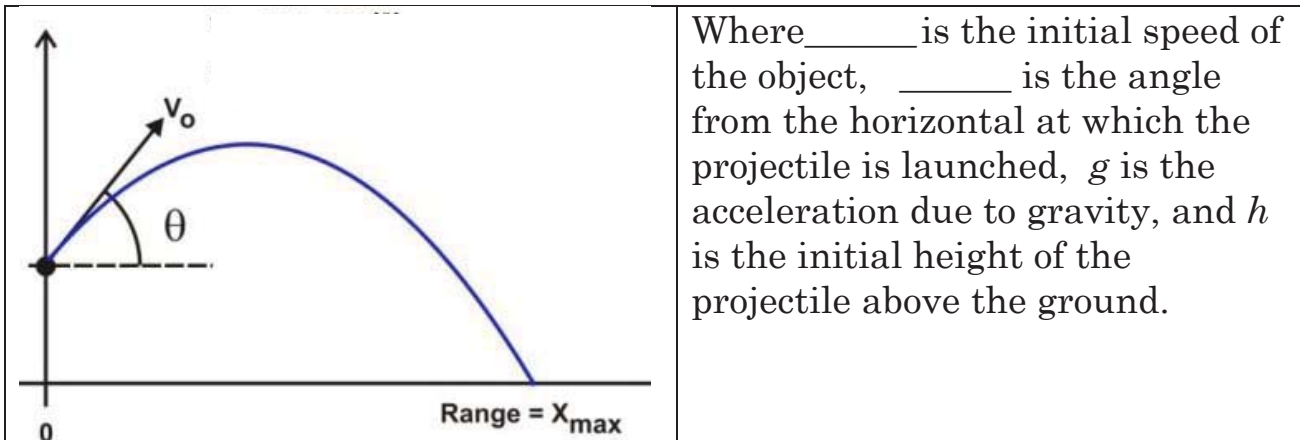
Ex. 3 $\begin{cases} x = 2\cos t \\ y = 3\sin t \end{cases}$ for $0 \leq t \leq \pi$.



Challenge: Research the cycloid and present findings to class.

Projectile Motion

Projectile motion is a form of motion where an object moves in parabolic path; the path that the object follows is called its trajectory. In projectile motion, the horizontal velocity remains constant. The vertical velocity increases due to gravity.



Where _____ is the initial speed of the object, _____ is the angle from the horizontal at which the projectile is launched, g is the acceleration due to gravity, and h is the initial height of the projectile above the ground.

Ignoring everything except the force of gravity, the path of the projectile is given by:

Horizontal Component

$$x = v_0 \cos(\theta)t$$

Vertical Component

$$y = -\frac{1}{2}gt^2 + v_0 \sin(\theta)t + h$$

Since the vertical component equation is a parabola, the vertex of the parabola gives us the time *when* the projectile reaches its maximum height and what the maximum height is.

Ex. 1: Herman throws a ball straight up with an initial speed of 40 feet per second from a height of 5 feet. (a) Find the parametric equations that model the motion of the ball as a function of time. (b) How long is the ball in the air? (c) When is the ball at its maximum height? (d) Determine the maximum height of the ball.

Ex. 2: Chris Carter hits a baseball with an initial speed of 130 feet per second at an angle of 25° to the horizontal. The baseball was hit at a height of 3 feet off the ground. (a) Find the parametric equations that model the motion of the ball as a function of time. (b) How long is the ball in the air? (c) When is the ball at its maximum height? (d) Determine the maximum height of the ball. (e) Determine the horizontal distance that the baseball travelled.

Challenge: If the ball travelled toward left field at Minute Maid Park, would it have been a homerun? (Minute Maid Park is known for being particularly hitter-friendly down the lines, especially in left field where it is only 315 ft. to the [Crawford Boxes](#), and the wall there is 19 feet tall.)

Parametric Equations

For the following, sketch the curve, indicate its orientation, and eliminate the parameter to get an equation involving just x and y .

1.
$$\begin{cases} x = 4t - 3 \\ y = 6t - 2 \end{cases} \text{ for } -1 \leq t \leq 1$$

2.
$$\begin{cases} x = t - 1 \\ y = 3 + 2t - t^2 \end{cases} \text{ for } 0 \leq t \leq 3$$

3.
$$\begin{cases} x = t^2 + 2t + 1 \\ y = t + 1 \end{cases} \text{ for } t \leq 1$$

4.
$$\begin{cases} x = 2t + 1 \\ y = 3\sqrt{t} \end{cases} \text{ for } 0 \leq t \leq 4$$

5.
$$\begin{cases} x = 4\cos t \\ y = 2\sin t \end{cases} \text{ for } -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

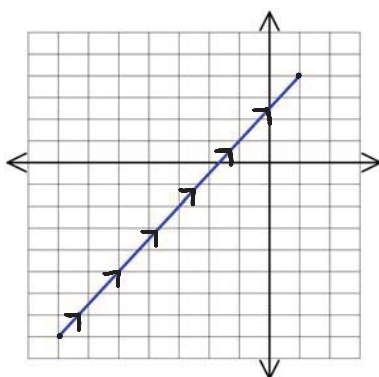
6.
$$\begin{cases} x = 3\cos t \\ y = 2\sin t + 1 \end{cases} \text{ for } \frac{\pi}{2} \leq t \leq 2\pi$$

Solve the following.

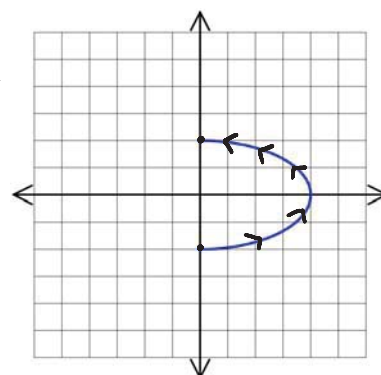
7. Ryan throws a ball straight up with an initial speed of 50 feet per second from a height of 6 feet. (a) Find the parametric equations that model the motion of the ball as a function of time. (b) How long is the ball in the air? (c) When is the ball at its maximum height? (d) Determine the maximum height of the ball.
8. Daryl throws a ball with an initial speed of 145 feet per second at an angle of 20° to the horizontal. The ball leaves Daryl's hand from a height of 5 feet. (a) Find the parametric equations that model the motion of the ball as a function of time. (b) How long is the ball in the air? (c) When is the ball at its maximum height? (d) Determine the maximum height of the ball. (e) Determine the horizontal distance that the ball travels.
9. Suppose that Sylvia hits a golf ball off a cliff 200 meters high with an initial speed of 45 meters per second at an angle of 40° to the horizontal. (a) Find the parametric equations that model the motion of the ball as a function of time. (b) How long is the ball in the air? (c) When is the ball at its maximum height? (d) Determine the maximum height of the ball. (e) Determine the horizontal distance that the ball travels. Use 9.8 m/s^2 for the acceleration due to gravity.

Answers—Parametric Equations

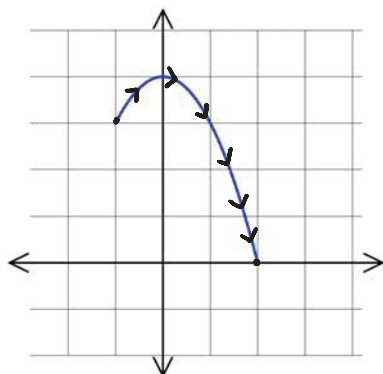
1. $y = \frac{3}{2}x + \frac{5}{2}$



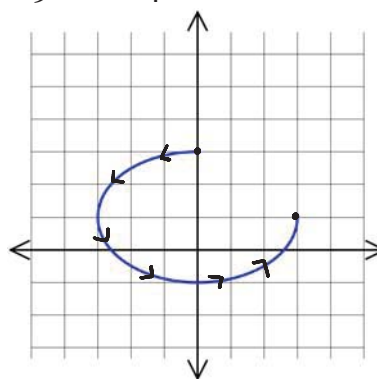
5. $\frac{x^2}{16} + \frac{y^2}{4} = 1$



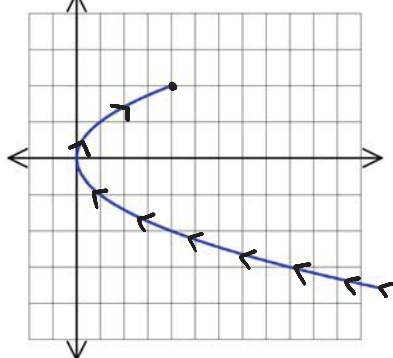
2. $y = -x^2 + 4$



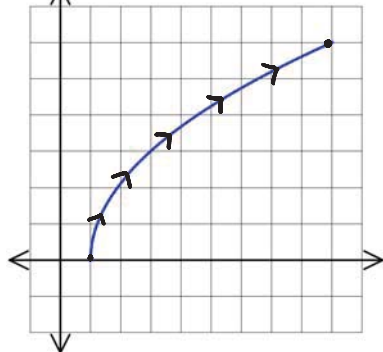
6. $\frac{x^2}{9} + \frac{(y-1)^2}{4} = 1$



3. $x = y^2$



4. $y = 3\sqrt{\frac{x-1}{2}}$



7. a) $y = -16t^2 + 50t + 6$; $x = 0$
 b) $t = 3.24$ seconds
 c) $t = 1.56$ seconds
 d) 45.06 feet

8. a) $y = -16t^2 + 145\sin(20)t + 5$
 $x = 145\cos(20)t$
 b) $t = 3.20$ seconds
 c) $t = 1.54$ seconds
 d) 43.43 feet
 e) 436.02 feet

9. a) $y = -4.9t^2 + 45\sin(40)t + 200$
 $x = 45\cos(40)t$
 b) $t = 9.99$ seconds
 c) $t = 2.95$ seconds
 d) 242.69 meters
 e) 344.38 meters

Partial Fraction Decomposition

Review of adding rational expressions (fractions):

$$\frac{3}{x+1} + \frac{1}{x-3}$$

Type 1: *Nonrepeated* linear factors

$$\frac{P(x)}{Q(x)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)} + \dots$$

Example #1: $\frac{4x-8}{x^2-2x-3}$

Process:

1. Factor the denominator.
2. Set up the equation.
3. Multiply the equation by the original denominator (LCD) to get rid of the fractions.
4. Let x be “convenient” values to solve for the constants (A , B , C , *etc.*)

Example #2: $\frac{3}{x^3-2x^2-8x}$

Type 2: **Repeated** linear factors

$$\frac{P(x)}{Q(x)} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3} + \dots$$

$$\frac{3x+5}{x^2(x+1)^3(x-3)} =$$

Example #3: $\frac{x+1}{x^2(x-2)^2}$

Process:

1. Factor the denominator.
2. Set up the equation.
3. Multiply the equation by the original denominator (LCD) to get rid of the fractions.
4. Let x be “convenient” values to solve for the constants (A , B , C , *etc.*)
5. Equate coefficients to solve for remaining constants.

Example #4: $\frac{x^2+x}{(x+2)(x-1)^2}$

Type 3: *Nonrepeated irreducible* quadratic factor(s)

$$\frac{P(x)}{Q(x)} = \frac{Ax + B}{ax^2 + bx + c}$$

Example #5: $\frac{x^3 + 5x^2 + 10}{x^2(x^2 + 2)}$

Example #6: $\frac{2x + 4}{x^3 - 1}$

Type 4: *Repeated irreducible* quadratic factor(s)

$$\frac{P(x)}{Q(x)} = \frac{Ax + B}{ax^2 + bx + c} + \frac{Cx + D}{(ax^2 + bx + c)^2} + \frac{Ex + F}{(ax^2 + bx + c)^3}$$

Example #7: $\frac{x^3 + x^2}{(x^2 + 4)^2}$

Partial Fractions

For the following, write the partial fraction decomposition of each rational expression.

1. $\frac{3x+2}{x^2-2x-24}$

2. $\frac{3x-7}{x^2-2x-3}$

3. $\frac{-7x+43}{3x^2+19x-14}$

4. $\frac{2x^2+4}{x^3+5x^2+4x}$

5. $\frac{3x+9}{(x-4)(x-3)^2}$

6. $\frac{2-9x}{(x-3)(2x-1)^2}$

7. $\frac{5x^2+3x-20}{x^2(x+4)}$

8. $\frac{3x+1}{(x-1)(x^2+1)}$

9. $\frac{3x^2+2x}{(x+2)(x^2+4)}$

10. $\frac{18x^2-8x-14}{(3x+4)(2x^2-3x+2)}$

11. $\frac{x^2+2x+3}{(x^2+4)^2}$

12. $\frac{-10x^4+x^3-19x^2+x-10}{x(x^2+1)^2}$

Answers—Partial Fractions

$$1. \frac{3x+2}{x^2-2x-24} = \frac{1}{x+4} + \frac{2}{x-6}$$

$$2. \frac{3x-7}{x^2-2x-3} = \frac{5}{2(x+1)} + \frac{1}{2(x-3)}$$

$$3. \frac{-7x+43}{3x^2+19x-14} = \frac{5}{3x-2} - \frac{4}{x+7}$$

$$4. \frac{2x^2+4}{x^3+5x^2+4x} = \frac{-2}{x+1} + \frac{3}{x+4} + \frac{1}{x}$$

$$5. \frac{3x+9}{(x-4)(x-3)^2} = \frac{-21}{(x-3)} - \frac{18}{(x-3)^2} + \frac{21}{x-4}$$

$$6. \frac{2-9x}{(x-3)(2x-1)^2} = \frac{-1}{x-3} + \frac{2}{2x-1} + \frac{1}{(2x-1)^2}$$

$$7. \frac{5x^2+3x-20}{x^2(x+4)} = \frac{-5}{x^2} + \frac{3}{(x+4)} + \frac{2}{x}$$

$$8. \frac{3x+1}{(x-1)(x^2+1)} = \frac{1-2x}{x^2+1} + \frac{2}{x-1}$$

$$9. \frac{3x^2+2x}{(x+2)(x^2+4)} = \frac{2x-2}{x^2+4} + \frac{1}{x+2}$$

$$10. \frac{18x^2-8x-14}{(3x+4)(2x^2-3x+2)} = \frac{4x-5}{2x^2-3x+2} + \frac{3}{3x+4}$$

$$11. \frac{x^2+2x+3}{(x^2+4)^2} = \frac{1}{(x^2+4)} + \frac{2x-1}{(x^2+4)^2}$$

$$12. \frac{-10x^4+x^3-19x^2+x-10}{x(x^2+1)^2} = \frac{-10}{x} + \frac{x}{(1+x^2)^2} + \frac{1}{1+x^2}$$

Sequences

Definition: A sequence is a function whose domain is the set of positive integers.

$n = \text{domain}$ Given the function: $f(n) = \frac{1}{n+1}$, evaluate the function for $n = 1, 2, 3, 4$

n th term notation: $\{d_n\} = \left\{ (-1)^{n-1} \left(\frac{n}{2n-1} \right) \right\}$. Find the first 4 terms.

Find the n th term:

$$\frac{1}{1 \cdot 2}, \frac{1}{2 \cdot 3}, \frac{1}{3 \cdot 4}, \frac{1}{4 \cdot 5}, \dots$$

$$2, -4, 6, -8, 10, \dots$$

$$\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \dots$$

Recursive Formulas

Fibonacci Sequence: 1, 1,

1. Given: $a_1 = 3$ and $a_n = 4 - a_{n-1}$, find the next 3 terms.

2. Given: $a_1 = -2$ and $a_n = n + 3a_{n-1}$, find the next 3 terms.

Challenge: Write the formula for the Fibonacci Sequence.

Sigma Notation

3. Find $\sum_{k=1}^5 (k - 2)$

4. Find $\sum_{k=1}^n (k + 1)^2$

5. Write the following sum using sigma notation: $1 + 3 + 5 + 7 + \dots + [2(12) - 1]$

6. Write the following sum using sigma notation: $\frac{2}{3} - \frac{4}{9} + \frac{8}{27} + \dots + (-1)^{12} \left(\frac{2}{3}\right)^{11}$

Sequences

Find the first 4 terms for the following:

1. $\{a_n\} = \{5n - 2\}$

2. $\{a_n\} = \left\{ \frac{\ln(n)}{n} \right\}$

3. $\{a_n\} = \{(-1)^{n-1} n^2\}$

A sequence is defined recursively. Find the first 4 terms for the following:

4. $a_1 = 2$ and $a_n = 3 + a_{n-1}$

5. $a_1 = 3$ and $a_n = 2a_{n-1} - 1$

6. $a_1 = 5$ and $a_n = \frac{a_{n-1}}{n}$

Write out each sum.

7. Find $\sum_{k=1}^4 (3k - 1)$

8. Find $\sum_{k=1}^5 (-1)^{k-1} \left(\frac{1}{2}\right)^k$

Write the following sum using sigma notation:

9. $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \dots + \frac{1}{3^{13}}$

10. $2^2 + 4^2 + 6^2 + \dots + 14^2$

Write the n th term of the following sequences:

11. $1, -2, 3, -4, 5, -6, \dots$

12. $1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} \dots$

Answers—Sequences

1. $a_1 = 3, a_2 = 8, a_3 = 13, a_4 = 18$

2. $a_1 = 0, a_2 = \frac{\ln 2}{2}, a_3 = \frac{\ln 3}{3}, a_4 = \frac{\ln 4}{4}$

3. $a_1 = 1, a_2 = -4, a_3 = 9, a_4 = -16$

4. $a_1 = 2, a_2 = 5, a_3 = 8, a_4 = 11$

5. $a_1 = 3, a_2 = 5, a_3 = 9, a_4 = 17$

6. $a_1 = 5, a_2 = \frac{5}{2}, a_3 = \frac{5}{6}, a_4 = \frac{5}{24}$

7. $\sum_{k=1}^4 (3k-1) = 2 + 5 + 8 + 11 = 26$

8. $\sum_{k=1}^5 (-1)^{k-1} \left(\frac{1}{2}\right)^k = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32}$

9. $\sum_{k=1}^{13} (-1)^{k-1} \left(\frac{1}{3}\right)^k$

10. $\sum_{k=1}^7 (2k)^2$

11. $a_n = (-1)^{n-1} n$

12. $a_n = \left(\frac{1}{5}\right)^{n-1}$

Arithmetic Sequences and Series

2, 5, 8, 11, 14,

The difference between successive terms is always the same.

This is called an ARITHMETIC SEQUENCE.

$$d = a_{n+1} - a_n$$

Recursive Formula:

$a_n = a_{n-1} + d$, where d is the _____

n th Term Formula

$a_n = a_1 + (n-1)d$, where a_1 = first term and n = term number

Series

The sum of a sequence is called a _____.

$$S_n = \frac{n}{2}(a_1 + a_n) \quad \text{or} \quad S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

Examples:

1. Find a formula for the n th term given $a_1 = 6$ and $d = -2$

2. Find the 80th term of $-1, 1, 3, 5, \dots$

3. Find a_1 , d and a_n , if the 8th term of an arithmetic sequence is 4 and the 18th term is -96 .

4. Find the sum of $7 + 1 - 5 - 11 - \dots - 299$

5. Find the sum: $\sum_{n=1}^{80} \left(\frac{1}{3}n + \frac{1}{2} \right)$

Arithmetic Sequences and Series

Find the common difference for each arithmetic sequence. Write none if not an arithmetic sequence.

1. $\{a_n\} = \{3n + 1\}$
2. $\{a_n\} = \{4 - 2n\}$
3. $\{a_n\} = \{(-1)^{n-1}n^2\}$

Find a formula for the n th term. What is the 41st term?

4. $a_1 = -12$ and $d = 3$

5. $a_1 = 25$ and $d = -2$

Find the indicated term for each arithmetic sequence.

6. 90th term of 1, -2, -5, ...
7. 70th term of $2\sqrt{5}, 4\sqrt{5}, 6\sqrt{5}, \dots$
8. 80th term: 8th term is 8; 20th term is 44
9. 80th term: 9th term is -5; 15th term is 31
10. 80th term: 5th term is -2; 13th term is 30

Find the sum:

11. $2 + 4 + 6 + \dots + 70$
12. $4 - 1 - 6 - \dots - 991$
13. $\sum_{n=1}^{80} (2n - 5)$
14. $\sum_{n=1}^{90} (3 - 2n)$

15. $\sum_{n=1}^{100} \left(6 - \frac{1}{2}n \right)$

Solve:

16. A brick staircase has a total of 30 steps. The bottom step requires 100 bricks. Each successive step requires two less bricks than the prior step. (a) How many bricks are required for the top step?
(b) How many bricks are required to build the staircase?

Answers—Arithmetic Sequences and Series

1. $d = 3$
2. $d = -2$
3. Not an arithmetic sequence
4. $a_n = 3n - 15, a_{41} = 108$
5. $a_n = 27 - 2n, a_{41} = -55$
6. $a_{90} = -266$
7. $a_{70} = 140\sqrt{5}$
8. $a_{80} = 224$
9. $a_{80} = 421$
10. $a_{80} = 298$
11. $S = 1260$
12. $S = -98700$
13. $S = 6080$
14. $S = -7920$
15. $S = -1925$
16. (a) 42 bricks
(b) 2130 total bricks

Geometric Sequences and Series

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$$

The ratio between successive terms is always the same. $r = \frac{a_{n+1}}{a_n}$

This is called a GEOMETRIC SEQUENCE.

$$-1, -2, -4, -8, \dots$$

Recursive Formula:

$$a_n = r \cdot a_{n-1}, \text{ where } r \text{ is the } \underline{\hspace{4cm}}$$

n th Term Formula

$$a_n = a_1 \cdot r^{n-1}, \text{ where } a_1 = \text{first term and } n = \text{term number}$$

Examples:

1. Is $\{5n^2 + 1\}$ a geometric sequence?
2. Find the 5th term of a geometric sequence if $a_1 = -2$ and $r = 4$.
3. Find the 20th term of 5, 10, 20, 40, ...

4. Find the 10th term of $\sqrt{2}, 2, 2\sqrt{2}, 4, \dots$

5. Find the 8th term of the geometric sequence, if $a_2 = 7$ and $r = \frac{1}{3}$.

6. Find the 10th term of the geometric sequence, if $a_3 = \frac{1}{3}$ and $a_6 = \frac{1}{81}$.

Series

$$S_n = a_1 \left(\frac{1-r^n}{1-r} \right) \text{ where } r \neq 0,1$$

Examples:

1. Find the sum: $\sum_{n=1}^6 4 \cdot 3^{n-1}$

2. Find the sum of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{256}$

Sum of an Infinite Series

Only possible if $|r| < 1$. NOT possible if $|r| \geq 1$

We say the series _____ if $|r| < 1$. The series _____ if $|r| \geq 1$

$$18, 6, 2, \frac{2}{3}, \frac{2}{9}, \dots$$

$$2, 6, 18, \dots$$

$$S_\infty = \left(\frac{a_1}{1-r} \right)$$

3. Find the sum, if possible: $\sum_{k=1}^{\infty} 2\left(\frac{3}{4}\right)^k$

4. Find the sum, if possible: $\sum_{k=1}^{\infty} 3\left(\frac{3}{2}\right)^{k-1}$

Geometric Sequences and Series

Find the common ratio for each geometric sequence.

1. $\{a_n\} = \{3^n\}$ 2. $\{a_n\} = \left\{-3\left(\frac{5}{2}\right)^n\right\}$

Find a formula for the n th term. What is the 5th term?

4. $a_1 = 2$ and $r = 3$ 5. $a_1 = 27$ and $r = -\frac{1}{3}$

Find the indicated term for each geometric sequence. Use exact values.

6. 7th term of $1, \frac{1}{2}, \frac{1}{4}, \dots$ 7. 8th term of $0.4, 0.04, 0.004, \dots$
8. 7th term: if $a_2 = 25$ and $a_5 = \frac{1}{5}$. 9. 8th term: if $a_3 = 2$ and $a_6 = \frac{2}{27}$.
10. 8th term: if $a_3 = 15$ and $a_5 = 45$.

Find the sum:

11. $6 + 2 + \frac{2}{3} + \frac{2}{9} + \dots + \frac{2}{6561}$
12. $1 + \sqrt{5} + 5 + 5\sqrt{5} + \dots + 3125$
13. $\sum_{n=1}^9 (-2)^{n-1}$
14. $\sum_{n=1}^7 64\left(-\frac{1}{2}\right)^{n-1}$
15. $\sum_{n=1}^{16} 2\left(\frac{4}{3}\right)^{n-1}$

Find the sum, if possible.

16. $\sum_{n=1}^{\infty} 5\left(\frac{1}{4}\right)^{n-1}$
17. $\sum_{n=1}^{\infty} 3\left(\frac{3}{2}\right)^{n-1}$
18. $\sum_{n=1}^{\infty} 3\left(\frac{2}{3}\right)^n$
19. $2 - \frac{1}{2} + \frac{1}{8} - \frac{1}{32} + \dots$

Solve:

20. Initially, a pendulum swings through an arc of 2 feet. On each successive swing, the length of the arc is 0.9 of the previous arc. a) What is the length of the arc of the 10th swing? b) After 15 swings, what total length will the pendulum have swung? c) When it stops, what total length will the pendulum have swung?
21. A ball is dropped from a height of 100 feet. Each time it strikes the ground, it bounces up 0.8 of the previous height. a) What height will the ball bounce up to after it strikes the ground for the third time? b) What total distance does the ball travel before it stops bouncing?

Answers—Geometric Sequences and Series

1. $r = 3$

2. $r = \frac{5}{2}$

3. There is no #3

4. $a_n = 2(3)^{n-1}; a_5 = 162$

5. $a_n = 27\left(\frac{-1}{3}\right)^{n-1}; a_5 = \frac{1}{3}$

6. $a_7 = \frac{1}{64}$

7. $a_8 = 4 \times 10^{-8} = 0.00000004$

8. $a_7 = \frac{1}{125}$

9. $a_8 = \frac{2}{243}$

10. $a_8 = 135\sqrt{3}$

11. $S \approx 8.99$

12. $S \approx 5652.37$

13. $S = 171$

14. $S = 43$

15. $S \approx 592.65$

16. $S = \frac{20}{3} \approx 6.66$

17. *not possible*

18. $S = 6$

19. $S = \frac{8}{5} = 1.6$

20. a) .775 ft.

b) 15.88 ft.

c) 20 ft.

21. a) 51.2 ft.

b) 900 ft.

Principle of Mathematical Induction

The Principle of Mathematical Induction, or PMI for short, is exactly that—a principle. It is a property of the natural numbers we either choose to accept or reject. In English, it says that if we want to prove that a formula works for all natural numbers n , we start by showing it is true for $n = 1$ (the 'base step') and then show that if it is true for a generic natural number k , it must be true for the next natural number, $k + 1$ (the 'inductive step').

The four steps of math induction are:

1. Show $P(1)$ is true.

Let $n = 1$ and work it out.

2. Assume $P(k)$ is true. (This is called the assumption statement)

Stick a k in for all n 's and say it is true.

3. Show $P(k) \rightarrow P(k + 1)$.

In math, the arrow \rightarrow means "implies" or "leads to"

Use $P(k)$ to show that $P(k + 1)$ is true

VERY important step! The third step is the only tricky part... And it's the most important step... You have to show EVERY little detail! Remember that you are proving something – which means that you have to spell out your entire argument.

4. End the proof.

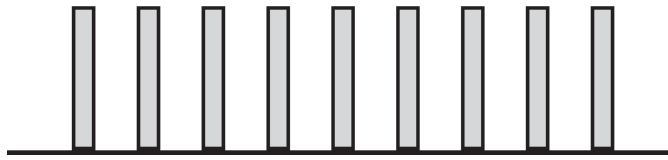
Write, thus $P(n)$ is true

Let's look at some dominoes...



Did you ever stack them so you could knock them all down? It's actually pretty fun and, if you've never done it, I highly recommend that you do.

Let's line up a row of dominoes...



There are four main parts to math induction...

1

Can we knock down the first domino?



Show $P(1)$ is true.

Yes!

2

Can we knock down a random domino somewhere in the middle?

Let's call it the k^{th} domino.



Yes!

Assume $P(k)$ is true.

3

If we knock down that k^{th} domino, will the next domino get knocked down too?

Show $P(k) \rightarrow P(k+1)$.



4

If we do all of the above, will all the dominoes fall?



YES!

Thus, $P(n)$ is true.

Prove: $1 + 3 + 5 + \dots + (2n - 1) = n^2$

1. Show $P(1)$ is true.

Let $n = 1$ and work it out.

2. Assume $P(k)$ is true.

Stick a k in for all n 's

3. Show $P(k) \rightarrow P(k + 1)$.

Add the next term on the left hand side of the assumption statement.
Re-place all k 's on the right hand side with $(k+1)$

4. Thus, $P(n)$ is true.

Prove: $3 + 5 + 7 + \dots + (2n + 1) = n(n + 2)$

Prove: $1 + 5 + 5^2 + \dots + 5^{n-1} = \frac{1}{4}(5^n - 1)$

Prove: $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4}n^2(n + 1)^2$

Principle of Mathematical Induction

Use the Principle of Mathematical Induction to prove the following:

1. $2 + 4 + 6 + \dots + 2n = n(n + 1)$

2. $3 + 4 + 5 + \dots + (n + 2) = \frac{1}{2}n(n + 5)$

3. $2 + 5 + 8 + \dots + (3n - 1) = \frac{1}{2}n(3n + 1)$

4. $1 + 5 + 9 + 13 + \dots + (4n - 3) = n(2n - 1)$

5. $1 + 2 + 2^2 + \dots + 2^{n-1} = 2^n - 1$

6. $1 + 4 + 4^2 + \dots + 4^{n-1} = \frac{1}{3}(4^n - 1)$

7. $4 + 3 + 2 + \dots + (5 - n) = \frac{1}{2}n(9 - n)$

8. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$

Factorials

Factorials

$$3! =$$

$$5! =$$

$$\frac{5!}{3!}$$

$$\frac{108!}{105!}$$

$$\frac{(n+2)!}{n!}$$

$$\frac{(n-1)!}{(n+1)!}$$

Combinations

$\binom{n}{j}$ This is read as " n choose j ". The number of combinations that can be made choosing j objects out of the set of n objects.

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

$$\binom{6}{4}$$

$$\binom{48}{20}$$

The Binomial Theorem

The binomial theorem is a formula for the expansion of $(a + b)^n$

$$(a + b)^1 = a + b$$

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$\binom{n}{j}$ This is read as " n choose j ". The number of combinations that can be made choosing j objects out of the set of n objects.

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

This info is a repeat of the material in the "Factorial" video. You can skip ahead.

$$\binom{6}{4}$$

$$\binom{48}{20}$$

Binomial Theorem

For any nonzero real numbers a and b , $(a + b)^n = \sum_{j=0}^n \binom{n}{j} a^{n-j} b^j$ for all natural numbers n .

Pascal's Triangle

You should write out Pascal's Triangle to 7 rows.

Expand and simplify: $(2x + 3)^5$

Expand and simplify: $(\sqrt{x} - 2\sqrt{y})^4$

Find the 6th term of $(3x + 2)^8$

Find the 7th term of $(x + x^2)^{12}$

Find the 8th term of $(k - \sqrt{2})^{10}$

Find the term containing y^3 in $(\sqrt{2} + y)^{12}$

Find the term containing no x in $\left(8x + \frac{1}{2x}\right)^8$

Find the term containing x^2 in $\left(\sqrt{x} + \frac{3}{\sqrt{x}}\right)^8$

Binomial Theorem

Evaluate the following:

1. $7!$

2. $12!$

3. $\frac{110!}{108!}$

4. $\binom{8}{5}$

5. $\binom{12}{3}$

Simplify the following:

6. $\frac{(n+1)!}{(n-1)!}$

7. $\frac{(n+3)!}{(n+1)!}$

8. $\frac{(n-2)!}{n!}$

Expand and simplify the following:

9. $(3x+1)^6$

10. $(\sqrt{x}-\sqrt{3})^4$

11. $(x^2+2y)^5$

12. Find the fifth term in the expansion of $(ab-1)^{20}$

13. Find the 6th term in the expansion of $(2x-5)^9$

14. Find the 4th term in the expansion of $\left(2+\frac{1}{2}x^4\right)^7$

15. Find the second term in the expansion of $\left(x-\frac{1}{x}\right)^{25}$

16. Find the term containing x^4 in the expansion of $(x+2y)^{10}$

17. Find the term containing b^8 in the expansion of $(a+b^2)^{12}$

18. Find the term containing x^4 in the expansion of $\left(x^2+\frac{3}{x}\right)^8$

19. Find the term that does not contain x in the expansion of $\left(5x+\frac{1}{2x}\right)^8$

20. Find the term that does not contain x in the expansion of $\left(x-\frac{1}{x^2}\right)^9$

21. Find the term containing x^4 in the expansion of $\left(x-\frac{2}{\sqrt{x}}\right)^{10}$

Answers—Binomial Theorem

1. 5040
2. 479001600
3. 11990
4. 56
5. 220
6. $n(n+1)$
7. $(n+3)(n+2)$
8. $\frac{1}{n(n-1)}$
9. $729x^6 + 1458x^5 + 1215x^4 + 540x^3 + 135x^2 + 18x + 1$
10. $x^2 - 4x\sqrt{3x} + 18x - 12\sqrt{3x} + 9$
11. $x^{10} + 10x^8y + 40x^6y^2 + 80x^4y^3 + 80x^2y^4 + 32y^5$
12. $4845a^{16}b^{16}$
13. $-6,300,000x^4$
14. $70x^{12}$
15. $-25x^{23}$
16. $13440x^4y^6$
17. $495a^8b^8$
18. $5670x^4$
19. 2734.375
20. -84
21. $3360x^4$

Domain of a Function

Definition of a Function:

A function f from a set A to a set B is a relation that assigns to element x in the set A exactly one element y in the set B . The set A is the domain (or set of x -values) of the function f , and the set B contains the range (or set of y -values.) No x value may be repeated in a function. We use the vertical line test to determine if a graph is a function.

Finding the domain of a function:

1. The **implied domain** is the set of all real numbers for which the expression is defined. The question one must ask when finding the domain is “Where is this function NOT defined?” For all polynomial functions the domain is all real numbers or expressed in interval notation: $(-\infty, \infty)$

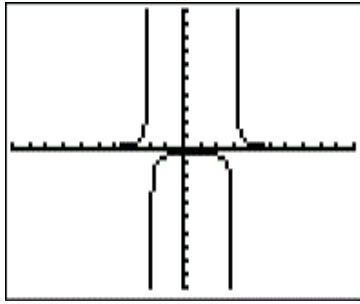
2. Rational Functions: This is a function that is comprised of a ratio of 2 polynomial functions. Thus, there is a denominator involved. We must always remember that **DIVISION BY ZERO IS UNDEFINED!!** To determine the domain of a rational function, set the denominator equal to zero and solve. These are the values that are NOT acceptable.

3. Radical Functions: This is a function that is underneath some type of radical. Remember when taking the **EVEN** root of a negative number, the answer is imaginary. Imaginary numbers are NOT acceptable for real valued functions. To determine the domain of a radical function, set the radicand greater than or equal to zero and solve. These are the values that **ARE** acceptable.

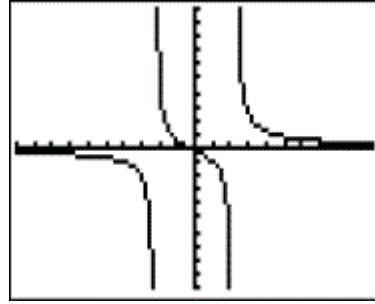
$$F(x) = \sqrt{f(x)}, \text{ where } f(x) \text{ is the radicand} \qquad f(x) \geq 0$$

(You will need your own paper to complete the following examples)

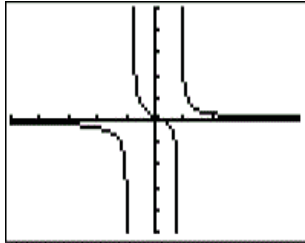
$$1. f(x) = \frac{2}{x^2 - x - 6}$$



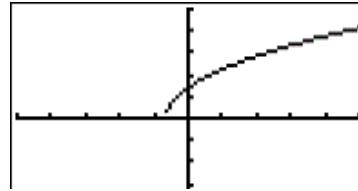
$$2. f(x) = \frac{3x+1}{x^2-5}$$



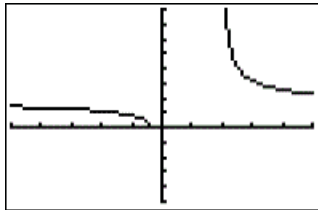
$$3. f(x) = \frac{4x-1}{8x^2-6}$$



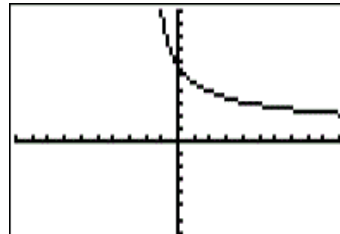
$$4. f(x) = \sqrt{3x+2}$$



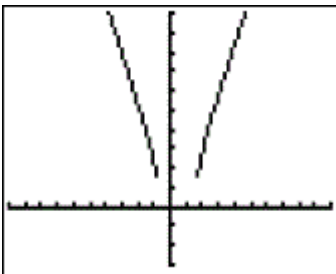
$$5. f(x) = \sqrt{\frac{3x+1}{x-2}}$$



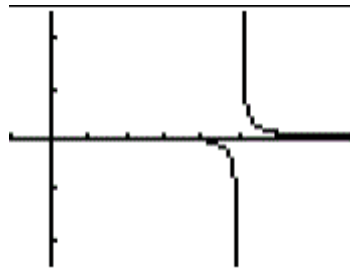
$$6. f(x) = \frac{10}{\sqrt{2x+3}}$$



$$7. f(x) = \sqrt{6x^2 - 5x - 6}$$



$$8. f(x) = \frac{\sqrt{x-4}}{x^2-25}$$



$$9. f(x) = \sqrt{\frac{x^2-5}{2x^2-7}}$$



The Difference Quotient

The difference quotient is defined by: $\frac{f(x+h) - f(x)}{h}, h \neq 0$

Find the difference quotient of the following functions:

1. $f(x) = -5x + 1$

2. $f(x) = 2x^2 - 3x + 5$

3. $f(x) = \frac{1}{x-5}$

Domain and Difference Quotient

Find the domain of the following (write answers in interval notation):

1. $f(x) = \frac{2x}{x^2 + 5x + 6}$

2. $f(x) = \frac{3x + 7}{x^2 - 6x - 27}$

3. $f(x) = \frac{x + 8}{2x^2 - 1}$

4. $f(x) = \frac{4x}{x^2 - 24}$

5. $f(x) = \sqrt{x + 5}$

6. $f(x) = \sqrt{3x - 7}$

7. $f(x) = \sqrt{\frac{12x - 24}{3x + 5}}$

8. $f(x) = \sqrt{\frac{-9x + 27}{5x - 2}}$

9. $f(x) = \frac{1}{\sqrt{3x + 1}}$

10. $f(x) = \frac{12x}{\sqrt{x - 5}}$

11. $f(x) = \frac{5x + 7}{\sqrt{x^2 - 9}}$

12. $f(x) = \sqrt{12x^2 + 11x - 5}$

13. $f(x) = \sqrt{x^2 + 5x + 6}$

14. $f(x) = \sqrt{x^2 - 3x - 4}$

15. $f(x) = \sqrt{9 - x^2}$

16. $f(x) = \sqrt{x^2 + 4}$

17. $f(x) = \sqrt{15x^2 - 4x - 3}$

18. $f(x) = \frac{\sqrt{x - 5}}{x^2 - 121}$

19. $f(x) = \frac{\sqrt{x + 6}}{x^2 - 100}$

20. $f(x) = \sqrt{\frac{x^2 - 36}{x - 2}}$

21. $f(x) = \sqrt{\frac{x^2 - 9}{2x^2 - 3}}$

22. $f(x) = \sqrt{\frac{3x^2 - 8}{x^2 - 12}}$

Find the difference quotient of the following functions:

1. $f(x) = 4x - 3$

2. $f(x) = -x^2 - 3x + 1$

3. $f(x) = 2x^2 + 5x - 4$

4. $f(x) = x^3 - 2x + 3$

5. $f(x) = \frac{1}{x + 3}$

Answers—Domain and Difference Quotient

- $(-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$ or $x \neq -3, -2$
- $(-\infty, -3) \cup (-3, 9) \cup (9, \infty)$ or $x \neq -3, 9$
- $\left(-\infty, -\sqrt{\frac{1}{2}}\right) \cup \left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right) \cup \left(\sqrt{\frac{1}{2}}, \infty\right)$ or $x \neq \pm\sqrt{\frac{1}{2}}$
- $(-\infty, -2\sqrt{6}) \cup (-2\sqrt{6}, 2\sqrt{6}) \cup (2\sqrt{6}, \infty)$ or $x \neq \pm 2\sqrt{6}$
- $[-5, \infty)$
- $\left[\frac{7}{3}, \infty\right)$
- $\left(-\infty, -\frac{5}{3}\right) \cup [2, \infty)$
- $\left(\frac{2}{5}, 3\right]$
- $\left(-\frac{1}{3}, \infty\right)$
- $(5, \infty)$
- $(-\infty, -3) \cup (3, \infty)$
- $\left(-\infty, -\frac{5}{4}\right] \cup \left[\frac{1}{3}, \infty\right)$
- $(-\infty, -3] \cup [-2, \infty)$
- $(-\infty, -1] \cup [4, \infty)$
- $[-3, 3]$
- $(-\infty, \infty)$
- $\left(-\infty, -\frac{1}{3}\right] \cup \left[\frac{3}{5}, \infty\right)$
- $[5, 11) \cup (11, \infty)$
- $[-6, 10) \cup (10, \infty)$
- $[-6, 2) \cup [6, \infty)$
- $(-\infty, -3] \cup \left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right) \cup [3, \infty)$
- $(-\infty, -2\sqrt{3}) \cup \left[-\sqrt{\frac{8}{3}}, \sqrt{\frac{8}{3}}\right] \cup (2\sqrt{3}, \infty)$

Difference Quotient:

- 4
- $-2x - h - 3$
- $4x + 2h + 5$
- $3x^2 + 3xh + h^2 - 2$
- $$\frac{-1}{(x+h+3)(x+3)}$$

Function Composition

$$f(x) = 3x - 1$$

x	$f(x)$
2	
5	
6	

$$g(x) = x^2 + 2$$

x	$g(x)$
2	
5	
6	

Find 1. $(f \circ g)(2)$

2. $(g \circ f)(2)$

3. $(f \circ f)(2)$

4. $(g \circ g)(2)$

5. $(f \circ g)(x)$

6. $(g \circ f)(x)$

1. $(f \circ g)(-3)$

2. $(f \circ g)(3)$

3. $(g \circ f)(-5)$

4. $(g \circ f)(1)$

5. $(f \circ g)(7)$

6. $(g \circ f)(3)$

7. $(f \circ g)(0)$

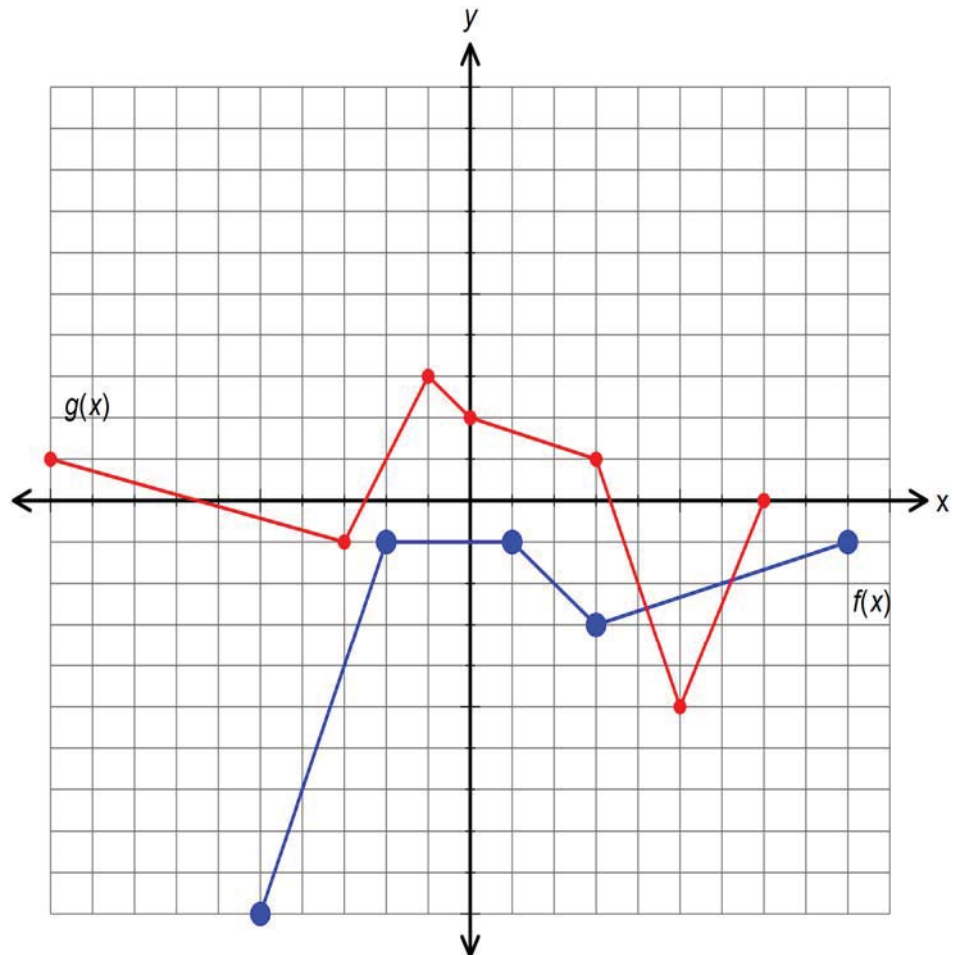
8. $(f \circ g)(-10)$

9. $(g \circ f)(3)$

10. $(g \circ f)(9)$

11. $(f \circ g)(-1)$

12. $(f \circ g)(5)$



Find the domain of a composition function:

Finding the domain of a composite function consists of two steps:

Step 1. Find the domain of the "inside" (input) function. If there are any restrictions on the domain, keep them.

Step 2. Construct the composite function (Simplify the expression). Find the domain of this new function. If there are restrictions on this domain, add them to the restrictions from Step 1. If there is an overlap, use the more restrictive domain (or the intersection of the domains).

Find $(f \circ g)(x)$ and its domain.

1. Given $f(x) = \frac{x}{x+3}$ and $g(x) = \frac{2}{x}$

2. Given $f(x) = x^2$ and $g(x) = \sqrt{x+2}$

3. Given $f(x) = \sqrt{x^2-1}$ and $g(x) = \sqrt{x-2}$

4. Given $f(x) = \frac{2x-1}{x-2}$ and $g(x) = \frac{x+4}{2x-5}$

5. Given $f(x) = \frac{1}{\sqrt{x-2}}$ and $g(x) = \frac{1}{x+2}$

Composite Functions

Use the table of values to evaluate each expression

x	$f(x)$	$g(x)$
0	7	9
1	6	5
2	5	6
3	8	2
4	4	1
5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

1. $(f \circ g)(8)$
2. $(f \circ g)(5)$
3. $(g \circ f)(5)$
4. $(g \circ f)(3)$
5. $(f \circ f)(4)$
6. $(f \circ f)(1)$
7. $(g \circ g)(2)$
8. $(g \circ g)(6)$

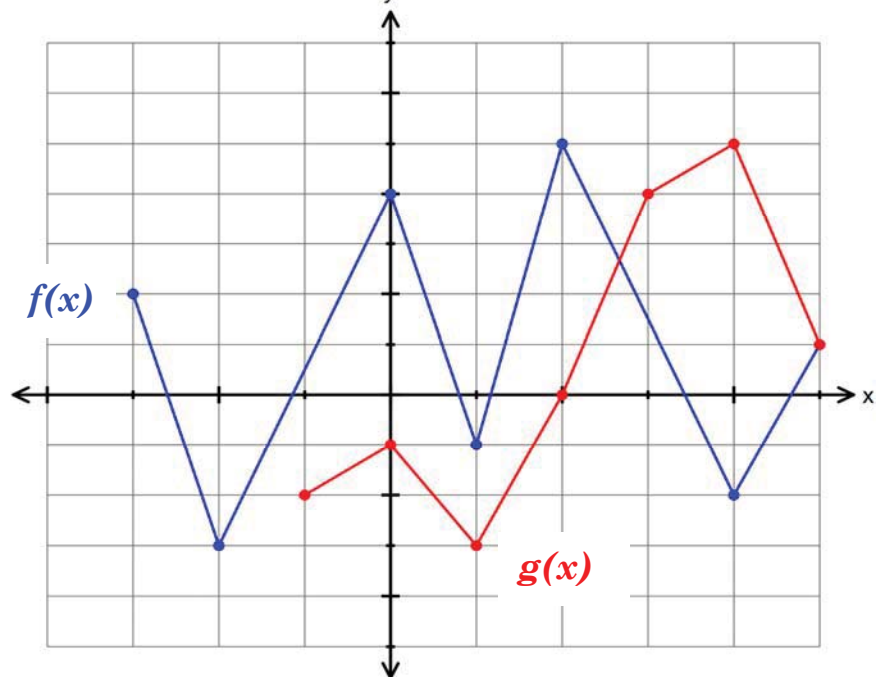
Given each pair of functions, calculate

(a) $(f \circ g)(4)$ (b) $(g \circ f)(2)$ (c) $(f \circ f)(1)$ (d) $(g \circ g)(0)$

9. $f(x) = \sqrt{x+4}$, $g(x) = 12 - x^2$ 10. $f(x) = \frac{1}{x+2}$, $g(x) = 4x + 3$

Use the graphs to evaluate the expressions below.

11. $(f \circ g)(3)$
12. $(f \circ g)(1)$
13. $(g \circ f)(1)$
14. $(g \circ f)(0)$
15. $(f \circ f)(5)$
16. $(f \circ f)(4)$
17. $(g \circ g)(0)$
18. $(g \circ g)(2)$



19. Given $f(x) = \frac{x}{x-1}$ and $g(x) = \frac{-4}{x}$, find $(f \circ g)(x)$ and the domain

Given $f(x) = \frac{1}{x+1}$ $g(x) = \frac{1}{x^2-3}$ $h(x) = \frac{1}{\sqrt{x+3}}$

20. Find $(f \circ g)(x)$ and the domain.

21. Find $(g \circ h)(x)$ and the domain.

22. Find $(f \circ f)(x)$ and the domain.

23. Find $(h \circ f)(x)$ and the domain.

24. Find $(h \circ g)(x)$ and the domain.

Answers—Composite Functions

1. 4

2. 9

3. 9

4. 4

5. 4

6. 2

7. 7

8. 3

9. a) 0

b) 6

c) $\sqrt{\sqrt{5}+4}$

d) -132

10. a) $\frac{1}{21}$

b) 4

c) $\frac{3}{7}$

d) 15

11. -2

12. 2

13. -2

14. 5

15. -1

16. -3

17. -2

18. -1

19. $(f \circ g)(x) = \frac{-4}{-4-x}$, Domain: $x \neq 0, -4$

20. $(f \circ g)(x) = \frac{x^2-3}{x^2-2}$, Domain: $x \neq \pm\sqrt{2}, \pm\sqrt{3}$

21. $(g \circ h)(x) = \frac{x+3}{-3x-8}$, Domain: $\left(-3, \frac{-8}{3}\right) \cup \left(\frac{-8}{3}, \infty\right)$

22. $(f \circ f)(x) = \frac{x+1}{x+2}$, Domain: $x \neq -1, -2$

23. $(h \circ f)(x) = \sqrt{\frac{x+1}{3x+4}}$, Domain: $\left(-\infty, \frac{-4}{3}\right) \cup (-1, \infty)$

24. $(h \circ g)(x) = \sqrt{\frac{x^2-3}{3x^2-8}}$, Domain: $(-\infty, -\sqrt{3}) \cup \left(-\sqrt{\frac{8}{3}}, \sqrt{\frac{8}{3}}\right) \cup (\sqrt{3}, \infty)$

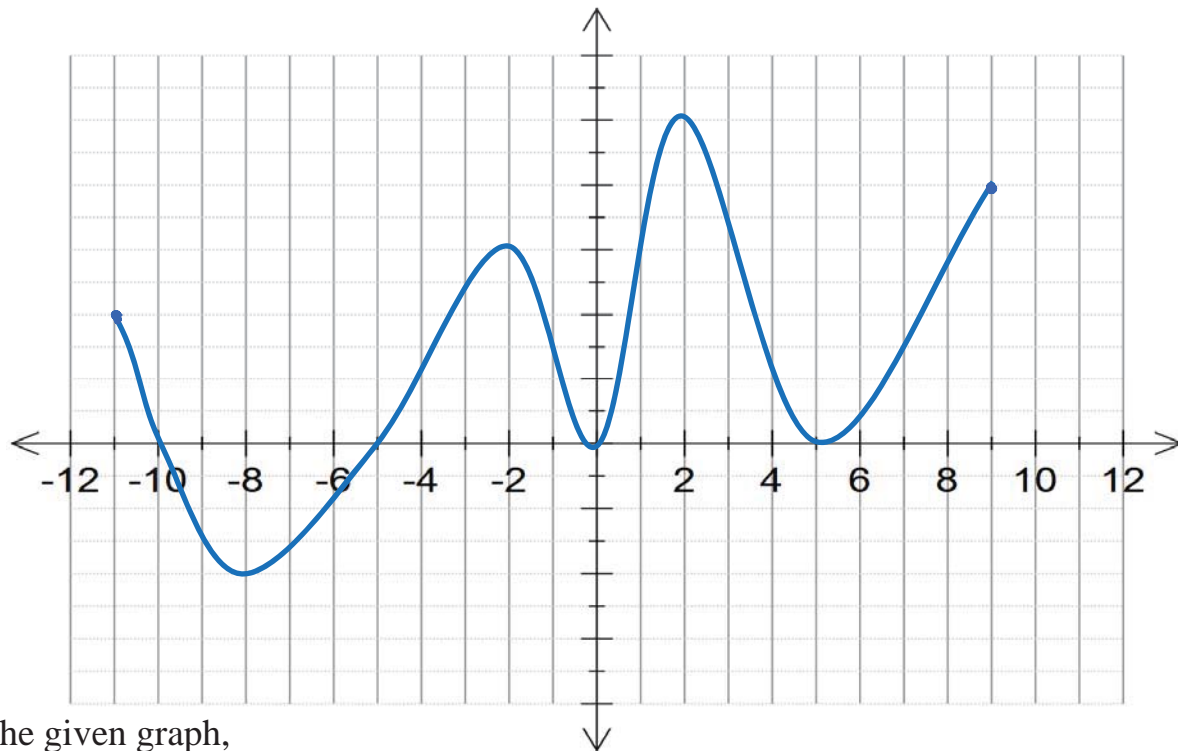
Properties of Functions

Increasing/Decreasing Intervals:

The part of the DOMAIN where y-values are increasing/ decreasing.

Local Maxima: A point where a function changes from increasing to decreasing is called a **local maximum**.

Local Minima: A point where a function changes from decreasing to increasing is called a **local minimum**.

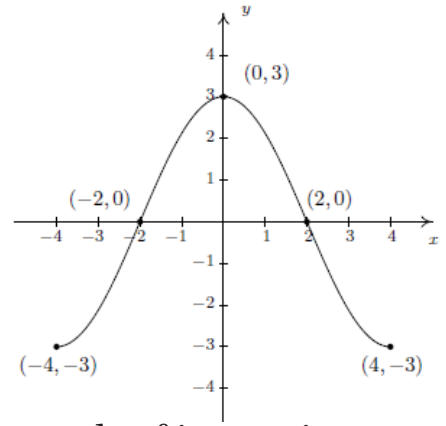
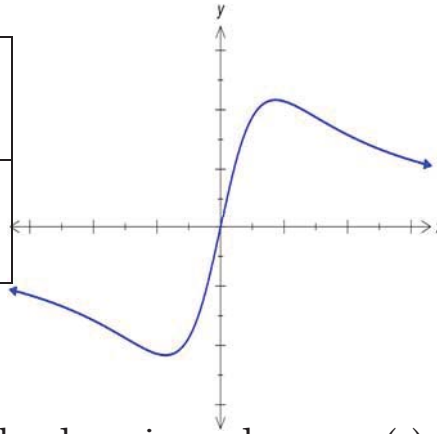


With the given graph,

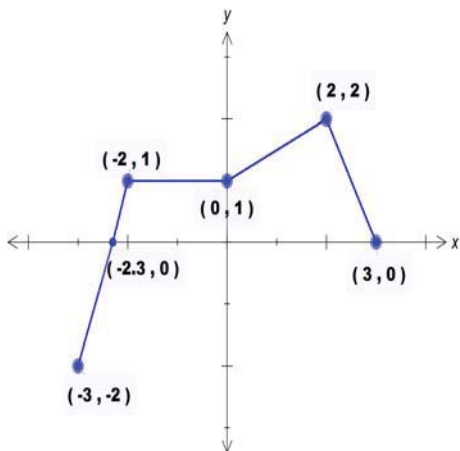
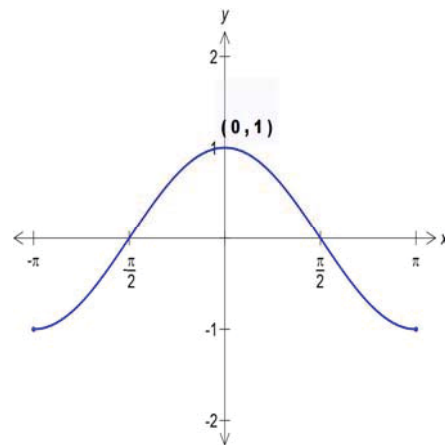
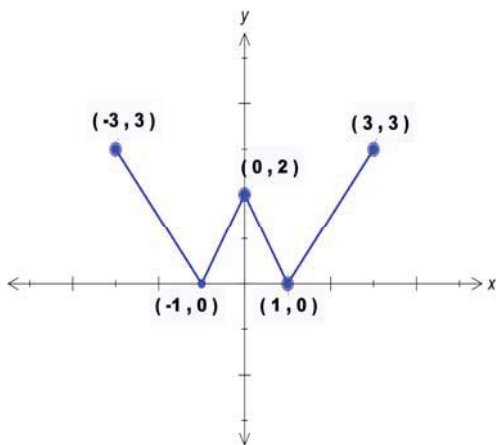
1. Determine the domain
2. Determine the range
3. Find the intervals where the function is increasing and where it is decreasing.
4. Find any local minima and maxima
5. Find all intercepts.
6. Determine $f(-8)$
7. Solve $f(x) = 10$
8. How many solutions are there to $f(x) = 6$?

Even Function is symmetric about the y-axis.

Odd Function is symmetric about the origin.

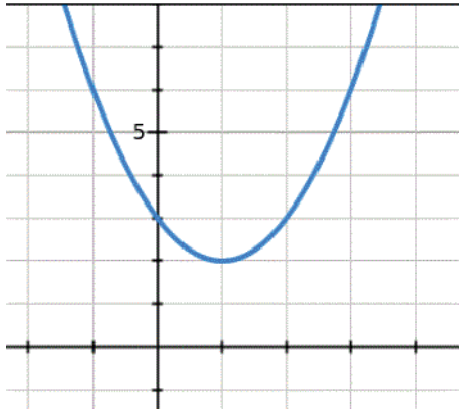


Find the (a) intercepts, (b) the domain and range, (c) intervals of increasing, decreasing, constant, (d) State whether the function is Even, Odd, or Neither



Average Rate of Change

$$f(x) = x^2 - 2x + 3$$



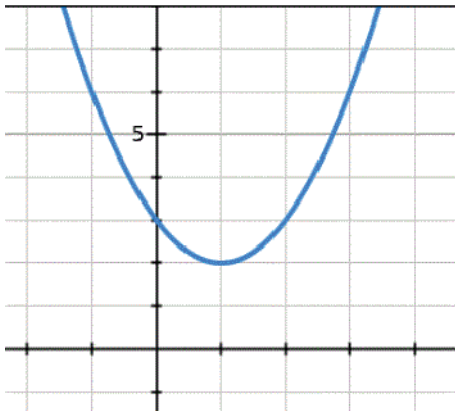
$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Find the average rate of change from

(a) -1 to 1

(b) 2 to 5

Find an equation of the secant line containing $(0, f(0))$ and $(3, f(3))$

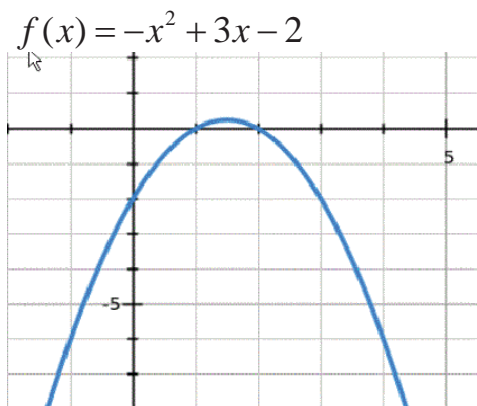


The slope of a secant line in general is the difference quotient:

$$m_{\text{sec}} = \frac{f(x + \Delta x) - f(x)}{\Delta x}, \quad \Delta x \neq 0$$

For the given function , $f(x) = -x^2 + 3x - 2$, find the following:

- The slope of the secant line in terms of x and Δx . Simplify your answer.
- Find m_{sec} for $\Delta x = 0.5, 0.1,$ and 0.01 at $x = 1$. What value does m_{sec} approach as Δx approaches 0?
- Find the equation for the secant line at $x = 1$ with $\Delta x = 0.01$
- On your own, find m_{sec} for $\Delta x = 0.01$ at $x = 3$. Is the slope a reasonable answer according to the graph?

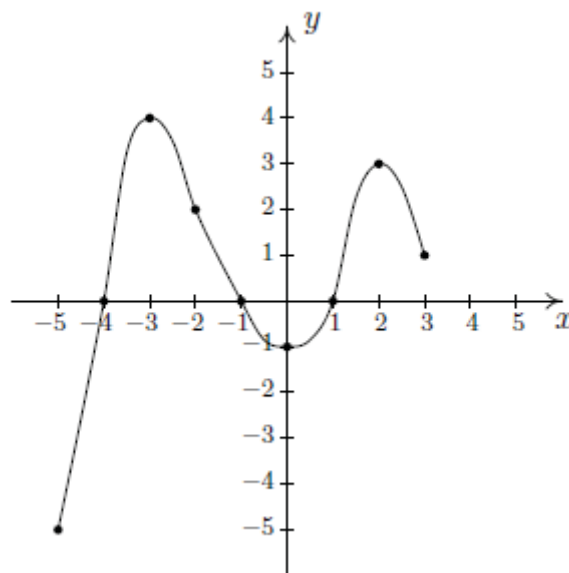


Note: As Δx approaches 0, the graph of the secant line is getting closer to becoming a tangent line. (touches the function at only one point)

Functions

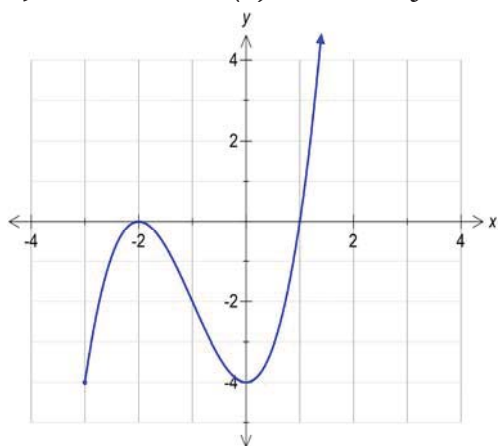
With the given graph,

1. Determine the domain.
2. Determine the range.
3. Find the intervals where the function is increasing and where it is decreasing.
4. Find any local minima and maxima.
5. Find all intercepts.
6. Determine $f(-2)$.
7. Solve $f(x) = 4$.
8. How many solutions are there to $f(x) = 2$?

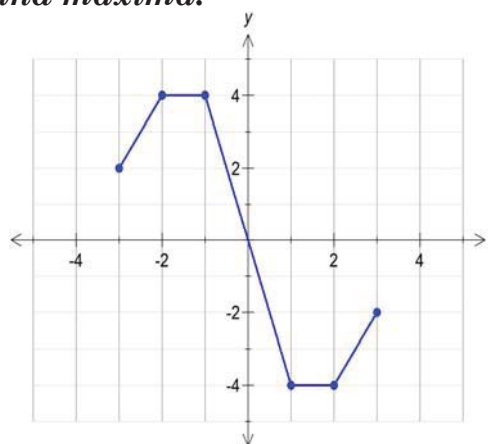


Find the (a) intercepts, (b) the domain and range, (c) intervals of increasing, decreasing, constant, (d) State whether the function is Even, Odd, or Neither. (e) Find any local minima and maxima:

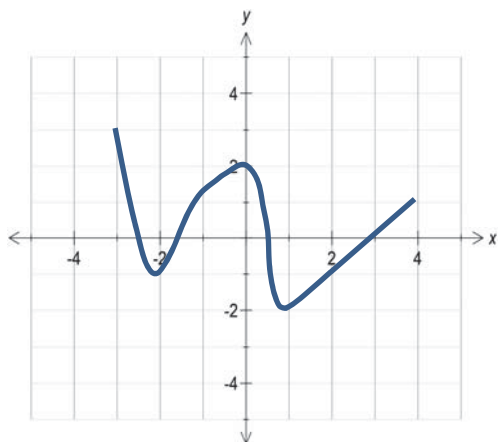
9.



10.



11.



Answers—Functions

1. $[-5, 3]$
2. $[-5, 4]$
3. Inc: $(-5, -3) \cup (0, 2)$ Dec.: $(-3, 0) \cup (2, 3)$
4. Minima: $(-5, -5), (0, -1), (3, 1)$, Maxima: $(-3, 4), (2, 3)$
5. $(-4, 0), (-1, 0), (1, 0), (0, -1)$
6. $f(-2) = 2$
7. $x = -3$
8. 4
9. a) $(-2, 0), (1, 0), (0, -4)$
b) D: $[-3, \infty)$ R: $[-4, \infty)$
c) Inc: $(-3, -2) \cup (0, \infty)$ Dec.: $(-2, 0)$
d) Neither
e) Minima: $(-3, -4), (0, -4)$, Maxima: $(-2, 0)$
10. a) $(0, 0)$
b) D: $[-3, 3]$ R: $[-4, 4]$
c) Inc: $(-3, -2) \cup (2, 3)$ Dec.: $(-1, 1)$ Con: $(-2, -1) \cup (1, 2)$
d) Odd
e) OMIT
11. a) $(-2.5, 0), (-1.8, 0), (.5, 0), (3, 0), (0, 2)$
b) D: $[-3, 4]$ R: $[-2, 3]$
c) Inc: $(-2, 0) \cup (1, 4)$ Dec.: $(-3, -2) \cup (0, 1)$
d) Neither
e) Minima: $(-2, -1), (1, -2)$, Maxima: $(-3, 4), (0, 2), (4, 1)$
12. a) -4 b) -8
13. a) 17 b) -1
14. $y = -4x + 1$
15. $y = -5x$
16. a) $m_{\text{sec}} = 2x + \Delta x + 2$
b) $m_{\text{sec}} = 4.5, m_{\text{sec}} = 4.1, m_{\text{sec}} = 4.01, m_{\text{sec}} = 4$
c) $y = 4.01x - 1.01$
17. a) $m_{\text{sec}} = 4x + 2\Delta x - 3$
b) $m_{\text{sec}} = 2, m_{\text{sec}} = 1.2, m_{\text{sec}} = 1.02, m_{\text{sec}} = 1$
c) $y = 1.02x - 1.02$

Piecewise Functions

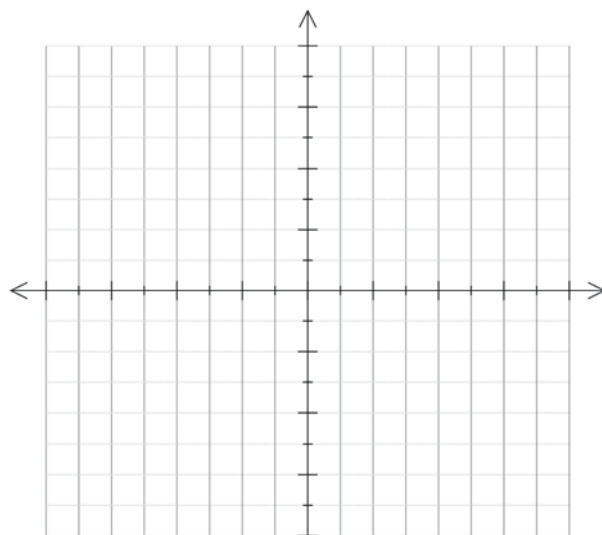
A **piecewise** function is a function in which the formula used depends upon the domain the input lies in. We notate this idea like:

$$f(x) = \begin{cases} \text{formula 1} & \text{if domain to use formula 1} \\ \text{formula 2} & \text{if domain to use formula 2} \\ \text{formula 3} & \text{if domain to use formula 3} \end{cases}$$

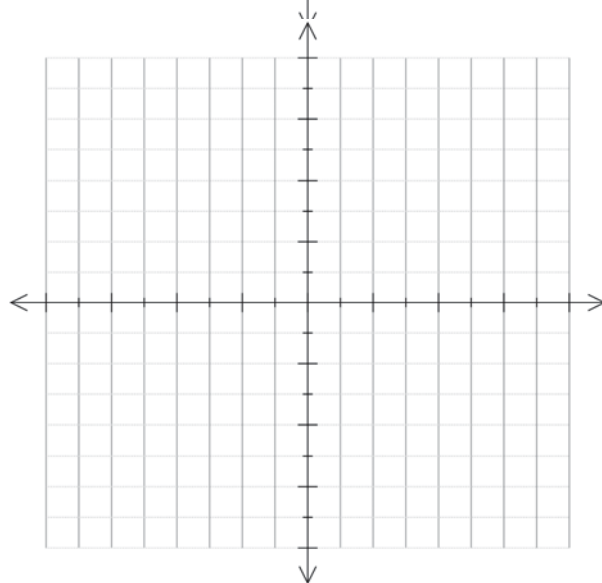
Graphing Piecewise Functions

To _____ a piecewise defined function, choose _____ values for _____, including the _____ of each domain, whether or not that the endpoint is _____ in the domain. Label each endpoint as _____ or not. Sketch the _____ of the function.

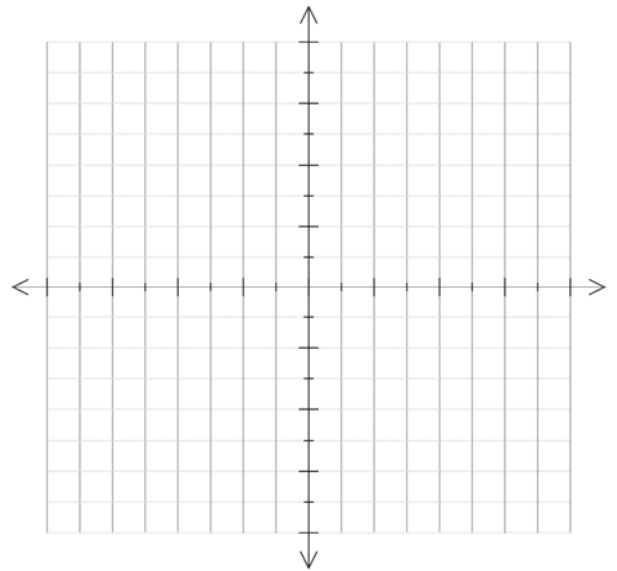
1. $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$



2. $f(x) = \begin{cases} x+3 & \text{if } -3 \leq x < 0 \\ 2 & \text{if } x = 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$



$$3. f(x) = \begin{cases} |x| & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$



Application

4. A museum charges \$5 per person for a guided tour with a group of 1 to 9 people, or a fixed \$50 fee for 10 or more people in the group. Set up a function relating the number of people, n , to the cost, C .

To set up this function, two different formulas would be needed. $C = 5n$ would work for n values under 10, and $C = 50$ would work for values of n ten or greater.

Function:

5. A cell phone company uses the function below to determine the cost, C , in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 25 + 10(g - 2) & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data, and the cost of using 4 gigabytes of data.

Piecewise Functions

Graph the following:

$$1. f(x) = \begin{cases} -2x + 3 & \text{if } x < 1 \\ 3x - 2 & \text{if } x \geq 1 \end{cases}$$

$$2. f(x) = \begin{cases} 3x & \text{if } x \geq 0 \\ -4 & \text{if } x < 0 \end{cases}$$

$$3. f(x) = \begin{cases} 2x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

$$4. f(x) = \begin{cases} 1 + x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

$$5. f(x) = \begin{cases} |x| & \text{if } -2 \leq x < 0 \\ -3 & \text{if } x = 0 \\ x^3 & \text{if } x > 0 \end{cases}$$

$$6. f(x) = \begin{cases} x + 3 & \text{if } -3 \leq x < 0 \\ 2 & \text{if } x = 0 \\ \sqrt{x} & \text{if } x > 0 \end{cases}$$

$$7. f(x) = \begin{cases} \sqrt[3]{x} + 1 & \text{if } x > 0 \\ -x^3 & \text{if } x \leq 0 \end{cases}$$

8. An on-line comic book retailer charges shipping costs according to the following formula $S(n) = \begin{cases} 1.5n + 2.5 & \text{if } 1 \leq n \leq 14 \\ 0 & \text{if } n \geq 15 \end{cases}$ where n is the number of comic books purchased and $S(n)$ is the shipping cost in dollars.

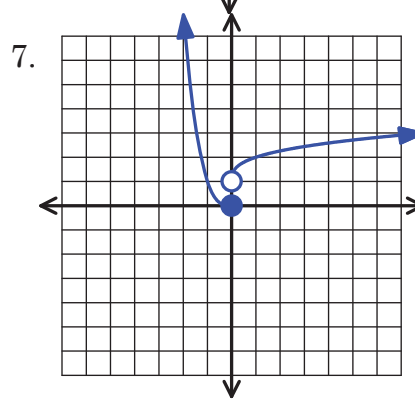
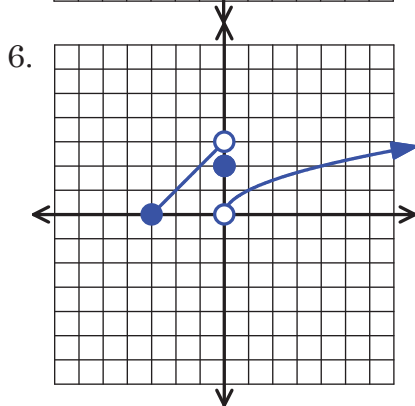
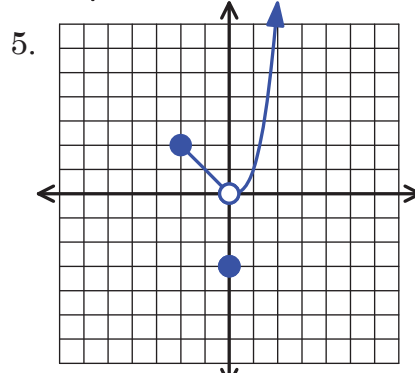
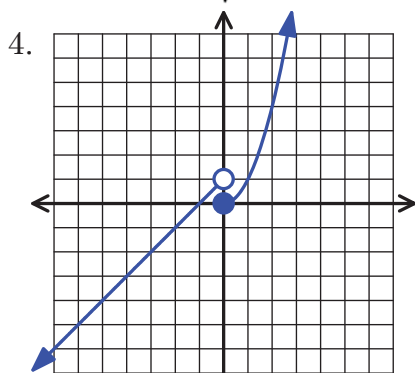
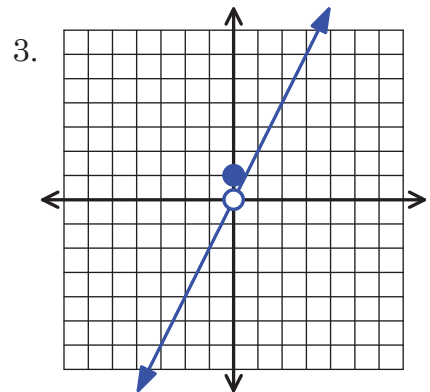
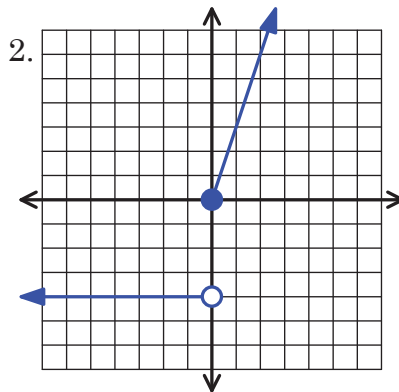
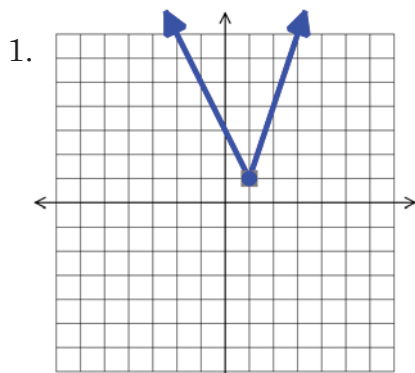
- (a) What is the cost to ship 10 comic books?
(b) What is the cost to ship 15 comic books?

9. The cost C (in dollars) to talk m minutes a month on a mobile phone plan is modeled by $C(m) = \begin{cases} 25 & \text{if } 0 \leq m \leq 1000 \\ 25 + 0.1(m - 1000) & \text{if } m > 1000 \end{cases}$

- (a) How much does it cost to talk 1750 minutes per month with this plan?
(b) How much does it cost to talk 20 hours a month with this plan?

10. At Pierce College during the 2009-2010 school year tuition rates for in-state residents were \$89.50 per credit for the first 10 credits, \$33 per credit for credits 11-18, and for over 18 credits the rate is \$73 per credit. Write a piecewise defined function for the total tuition, T , at Pierce College during 2009-2010 as a function of the number of credits taken, c . Be sure to consider a reasonable domain and range.

Answers—Piecewise Functions



8. (a) \$17.50 (b) \$0

9. (a) \$100 (b) \$45

10.
$$T(c) = \begin{cases} 89.50c & \text{if } c \leq 10 \\ 33c & \text{if } 11 \leq c \leq 18 \\ 73c & \text{if } c > 18 \end{cases}$$

Asymptotes and Holes

How to find the asymptotes and holes of a rational function:

1. If possible, factor the numerator and denominator. Reduce to lowest terms.
2. **Holes:** These will only exist if there is a common factor in the numerator and denominator.
Set the common factor equal to zero and solve for x .
Then substitute this x -value into the reduced function to find the y -value.

$$f(x) = \frac{4x^2 - 1}{2x^2 + 5x - 3}$$

3. **Vertical Asymptotes:** These exist where the reduced function is undefined.
Set the reduced denominator equal to zero and solve for x .
Draw these using dashed/dotted lines on the graph.
4. **Horizontal Asymptotes:** These exist if the degree of the numerator is less than or equal to the degree of the denominator.
If the degree of the numerator is less than the degree of the denominator, the horizontal asymptote will ALWAYS be $y = 0$
If the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote will ALWAYS be $y = \frac{\text{leading coefficient}}{\text{leading coefficient}}$

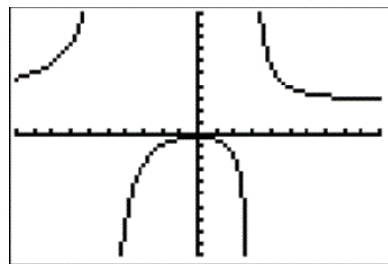
5. **Oblique Asymptotes**: These exist if the degree of the numerator is greater than the degree of the denominator.

$$f(x) = \frac{x^2 - 1}{x + 2}$$

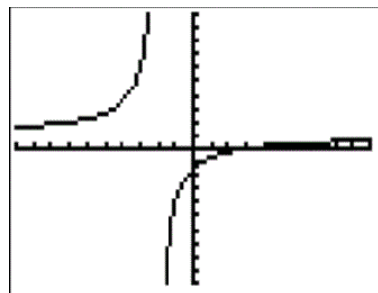
Graphing Rational Functions

1. Find all vertical asymptotes. Draw these as dashed/dotted lines on graph.
2. Find any horizontal/oblique asymptotes. Draw these as dashed/dotted lines on graph.
Determine points, if any, at which the graph intersects this asymptote.
Set the function equal to the asymptote and solve for x .
3. Find any holes. Plot these points as "open" circles.
4. Find all intercepts. Plot these points.
y-intercept: Let $x = 0$ and solve for y .
x-intercept(s): Let $f(x) = 0$ and solve for x .
5. Use graphing calculator and additional points to complete the graph.

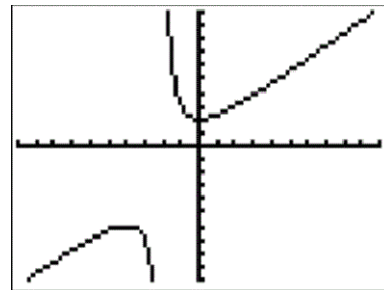
Example 1: $f(x) = \frac{3(x^2 + 1)}{x^2 + 2x - 15}$



Example 2: $f(x) = \frac{x^2 + 2x - 15}{x^2 + 7x + 10}$



Example 3: $f(x) = \frac{x^3 - 8}{x^2 - 4}$



Rational Functions

Find all asymptotes and holes of the following functions:

1. $f(x) = \frac{2x^2 + 5}{2 - x - x^2}$

2. $f(x) = \frac{x - 2}{x^2 - 4}$

3. $f(x) = \frac{3x^2 + 2}{x^2 + 4x - 5}$

4. $f(x) = \frac{2x^2 - 3x - 20}{x - 5}$

5. $f(x) = \frac{2x^2 - 5x - 12}{3x^2 - 11x - 4}$

6. $f(x) = \frac{6x^2 + 7x - 5}{3x + 5}$

7. $f(x) = \frac{8x^2 + 26x - 7}{4x - 1}$

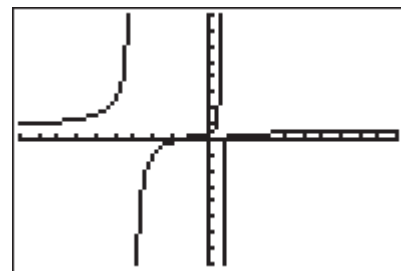
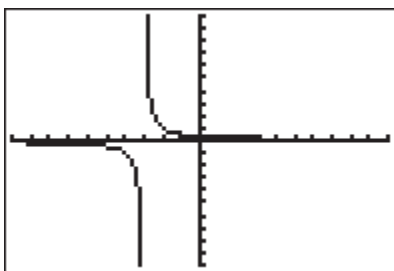
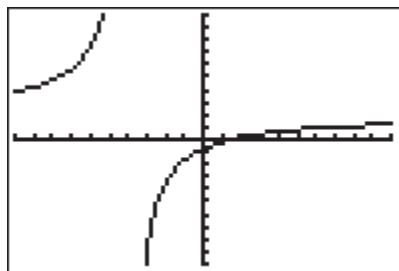
For the following functions: (a) Find all asymptotes. (b) Determine points, if any, at which the graph intersects any asymptote.

(c) Find any holes. (d) Find all intercepts. (e) Graph the function.

8. $f(x) = \frac{2x - 3}{x + 4}$

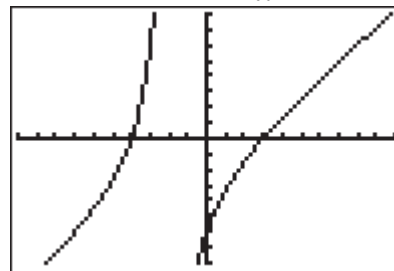
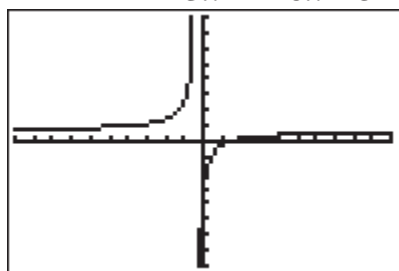
9. $f(x) = \frac{x - 3}{x^2 - 9}$

10. $f(x) = \frac{2x^2 + x - 3}{3x^2 + 10x - 8}$



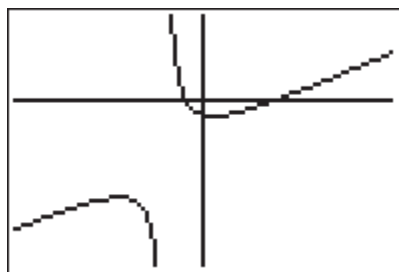
11. $f(x) = \frac{2x^2 + 7x - 15}{3x^2 + 16x + 5}$

12. $f(x) = \frac{x^2 + x - 12}{x + 2}$



13. $f(x) = \frac{x^2 - 3x - 4}{x + 2}$

14. $f(x) = \frac{x^3 - 1}{x^2 - 9}$



Answers— Rational Functions

1. $x = -2, 1$, $y = -2$, no holes
2. $x = -2$, $y = 0$, Hole: $\left(2, \frac{1}{4}\right)$
3. $x = -5, 1$, $y = 3$, no holes
4. $x = 5$, $y = 2x + 7$, no holes
5. $x = -\frac{1}{3}$, $y = \frac{2}{3}$, Hole: $\left(4, \frac{11}{13}\right)$
6. No vertical, $y = 2x - 1$, Hole: $\left(-\frac{5}{3}, -\frac{13}{3}\right)$
7. No vertical, $y = 2x + 7$, Hole: $\left(\frac{1}{4}, \frac{15}{2}\right)$
8. a) $x = -4$, $y = 2$, b) none c) no holes d) $\left(0, -\frac{3}{4}\right), \left(\frac{3}{2}, 0\right)$
9. a) $x = -3$, $y = 0$, b) none c) $\left(3, \frac{1}{6}\right)$ d) $\left(0, \frac{1}{3}\right)$
10. a) $x = \frac{2}{3}, -4$, $y = \frac{2}{3}$, b) $\left(\frac{7}{17}, \frac{2}{3}\right)$ c) no holes d) $\left(0, \frac{3}{8}\right), \left(-\frac{3}{2}, 0\right), (1, 0)$
11. a) $x = -\frac{1}{3}$, $y = \frac{2}{3}$, b) none c) $\left(-5, \frac{13}{14}\right)$ d) $(0, -3), \left(\frac{3}{2}, 0\right)$
12. a) $x = -2$, $y = x - 1$, b) none c) no holes d) $(0, -6), (-4, 0), (3, 0)$
13. a) $x = -2$, $y = x - 5$, b) none c) no holes d) $(0, -2), (4, 0), (-1, 0)$
14. a) $x = -3, 3$, $y = x$, b) $\left(\frac{1}{9}, \frac{1}{9}\right)$ c) no holes d) $\left(0, \frac{1}{9}\right), (1, 0)$

Properties of Logarithms

As you may recall from previous math courses, logarithmic functions are

_____ of exponential functions. Therefore, logs inherit analogs of all of the properties of exponents.

Definition: The **logarithm** of a number is the exponent to which another fixed value, the base, must be raised to produce that number.

$$b^a = c \text{ if and only if } \log_b c = a$$

The **natural logarithm** is the logarithm to the base e , where e is an irrational and transcendental constant approximately equal to 2.718281828

$$e^a = c \text{ if and only if } \ln c = a$$

Properties: Let $M, N, b > 0$, and $b \neq 1$.

- Inverse Property says that $\log_b b^x = x$ for all x and $b^{\log_b x} = x$ for all $x > 0$
- Product Rule states $\log_b M + \log_b N = \log_b MN$
- Quotient Rule states $\log_b M - \log_b N = \log_b \frac{M}{N}$
- Power Rule states $p \log_b M = \log_b M^p$
- Change of base rule $\log_a x = \frac{\log_b x}{\log_b a}$

Evaluate the following expressions:

1. $\log_2 2^{-13}$

2. $\ln e^{\sqrt{2}}$

3. $e^{\ln 8}$

4. $\log_6 9 + \log_6 4$

5. $5^{\log_5 6 + \log_5 7}$

6. $\log_2 6 \cdot \log_6 8$

Use properties of logs to expand the expression and simplify:

$$7. \log_5 \left(\frac{\sqrt[3]{x^2 + 1}}{x^2 - 1} \right)$$

$$8. \ln \left(\frac{5x^2 \sqrt{1-x}}{4(x-1)^2} \right)$$

$$9. \ln \left(\frac{2x+1}{3x-5} \right)^3$$

Use the properties of logarithms to write the expression as a single logarithm.

$$10. 3\ln(3x+1) - 2\ln(2x-1) - \frac{1}{2}\ln x$$

Express y as a function of x :

$$11. \ln y = \ln(x + C)$$

$$12. \ln y = \ln x - \ln(x + 1) + \ln C$$

Logarithmic Equations

Goal: Get equation in the form of $\log_b(y)=x$ and re-write as $b^x=y$

1. Combine logs using log properties.
2. Re-write the equation in exponential form to get rid of the log:
3. Solve for variable.
4. Check your answer back into the original equation.
(You can only take the log of numbers that are greater than zero.)

Solve the following. Write an exact answer and an approximate decimal solution.
Round to 3 decimal places.

1. $\ln(x + 5) = 4$

2. $3 + \ln\sqrt{x - 4} = 7$

3. $\log_6(x + 4) + \log_6(x + 3) = 1$

4. $4\ln(x - 1) = \ln e^3 - 1$

$$5. \ln(x-3) + \ln x - \ln(x+2) = 2$$

Logarithms

Evaluate the following expressions:

1. $\log_3 3^{25}$

2. $\ln e^{-3}$

3. $3^{\log_3 12}$

4. $\log_8 16 - \log_8 2$

5. $4^{\log_4 10 + \log_4 2}$

6. $\log_3 8 \cdot \log_8 27$

7. $e^{\log_{e^2} 16}$

Use properties of logs to expand the expression and simplify:

8. $\ln \left(\frac{\sqrt{2x+1}}{x+5} \right)$

9. $\log_2 \left(\frac{4\sqrt[3]{x^2}}{y\sqrt{z}} \right)$

10. $\log_6 \left(\frac{216}{x^3 y} \right)^4$

11. $\ln \left(\frac{\sqrt{1+2x}}{(x-1)^3} \right)$

Use the properties of logarithms to write the expression as a single logarithm.

12. $2\ln(x+1) - \ln(x+5) - \ln(x-1)$

13. $\ln \left(\frac{2x}{x-3} \right) + \ln \left(\frac{x+3}{x} \right) - \ln(x^2 - 9)$

Express y as a function of x :

14. $\ln y = \ln x + \ln C$

15. $\ln y = -2x + \ln C$

16. $\ln(y-3) = 4x + \ln C$

Solve the following. Write an exact answer and an approximate decimal solution. Round to 3 decimal places.

17. $\ln(x-7) = 2$

24. $\log_7(x+1) + \log_7(x-5) = 1$

18. $\ln(4x-1) = -5$

25. $\ln(x-6) + \ln(x-4) - \ln(x) = 2$

19. $\ln \sqrt{x+3} = 1$

26. $3\ln(x-1) = 5 - \ln e^2$

20. $\ln(x+5) + \ln(x) = 2$

27. $\ln(x+4) = \ln(x) + \ln(4)$

21. $\ln(x-1) + \ln(x+1) = 3$

28. $\ln(5x+1) = \ln(2x+3) + \ln(2)$

22. $\ln(x+2) - \ln(x-5) = 3$

23. $\log_4(3x-1) - \log_4(x+1) = 1$

Answers—Logarithms

1. 25

2. -3

3. 12

4. 1

5. 20

6. 3

7. 4

8. $\frac{1}{2}\ln(2x+1) - \ln(x+5)$

9. $2 + \frac{2}{3}\log_2 x - \log_2 y - \frac{1}{2}\log_2 z$

10. $12 - 12\log_6 x - 4\log_6 y$

11. $\frac{1}{2}\ln(2x+1) - 3\ln(x-1)$

12. $\ln \frac{(x+1)^2}{(x+5)(x-1)}$

13. $\ln \frac{2x(x+3)}{x(x-3)(x^2-9)} = \ln \frac{2x}{x(x-3)^2}$

14. $y = Cx$

15. $y = Ce^{-2x}$

16. $y = Ce^{4x} + 3$

17. $e^2 + 7 \approx 14.389$

18. $\frac{e^{(-5)} + 1}{4} \approx 0.252$

19. $e^2 - 3 \approx 4.389$

20. $\frac{-5 + \sqrt{25 + 4e^2}}{2} \approx 1.193$

21. $\sqrt{1 + e^3} \approx 4.592$

22. $\frac{-2 - 5e^3}{1 - e^3} \approx 5.367$

23. no solution

24. 6

25. $\frac{(10+e^2) + \sqrt{(10+e^2)^2 - 96}}{2} \approx 15.877$

26. $e+1 \approx 3.718$

27. $\frac{4}{3} \approx 1.333$

28. 5

Exponential Equations

An exponential equation has the variable in the exponent. We make use of the following logarithmic property:

$$p \log_a M = \log_a M^p \text{ or } p \ln M = \ln M^p$$

Process:

1. Isolate the base.
2. Take the log or ln of both sides of the equation.
3. Use the log property (above) to re-write the exponents as coefficients.
4. Solve.
5. Use a calculator to approximate the solution.

Solve the following:

1. $7^x = 60$

2. $40 = 8^{2x-1}$

3. $e^{2x+1} - 7 = 13$

4. $3 = 8(6)^{3-x}$

5. $4^{1-x^2} = 3^{x^2}$

6. $3^{x+1} = 7^{2x-5}$

Exponential Models

In Biology, **The Law of Uninhibited Growth** states as its premise that the instantaneous rate at which a population increases at any time is directly proportional to the population at that time. In other words, an amount Q varies with time t according to the function:

$$Q(t) = Q_0 e^{kt} \quad (0 \leq t < \infty)$$

where Q_0 is the amount of substance that is initially present ($t = 0$) and k is the constant of proportionality and is called the *growth constant*. If k is positive, then we have growth of quantity and if k is negative then we have a decline in quantity.

1. The growth rate of the bacterium *Escherichia coli*, a common bacterium found in the human intestine, is proportional to its size. Under ideal laboratory conditions, when this bacterium is grown in a nutrient broth medium, the number of cells in a culture doubles approximately every 20 minutes.
 - a. If the initial cell population is 100, determine the function $Q(t)$ that expresses the exponential growth of the number of cells of this bacterium as a function of time.
 - b. How long will it take for a colony of 100 cells to increase to a population of 1 million?

2. The population of Brazil at the beginning of 1990 was 150 million. Assume that the population continues to grow at the rate of approximately 1.2%/year.
- Find the function $Q(t)$ that expresses the Brazil's population (in millions) as a function of time t (in years), with $t = 0$ corresponding to the beginning of 1990
 - If Brazil's population continues to grow at the rate of approximately 1.2%/year, find the length of time t required for the population to double in size. (Round your answers to two decimal places.)
 - Using the time t found in part (b), what would be Brazil's population if the growth rate were reduced to 0.8%/year? (Round your answers to two decimal places.)

3. If the temperature is constant, then the atmospheric pressure P (in millibars) varies with altitude above sea level h in accordance with the law:

$$P(h) = P_0 e^{kh}$$

Where P_0 the atmospheric pressure is at sea level and k is a constant. If the atmospheric pressure is 1013 millibars at sea level and the pressure at an altitude of 20 km is 90 millibars, find the atmospheric pressure at an altitude of 50 km. (round answer to 3 decimal places.)

4. The half-life of the plutonium isotope is 24,360 years.
If 10 grams of plutonium is released into the atmosphere by a nuclear accident, how much of the plutonium will still be present after 55,000 years?
5. A piece of wood is found to contain 30% of the C-14 that it originally had. When did the tree die from which the wood came? Use 5730 years as the half-life of carbon-14 (C-14). Round answer to nearest whole number.

Exponential Equations

Solve the following. Write an exact answer and an approximate decimal solution. Round to 3 decimal places.

1. $4^x = 21$

2. $3^{x+1} = 16$

3. $9^{x-2} - 17 = 6$

4. $3e^{5x} = 25$

5. $4e^{2x-3} = 120$

6. $3(2 + 9^{2x}) = 11$

7. $e^{3-4x} = 23$

8. $3^{x+1} = 7^{2x-5}$

9. $3e^{4x+1} + 1 = 19$

10. $2^{8x+2} = 3^{5x-2}$

11. $5^{3x+1} = 7^{5x+4}$

Solve the following. Write an exact answer and an approximate decimal solution. Round to 3 decimal places.

12. Use of a new social networking website has been growing exponentially, with the number of new members doubling every 5 months.
- If the site currently has 120,000 users, determine the function $Q(t)$ that expresses the exponential growth of the number of users of this site as a function of time.
 - How many users will the site have in 12 months?
 - How long will it take for the site to have 2 million users?
13. The population of Houston at the beginning of 2011 was 2.1 million. Assume that the population continues to grow at the rate of approximately 1.6%/year.
- Find the function $Q(t)$ that expresses the Houston's population (in millions) as a function of time t (in years), with $t = 0$ corresponding to the beginning of 2011
 - If Houston's population continues to grow at the rate of approximately 1.6%/year, find the length of time t required for the population to double in size. (Round your answers to two decimal places.)
14. The half-life of Radium-226 is 1590 years. If a sample initially contains 200 mg, how many milligrams will remain after 1000 years?
15. A piece of wood is found to contain 40% of the C-14 that it originally had. When did the tree die from which the wood came? Use 5730 years as the half-life of carbon-14 (C-14). Round answer to nearest whole number.

Answers—Exponential Equations

1. $x = \frac{\ln 21}{\ln 4} \approx 2.196$

2. $x = \frac{\ln 16}{\ln 3} - 1 \approx 1.524$

3. $x = \frac{\ln 23}{\ln 9} + 2 \approx 3.427$

4. $x = \frac{1}{5} \ln\left(\frac{25}{3}\right) \approx 0.424$

5. $x = \frac{3 + \ln 30}{2} \approx 3.201$

6. $x = \frac{\ln\left(\frac{5}{3}\right)}{2 \ln 9} \approx 0.116$

7. $x = \frac{3 - \ln 23}{4} \approx -0.034$

8. $x = \frac{-\ln 3 - 5 \ln 7}{\ln 3 - 2 \ln 7} \approx 3.877$

9. $x = \frac{-1 + \ln 6}{4} \approx 0.198$

10. $x = \frac{-2 \ln 3 - 2 \ln 2}{8 \ln 2 - 5 \ln 3} \approx -68.760$

11. $x = \frac{4 \ln 7 - \ln 5}{3 \ln 5 - 5 \ln 7} \approx -1.260$

12. (a) $Q(t) = 120000e^{.139t}$

(b) about 636,187

(c) about 20 months

13. (a) $Q(t) = 2.1e^{.016t}$

(b) about 43 years

14. 129.33mg

15. 7575 years ago

Factoring Review

Factor Completely:

1. $4(2x+1)^{3/2}(3x-1)^{1/3} + 3(2x+1)^{1/2}(3x-1)^{4/3}$

2. $2(4x-1)^{1/2}(3x+5)^{-1/3} + 2(4x-1)^{-1/2}(3x+5)^{2/3}$

Solve:

3. $\frac{6\sqrt{x+1} - (3x+2)(x+1)^{-1/2}}{4(x+1)} = 0$

$$4. \frac{(x+2)^{3/4}(x+3)^{-2/3} + (x+3)^{1/3}(x+2)^{-1/4}}{\left[(x+2)^{3/4}\right]^2} = 0$$

$$5. \frac{2(3x-1)^{1/3} - (2x+1)\left(\frac{1}{3}\right)(3x-1)^{-2/3}(3)}{(3x-1)^{2/3}} = 0$$

Factoring Review

Solve the following:

$$1. \frac{1}{2}x(x+3)^{-1/2} + (x+3)^{1/2} = 0$$

$$2. \frac{1}{2}x(4-x^2)^{-1/2}(-2x) + (4-x^2)^{1/2} = 0$$

$$3. \frac{\frac{1}{2}(x^2-1)(x^2+3)^{-1/2}(2x) - (x^2+3)^{1/2}(2x)}{(x^2-1)^2} = 0$$

$$4. \frac{(x^2-4)^2(-10) - (-10x)(2)(x^2-4)(2x)}{(x^2-4)^4} = 0$$

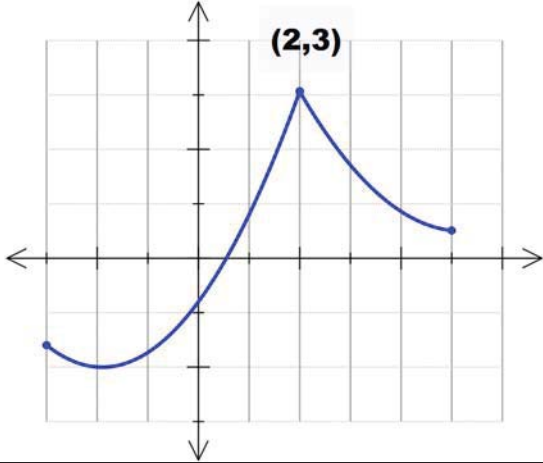
$$5. (x-2)^{1/3}\left(\frac{2}{3}\right)(x+1)^{-1/3} + (x+1)^{2/3}\left(\frac{1}{3}\right)(x-2)^{-2/3} = 0$$

Limits from a Graph

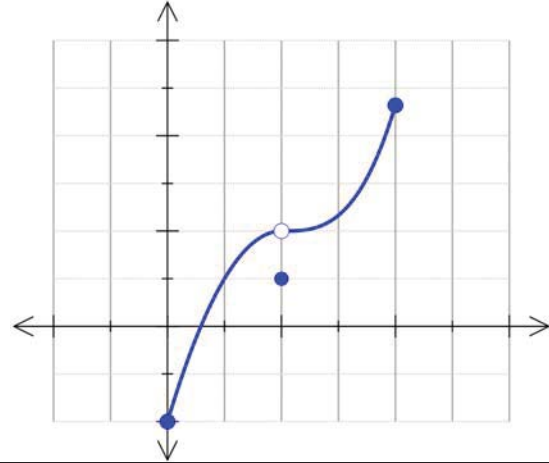
When thinking of a limit, you should ask yourself:

"What is the value of y getting close to as x is getting close to a given value?"

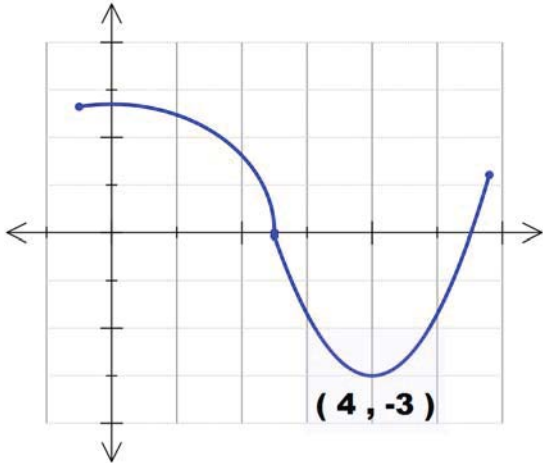
1. $\lim_{x \rightarrow 2} f(x)$



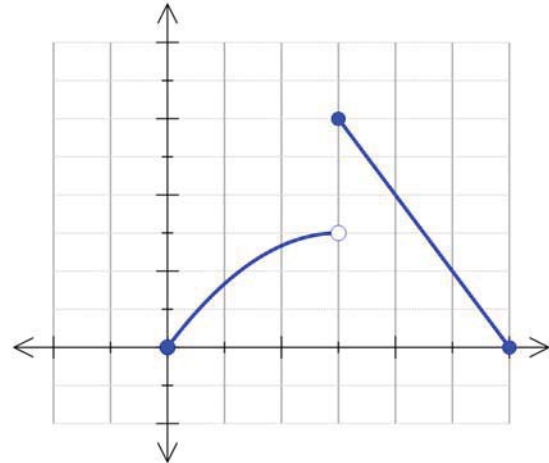
2. $\lim_{x \rightarrow 2} f(x)$



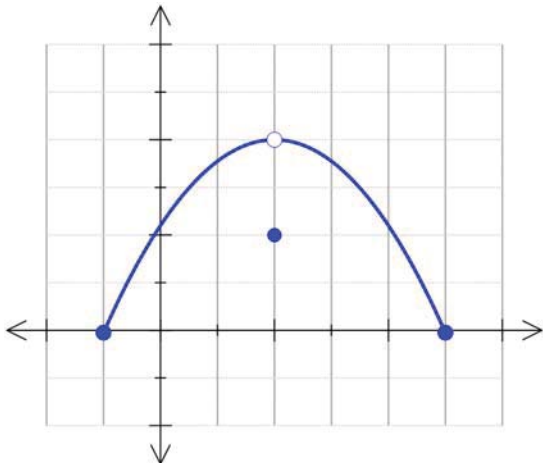
3. $\lim_{x \rightarrow 4} f(x)$



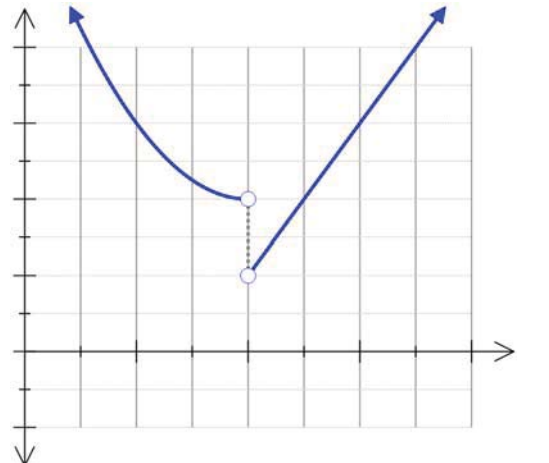
4. $\lim_{x \rightarrow 3} f(x)$



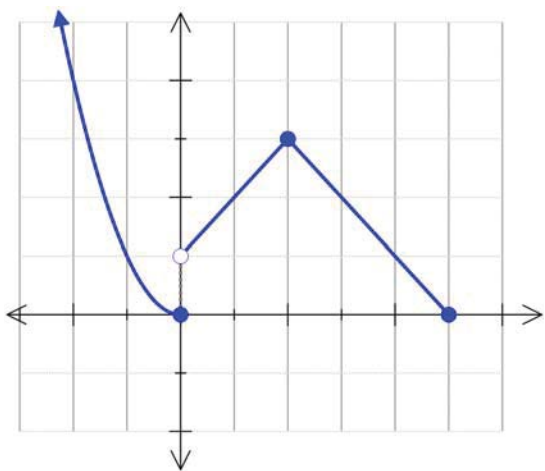
5. $\lim_{x \rightarrow 2} f(x)$



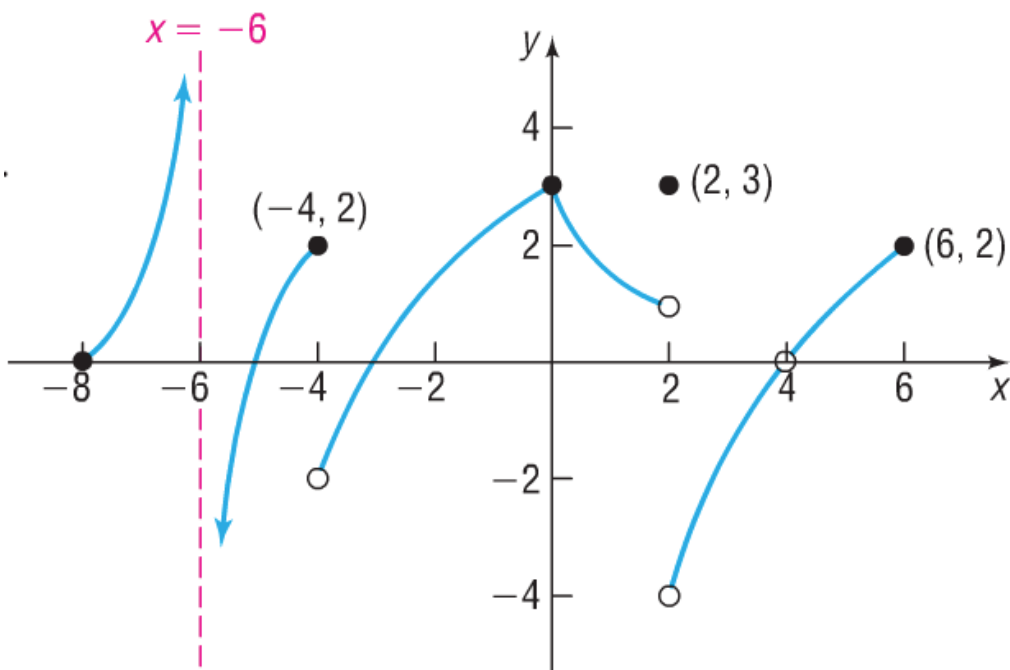
6. $\lim_{x \rightarrow 4} f(x)$



7. a. $\lim_{x \rightarrow 0} f(x)$ b. $\lim_{x \rightarrow 2} f(x)$

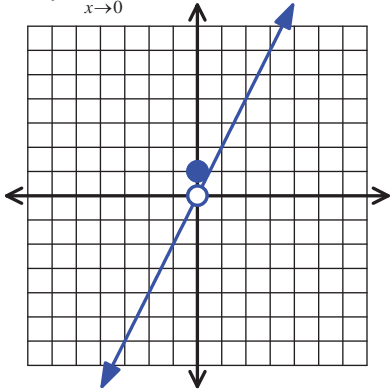


8. a. $\lim_{x \rightarrow 4} f(x)$ b. $\lim_{x \rightarrow 2} f(x)$ c. $\lim_{x \rightarrow 0} f(x)$ d. $\lim_{x \rightarrow -4} f(x)$ e. $\lim_{x \rightarrow -6} f(x)$

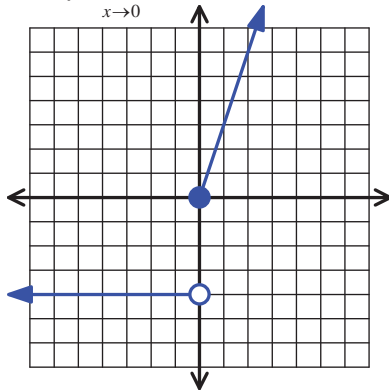


Limits

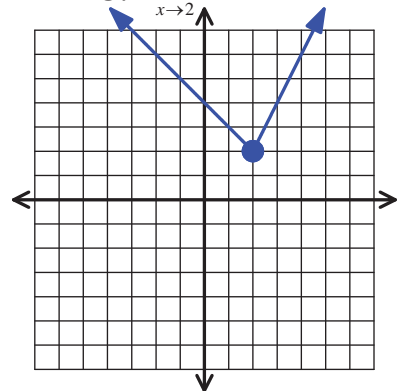
1. $\lim_{x \rightarrow 0}$



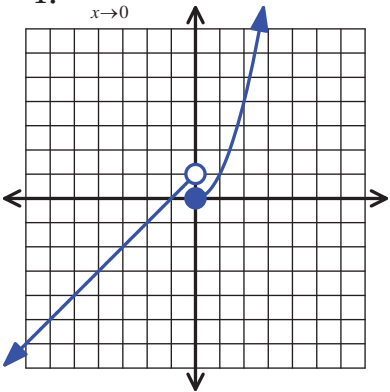
2. $\lim_{x \rightarrow 0}$



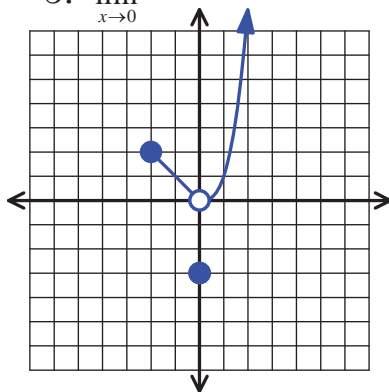
3. $\lim_{x \rightarrow 2}$



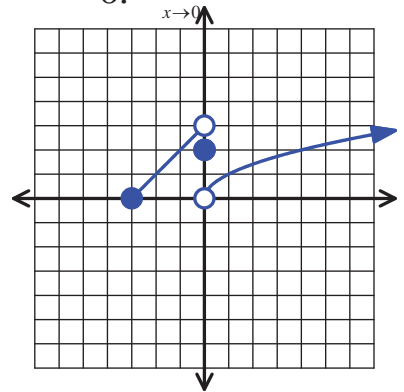
4. $\lim_{x \rightarrow 0}$



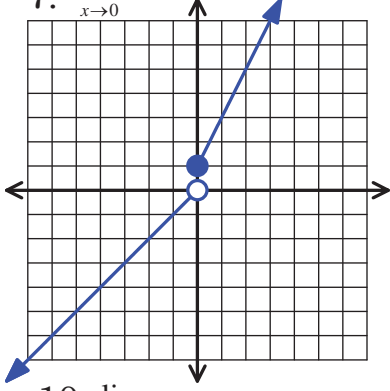
5. $\lim_{x \rightarrow 0}$



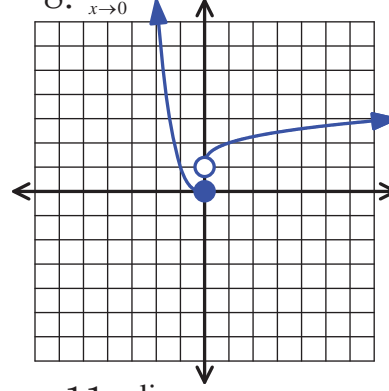
6. $\lim_{x \rightarrow 0}$



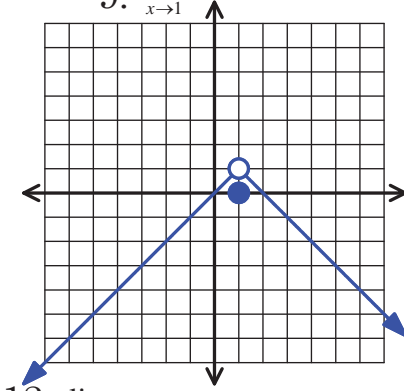
7. $\lim_{x \rightarrow 0}$



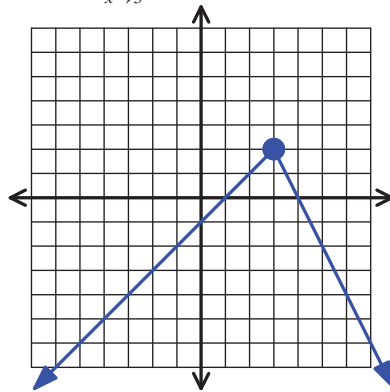
8. $\lim_{x \rightarrow 0}$



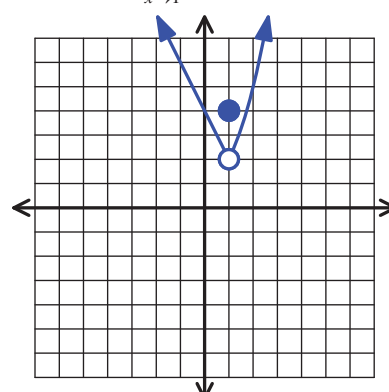
9. $\lim_{x \rightarrow 1}$



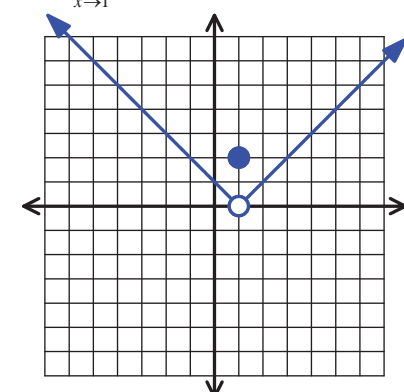
10. $\lim_{x \rightarrow 3}$



11. $\lim_{x \rightarrow 1}$



12. $\lim_{x \rightarrow 1}$



Answers—Factoring and Limits

Factoring:

1. $x = -2$
2. $x = \pm\sqrt{2}$
3. $x = 0, \pm i\sqrt{7}$
4. $x = \pm \frac{2i}{\sqrt{3}}$
5. $x = 1$

Limits:

1. 0
2. DNE
3. 2
4. DNE
5. 0
6. DNE
7. DNE
8. DNE
9. 1
10. 2
11. 2
12. 0

