Prefix-reversal Gray codes

Alexey Medvedev

Central European University, Budapest, Hungary,
Sobolev Institute of Mathematics, Novosibirsk, Russia

joint work with

Elena Konstantinova, Sobolev Institute of Mathematics

Department of Mathematics and its Applications

CEU, 25.02.2015

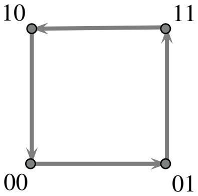
Binary Reflected Gray code

Hamming cube H_n [F. Gray, (1953), U.S. Patent 2,632,058]

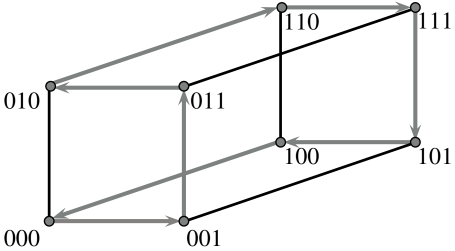
The first Gray code was introduced relative to binary strings

$n = 2$: 00 01 | 11 10

$n = 3$: 000 001 011 010 | 110 111 101 100



H_2



H_3

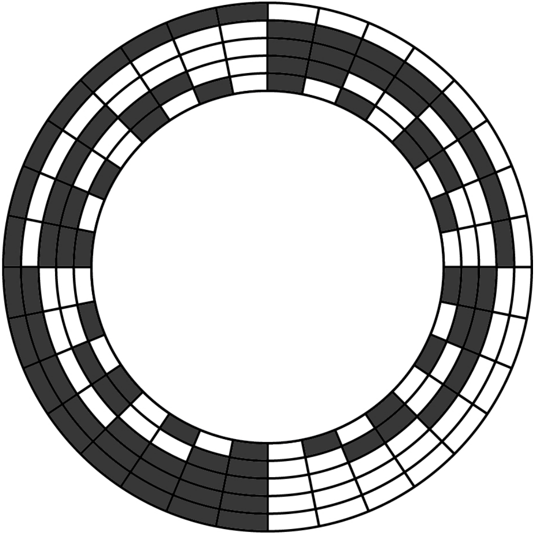
Gray codes are useful

The Gray codes are used in many applications in

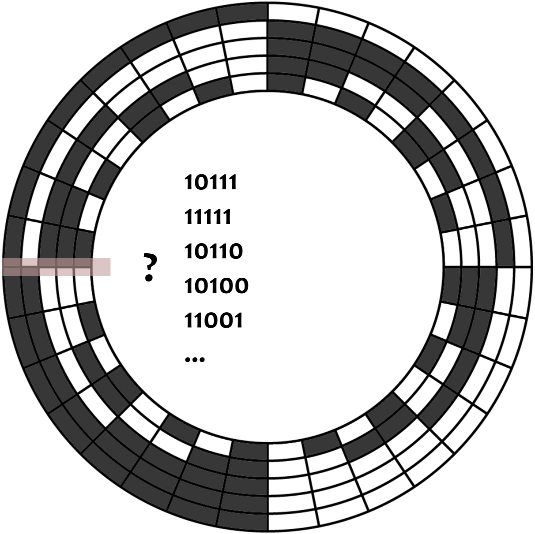
- *mathematics;*
- *computer science;*
- *electrical engineering;*
- *data communications;*
- *etc.*

Example: HDD

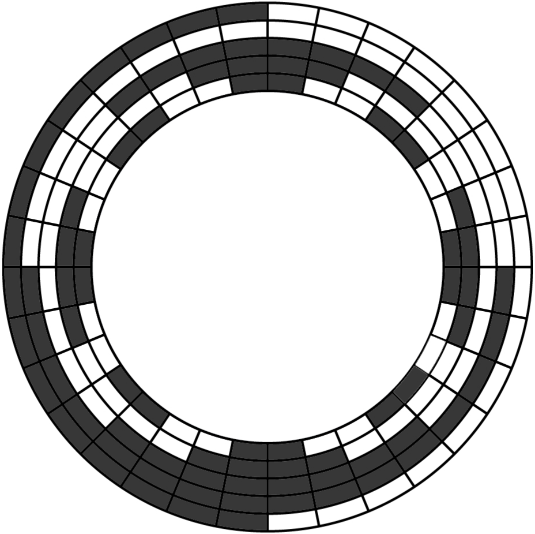
Example: spinning wheel



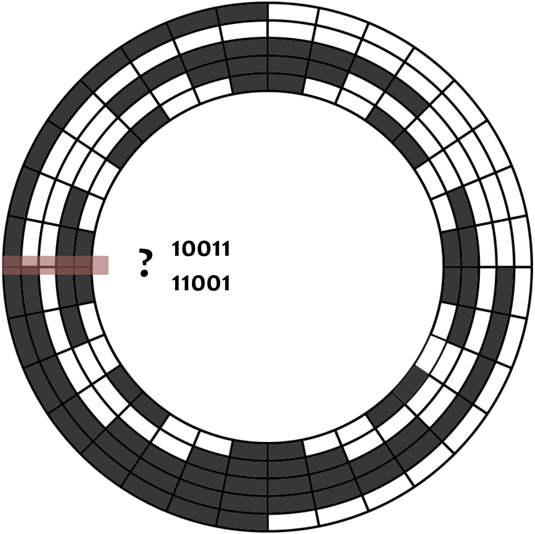
Example: spinning wheel



Example: spinning wheel



Example: spinning wheel



Gray codes

Combinatorial Gray codes [J. Joichi et al., (1980)]

A combinatorial Gray code is now referred as a method of generating combinatorial objects so that successive objects differ in some pre-specified, usually small, way.

[D.E. Knuth, The Art of Computer Programming, Vol.4 (2010)]

Knuth recently surveyed combinatorial generation:

Gray codes are related to

efficient algorithms for exhaustively generating combinatorial objects.

(tuples, permutations, combinations, partitions, trees)

Example: generating permutations

Steinhaus-Johnson-Trotter algorithm, (1964)

List all the $n!$ permutations, such that the successive permutations differ by transposition of two **adjacent** *elements.*

[1234]	[3124]	[2314]
[1243]	[3142]	[2341]
[1423]	[3412]	[2431]
[4123]	[4312]	[4231]
[4132]	[4321]	[4213]
[1432]	[3421]	[2413]
[1342]	[3241]	[2143]
[1324]	[3214]	[2134]

Generating permutations in Sym_4

Example: generating permutations

Steinhaus-Johnson-Trotter algorithm, (1964)

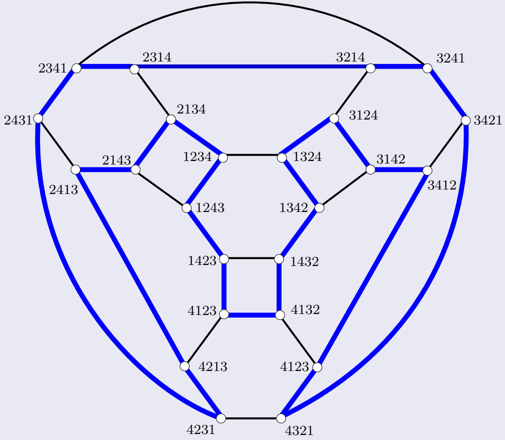


Figure: Hamilton cycle in $Cay(Sym_4, \{(1\,2), (2\,3), (3\,4)\})$

Relation between codes and graphs

Define the graph $\Gamma = (V, E)$, *where* V *– the set of combinatorial objects and* $(u, v) \in E$ *iff* u *and* v *differ in "pre-specified small way". Then*

- *the Hamilton* **path** *in* $\Gamma \sim$ *Gray code on* V*;*
- *the Hamilton* **cycle** *in* $\Gamma \sim$ **cyclic** *Gray code on* V*.*

AntiExample: generating permutations

Symmetric group Sym_n [R. Eggleton, W. Wallis, (1985); D. Rall, P. Slater, (1987)]

The group of permutations:

Q: Is it possible to list all permutations in a list so that each one differs from its predecessor in every position?

A: YES!

[1234]	[3124]	[2314]
[4123]	[4312]	[4231]
[2341]	[1243]	[3142]
[3412]	[2431]	[1423]
[1324]	[3214]	[2134]
[4132]	[4321]	[4213]
[3241]	[2143]	[1342]
[2413]	[1432]	[3421]

Generating permutations in Sym_4

Gray codes: generating permutations

[S. Zaks, (1984)]

Zaks' algorithm:

each successive permutation is generated by reversing a suffix of the preceding permutation.

Describe in terms of prefixes:

- *Start with* $I_n = [12\dots n]$*;*
- *Let* ζ_n *be the sequence of sizes of these prefixes defined by recursively as follows:*

$$\begin{aligned}\zeta_2 &= 2\\ \zeta_n &= (\zeta_{n-1}\, n)^{n-1}\,\zeta_{n-1},\ n > 2,\end{aligned}$$

  where a sequence is written as a concatenation of its elements;
- *Flip prefixes according to the sequence.*

Zaks' algorithm: examples

If $n = 2$ then $\zeta_2 = 2$ and we have:

$$[\underline{12}] \quad [21]$$

If $n = 3$ then $\zeta_3 = 23232$ and we have:

$$[\underline{12}3] \quad [\underline{31}2] \quad [\underline{23}1]$$

$$[\underline{213}] \quad [\underline{132}] \quad [321]$$

If $n = 4$ then $\zeta_4 = 23232423232423232423232$ and we have:

$$[\underline{12}34] \quad [\underline{41}23] \quad [\underline{34}12] \quad [\underline{23}41]$$

$$[\underline{213}4] \quad [\underline{142}3] \quad [\underline{431}2] \quad [\underline{324}1]$$

$$[\underline{31}24] \quad [\underline{24}13] \quad [\underline{13}42] \quad [\underline{42}31]$$

$$[\underline{132}4] \quad [\underline{421}3] \quad [\underline{314}2] \quad [\underline{243}1]$$

$$[\underline{23}14] \quad [\underline{12}43] \quad [\underline{41}32] \quad [\underline{34}21]$$

$$[\underline{3214}] \quad [\underline{2143}] \quad [\underline{1432}] \quad [4321]$$

Greedy Gray code: generating permutations

[A. Williams, J. Sawada, (2013)]

Describe in terms of prefixes:

- *Start with* $I_n = [12 \dots n]$*;*
- *Take the largest size prefix we can flip not repeating a created permutation;*
- *Flip this prefix.*

Example: for $n = 4$ then we have

$$[\overline{1234}]\ [\overline{432}1]\ [\overline{2341}]\ [\overline{143}2]\ [\overline{3412}]\ [\overline{214}3]\ [\overline{4123}]\ [\overline{32}14]$$

$$[\overline{2314}]\ [\overline{413}2]\ [\overline{3142}]\ [\overline{241}3]\ [\overline{1423}]\ [\overline{324}1]\ [\overline{4231}]\ [\overline{13}24]$$

$$[\overline{3124}]\ [\overline{421}3]\ [\overline{1243}]\ [\overline{342}1]\ [\overline{2431}]\ [\overline{134}2]\ [\overline{4312}]\ [\overline{21}34]$$

Prefix–reversal Gray codes: generating permutations

Each 'flip' is formally known as prefix–reversal.

The Pancake graph P_n

is the Cayley graph on the symmetric group Sym_n with generating set $\{r_i \in Sym_n, 2 \leqslant i \leqslant n\}$, where r_i is the operation of reversing the order of any substring $[1, i]$, $1 < i \leqslant n$, of a permutation π when multiplied on the right, i.e., $[\pi_1 \dots \pi_i \pi_{i+1} \dots \pi_n] r_i = [\pi_i \dots \pi_1 \pi_{i+1} \dots \pi_n]$.

Cycles in P_n [A. Kanevsky, C. Feng, (1995); J.J. Sheu, J.J.M. Tan, K.T. Chu, (2006)]

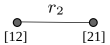
All cycles of length ℓ, where $6 \leqslant \ell \leqslant n!$, can be embedded in the Pancake graph $P_n, n \geqslant 3$, but there are no cycles of length $3, 4$ or 5.

Pancake graphs: hierarchical structure

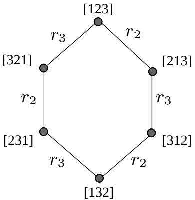
P_n consists of n copies of $P_{n-1}(i) = (V^i, E^i)$, $1 \leqslant i \leqslant n$, where the vertex set V^i is presented by permutations with the fixed last element.

P_1

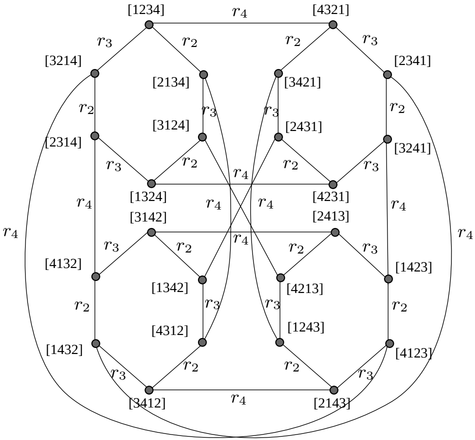
P_2



P_3

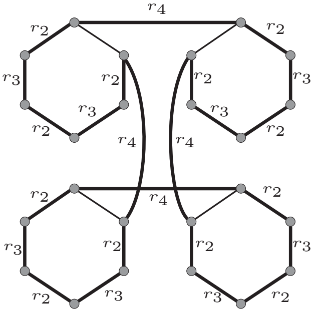


P_4



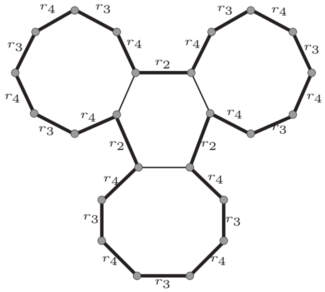
Both algorithms are based on independent cycles in P_n.

Zaks' prefix–reversal Gray code: $(r_2\, r_3)^3$ – flip the minimum number of topmost pancakes that gives a new stack.



(a) Zaks' code in P_4

Williams' prefix–reversal Gray code: $(r_n\, r_{n-1})^n$ – flip the maximum number of topmost pancakes that gives a new stack.



(b) Williams' code in P_4

Independent cycles in P_n

Theorem 1. (K., M.)

The Pancake graph $P_n, n \geqslant 4$, contains the maximal set of $\frac{n!}{\ell}$ independent ℓ–cycles of the canonical form

$$C_\ell = (r_n\, r_m)^k, \tag{1}$$

where $\ell = 2\,k$, $2 \leqslant m \leqslant n-1$ and

$$k = \begin{cases} O(1) & \text{if} \quad m \leqslant \lfloor \frac{n}{2} \rfloor; \\ O(n) & \text{if} \quad m > \lfloor \frac{n}{2} \rfloor \quad \text{and} \quad n \equiv 0 \pmod{n-m}; \\ O(n^2) & \text{else.} \end{cases} \tag{2}$$

Corollary

The cycles presented in Theorem 1 have no chords.

Hamilton cycles based on small independent even cycles

Hamilton cycle or path in $P_n \Rightarrow$ PRGC

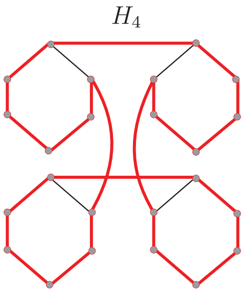
Definition

The Hamilton cycle H_n **based on independent ℓ–cycles** *is called a Hamilton cycle in P_n, consisting of paths of lengths $l = \ell - 1$ of independent cycles, connected together with external to these cycles edges.*

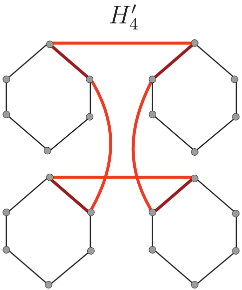
Hamilton cycles based on small independent even cycles

Definition

The fastening cycle H'_n to the Hamilton cycle H_n based on independent cycles is defined on unused edges of H_n and the same external edges.



(c) Hamilton cycle H_4 in P_4



(d) Fastening cycle H'_4 in P_4

Hamilton cycles based on the independent cycles in P_4

Theorem

In the Pancake graph P_4 there are only four Hamilton cycles based on the maximal set independent cycles.

Proof. The collection of all possible maximal sets of independent cycles of the same form in P_4 is presented below by the following table:

6–cycles	8–cycles	12–cycles
$C_6 = (r_3\, r_2)^3$	$C_8^1 = (r_4\, r_2)^4$ $C_8^2 = (r_4\, r_3)^4$	$C_{12}^1 = (r_2 r_3 r_4 r_3 r_2 r_4)^2$ $C_{12}^2 = (r_3 r_2 r_4 r_2 r_3 r_4)^2$

Hamilton cycles based on the independent cycles in P_4

Theorem

In the Pancake graph P_4 there are only four Hamilton cycles based on the maximal set independent cycles.

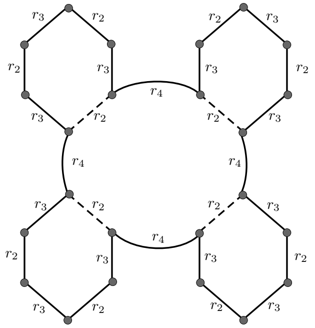
Proof. All possible cases of Hamilton cycles based on the independent cycles in P_4 are presented in the table below:

H_4^i	$\overline{H_4^i}$	Description
$H_4^1 = ((r_2\, r_3)^2 r_2\, r_4)^4$	$\overline{H_4^1} = (r_4 r_3)^4$	Zaks' Hamiltonian cycle;
$H_4^2 = ((r_3\, r_2)^2 r_3\, r_4)^4$	$\overline{H_4^2} = (r_4 r_2)^4$	based on independent cycles C_6;
$H_4^3 = ((r_4\, r_3)^3 r_4\, r_2)^3$	$\overline{H_4^3} = (r_3 r_2)^3$	Williams' Hamiltonian cycle;
$H_4^4 = ((r_4\, r_2)^3 r_4\, r_3)^3$	$\overline{H_4^4} = (r_2 r_3)^3$	based on independent cycles C_8.

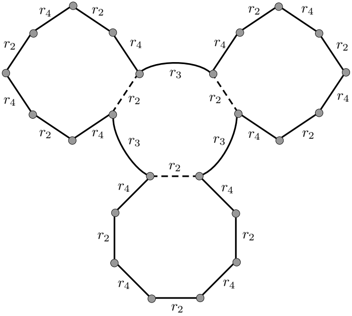
Hamilton cycles based on the independent cycles in P_4

Theorem

In the Pancake graph P_4 there are only four Hamilton cycles based on the maximal set independent cycles.



(e) Hamiltonian cycle $(H_4^2, \overline{H_4^2})$ in P_4



(f) Hamiltonian cycle $(H_4^4, \overline{H_4^4})$ in P_4

Non-existence of Hamilton cycles

Suppose the fastening cycle H'_n has form $(r_m\, r_j)^t$, where $m \in \{2, \ldots, n\}$, $r_j \in PR \backslash \{r_m\}$.

Theorem 2. (K., M.)

The only Hamilton cycles H_n based on independent cycles from Theorem 1 with the fastening cycle H'_n of form $(r_m\, r_j)^t$, where $m \in \{2, \ldots, n\}$, are Zaks', Greedy and Hamilton cycle based on $(r_4\, r_2)^4$ in P_4.

Proof. $H'_n = (r_m\, r_j)^t \Rightarrow H'_n$ has form from Theorem 1. Thus, the following inequality should hold

$$2\frac{n!}{L_{\mathrm{max}}} \leqslant L_{\mathrm{max}}, \tag{3}$$

where L_{max} is the maximal length of cycles from Theorem 1.

Non-existence of Hamilton cycles

The length L_{max} can be estimated as

$$L_{\text{max}} \leqslant n(n+2),$$

and therefore

$$2n! \leqslant L_{\text{max}}^2,$$

$$n! \leqslant \frac{1}{2}n^2(n+2)^2.$$

The inequality does not hold starting from $n = 7$. For n from 4 to 6 it is easy to verify using the exact lengths that inequality holds only for $n = 4$.
□

Non-existence of Hamilton cycles

Suppose the fastening cycle H'_n has form $H'_n = (r_m\, r_\xi)^t$, where by r_ξ we mean that every second reversal may be different from previous. Another way of thinking of it is to treat r_ξ as a random variable taking values in $PR \backslash \{r_n, r_m\}$ with some distribution.

Theorem 3. (K., M.)

The only Hamilton cycles H_n based on independent cycles from Theorem 1 with the fastening cycle H'_n of form $(r_m\, r_\xi)^t$, where $m \neq \{n, n-2\}$ and $r_\xi \in$ PR$\backslash\{r_n,\, r_m\}$ is Greedy Hamilton cycle in P_n.

Proof is based on structural properties of the graph, hierarchical structure and length's argument above.

Remark. Existence in the case $m = n - 2$ is only unresolved when $\ell = O(n)$.

Hamilton cycles based on small independent even cycles

Open problem

Suppose the fastening cycle H_n' *has form* $H_n' = (r_\eta \, r_\xi)^t$, *where* $r_\eta \in \{r_n, r_m\}$ *and* $r_\xi \in PR \backslash \{r_n, r_m\}$.

PRGC: hierarchical construction

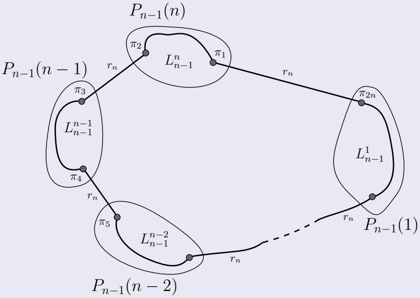
Hierarchical construction

Suppose we know a bunch of Hamilton cycle constructions in graph P_{n-1}. Then the PRGC can be constructed using the fastening $2n$–path passing through all copies of P_{n-1} in P_n exactly once.

Example:

Zaks' construction:

$$H_n'^1 = (r_n\, r_{n-1})^n$$



Thank you for your attention!