Preparing High School Geometry Teachers to Teach the Common Core

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Postulates and Theorems for Connecticut Core Geometry
(Plane Geometry: Units 1-5)

Theorems and Postulates that are part of the formal system are in **bold.**

Unit and Investigation	Postulate or Theorem	Comments	Definitions or Undefined terms
1-1	Pythagorean Theorem	Informal Dissection Proof	point, line, plane
1-1	Distance Formula	Coordinate Proof Based on Pythagorean Theorem	
1-2	Parallel Line Slope Theorem: If two lines in the coordinate plane have the same slope, then they are parallel.	Assumed without proof from Algebra 1	
1-2	Midpoint Formula: If the coordinates of the endpoints of a segment in a plane are (x_1, y_1) and (x_2, y_2), then the midpoint of the segment has coordinates $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.	Informal coordinate proof	segment, ray, vector, translation
1-2	**Line Separation Postulate:** Every point on a line divides the line into two rays with only their endpoint in common.		opposite rays
1-3	Perpendicular Lines Slope Theorem: If the slopes of two lines are opposite reciprocals, then the lines are perpendicular.	Assumed without proof from Algebra 1	Angle, angle measure, rotation

1-4	**Plane Separation Postulate:** If a line *l* lies in a plane then every point in the plane not on *l* lies on one side or the other side of *l*. If *B* lies on the opposite side of *l* from *A* then the line segment $\overline{AB}$ intersects line *l* in one point.	Implicit. Needed in proofs of SAS, ASA, and SSS congruence theorems.	reflection
1-5			Glide reflection
1-6			Mirror line = line of symmetry Rotational symmetry
1-7	**Isometry Postulate** (applies to Translations, Rotations, Reflections and Glide Reflections): All isometries preserve distance and angle measure. Pairs of parallel lines are mapped onto parallel lines. Midpoints of segments are mapped onto midpoints.	This and special properties of translations, reflections, and rotations, will be discovered by students during the course of Investigations 2-4 and codified in Activity 1.7.3.	Shear, dilation, stretch Isometry: a transformation that preserves distances and angle measures. The isometries are translation, rotation, reflection, and the composition of two or more of these transformations.
1-7	**Reflection Postulate**: Under a reflection about line *l* every point on *l* is mapped onto itself. A line that is parallel to *l* is mapped onto another line that is parallel to *l*. **Translation Postulate**: Under a translation by the vector from *X* to *Y*, a line parallel to $\overrightarrow{XY}$ is mapped onto itself. A line that is not parallel to $\overrightarrow{XY}$ is mapped onto another line that is parallel to itself. **Rotation Postulate**: Under a rotation about a point *P*, the point *P* is mapped onto itself. A line through point *P* is mapped onto another line through *P*. If the angle of rotation is 180° a line through *P* is mapped onto itself.		

2-1	Corresponding parts of congruent triangles are congruent (CPCTC); follows immediately from our definition of congruence.	G-CO-7: Necessary condition for congruent triangles is that *all* pairs of corresponding parts are congruent. Sufficient conditions are given in the SAS, ASA, and SSS congruence theorems.	Congruent figures: Two figures are congruent if one is the image of the other under an isometry.
2-2	**Point-Line Postulate** Between two points (it is possible to construct) exactly one line. Two lines intersect in at most one point.	The second half can actually be proved from the first.	
2-2	**SAS Congruence Theorem:** If two sides and the included angle of one triangle are congruent to two sides and the included angle of a second triangle, then the two triangles are congruent to each other.	Transformational proof (in spirit of Euclid's I.4)	
2-2	**ASA Congruence Theorem:** If two angles and the included side of one triangle are congruent to two angles and the included side of a second triangle, then the two triangles are congruent to each other.	Transformational proof	
2-3	**Isosceles Triangle Theorem:** If two sides of a triangle are congruent, then the angles opposite these sides are congruent.	Transformational proof. An alternative proof requires postulating the existence of the bisector of an angle.	Isosceles, equilateral, scalene triangles
2-3	**Isosceles Triangle Converse:** If two angles of a triangle are congruent, then the sides opposite these angles are congruent.	Transformational proof. An alternative proof requires postulating the existence of the midpoint of a segment and would have to be delayed until SSS is proved in Unit 2 Investigation 4.	

2-4	**SSS Congruence Theorem:** If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent.	Transformational proof. Form a kite *ACBC'* and use Isosceles Triangle Theorem to show congruence of angles at *C* and *C'*. Then use SAS.	Kite: a quadrilateral with two (distinct) pairs of congruent adjacent sides
2-5	**Linear Pair Postulate:** if two angles form a linear pair, then they are supplementary.	Needed for Vertical Angles Theorem	Linear pair, supplementary angles, vertical angles
2-5	**Vertical Angles Theorem:** If two lines intersect, pairs of vertical angles are congruent.		
2-5	**Parallel Postulate:** Through a point not on a given line there is exactly one line that can be drawn parallel to the given line.	Playfair's Axiom	
2-5	**Parallel Lines Corresponding Angles Theorem:** If two parallel lines are cut by a transversal, then pairs of corresponding angles are congruent.	Transformational proof based on Parallel Postulate (Some teachers may want to make this a postulate)	Angles formed by two lines and a transversal: corresponding, alternate interior, alternate exterior, same side interior, same side exterior
2-5	**Parallel Lines Alternate Interior Angles Theorem:** If two parallel lines are cut by a transversal, then pairs of alternate interior angles are congruent.	Use corresponding angles and vertical angles.	
2-5	**Parallel Lines Alternate Exterior Angles Theorem:** If two parallel lines are cut by a transversal, then pairs of alternate exterior angles are congruent.		

Activity 1.7. 3 Properties of Isometries

Consider the three basic isometries: translation, rotation, and reflection.

Work with a partner. Use Geogebra or coordinates if you need to check your responses.

Part I. The Geogebra file ctcoregeomACT1731.ggb may be used for experiments in part I.

Determine which transformations have these properties:

1. When segment $\overline{AB}$ is mapped onto segment $\overline{A'B'}$, $AB = A'B'$.

   a. Is this a property of all translations? _________

   b. Is this a property of all rotations? ___________

   c. Is this a property of all reflections? __________

2. When $\angle ABC$ is mapped onto $\angle A'B'C'$, m $\angle ABC$ = m $\angle A'B'C'$

   a. Is this a property of all translations? _________

   b. Is this a property of all rotations? ___________

   c. Is this a property of all reflections? __________

3. Suppose $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. When lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AB}$ are mapped onto lines $\overleftrightarrow{A'B'}$ and $\overleftrightarrow{C'D'}$, $\overleftrightarrow{A'B'} \parallel \overleftrightarrow{C'D'}$.

   a. Is this a property of all translations? _________

   b. Is this a property of all rotations? ___________

   c. Is this a property of all reflections? __________

4. Suppose M is the midpoint of $\overline{AB}$. When points A, B, and M are mapped onto points A', B', and M', M' is the midpoint of $\overline{A'B'}$.

   a. Is this a property of all translations? _________

   b. Is this a property of all rotations? ___________

   c. Is this a property of all reflections? __________

Part II. The Geogebra file ctcoregeomACT1732 may be used for experiments in part II.

5. When is a line mapped onto itself?

   a. Suppose you have a translation by a vector from X to Y. Which lines are mapped onto themselves under this transformation?

   b. Suppose you have a 180° rotation about a point P? Which lines are mapped onto themselves under this transformation?

   c. Suppose you have a rotation about point P, but the angle of rotation is not a multiple of 180°. Are there any lines that are mapped onto themselves under this transformation?

   d. Suppose you have a reflection about line l. Which line is mapped onto itself under this transformation?

6. When is a point mapped onto itself?

   a. If you have a translation, are there any points that are mapped onto themselves? If so, which ones?

   b. If you have a rotation, are there any points that are mapped onto themselves? If so, which ones?

   c. If you have a reflection, are there any points that are mapped onto themselves? If so, which ones?

7. Which transformation or transformations will map a line l onto a line that is parallel to l?

   a. Does this work for any translations? If so, which ones?

   b. Does this work for any rotations? If so, which ones?

   c. Does this work for any reflections? If so, which ones?

Part III.
Summarize the properties you have discovered by filling in the blanks.

8. All isometries preserve distance and ___________ measure. Pairs of parallel lines are mapped onto parallel lines. Midpoints of segments are mapped onto ______________.

9. Under a translation by the vector from X to Y, a line parallel to $\overleftrightarrow{XY}$ is mapped onto ________. A line that is not parallel to $\overleftrightarrow{XY}$ is mapped onto another line that is _____________ to $\overleftrightarrow{XY}$

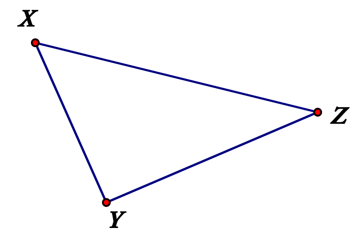
10. Under a rotation about a point P, the point P is mapped onto __________. A line through point P is mapped onto another line through _________. If the angle of rotation is 180° a line through P is mapped onto _____________.

11. Under a reflection about line l every point on l is mapped onto ___________. A line that is parallel to l is mapped onto another line that is ____________ to l.

Activity 2.2.1 SAS Congruence

In this activity you will discover and prove our first theorem about congruent triangles.

1. Included angles. For each pair of sides in ΔXYZ, name the included angle.



Sides: $\overline{XY}$ and $\overline{YZ}$ Included Angle: $\angle$ _________

Sides: $\overline{XY}$ and $\overline{ZX}$ Included Angle: $\angle$ _________

Sides: $\overline{ZX}$ and $\overline{YZ}$ Included Angle: $\angle$ _________

2. Experiment. Work with one other student. You will each draw one triangle using a ruler and protractor as described below:

a. Agree upon the measure of two sides of the triangle and the included angle.

Our two sides measure _______________ and _______________.

Our included angle measures ________________.

b. Now draw your triangles. Start by drawing the angle and then measure the sides. Cut one triangle out and place it on the other. What do you notice?

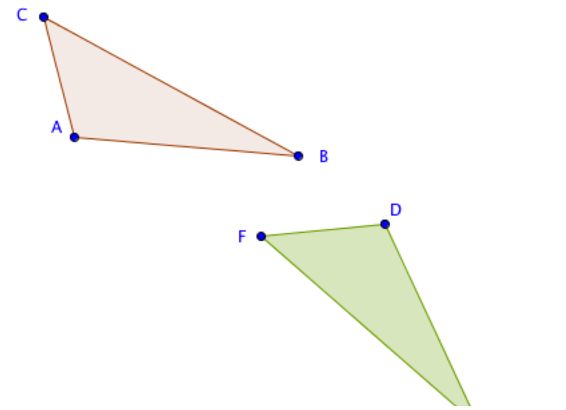
c. Formulate a conjecture: If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle then

____.

3. Proving the SAS Congruence Theorem. Study this proof and fill in the blanks.

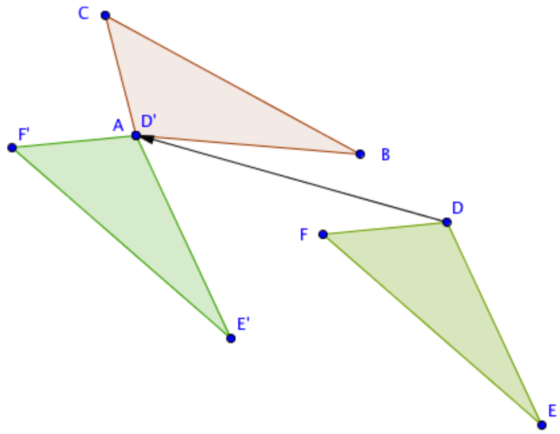
Given ΔABC and ΔDEF with
$AB = DE$,
$AC = DF$, and
$m\angle BAC = m\angle EDF$

Prove $\Delta ABC \cong \Delta DEF$.



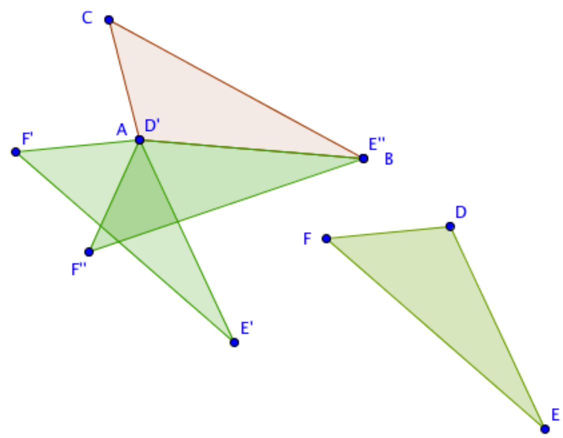
Step 1. If A and D do not coincide then translate ΔDEF by the vector from D to A.

Now D' is the same point as _______.



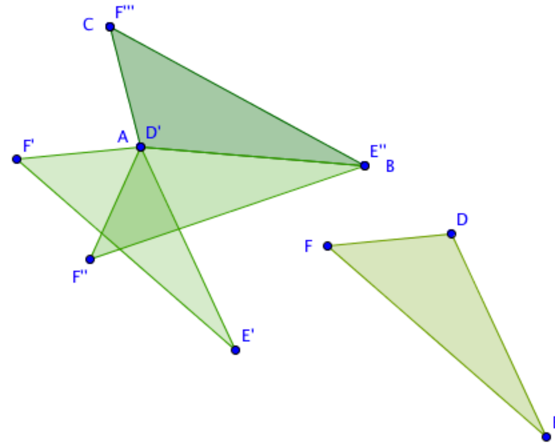
Step 2. If $\overline{AB}$ and $\overline{D'E'}$ do not coincide then rotate $\Delta D'E'F'$ about point A through $\angle E'AB$

Step 3. $\overline{D''E''}$ will now coincide with $\overline{AB}$ since we were given that $AB =$ _______.



Step 4. If F'' is on the opposite side of $\overleftrightarrow{AB}$ from C, reflect $\Delta D''E''F''$ over $\overleftrightarrow{AB}$.

D'' is the same as which point? _____



Step 5. $\angle E'''D'''F'''$ will now coincide with $\angle BAC$ since we were given $m\angle BAC = m\angle EDF$, and $\overline{D'''F'''}$ will coincide with $\overline{AC}$ since we were given $AC = DF$.

Step 6. Therefore F''' coincides with C and E''' coincides with B.

How many lines can be drawn from point B to point C?

____

Explain why $\overline{E'''F'''}$ must coincide with $\overline{BC}$:

____

Step 7. Since ΔABC is the image of ΔDEF under an isometry, $\Delta ABC \cong \Delta DEF$.

4. Now state the theorem you have just proved. We will call this the SAS Congruence Theorem.

If ____,

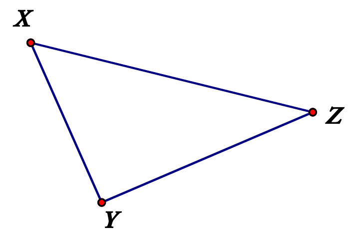
then ____.

5. What properties of transformations were used in the proof?

6. Do you think this proof would work if the angle were not included between the two sides? Explain your thinking.

Activity 2.2.1 SAS Congruence

In this activity you will discover and prove our first theorem about congruent triangles.



1. Included angles. For each pair of sides in ΔXYZ, name the included angle.

Sides: $\overline{XY}$ and $\overline{YZ}$ Included Angle: $\angle$ _________

Sides: $\overline{XY}$ and $\overline{ZX}$ Included Angle: $\angle$ _________

Sides: $\overline{ZX}$ and $\overline{YZ}$ Included Angle: $\angle$ _________

2. Experiment. Work with one other student. You will each draw one triangle using a ruler and protractor as described below:

a. Agree upon the measure of two sides of the triangle and the included angle.

Our two sides measure _______________ and _______________.

Our included angle measures ________________.

b. Now draw your triangles. Start by drawing the angle and then measure the sides. Cut one triangle out and place it on the other. What do you notice?

c. Formulate a conjecture: If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle then

____.

3. Proving the SAS Congruence Theorem. Study this proof and fill in the blanks.

Given ΔABC and ΔDEF with
$AB = DE$,
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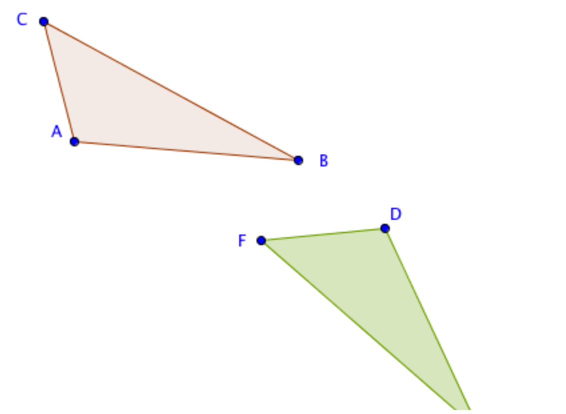
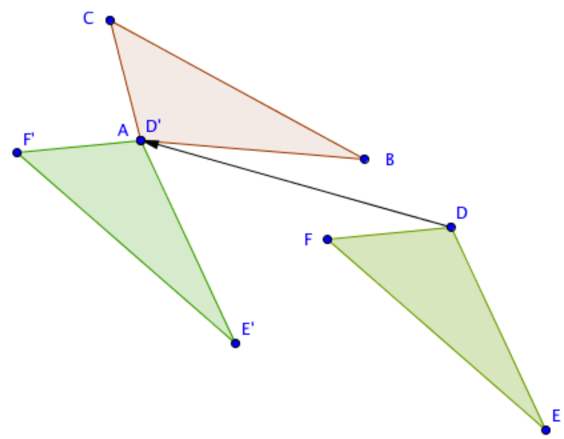


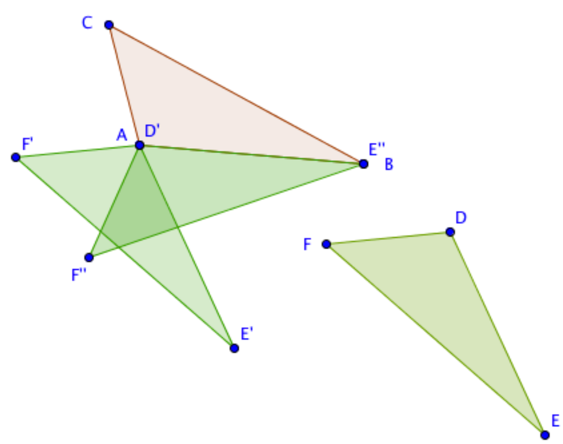
Step 1. If A and D do not coincide then translate ΔDEF by the vector from D to A.

Now D' is the same point as _______.



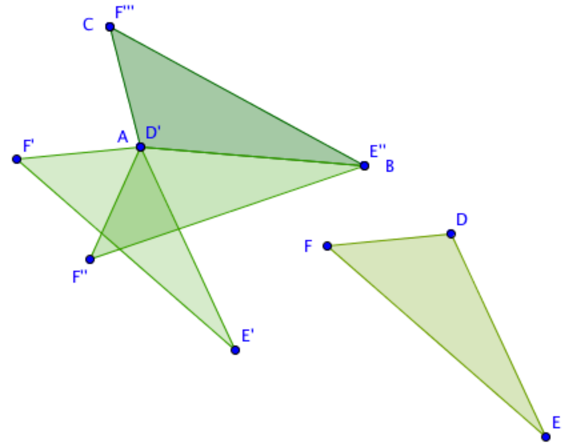
Step 2. If $\overline{AB}$ and $\overline{D'E'}$ do not coincide then rotate $\Delta D'E'F'$ about point A through $\angle E'AB$

Step 3. $\overline{D''E''}$ will now coincide with $\overline{AB}$ since we were given that $AB =$ _______.



Step 4. If F'' is on the opposite side of $\overleftrightarrow{AB}$ from C, reflect $\Delta D''E''F''$ over $\overleftrightarrow{AB}$.

D'' is the same as which point? _____



Step 5. $\angle E'''D'''F'''$ will now coincide with $\angle BAC$ since we were given $m\angle BAC = m\angle EDF$, and $\overline{D'''F'''}$ will coincide with $\overline{AC}$ since we were given $AC = DF$.

Step 6. Therefore F''' coincides with C and E''' coincides with B.

How many lines can be drawn from point B to point C?

____

Explain why $\overline{E'''F'''}$ must coincide with $\overline{BC}$:

____

Step 7. Since ΔABC is the image of ΔDEF under an isometry, $\Delta ABC \cong \Delta DEF$.

4. Now state the theorem you have just proved. We will call this the SAS Congruence Theorem.

If ____,

then ____.

5. What properties of transformations were used in the proof?

6. Do you think this proof would work if the angle were not included between the two sides? Explain your thinking.

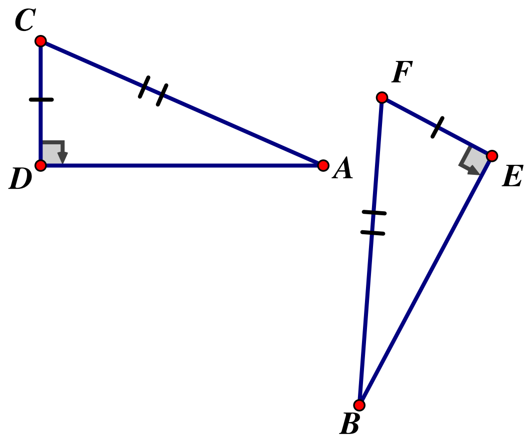
Activity 5.4.3 Two Theorems Involving Right Triangles

This activity asks that you show how to use transformations to prove the **Hypotenuse Leg Congruence Theorem:** If two right triangles have congruent hypotenuses and a pair of corresponding legs congruent, then the triangles are congruent.

Given ΔADC and ΔBEF have right angles at D and E. Also $AC = BF$ and $CD = FE$.

We begin with two cases depending on the orientation of the triangles.

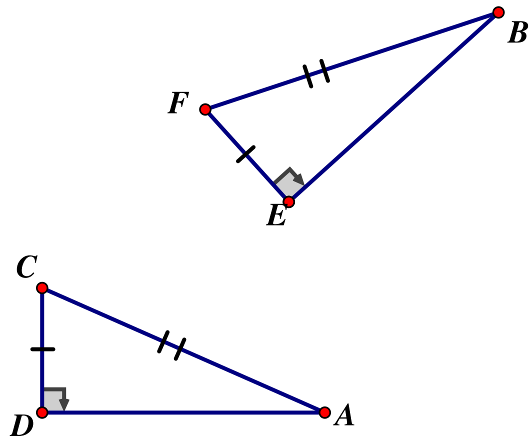
1. *Case One* involving only a translation and a rotation. Show the required transformations for the steps below:



   a. Use a translation from point D to point E to move ΔADC so that the image shares a vertex with E

   b. Find an angle of rotation so that the image of CD lines up with $\overline{FE}$.

2. *Case Two* involving a reflection as well as translations and rotations. Show the required transformations for the steps below:

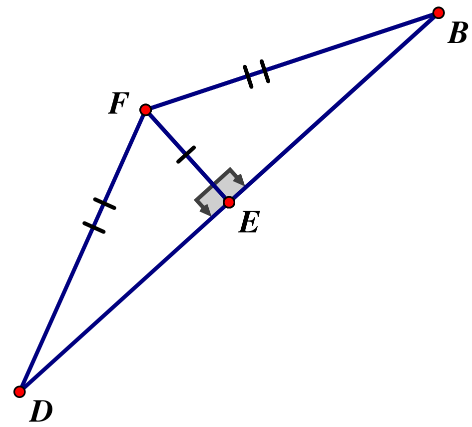


a. Use a translation from the vertex of the right angle in ΔADC to the vertex of the right angle of $\Delta\ BEF$ to translate $\Delta\ DAC$.

b. Then holding the vertex D of the translated vertex of the new triangle, find the angle required to rotate the triangle so that $\overline{CD}$ coincides with $\overline{FE}$.

c. Use $\overleftrightarrow{EF}$ as a mirror line to reflect the triangular image from step b.

3. Your resulting picture should look something like this:



   a. Why does point E lie on $\overline{BD}$?

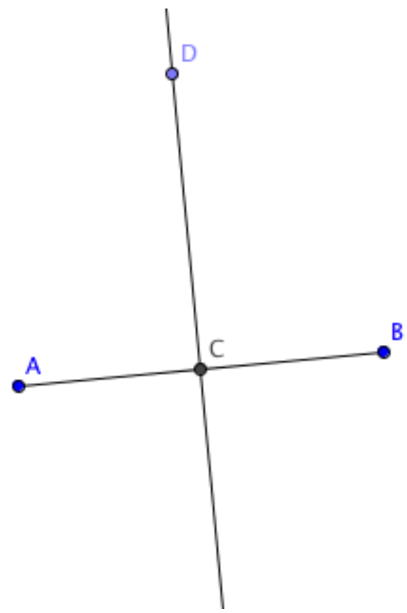
   b. Now we have a new triangle formed. It has two congruent sides $\overline{EB}$ and $\overline{FD}$. Explain why $\angle FDE \cong \angle EBF$.

   c. We know that in the small triangles the right angles are congruent and that the leg $\overline{FE}$ is now common to both triangles. Why are ΔDEF and ΔBEF congruent?

We have now proved that if two right triangles have congruent hypotenuses and a pair of corresponding legs congruent, the triangles are congruent. We may abbreviate this as the **HL Congruence Theorem**.

Activity 5.2.1a The Perpendicular Bisector as a Locus of Points

Use a ruler and protractor to make measurements on the figure below.



1. Measure the distances CA and CB. What do you notice?
2. C is the ____________ of $\overline{AB}$.
3. Measure $\angle DCB$. What do you notice?
4. $\overleftrightarrow{DC}$ is __________________to $\overline{AB}$.
5. $\overleftrightarrow{DC}$ is the _______________ _______________ of $\overline{AB}$.
6. Measure the distances DA and DB. What do you notice?
7. Now place more points E, F, and G on $\overleftrightarrow{DC}$.
8. Measure these distances

   EA = __________ EB = ____________

   FA = __________ FB = _____________

   GA = __________ GB = _____________

   What do you notice?
9. Make a conjecture about all points that lie on $\overleftrightarrow{DC}$
10. Now place a point H in the plane that is not on $\overleftrightarrow{DC}$. Measure HA and HB. What do you notice?
11. Try to find a point J in the plane that is not on $\overleftrightarrow{DC}$ so that $JA = JB$. What do you notice?
12. Make a conjecture about all points that are equidistant from points A and B.

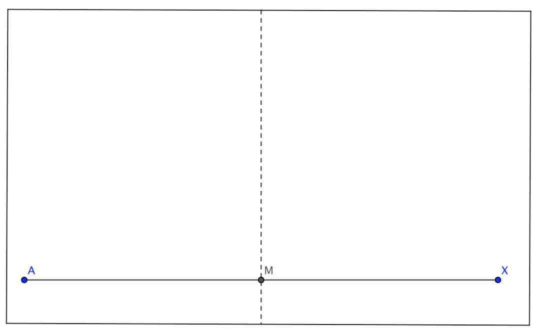
Activity 5.7.1a Exploring the Parabola as a Locus of Points

Materials Needed: A piece of parchment paper, Pencil, Ruler

1. Use your ruler to draw a long segment that appears to be parallel to the bottom edge of your parchment paper. Place this segment at most 2 inches away from the bottom of the paper. (Don't worry if this segment isn't "perfectly parallel" to the bottom edge of the paper.)

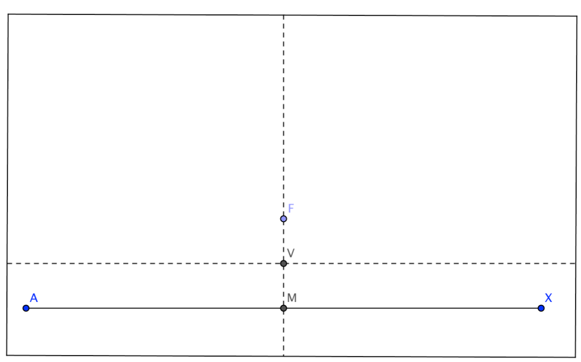
2. Plot and label the left-most endpoint of your segment as A. Plot and label the right-most endpoint of your segment as X.

3. Fold point A on top of point X. (This will create a vertical crease that serves as the perpendicular bisector of $\overline{AX}$.) Label the midpoint of $\overline{AX}$ as M. (See figure.)



4. Plot and label a point F anywhere along the vertical crease you created in step (3) above. If possible, try to keep the distance between this point F and M less than 3 inches.

5. Fold point M onto point F and crease sharply. (This should create a horizontal crease that is parallel to $\overline{AX}$). Label the intersection of this horizontal crease and the vertical crease you formed in step (3) as V. (See figure.)



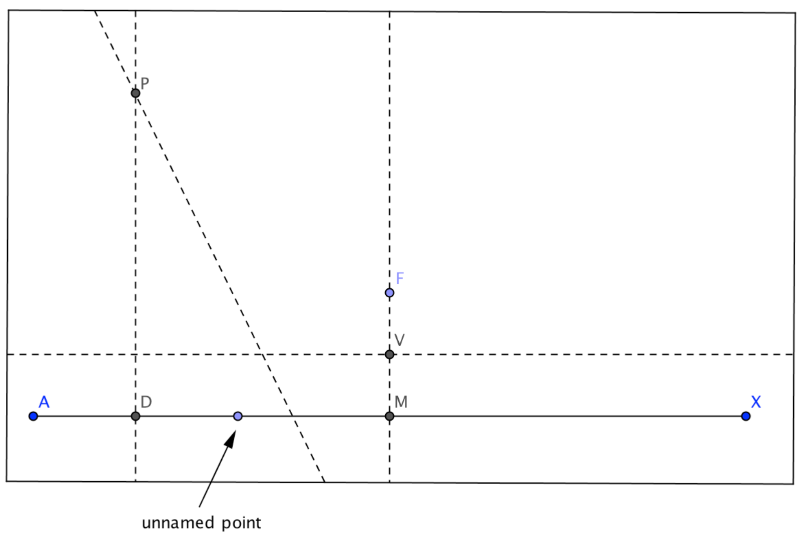
6. This is the most important step here, so please read carefully!

a) Plot any point (leave it unnamed) on $\overline{AX}$ that lies either to the left or right side of *M*. Fold point *A* directly on top of this point. Crease sharply. This should create a vertical crease on your paper (on the left side of *M*).

b) Plot and label a point *D* at the intersection of $\overline{AX}$ and vertical crease you just formed.

c) Fold point *D* onto point *F*. Crease sharply. This will create a diagonal crease somewhere on your paper.

d) Plot and label a point *P* at which this diagonal crease you formed in step (c) above and the vertical crease you formed in (a) above intersect. (See figure.)



e) Now fold point *X* on top of this unnamed point you created in step (6a). This will again create a vertical crease somewhere on your paper (to the right side of *M*). Then repeat steps (6b) – (6d) only.

7. Take your ruler and quickly measure the lengths *FP* and *PD* (on the left side of *M*.) What do you notice?

Why is your observation in step (8) above true? Can you think of a previously learned theorem that helps support your observation?

8. Repeat the **entire step (6)** at least 15 more times. (If you have more time, please continue. The more folds you make, the easier your observations will be later on.)

9. How would you describe (classify) the locus (set) of points you plotted by completing step (7) numerous times? (What does this set of points look like?)

10. Now even though $\overline{AX}$ was a segment, we could keep generating more points (all with the label *P*) if we had parchment paper big enough. So, as we complete this formal definition below, consider the segment with endpoints *A* and *X* to be a LINE instead.

After class discussion:

11. Use your observation to help Complete the following sentence definition:

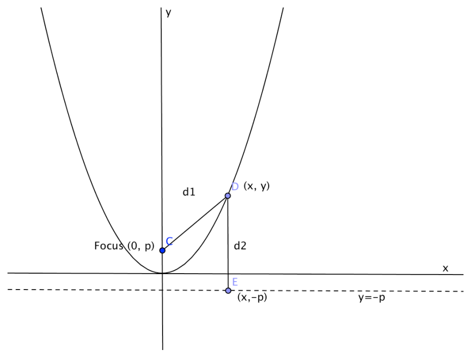
A _________________________ is a locus of points in a plane that are

___________________________ from a fixed __________________ (called the

_______________) and a given line (called the ________________________).

Activity 5.7.2a Deriving the Equation of a Parabola in Standard Form

Let's derive an equation of a Parabola with vertex (0, 0) and a horizontal directrix. Let the directrix lie below the *x*-axis at $y = -p$. The focus will then lie above the *x*-axis at the point $(0, p)$.



The distance to the focus from any point on the locus is $d_1 = \sqrt{(x-0)^2 + (y-p)^2}$

The distance to the directrix from any point on the locus is $d_2 = |y - (-p)| = \sqrt{(y-(-p))^2}$
(Writing d_2 as $\sqrt{(y-(-p))^2}$ ensures that the distance is positive.)

By the definition of parabola as a locus of points, $d_1 = d_2$

Therefore $\sqrt{(x)^2 + (y-p)^2} = \sqrt{(y-(-p))^2}$

Square both sides

$$(x)^2 + (y-p)^2 = (y+p)^2$$

$$x^2 + y^2 - 2py + p^2 = y^2 - 2py + p^2$$

This gives us the general equation: $x^2 = 4py$, where $(0, p)$ are the coordinates of the focus, (x, y) is a point on the parabola, and $y = -p$ is the equation of the directrix.