PRICING AND HEDGING LOAN PREPAYMENT RISK

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1. INTRODUCTION

Many financial products contain prepayment options. Loan contracts are often structured to provide the borrower with the option to prepay the loan at any time, or on specific dates, prior to the maturity date of the loan. These options are important aspects of these financial products. The most commonly encountered investment product with these feature is the mortgage backed security. Investment contracts issued by life insurance companies contain similar options where the policyholder is allowed to surrender the policy with no surrender charge.

This paper analyses loan contracts with such a prepayment option. The loan contracts are fixed interest rate loans with no restrictions on the pattern of prepayments. They can take the form of credit-foncier level repayment contracts, interest only and bullet principal repayment contracts or any other contractual repayment structure. The method that is used to price these contracts set out in this paper is a general algorithm based approach which is not dependent on the structure of the loan cash flows. The algorithm allows for stochastic interest rates and incorporates a one-factor term structure of interest rates model. The algorithm is arbitrage free in the sense that the parameters of the one-factor term structure model are chosen to ensure that prices of zero coupon bonds at traded maturities are priced so that the algorithm produces values equal to the market prices of such bonds on the valuation date.

This paper does not develop new theoretical results. It aims to illustrate the practical implementation of some of the techniques of modern financial mathematics and economics as developed for the analysis of interest rate options. Section 1 of the paper details the nature of the prepayment risk in these loan contracts. Section 2 sets out the algorithms that are the basis of the implementation of a one-factor arbitrage free term structure model and shows how such a model is used to value the prepayment risk in such loan contracts. Section 3 sets out the formulae and algorithms for determining the conventional risk statistics used in the management of a portfolio of loan contracts with prepayment options. These include the delta, gamma, theta, duration and convexity of the loan cash flows. Section 4 briefly discusses some issues related to the management of a portfolio of these loan contracts and how the prepayment and interest rate risk of these loans might be hedged in financial markets.

2. PREPAYMENT RISK

This paper will consider prepayment risk from the point of view of an issuer of loans with early repayment options. Prepayment risk arises in such a loan contract when the borrower is given the option to prepay a fixed interest rate loan prior to the maturity date of the loan without penalty. In this paper the loan contract analysed is a fixed interest rate, fixed term loan. It is assumed that under the terms of the loan agreement the borrower can repay the loan for the balance outstanding regardless of current market interest rates at any time during the term of the loan.

The prepayment option reflects the difference between the value of the outstanding loan repayments at the interest rate at the time of prepayment for the remaining term of the loan less the amount of the loan then outstanding (which is the value of the outstanding loan repayments at the original loan interest rate). If interest rates have fallen then the payoff from the prepayment option would be positive. The option would be "in the money". Similarly if rates have risen then the prepayment option would be out of the money.

3. VALUATION OF LOAN CONTRACTS WITH PREPAYMENT OPTIONS

The loan contract is the equivalent of a fixed rate loan with an option to repay early. The option is a call option held by the customer on the loan contract with an exercise price equal to the loan outstanding. A rational exercise policy for the prepayment option would be to exercise the call option on the loan contract only when the difference between the value of the loan at the prevailing interest rates on any future date and the loan amount outstanding exceeds the value of the prepayment option on that date assuming a rational exercise policy for the remaining term of the loan. Otherwise the prepayment option should not be exercised since it is worth more "alive" than exercised.

This is equivalent to an American style option to swap the fixed rate loan for a floating rate loan for a term equal to the remaining term of the original loan. Both the term and the face value, for reducing balance loans, of this swap reduce through time. Because of the reducing term of the loan this option to swap is not equivalent to a swaption on a fixed term swap. For interest only loans it is equivalent to an American style option on a physical bond with an original maturity equal to the original term of the loan.

For the loan contract it can be expected that borrowers will not follow this rational exercise policy. Some borrowers will prepay when it is not economically rational to do so and not all borrowers will prepay even when it is economically rational to do so. Such departures from a rational exercise policy arise because of market frictions such as transaction costs and also because of the occurrence of events such as death. In this paper the term "non-rational" is to indicate a departure from the assumed rational behaviour described above.

The important point with this contract is that this non-rational exercise of the option provides positive value to the lender in all circumstances as does not exercising when it is economically rational to do so. If the loan can be issued for the full cost of the rational prepayment option then the lender need not consider an allowance for the non rational early prepayments in the pricing and could then allow profits from such prepayments to be recognised as they occur. It would also need to recognise profits from non-exercise of the prepayment option when it is economically rational to do so. If however the early exercise option, including the value of the non rational exercise, is priced into the loan contract on a competitive basis then allowance for the rate of non rational exercise will need to be made. In these circumstances the important point to note is that non rational prepayment is difficult to hedge precisely and is also difficult to predict. This is the problem with mortgage backed securities where the market prices reflect the prepayment option including the non rational prepayment value.

3.1. Valuation of Rational Repayment Option

The algorithm set out here for valuation and analysis of the rational prepayment option is based on a technique developed by Jamshidian (1991). The algorithm is fast and efficient and allows the valuation of a range of interest rate related options. The basic approach is set out in this section with the algorithms and an example is used to illustrate the implementation of the algorithms.

To begin, express the one period spot interest rate at time t, denoted by $r(t)$, as a function of a random variable $z(t)$. The $z(t)$ will take on values called "states" at different future times based on a lattice. The simplest case to use is the binomial lattice and since this has accepted usage this will be the basis adopted. Alternative lattice structures do have potential computational advantages but this is unlikely to be an issue of concern with the algorithm recommended for use in the computations in this paper.

Start at time zero with state $s(0,0)$ equal to 0. Allow the state to jump up to $+1$ at time 1 or down to -1 at time 1. These new states will be referred to as nodes on the lattice. From each node allow the state to increase by 1 or to decrease by 1.

The algorithm to construct the state lattice is

$$s(0,0) = 0$$

for $t = 1, \dots, n$

$$s(t,t) = s(t-1, t-1) + 1$$

for $j = 0, \dots, t-1$

$$i = 2j - t$$

$$s(t,i) = s(t-1, i+1) - 1$$

This state lattice is constructed for the maximum time period to be used in the valuation. For example using a monthly time interval will require 60 time intervals for a five year loan. A monthly interval should be accurate enough for many applications. In general the number of intervals should be an input variable. Hence if M is the maximum time (in years) for construction of the interest rate lattice and n is the number of time intervals into which this period is to be divided then each time interval is of length $h = n/M$ years. For 5 years and 60 time intervals then the length of each is $5/60 = 1/12$ of a year or one month.

The probabilities of an up or down change in the state lattice are taken to be 0.5. This is used since it allows the fastest computation of values. Under this assumption the expected value of the state at time t is zero and the variance of the state is t.

As an example of the construction of the lattice consider a twelve month loan.

Table 1 illustrates the lattice to use for such an example. The values for t are given by the column number and the indicator i is given by the row number. Hence the value for the states at time 9 are given in column 10 since the first column represents t equal to 0. The values $s(9, i)$ range from 9 to -9 in increments of -2.

Table 1 - Lattice of States $s(t, i)$

0	1	2	3	4	5	6	7	8	9	10	11	12
	-1	0	1	2	3	4	5	6	7	8	9	10
		-2	-1	0	1	2	3	4	5	6	7	8
			-3	-2	-1	0	1	2	3	4	5	6
				-4	-3	-2	-1	0	1	2	3	4
					-5	-4	-3	-2	-1	0	1	2
						-6	-5	-4	-3	-2	-1	0
							-7	-6	-5	-4	-3	-2
								-8	-7	-6	-5	-4
									-9	-8	-7	-6
										-10	-9	-8
											-11	-10
												-12

The one period spot rates (r(t)) will be assumed to be semi-annual compounding p.a. rates. These r(t) are functions of the state s(t,i), the assumed one period volatility of spot interest rates and the median future spot interest rate. The choice of the function of r(t) determines the limiting distribution of future one period spot interest rates. Two common alternatives are the normal and the lognormal formulae. Others are possible.

The normal distribution specification is

$$r(t, i) = f(t) + (\sigma_N(t)/100)s(t, i)\sqrt{h}$$

and the lognormal specification is

$$r(t, i) = f(t) \exp\left\{(\sigma_L(t)/100)s(t, i)\sqrt{h}\right\}$$

where $f(t)$ is the median interest rate, $\sigma_N(t)$ is the one period spot rate volatility in absolute terms, $\sigma_L(t)$ is the one period spot rate volatility in percentage terms, and h is the length of the time interval used.

There are advantages and disadvantages in using either of these models. There are also other models that can incorporate mean reversion and other distributions. For the illustrative example the lognormal model is used.

The reasons for using the lognormal model are as follows:

- it does not allow negative interest rates as is possible under the normal model.
- it allows the yield curve to move in a non-parallel fashion unlike the normal model which implies parallel moves.
- the implied volatility curve for zero coupon bonds derived from the resulting spot interest rates has higher volatilities for short term bonds than for long term bonds unlike the normal model which has approximately constant volatility for different term zero coupons. Higher volatility in short term interest rates is an observed empirical fact for interest rates.
- the volatility parameter for the lognormal model is percentage yield volatility of the time t maturity one period forward interest rate and can be estimated from prices for options on forward interest rates.

Note also that it is yield volatility and not price volatility used in the formula. If options data gives price volatility then this must be converted into percentage yield volatility to use in the lognormal model. If the normal model is used then price volatility must be converted into absolute dollar yield volatility. Appendix One give details on how to convert from one volatility to another. Volatility can be interpolated from a forward rate volatility curve or assumed constant for all periods for ease of computation.

Having selected the formula for $r(t,i)$ this is then used to derive present value factors for the nodes of the lattice to value cash flow. These present value factors are one period factors that are applied to the average value of the cash flows which occur at the up and down states at the end of each period. If $r(t,i)$ is a p.a. semi-annual compounding rate then the present value factor at node (t,i) is

$$p(t,i) = \left[1 + r(t,i)/200\right]^{(-2h)}$$

where h is the time interval.

This formula is readily adapted for other compounding frequencies for $r(t,i)$.

The choice of the $r(t,i)$ values must be made so that the value (yield) of zero coupon bonds maturing at the end of each time interval h when determined using the state lattice and the $p(t,i)$ factors is equal to the current market price (yield) of those bonds. In order to do this as efficiently as possible it is necessary to determine current prices of single dollar cash flows at each node in the lattice. These are referred to as "state contingent" prices.

Denote $G(n,j,t,i)$ as the price at node (n,j) of a security which has a cash-flow of 1 at node (t,i) and zero everywhere else. It is only necessary to calculate $G(0,0,t,i)$, the current price of 1 paid at node (t,i), since such a lattice of numbers can be used to value any cash flows by setting out the cash flows at node (t,i) then multiplying these cash flows by the prices $G(0,0,t,i)$ and summing the values.

Take as an input the zero coupon yield curve $y(t)$ for zero coupon bonds maturing at the end of each of the time intervals in the lattice. This would be monthly out to five years in the example set out in this paper. Interpolation is used where necessary. The price of a unit face value zero coupon bond maturing at the end of time interval t in the lattice with a p.a. semi-annual yield of $y(t)\%$ is given by

$$P(t) = \left\{1 + y(t)/200\right\}^{(-2th)}$$

The valuation lattice is constructed in an iterative manner starting at node $(0,0)$ in the lattice. The input is

- the state lattice (derived above).
- the zero coupon yield curve $y(t)$ which is used to determine the price of zero coupon bonds $P(t)$ (formula given above).
- the one period forward rate percentage volatilities for each period in the lattice.

For the example this data is taken as follows:

t	1	2	3	4	5	6	7	8	9	10	11	12
$y(t)$	6.65	6.58	6.53	6.5	6.44	6.41	6.39	6.39	6.38	6.4	6.42	6.45
$\sigma(t)$	21	21	21	21	21	21	21	21	21	21	21	21

These zero coupon bond yields give the following zero coupon bond prices for each maturity.

1	0.994	0.989	0.984	0.978	0.973	0.968	0.963	0.958	0.953	0.948	0.943	0.938

The output is

- the current state contingent prices $G(0,0,t,i)$ for each node in the lattice.
- the one period discount factors $p(t,i)$ at each node in the lattice.
- the implied one period spot interest rates at each node in the lattice calculated from the $p(t,i)$ factors. This need not be produced since it is not meaningful in terms of expected future interest rates except in the sense that it represents market expectations on the assumption that the lognormal model (or other model used) is the correct model for interest rates. These rates are a product of fitting the interest rate model to market data in the form of the zero coupon yield curve.
- the median forward interest rates for each period.

For the example the following $G(0,0,m,j)$ values are produced.

Values of state contingent securities $G(0,0,m,j)$

1	0.497282	0.24723	0.12288	0.06106	0.03033	0.015064	0.00747	0.00370	0.00184	0.00091	0.00045	0.0002
	0.49728	0.49463	0.36891	0.24449	0.15188	0.09054	0.05245	0.02975	0.01661	0.00915	0.00498	0.0026
		0.24739	0.36914	0.36709	0.30417	0.22675	0.15770	0.10441	0.06664	0.04132	0.02503	0.0148
			0.12312	0.24495	0.30455	0.30281	0.26336	0.20932	0.15593	0.11055	0.07540	0.0498
				0.06129	0.15245	0.22745	0.26385	0.26224	0.23452	0.19407	0.15136	0.1126
					0.03052	0.09110	0.15858	0.21023	0.23509	0.23356	0.21261	0.1808
						0.01520	0.05294	0.10532	0.15709	0.19516	0.21327	0.2117
							0.00757	0.03014	0.06747	0.1118	0.15278	0.1821
								0.00377	0.01690	0.04202	0.07659	0.1141
									0.00188	0.00935	0.02559	0.0508
										0.00093	0.00513	0.0153
											0.00046	0.0027
												0.0002

The values in the table have been derived by selecting the median forward rate used in the formula for r(t,i) until the sum of the G(0,0,m,j) values equals the price of the zero coupon bond for maturity m. For example, if the values in this table for column 4 are summed then a total of 0.98925 is obtained which equals the zero coupon bond price for maturity 2. The procedure begins at time 0 and works forward using forward induction. At each stage of the algorithm the forward discount factors are derived. These are set out in the following table for the 12 month example.

Values of $p(t, i)$

0.9945	0.99436	0.99409	0.99376	0.99360	0.99316	0.99274	0.99217	0.99182	0.99097	0.99038	0.9895
	0.99499	0.99475	0.99446	0.99432	0.99392	0.99355	0.99304	0.99273	0.99197	0.99145	0.9907
		0.99534	0.99508	0.99495	0.99460	0.99427	0.99382	0.99354	0.99286	0.99240	0.9917
			0.99563	0.99552	0.99521	0.99492	0.99451	0.99427	0.99366	0.99325	0.9926
				0.99602	0.99575	0.99549	0.99513	0.99491	0.99437	0.99400	0.9934
					0.99623	0.99599	0.99567	0.99548	0.99500	0.99467	0.9942
						0.99645	0.99616	0.99599	0.99556	0.99527	0.9948
							0.99659	0.99644	0.99606	0.99580	0.9954
								0.99684	0.99651	0.99627	0.9959
									0.99690	0.99669	0.9964
										0.99707	0.9968
											0.9971

Corresponding to each $p(t, i)$ value is a corresponding one period interest rate. These have been determined for the example and expressed as semi-annual compounding rates in the following table.

One period spot rates % p.a. semi-annual compounding

6.65	6.90410	7.23399	7.64771	7.84784	8.40330	8.92705	9.65311	10.0923	11.1838	11.9341	12.968
	6.11578	6.408	6.77448	6.95176	7.44380	7.90774	8.55090	8.93999	9.90685	10.5714	11.487
		5.67632	6.00095	6.158	6.59385	7.00482	7.57454	7.91921	8.77567	9.36441	10.175
			5.31575	5.45486	5.84095	6.205	6.70966	7.01498	7.77365	8.29516	9.0139
				4.83202	5.17402	5.49650	5.94354	6.214	6.88604	7.34800	7.9847
					4.58324	4.86890	5.26490	5.50447	6.09978	6.509	7.0730
						4.31296	4.66374	4.87596	5.40329	5.76579	6.2654
							4.13123	4.31921	4.78634	5.10744	5.5500
								3.82604	4.23982	4.52426	4.9163
									3.75571	4.00767	4.3549
										3.55007	3.8577
											3.4172

The median future spot rates are solved to produce the market price of the zero coupon bonds. The median rates for the example are as follows.

One period spot rates % p.a. semi-annual compounding

6.65	6.498	6.408	6.376	6.158	6.206	6.205	6.315	6.214	6.481	6.509	6.657

The algorithm for determining these is as follows. Begin by initialising the time 0 values.

$G(0,0,0,0) = 1$
$p(0,0) = P(1)$

Carry out the following for each successive value of $t = 1$ to $n - 1$:

- begin with date t
- estimate the median interest rate using the previous period's rate $f(t) = f(t-1)$
- calculate the time t, state t spot rate

$$r(t,t) = f(t)\exp\left\{(\sigma_L(t)/100)s(t,t)\sqrt{h}\right\}$$

- calculate the discount factor corresponding to $r(t,t)$

$$p(t,t) = \left[1 + r(t,t)/200\right]^{(-2h)}$$

- calculate the state contingent value of 1 at time t using the forward relationship

$$G(0,0,t,t) = 0.5p(t-1,t-1)G(0,0,t-1,t-1)$$

- calculate the contribution to the next period zero coupon bond price implied by the discount factor and state contingent price

$$P^*(t+1) = G(0,0,t,t)p(t,t)$$

- repeat this for all possible values of the state for each time period summing the contribution to the next period zero coupon bond price from the calculated discount factor and state contingent price derived from the forward relationship for state contingent prices for $j = 1, \ldots, t-1$
  $i = t - 2j$

$$r(t,i) = f(t)\exp\left\{(\sigma_L(t)/100)s(t,i)\sqrt{h}\right\}$$

$$p(t,i) = \left[1 + r(t,i)/200\right]^{(-2h)}$$

$$G(0,0,t,i) = 0.5\Big\{p(t-1,i-1)G(0,0,t-1,i-1)+$$
$$+\,p(t-1,i+1)G(0,0,t-1,i+1)\Big\}$$

$$P^*(t+1) = P^*(t+1) + G(0,0,t,i)p(t,i)$$

- finally, complete the process for state $-t$

$$G(0,0,t,-t) = 0.5p(t-1,-t+1)G(0,0,t-1,-t+1)$$

$$r(t,-t) = f(t)\exp\left\{(\sigma_L(t)/100)s(t,-t)\sqrt{h}\right\}$$

$$p(t,-t) = \left[1 + r(t,-t)/200\right]^{(-2h)}$$

$$P^*(t+1) = P^*(t+1) + G(0,0,t,-t)p(t,-t)$$

Now check to see if $P^*(t+1)$ is within 10^{-6} of $P(t+1)$ from the zero coupon yield curve given by

$$P(t+1) = \left\{1 + y(t+1)/200\right\}^{(-2\{t+1\}h)}.$$

If not alter $f(t)$ using either the secant method or Newton-Raphson and repeat. Once $P^*(t+1)$ has converged to $P(t+1)$ proceed to the next value of t. Convergence should take only two iterations at most.

At the end of this procedure the values of $p(t,i)$ and $G(0,0,t,i)$ for all values of t,i in the lattice will have been derived. These values are all that are required to value the rational prepayment option.

The steps involved in the valuation are to firstly determine the current values of the loan repayments at each node of the lattice. To do this the contractual loan cash flows are set out in a lattice. Denote the loan cash flow at node t,i in the lattice as $c(t,i)$. The value of the loan is derived by stepping back recursively through the lattice. Denote the value of the loan at node t,i as $v(t,i)$. The loan values are determined using the following algorithm.

Initialise
for $j = 0$ to n

$$i = 2j - n$$
$$v(n, i) = 0$$

calculate each value of $v(t, i)$ recursively using
for $t = n - 1$ to 0

$$j = 0 \text{ to } t$$
$$i = 2j - n$$
$$v(t,i) = 0.5\Big\{v(t+1,i+1) + c(t+1,i+1) + \\ + v(t+1,i-1) + c(t+1,i-1)\Big\}p(t,i)$$

This gives the lattice of loan values.

The example will use a 12 month interest only fixed rate loan with an interest rate of 6.449% p.a. (semi-annual compounding) for an amount of 10,000. Monthly interest repayments for such a loan will be 53.036 with repayment of the 10,000 in a bullet payment in 12 months. The loan rate of 6.449% p.a. (semi-annual) was chosen since this is the "arbitrage-free" interest rate for such a loan based on the one factor term structure model used in this example. Hence the value of such a loan using the algorithm set out above is 10,000. The loan cash flows and the state contingent values of the loan are set out in the following tables.

Loan cash flow lattice c(i,j)

0	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	10053.
	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	10053.
		53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	10053.
			53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	10053.
				53.036	53.036	53.036	53.036	53.036	53.036	53.036	53.036	10053.
					53.036	53.036	53.036	53.036	53.036	53.036	53.036	10053.
						53.036	53.036	53.036	53.036	53.036	53.036	10053.
							53.036	53.036	53.036	53.036	53.036	10053.
								53.036	53.036	53.036	53.036	10053.
									53.036	53.036	53.036	10053.
										53.036	53.036	10053.
											53.036	10053.
												10053.

Value of loan cash flows $v(t,i)$

10000	9967.96	9939.11	9914.29	9894.43	9878.05	9868.34	9865.44	9871.50	9884.69	9910.94	9948.32	0
	10035.3	10003.8	9975.93	9952.47	9932.10	9917.67	9909.32	9908.94	9914.89	9932.46	9959.89	0
		10061.7	10031.0	10004.3	9980.45	9961.80	9948.56	9942.41	9941.88	9951.68	9970.22	0
			10080.3	10050.7	10023.6	10001.2	9983.61	9972.30	9965.98	9968.83	9979.44	0
				10092.2	10062.2	10036.4	10014.8	9998.96	9987.47	9984.13	9987.65	0
					10096.6	10067.8	10042.7	10022.7	10006.6	9997.75	9994.97	0
						10095.7	10067.6	10043.9	10023.6	10009.8	10001.4	0
							10089.7	10062.7	10038.8	10020.6	10007.2	0
								10079.5	10052.3	10030.2	10012.4	0
									10064.4	10038.8	10017.0	0
										10046.4	10021.0	0
											10024.6	0
												0

The values of the loan cash flows have been derived by averaging the next periods loan values plus the loan cash flows for the two states that originate from the node and multiplying this by the one period discount factor for that node.

The next step is to generate a lattice of balances outstanding under the original loan yield rate. These are the exercise prices of the prepayment option and will be denoted by $b(t,i)$. Note that these do not vary for differing values of i since the loan outstanding which is to be prepaid is based on the value of the repayments at the original loan interest rate.

Loan outstanding lattice $b(t,i)$

10000	10000	10000	10000	10000	10000	10000	10000	10000	10000	10000	10000	0
	10000	10000	10000	10000	10000	10000	10000	10000	10000	10000	10000	0
		10000	10000	10000	10000	10000	10000	10000	10000	10000	0000	0
			10000	10000	10000	10000	10000	10000	10000	10000	10000	0
				10000	10000	10000	10000	10000	10000	10000	10000	0
					10000	10000	10000	10000	10000	10000	10000	0
						10000	10000	10000	10000	10000	10000	0
							10000	10000	10000	10000	10000	0
								10000	10000	10000	10000	0
									10000	10000	10000	0
										10000	10000	0
											10000	0
												0

The cash flow on rational early prepayment is the difference between the value of the loan and the balance outstanding provided that this is positive. These values are denoted by $o(t,i)$ and they are determined as follows.

For all t and i

$$o(t,i) = \text{maximum}\left(v(t,i) - b(t,i), 0\right).$$

These values for the example are set out in the following table.

Early exercise cash flows $o(t,i)$

0.0042	0	0	0	0	0	0	0	0	0	0	0	0
	35.3030	3.82141	0	0	0	0	0	0	0	0	0	0
		61.7228	31.0821	4.39401	0	0	0	0	0	0	0	0
			80.3743	50.7933	23.6515	1.22811	0	0	0	0	0	0
				92.2181	62.2178	36.4159	14.8888	0	0	0	0	0
					96.6164	67.7972	42.7774	22.7389	6.62869	0	0	0
						95.7628	67.6269	43.9164	23.6899	9.88974	1.48518	0
							89.7541	62.7706	38.8768	20.6868	7.27833	0
								79.5469	52.3878	30.2902	12.4296	0
									64.4022	38.8280	17.0084	0
										46.4154	21.0765	0
											24.6899	0
												0

Notice that for lower future spot interest rates the prepayment option is more "in the money". Because the prepayment option is assumed to be exercisable at any time this table does not give the state contingent values of the prepayment option. At each node it is necessary to check if the prepayment option is worth more if left unexercised and a rational exercise policy followed for the remaining term of the loan.

The value of the rational early prepayment option is determined by stepping back through the rational early prepayment cash flow lattice allowing for the possibility that the value of the prepayment option at any time point is worth more than the value that would be received by prepaying at that time. To do this denote the rational early prepayment option value by $ov(t,i)$. The required value is $ov(0,0)$. The following algorithm is then used to determine $ov(t,i)$.
Initialise
for $j = 0$ to n

$$i = 2j - n$$
$$ov(n,i) = 0$$

Calculate the option value recursively
for $t = n - 1$ to 0

$$j = 0 \text{ to } t$$
$$i = 2j - n$$
$$ov(t,i) = 0.5\Big\{ov(t+1, i+1) + ov(t+1, i-1)\Big\}p(t,i)$$

if $o(t,i)$ is greater than $ov(t,i)$ then set $ov(t,i) = o(t,i)$.

Prepayment option values $ov(t,i)$

26.422	12.4014	4.79104	1.38787	0.25693	0.02238	0	0	0	0	0	0	0
	40.7326	20.1524	8.25111	2.53623	0.49478	0.04506	0	0	0	0	0	0
		61.7228	32.2662	14.0578	4.60664	0.95055	0.09072	0	0	0	0	0
			80.3743	50.7933	23.6515	8.31267	1.82132	0.18256	0	0	0	0
				92.2181	62.2178	36.4159	14.8888	3.48017	0.36724	0	0	0
					96.6164	67.7972	42.7774	22.7389	6.62869	0.73864	0	0
						95.7628	67.6269	43.9164	23.6899	9.88974	1.48518	0
							89.7541	62.7706	38.8768	20.6868	7.27833	0
								79.5469	52.3878	30.2902	12.4296	0
									64.4022	38.8280	17.0084	0
										46.4154	21.0765	0
											24.6899	0
												0

Note that whenever $o(t,i)$ exceeds $ov(t,i)$ then it is optimal to exercise the early prepayment option in full and repay the whole loan. Otherwise the option value is greater if kept alive. The rational exercise of the option involves 100% exercise whenever it is optimal to do so.

For the example the prepayment option value equals 26.422 per 10,000 face value for the interest only loan.

3.2. Valuing the non-rational exercise (and non-exercise) of the prepayment option

Non-rational exercise of the option can be incorporated into the calculations. This is done most easily by assuming a rate of prepayment as a proportion of the loan outstanding at each time period which is independent of the current interest rate. These rates will be denoted by q(t). These prepayment proportions are used to determine the altered principal repayment cash flows. The altered balance outstanding on the loan is determined allowing for the early prepayment proportions.

This is used to establish the altered interest payments at the original loan interest rate. The sum of the principal repayments and interest repayments allowing for previous early prepayments is the altered loan cash flows $c(t, i)$.

The value of the loan at each node of the lattice is determined in the same way as before using the altered loan cash flows. The balance outstanding of the loan is determined either by valuing the altered outstanding repayments at the original loan interest rate or by determining a survivorship proportion of the principal outstanding using the rates of prepayment $q(t)$.

The early prepayment option value can be determined using the lattice approach by assuming that the remaining principal outstanding at the nodes where it is optimal to exercise the prepayment option is repaid in full. The lattice approach as covered will also require an assumption of path independence of the non-rational prepayments. Such a calculation will incorporate the value of exercise of the option when it is not economically rational to do so assuming that the rate of prepayment is not dependent on interest rates. It does not allow for the value of the non-exercise of the option when it is economically rational to do so.

If an allowance for non rational exercise is to be incorporated into the pricing of the loan then estimates of the prepayment rates will be required.

The US experience is a guide. The experience in the mortgage backed security market is referred to in Bartlett (1989). This reference indicates that prepayment rates vary with the original loan interest rate with higher original interest rate loans having higher prepayment rates. An increase of 4% in the original interest rate can mean a four times higher prepayment rate. A standard that is used in the US market has prepayment rates commencing at 0% in month 0 and increasing by 0.2% monthly to 6% in month 30 and a constant 6% thereafter. There is also a lag in the time period from when interest rates increase to when loans prepay early of around three months. Prepayments are seasonal reflecting the timing of house sales which are higher in summer and spring than in winter. Loans are not very sensitive to prepayment when rates rise during the first 2.5 years of loan issue.

If prepayments are to be incorporated this suggests that the prepayment rate should vary by

- the original loan interest rate
- time since the loan was issued
- the interest rate three months previously

- the month of the year (seasonal)

and accurate estimation of such rates will be important since they will not be possible to hedge directly.

If such an allowance for prepayments were to be made then the value of the loan outstanding and the balance outstanding become dependent on the interest rate path through the valuation lattice and the option valuation becomes path-dependent. This is a computationally more intensive valuation than the non-path dependent case outlined so far. The conceptual requirements are however the same but the state lattice at each time point will require a separate node for each distinct path through the lattice. In this case a down movement followed by an up movement in the state lattice is different to an up movement followed by a down movement.

The path independent state lattice has n+1 states at the maturity date of the loan where there are n periods in the lattice. For a monthly time interval and a maximum term of 5 years produces 61 state nodes at the maturity date. The path dependent state lattice will have 2^n state nodes at maturity since the states will double over each time interval in the lattice resulting in 2^{61} ultimate states. This presents a computational problem. The way around this is to use simulation to value the option.

The simulation approach would proceed as follows. The loan yield rate could be modelled as a normal or lognormal random variable. Mean-reversion could readily be incorporated. To do this generate standard normal random variables one for each period for the number of simulations to be performed. If 100 simulations are to be carried out then using monthly intervals over a five year period will require 5900 standard normal random variables. The number of simulations required for accuracy would be determined by valuing options for which the value was known, such as traded bond options or swaptions, and checking how close the simulated value was to the accurate value. The number of simulations required for any given accuracy could be reduced using Monte-Carlo techniques such as control-variate techniques.

It is convenient to assume a flat yield curve in the simulation calculations. It would be possible to simulate values for one period forward rates for all the maturities of the yield curve required in the calculations. This would require a much higher number of simulated values and a longer computation time.

Generate the interest rates using the following formulae. The current interest rate level will be $r(0)$ determined from fitting the model parameters to the current yield curve and other market data. Generate the interest rate level at the end of each of the required months using

Normal distribution:

$$r(t) = r(t-1) + \sigma_N \sqrt{h e(t)}$$

where $e(t)$ is normal $(0, 1)$ random number, h is the time interval in years (for monthly values this is $\sqrt{(1/12)} = .288675$) and σ_N is the annual volatility of the one period interest rate in absolute terms.
Log-normal distribution:

$$r(t) = r(t-1)(1 + \sigma_L \sqrt{h e(t)})$$

where σ_L is the percentage volatility of the interest rate so that

$$\sigma_N = r(t) \times \sigma_L .$$

Mean-reversion:

$$r(t) = r(t-1) + k(\mu - r(t-1)) + \sigma_L \sqrt{h r(t-1) e(t)}$$

where μ is the long term mean interest rate, and k is the speed with which the current interest rate tends to move to the long-term interest rate.

For each month generate the market interest rate $r(t)$. Determine the proportion of the principal outstanding to be repaid $q(t, r(t))$. It should consist of

- a monthly proportion varying with the time since issue of the loan and the month of the year (to allow for seasonal effects), and
- a proportion varying with the difference between the current rate $r(t)$ and the original loan rate.

Value the contractual repayments, reduced by the proportion of the loan repaid, at the market interest rate $r(t)$ to obtain the value of the loan $v(t)$. Value the outstanding contractual repayments at the original loan interest rate y to get the balance outstanding $b(t)$. Determine the cash flow from the early prepayment option (could be a cost or benefit) by multiplying the proportion repaying by the difference between the value of the loan $v(t)$ and the loan outstanding $b(t)$. Denote this by $o(t)$ so that $o(t) = q(t, r(t))\{v(t) - b(t)\}$.

Determine the new principal outstanding by reducing for contractual repayments of principal and the proportion who early repay. Determine the reduced contractual repayments.

Continue for each month to the maturity month of the loan.

Present value the early prepayment option cash flows $o(t)$ using the $r(t)$ rates generated for each of the months. If $r(t)$ is a semi-annual compounding rate then the procedure would obtain the value $ov(0)$ for this simulation as follows:

Initialise $ov(n) = 0$
for $i = n - 1$ to 0

$$ov(i) = o(i) + ov(i-1)^*(1 + r(i-1)/200)^{(-2/12)}$$

Repeat this procedure for the desired number of simulations. To determine the prepayment option cost simply calculate the average of the simulated option values $ov(0)$ determined from each of the simulations.

The same random numbers would be used for loans of different terms to produce consistent values. The random numbers for the term of the loan would be selected from the appropriate random number paths.

Models which allow for prepayments for mortgage backed securities have been set out in a number of studies. Examples of US studies include Green and Shoven (1986) and Eduardo S. Schwartz and Walter N. Torous (1989). Prepayment models used in practice are often considered to be proprietary even though the basic form of such models is standard.

The basic form of these models allow for the following components:

(a) a proportion of loan amounts outstanding being prepaid which varies through time and is assumed to be independent of interest rates. This proportion is assumed to increase to a maximum and then remain constant or decline slowly;

(b) an increase in this proportion whenever the difference between the original loan rate and an indicator of market interest rates increases above a threshold margin. This threshold margin reflects refinancing costs;

(c) a reduction in the proportion to the time dependent (non-interest sensitive) prepayment proportions whenever the difference between the original loan rate and an indicator of market interest rates exceeds a "burnout" level beyond which it is assumed that all interest sensitive loans will have prepaid.

The indicator of market interest rates is usually taken as the current fixed interest refinancing rate for a similar loan or a previous value of such a market rate such as the minimum rate since issue of the loan or the rate three months previously. Another alternative is an average of several previous months market interest rates.

The usual form of these prepayment models can be written as follows:

$$q(t,r) = q(t)k(r)$$

where $q(t)$ is the interest rate independent prepayment proportion which depends only on the time since issue of the loan and $k(r)$ is a function of the market interest rates which gives the proportionate increase in $q(t)$ resulting from the financial incentive to refinance for those borrowers who are considered to be interest sensitive.

There are a multitude of formulae that can be used for $q(t)$ and $k(r)$ and the aim should be to select a formula that is capable of modelling a range of possibilities and which depends on as few parameters as possible.

All of these prepayment models apply to a pool of loans so that they are considered to be the average percentages of loans which prepay as a percentage of the balance outstanding for a large group of similar loans which were issued at the same time and for the same fixed interest rate. They do not apply directly to a single loan since in most cases a single loan will either fully prepay or will continue with the contractual payments. Pricing and hedging of such contracts will be dependent on a large enough volume of similar loans being issued so that the prepayment functions can be considered as expected values. For smaller volumes of similar loans the prepayment percentages will vary from the model and it is important to analyse the sensitivity of any pricing or hedging to the prepayment model in the light of the expected statistical variation in such prepayment rates.

The form of the prepayment model should ideally be estimated from available data. It will be important to examine the sensitivity of any pricing to the prepayment assumption before adopting a particular model. The following formula captures a range of prepayment patterns. Use $q(t,r) = q(t)k(r)$ with

$$q(t) = (gp)(gt)^{(p-1)}/\{1+(gt)^p\}$$

where g and p are parameters and t is the number of months since issue of the loan ($t = 0, 1, \ldots, 59$ for a 5 year loan).

Parameters of $g = 0.008$ and $p = 1.3$ give $q(t)$ proportions similar to those of the PSA standard for USA mortgage backed securities. Values of $g = 0.013$ and $p = 1.9$ give proportions approximately twice those of the PSA standard. The $q(t)$ proportions can be calculated as multiples of the PSA standard by following these two steps:

(i) convert them to annual equivalents using the formula

$$cpr(t) = \left\{1 - [1 - q(t)]^{12}\right\}$$

and

(ii) multiply by

$500/t$ for $t < 30$

or 16.67 for $t \geq 30$.

The prepayment proportions should be examined visually in a graph of $q(t)$ versus time to compare one set of assumptions against another.

The form of $k(r)$ can be specified in many ways. The following is one possible approach. Other specifications of $k(r)$ are possible.

Define the following inputs:

(i) $a(t)$ as the moving average of the market refinancing fixed rates $r(t)$ generated using the simulation model for the previous x months. Select x equal to six months in order to approximate the three month lag typically found in studies of interest rate sensitivity of prepayments. For the first six months $a(t)$ will be the average of all of the monthly rates available until a period of six months has passed.

(ii) r_l as the threshold point equal to the difference required between the original loan rate y and the average of market rates at time t, $a(t)$, in order for prepayments to be influenced by falling interest rates.

(iii) r_h as the burnout point equal to the difference required between the original loan rate and the six month average of market rates at which point all interest sensitive prepayments will be assumed to have occurred.

(iv) $b = \ln(1 + k/100)$ where ln is the natural logarithm and k is the maximum percentage increase in the prepayment proportion assumed to occur at the burnout point.

Calculate $k(r)$ as follows:

if the original loan rate minus the moving average market rate at time t is greater than the threshold point and less than the burnout point (i.e. $r_l < \{y - a(t)\} < r_h$), and if the current moving average market rate is lower than the minimum of all of its previous values (i.e.

$a(t) >$ minimum $\{a(i), i < t\})$ then

$$k(r) = \exp\left[b\{(y - a(t))/r_h\}\right]$$

otherwise

$$k(r) = 1.$$

Suitable values for the input parameters need to be determined based on any repayment experience available. It would also be possible to model a slowing down of the rate of prepayments whenever prepayments over the life of the loans issued at a particular time have been higher than expected. To do this the prepayment proportion $q(r, t)$ would be multiplied by the factor

$$f(r, t) = \exp\Big(- c\{\ln(ob(t)/sb(t))\}\Big)$$

where $ob(t)$ is the actual outstanding balance outstanding on the loans issued and $sb(t)$ is the scheduled balance outstanding. This factor is included in some of the US studies of mortgage backed securities.

The simplest procedure for determining the parameters is to select the values of g, p, b, r_l, r_h which fit the available or expected loan experience data "best". This loan experience could be based on forecast experience or on historical data. The historical data required would be

- the month of issue
- the fixed interest rate
- the actual balances outstanding for each month since issue
- the contractual balances outstanding had the loan followed the contractual repayment pattern for each month since issue
- the market interest rate for each month since issue.

The best fit could be determined using a number of techniques of which least squares would be the most straightforward. US studies have used other statistical techniques such as maximum likelihood.

If this data is readily available and it appears that the prepayment assumption has a significant financial effect on the costing then it would be advisable to analyse any available data. This would be most useful for estimating the $q(t)$ proportions rather than the sensitivity of prepayments to interest rates given by $k(r)$.

4. Risk statistics

As with any portfolio of interest sensitive financial contracts it is useful for the management of the portfolio to determine the sensitivities of the portfolio value to the underlying factors on which the value is based. For options these sensitivities, or risk statistics, are the delta, gamma and theta. Since this loan contract is an interest rate related instrument it is also useful to evaluate interest rate related risk statistics such as duration and convexity.

The prepayment option is an American style option and to evaluate risk statistics it is necessary to use a numerical technique. The algorithms for determining the risk statistics for the rational prepayment option and the risk statistics for the prepayment option incorporating the non-rational exercise and non-exercise of the prepayment option are set out here. The underlying asset is taken to be the loan.

4.1. Option delta

For the rational prepayment option calculate the delta using the following lattice values:

$$\text{approximate delta } = ov(2,2) - ov(2,1)/\big(v(2,2) - v(2,0)\big)\,.$$

For the non-rational simulation option value:

$$\text{approximate delta } = ov^*(0) - ov(0)/\{v^* - v\}$$

where $v^* = v(1.0001)$ is the current value of the loan v increased by .01% and ov^* is the option value using the identical random numbers $e(t)$ for ov but with a starting market interest rate $r^*(0)$ determined by equating the value of the outstanding contractual loan prepayments to v^*.

4.2. Option Gamma

For the rational prepayment option use the lattice values:

$$\begin{aligned}\text{approximate gamma } &= \Big[\big\{ov(2,2) - ov(2,0)/(v(2,2) - v(2,0))\big\}+ \\ &\quad - \big\{ov(2,0) - ov(2,-2)/(v(2,0) - v(2,-2))\big\}\Big]/h\end{aligned}$$

where $h = \{v(2,2) - v(2,-2)\}/2$.

For the non-rational simulation option value:

$$\text{approximate gamma} = \{ov^*(0) - 2ov(0) + ov^{**}(0)\}/\{v^* - v\}^2$$

where ov^* and v^* are as for the option delta and $v^{**} = 0.9999v$ so that ov^{**} is the option value corresponding to a starting market interest rate which equates the outstanding contractual loan repayments, adjusted for any previous early proportion of prepayments, to v^{**} using the same set of random numbers used to calculate the option value.

4.3. Option theta

For the rational prepayment option use the lattice values:

$$\text{approximate theta} = \{ov(2,0) - ov(0,0)\}/2h$$

where $h = 1/12$.

For the non-rational simulation option:

$$\text{approximate theta} = \{ov^*(1) - ov(0)\}/h$$

where $ov^*(1)$ is the value of the option at the end of the following month when the loan term will be reduced by one month and allowance is made for the contractual repayments due over the next month. The same random numbers are used in the calculations as when calculating $ov(0)$.

4.4. Option vega

For the rational prepayment option it will be necessary to relate the option sensitivity to the spot rate volatility which is an input in the calculations. It is not possible to derive the option vega from the lattice values and it must be approximated by recalculation of the option value for a small change in the input volatilities. The same approximation can be used in the simulation calculations.

$$\text{approximate vega} = ov(\sigma 1.01) - ov(0)/(\sigma .01)$$

where $ov(\sigma 1.01)$ is the option value calculated using a volatility of 1.01 times the volatility used to derive the option value $ov(0)$.

4.5. Option duration

Option deltas given above measure the sensitivity of the option value to changes in the value of the underlying loan. They can be interpreted as hedge ratios if the option is to be replicated using loan instruments with the same cash flow and value characteristics. Hedging could also be considered using financial instruments with different cash flow characteristics to the loan but with similar sensitivity to general levels of interest rates. The option duration is a measure of the sensitivity of the change in the value of the option to changes in interest rates used to determine the value of the loan.

The loan option duration can be estimated using the formula

$$\text{prepayment option duration} = D \text{ delta } \{v(0,0)/ov(0,0)\}$$

where D is the modified duration of the outstanding loan repayments.

It can also be approximated using the formula

$$\{ov(r+) - ov\}/(0.0001^{*}ov)$$

where $ov(r+)$ is the option value calculated for a 1 basis point increase in the interest rates input into the option calculations.

4.6. Option Convexity

The prepayment option convexity can also be approximated numerically by recalculating the option value for a 1 basis point decrease in interest rates as well as a 1 basis point increase and using the approximate formula

$$\big\{ov(r+) - 2ov + ov(r-)\big\}/\big\{(0.0001)^2 ov\big\}$$

5. Hedging

The ideal hedge instrument for the prepayment option is effectively an American style call option on an amortising reducing term swap. Any alternative hedging strategies will involve either over-hedging or under-hedging. The amortisation of the swap would need to correspond to the prepayment pattern of the loan. In the case of interest only loans the ideal hedge for the prepayment option would be an American style physical bond option for the term of the loan.

The ideal hedge instrument might not exist and so it is necessary to consider alternate ways of hedging the option. The actual hedging instruments can also be used to price the option since the market price of the hedge instruments which perfectly hedge the option will be the cost of the prepayment option.

In order to evaluate alternative hedging methods it is necessary to consider the pricing basis as well. If the prepayment option is to be priced on a rational exercise basis and this is the price charged to the borrower then the rational option cash flows determined using the lattice approach are the appropriate basis for determining the required hedging. If the non-rational value of the option is to be estimated and charged then it will be necessary to attempt to hedge the expected cash flows based on assumed exercise proportions and the accuracy of the hedge will depend on the accuracy of the estimate of the proportions prepaying. Some assessment of the basis risk of any hedging strategy will also need to be made.

A simple approach to hedging the prepayment option would be to use an immunisation or dynamic hedging approach. Hedge instruments would be chosen to match the prepayment option duration and convexity. Each month the portfolio of hedge instruments would be rebalanced to match the altered duration and convexity of the prepayment option. This strategy would have a high level of basis risk since it would rely on the accuracy of the estimates of interest rate sensitivity (in the form of duration and convexity) of the hedge instruments and the prepayment option. In theory any interest sensitive hedge instrument for which an estimate of the duration and convexity could be calculated could be used. This would include bond options, swaptions, caps and floors. The selection of hedge instrument would be based on liquidity and depth of the market for the instrument, transactions costs and other market related factors.

A better approach would be to attempt to match the cash flows of the prepayment option more exactly. Although the prepayment option is American style, most over the counter interest rate options are European style. A portfolio of European style options will not provide a good match to the prepayment option cash flows unless it is possible to sell the European options when it is optimal to exercise early. Even so the sale of the European style options will be for less than the payoff on early exercise of the prepayment option since American style options will usually have higher values than European options.

Even when American style options are available these are not usually on an underlying instrument which is equivalent to the loan. Using American style options will involve constructing a portfolio of options on different underlying instruments. Such a portfolio of options is unlikely to exactly replicate the prepayment option since it will not recognise the interdependencies of the underlying instruments that is needed when they are put together to form the loan cash flows. Purchasing a portfolio of options will involve paying too much for the hedge.

6. Conclusions

This paper has set out the computational algorithms required to value and analyse the prepayment risk of certain loan contracts. It has used a simple example to illustrate the techniques. The computational algorithms have been presented in a form that can be readily implemented rather than in the mathematical form that these term structure models are usually presented.

These models are relatively easy to implement, as is hopefully demonstrated in this paper, and should be used to assess interest rate related options in assets and liabilities.

7. Appendix One
Price and Yield Volatility

Let $P(r)$ be the price of security at yield r. The current values of P and r are known. The absolute volatility of P and r will be denoted by $\sigma(P)$ and $\sigma(r)$ respectively. The percentage volatility of P and r are simply $\sigma(P)/P$ and $\sigma(r)/r$. The relationship between price and yield volatility can be approximated by using Ito's Lemma and assuming a diffusion process for both P and r. In this case

$$dP = u(P)dt + \sigma(P)dZ$$

and

$$dr = u(r)dt + \sigma(r)dZ$$

and since $P = P(r)$ then

$$dP = \{P_r u(r) + 1/2 P_{rr}\sigma^2(r)\}dt + P_r\sigma(r)dZ$$

where P_r denotes the partial derivative of the price with respect to the interest rate.

It then follows that

$$\sigma(P) = P_r \sigma(r)$$

and since the modified duration of a security is defined as $-P_r/P$ this gives

$$\sigma(P) = PD\sigma(r)$$

where D is the modified duration of the security.
Hence
absolute price volatility = price × modified duration × absolute yield volatility.

We also have

$$\sigma(P)/P = D\{\sigma(r)/r\}r$$

or
percentage price volatility = modified duration × percentage yield volatility × yield

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