Chapter 6: Numeration Systems and Number Theory Prime Factorization
Book Title: Mathematical Excursions
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Prime Factorization

The **prime factorization** of a composite number is a factorization that contains only prime numbers. Many proofs in number theory make use of the following important theorem.

The Fundamental Theorem of Arithmetic

Every composite number can be written as a unique product of prime numbers (disregarding the order of the factors).

To find the prime factorization of a composite number, rewrite the number as a product of two smaller natural numbers. If these smaller numbers are both prime numbers, then you are finished. If either of the smaller numbers is not a prime number, then rewrite it as a product of smaller natural numbers. Continue this procedure until all factors are primes. In Example 4 we make use of a *tree diagram* to organize the factorization process.

Example 4

Find the Prime Factorization of a Number

Determine the prime factorization of the following numbers.

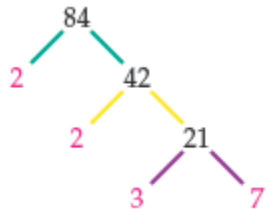
a. 84

b. 495

c. 4004

Solution

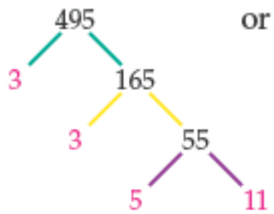
a. The following tree diagrams show two different ways of finding the prime factorization of 84, which is $2 \cdot 2 \cdot 3 \cdot 7 = 2^2 \cdot 3 \cdot 7$. Each number in the tree is equal to the product of the two smaller numbers below it. The numbers (in red) at the extreme ends of the branches are the prime factors.



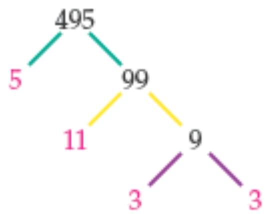
or

$84 = 2^2 \cdot 3 \cdot 7$

b.

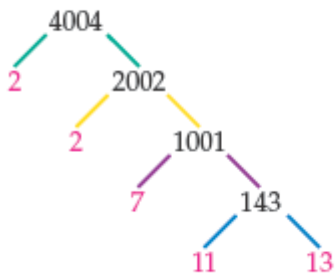


or



$495 = 3^2 \cdot 5 \cdot 11$

c.



$4004 = 2^2 \cdot 7 \cdot 11 \cdot 13$

Take Note

The TI-83/84 program listed in The Truth Table for the Conditional $p \rightarrow q$ can be used to find the prime factorization of a given natural number less than 10 billion.

Take Note

The following compact division procedure can also be used to determine the prime factorization of a number.

$$\begin{array}{r|l} 2 & 4004 \\ \hline 2 & 2002 \\ \hline 7 & 1001 \\ \hline 11 & 143 \\ \hline & 13 \end{array}$$

In this procedure, we use only prime number divisors, and we continue until the last quotient is a prime number. The prime factorization is the indicated product of all the prime numbers, which are shown in red.

Check Your Progress 4

Determine the prime factorization of the following numbers.

a. 315

b. 273

c. 1309

Math Matters

Srinivasa Ramanujan

Srinivasa Ramanujan (1887–1920)

The Granger Collection, NY

On January 16, 1913, the 26-year-old Srinivasa Ramanujan (Rä-mä'noo-jûn) sent a letter from Madras, India, to the illustrious English mathematician G.H. Hardy. The letter requested that Hardy give his opinion about several mathematical ideas that Ramanujan had developed. In the letter Ramanujan explained, “I have not trodden through the conventional regular course which is followed in a University course, but I am striking out a new path for myself.” Much of the mathematics was written using unconventional terms and notation; however, Hardy recognized (after many detailed readings and with the help of other mathematicians at Cambridge University) that Ramanujan was “a mathematician of the highest quality, a man of altogether exceptional originality and power.”

On March 17, 1914, Ramanujan set sail for England, where he joined Hardy in a most unusual collaboration that lasted until Ramanujan returned to India in 1919. The following famous story is often told to illustrate the remarkable mathematical genius of Ramanujan.

> After Hardy had taken a taxicab to visit Ramanujan, he made the remark that the license plate number for the taxi was “1729, a rather dull number.” Ramanujan

immediately responded by saying that 1729 was a most interesting number, because it is the smallest natural number that can be expressed in two different ways as the sum of two cubes.

$$1^3 + 12^3 = 1729$$

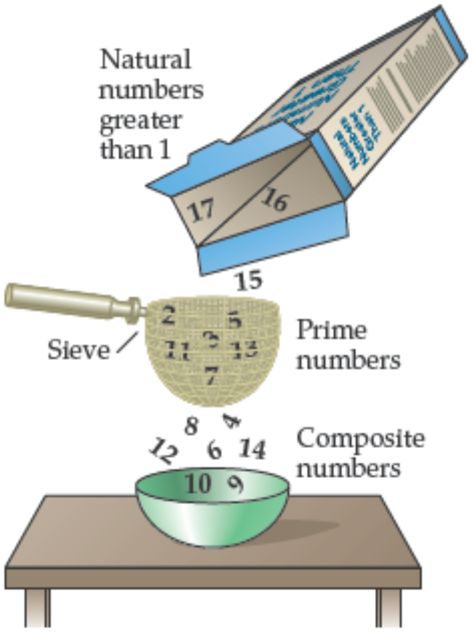
and

$$9^3 + 10^3 = 1729$$

AP/Wide World photos

An interesting biography of the life of Srinivasa Ramanujan is given in *The Man Who Knew Infinity: A Life of the Genius Ramanujan* by Robert Kanigel. *

It is possible to determine whether a natural number n is a prime number by checking each natural number from 2 up to the largest integer not greater than $\sqrt{n}$ to see whether each is a divisor of n. If none of these numbers is a divisor of n, then n is a prime number. For large values of n, this division method is generally time consuming and tedious. The Greek astronomer and mathematician Eratosthenes (ca. 276–192 BC) recognized that multiplication is generally easier than division, and he devised a method that makes use of multiples to determine every prime number in a list of natural numbers. Today we call this method the *sieve of Eratosthenes.*



Take Note

To determine whether a natural number is a prime number, it is necessary to consider only divisors from 2 up to the square root of the number because every composite number n has at least one divisor less than or equal to $\sqrt{n}$. The proof of this statement is outlined in Exercise 77 of this section.

To sift prime numbers, first make a list of consecutive natural numbers. In Table 6.14, we have listed the consecutive counting numbers from 2 to 100.

- Cross out every multiple of 2 larger than 2. The next smallest remaining number in the list is 3. Cross out every multiple of 3 larger than 3.
- Call the next smallest remaining number in the list k. Cross out every multiple of k larger than k. Repeat this step for all $k < \sqrt{100}$.

Table 6.14

The Sieve Method of Finding Primes

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~

~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

Point of Interest

In article 329 of *Disquisitiones Arithmeticae*, Gauss wrote:

> “The problem of distinguishing prime numbers from composite numbers and of resolving the latter into their prime factors is known to be one of the most important and useful in arithmetic …. The dignity of the science itself seems to require that every possible means be explored for the solution of a problem so elegant and so celebrated.” (*Source:* Paulo Ribenboim, *The Little Book of Big Primes,* New York: Springer-Verlag, 1991)

The numbers in blue that are not crossed out are prime numbers. Table 6.14 shows that there are 25 prime numbers smaller than 100.

Over 2000 years ago, Euclid proved that the set of prime numbers is an infinite set. Euclid’s proof is an *indirect proof* or a *proof by contradiction.* Essentially, his proof shows that for any finite list of prime numbers, we can create a number T, as described below, such that any prime factor of T can be shown to be a prime number that is not in the list. Thus there must be an infinite number of primes because it is not possible for all of the primes to be in any finite list.

Euclid’s Proof Assume that *all* of the prime numbers are contained in the list $p_1, p_2, p_3, \ldots, p_r$. Let $T = (p_1 \cdot p_2 \cdot p_3 \cdots p_r) + 1$. Either T is a prime number or T has a prime divisor. If T is a prime, then it is a prime that is not in our list and we have reached a contradiction. If T is not prime, then one of the primes $p_1, p_2, p_3, \ldots, p_r$ must be a divisor of T. However, the number T is not divisible by any of the primes $p_1, p_2, p_3, \ldots, p_r$ because each p_i divides $p_1 \cdot p_2 \cdot p_3 \cdots p_r$ but does not divide 1. Hence any prime divisor of T, say p, is a prime number that is not in the list $p_1, p_2, p_3, \ldots, p_r$. So p is yet another prime number, and $p_1, p_2, p_3, \ldots, p_r$ is not a complete list of all the prime numbers.

We conclude this section with two quotations about prime numbers. The first is by the illustrious mathematician Paul Erdös (1913–1996), and the second by the mathematics professor Don B. Zagier of the Max-Planck Institute, Bonn, Germany.

> It will be millions of years before we'll have any understanding, and even then it won't be a complete understanding, because we're up against the infinite.—P. Erdös, about prime numbers in *Atlantic Monthly*, November 1987, p. 74. *Source:* http://www.mlahanas.de/Greeks/Primes.htm.

In a 1975 lecture, D. Zagier commented,

> There are two facts about the distribution of prime numbers of which I hope to convince you so overwhelmingly that they will be permanently engraved in your hearts. The first is that, despite their simple definition and role as the building blocks of the natural numbers, the prime numbers belong to the most arbitrary and ornery objects studied by mathematicians: they grow like weeds among the natural numbers, seeming to obey no other law than that of chance, and nobody can predict where the next one will sprout. The second fact is even more astonishing, for it states just the opposite: that the prime numbers exhibit stunning regularity, that there are laws governing their behavior, and that they obey these laws with almost military precision. (*Source:* Don B. Zagier, "The first 50 million prime numbers," *The Mathematical Intelligencer*, 1977)

Don B. Zagier

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