Primer on Geometric Algebra
for introductory mathematics and physics

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Galileo's Manifesto for Science

"To be placed on the title-page of my collected works:

> "Here it will be perceived from innumerable examples what is the use of mathematics for judgment in the natural sciences, and how impossible it is to philosophize correctly without it . . .
>
> "Philosophy is written in that great book which ever lies before our eyes — I mean the universe — but we cannot understand it if we do not first learn the language and characters in which it is written. This language is mathematics, and the characters are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth."

Primer on Geometric Algebra

OUTLINE

I. Prolog: On optimizing the design of introductory mathematics.

Physics teachers are universally dismayed by the paltry understanding of mathematics that students bring from their mathematics courses. Blame is usually laid on faulty teaching. But I hold that the crux of the problem is deeply embedded in the curriculum. From the perspective of a practicing scientist, *the mathematics taught in high school and college is fragmented, out of date and inefficient*!

The central problem is found in high school geometry. Many schools are dropping the course as irrelevant. But that would be a terrible mistake for reasons already clear to Galileo at the dawn of science.

- Geometry is the starting place for physical science, the foundation for mathematical modeling in physics and engineering and for the science of measurement in the real world.

- Synthetic methods employed in the standard geometry course are centuries out of date; they are computationally and conceptually inferior to modern methods of analytic geometry, so they are only of marginal interest in real world applications.

- A reformulation of Euclidean geometry with modern vector methods centered on kinematics of particle and rigid body motions will simplify theorems and proofs, and vastly increase applicability to physics and engineering.

A basic pedagogical principle: *The depth and extent of student learning is critically dependent on the quality of the available mathematical tools.*

Therefore, we can expect a well-designed curriculum based on vector methods to produce significant improvements in the depth, breadth and usefulness of student learning. Further enhancements can be expected from software that facilitates application of vector methods.

Whether or not the high school geometry course can be reformed in practice, the course content deserves to be reformed to make it more useful in applications.

Objective of this workshop: To demonstrate with specific examples how **geometric algebra** unifies high school geometry with algebra and trigonometry and thereby simplifies and facilitates applications to physics and engineering.

References for further study:

D. Hestenes, "Oersted Medal Lecture 2002: Reforming the mathematical language of physics, " Am. J. Phys. **71:** 104-121 (2003).

D. Hestenes, *New Foundations for Classical Physics* (Kluwer, Dordrecht, 1986, 2nd ed. 1999)

Website <modelingnts.la.asu.edu>

II. Standard algebraic tools for linear geometry: Vector Addition and scalar multiplication.

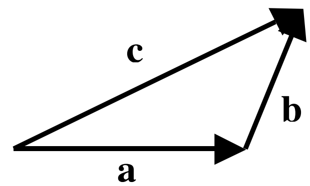
The term *scalar* refers to a *real number or variable,* with properties taken for granted here.

The concept of *vector* is defined by algebraic rules for combining vectors.

In addition, geometric meaning is ascribed to vectors by depicting them as directed line segments.

Thus vector **a** is depicted by 

A. Rules for vector addition.



- **Closure** (The sum of vectors is also a vector)

  a + b = c

Exercise: As appropriate, sketch geometric depictions for the following algebraic rules:

- **Commutative**

  a + b = b + a

- **Associative**

  (a + b) + c = a + (b + c)

- **Additive inverse and zero vector** (depicted by a point)

  a + (–a) = 0

  a + 0 = a

- **Subtraction**

  a – b = a + (–b)

B. Rules for multiplication by scalars (denoted by Greek letters and/or italics)

- **Additive and multiplicative identities**

  $1\mathbf{a} = \mathbf{a}, \qquad (-1)\mathbf{a} = -\mathbf{a}, \qquad 0\mathbf{a} = \mathbf{0}$

- **Distributive**

  $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$

  $(\alpha+\beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$

Example: Repeated vector addition as scalar multiplication

$$\mathbf{a} + \mathbf{a} + \mathbf{a} = (1+1+1)\mathbf{a} = 3\mathbf{a}$$

- **Associative**

  $\alpha(\beta\mathbf{a}) = (\alpha\beta)\mathbf{a}$

- **Commutative**

  $\alpha\mathbf{a} = \mathbf{a}\alpha$

- **Magnitude and direction:** Every vector **a** has a unique scalar *magnitude* $a = |\mathbf{a}|$ and (if $\mathbf{a} \neq 0$) a *direction* $\hat{\mathbf{a}}$ so that

  $\mathbf{a} = a\hat{\mathbf{a}}$

- **Collinearity.** Nonzero vectors **a** and **b** are said to be *collinear* or *linearly dependent* if there is a scalar β such that

  $\mathbf{b} = \beta\mathbf{a}$

- **Linear independence.** Nonzero vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are said to be *linearly independent* if

$$\mathbf{x}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \ldots + \alpha_n\mathbf{a}_n$$

is not zero for any combination of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ (not all zero). The scalars $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ are said to be *coordinates* for the vector $\mathbf{x}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ with respect to the *basis* $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$. The set $\{\mathbf{x}(\alpha_1, \alpha_2, \ldots, \alpha_n)\}$ for all values of the coordinates is an *n-dimensional vector space*.

C. Parametric equations

Exercise: As appropriate, identify and sketch the indicated geometric figures below. When is it necessary to designate a particular point by the zero vector?

- **Line:** $\mathbf{x}(\alpha) = \alpha\mathbf{a} + \mathbf{b}$
  Line segment for $0 \le \alpha \le 1$

- **Plane:** $\mathbf{x}(\alpha,\beta) = \alpha\mathbf{a} + \beta\mathbf{b} + \mathbf{c}$

  Linear constraints: Sketch lines for $\alpha = 1, 2, 3$ and then for $\beta = 1, 2, 3$.

  Quadratic constraints: (Choose the most symmetrical parametric form)

  1. $\alpha^2 + \beta^2 = 1$
  2. $\beta = \alpha^2$
  3. $\alpha^2 - \beta^2 = 1$

  Solutions

  1. $\mathbf{x}(\theta) = \mathbf{a}\cos\theta + \mathbf{b}\sin\theta + \mathbf{c}$
  2. $\mathbf{x}(\alpha) = \mathbf{a}\alpha + \mathbf{b}\alpha^2 + \mathbf{c}$
  3. $\mathbf{x}(\theta) = \mathbf{a}\cosh\theta + \mathbf{b}\sinh\theta + \mathbf{c}$

News Release: Physics Education Research in a large state university found that, after completing a semester of introductory physics, most students were unable to carry out graphical vector addition in two dimensions. The more complex skills of coordinating scalar multiplication with vector addition were not investigated. [Nguyen & Meltzer, *AJP* **71**: 630-638 (2003)]

Question: What are likely reasons for this unacceptable failure of mathematics instruction?

Answers:

- Failure of the math curriculum to provide timely instruction in vector methods.
- Over reliance on coordinate methods in most courses.
- Vectors are only sporadically employed and usually with orthogonal bases, so students have little opportunity to develop fluency with the general features of vector algebra listed above.
- Students are unclear about the geometric interpretation of vectors (see below)
- Vector algebra is incomplete without rules for multiplying vectors that encode information about magnitudes and relative direction (see GA below).

PART I. Introduction to Geometric Algebra and Basic Applications

III. Defining and Interpreting the Geometric Product

The algebraic properties of vector addition and scalar multiplication are insufficient to characterize the geometric concept of a vector as a directed line segment, because they fail to encode the properties of magnitude and relative direction. That deficiency is corrected by defining suitable algebraic rules for multiplying vectors.

Geometric product: The product **ab** for *vectors* **a, b, c** is defined by the rules

- **Associative:** $(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{bc})$
- **Left distributive:** $\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac}$
- **Right distributive:** $(\mathbf{b} + \mathbf{c})\mathbf{a} = \mathbf{ba} + \mathbf{ca}$
- **Euclidean metric:** $\mathbf{a}^2 = a^2$,

where $a = |\mathbf{a}|$ is a positive *scalar* (= real number) called the *magnitude* of **a**.

In terms of the *geometric product* **ab** we can define two other products, a symmetric *inner product*

(1) $\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{ab} + \mathbf{ba}) = \mathbf{b} \cdot \mathbf{a}$,

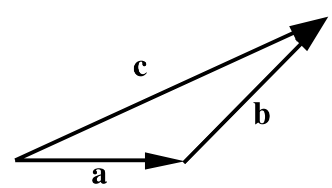
and an antisymmetric *outer product*

(2) $\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{ab} - \mathbf{ba}) = -\mathbf{b} \wedge \mathbf{a}$

Adding (1) and (2) we obtain the fundamental formula

(3) $\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$

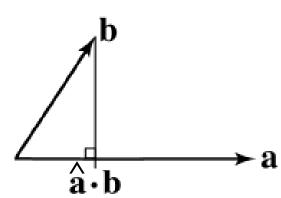
called the *expanded form* for the geometric product. Our next task is to provide geometric interpretations for these three products.



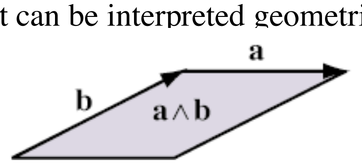
Problem: For a triangle defined by the vector equation **a** + **b** = **c**, derive the standard *Law of Cosines:*

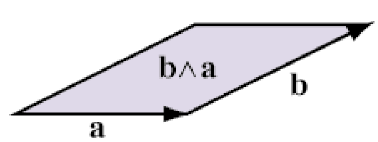
$$a^2 + b^2 + 2\mathbf{a} \cdot \mathbf{b} = c^2,$$

and so prove that the inner product $\mathbf{a} \cdot \mathbf{b}$ is always scalar-valued. Therefore, the inner product can be given the usual geometric interpretation as a perpendicular projection of one line segment on the direction of another.

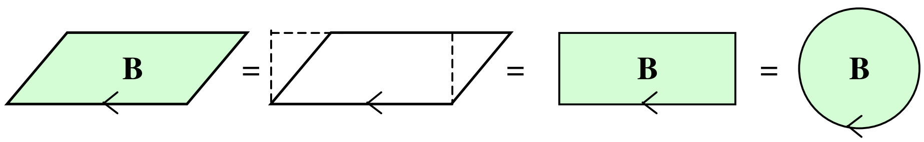


The outer product $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$ generates a new kind of geometric quantity called a **bivector**, that can be interpreted geometrically as **directed area** in the plane of **a** and **b.**





We have shown that the geometric product interrelates three kinds of algebraic entities: *scalars* (0-vectors), *vectors* (1-vectors), and *bivectors* (2-vectors) that can be interpreted as geometric objects of different dimension. Geometrically, scalars represent 0-dimensional objects, because they have magnitude and orientation (sign) but no direction. Vectors represent directed line segments, which are1-dimensional objects. Bivectors represent *directed plane segments*, which are 2-dimensional objects. It may be better to refer to a bivector $\mathbf{B} = B\hat{\mathbf{B}}$ as a *directed area*, because its magnitude $B = |\mathbf{B}|$ is the ordinary area of the plane segment and its direction $\hat{\mathbf{B}}$ represents the plane in which the segment lies, just as a unit vector represents the direction of a line. The shape of the plane segment is not represented by any feature of **B**, as expressed in the following equivalent geometric depictions (with *clockwise* orientation):



Prove the following:

Given any non-zero vector **a** in the plane of bivector **B**, one can find a vector **b** such that

$\mathbf{B} = \mathbf{ba} = -\mathbf{ab}$,

$\mathbf{B}^2 = -|\mathbf{B}|^2 = -a^2b^2$ **,**

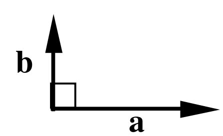
$\mathbf{aB} = -\mathbf{Ba}$, that is, **B** anticommutes with every vector in the plane of **B**.

Every vector **a** has a multiplicative inverse: $\mathbf{a}^{-1} = \dfrac{1}{\mathbf{a}} = \dfrac{\mathbf{a}}{a^2}$

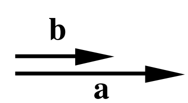
that is, geometric algebra makes it possible to divide by vectors.

Prove the following theorems about *the geometric meaning of commutivity and anticommutivity:*

$\mathbf{a}\bullet\mathbf{b} = 0 \quad \Leftrightarrow \quad \mathbf{ab} = -\mathbf{ba}$ **Orthogonal vectors anticommute!!**



$\mathbf{a} = \lambda\mathbf{b} \quad \Leftrightarrow \quad \mathbf{a} \wedge \mathbf{b} = 0 \quad \Leftrightarrow \quad \mathbf{ab} = \mathbf{ba}$ **Collinear vectors commute!!**



The problem remains to assign geometric meaning to the quantity **ab** without expanding it into inner and outer products.

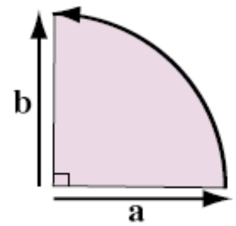
IV. Rotors and rotations in the Euclidean plane

Let **i** denote the *unit bivector for the plane*. As proved above, $\mathbf{i}^2 = -1$, so **i** has the properties of the unit imaginary in the complex number system. However, by relating it multiplicatively to verctors, GA endows **i** with two new **geometric interpretations**:

• **Operator interpretation:** We can assign an orientation to **i** so that left multiplication by **i** rotates vectors in its plane through a right angle. Thus, any vector **a** is rotated into a vector **b** given by the equation

$$\mathbf{ia} = -\mathbf{ai} = \mathbf{b},$$

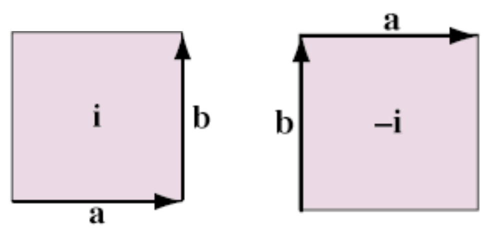
with the action of **i** depicted by the directed arc in the diagram:



• **Object interpretation:** Bivector **i** represents a unit plane segment, as expressed by the following equations and depictions (for $\mathbf{a}^2 = \mathbf{b}^2 = 1$):

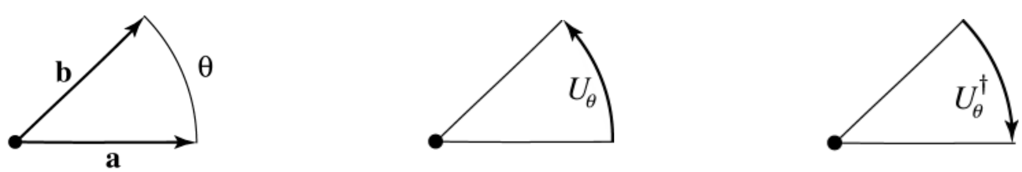
$\mathbf{i} = \mathbf{ba}^{-1}$ (counterclockwise sense)

$-\mathbf{i} = \mathbf{a}^{-1}\mathbf{b}$ (clockwise sense)



The *operator interpretation* of **i** generalizes to the concept of **rotor** U_θ, the entity produced by the product **ba** of unit vectors with relative direction θ.

Rotor $U_\theta = \mathbf{ba}$ is *depicted* as a *directed arc* on the unit circle. **Reverse** $U_\theta^\dagger = \mathbf{ab}$.

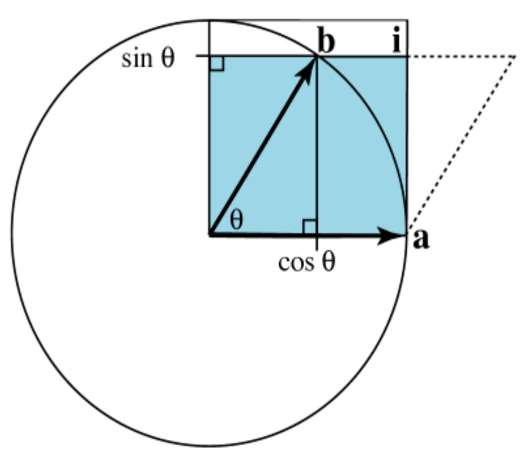


Defining sine and cosine functions from products of unit vectors.

$$\mathbf{a}^2 = \mathbf{b}^2 = 1, \quad \mathbf{i}^2 = -1$$

$$\mathbf{b} \cdot \mathbf{a} \equiv \cos\theta$$

$$\mathbf{b} \wedge \mathbf{a} \equiv \mathbf{i}\sin\theta$$

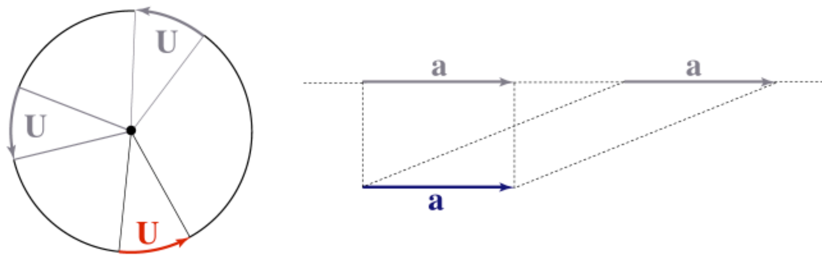


Rotor:

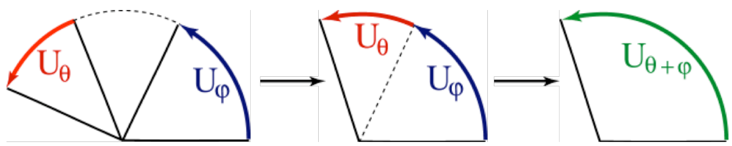
$$U_\theta = \mathbf{ba} = \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \wedge \mathbf{a} = \cos\theta + \mathbf{i}\sin\theta \equiv e^{\mathbf{i}\theta}$$

Rotor equivalence of directed arcs is like

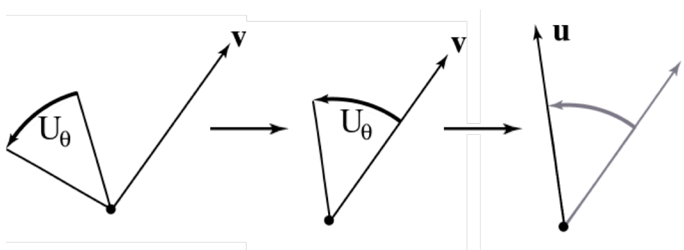
vector equivalence of directed line segments:



Product of rotors is equivalent to addition of angles: $U_\theta U_\varphi = U_{\theta+\varphi}$ or $e^{\mathbf{i}\theta}e^{\mathbf{i}\varphi} = e^{\mathbf{i}(\theta+\varphi)}$.



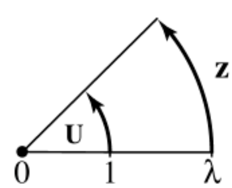
Rotor-vector product = vector: $U_\theta \mathbf{v} = e^{\mathbf{i}\theta}\mathbf{v} = \mathbf{u}$



Thus rotor algebra represents the algebra of 2d rotations !

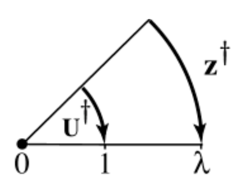
The concept of rotor generalizes to the concept of

complex number *interpreted as a directed arc.*



$z = \lambda U = \lambda e^{\mathbf{i}\theta} = \mathbf{ba}$

Reversion = complex conjugation



$z^\dagger = \lambda U^\dagger = \lambda e^{-\mathbf{i}\theta} = \mathbf{ab}$

$zz^\dagger = \lambda^2 = (\mathbf{ba})(\mathbf{ab}) = \mathbf{a}^2\mathbf{b}^2 = |z|^2$

Modulus: $|z| = \lambda = |\mathbf{a}||\mathbf{b}|$

Relation to standard complex number notation: $z = \mathrm{Re}\,z + \mathbf{i}\,\mathrm{Im}z = \mathbf{ba}$

with $\mathrm{Re}\,z = \frac{1}{2}(z + z^{\dagger}) = \mathbf{b}\cdot\mathbf{a}$ $\quad \mathbf{i}\,\mathrm{Im}\,z = \frac{1}{2}(z - z^{\dagger}) = \mathbf{b}\wedge\mathbf{a}$

(Note: this representation of complex numbers in a real GA is a special case of spinors for 3d).

Note the new GA

Roots of unity: Two roots of –1: $\sqrt{-1} = \pm\mathbf{i}$

Many roots of +1: $\sqrt{+1} = \hat{\mathbf{a}}$, 1.

V. Vector identities and plane trigonometry with GA

Exercise: Prove the following vector identities and show that they are equivalent to trig identities for unit vectors in a common plane.

(a) $(\mathbf{a}\cdot\mathbf{b})^2 - (\mathbf{a}\wedge\mathbf{b})^2 = \mathbf{a}^2\mathbf{b}^2$

(b) $(\mathbf{a}\wedge\mathbf{b})(\mathbf{b}\wedge\mathbf{c}) = \mathbf{b}^2\mathbf{a}\cdot\mathbf{c} - (\mathbf{a}\cdot\mathbf{b})\mathbf{b}\cdot\mathbf{c}$

Hint: Expand **abbc** in two different ways

Exercise: Use rotor products to derive the trigonometric double angle formulas:

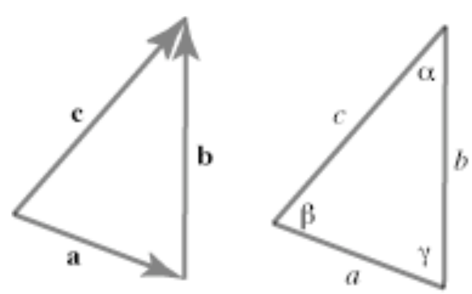
(c) $\cos(\theta + \varphi) =$ **??**

(b) $\sin(\theta + \varphi) =$??

Plane (Euclidean) Trigonometry can be reduced to two basic problems:

(a) Solving a triangle

(b) Solving the circle (composition of rotations).



A triangle relates six scalars: 3 sides (S) & 3 angles (A).

Given 3 of these scalars
(SSS, SAS, SSA, ASA, AAS, AAA)

"**solving the triangle**" consists of determining the other 3 sides.

Laws of the triangle follow directly from the geometric product:

(a) Law of cosines (scaled projection)

$$a^2 + b^2 - 2ab\cos\gamma = c^2$$

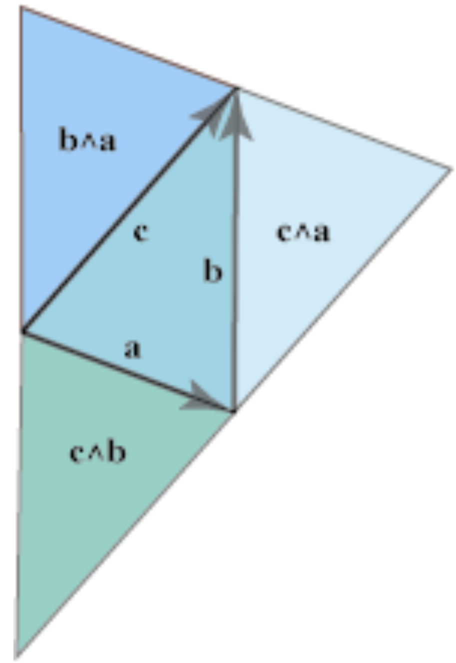
(b) Law of sines (areas)

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

(c) Angle laws:

- Complementary angles (interior/exterior)
- Sum of interior angles
- Sum of exterior angles

Exercise: Derive these laws with GA and interpret with Diagrams, such as the one on the right:



VI. Modeling real objects and motions with vectors.

News Bulletin: The *World Health Organization* has announced a world-wide epidemic of the *Coordinate Virus* in mathematics and physics courses at all grade levels. Students infected with the virus exhibit compulsive *vector avoidance* behavior, unable to conceive of a vector except as a list of numbers, and seizing every opportunity to replace vectors by coordinates. At least two thirds of physics graduate students are severely infected by the virus, and half of those may be permanently damaged so they will never recover. The most promising treatment is a strong dose of *Geometric Algebra*.

It may be surprising that the concept of vector is so difficult for students, since *intuitive notions of direction and distance* are essential for navigating the everyday world. Surely these intuitions need to be engaged in learning the algebraic concept of vector, as they are essential for applications. The necessary engagement occurs only haphazardly in conventional instruction, and that is evidently insufficient for most students.

One barrier to developing the vector concept is the fact that the **correspondence between vector and directed line segment has *many different interpretations*** in modeling properties of real objects and their motions**:**

- *Abstract depiction* of vectors as manipulatable arrows has no physical interpretation, though it can be intuitively helpful in developing an abstract geometric interpretation.

- *Vectors as points* designate *places* in a Euclidean space or with respect to a physical reference frame. Requires designation of a distinguished point (the *origin*) by the zero vector.
- *Position vector* **x** for a particle which can "move" along a *particle trajectory* $\mathbf{x} = \mathbf{x}(t)$ must be distinguished from places which remain fixed.
- *Kinematic vectors,* such as *velocity* $\mathbf{v} = \mathbf{v}(t)$ and *acceleration* are "tied" to particle position $\mathbf{x}(t)$. Actually, they are vector fields defined along the whole trajectory.
- *Dynamic vectors* such as momentum and force representing particle interactions.
- *Rigid bodies*. It is often convenient to use a vector **a** as a 1d geometric model for a rigid body like a rod or a ruler. Its magnitude $a = |\mathbf{a}|$ is then equal to the length of the body, and its direction $\hat{\mathbf{a}}$ represents the body's orientation or, better, its *attitude* in space. The endpoints of **a** correspond to ends of the rigid body, as expressed in the following equation

  $$\mathbf{x}(\alpha) \;=\; \mathbf{x}_0 + \alpha\mathbf{a} \qquad \{0 \le \alpha \le 1\}$$

  for the position vectors of a continuous distribution of particles in the body. Note the crucial distinction between curves (and their parametric equations) that represent particle paths and curves that represent geometric features of physical bodies.

Exercises with *Barycentric coordinates*:

(1) Discuss and sketch values of the parametric equation $\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b}$ with the constraint $\alpha + \beta = 1$.

(2) For the parametric equation $\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$ with $\alpha + \beta + \gamma = 1$, discuss values of the parameters that give vertices, edges and interior points of a triangle.

(3) Discuss the relation of barycentric coordinates to center of mass

VII. High school geometry with geometric algebra

Nonparametric equation for a line $\{\mathbf{x}\}$ through point **a** with direction **u:**

$$(\mathbf{x} - \mathbf{a}) \wedge \mathbf{u} = 0\,.$$

Exercises:

(1) Sketch the line.

(2) Derive an equivalent parametric equation for the line with $\mathbf{u}^2 = 1$.

(3) Find the directed distance **d** from the origin **0** to the line and sketch.

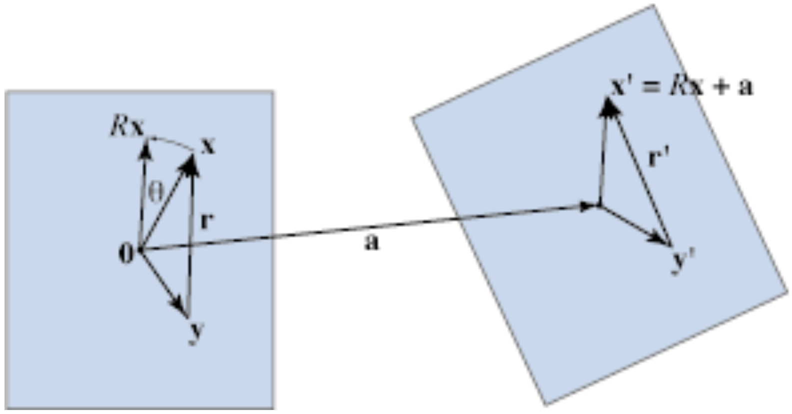
(4) Find the directed distance from an arbitrary point **y** to the line.

Hints:

(2) Write $(\mathbf{x} - \mathbf{a}) \cdot \mathbf{u} = \alpha$ **and solve for** $\mathbf{x} = \alpha\mathbf{u} + \mathbf{a}$

(3) $\mathbf{x}\wedge\mathbf{u}=\mathbf{a}\wedge\mathbf{u}=\mathbf{du}$. Sketch the directed areas and solve for **d**.

Rigid displacements: Congruence and measurement



Equations for a **rigid displacement of particles** in a body or points in a reference frame.

$$\mathbf{x} \;\rightarrow\; \mathbf{x}' = R\mathbf{x} + \mathbf{a} \qquad\qquad R = e^{\mathbf{i}\theta}$$

$$\mathbf{y} \;\rightarrow\; \mathbf{y}' = R\mathbf{y} + \mathbf{a} \qquad\qquad R^\dagger = e^{-\mathbf{i}\theta}$$

Rigid displacement of an interval

$$\mathbf{r} = \mathbf{x} - \mathbf{y} \;\rightarrow\; \mathbf{r}' = \mathbf{x}' - \mathbf{y}' = R(\mathbf{x} - \mathbf{y}') \qquad \text{or} \qquad \mathbf{r}' = R\mathbf{r} = \mathbf{r}R^\dagger$$

Invariants of rigid displacements: Euclidean distance: $(\mathbf{x}' - \mathbf{y}')^2 = (\mathbf{x} - \mathbf{y})^2$

VIII. Basic kinematic models of particle motion

Kinematics: The geometry of motion

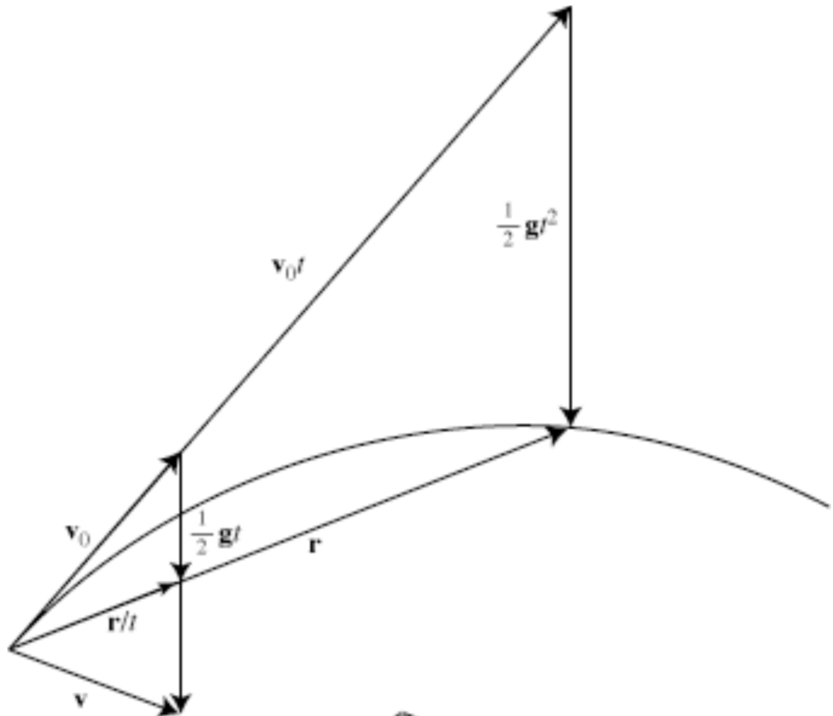
A. Free particle. Sketch a *motion map* for the algebraic model:

$$\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{v}t$$

[Note that including the origin or coordinates in the map introduces arbitrary and unnecessary complications.]

Derive a nonparametric equation for this model, and relate it to angular momentum.

B. Constant acceleration model.
(without coordinates!!)



$$\frac{d\mathbf{v}}{dt} = \mathbf{g} \quad \Rightarrow \quad \mathbf{v} = \mathbf{v}_0 + \mathbf{g}t$$

(*hodograph*)

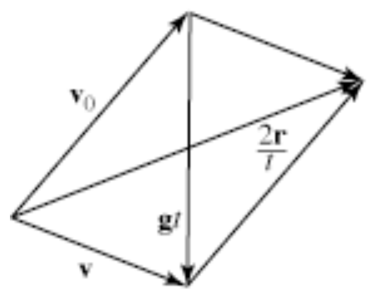
$$\frac{d\mathbf{x}}{dt} = \mathbf{v} \quad \Rightarrow \quad \mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0 t + \tfrac{1}{2}\mathbf{g}t^2$$

(*trajectory*)

Vector algebraic model:

$$\tfrac{1}{2}(\mathbf{v} + \mathbf{v}_0) = \frac{\mathbf{r}}{t} \equiv \overline{\mathbf{v}}$$

$$\mathbf{v} - \mathbf{v}_0 = \mathbf{g}t$$



Reduces all projectile problems to

"*solving a parallelogram!!*"

Problem: Determine
(a) the range r of a target sighted in a direction $\hat{\mathbf{r}}$ that has been hit by a projectile launched with velocity $\mathbf{v}_0$;
(b) launching angle for maximum range;
(c) time of flight

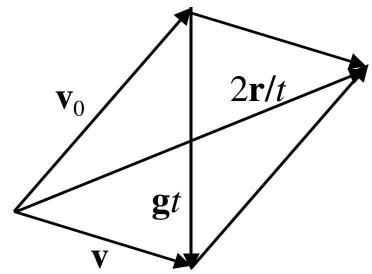
Simplest case: $\hat{\mathbf{r}}$ is horizontal.

Hint: Consider $(\mathbf{v} - \mathbf{v}_0)(\mathbf{v} + \mathbf{v}_0)$

General case: Elevated target.

Complicated solution with rectangular coordinates published in AJP.
Much simpler GA solution in NFCM.

Solving a parallogram with GA:



$$\mathbf{v} - \mathbf{v}_0 = \mathbf{g}t \qquad\qquad \mathbf{v} + \mathbf{v}_0 = \frac{2\mathbf{r}}{t}$$

We eliminate t by multiplication, to get:

$$(\mathbf{v} - \mathbf{v}_0)(\mathbf{v} + \mathbf{v}_0) = 2\mathbf{rg}$$

Expanding, we obtain

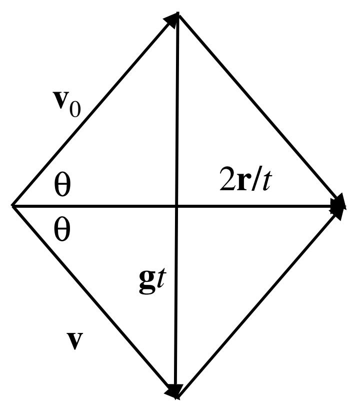
$$v^2 - {v_0}^2 + \underbrace{\mathbf{v}\mathbf{v}_0 - \mathbf{v}_0\mathbf{v}}_{2\mathbf{v}\wedge\mathbf{v}_0} = 2(\mathbf{r}\cdot\mathbf{g} + \mathbf{r}\wedge\mathbf{g})$$

Separately equating scalar and bivector parts, we get:

$$v^2 - {v_0}^2 = 2\mathbf{r}\cdot\mathbf{g} \qquad\qquad \mathbf{v}\wedge\mathbf{v}_0 = \mathbf{r}\wedge\mathbf{g}$$

For $\mathbf{r} = r\hat{\mathbf{r}}$ horizontal, we have

$$\mathbf{r}\bullet\mathbf{g} = 0 \quad \Rightarrow \quad v_0^2 = v^2.$$



So $\quad |\mathbf{v}\wedge\mathbf{v}_0| = |\mathbf{r}\wedge\mathbf{g}| = rg$

gives $\quad v_0^2 \sin 2\theta = rg$

Hence the *range formula*: $\boxed{r = \frac{v_0^2}{g}\sin 2\theta}$

The general GA solution for an elevated target and time of flight is derived in NFCM.

C. Circular motion. Motion on a circle of radius $r = |\mathbf{r}|$ centered at point **c**:

Trajectory: $\mathbf{x} = \mathbf{c} + \mathbf{r}(t)$ [Sketch motion map including unit circle]

Radius vector: $\mathbf{r} = e^{\mathbf{i}\theta(t)}\mathbf{r}_0$

Velocity: $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\Omega}\mathbf{r} = \omega\mathbf{i}\mathbf{r}$

Angle: $\theta > 0$ for counterclockwise motion **Angular speed:** $\omega = \frac{d\theta}{dt}$

Path length: $s = r\theta$ **speed** $v = r\omega$

Rotor: $R = e^{\mathbf{i}\theta}$ **Angular velocity:** $\boldsymbol{\Omega} = \mathbf{i}\omega$

Rotor eqn. of motion: $\frac{dR}{dt} = \boldsymbol{\Omega}R$

UCM $\Rightarrow$ $\theta = \omega t$ $\Rightarrow$ $\mathbf{r} = e^{\mathbf{i}\omega t}\mathbf{r}_0$

D. General kinematic theorem.

x = x(s) = parametric eqn. for *particle path*. [Sketch general motion map]

x = x(t) = parametric eqn. for *particle trajectory*.

Path length: *s = s(t)*

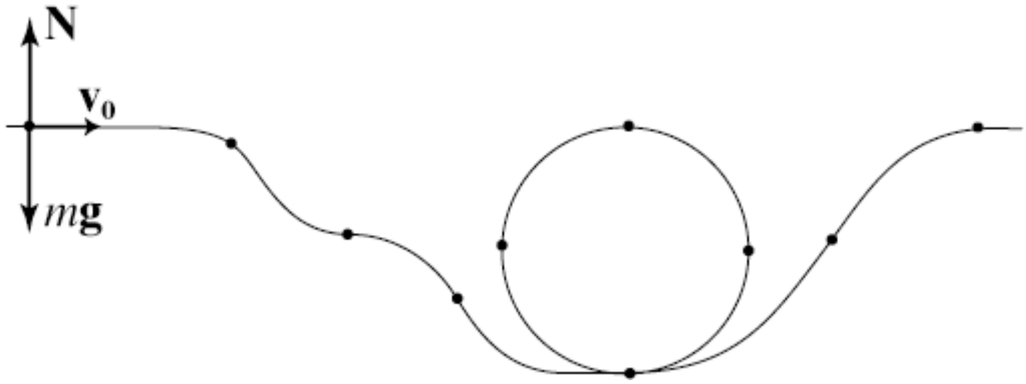
Speed: $v = \frac{ds}{dt}$ **Tangent direction:** $\hat{\mathbf{v}} = \frac{d\mathbf{x}}{ds} = e^{\mathbf{i}\theta}\hat{\mathbf{v}}_0$

Thm: Velocity: $\mathbf{v} = \hat{\mathbf{v}}v$ is tangent to the particle path (trajectory)

Acceleration: $\frac{d\mathbf{v}}{dt} = \hat{\mathbf{v}}\frac{dv}{dt} + \mathbf{i}\mathbf{v}\frac{d\theta}{dt} = \left(\frac{dv}{dt} \pm \mathbf{i}\frac{v^2}{r}\right)\hat{\mathbf{v}}$ points inside the curving path

and the radius *r* of the *osculating circle* is defined by $v = \frac{ds}{dt} = r\frac{d|\theta|}{dt}$

Exercise: A particle slides on the frictionless track below subject to a force $m\mathbf{g} + \mathbf{N} = m\frac{d\mathbf{v}}{dt}$ that keeps it on the track. Sketch its velocity **v** and acceleration **a** at the indicated points:



D. General Keplerian motion (under an inverse square force). Full treatment in NFCM

(1) Dynamical model: $m\dfrac{d\mathbf{v}}{dt} = -\dfrac{k}{r^2}\hat{\mathbf{r}}$

(2) Problem: Reduce to **algebraic model** by finding constants of motion

(a) **Angular momentum:** $\mathbf{L} = m\mathbf{r}\wedge\mathbf{v} = mr^2\hat{\mathbf{r}}\dfrac{d\hat{\mathbf{r}}}{dt}$

(b) **Eccentricity vector $\boldsymbol{\varepsilon}$** in: $\mathbf{L}\mathbf{v} = k(\hat{\mathbf{r}} + \boldsymbol{\varepsilon})$

Hints: (a) $m\mathbf{r}\wedge\dfrac{d\mathbf{v}}{dt} = \dfrac{d}{dt}(m\mathbf{r}\wedge\mathbf{v}) = \mathbf{r}\wedge\mathbf{f}(\mathbf{r}) = 0$ (Central force)

(b) $\mathbf{L}\dfrac{d\mathbf{v}}{dt} = -\dfrac{k\mathbf{L}}{mr^2}\hat{\mathbf{r}} = k\dfrac{d\hat{\mathbf{r}}}{dt}$

(3) Model analysis. Derive the following algebraic features of the model:
[Details are fully worked out in NFCM]

(a) **Energy** $E = \frac{1}{2}m\mathbf{v}^2 - \dfrac{k}{r}$ is constant and related to eccentricity by

$$\varepsilon^2 - 1 = \frac{2L^2E}{mk^2}$$

(b) **Orbit** $r = |\mathbf{r}| = r(\hat{\mathbf{r}})$ for $\mathbf{L} \neq 0$.

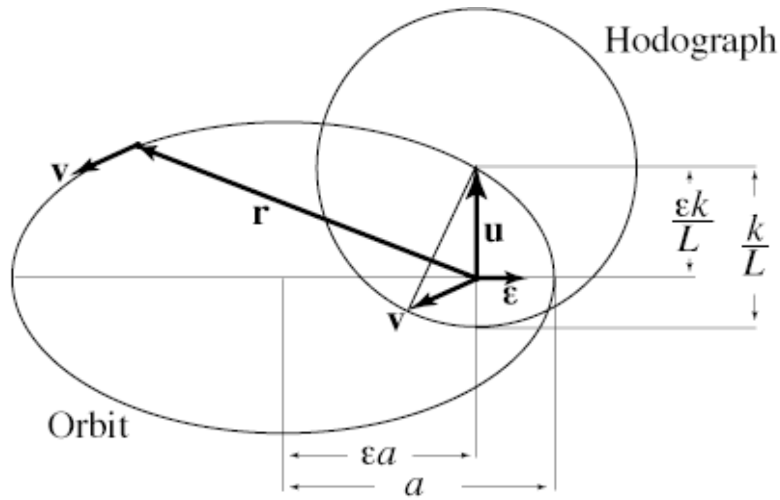
$r = \dfrac{\pm\ell}{1+\boldsymbol{\varepsilon}\cdot\hat{\mathbf{r}}}$ where $\ell = \dfrac{L^2}{m|k|}$ and ± for attractive/repulsive force

[Note: orbit is not a trajectory, which gives position as a function of time.]

(c) **Hodograph** $\mathbf{v} = \mathbf{v}(\hat{\mathbf{r}}) = \frac{k}{\mathbf{L}}(\hat{\mathbf{r}} + \boldsymbol{\varepsilon})$ (orbit in velocity space)

Show this is a circle by deriving the non-parametric eqn. $(\mathbf{v} - \mathbf{u})^2 = \frac{k^2}{L^2}$.

(d) **Classification of orbits:** (by energy, eccentricity & hodograph center **u**)



Elliptical orbit and hodograph

(e) **Initial value problem:** Determine **L** and **ε** given one value of **v** and $\hat{\mathbf{r}}$.

(4) Scattering problem: Given asymptotic initial velocity and angular momentum, find $\mathbf{v}_f = \mathbf{v}_{\text{final}}$

(a) **Asymptotic conditions:**

$$E = \tfrac{1}{2}m\mathbf{v}^2 - \frac{k}{r} \xrightarrow[r\to\infty]{} \tfrac{1}{2}mv_0^2 \qquad \Rightarrow \quad v_o = |\mathbf{v}_0| = |\mathbf{v}_f| = \left(\frac{2E}{m}\right)^{\frac{1}{2}}$$

$$\hat{\mathbf{r}} \wedge \mathbf{v} = \frac{\mathbf{L}}{mr} \xrightarrow[r\to\infty]{} 0 \qquad \Rightarrow \quad \hat{\mathbf{r}}_0 = -\hat{\mathbf{v}}_0, \qquad \hat{\mathbf{r}}_f = \hat{\mathbf{v}}_f$$

(b) **Eccentricity conservation:** $\mathbf{L}\mathbf{v}_2 - k\hat{\mathbf{r}}_2 = \mathbf{L}\mathbf{v}_1 - k\hat{\mathbf{r}}_1$

$$\Rightarrow \quad (\mathbf{L}v_0 - k)\hat{\mathbf{v}}_f = (\mathbf{L}v_0 + k)\hat{\mathbf{v}}_0$$

$$\Rightarrow \quad \mathbf{v}_f = \left(\frac{\mathbf{L}v_0 + k}{\mathbf{L}v_0 - k}\right)\mathbf{v}_0$$

This solves the scattering problem completely**!!**

Remaining problem is to reformulate it in terms of observed quantities:

Scattering angle Θ and impact parameter b defined as follows:

$$\mathbf{v}_f = \mathbf{v}_0 e^{\mathbf{i}\Theta} = e^{-\mathbf{i}\Theta}\mathbf{v}_0$$

$$L = bmv_0 = \frac{2bE}{v_0} \qquad \hat{\mathbf{L}} = \mathbf{i} \quad (k > 0) \quad \text{or} \quad \hat{\mathbf{L}} = -\mathbf{i} \quad (k < 0)$$

$$e^{\mathbf{i}\Theta} = \frac{\mathbf{L}v_0 - k}{\mathbf{L}v_0 + k} = \frac{2Eb\mathbf{i} - |k|}{2Eb\mathbf{i} + |k|} \quad \Rightarrow \quad b = \left|\frac{k}{2E}\right| \cot \tfrac{1}{2}\Theta$$

$\Rightarrow$ **Rutherford scattering formula** for coulomb scattering!

IX. Planar rigid body motion and reference frames.

A **reference system** assigns each particle a definite **position** with respect to a given rigid body (or **reference frame**). The set of all possible **position vectors** is a 3d Euclidean vector space called the **position space** of the frame.

Time dependent rigid displacement of one reference frame with respect to another is completely specified by a time dependent

rotor $R_t = e^{\mathbf{i}\theta(t)}$ and *translation vector* $\mathbf{a} = \mathbf{a}(t)$

Accordingly, a particle path $\mathbf{x} = \mathbf{x}(t)$ in the "*unprimed frame*" is mapped into

a particle path $\mathbf{x}'(t) = R_t\mathbf{x}(t) + \mathbf{a}(t)$ in the "*primed frame*."

Problem: Suppressing the time argument and writing $\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}$, $\dot{R} = \frac{dR}{dt} = \mathbf{\Omega}R$,
derive the following equations relating velocities and accelerations in the two frames aligned at time t.:

$$\dot{\mathbf{x}}' = \dot{\mathbf{x}} + \mathbf{\Omega}\mathbf{x} + \dot{\mathbf{a}}$$

$$\ddot{\mathbf{x}}' = \ddot{\mathbf{x}} + 2\mathbf{\Omega}\dot{\mathbf{x}} + (\dot{\mathbf{\Omega}} + \mathbf{\Omega}^2)\mathbf{x} + \dot{\mathbf{a}}$$

Newton's 1st Law (implicitly) defines an **inertial frame** as a rigid body with respect to which every free particle has constant velocity.

Principle of Relativity requires that the laws of physics are the same in all inertial systems. [First formulated by Galileo and incorporated by Newton as a corollary in his theory].

Problem: Apply this to Newton's 2^{nd} law to prove that any two inertial frames are related by a **Galilean transformation**

$$\mathbf{x} \quad \to \quad \mathbf{x}' = R\mathbf{x} + \mathbf{c} + \mathbf{u}t \qquad \text{where } R, \mathbf{c} \text{ and } \mathbf{u} \text{ are constant.}$$

Derive therefrom the Galilean **velocity addition theorem:** $\mathbf{v}' = \mathbf{v} + \mathbf{u}$.

Exercise: Inside a cable car climbing a slope with constant velocity $\mathbf{v}_0$ an object is dropped from rest. Derive eqns. for the trajectory within the car and with respect to the earth outside.

Problem: Discuss invariance of Newton's first and second laws with respect to **Galilean time translation and scaling**: $t \quad \to \quad t' = \alpha t + \beta$, where α and β are constants.

X. The **Zeroth Law of physics** defines the fundamental presumptions about space, time and existence of real entities that underlie all of physics.

Different versions of the Zeroth Law define different physical theories.
Einstein's Special Theory of Relativity amounts to a modification of the Zeroth Law implicit in Newtonian theory.

Tenets of the Newtonian Zeroth Law:

- Any material object can be modeled as a particle or body {system of particles}
- At any time, every particle has a definite position $\mathbf{x}$ in the 3d Euclidean *position space* of a given *reference system*.
- Particle motion is represented by a continuous trajectory $\mathbf{x}(t)$ in position space. (Time is measured by comparison with a standard moving object called a *clock*.)

Newton's First Law defines an inertial system and a uniform time scale.

The remaining four Laws of Newtonian physics define particle interactions and dynamics.

[Ref. Modeling Games in the Newtonian World, *AJP* **60:** 732 (1992)]

PART II. Special Relativity with Geometric Algebra

[Ref. NFCM, 2nd Ed. (1999), Chap. 9]

XI. Defining Spacetime

Einstein (1905) recognized that Newtonian mechanics is inconsistent with Electromagnetic Theory, and he traced the difficulty to the Newtonian concept of time. He resolved this problem by adopting two principles:

1. **Principle of Relativity.** Einstein adopted this principle from Newtonian theory, but raised its status from a mere corollary to a basic principle.
2. **Invariance of the speed of light.** Einstein assumed that the speed of light c has the same value in all inertial systems.

Minkowsky (1908) incorporated these principles into a new conceptual fusion of space and time that can be defined with GA by the following assumptions:

1. In a given inertial system, the time t and place **x** is of an **event** is represented as a single point $X = ct + \mathbf{x}$ in a 4-dimensional space called **spacetime** (see spacetime maps below).

2. The spacetime **interval**

   $$\Delta X = X_2 - X_1 = c(t_2 - t_1) + (\mathbf{x}_2 - \mathbf{x}_2) = c\Delta t + \Delta\mathbf{x}$$

   between events X_2 and X_1 has an *invariant magnitude* $|\Delta X|$ called the **proper distance** between the events and given by

   $$\Delta X \Delta\tilde{X} = \varepsilon|\Delta X|^2 = (c\Delta t)^2 - (\Delta\mathbf{x})^2 ,$$

   where $\Delta\tilde{X} = c\Delta t - \Delta\mathbf{x}$, and the **signature** ε of the interval has the value 1, 0 or –1, and, respectively, the interval is said to be *timelike, lightlike*, or *spacelike*.

3. "Invariance of the interval" means that

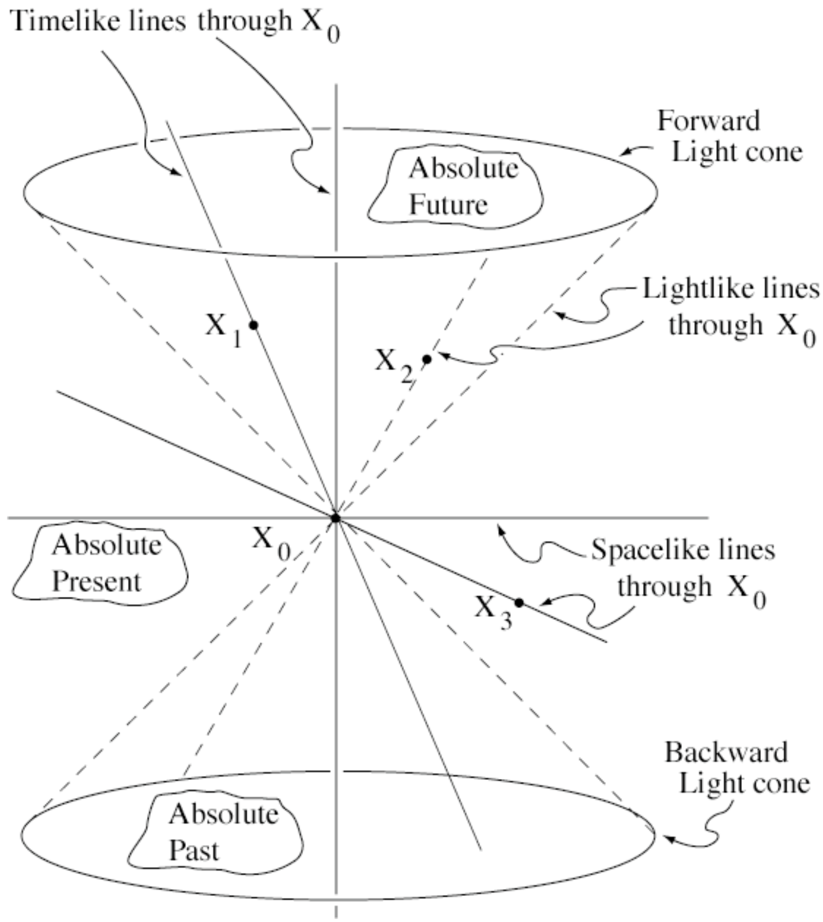
   $$\Delta X \Delta\tilde{X} = \varepsilon|\Delta X|^2 = \varepsilon|\Delta X'|^2 = \Delta X' \Delta\tilde{X}'$$

   wnere $\Delta X' = X_2' - X_1' = c\Delta t' + \Delta\mathbf{x}'$ is the same interval represented in some other inertial system.

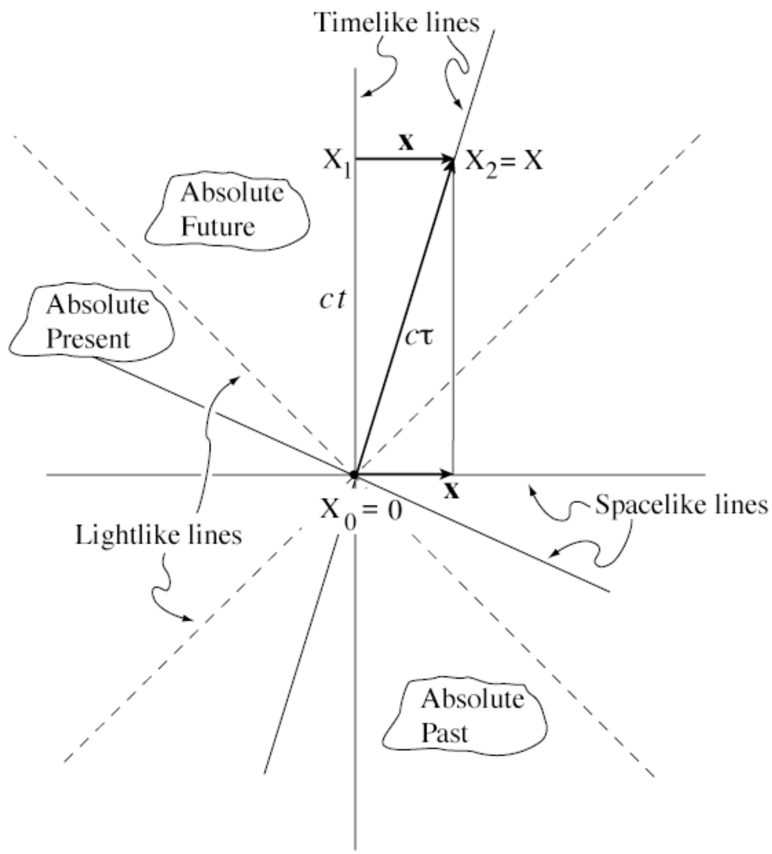
Problem: Prove that spacetime interval invariance implies that the speed of light c has the same constant value in all inertial systems.

XII. Spacetime Maps

Spacetime maps are essential for physical interpretation of GA equations in relativity theory.



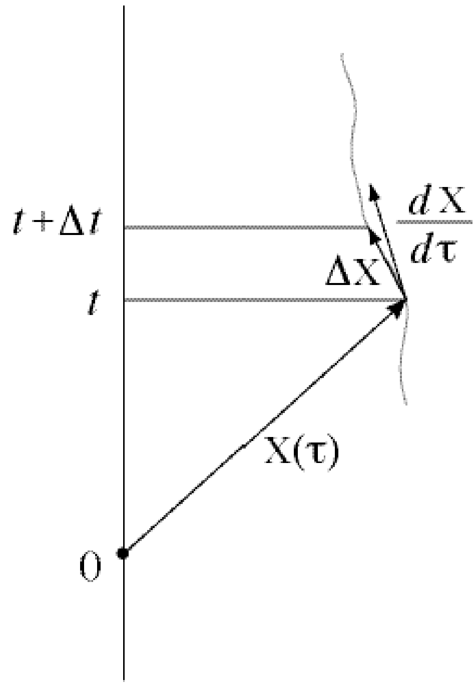
This is a spacetime map showing the lightcone for an event X_0. Events X_k (k = 1, 2, 3) lie on straight lines passing through X_0. Intervals $\Delta X_k = X_k - X_0$ are said to be *timelike, lightlike* or spacelike, respectively, as they lie inside, on, or outside the invariant lightcone.



This is a spacetime map of events in a timelike plane showing the position vector $\mathbf{x} = X_2 - X_1$ for the event $X = ct + \mathbf{x}$ with respect to a given inertial system.

A differentiable curve in spacetime is said to be timelike, lightlike or spacelike, if its tangent is proportional to, respectively, a timelike, lightlike or spacelike interval.

A. Particle history and proper velocity.



The **history** of a material particle is a timelike curve $X = X(\tau)$, as illustrated in the figure.

The path length $\Delta\tau$ of an interval ΔX along the history is the **proper time** interval registered by a clock traveling with the particle. Hence,

$$\Delta X \Delta \tilde{X} = |\Delta X|^2 \approx (c\Delta\tau)^2$$

and, in the limit, $\quad |dX| = c\,d\tau$

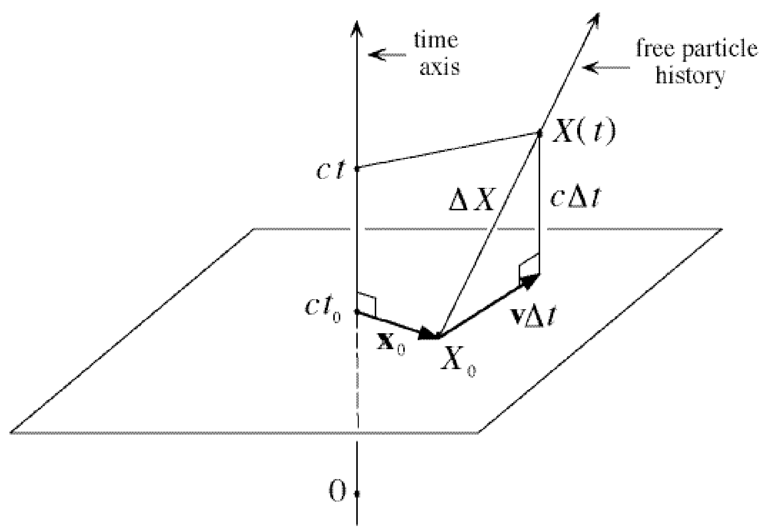
The **proper velocity** $V = V(\tau)$ of a particle is defined by

$$V = \frac{1}{c}\frac{dX}{d\tau} = \frac{dt}{d\tau} + \frac{1}{c}\frac{d\mathbf{x}}{d\tau},$$

which must be distinguished from the **relative velocity**

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} \text{ in a given inertial system.}$$

Exercise: Prove $\quad V = \gamma\left(1 + \frac{\mathbf{v}}{c}\right), \qquad \gamma = \frac{dt}{d\tau} = \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}$



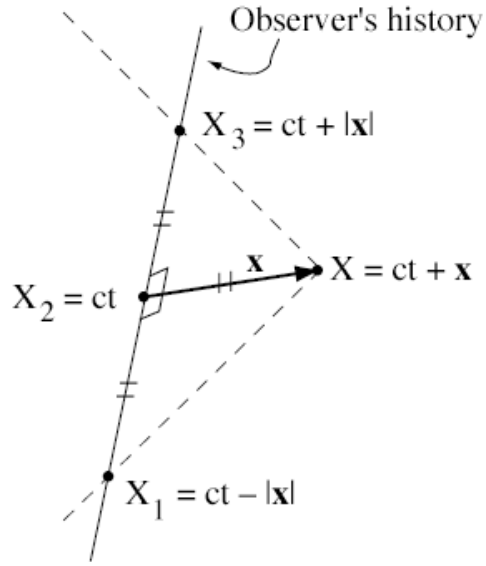
The proper velocity V of a **free particle** is constant, so its history is a straight line given by the equation

$$X(\tau) - X(0) = Vc\tau$$

The time axis of an inertial system is the history of a free particle at rest at the origin so that proper time τ can be identified with "observer time" t, and the position vector $\mathbf{x}$ for an event $X = ct + \mathbf{x}$ is the "directed distance" from the particle history to the event. Similarly,

the time axis of ***every*** *inertial system can be identified with the history of a free particle.*

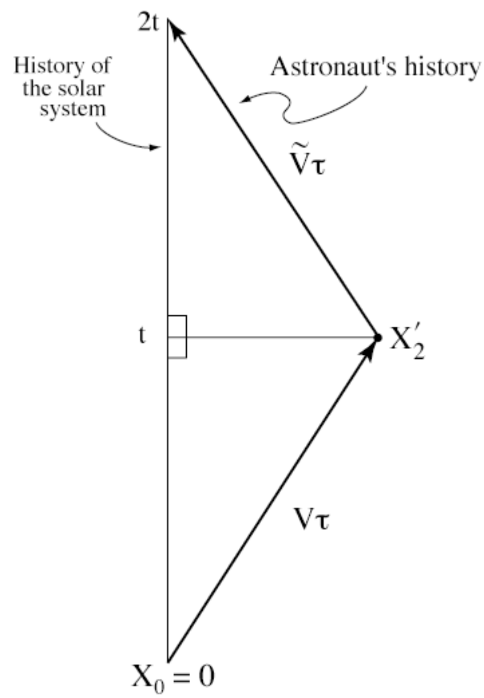
For a given observer (free particle) the position **x** and time *t* of a given event *X* can (in principle) be determined by *radar ranging* with light signals, as illustrated in the figure. Note that the event *X* is simultaneous with event $X_2 = ct$. This geometric construction *is Einstein's operational procedure for* ***synchronizing clocks*** *at distant events*.



VIII. Spacetime Trig.

The most basic measurements in surveying spacetime events can be reduced to solving triangles in a timelike plane, as already shown in the operational procedure for synchronizing clocks.

A. Time dilation and desynchronization.



The basic idea is illustrated by the parable of the twins. As illustrated in the spacetime map for their histories, the astronaut twin travels to a distant star with velocity

$$V = \gamma\left(1 + \frac{\mathbf{v}}{c}\right),$$

and returns with velocity

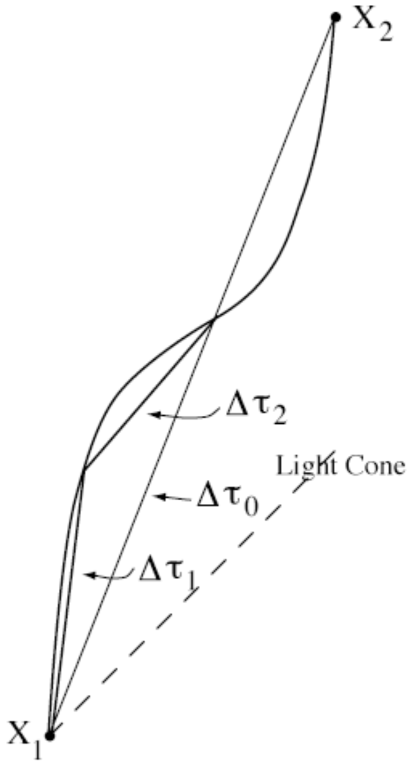
$$\tilde{V} = \gamma\left(1 - \frac{\mathbf{v}}{c}\right).$$

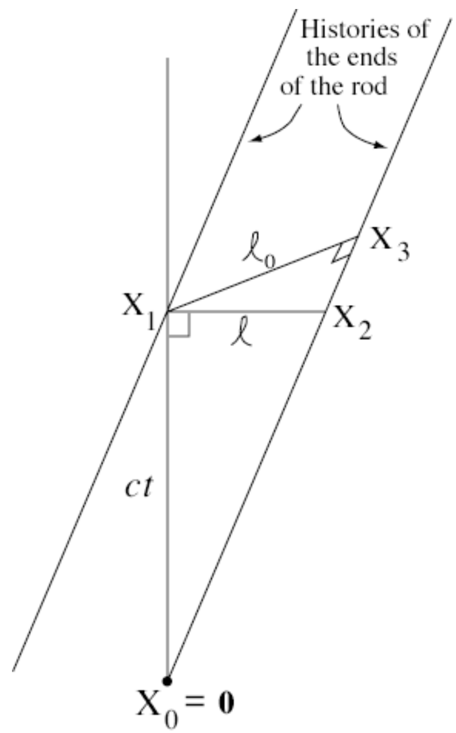
The algebraic equation for the triangle is

$$2t = V\tau + \tilde{V}\tau$$

Exercises:

(1) Compare ages of the twins when the trip is over. Discuss implications of this result.
(2) Prove that the longest path between two points separated by a timelike interval is a straight line. (See next figure)





B. Lorentz contraction: A rod of rest length l_0 moves with velocity **v** who measures its length as l at time t, as shown in the spacetime map.

Exercise: Reasoning from similar triangles in the map,
derive the Lorentz contraction formula

$$l_0 = \gamma l$$

and discuss its physical meaning.

C. Doppler shift.

A distant source with velocity $V = \gamma\left(1 + \frac{\mathbf{v}}{c}\right)$ emits light signals with frequency $f' = \frac{\omega'}{2\pi} = \frac{1}{\Delta t'}$ that are received with frequency $f = \frac{\omega}{2\pi} = \frac{1}{\Delta t}$, as shown in the figure.

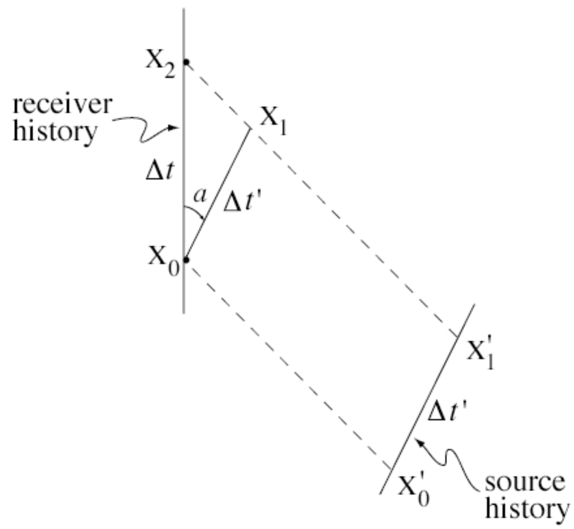
Exercise: From the figure derive the equation

$$\lambda(X_2 - X_1) = D - V\ ,$$

where λ is a scale factor and

$$D \equiv f'/f$$

is the **Doppler factor**.



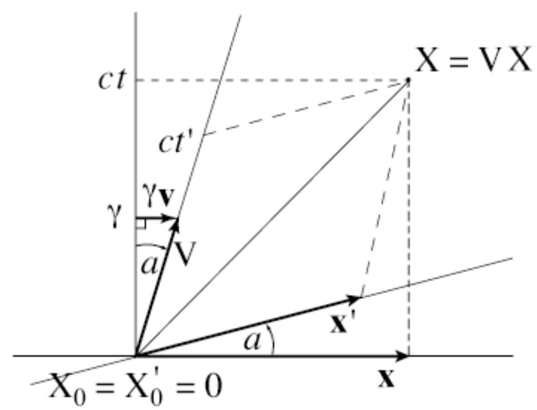
Derive and discuss the result

$$D = \frac{f'}{f} = \gamma(1 \pm v/c) = \sqrt{\frac{c \pm v}{c \mp v}} = \frac{1}{\gamma(1 \mp v/c)}$$

XIV. Lorentz transformations.

Two different inertial systems (primed and unprimed) with a common origin $X_0 = X_0' = 0$ assign different labels $X' = ct' + \mathbf{x}'$ and $X = ct + \mathbf{x}$ to each spacetime event. If $V = \gamma\left(1 + \frac{\mathbf{v}}{c}\right)$ is the proper velocity of the primed frame with respect to the unprimed frame, then, in the timelike plane containing both time axes, the labels are related by the *Lorentz transformation*:

(1) $\quad X = VX' \quad$ or $\quad X' = \tilde{V}X$



Exercise: Derive therefrom the standard Relations between times and positions:

$$t' = \gamma\left(t - \frac{\mathbf{v}\cdot\mathbf{x}}{c^2}\right)$$
$$\mathbf{x}' = \gamma(\mathbf{x} - \mathbf{v}t)$$

Velocity composition:

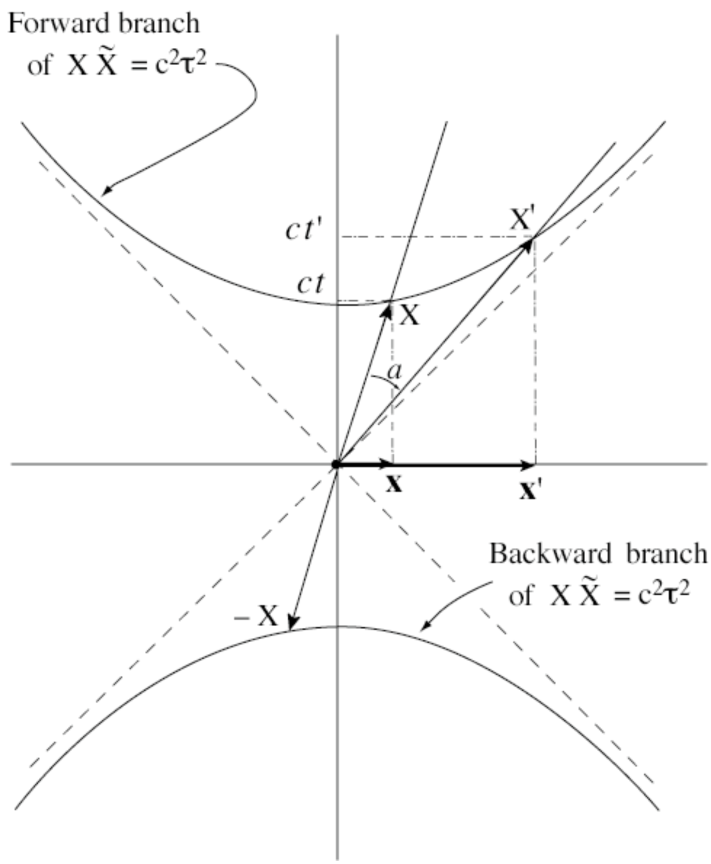
Let $X' = X'(\tau)$ and $X = X(\tau)$ represent the history of a particle with proper velocity $U' = \gamma_{u'}\left(1 + \frac{\mathbf{u}'}{c}\right)$ and $U = \gamma_u\left(1 + \frac{\mathbf{u}}{c}\right)$ in the two inertial systems.

Exercise: Derive and interpret the relativistic **velocity composition law:** $\quad U' = \tilde{V}U$.
Therefrom, derive the corresponding composition laws for time dilations and relative velocities:

$$\gamma_{u'} = \gamma_u\gamma_v\left(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right), \qquad \mathbf{u}' = \frac{\mathbf{u} - \mathbf{v}}{1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}$$

Active Lorentz transformations.

The Lorentz transformation $X = V X'$ is interpreted passively as a change of labels for a single event. Alternatively, it can be interpreted actively as a relation between two different events that are the same proper distance $|X'| = |X| = c\tau$ from the origin.



Advanced Exercise: Interpret the active Lorentz transformation in the figure as a hyperbolic rotation of proper velocity. The rotation angle $a = |\mathbf{a}|$ is the arc length on a unit hyperbola, as expressed by

$$V = \gamma\left(1 + \frac{\mathbf{v}}{c}\right) = e^{\mathbf{a}} = \cosh \mathbf{a} + \sinh \mathbf{a}$$

$$\cosh \mathbf{a} = \cosh a = \gamma = \frac{t}{\tau} \qquad \sinh \mathbf{a} = \hat{\mathbf{v}} \sinh a = \gamma \frac{\mathbf{v}}{c} = \frac{\mathbf{x}}{c\tau}$$

Construct a diagram to express velocity composition as a product of hyperbolic rotations:

$e^{\mathbf{a}} e^{\mathbf{b}} = e^{\mathbf{a}+\mathbf{b}}$ and compare with the product of Euclidean rotations: $e^{\mathbf{i}\theta} e^{\mathbf{i}\varphi} = e^{\mathbf{i}(\theta+\varphi)}$

XV. Energy & momentum are unified by special relativity

A. The **proper momentum** P for a material particle with rest mass m and velocity $V = \gamma\left(1 + \frac{\mathbf{v}}{c}\right)$ is defined by

$$P = mcV = \frac{E}{c} + \mathbf{p}$$

Exercise: Derive expressions for

Mass: $m^2c^4 = E^2 - \mathbf{p}^2c^2$

Momentum: $\mathbf{p} = m\gamma\mathbf{v} = \frac{m\mathbf{v}}{\sqrt{1 - \mathbf{v}^2/c^2}} = m\frac{d\mathbf{x}}{d\tau} = m\gamma\frac{d\mathbf{x}}{dt}$

Energy: $E = mc^2\gamma = \dfrac{mc^2}{\sqrt{1-\mathbf{v}^2/c^2}} = mc^2 + K$

Kinetic energy: $K = (\gamma - 1)mc^2 \approx \frac{1}{2}m\mathbf{v}^2 + \frac{1}{8}m\dfrac{\mathbf{v}^4}{c^2} + \ ...$

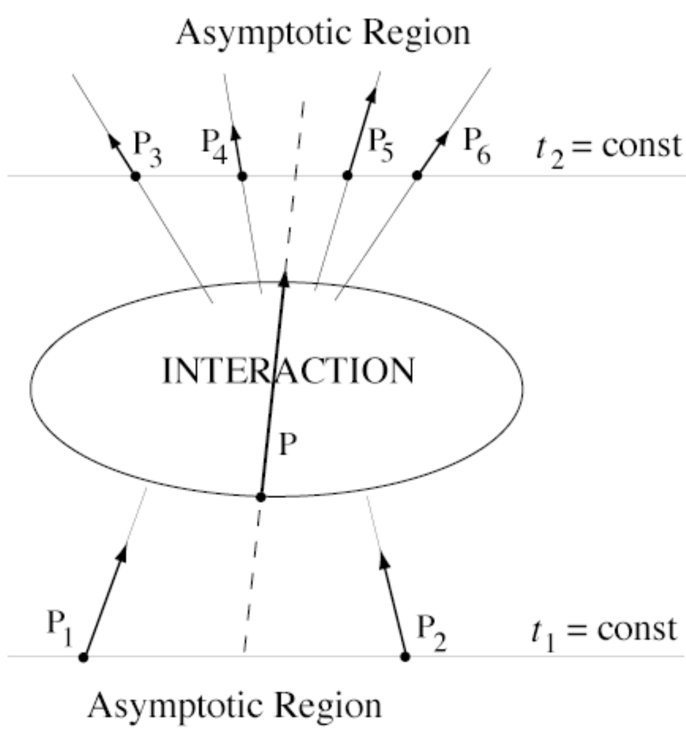
B. The **photon** is a massless particle with proper momentum $P = \dfrac{E}{c} + \mathbf{p}$, where the energy is given by Planck's Law: $E = \hbar\omega = hf$.

Exercise: Show that $P = \hbar(\omega + \mathbf{k})$, where $\omega^2 = \mathbf{k}^2$ and $\mathbf{p} = \hbar\mathbf{k}$

C. Energy-momentum conservation

The total proper momentum P for an isolated system of particles is conserved:

$$P = \sum_{\text{before}} P_k = \sum_{\text{after}} P_k$$



Examples:

(1) Compton effect: $\gamma + e^- \rightarrow \gamma + e^-$

Conservation: $P_1 + P_2 = P_3 + P_4$

Photons: $P_1\tilde{P}_1 = 0 = P_3\tilde{P}_3$, $\quad p = |\mathbf{p}| = \dfrac{E}{c} = hf = \dfrac{h}{\lambda}$

Electrons: $P_2\tilde{P}_2 = m^2c^2 = P_4\tilde{P}_4$

Electron initially at rest: $P_2 = \dfrac{E_2}{c} = mc$

Photons: $P_1 = \frac{E_1}{c}(1 + \hat{\mathbf{p}}_1)$, $P_3 = \frac{E_3}{c}(1 + \hat{\mathbf{p}}_3)$

Problem: Determine the shift in photon frequency due to the scattering.

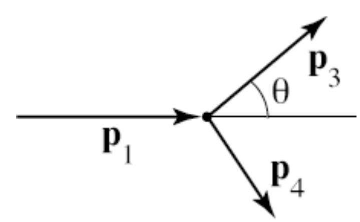
$$P_4 \tilde{P}_4 = [(P_1 - P_3) + P_2][(\tilde{P}_1 - \tilde{P}_3) + \tilde{P}_2]$$

$$m^2 c^2 = -2\langle P_1 \tilde{P}_3 \rangle + 2\langle (P_1 - P_3)\tilde{P}_2 \rangle + m^2 c^2$$

$$\langle (P_1 - P_3)\tilde{P}_2 \rangle = (E_1 - E_3) mc$$

$$c^2 \langle P_1 \tilde{P}_3 \rangle = E_1 E_3 - c^2 \mathbf{p}_1 \cdot \mathbf{p}_3 = E_1 E_3 (1 - \hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_3)$$

Scattering angle: $\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{p}}_3 = \cos\theta$



$$E_1 E_3 (1 - \cos\theta) = (E_1 - E_3) mc^2$$

$$\frac{h^2 c^2}{\lambda_1 \lambda_2}(1 - \cos\theta) = \left(\frac{hc}{\lambda_1} - \frac{hc}{\lambda_2} \right) mc^2$$

$$\lambda_2 - \lambda_1 = \Delta\lambda = \frac{h}{mc}(1 - \cos\theta)$$ **Compton's formula**

(2) Pion decay: $\pi^- \rightarrow \mu^- + \nu$

$$P = P_1 + P_2$$

XVI. Universal laws for spacetime physics

Zeroth Law: Make your own formulation to supercede the Newtonian version!

First Law: *The history of a free particle is a straight line.*