Principia Mathematica endeavors to demonstrate that the convenient quantification theory of its section *10 can be recovered in the system of *9 which works by employing universal and existential generalizations on tautologies, switching quantifiers and then pushing them into subordinate occurrences by the definitions of *9 see, *Principia Mathematica*, p. 129). A rigorous demonstration of this, however, requires induction in the meta-language on the length of a *wff* (Landini 2007). Whitehead and Russell knew this, but preferred not to engage in proofs by induction until section *90 which introduced and defined the *ancestral* relation (see *Principia Mathematica*, p. 129). Logical notions, including the use of numerical notions and other mathematical notions, must be used to set out any formal system for logic. There is nothing circular about that so long as one does not confuse logic and its applications with a formal axiomatic system. Hence, there is nothing circular in using proof by induction to set out and legitimate the formal system of quantification theory.

In the early 21st century, the quantification schemas of *Principia Mathematica's* section *10 are more familiar to logicians. We find the following:

***10.1** $(x^t)\varphi x^t \supset \varphi y^t$

where y^t is free for x^t in the *wff* φ.

***10.12** $(x^t)(p \vee \varphi x^t)\ .\supset.\ p \vee (x^t)\varphi x^t$

where x^t does not occur free in *p* and *p* may contain quantifiers. The wonderful intuition behind *Principia's* system *9 is that for any given instance of *10.1 or *10.2, we can imagine finding a proof of it in the system of *9 by working first in reverse order. First move all its quantifiers to initial placement by means of the definitions of *9 and then, by creativity, find some tautology to generalize. The success of section *9 does not assure decidability of quantification theory and of course, in 1936 Church proved that it is not decidable. But it was an enthusiasm over viability of system *9 that Wittgenstein and Ramsey concluded that quantification theory of any simple type consists in generalized tautologies and is decidable (see, Wittgenstein, 1914, p. 126). Wittgenstein, made a leap of faith, making the decidability of quantification theory a characteristic feature of the conception of logic advanced in his *Tractatus*. Ramsey was more measured and, unlike Wittgenstein, did not mean to connote decidability in saying that quantification theory consists of generalized tautologies (see, Ramsey, 1931, p. 5). Russell was wholly silent on the matter.

Interestingly, in *Principia Mathematica's* second edition, Russell offers Appendix A which sketches, together with the new introduction, a system of deduction for a new Section *8. This system is intended to replace Section *9 of the first edition. The purpose of Section *8 is to offer a remarkable new system of deduction without free variables. Unfortunately, Russell sets out section *8 using typical ambiguity and he also couches it in a system that adopts Nicod's axioms for a language whose sole proposition sign is the Sheffer stroke. This adds yet another layer of complexity that tends to hide the remarkable new result of quantification theory without free variables that is quite independent of the use of Nicod's work and would have been better stated separately. In any case, let us set it out with type indices restored. With the Sheffer stroke, no special definitions are needed for the negation of a quantifier. Where φ and ψ are any *wffs*, quantifier-free or otherwise, the stroke Sheffer definitions are these:

$\sim\varphi =_{df} \varphi \mid \varphi$

$\varphi \vee \psi =_{df} \sim\varphi \mid \sim\psi$

$\varphi \bullet \psi =_{df} \sim(\varphi \mid \psi)$

$\varphi \supset \psi =_{df} \varphi \mid \sim\psi$

The new system of *8, suggests that perhaps Russell came to realize that he needed a correction of the system of *9 because of its absence of *9.0*x* and *9.0*y*. With the Sheffer stroke as the only primitive propositional sign, and where *p* is quantifier-free, Russell offers the following definitions for the *8 quantification theory without free variables:

*8.01 $(x^t)\varphi x^t \mid p$ =df $(\exists x^t)(\varphi x^t \mid p)$
*8.011 $(\exists x^t)\varphi x^t \mid p$ =df $(x^t)(\varphi x^t \mid p)$
*8.012 $p \mid (x^t)\varphi x^t$ =df $(\exists x^t)(p \mid \varphi x^t)$
*8.013 $p \mid (\exists x^t)\varphi x^t$ =df $(x^t)(p \mid \varphi x^t)$

For cases where *p* is not a quantifier-free *wff*, Russell adopts the plan of defining so that the quantifier on the left of the stroke is moved to the front, and then the one on the right is moved to the front (1925*PM*, p. 635). As axioms, Russell has the following which parallel *9.1 and *9.11:

*8.1 $(\exists x^t)(\exists y^t)(\varphi z^t \mid (\varphi x^t \mid \varphi y^t))$

*8.11 $(\exists x^t)(\varphi x^t \mid (\varphi y^t \mid \varphi z^t))$

The system of *8 explicitly adopts the inference rule *Switch* (that is, *8.13). We have borrowed it on behalf of *Principia Mathematica's* system *9 since it is clearly intended there. Since the system of *8 is to work without free variables, Russell also adopts the following inference rule

*8.12 From $(x^t)\varphi x^t$ and $(x^t)(\varphi x^t \supset \psi x^t)$, infer $(x^t)\psi x^t$.

This rule needs a slight modification because the formal system of *8 does not allow vacuous quantifiers. But, this is a small omission that is easily corrected (see, Landini, 2005).

The system of *8 is the *only* innovation properly endorsed in the second edition of *Principia Mathematica*. It is important to understand that *8 was not intended merely as a replacement that does exactly what is done by *9 using the Sheffer stroke. It is designed, quite unlike *9, to perform quantification theory *without* free variables. The system of *9 could easily have used the Sheffer stroke as its primitives instead of having the primitive signs '$\sim$' and '$\vee$'. The Sheffer stroke is completely irrelevant to conducting quantification theory without free variables. In any case, the new system of *8 preceded Quine's *Mathematical Logic* (1940) by fifteen years, which sets forth a theory of deduction without free variables.

In order to advance a quantification theory without free variables, Russell requires a definition of the universal closure of a *wff*, since there are several ways to universally close a *wff*. Unfortunately, Russell does not choose one, but we can easily supply one so that for any *wff* $(x_1^{t_1}, \ldots, x_n^{t_n})$ the intended universal closure is to be this:

$$(x_n^{t_n}), \ldots, (x_1^{t_1})\ f(x_1^{t_1}, \ldots, x_n^{t_n})$$

In the system of *8, the closure of *a wff* such as $f\{\varphi!x\}$ is $(\varphi)(x)f\{\varphi!x\}$. We must respect the fact that *f* and φ are schematic for *wffs* and that the bindable object-language predicate variables 'φ!' are always decorated with the exclamation '!'. Russell intends his discussion of *8 to be applied to the original system of *Principia Mathematica's* first edition and thus it is not intended to upturn anything of significance in the original system. Letters φ are not predicate variables, but are rather used schematically in φ*x* for *wffs* of the object-language. For example, since φ is schematic for a *wff* the closure of $(x)\varphi x \supset \varphi y$ is not

$$(\varphi)(y)(\ (x)\varphi!x \supset \varphi!y)$$

but rather

$$(\psi_1),\ldots, (\psi_n)(z_1),\ldots,(z_m)(y)(\ (x)\varphi x \supset \varphi y)$$

where $\psi_1!$,…, $\psi_n!$ and z_1, …, z_m are all the free variables besides *x* occurring free in the *wff* φ. Unfortunately, Russell writes in the introduction to the second edition (1925, *xiv*) that instead of '├. (*p*). *fp*' we are to write '├. (φ, *x*). *f*(φ*x*)'. This is quite infelicitous. Obviously, there are no propositions and thus no legitimate propositional variables are syntactically well-formed. And as we have just noted, bound predicate variables in the syntax of *Principia Mathematica's* first edition must come with the exclamation. Thus '├. (φ, *x*). *f*(φ!*x*)' would be the proper universal closure. The best explanation of this *faux pas* is that Russell intended to use the new system of *8 with the experiment he conducted in its new Introduction and Appendix B—an experiment which explores the benefits and weaknesses of altering *Principia Mathematica's* grammar and adopting a Wittgenstein suggestion of accepting a sweeping axiom schema of extensionality. This must all be kept completely separate.

The system of *8 is interestingly different from that of Quine, which uses vacuous quantifiers to arrive at a system of deduction without free variables. Quine regards

$$(x)(p\ \&\ {\sim}p)$$

to be true when *x* is not free in *p*. Russell's systems *8 and *9 render it ill-formed. The difference is unimportant until one endeavors to extend the systems to develop quantification theory without free variables and without any existential theorems for individuals (of lowest type). In *Principia Mathematica*, Russell tagged *9.1 (*10.1) and the existence of free variables as the source of its commitment to the existence of at least one individual of lowest type (pp. 20, 226). In his *Introduction to Mathematical Philosophy* (1919), Russell regards this as a "defect of logical purity" (p. 203). Originally he hoped that his system *8 might achieve such purity. Unfortunately, it does not. But it is a first step. The system of *8 can be extended to reach this result (see, Landini, 2005). Quine also extended his system of deduction without free variables to reach such purity. But Quine's system regards,

$p \mathbin{.\supset.} (\varphi)\,(x)\,\varphi!x \supset p$

as logically true (in fact, a tautology). In the Russellian system this is certainly not a tautology since it is, by the definitions of *8, an existential theorem—namely:

$(\exists\varphi)\,(\exists x)(\,p \mathbin{.\supset.} \varphi!x \supset p\,).$

The extension of *8 to avoid existential theorems (concerning individuals of lowest type) embraces different theorems than does Quine's extension (using his vacuous quantifiers) of his system. That is a remarkable distinction between the two systems and shows that achieving logical purity is not the trivial matter that Quine thinks it to be.

Scope is very important in *Principia Mathematica*. It has often been charged that the order of application of its many definitions is not determinate (see, Gödel, 1944, p. 126). This charge is mistaken. We have only to take Whitehead and Russell at their word when they say that definite descriptions and class expressions are *not* genuine terms of the theory. The main definitions for the theory of definite descriptions are these:

*14.01 $[\iota x\varphi x][\,\psi(\iota x\varphi x)] =\text{df}\ (\exists x)(\,\varphi z \equiv_z z = x \mathbin{.\&.} \psi x)$

*14.02 $E!(\iota x\varphi x) =\text{df}\ (\exists x)(\,\varphi z \equiv_z z = x).$

It is natural as well to add:

*14.01a $[\iota x\varphi x][\,\psi \text{x}] =\text{df}\ (\exists x)(\,\varphi z \equiv_z z = x \mathbin{.\&.} \psi x),$

But *14.01, rather than *14.01a, is needed to facilitate *Principia Mathematica's* convention that one may drop the scope marker when *smallest* scope is intended. When we encounter an expression such as

$\psi(\iota x\varphi x)$

with a schematic letter ψ for a *wff* and without a scope marker, we are not in a position to assign any scope. We simply know that *Principia Mathematica* tells us that the smallest possible scope is intended. Thus we are not in a position to restore the scope marker until a *wff* is assigned to the schema that makes smallest scope clear. Consider, for example, a case such as this:

$\sim f!(\iota x\varphi x) \vee \chi!(\iota x\varphi x).$

The scope is clear because *f*! and χ! are both object-language bindable predicate variables. The intended restoration is this:

$\sim[\iota x\varphi x][f!(\iota x\varphi x)] \vee [\iota x\varphi x][\chi!(\iota x\varphi x)]$.

These are the smallest scopes.

When multiple definite description expressions occur, *Principia Mathematica* adopts the convention that the left most is to be eliminated first. It should be noted that in a case such as

$\psi!(\iota x\varphi x, \iota x\varphi x)$

Principia Mathematica intends

$[\iota x\varphi x][\psi!xx]$

that is, $(\exists x)(\ \varphi z \equiv_z z = x\ .\&.\ \psi!(x, x)\ )$.

Though no provision is made, it is easy to amend the convention so that in a case such as,

$\psi!\ (\iota x\varphi x, \iota y\varphi y)$

the intended scope is this:

$[\iota x\varphi x][\iota y\varphi y][\psi!xy]$

that is, $(\exists x)(\ \varphi z \equiv_z z = x\ .\&.\ (\exists y)(\ \varphi z \equiv_z z = y\ .\&.\ \psi!(x, y)\ )$.

In all this, it is very important to remind ourselves that definite descriptions are not terms of the object language. It follows that any definition formed with *individual* variables cannot be applied to definite description expressions and class expressions. For example, consider:

$x = y =df_{*13.02} (\psi)(\psi!x \supset \psi!y)$.

Now consider the expression,

$\iota x\varphi x = y$.

There is no problem of scope. Definition *13.01 cannot apply since $\iota x\varphi x$ is not a term of the language of *Principia Mathematica*. Thus, we have

$[\iota x\varphi x][\iota x\varphi x = y]$.

This is the smallest scope possible. We can next apply *14.01, and we arrive at this:

$(\exists x)(\ \varphi z \equiv_z z = x\ .\&.\ x = y\ )$.

Only now can definition *13.01 be applied. We arrive at:

$(\exists x)(\ \varphi z \equiv_z z = x\ .\&.\ (\psi)(\psi!x \equiv \psi!y)\ )$.

In a case such as,

$\iota x\varphi x = \iota x\varphi x$,

the scope is again quite clear. We have

$(\exists x)(\ \varphi z \equiv_z z = x\ .\&.\ x = x\ )$.

A similar point holds for the following definition

$x \neq y =df_{*13.02} \sim(x = y)$.

Consider the expression $\iota x\varphi x \neq y$. Since definition *13.02 cannot apply, the smallest scope possible is the following:

$[\iota x\varphi x][\iota x\varphi x \neq y]$.

Next, by applying *14. 01, we get:

$(\exists x)(\ \varphi z \equiv_z z = x\ .\&.\ x \neq y\ )$.

Only now can we apply *13.02. It should be noted that not all contexts formed from the formal language of *Principia Mathematica* are extensional. That is because the formal language has the identity sign and admits *wffs* such as $\varphi! = \psi!$, that is, $x^{(t)} = y^{(t)}$. There are no other primitive non-extensional contexts. The general schema for the equivalence of primary and secondary scope of definite descriptions of individuals (of any type) is this:

$E!(\iota x\varphi x)\ .\supset.\ f\{[\iota x\varphi x][\chi(\iota x\varphi x)]\} \equiv [\iota x\varphi x][f\{\chi(\ \iota x\varphi x)\}]$,

where χ is truth-functional. Matters are more complicated for expression of definite descriptions of classes as we shall see.

Issues pertaining to scope also arise in connection definitions emulating class expressions in *Principia Mathematica*. The definitions for expressions of classes of individuals (of any type) are as follows:

*20.01 $[\ \hat{z}\psi z][\ f\{\hat{z}\psi z\}] =\text{df}\ (\exists\varphi)(\ \varphi x \equiv_x \psi x)\ .\&.\ f\ \{\varphi!\})$

*20.02 $x \in \varphi! =\text{df}\ \varphi!x$

Theses apply only for expressions of classes of individuals (of any type). Classes are not individuals of any type. Hence, for expressions for classes of classes of individuals (of any type), we find the following new definitions:

*20.07 $(\alpha)f\alpha\ =\text{df}\ (\varphi)\ f\{\hat{z}\varphi!z\}$

*20.071 $(\exists\alpha)\ f\alpha =\text{df}\ (\exists\varphi)\ f\{\hat{z}\varphi!z\}$

*20.08 $[\ \hat{\alpha}\ \psi\alpha][\ f\{\hat{\alpha}\ \psi\alpha\}] =\text{df}\ (\exists\Sigma)(\ \Sigma!\alpha \equiv_\alpha \psi\alpha)\ .\&.\ f\ \{\Sigma!\ \hat{\alpha}\})$

*20.081 $\alpha \in \Sigma! =\text{df}\ \Sigma!\alpha$

The essential scope markers are restored to *20.01 and *20.08. It is no less essential to the definitions that scope markers be absent in *20.07 and *20.071, for here *Principia Mathematica* intends that we are to take the *smallest* scope possible. The definitions *20.01 and *20.02 emulate classes of individuals of any type. They do not emulate classes of classes. For that, the other definitions introducing lower-case Greek letters are needed.

Definitions involving individual variables, though they are typically ambiguous, cannot be applied to expressions involving class expressions such as $\hat{x}\varphi x$. We must apply definitions such as *20.01

first. This at once dispenses with Gödel's concern that the order of elimination of the class expression $\hat{x}\varphi!x$ in

$$\varphi!\hat{x} = \hat{x}\varphi!x$$

is not determinate. But, we now see that since definition *13.01 cannot apply to class expressions the order is determinate just as it was in the case of definite descriptions.

It is important to observe, however, that definitions in *Principia Mathematica* made with lower-case Greek cannot be applied to definite descriptions, not even definite descriptions of the form $\iota\alpha f\alpha$ for classes. Consider, for example the following definition:

$$*24.03\ \exists!\alpha =df\ (\exists x)(x \in \alpha)$$

Here the free lower-case Greek α stands in for a class expression of the form $\hat{z}\varphi z$. Thus, we have:

$$\exists!\hat{x}\varphi x =df\ (\exists x)(x \in \hat{z}\varphi z).$$

All the same, we have:

$$\exists!\iota\alpha f\alpha =df\ [\iota\alpha f\alpha][\ (\exists x)(x \in \iota\alpha f\alpha\ )].$$

We are not permitted to apply *24.03. We must first apply *14.01 to arrive at

$$(\exists\alpha)(f\beta \equiv_\beta \beta = \alpha\ .\&.\ \exists!\alpha).$$

Only then are we permitted to apply *24.03 to arrive at

$$(\exists\alpha)(f\beta \equiv_\beta \beta = \alpha\ .\&.\ (\exists x)(x \in \alpha)).$$

The following theorem, which shows an equivalence, is thus rather important

$$\iota\alpha f\alpha = \hat{z}\varphi z \supset \exists!\iota\alpha f\alpha \equiv \exists!\hat{x}\varphi x.$$

Unfortunately, *Principia Mathematica* never stops to prove this.

This sometimes generates curious situations in *Principia Mathematica*. Observe that the following is quite readily proved:

$$*24.54\ \ \exists!\hat{x}\varphi x \equiv\ \hat{x}\varphi x \neq \Lambda.$$

The analog for definite descriptions is also provable:

$$\exists!\iota\alpha f\alpha \equiv\ \iota\alpha f\alpha \neq \Lambda.$$

Now we have

$$(\iota\alpha)(\alpha\vec{R}y) = \hat{z}(zRy).$$

Observe as well that the following are not contradictories in *Principia*:

$$\hat{z}(zRy) = \Lambda$$

$$\hat{z}(zRy) \neq \Lambda.$$

Expanding the class symbols they are, respectively,

$$\hat{z}(zRy) = \Lambda =df\ \ (\exists\Gamma)(\Gamma!z \equiv_z zRy\ .\&.\ (\exists\theta)(\theta!z \equiv_z z \neq z\ .\&.\ \Gamma!\hat{z} = \theta!\hat{z}))$$

$$\hat{z}(zRy) \neq \Lambda =df\ \ (\exists\Gamma)(\Gamma z \equiv_z zRy\ .\&.\ (\exists\theta)(\theta z \equiv_z z \neq z\ .\&.\ \Gamma!\hat{z} \neq \theta!\hat{z}\ )).$$

Nonetheless, the following are contradictories:

$\exists!(\iota\alpha)(\alpha \vec{R}\, y)$

$\sim\exists!(\iota\alpha)(\alpha \vec{R}\, y)$.

Whitehead and Russell's comments after *32.121 suggest otherwise, but they seem just mistaken.

It is worth noting that while definite descriptions must be eliminated before applying definitions, in situations of class expressions it is the definitions made with lower-case Greek that are to apply first. Consider, for example:

*22.03 $\alpha \cup \beta$ =df $\hat{x}(\, x \in \alpha\ v\ x \in \beta)$.

It is not permitted to apply *20.01 to

$\hat{x}\varphi x \cup \beta$

for it is impossible to apply *22.03 to the clause $\Gamma!\hat{z} \cup \beta$ in

$(\exists\Gamma)(\Gamma!z \equiv_z \varphi z$.&. $\Gamma!\hat{z} \cup \beta\,)$.

The order of definitions in *Principia Mathematica* is clear enough.

Non-extensionality cannot come into play with definite descriptions of individuals of any type. But it can come into play with definite descriptions of classes. For definite descriptions of the form $\iota\alpha f\alpha$, the conditions under which primary and secondary scopes are equivalent is this:

*14.3_α $E!(\iota\alpha\varphi\alpha)$.⊃. $f\{[\iota\alpha\varphi\alpha][\chi(\iota\alpha\varphi\alpha) \ \equiv [\iota\alpha\varphi\alpha)][f\{\chi(\, \iota\alpha\varphi\alpha))\}]$,

where f is truth-functional and χ is extensional. This parallels the situation of class expressions:

*$14.3_{\alpha\alpha}$ $E!(\hat{z}\varphi z)$.⊃. $f\{[\hat{z}\varphi z][\chi(\, \hat{z}\varphi z)]\} \equiv [\hat{z}\varphi z][f\{\chi(\, \hat{z}\varphi z)\}]$,

where f is truth-functional and χ is extensional. Of course in each of these cases, *12.n assures $E!(\iota\alpha\varphi\alpha)$ and $E!(\iota\alpha\varphi\alpha)$.

There is a bit of a surprise here that differs from the case of definite descriptions of individuals. Even with *12.n, primary scopes of class expression do not always entail secondary scopes. Consider the following:

$[\hat{y}\varphi!y] \sim [\hat{y}\varphi!y = \psi!\hat{z}\ ] \supset_\varphi \sim[\hat{y}\varphi y][\hat{y}\varphi y = \psi!\hat{z}\ ]$.

By definition *20.01 we have:

$(\exists\Gamma)(\Gamma!z \equiv_z \varphi z$.&. $\sim(\Gamma!\hat{z} = \psi!\hat{z})) \supset_\varphi \sim(\exists\Gamma)(\Gamma z \equiv_z \varphi z$.&. $\Gamma!\hat{z} = \psi!\hat{z}\ )$.

We can have a situation where

$(\exists\Gamma)(\Gamma!z \equiv_z \varphi z$.&. $\sim(\Gamma!\hat{z} = \psi!\hat{z}))$

is true. But $\sim(\exists\Gamma)(\Gamma z \equiv_z \varphi z$.&. $\Gamma!\hat{z} = \psi!\hat{z}\ )$ is false since clearly there is some $\Gamma!\hat{z}$, namely $\psi!\hat{z}$ itself which is such that $\Gamma!\hat{z} = \psi!\hat{z}$. This completes the proof.

A very important feature is the *absence* of the scope markers in *20.07 and *20.071. This is to assure that smallest scopes are taken. This feature is central to the viability of the theory of classes of classes. Definition *20.07 must *not* be taken as

$(\alpha)f\alpha \ =df\, (\varphi)[\hat{z}\varphi!\, z][f\{\, \hat{z}\varphi!z\}]$.

Were it to be taken this way, the theory of classes would collapse. To see this, we have only to examine what happens in the proof of

$\hat{z}\Gamma!z \in \hat{\alpha}\psi\alpha \,.\!\equiv\!.\, \psi(\hat{z}\Gamma!z)$.

By applying definitions *20.08 and *20.081 to the left-hand side we get:

$(\exists\Sigma)(\Sigma!\alpha \equiv_{\alpha} \psi\alpha \,.\&.\, \Sigma!(\hat{z}\Gamma!z)\ )$.

Since Σ! is a genuine predicate variable we know the scope is

$(\exists\Sigma)(\Sigma!\alpha \equiv_{\alpha} \psi\alpha \,.\&.\, [\hat{z}\Gamma!z][\Sigma!(\hat{z}\Gamma!z)]\ )$.

Ok, now focus on

$\Sigma!\alpha \equiv_{\alpha} \psi\alpha$.

The question is how to apply *20.07 to this so as to prove $\psi\,(\hat{z}\Gamma!z)$. It is important that *20.07 is not interpreted to demand the following primary scope:

$(\varphi)(\ [\hat{z}\varphi!z][\ \Sigma!(\hat{z}\varphi!z) \equiv \psi(\hat{z}\varphi!z)\ ])$

If it demanded such a scope, the proof stalls. But when properly understood, *20.07 tells us that we do not know the scope $\hat{z}\varphi!z$ has in $(\varphi)f(\hat{z}\varphi!z)$ until after $f\,(\ldots)$ is assigned; and it tells us that when it is assigned, we are to take the smallest scope possible. Thus *20.07 yields

$(\varphi)(\ \Sigma!(\hat{z}\varphi!z) \equiv \psi(\hat{z}\varphi!z))$.

Now since Σ! is a genuine predicate variable, we know that the smallest scope of $\hat{z}\varphi!z$ in $\Sigma!(\hat{z}\varphi!z)$ is primary. Hence, we get:

$(\varphi)(\ [\hat{z}\varphi!z][\Sigma!(\hat{z}\varphi!z)] \equiv \psi(\hat{z}\varphi!z))$.

So, by universal instantiation

$[\hat{z}\Gamma!z][\Sigma!(\hat{z}\Gamma!z)] \equiv \psi(\hat{z}\Gamma!z))$.

Hence, since we have $[\hat{z}\Gamma!z][\Sigma!(\hat{z}\Gamma!z)]$ we readily arrive at what we want, namely, $\psi(\hat{z}\Gamma!z)$.