Probability Exam Questions with Solutions

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This note gives a large number of exam problems for a first course in probability. Fully worked-out solutions of these problems are also given, but of course you should first try to solve the problems on your own!

[1]The exam questions accompany the basic probability material in the chapters 7-14 of my textbook *Understanding Probability*, third edition, Cambridge University Press, 2012. The simulation-based chapters 1-6 of the book present many probability applications from everyday life to help the beginning student develop a feel for probabilities. Moreover, the book gives an introduction to discrete-time Markov chains and continuous-time Markov chains in the chapters 15 and 16.

Chapter 7

7E-1 A bridge hand in which there is no card higher than a nine is called a Yarborough. Specify an appropriate sample space and determine the probability of Yarborough when you are randomly dealt 13 cards out of a well-shuffled deck of 52 cards.

7E-2 Five dots are placed at random on a 8×8 grid in such a way that no cell contains more than one dot. Specify an appropriate sample space and determine the probability that no row or column contains more than one dot.

7E-3 Five people are sitting at a table in a restaurant. Two of them order coffee and the other three order tea. The waiter forgot who ordered what and puts the drinks in a random order for the five persons. Specify an appropriate sample space and determine the probability that each person gets the correct drink

7E-4 A parking lot has 10 parking spaces arranged in a row. There are 7 cars parked. Assume that each car owner has picked at a random a parking place among the spaces available. Specify an appropriate sample space and determine the probability that the three empty places are adjacent to each other.

7E-5 Somebody is looking for a top-floor apartment. She hears about two vacant apartments in a building with 7 floors en 8 apartments per floor. What is the probability that there is a vacant apartment on the top floor?

7E-6 You choose at random two cards from a standard deck of 52 cards. What is the probability of getting a ten and hearts?

7E-7 A box contains 7 apples and 5 oranges. The pieces of fruit are taken out of the box, one at a time and in a random order. What is the probability that the bowl will be empty after the last apple is taken from the box?

7E-8 A group of five people simultaneously enter an elevator at the ground floor. There are 10 upper floors. The persons choose their exit floors independently of each other. Specify an appropriate sample space and determine the probability that they are all going to different floors when each person randomly chooses one of the 10 floors as the exit floor. How does the answer change when each person chooses with probability $\frac{1}{2}$ the 10th floor as the exit floor and the other floors remain equally likely as the exit floor with a probability of $\frac{1}{18}$ each.

7E-9 Three friends and seven other people are randomly seated in a row. Specify an appropriate sample space to answer the following two questions.
(a) What is the probability that the three friends will sit next to each other?

(b) What is the probability that exactly two of the three friends will sit next to each other?

7E-10 You and two of your friends are in a group of 10 people. The group is randomly split up into two groups of 5 people each. Specify an appropriate sample space and determine the probability that you and your two friends are in the same group.

7E-11 You are dealt a hand of four cards from a well-shuffled deck of 52 cards. Specify an appropriate sample space and determine the probability that you receive the four cards J, Q, K, A in any order, with suit irrelevant.

7E-12 You draw at random five cards from a standard deck of 52 cards. What is the probability that there is an ace among the five cards and a king or queen?

7E-13 Three balls are randomly dropped into three boxes, where any ball is equally likely to fall into each box. Specify an appropriate sample space and determine the probability that exactly one box will be empty.

7E-14 An electronic system has four components labeled as 1, 2, 3, and 4. The system has to be used during a given time period. The probability that component i will fail during that time period is f_i for $i = 1, \ldots, 4$. Failures of the components are physically independent of each other. A system failure occurs if component 1 fails or if at least two of the other components fail. Specify an appropriate sample space and determine the probability of a system failure.

7E-15 The Manhattan distance of a point (x, y) in the plane to the origin $(0, 0)$ is defined as $|x| + |y|$. You choose at random a point in the unit square $\{(x, y) : 0 \leq x, y \leq 1\}$. What is the probability that the Manhattan distance of this point to the point $(0, 0)$ is no more than a for $0 \leq a \leq 2$?

7E-16 You choose at random a point inside a rectangle whose sides have the lengths 2 and 3. What is the probability that the distance of the point to the closest side of the rectangle is no more than a given value a with $0 < a < 1$?

7E-17 Pete tosses $n + 1$ fair coins and John tosses n fair coins. What is the probability that Pete gets more heads than John? Answer this question first for the cases $n = 1$ and $n = 2$ before solving the general case.

7E-18 Bill and Mark take turns picking a ball at random from a bag containing four red balls and seven white balls. The balls are drawn out of the bag without replacement and Mark is the first person to start. What is the probability that Bill is the first person to pick a red ball?

7E-19 Three desperados A, B and C play Russian roulette in which they take turns pulling the trigger of a six-cylinder revolver loaded with one bullet. Each time the magazine is spun to randomly select a new cylinder to fire as long the deadly shot has not fallen. The desperados shoot according to the order $A, B, C, A, B, C, \ldots$. Determine for each of the three desperados the probability that this desperado will be the one to shoot himself dead.

7E-20 A fair coin is tossed 20 times. The probability of getting the three or more heads in a row is 0.7870 and the probability of getting three or more heads in a row *or* three or more tails in a row is 0.9791.[2] What is the probability of getting three or more heads in a row *and* three or more tails in a row?

7E-21 The probability that a visit to a particular car dealer results in neither buying a second-hand car nor a Japanese car is 55%. Of those coming to the dealer, 25% buy a second-hand car and 30% buy a Japanese car. What is the probability that a visit leads to buying a second-hand Japanese car?

7E-22 A fair die is repeatedly rolled and accumulating counts of 1s, 2s, …, 6s are recorded. What is an upper bound for the probability that the six accumulating counts will ever be equal?

7E-23 A fair die is rolled six times. What is the probability that the largest number rolled is r for $r = 1, \ldots, 6$?

7E-24 Mr. Fermat and Mr. Pascal are playing a game of chance in a cafe in Paris. The first to win a total of ten games is the overall winner. Each of the two players has the same probability of $\frac{1}{2}$ to win any given game. Suddenly the competition is interrupted and must be ended. This happens at a moment that Fermat has won a games and Pascal has won b games with $a < 10$ and $b < 10$. What is the probability that Fermat would have been the overall winner when the competition would not have been interrupted? *Hint*: imagine that another $10 - a + 10 - b - 1$ games would have been played.

7E-25 A random number is repeatedly drawn from $1, 2, \ldots, 10$. What is the probability that not all of the numbers $1, 2, \ldots, 10$ show up in 50 drawings?

7E-26 Three couples attend a dinner. Each of the six people chooses randomly a seat at a round table. What is the probability that no couple sits together?

7E-27 You roll a fair die six times. What is the probability that three of the six possible outcomes do not show up and each of the other three possible

[2] These probabilities can be computed by using an absorbing Markov chain discussed in Chapter 15 of the book.

outcomes shows up two times? What is the probability that some outcome shows up at least three times?

7E-28 In a group of n boys and n girls, each boy chooses at random a girl and each girl chooses at random a boy. The choices of the boys and girls are independent of each other. If a boy and a girl have chosen each other, they form a couple. What is the probability that no couple will be formed?

7E-29 Twelve married couples participate in a tournament. The group of 24 people is randomly split into eight teams of three people each, where all possible splits are equally likely. What is the probability that none of the teams has a married couple?

7E-30 An airport bus deposits 25 passengers at 7 stops. Each passenger is as likely to get off at any stop as at any other, and the passengers act independently of one another. The bus makes a stop only if someone wants to get off. What is the probability that somebody gets off at each stop?

7E-31 Consider a communication network with four nodes n_1, n_2, n_3 and n_4 and five directed links $l_1 = (n_1, n_2)$, $l_2 = (n_1, n_3)$, $l_3 = (n_2, n_3)$, $l_4 = (n_3, n_2)$, $l_5 = (n_2, n_4)$ and $l_6 = (n_3, n_4)$. A message has to be sent from the source node n_1 to the destination node n_4. The network is unreliable. The probability that the link l_i is functioning is p_i for $i = 1, \ldots, 5$. The links behave physically independent of each other. A path from node n_1 to node n_4 is only functioning if each of its links is functioning. Use the inclusion-exclusion formula to find the probability that there is some functioning path from node n_1 to node n_4. How does the expression for this probability simplify when $p_i = p$ for all i?

Chapter 8

8E-1 Three fair dice are rolled. What is the probability that the sum of the three outcomes is 10 given that the three dice show different outcomes?

8E-2 A bag contains four balls. One is blue, one is white and two are red. Someone draws together two balls at random from the bag. He looks at the balls and tells you that there is a red ball among the two balls drawn out. What is the probability the the other ball drawn out is also red?

8E-3 A fair coin is tossed n times. What is the probability of heads on the first toss given that r heads were obtained in the n tosses?

8E-4 A hand of 13 cards is dealt from a standard deck of 52 cards. What is the probability that it contains more aces than tens? How does this probability change when you have the information that the hand contains at least one ace?

8E-5 In a high school class, 35% of the students take Spanish as a foreign language, 15% take French as a foreign language, and 40% take at least one of these languages. What is the probability that a randomly chosen student takes French given that the student takes Spanish?

8E-6 Let A and B be independent events. Denote by the events A^c and B^c the complements of the events A and B. Verify that the events A and B^c are independent. Conclude directly from this result that the events A^c and B^c are also independent.

8E-7 Fifty different numbers are arranged in a matrix with 5 rows and 10 columns. You pick at random one number from the matrix. Let A be the event that the number comes from an odd-numbered row and B be the event that the number comes from the first five columns. Are the events A and B independent?

8E-8 Consider again the Problems 7E-3, 7E-5, 7E-10 and 7E-11. Use conditional probabilities to solve these problems.

8E-9 A bowl contains four red and four blue balls. As part of drawing lots, you choose four times two balls at random from the bowl without replacement. What is the probability that one one red and one blue ball are chosen each time?

8E-10 There are three English teams among the eight teams that have reached the quarter-finals of the Champions League soccer. What is the probability that the three English teams will avoid each other in the draw if the teams are paired randomly?

8E-11 A jar contains three white balls and two black balls. Each time you pick at random one ball from the jar. If it is a white ball, a black ball is inserted instead; otherwise, a white ball is inserted instead. You continue until all balls in the jar are black. What is the the probability that you need no more than five picks to achieve this?

8E-12 You are among N players that will play a competition. A lottery is used to determine the placement of each player. You have an advantage. Two tickets with your name are put in a hat, while for each of the other players only one ticket with her/his name is put in the hat. The hat is well shaken and tickets are drawn one by one from the hat. The order of names appearing determines the placement of each player. What is the probability that you will get assigned the nth placement for $n = 1, 2, \ldots, N$?

8E-13 Twenty-five people choose each at random a number from $1, 2, \ldots, 100$, independently of each other. Next the chosen numbers are announced one by one. The first person (if any) who announces a number that has been announced before wins a bonus. Which person has the largest probability to win the bonus?

8E-14 In a poker game with three players A, B and C, the dealer is chosen by the following procedure. In the order A, B, C, A, B, C, $\ldots$, a card from a well-shuffled deck is dealt to each player until someone gets an ace. This first player receiving an ace gets to start the game as dealer. Do you think that everyone has an equal chance to become dealer?

8E-15 A drunkard removes two randomly chosen letters of the message HAPPY HOUR that is attached on a billboard in a pub. His drunk friend puts the two letters back in a random order. What is the probability that HAPPY HOUR appears again?

8E-16 A professor gives only two types of exams, "easy"and " hard". You will get a hard exam with probability 0.80. The probability that the first question on the exam will be marked as difficult is 0.90 if the exam is hard and is 0.15 otherwise. What is the probability that the first question on your exam is marked as difficult. What is the probability that your exam is hard given that the first question on the exam is marked as difficult?

8E-17 Bill and Mark play a series of games until one of the players has won two games more than the other player. Any game is won by Bill with probability p and by Mark with probability $q = 1 - p$. The results of the games are independent of each other. What is the probability that Bill will be the winner of the match?

8E-18 Somebody puts eight balls into a bowl. The balls have been colored independently of each other and each ball has been colored red or white with equal probabilities. This all happens unseen to you. Then you see that two red balls are added to bowl. Next five balls ball are taken at random from the bowl and are shown to you. All these five balls are white. What is the probability that all the other five balls in the bowl are red?

8E-19 Your friend has chosen at random a card from a standard deck of 52 cards but keeps this card concealed. You have to guess what card it is. Before doing so, you can ask your friend either the question whether the chosen card is red or the question whether the card is the ace of spades. Your friend will answer truthfully. What question would you ask?

8E-20 Player 1 tosses $N + 1$ times a fair coin and player 2 tosses N times a fair coin. Player 1 wins the game if player 1 tosses more heads than player 2; otherwise, player 2 wins.
(a) What is the probability of a tie after N tosses?
(b) What is the probability that player 1 will win the game?

8E-21 A jar contains five blue balls and five red balls. You roll a fair die once. Next you randomly draw (without replacement) as many balls from the jar as the number of points you have rolled with the die.
(a) What is the probability that all of the balls drawn are blue?
(b) What is the probability that the number of points shown by the die is r given that all of the balls drawn are blue?

8E-22 A tennis tournament is arranged for 8 players. It is organized as a knockout tournament. First, the 8 players are randomly allocated over four groups of two players each. In the semi-finals the winners of the groups 1 and 2 meet each other and the winners of the groups 3 and 4. In any match either player has a probability 0.5 of winning. John and Pete are among the 8 players. What is the probability that they meet each other in the semi-finals? What is the probability that they meet each other in the final?

8E-23 Consider again Problem 7E-24. Show how this problem can be solved by a recursive approach.

8E-24 A biased coin is tossed repeatedly. The probability that a toss of the coin results in heads is p with $0 < p < 1$.
(a) Give a recursion for the probability that the total number of heads after n tosses is even.
(b) Give a recursion for the probability that a sequence of n tosses does not show five or more consecutive heads.

8E-25 A lottery organization distributes one million tickets every week. At one end of the ticket, there is a visible printed number consisting of six digits, say 070469. At the other end of the ticket, another six-digit number is printed, but this number is hidden by a layer of scratch-away silver paint. The ticket holder scratches the paint away to reveal the underlying number. If the number is the same as the number at the other end of the ticket, it is a winning ticket. The two six-digit numbers on each of the one million tickets printed each week are randomly generated in such a way that no two tickets are printed with the same visible numbers or the same hidden numbers. Assume that in a particular week only one half of the tickets printed are sold. What is the probability of exactly r winners in that week for $r = 0, 1, \ldots$?

8E-26 In a binary transmission channel, a 1 is transmitted with probability 0.8 and a 0 with probability 0.2. The conditional probability of receiving a 1 given that a 1 was sent is 0.95, the conditional probability of receiving a 0 when a 0 was sent is 0.99. What is the probability that a 1 was sent when receiving a 1? Use Bayes' formula in odds form to answer this question.

8E-27 On the island of liars each inhabitant lies with probability $\frac{2}{3}$. You overhear an inhabitant making a statement. Next you ask another inhabitant whether the inhabitant you overheard spoke truthfully. Use Bayes' rule in odds form to find the probability that the inhabitant you overheard indeed spoke truthfully given that the other inhabitant says so.

8E-28 An oil explorer performs a seismic test to determine whether oil is likely to be found in a certain area. The probability that the test indicates the presence of oil is 90% if oil is indeed present in the test area and the probability of a false positive is 15% if no oil is present in the test area. Before the test is done, the explorer believes that the probability of presence of oil in the test area is 40%. Use Bayes' rule in odds form to revise the value of the probability of oil being present in the test area given that the test gives a positive signal.

8E-29 An isolated island is ruled by a dictator. Every family on the island has two children. Each child is equally likely a boy or a girl. The dictator has decreed that each first girl (if any) born to the family must bear the name Mary Ann (the name of the beloved mother-in-law of the dictator). Two siblings never have the same name. You are told that a randomly chosen family that is unknown to you has a girl named Mary Ann. What is the probability that this family has two girls?

The dictator has passed away. His son, a womanizer, has changed the rules. For each first girl born to the family a name must be chosen at random from 10 specific names including the name Mary Ann, while for each second girl born to the family a name must be randomly chosen from the remaining 9 names. What is now the probability that a randomly chosen family has two girls when you are told that this family has a girl named Mary Ann? Can you intuitively explain why this probability is not the same as the previous probability?

8E-30 A family is chosen at random from all three-child families. What is the probability that the chosen family has one boy and two girls if the family has a boy among the three children? Use Bayes' rule in odds form to answer this question.

8-31 (a) A box contains 10,000 coins. One of the coins has heads on both sides but all the other coins are fair coins. You choose at random one of the coins. Use Bayes' rule in odds form to find the probability that you have chosen the two-headed coin given that the first 15 tosses all have resulted in heads. What is the answer when you would have obtained 25 heads in a row in the first 25 tosses?
(b) A box contains $r + 1$ coins $i = 0, 1, \ldots, r$. Coin i lands heads with

probability $\frac{i}{r}$ for $i = 0, 1, \ldots, r$. You choose at random one of the coins. Use Bayes' rule in odds form to find the probability that you have chosen coin s given that each of the first n tosses has resulted in heads.

8E-32 Your friend has generated two random numbers from $1, \ldots, 10$, independently of each other. Use Bayes' rule in odds form to answer the following two questions.
(a) What is the probability that both numbers are even given the information that there is an even number among the two numbers?
(b) What is the probability that both numbers are even given the information that the number 2 is among the two numbers?

8E-33 Your friend has fabricated a loaded die. In doing so, he has first simulated a number at random from 0.1, 0.2, 0.3, and 0.4. He tells you that the die is loaded in such a way that any roll of the die results in the outcome 6 with a probability which is equal to the simulated number. Next the die is rolled 300 times and you are informed that the outcome 6 has appeared 75 times. What is the posterior distribution of the probability that a single roll of the die gives a 6?

8E-34 Your friend is a basketball player. To find out how good he is in free throws, you ask him to shoot 10 throws. You assume the three possible values 0.25, 0.50 and 0.75 for the success probability of the free shots of your friend. Before the 10 throws are shot, you believe that these three values have the respective probabilities 0.2, 0.6 and 0.2. What is the posterior distribution of the success probability given that your friend scores 7 times out of the 10 throws?

Chapter 9

9E-1 Five men and five women are ranked according to their scores on an exam. Assume that no two scores are the same and all possible rankings are equally likely. Let the random variable X be the highest ranking achieved by a women. What is the probability mass function of X?

9E-2 Accidentally, two depleted batteries got into a set of five batteries. To remove the two depleted batteries, the batteries are tested one by one in a random order. Let the random variable X denote the number of batteries that must be tested to find the two depleted batteries. What is the probability mass function of X?

9E-3 You roll a fair dice twice. Let the random variable X be the product of the outcomes of the two rolls. What is the probability mass function of X? What are the expected value and the standard deviation of X?

9E-4 In a lottery a four-digit number is chosen at random from the range $0000 - 9999$. A lottery ticket costs \$2. You win \$50 if your ticket matches the last two digits but not the last three, \$500 if your ticket matches the last three digits but not all four, and \$5,000 if your ticket matches all four digits. What is the expected payoff on a lottery ticket? What is the house edge of the lottery?

9E-5 The following dice game is offered to you. You may simultaneously roll one red die and three blue dice. The stake is \$1. If none of the blue dice matches the red die, you lose your stake; otherwise, you get anyway paid $k+1$ dollars if exactly k of the blue dice match the red die. In the case that exactly one blue die matches the red die, you get paid an additional \$0.50 if the other two blue dice match. What is the expected payoff of the game?

9E-6 The following game is offered. There are 10 cards face-down numbered 1 through 10. You can pick one card. Your payoff is \$0.50 if the number on the card is less than 5 and is the dollar value on the card otherwise. What are the expected value and the standard deviation of your payoff?

9E-7 A fair die is rolled six times. What are the expected value and the standard deviation of the smallest number rolled?

9E-8 Eleven closed boxes are put in random order in front of you. One of these boxes contains a devil's penny and the other ten boxes contain given dollar amounts $a_1, \ldots, a_{10}$. You may open as many boxes as you wish, but they must be opened one by one. You can keep the money from the boxes you have opened as long as you have not opened the box with the devil's penny. Once you open this box, the game is over and you lose all the money gathered so far. What is a good stopping rule to maximize the expected value of your gain?

9E-9 You play a sequence of s games, where $s \geq 2$ is fixed. The outcomes of the various games are independent of each other. The probability that you will win the kth game is $\frac{1}{k}$ for $k = 1, 2, \ldots, s$. You get one dollar each time you win two games in a row. What is the expected value of the total amount you will get?

9E-10 You toss a biased coin with probability p of heads, while your friend tosses at the same time a fair coin. What is the probability distribution of the number of tosses until both coins simultaneously show the same outcome?

9E-11 You distribute randomly 25 apples over 10 boxes. What is the expected value of the number of boxes that will contain exactly k apples for $k = 0, 1, \ldots, 25$?

9E-12 You have a thoroughly shuffled deck of 52 cards. Each time you choose one card from the deck. The drawn card is put back in the deck and all 52 cards are again thoroughly shuffled. You continue this procedure until you have seen all four different aces. What are the expected value and the standard deviation of the number of times you have to draw a card until you have seen all four different aces?

9E-13 A group of m people simultaneously enter an elevator at the ground floor. Each person randomly chooses one of the r floors $1, 2, \ldots, r$ as the exit floor, where the choices of the persons are independent of each other. The elevator only stops on a floor if at least one person wants to exit on that floor. No other people enter the elevator at any of the floors $1, 2, \ldots, r$. What are the expected value and the standard deviation of the number of stops the elevator will make?

9E-14 (a) An integer is repeatedly drawn at random from $1, 2, \ldots, 10$. What are the expected value and the standard deviation of the number of integers from $1, 2, \ldots, 10$ that do not show up in 20 drawings?
(b) In each drawing of the Lotto 6/45 six different integers are randomly chosen from $1, 2, \ldots, 45$. What are the expected value and the standard deviation of the number of integers from $1, 2, \ldots, 45$ that do not show up in 15 drawings?

9E-15 Take a random permutation of the integers $1, 2, \ldots, n$. Let us say that the integers i and j with $i \neq j$ are switched if the integer i occupies the jth position in the random permutation and the integer j the ith position. What is the expected value of the total number of switches?

9E-16 Twelve married couples participate in a tournament. The group of 24 people is randomly split into eight teams of three people each, where all possible splits are equally likely. What is the expected value of the number of teams with a married couple?

9E-17 Suppose you know that the hands of you and your bridge partner contain eight of the 13 spades in the deck. What is the probability of 3-2 split of the remaining five spades in the bridge hands of your opponents?

9E-18 You choose at random an integer from $1, 2, \ldots, 6$. Next you roll a fair die until you get an outcome that is larger than or equal to the randomly chosen integer. What is the probability mass function of the number of times you will roll the die? What are the expected value and the standard deviation of the number of times you will roll the die?

9E-19 You have two coins. One coin is fair and the other is biased with probability p of heads. The first toss is done with the fair coin. At the subsequent tosses the fair coin is used if the previous toss resulted in heads and the biased coin is used otherwise. What is the expected value of the number of heads in r tosses for $r = 1, 2, \ldots$?

9E-20 A bag contains R red balls and W white balls. Each time you take one ball out of the bag at random and without replacement. You stop as soon as all red balls have been taken out of the bag. What is the expected number of white balls remaining in the bag when you stop?

9E-21 Let the random variable X be defined by $X = YZ$, where Y and Z are independent random variables each taking on the values -1 and 1 with probabilities 0.5. Verify that X is independent of both Y and Z, but not of $Y + Z$.

9E-22 Let X and Y be independent random variables, where X is binomially distributed with parameters n and p and Y is binomially distributed with parameters m and p.
(a) Explain in terms of Bernoulli experiments that $X + Y$ is binomially distributed with parameters $n + m$ and p. Next give a formal proof.
(b) Verify that for fixed k the probabilities $P(X = j \mid X + Y = k)$ for $j = 0, \ldots, k$ constitute a hypergeometric distribution.

9E-23 A radioactive source emits particles toward a Geiger counter. The number of particles that are emitted in a given time interval is Poisson distributed with expected value λ. An emitted particle is recorded by the counter with probability p, independently of the other particles. Let the random variable X be the number of recorded particles in the given time interval and Y be the number of unrecorded particles in the time interval. What are the probability mass functions of X and Y? Are X and Y independent?

9E-24 (a) The random variable X is Poisson distributed with expected value λ. Verify that $E[\lambda g(X+1) - Xg(X)] = 0$ for any bounded function $g(x)$ on the integers $0, 1, \ldots$.
(b) Let X be a random variable on the integers $0, 1, \ldots$ and $\lambda > 0$ a given number. Prove that X has a Poisson distribution with expected value λ if $E[\lambda g(X + 1) - Xg(X)] = 0$ for any bounded function $g(x)$ on the integers $0, 1, \ldots$. *Hint*: take the function $g(x)$ defined by $g(r) = 1$ and $g(x) = 0$ for $x \neq r$ with r a fixed nonnegative integer.

9E-25 You first roll a fair die once. Next you roll the die as many times as the outcome of this first roll. Let the random variable X be the total number of sixes in all the rolls of the die, including the first roll. What is the probability mass function of X?

9E-26 The following game is played in a particular carnival tent. You pay one dollar to draw blindly three balls from a box without replacement. The box contains 10 balls and four of those balls are gold-colored. You get back your original one-dollar stake if you draw exactly two gold-colored balls, while you win 10 dollars and get back your original one-dollar stake if you draw three gold-colored balls; otherwise, you get nothing back. What is the house advantage for the game?

9E-27 In a close election between two candidates A and B in a small town the winning margin of candidate A is 1,422 to 1,405 votes. However, 101 votes are illegal and have to be thrown out. Assuming that the illegal votes are not biased in any particular way and the count is otherwise reliable, what is the probability the removal of the illegal votes changes the result of the election?

9E-28 A bowl contains n red balls and m white balls. You randomly pick without replacement one ball at a time until you have r red balls. What is the probability that you need k draws?

9E-29 You and your friend both draw a random number from $1, 2, \ldots, 10$ at the same time and independently of each other. This procedure is repeated until you have drawn one of the four numbers $1, \ldots, 4$ or your friend has drawn one of the six numbers $5, \ldots, 10$. The first player to get one of his marked numbers is the winner with the convention that you are the winner if a tie occurs. What is your probability of winning the game? What is the probability mass function of the length of the game?[3]

9E-30 The G–50 airplane is at the end of its lifetime. The remaining operational lifetime of the plane and is 3, 4 or 5 years each with probability $\frac{1}{3}$. A decision must be made how many spare parts of a certain component to produce. The demand for spare parts of the component is Poisson distributed with an expected value of λ units per year for each year of the remaining lifetime of the plane, where the demands in the various years are independent of each other. It is decided to produce Q units of the spare part. What is the probability that the production size will not be enough to cover the demand? What is the expected value of the shortage? What is the expected value of the number of units left over at the end of the operational lifetime of the plane?

9E-31 You have bought 10 young beech trees. They come from a garden center and were randomly chosen from a collection of 100 trees consisting of 50 trees from tree-nurseryman A and 50 trees from tree-nurseryman B. Ten percent of the trees from tree-nurseryman A and five percent of the trees

[3]This problem is based on D. Neal and F. Polivka, A geometric gambling game, *The Mathematical Spectrum*, Vol. 43 (2010), p. 20-25.

from tree-nurseryman B do not know grow well. What is the probability that no more than one of your ten trees will not grow well?

9E-32 In the lotto 6/49 six different numbers are drawn at random from $1, 2, \ldots, 49$. What is the probability that the next drawing will have no numbers common with the last two two drawings?

9E-33 Bill and Matt choose each five different numbers at random from the numbers $1, 2, \ldots, 100$. What is the expected number of common numbers in their choices? What is the probability that the choices of Bill and Matt have a number in common?

9E-34 You are offered the following game. You can repeatedly pick at random an integer from $1, \ldots, 25$. Each pick costs you one dollar. If you decide to stop, you get paid the dollar amount of your last pick. What strategy should you use to maximize your expected net payoff?

Chapter 10

10E-1 The density function of the continuous random variable X is given by $f(x) = c(x + \sqrt{x})$ for $0 < x < 1$ and $f(x) = 0$ otherwise. What is the constant c? What is probability density of $\frac{1}{X}$?

10E-2 The radius of a circle is uniformly distributed on $(0, 1)$. What is the probability density of the area of the circle?

10E-3 You choose at random a point inside a rectangle whose sides have the lengths 2 and 3. Let the random variable X be the distance from the point to the closest side of the rectangle. What is the probability density of X? What are the expected value and the standard deviation of X?

10E-4 Liquid waste produced by a factory is removed once a week. The weekly volume of waste in thousands of gallons is a continuous random variable with probability density function $fx) = 105x^4(1-x)^2$ for $0 < x < 1$ and $f(x) = 0$ otherwise. How to choose the capacity of a storage tank so that the probability of overflow during a given week is no more than 5%?

10E-5 Consider again Problem 10E-4. Assume that the storage tank has a capacity of 0.9 expressed in thousands of gallons. The cost of removing $x > 0$ units of waste at the end of the week is $1.25 + 0.5x$. Additional costs $5 + 10z$ are incurred when the capacity of the storage tank is not sufficient and an overflow of $z > 0$ units of waste occurs during the week. What are the expected value and the standard deviation of the weekly costs?

10E-6 You have to make an one-time business decision how much stock to order in order to meet a random demand during a single period. The demand is a continuous random variable X with a given probability density $f(x)$. Suppose you decide to order Q units. What is the probability that the initial stock Q will not be enough to meet the demand? What is the expected value of the stock left over at the end of the period? What is the expected value of demand that cannot be satisfied from stock?

10E-7 Let Q be a fixed point on the circumference of a circle with radius r. Choose at random a point P on the circumference of the circle and let the random variable X be the length of the line segment between P and Q. What are the expected value and the standard deviation of X?

10E-8 An insurance policy for water damage pays an amount of damage up to \$450. The amount of damage is uniformly distributed between \$250 and \$1,250. The amount of damage exceeding \$450 is covered by a supplement policy up to \$500. Let the random variable Y be the amount of damage paid by the supplement policy. What are the expected value and the probability distribution function of Y?

10E-9 An expensive item is being insured against early failure. The lifetime of the item is normally distributed with an expected value of seven years and a standard deviation of two years. The insurance will pay a dollars if the item fails during the first or second year and $\frac{1}{2}a$ dollars if the item fails during the third or fourth year. If a failure occurs after the fourth year, then the insurance pays nothing. How to choose a such that the expected value of the payment per insurance is \$50?

10E-10 Let Θ be a randomly chosen angle in $(0, \frac{\pi}{4})$. The random variable Y is defined as the y-coordinate of the point at which the ray through the origin at angle Θ intersects the line $x = 1$ in the plane. What are the expected value and the standard deviation of the area of the triangle with the corner points $(0, 0)$, $(1, 0)$ and $(1, Y)$? What is the probability density of this area of this triangle?

10E-11 A shot is fired at a very large circular target. The horizontal and vertical coordinates of the point of impact are independent random variables each having a standard normal density. Here the center of the target is taken as the origin. What is the density function of the distance from the center of the target to the point of impact? What are the expected value and the mode of this distance?

10E-12 Let X and Y be independent random variables each having the standard normal distribution. Consider the circle centered at the origin and passing through the point (X, Y). What is the probability density of the area of the circle? What is the expected value of this area?

10E-13 Let $X, Y)$ be a randomly chosen point on the circumference of the unit circle having $(0, 0)$ as center. What is the expected length of the line segment between the points (X, Y) and $(1, 0)$? *Hint*: note that X is distributed as $\cos(\Theta)$, where Θ is uniformly distributed on $(0, 2\pi)$.

10E-14 Choosing at random a point in $(0, 1)$ divides this interval into two subintervals. What is the expected value of the subinterval covering a given point s with $0 < s < 1$?

10E-15 A service station has a slow server (server 1) and a fast server (server 2). Upon arrival at the station, you are routed to server i with probability p_i for $i = 1, 2$, where $p_1 + p_2 = 1$. The service time at server i is exponentially distributed with parameter μ_i for $i = 1, 2$. What is the probability density of your service time at the station?

10E-16 Use first principles to answer the following questions.
(a) The random variable X has a standard normal distribution. What is the probability density of the random variable $Y = X^2$?
(b) The random variable X has a standard normal distribution. What is the probability density of the random variable $Y = \sqrt{|X|}$?

10E-17 The random variable X has the probability density function $f(x) = \frac{8}{\pi}\sqrt{x(1-x)}$ for $0 < x < 1$ and $f(x) = 0$ otherwise. What is the probability density function of the random variable $Y = 2X + 1$?

10E-18 The random variable X has the Cauchy density $f(x) = \frac{1}{\pi(1+x^2)}$ for $-\infty < x < \infty$. Use the transformation rule for random variables to find the probability density of the random variable $Y = \frac{1}{X}$.

10E-19 Let X be a continuous random variable with probability density function $f(x)$. How would you define the conditional expected value of X given that $X \leq a$? What is $E(X \mid X \leq a)$ when X is exponentially distributed with parameter λ?

10E-20 You wish to cross a one-way traffic road on which cars drive at a constant speed and pass according to independent interarrival times having an exponential distribution with an expected value of $1/\lambda$ seconds. You can only cross the road when no car has come round the corner since c time seconds. What is the probability distribution of the number of passing cars before you can cross the road when you arrive at an arbitrary moment? What property of the exponential distribution do you use?

10E-21 The amount of time needed to wash a car at a car washing station is exponentially distributed with an expected value of 15 minutes. You arrive at a car washing station, while the washing station is occupied and one other car is waiting for a washing. The owner of this car informs you that the car in the washing station is already there for 10 minutes. What is the probability that the car in the washing station will need no more than five other minutes? What is the probability that you have to wait more than 20 minutes before your car can be washed?

10E-22 You simulate a random observation from an exponentially distributed random variable X with expected value 1. What is the probability that the closest integer to the random observation is odd? What is this probability if the random observation is larger than a given even integer r? Can you explain why the two probabilities are the same?

10E-23 A crucial component of a reliability system operates in a good state during an exponentially distributed time with expected value $1/\mu$. After leaving the good state, the component enters a bad state. The system can still function properly in the bad state during a fixed time $a > 0$, but a failure of the system occurs after this time. The component is inspected every T time units, where $T > a$. It is replaced by a new one when the inspection reveals that the component is not in the good state. What is the probability the replacement of a particular component is because of a system failure? What is the expected time between two replacements?

10E-24 In a video game with a time slot of fixed length T, signals are generated according to a Poisson process with rate λ, where $T > \frac{1}{\lambda}$. During the time slot you can push a button only once. You win if at least one signal occurs in the time slot and you push the button at the occurrence of the last signal. Your strategy is to let pass a fixed time s with $0 < s < T$ and push the button upon the first occurrence of a signal (if any) after time s. What is your probability of winning the game? What value of s maximizes this probability?

10E-25 Cars pass through an out-of-the way village according to a Poisson process. The probability of one or more cars passing through the village during one hour is 0.64. What is the probability of a car passing through the village during the next half hour?

10E-26 Two instruments are used to measure the unknown length of a beam. If the true length of the beam is l, the measurement error made by the first instrument is normally distributed with mean 0 and standard deviation $0.006l$ and the the measurement error made by the first instrument is normally distributed with mean 0 and standard deviation $0.004l$. The two measurement errors are independent of each other. What is the probability that the average value of the two measurements is within 0.5% of the actual length of the beam?

10E-27 The lifetimes of two components in an electronic system are independent random variables X_1 and X_2, where X_i has a normal distribution with an expected value of μ_i time units and a standard deviation of σ_i time units. What is the probability that the lifetimes of the two components expire within a time units from each other?

10E-28 A space mission will take 150 days. A number of copies of a daily-use appliance must be taken along. The amount of time the appliance can be used is exponentially distributed with an expected value of two days. What is the probability mass function of the number of copies of the appliance to be used when an infinite supply would be available? Use the normal approximation to find how many copies of the appliance should be stocked so that the probability of a shortage during the mission is no more than 10^{-3}. Compare the approximate result to the exact result.

10E-29 A new casino has just been opened. The casino owner makes the following promotional offer to induce gamblers to play at his casino. People who bet \$10 on red get half their money back if they lose the bet, while they get the usual payout of \$20 if they win the bet. This offer applies only to the first 2,500 bets. In the casino European roulette is played so that a bet on red is lost with probability $\frac{19}{37}$ and is won with probability $\frac{18}{37}$. Use the normal distribution to approximate the probability that the casino owner will lose more than 6,500 dollars on the promotional offer.

10E-30 Consider again Problem 10E-29. Assume now that the casino owner makes an offer only to the most famous gambler in town. The casino owner lends \$1,000 to the gambler and proposes him to make 100 bets on red with this starting capital. The gambler is allowed to stake any amount of his bankroll at any bet. The gambler gets one fourth of the staked money back if he loses the bet, while he gets double the staked money back if he wins the bet. As reward, the gambler can keep to himself any amount in excess of \$1,000 when stopping after 100 bets. Suppose that the gambler decides to stake each time a fixed percentage of 5% of his current bankroll. Use the normal distribution to approximate the probability that the gambler takes home more than d dollars for $d = 0$, 500, 1,000 and 2,500. *Hint*: consider the logarithm of the size of the bankroll of the gambler.[4]

10E-31 A battery comes from supplier 1 with probability p_1 and from supplier 2 with probability p_2, where $p_1 + p_2 = 1$. A battery from supplier i has an exponentially distributed lifetime with expected value $1/\mu_i$ for $i = 1, 2$. The

[4]This problem is related to Kelly betting discussed in the Sections 2.7 and 5.9 of the book.

battery has already lasted s time units. What is the probability that the battery will last for another t time units?

10E-32 The failure rate function of the lifetime of a vacuum tube is $r(x) = \left(\mu_1 e^{-\mu_1 x} + \mu_2 e^{-\mu_2 x} - (\mu_1 + \mu_2) e^{-(\mu_1+\mu_2)x}\right) / \left(e^{-\mu_1 x} + e^{-\mu_2 x} - e^{-(\mu_1+\mu_2)x}\right)$ for $x > 0$, where $0 < \mu_1 < \mu_2$. Verify that the function $r(x)$ has a bathtub shape and determine the corresponding probability distribution function.

Chapter 11

11E-1 (a) A fair coin is tossed three times. Let X be the number of heads among the first two tosses and Y be the number of heads among the last two tosses. What is the joint probability mass function of X and Y? What is $E(XY)$?
(b) You have two fair coins. The first coin is tossed five times. Let the random variable X be the number of heads showing up in these five tosses. The second coin is tossed X times. Let Y be the number of heads showing up in the tosses of the second coin. What is the joint probability mass function of X and Y? What is $E(X + Y)$?

11E-2 In the final of the World Series Baseball, two teams play a series consisting of at most seven games until one of the two teams has won four games. Two unevenly matched teams are pitted against each other and the probability that the weaker team will win any given game is equal to 0.45. Let X be equal to 1 if the stronger team is the overall winner and X be equal to 0 otherwise. The random variable Y is defined as the number of games the final will take. What is the joint probability mass function of X and Y?

11E-3 A standard deck of 52 cards is thoroughly shuffled and laid face-down. You flip over the cards one by one. Let the random variable X_1 be the number of cards flipped over until the first ace appears and X_2 be the number of cards flipped over until the second ace appears. What is the joint probability mass function of X_1 and X_2? What are the marginal distributions of X_1 and X_2?

11E-4 You roll a fair die once. Let the random variable N be the outcome of this roll. Two persons toss each N fair coins, independently of each other. Let X be the number of heads obtained by the first person and Y be the number of heads obtained by the second person. What is the joint probability mass function of X and Y? What is the numerical value of $P(X = Y)$?

11E-5 You simultaneously roll d fair dice. Let the random variable X be the outcome of the highest scoring die and Y be the outcome of the second-highest scoring die with the convention that the second-highest score equals the highest score in the case that two or more dice yield the highest score. What is the joint probability mass function of X and Y?

11E-6 The joint probability mass function of the lifetimes X and Y of two connected components in a machine can be modeled by $p(x, y) = \frac{e^{-2}}{x!(y-x)!}$ for $x = 0, 1, \ldots$ and $y = x, x + 1, \ldots$.
(a) What are the marginal distributions of X and Y?
(b) What is the joint probability mass function of X and $Y - X$? Are X and $Y - X$ independent?
(c) What is the correlation between X and Y?

11E-7 A fair coin is rolled six times. Let X be the number of times a 1 is rolled and Y be the number of times a 6 is rolled. What is the joint probability mass function of X and Y? What is the correlation coefficient of X and Y?

11E-8 The joint density function of the continuous random variables X and Y is given by $f(x, y) = cxy$ for $0 < y < x < 1$ and $f(x, y) = 0$ otherwise. What is the constant c? What are the marginal densities $f_X(x)$ and $f_Y(y)$?

11E-9 The joint density function of the random variables X and Y is given by $f(x, y) = x + y$ for $0 \leq x, y \leq 1$ and $f(x, y) = 0$ otherwise. Consider the circle centered at the origin and passing through the point (X, Y). What is the probability that the circumference of the circle is no more than 2π?

11E-10 A stick is broken into three pieces at two randomly chosen points on the stick. What is the probability that no piece is longer than half the length of the stick?

11E-11 There are two alternative routes for a ship passage. The sailing times for the two routes are random variables X and Y that have the joint density function $f(x, y) = \frac{1}{10} e^{-\frac{1}{2}(y+3-x)}$ for $5 < x < 10$, $y > x - 3$ and $f(x, y) = 0$ otherwise. What is $P(X < Y)$?

11E-12 The joint density function of the random variables X and Y is given by $f(x, y) = xe^{-x(y+1)}$ for $x, y > 0$ and $f(x, y) = 0$ otherwise. What is the density function of the random variable XY?

11E-13 The joint density function of the random variables X and Y is given by $f(x, y) = \frac{1}{2}(x + y)e^{-(x+y)}$ for $x, y > 0$ and $f(x, y) = 0$ otherwise. What is the density function of the random variable $X + Y$?

11E-14 The lifetimes X and Y of two components in a machine have the joint density function $f(x, y) = \frac{1}{4}(2y + 2 - x)$ for $0 < x < 2$, $0 < y < 1$ and $f(x, y) = 0$ otherwise.
(a) What is the probability density of the time until neither of two components is still working?

(b) What is the probability distribution of the amount of time that the lifetime X survives the lifetime Y?

11E-15 An unreliable electronic system has two components hooked up in parallel. The lifetimes X and Y of the two components have the joint density function $f(x, y) = e^{-(x+y)}$ for $x, y \geq 0$. The system goes down when both components have failed. The system is inspected every T time units. At inspection any failed unit is replaced. What is the probability that the system goes down between two inspections? What is the expected amount of time the system is down between two inspections?

11E-16 An electronic device contains two circuits. The second circuit is a backup for the first and is switched on only when the first circuit has failed. The electronic device goes down when the second circuit fails. The continuous random variables X and Y denote the lifetimes of the first circuit and the second circuit and have the joint density function $f(x,\, y) = 24/(x + y)^4$ for $x, y > 1$ and $f(x, y) = 0$ otherwise. What is the expected value of the time until the electronic device goes down? What is the probability density function of this time?

11E-17(a) The joint density function $f(a, b)$ of the random variables A and B is given by $f(a, b) = a+b$ for $0 < a, b < 1$ and $f(a, b) = 0$ otherwise. What is the probability that the equation $Ax^2 + Bx + 1 = 0$ has two real roots?
(b) The joint density function $f(a, b, c)$ of the random variables A, B, and C is given by $f(a, b, c) = \frac{2}{3}(a + b + c)$ for $0 < a, b, c < 1$ and $f(a, b, c) = 0$ otherwise. What is the probability that the equation $Ax^2 + Bx + C = 0$ has two real roots?

11E-18 Choose three random numbers X_1, X_2 and X_3 from $(0, 1)$, independently of each other. What is the probability $P(X_1 > X_2 + X_3)$? What is the probability that the largest of the three random numbers is greater than the sum of the other two?

11E-19 Let $X_1, X_2, \ldots, X_n$ be independent random variables that are uniformly distributed on $(0, 1)$. What is $P(X_1 + X_2 + \cdots + X_n \leq 1)$? Answer this question first for $n = 2$ and $n = 3$.

11E-20 The random variables X and$Y$ are independent and uniformly distributed on $(0, 1)$. Let $V = X + Y$ and $W = \frac{X}{Y}$. What is the joint density of V and W? Are V and W independent?

11E-21 The random variables V and W are defined by $V = Z_1^2 + Z_2^2$ and $W = Z_1^2 - Z_2^2$, where Z_1 and Z_2 are independent random variables each having the standard normal distribution. What is the joint density function of V and W? Are V and W independent?

11E-22 The random variables X and Y are independent and exponentially distributed with parameter μ. Let $V = X + Y$ and $W = \frac{X}{X+Y}$. What is the joint density of V and W? Prove that V and W are independent.

11E-23 The continuous random variables X and Y have the joint density function $f(x,\, y) = cxe^{-\frac{1}{2}x(1+y^2)}$ for $x, y > 0$ and $f(x,\, y) = 0$ otherwise. What is the constant c? What are the marginal densities of X and Y? Show that the random variables $Y\sqrt{X}$ and X are independent. *Hint*: Use the fact that the gamma density $\lambda^{\alpha}x^{\alpha-1}e^{-\lambda x}/\Gamma(\alpha)$ integrates to 1 over $(0, \infty)$ for any $\lambda, \alpha > 0$ and note that $\Gamma(1.5) = \frac{1}{2}\sqrt{\pi}$.

11E-24 The continuous random variables X and Y have the joint density function $f(x,\, y) = 6(x - y)$ for $0 < y < x < 1$ and $f(x,\, y) = 0$ otherwise. Determine the correlation coefficient of X and Y.

11E-25 John and Pete are going to throw the discus in their very last turn at a champion game. The distances thrown by John and Pete are independent random variables D_1 and D_2 that are $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ distributed. What the best linear prediction of the distance thrown by John given that the difference between the distances of the throws of John and Pete is d?

Chapter 12

12E-1 Let Z_1 and Z_2 be independent random variables each having the standard normal distribution. Define the random variables X and Y by $X = Z_1 + 3Z_2$ and $Y = Z_1 + Z_2$. Argue that the joint distribution of (X, Y) is a bivariate normal distribution. What are the parameters of this distribution?

12E-2 Let the random vector $(X,\, Y)$ have the standard bivariate normal distribution with correlation coefficient $\rho = -0.5$. What are the values of a for which the random variables $V = aX + Y$ and $W = X + aY$ are independent?

12E-3 Let the random vector $(X,\, Y)$ have the standard bivariate normal distribution with correlation coefficient ρ with $-1 < \rho < 1$. Let Z be an $N(0, 1)$ distributed random variable that is independent of the random variable X. Verify that the random vector $(X,\, \rho X + \sqrt{1 - \rho^2} Z)$ has the same standard bivariate normal distribution as $(X,\, Y)$.

12E-4 Let the random vector $(X,\, Y)$ have the standard bivariate normal distribution with correlation coefficient ρ with $-1 < \rho < 1$. What are the probabilities $P(Y > X \mid X > 0)$ and $P(Y/X \leq 1)$?

Month	Observed	Expected
Jan	60,179	61,419.5
Feb	54,551	55,475.7
Mar	59,965	61,419.5
Apr	57,196	59,438.2
May	59,444	61,419.5
Jun	59,459	59,438.2
Jul	62,166	61,419.5
Aug	60,598	61,419.5
Sep	62,986	59,438.2
Oct	64,542	61,419.5
Nov	60,745	59,438.2
Dec	61,334	61,419.5

12E-5 Let $\mathbf{X} = (X_1, \ldots, X_n)$ and $\mathbf{Y} = (Y_1, \ldots, Y_m)$ be independent random vectors each having a multivariate normal distribution. Prove that the random vector $(\mathbf{X}, \mathbf{Y})$ has also a multivariate normal distribution.

12E-6 In the table the column "Observed" gives the observed frequencies of birth months for the children born in England and Wales in the year 2010. The column "Expected" gives how many births could have been expected in each month under the hypothesis that all birth dates are equally likely. Use the chi-square test to make clear that the assumption of equally likely birth dates is not satisfied in reality.[5]

12E-7 In a famous physics experiment performed by Rutherford, Chadwick and Ellis in 1920, the number α-particles emitted by a piece of radioactive material were counted during 2,608 time intervals of each 7.5 seconds. There were 57 intervals with zero particles, 203 intervals with 1 particle, 383 intervals with 2 particles, 525 intervals with 3 particles, 532 intervals with 4 particles, 408 intervals with 5 particles, 273 intervals with 6 particles, 139 intervals with 7 particles, 45 intervals with 8 particles, 27 intervals with 9 particles, 10 intervals with 10 particles, 4 intervals with 11 particles, 0 intervals with 12 particles, 1 interval with 13 particles, and 1 interval with 14 particles. Use a chi-square test to investigate how closely the observed frequencies conform to Poisson frequencies.

12E-8 Vacancies in the U.S. Supreme Court over the 78-years period 1933-

[5]For birthday problems it can be shown the probability of matching birthdays gets larger when the assumption that each day of the year is equally likely as birthday is dropped.

2010 have the following history: 48 years with 0 vacancies, 23 years with 1 vacancy, 7 years with 2 vacancies, and 0 years with ≥ 3 vacancies. Test the hypothesis that the number of vacancies per year follows a Poisson distribution.

Chapter 13

13E-1 In the final of the World Series Baseball, two teams play a series consisting of at most seven games until one of the two teams has won four games. Two unevenly matched teams are pitted against each other and the probability that the weaker team will win any given game is equal to 0.45. What is the conditional probability mass function of the number of games played in the final given that the weaker team has won the final?

13E-2 Let $P(X = n, Y = k) = \binom{n}{k}\left(\frac{1}{6}\right)^k\left(\frac{1}{3}\right)^{n-k}$ for $k = 0, 1, \ldots, n$ and $n = 1, 2, \ldots$ be the joint probability mass function of the random variables X and Y. What is the conditional distribution of Y given that $X = n$?

13E-3 A fair die is repeatedly rolled. Let the random variable X be the number of rolls until the face value 1 appears and Y be the number of rolls until the face value 6 appears. What are $E(X \mid Y = 2)$ and $E(X \mid Y = 20)$?

13E-4 The continuous random variables X and Y satisfy $f_Y(y \mid x) = \frac{1}{x}$ for $0 < y < x$ and $f_Y(y \mid x) = 0$ otherwise. The marginal density function of X is given by $f_X(x) = 2x$ for $0 < x < 1$ and $f_X(x) = 0$ otherwise. What is the conditional density $f_X(x \mid y)$? What is $E(X \mid Y = y)$?

13E-5 The random variables X and Y have the joint density function $f(x, y) = e^{-y}$ for $0 \leq x \leq y$ and $f(x, y) = 0$ otherwise. Describe a method to simulate a random observation from $f(x, y)$.

13E-6 Consider again Problem 11E-16. What is the expected value of the lifetime of the second circuit given that the first circuit has failed after s time units? What is the probability that the second circuit will work more than v time units given that the first circuit has failed after s time units?

13E-7 Eleven closed boxes are put in random order in front of you. One of these boxes contains a devil's penny and the other ten boxes contain given dollar amounts $a_1, \ldots, a_{10}$. You may mark as many boxes as you wish. The marked boxes are opened. You win the money from these boxes if the box with the devil's penny is not among the opened boxes; otherwise, you win nothing. How many boxes should you mark to maximize your expected return?

13E-8 (a) You first toss a fair coin five times. Next you toss the coin as many times as the number of heads showing up in these five tosses. Let the random variable X be the number of heads in all tosses of the coin, including the first five tosses. Use conditional expectations to find the expected value of X.
(b) You first roll a fair die once. Next you roll the die as many times as the outcome of this first roll. Let the random variable X be the total number of sixes in all the rolls of the die, including the first roll. Use conditional expectations to find the expected value of X.

13E-9 The random variables X and Y have a joint density function. The random variable Y is positive with $E(Y) = 1$ and $\sigma^2(Y) = 2$. The conditional distribution of X given that $Y = y$ is the uniform distribution on $(1-y, 1+y)$ for any y. What are $E(X)$ and $\sigma^2(X)$?

13E-10 Let X_1 and X_2 be independent random variables each having a geometric distribution with parameter p. What is the conditional probability mass function of X_1 given that $X_1 + X_2 = r$?

13E-11 You draw at random a number p from the interval $(0, 1)$. Next you toss n times a coin with probability p of heads. What is the probability mass function of the number of times that heads will appear? *Hint*: Use the fact that the beta integral $\int_0^1 x^{r-1}(1-x)^{s-1}\,dx$ is equal to $(r-1)!(s-1)!/(r+s-1)!$ for positive integers r and s.

13E-12 Let X_1 and X_2 be independent random variables each having an exponential distribution with parameter μ. What is the conditional probability density function of X_1 given that $X_1 + X_2 = s$?

13E-13 Let Θ and R be independent random variables, where Θ is uniformly distributed on $(-\pi, \pi)$ and R is a positive random variable with density function $re^{-\frac{1}{2}r^2}$ for $r > 0$. Define the random variables V and W by $V = R\cos(\Theta)$ and $W = R\sin(\Theta)$. What is the conditional density function of V given that $W = w$? What is $E(V \mid W = w)$?

13E-14 Let $X_1, X_2, \ldots$ be independent random variables that are uniformly distributed on $(0, 1)$. The random variable N is defined as the smallest $n \geq 2$ for which $X_n > X_1$. What is the probability mass function of N?

13E-15 Let X, Y and Z be independent random variables each having a Poisson distribution with expected value λ. Use the law of conditional probability to find the joint probability mass function of $V = X + Y$ and $W = X + Z$?

13E-16 Suppose N cars start in a random order along an infinitely long one-lane highway. They are all going at different but constant speeds and cannot pass each other. If a faster car ends up behind a slower car, it must slow down to the speed of the slower car. Eventually the cars will clump up in traffic jams. Use a recursion to find the expected number of clumps of cars? A clump is a group of one or more cars. *Hint*: consider the position of the slowest car.

13E-17 Suppose that the random variables X and Y have a joint density function $f(x, y)$. Prove that $\text{cov}(X, Y) = 0$ if $E(X \mid Y = y)$ does not depend on y.

13E-18 Let U_1 and U_2 be two independent random variables that are uniformly distributed on $(0, 1)$. How would you define the conditional densities of U_1 and U_2 given that $U_1 > U_2$? What are $E(U_1 \mid U_1 > U_2)$ and $E(U_2 \mid U_1 > U_2)$?

13E-19 Let X and Y be two independent random variables that have the same exponential density function with expected value $\frac{1}{\lambda}$. What are $E(X \mid X > Y)$ and $E(Y \mid X > Y)$?

13E-20 Suppose that the random variable B has the standard normal density. What is the conditional probability density function of the sum of the two roots of the quadratic equation $x^2 + 2Bx + 1 = 0$ given that the two roots are real?

13E-21 The transmission time of a message requires a geometrically distributed number of time slots, where the geometric distribution has parameter a with $0 < a < 1$. In each time slot one new message arrives with probability p and no message arrives with probability $1 - p$. What are the expected value and the standard deviation of the number of newly arriving messages during the transmission time of a message?

13E-22 In a buffer there are a geometrically distributed number of messages waiting to be transmitted over a communication channel, where the parameter p of the geometric distribution is known. Your message is one of the waiting messages. The messages are transmitted one by one in a random order. Let the random variable X be the number of messages that are transmitted before your message. What are the expected value and the standard deviation of X?

13E-23 The following game is offered in a particular carnival tent. The carnival master has a red and a blue beaker each containing 10 balls numbered as $1, \ldots, 10$. He shakes the beakers thoroughly and picks at random one ball from each beaker. Then he tells you the value r of the ball picked from the red beaker and asks you to guess whether the unknown value b of the ball picked from the blue beaker is larger than r or smaller than r. If you guess correctly, you get b dollars. If $r = b$, you get $\frac{1}{2}b$ dollars. If you are wrong about which is larger, you get nothing. You have to pay \$4.50 for the privilege of playing the game. Is this a fair game?[6]

13E-24 A bin contains N strings. You randomly choose two loose ends and tie them up. You continue until there are no more free ends. What is the expected number of loops you get?

13E-25 (**a**) You sample a random observation of a Poisson distributed random variable with expected value 1. The result of the random draw determines the number of times you toss a fair coin. What is the probability distribution of the number of heads you will obtain?
(**b**) You perform the following experiment. First you generate a random number from $(0, 1)$. Then you simulate an integer by taking a random observation from a Poisson distribution whose expected value is given by the random number you have generated. Let the random variable X be the integer you will obtain from this experiment. What is the probability mass function of X?

13E-26 You simulate a random observation from the random variable X with the gamma density $x^{r-1}[(1-p)/p]^r e^{-(1-p)x/p}/\Gamma(r)$, where r and p are given positive numbers. Then you generate an integer by taking a random observation from a Poisson distribution whose expected value is given by the number you have simulated from the gamma density. Let the random variable N denote the generated integer. What is the probability mass function of N?

13E-27 Let the random variable Z have the standard normal distribution. Describe how to draw a random observation from $|Z|$ by using the acceptance-rejection method with the envelope density function $g(x) = e^{-x}$? How do you get a random observation from Z when you have simulated a random observation from $|Z|$?

13E-28 Your friend has fabricated a loaded die. In doing so, he has first simulated a number at random from the interval $(0.1, 0.4)$. He tells you that the die is loaded in such a way that any roll of the die results in the outcome 6 with a probability which is equal to the simulated number. Next the die is rolled 300 times and you are informed that the outcome 6 has appeared 75 times. What is the posterior density of the probability that a single roll of the die gives a 6?

13E-29 Your friend is a basketball player. To find out how good he is in free throws, you ask him to shoot 10 throws. Your priority density $f_0(\theta)$ of the

[6]This problem is inspired by the Numberplay Blog of the New York Times, March 4, 2013.

success probability of the free throws of your friend is a triangular density on $(0.25, 0.75)$ with mode at 0.50. That is, $f_0(\theta) = 16(\theta - 0.25)$ for $0.25 < \theta \leq 0.50$, $f_0(\theta) = 16(0.75 - \theta)$ for $0.50 < \theta < 0.75$, and $f_0(\theta) = 0$ otherwise. What is the posterior density function $f(\theta)$ of the success probability given that your friend scores 7 times out of the 10 throws? What value of θ maximizes the posterior density? Give a 95% Bayesian confidence interval for the success probability θ.

13E-30 A job applicant will have an IQ test. The prior density of his IQ is the $N(\mu_0, \sigma_0^2)$ density with $\mu_0 = 100$ and $\sigma_0 = 15$. If the true value of the IQ of the job applicant is x, then the test will result in a score that has the $N(x, \sigma_1^2)$ distribution with $\sigma_1 = 7.5$. The test results in a score of 123 points for the job applicant. What is the posterior density $f(\theta \mid \text{data})$ of the IQ of the job applicant? Give the value of θ for which the posterior density is maximal and give a 95% Bayesian confidence interval for θ.

Chapter 14

14E-1 Let $G_X(z)$ be the generating function of the nonnegative, integer-valued random variable X. Verify that $P(X \text{ is even }) = \frac{1}{2}(G_X(-1) + 1)$.

14E-2 You first roll a fair die once. Next you roll the die as many times as the outcome of this first roll. Let the random variable S be the sum of the outcomes of all the rolls of the die, including the first roll. What is the generating function of S? What are the expected and standard deviation of S?

14E-3 You perform a sequence of independent Bernoulli trials with success probability p. Let the random variable X be the number of trials needed to obtain three successes in a row or three failures in a row. Determine the generating function of X. *Hint*: let X_1 (X_2) be the additional number of trials to obtain three successes in a row or three failures in a row when the first trial results in a success (failure).

14E-4 Suppose that the random variable X has the so-called logistic density function $f(x) = e^x/(1+e^x)^2$ for $-\infty < x < \infty$. What is the interval on which the moment-generating $M_X(t)$ is defined? Use $M_X(t)$ to find the expected value and variance of X.

14E-5 Suppose that the moment-generating function $M_X(t)$ of the continuous random variable X has the property $M_X(t) = e^t M_X(-t)$ for all t. What is $E(X)$? Do you think that this property determines the density of X?

14E-6 You do not know the probability distribution of a random variable X, but you do know its mean μ and standard deviation σ. Use Chebyshev's

inequality to answer the following two questions.
(a) What is a value for k such that the random variable X falls in the interval $(\mu - k\sigma, \mu + k\sigma)$ with a probability of at least p?
(b) Prove that $P(X > \mu + a) \leq \frac{\sigma^2}{a^2+\sigma^2}$ for any constant $a > 0$.

14E-7 (a) What is the moment-generating function of the random variable X having the uniform distribution on the interval $(-1, 1)$?
(b) Let $X_1, X_2, \ldots, X_n$ be independent random variables that are uniformly distributed on the interval $(-1, 1)$. Verify the Chernoff bound $P\big(\frac{1}{n}(X_1 + X_2 + \cdots + X_n) \geq c\big) \leq e^{-\frac{3}{2}c^2 n}$ for $c > 0$.

14E-8 Let X be a Poisson distributed random variable with expected value λ. Give a Chernoff bound for $P(X \geq c)$ when $c > \lambda$.

14E-9 In a branching process with one ancestor, the number of offspring of each individual has the shifted geometric distribution $\{p_k = p(1-p)^k,\ k = 0, 1, \ldots\}$ with parameter $p \in (0, 1)$. What is the probability of extinction as function of p?

14E-10 At a production facility orders arrive one at a time, where the interarrival times of the orders are independent random variables each having the same distribution with expected value μ. A production is started only when N orders have accumulated. The production time is negligible. A fixed cost of $K > 0$ is incurred for each production setup and holding costs are incurred at the rate of hj when j orders are waiting to be processed. Identify a regenerative stochastic process and find the long-run average cost per unit time as function of N. What value of N minimizes the long-run average cost per unit time?[7]

14E-11 Messages arriving at a communication channel according to a Poisson process with rate λ are temporarily stored in a buffer with ample capacity. The buffer is emptied every T time units, where $T > 0$ is fixed. There is a fixed cost of $K > 0$ for emptying the buffer. A holding cost at rate $h > 0$ per unit time is incurred for each message in the buffer. Identify a regenerative stochastic process and find the long-run average cost per unit time as function of T. What value of T minimizes the long-run average cost per unit time?

14E-12 A canal touring boat departs for a tour through the canals of Amsterdam every T minutes with T fixed. Potential customers pass the point of departure according to a Poisson process with rate λ. A potential customer who sees that the boat leaves s minutes from now joins the boat with probability $e^{-\mu s}$ for $0 \leq s \leq T$. Assume that a fixed cost of $K > 0$ is incurred for

[7] The problems 14E-10 to 14E-14 are a bonus and illustrate the power of the strong law of large numbers through the renewal-reward theorem.

each round trip and that a fixed amount $R > 0$ is earned for each passenger. Identify a regenerative stochastic process to find the long-run average net reward per unit time as function of T. What value of T maximizes this long-run average net reward? *Hint*: use the fact the arrival time of the nth potential customer has the density $\lambda^n t^{n-1} e^{-\lambda t}/(n-1)!$ and use conditioning to find the probability that the nth arrival joins the first trip of the touring boat.

14E-13 Customers asking for a single product arrive according to a Poisson process with rate λ. Each customer asks for one unit of the product. Each demand which cannot be satisfied directly from stock on hand is lost. Opportunities to replenish the inventory occur according to a Poisson process with rate μ. This process is assumed to be independent of the demand process. For technical reasons a replenishment can only be made when the inventory is zero. The inventory on hand is raised to the level Q each time a replenishment is done. Use the renewal reward theorem to find the long-run fraction of time the system is out of stock and the long-run fraction of demand that is lost. Verify that these two fractions are equal to each other.

14E-14 Suppose that jobs arrive at a work station according to a Poisson process with rate λ. The work station has no buffer to store temporarily arriving jobs. An arriving job is accepted only when the work station is idle, and is lost otherwise. The processing times of the jobs are independent random variables having a common probability distribution with finite mean β. Use the renewal-reward theorem to find the long-run fraction of time the work station is busy and the long-run fraction of jobs that are lost. Conclude that these two fractions are equal to each other and that they use the probability distribution of the processing time only through its expected value.

Solutions of the Probability Exam Questions

Here we give the fully worked-out solutions of all of the probability exam questions.

Chapter 7

7E-1 The choice of the sample space depends on whether we care about the order in which the cards are chosen. If we consider the order in which the 13 cards are chosen as being relevant, we take an ordered sample space. Let us assume that the 52 cards of the deck as $1, 2, \ldots, 52$ before the deck is thoroughly shuffled. Imagine that the 13 cards of the bridge hand are chosen one at a time. The ordered sample space is the set of outcomes all possible ordered 13-tuples $(i_1, \ldots, i_{13})$, where i_k is the number of the kth chosen card. The sample space has $52 \times 51 \times \cdots \times 40$ equally likely outcomes. There are $32 \times 31 \times \cdots \times 20$ outcomes for which there is no card above a nine among the 13 cards chosen. Hence the probability of a Yarborough is

$$\frac{52 \times 51 \times \cdots \times 40}{32 \times 31 \times \cdots \times 20} = 0.000547.$$

Alternatively, this probability can be computed by using an unordered sample space. The order in which the 13 cards are chosen is not relevant in an unordered sample space. Each outcome of this sample space is a set of 13 different cards from the deck of 52 cards. In a set we don't care which element is first, only which elements are actually present. The number of ways you can choose a set of 13 different cards from a deck of 52 cards is given by the binomial coefficient $\binom{52}{13}$. Hence the unordered sample space has $\binom{52}{13}$ equally likely outcomes. The number of outcomes with no card above a nine is $\binom{32}{13}$. This leads to the same value for the desired probability of a Yarborough:

$$\frac{\binom{32}{13}}{\binom{52}{13}} = 0.000547.$$

This probability says that the odds against a Yarborough are 1827 to 1. A Yarborough is named after Lord Yarborough (1809–1862). The second Earl of Yarborough would offer his whist-playing friends a wager of 1,000 pounds to 1 against them picking up such a hand. You see that the odds were on his side. There is no record that he ever paid off.

7E-2 Take an ordered sample space whose outcomes are given by all possible orderings of five different cells. The total number of outcomes of the sample space is $64 \times 63 \times 62 \times 61 \times 60$ and all outcomes are equally probable. The number of outcomes for which no row or column contains more than one dot is $64 \times 49 \times 36 \times 25 \times 16$. Hence the desired probability is

$$\frac{64 \times 49 \times 36 \times 25 \times 16}{64 \times 63 \times 62 \times 61 \times 60} = 0.0494.$$

Alternatively, the desired probability can be computed by using an unordered sample space. Take the sample space whose outcomes are given by all unordered sets of five different cells. The total number of outcomes of the sample space is $\binom{64}{5}$. Noting that you can choose five different rows in $\binom{8}{5}$ ways, it follows that the number of choices of five different cells for which no row or column contains more than one dot is $\binom{8}{5} \times 8 \times 7 \times 6 \times 5 \times 4$. This gives the same result

$$\frac{\binom{8}{5} \times 8 \times 7 \times 6 \times 5 \times 4}{\binom{64}{5}} = 0.0494$$

for the desired probability.

7E-3 Take an ordered sample space with as outcomes all possible 5! orderings of the five people, where the first two people in the ordering get tea from the waiter and the other three get coffee. The number of orderings in which the first two people have ordered coffee and the other people have ordered tea is $2! \times 3!$. Hence the desired probability is $\frac{2! \times 3!}{5!} = 0.1$. Alternatively, we can take an unordered sample space whose outcomes are given by all possible choices of two people from the five people, where these two people get coffee. This leads to the same probability of $1/\binom{5}{2} = 0.1$.

7E-4 Take an unordered sample space whose outcomes are given by all possible sets of the three different parking places out of the ten parking places. The number of outcomes of the sample space is $\binom{10}{3}$. Each outcome of the sample space gets assigned the same probability of $1/\binom{10}{3}$. The number of outcomes with three adjacent parking places is 8. Hence the desired probability is

$$\frac{8}{\binom{10}{3}} = \frac{1}{15}.$$

7E-5 Imagine that the $7 \times 8 = 56$ apartments are numbered as $1, 2, \ldots, 56$. Take as sample space the set of all unordered pairs $\{i, j\}$ of two distinct numbers from $1, 2, \ldots, 56$, where each pair corresponds to two vacant apartments. The sample space has $\binom{56}{2} = 1{,}540$ equally likely elements. The number of elements for which there is one apartment vacant on the top floor is $\binom{8}{1} \times \binom{48}{1} = 384$ and the number of elements for which there are two apartments vacant on the top floor is $\binom{8}{2} = 28$. Hence the desired probability is

$(384 + 28)/1{,}540 = 0.2675$. Alternatively, an ordered sample space can be used to answer the question. Take as ordered sample the set of all 56! permutations of the apartments in the building, where the first two elements in the permutation refer to the vacant apartments. The number of permutations with no vacancy on the top floor is $48 \times 47 \times 54!$. Hence the probability that there is no vacancy on the top floor is $\frac{48\times 47\times 54!}{56!} = 0.7325$.

7E-6 Take as sample space the set of all unordered pairs of two distinct cards. The sample space has $\binom{52}{2} = 1{,}326$ equally likely elements. There are $\binom{1}{1} \times \binom{51}{1} = 51$ elements with the ten of hearts, and $\binom{3}{1} \times \binom{12}{1} = 36$ elements with hearts and a ten but not the ten of hearts. Hence the desired probability is $(51 + 36)/(1{,}326) = 0.0656$.

7E-7 Imagine that the 12 pieces of fruit are numbered as $1, 2, \ldots, 12$. There are 12! possible orders at which the 12 pieces of fruit can be taken out of the box. There are $7 \times 11!$ orders in which the last element is an apple. Hence the probability that the bowl will be empty after the last apple is taken from the box is equal to

$$\frac{7 \times 11!}{12!} = \frac{7}{12}.$$

7E-8 A same sample space can be used to answer both questions. Take an ordered sample space whose outcomes are given by all possible 5-tuples $(i_1, i_2, \ldots, i_5)$, where i_k is the exit floor for person k for $k = 1, \ldots, 5$. The sample space has 10^5 outcomes. Let us first consider the case that the floors are equally likely to be chosen as exit floor. Then the probability $\frac{1}{10^5}$ is assigned to each element of the sample outcomes. The number of outcomes for which the five persons are all going to different floors is $10 \times 9 \times 8 \times 7 \times 6$. Hence the probability that they are all going to different floors is

$$\frac{10 \times 9 \times 8 \times 7 \times 6}{10^5} = 0.3024.$$

For the case that the 10 floors are not equally likely to be chosen as the exit floor, the probability $\left(\frac{1}{2}\right)^r \left(\frac{1}{18}\right)^{5-r}$ is assigned to each outcome $(i_1, i_2, \ldots, i_5)$ for which exactly r of the components are equal to 10. The number of outcomes for which the five persons are all going to different floors and floor 10 is not chosen as the exit floor is equal to $9 \times 8 \times 7 \times 6 \times 5 = 15{,}120$, while the number of outcomes for which the five persons are all going to different floors and exactly one person chooses floor 10 as the exit floor is equal to $\binom{5}{1} \times 9 \times 8 \times 7 \times 6 = 15{,}120$. Then the probability that the persons are all going to different floors is

$$15{,}120 \times \left(\frac{1}{18}\right)^5 + 15{,}120 \times \frac{1}{2}\left(\frac{1}{18}\right)^4 = 0.0800.$$

7E-9 Take as sample space the set of all possible orderings of the ten people on the ten seats. The sample space has 10! equally likely outcomes.
(a) The number of ways to choose three adjacent seats from the ten seats is 8. Hence the number of outcomes for which the three friends are seated next to each other is $8 \times 3! \times 7!$ and so the probability that the three friends will sit next to each other is

$$\frac{8 \times 3! \times 7!}{10!} = \frac{1}{15}.$$

(b) The number of ways to choose two friends from the three friends is $\binom{3}{2}$. The number of outcomes for which the two chosen friends sit next to each other at one of the ends of the row and the third friend has a non-adjacent seat is $2 \times 7 \times 2! \times 7!$, while the number of outcomes for which the two chosen friends sit next to each other but not at one of the ends of the row and the third friend has a non-adjacent seat is $7 \times 6 \times 2! \times 7!$. Hence the probability that exactly two of the three friends will sit next to each other is

$$\frac{\binom{3}{2}\left[2 \times 7 \times 2! \times 7! + 7 \times 6 \times 2! \times 7!\right]}{10!} = \frac{7}{15}.$$

7E-10 Take an ordered sample space whose outcomes are given by the 10! possible orderings of the 10 people, where the first five people in the ordering form the first group and the other five people form the second group. Each element of the sample space is equally likely and so the desired probability is

$$\frac{5 \times 4 \times 3 \times 7! + 5 \times 4 \times 3 \times 7!}{10!} = \frac{1}{6}.$$

Alternatively, by taking an unordered sample space whose outcomes are given by all possible sets of five people for the first group, the probability can be computed as

$$\frac{\binom{7}{2} + \binom{7}{5}}{\binom{10}{5}} = \frac{1}{6}.$$

7E-11 This problem is easiest solved by using an unordered sample space. Take as sample space the set of all possible sets of four cards from the 52 cards. The sample space has $\binom{52}{4}$ equally likely elements. The number sets in which the four cards are J, Q, K, A is $\binom{4}{1} \times \binom{4}{1} \times \binom{4}{1} \times \binom{4}{1}$. Hence the desired probability is

$$\frac{\binom{4}{1} \times \binom{4}{1} \times \binom{4}{1} \times \binom{4}{1}}{\binom{52}{4}} = 9.46 \times 10^{-4}.$$

You can also use an ordered sample space. Take an ordered sample space whose outcomes are given by all 52! possible orderings of the 52 cards. There are $4^4 \times 4! \times 48!$ orderings for which the first four cards are J, Q, K, A in any order, with suit irrelevant. This gives the same probability

$$\frac{4^4 \times 4! \times 48!}{52!} = 9.46 \times 10^{-4}.$$

7E-12 Let A be the event that there is no ace among the five cards and B be the event that there is neither a king nor a queen among the five cards. The desired probability is given by

$$\begin{aligned} 1 - P(A \cup B) &= 1 - \big[P(A) + P(B) - P(AB)\big] \\ &= 1 - \left[\frac{\binom{48}{5}}{\binom{52}{5}} + \frac{\binom{44}{5}}{\binom{52}{5}} - \frac{\binom{40}{5}}{\binom{52}{5}}\right] = 0.1765. \end{aligned}$$

7E-13 Label both the balls and the boxes as 1, 2, and 3. Take an ordered sample space whose elements are all possible three-tuples (b_1, b_2, b_3), where b_i is the label of the box in which ball i is dropped. The sample space has 3^3 equally likely elements. Let A_i be the event that only box i is empty. The events A_1, A_2 and A_3 are mutually disjoint and have the same probability. Hence the desired probability is $P(A_1 \cup A_2 \cup A_3) = 3P(A_1)$. To find $P(A_1)$, note that the number of elements for which box 2 (3) contains two balls and box 3 (2) contains one ball is $\binom{3}{2} \times 1 = 3$. Hence the number of elements for which only box 1 is empty is $2 \times 3 = 6$. This gives $P(A_1) = \frac{6}{27}$ and so the probability that exactly one box will be empty is $\frac{2}{3}$.

7E-14 The sample space consists of all four-tuples $(\delta_1, \delta_2, \delta_3, \delta_4)$ with δ_i is 0 or 1, where $\delta_i = 0$ if component i has failed and $\delta_i = 1$ otherwise. The probability $r_1 r_2 r_3 r_4$ is assigned to each outcome $(\delta_1, \delta_2, \delta_3, \delta_4)$, where $r_i = f_i$ if $\delta_i = 0$ and $r_i = 1 - f_i$ if $\delta_i = 1$. It is easiest to compute the complementary probability of no system failure. Let A be the event that none of the four components has failed and B be the event that only one of the components 2, 3, or 4 has failed. Then $P(A \cup B)$ gives the probability of no system failure. The events A and B are mutually disjoint and so $P(A \cup B) = P(A) + P(B)$. Obviously, $P(A) = (1-f_1)(1-f_2)(1-f_3)(1-f_4)$ and $P(B) = (1-f_1)f_2(1-f_3)(1-f_4) + (1-f_1)(1-f_2)f_3(1-f_4) + (1-f_1)(1-f_2)(1-f_3)f_4$. Hence the probability that the system will fail is given by

$$\begin{aligned} &1 - \big[(1-f_1)(1-f_2)(1-f_3)(1-f_4) + (1-f_1)f_2(1-f_3)(1-f_4) \\ &+ (1-f_1)(1-f_2)f_3(1-f_4) + (1-f_1)(1-f_2)(1-f_3)f_4\big]. \end{aligned}$$

7E-15 The sample space of this experiment is the set $\{(x, y) : 0 \le x, y \le 1\}$. The probability $P(A)$ assigned to each subset A of the unit square is the area of the set A. For fixed a with $0 < a < 2$, let A be the subset of points (x, y) in the unit square that satisfy $x + y \leq a$. The area of the set A is given by $\frac{1}{2}a^2$ for $0 < a \leq 1$ and by $1 - \frac{1}{2}(2-a)^2$ for $1 \leq a < 2$ (draw a picture). *Note*: the probability of a Manhattan distance of no more than a is also equal to $\frac{1}{2}a^2$ for $0 < a \leq 1$ and to $1 - \frac{1}{2}(2-a)^2$ for $1 \leq a < 2$ when the point is randomly chosen in the square $\{(x, y) : -1 \leq x, y \leq 1\}$, as follows by using a symmetry argument.

7E-16 The sample space is the set $\{(x, y) : 0 \leq x \leq 3,\ 0 \leq y \leq 2\}$. Let A be the subset of points from the rectangle for which the distance to the closest side of the rectangle is larger than a. Then A is a rectangle whose sides have the lengths $3 - 2a$ and $2 - 2a$ (it is helpful to make a picture) and so the area of A is $(3 - 2a)(2 - 2a)$. It now follows that the desired probability is

$$\frac{6 - (3 - 2a)(2 - 2a)}{6} = \frac{5}{3}a - \frac{2}{3}a^2.$$

7E-17 For the case of $n = 1$, the sample space consists of the eight 3-tuples (H, H, H), (H, H, T), (H, T, H), (H, T, T), (T, H, H), (T, H, T), (T, T, H), and (T, T, T), where the first two components refer to the outcomes of the two coins of Pete. This gives that the probability of Pete getting more heads than John is $\frac{4}{8} = \frac{1}{2}$ for $n = 1$. By considering all possible 5-tuples of H's and T's for the case of $n = 2$, the value $\frac{16}{32} = \frac{1}{2}$ is found for the probability of Pete getting more heads than John. For the general case the sample space consists of the 2^{2n+1} possible $2n + 1$-tuples of H's and T's. The probability of Pete getting more heads than John is then given by

$$\frac{1}{2^{2n+1}} \sum_{k=0}^{n} \binom{n}{k} \sum_{j=k+1}^{n+1} \binom{n+1}{j}.$$

Evaluating this expression for several values of n gives each time the value $\frac{1}{2}$. A very simple probabilistic argument can be given for the result that the probability of Pete getting more heads than John is always equal to $\frac{1}{2}$. Let A be the event that Pete gets more heads than John and B be the event that Pete gets more tails than John. Since Pete has only coin more than John, the events A and B are disjoint. In view of the fact that the total number of coins is odd, it also holds that the union of A and B is the complete sample space. Hence $P(A \cup B) = P(A) + P(B) = 1$. By a symmetry argument $P(A) = P(B)$ and so $P(A) = \frac{1}{2}$ as was to be verified.

7E-18 Label the 11 balls as $1, 2, \ldots, 11$. Think of the order in which the balls are drawn out of the bag as a permutation of $1, 2, \ldots, 11$. The sample space is the set of all permutations of the numbers $1, 2, \ldots, 11$. All ordered outcomes are equally likely. Let A_i be the event that a red ball appears for the first time at the ith drawing. The events A_i are mutually disjoint and so the probability that Bill is the first person to pick a red ball is $\sum_{k=1}^{4} P(A_{2k})$. The set A_i contains $\binom{7}{i-1} \times (i-1)! \times 4 \times (7-(i-1)+3)!$ outcomes. Hence

$$P(A_i) = \frac{\binom{7}{i-1} \times (i-1)! \times 4 \times (7-(i-1)+3)!}{11!} \quad \text{for } i = 1, 2, \ldots 8.$$

This leads to the value $\frac{13}{33}$ for the probability that Bill is the first person to pick a red ball.

7E-19 Take the same sample space as in the solution of Example 7.6 in the book. Let A_i be the event that the deadly shot falls at the ith trial. Then $P(A_i) = \left(\frac{5}{6}\right)^{i-1} \frac{1}{6}$ for $i = 1, 2, \ldots$. The events A_i are mutually exclusive. The probability that desperado A will shoot himself dead is

$$P\left(\bigcup_{k=0}^{\infty} A_{1+3k}\right) = \frac{1}{6} \sum_{k=0}^{\infty} \left(\frac{5}{6}\right)^{3k} = \frac{1/6}{1-(5/6)^3} = 0.3956$$

The probability that desperado B will shoot himself dead is

$$P\left(\bigcup_{k=0}^{\infty} A_{2+3k}\right) = \frac{1}{6} \sum_{k=0}^{\infty} \left(\frac{5}{6}\right)^{1+3k} = \frac{5/36}{1-(5/6)^3} = 0.3297.$$

The probability that desperado C will shoot himself dead is

$$P\left(\bigcup_{k=0}^{\infty} A_{3+3k}\right) = \frac{1}{6} \sum_{k=0}^{\infty} \left(\frac{5}{6}\right)^{2+3k} = \frac{25/216}{1-(5/6)^3} = 0.2747.$$

7E-20 Let A be the event of getting three or more heads in a row in 20 tosses of the coin and B be the event of getting three or more tails in a row in 20 tosses of the coin. Then $P(A) = P(B) = 0.7870$ and $P(A \cup B) = 0.9791$. Using the relation $P(A \cup B) = P(A) + P(B) - P(AB)$, it follows that the desired probability is given by

$$P(AB) = 0.7870 + 0.7870 - 0.9791 = 0.6049.$$

7E-21 Let A be the event that a second-hand car is bought and B be the event that a Japanese car is bought. The desired probability is $P(A \cap B)$. Denoting by E^c the complement of even E, it follows that

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) = P(A) + P(B) - [1 - P((A \cup B)^c)] \\ &= P(A) + P(B) - [1 - P(A^c \cap B^c)] \\ &= 0.25 + 0.30 - [1 - 0.55] = 0.10. \end{aligned}$$

7E-22 Let A_n be the event that in the first $6n$ rolls of the die each of the six possible outcomes occurs n times. Then $P(\cup_{n=1}^{\infty} A_n)$ is an upper bound for the probability that the six accumulating counts will ever be equal. For each n,

$$P(A_n) = \frac{\binom{6n}{n}\binom{5n}{n}\cdots\binom{6n}{n}}{6^{6n}} = \frac{(6n)!}{(n!)^6\, 6^{6n}}$$

Using the fact that $P(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n)$, an upper bound for the desired probability is given by

$$\sum_{n=1}^{\infty} P(A_n) = 0.02251,$$

where the infinite sum has been numerically calculated.

7E-23 Take an ordered sample space whose outcomes are given by all 6^6 possible sequences of the integers $1, \ldots, 6$ to the length 6. All outcomes of the sample space are equally likely. Let A_r be the event that the largest number rolled is r. Also, let B_k the event that the largest number rolled is smaller than or equal to k for $k = 1, \ldots, 6$. Then $B_r = B_{r-1} \cup A_r$. The events B_{r-1} and A_r are disjoint and so $P(B_r) = P(B_{r-1}) + P(A_r)$. For any r, $P(B_r) = \frac{r^6}{6^6}$. Hence

$$P(A_r) = \frac{r^6}{6^6} - \frac{(r-1)^6}{6^6} \qquad \text{for } r = 1, \ldots, 6.$$

An alternative way to obtain $P(A_r)$ is as follows. For fixed r, let E_j be the event that j of the six rolls of the die give the outcome r and the other $6 - j$ rolls give an outcome less than r. Then $A_r = \cup_{j=1}^{6} E_j$. The events $E_1, \ldots, E_6$ are mutually exclusive and so $P(A_r) = \sum_{j=1}^{6} P(E_j)$. This gives the alternative expression

$$P(A_r) = \sum_{j=1}^{6} \frac{\binom{6}{j} \times 1^j \times (r-1)^{6-j}}{6^6}.$$

The probability $P(A_r)$ has the numerical values 2.14×10^{-5}, 0.0014, 0.0143, 0.0722, 0.2471, and 0.6651 for $r = 1$, 2, 3, 4, 5, and 6.

7E-24 Some reflection shows that the desired probability of Fermat being the overall winner is the probability that Fermat wins at least $10 - a$ games of the additional $19 - a - b$ games. Take as sample space all possible sequences of "ones" and "zeros " to a length of $19 - a - b$, where a win of Fermat is recorded as a "one" and a loss as a "zero". Assign to each element of the sample space the same probability of $\frac{1}{2^{19-a-b}}$. Let A_k be the event that Fermat wins exactly k of these games. The set A_k contains $\binom{19-a-b}{k}$ elements.The events A_k are disjoint and so the desired probability is

$$\sum_{k=10-a}^{19-a-b} P(A_k) = \sum_{k=10-a}^{19-a-b} \binom{19-a-b}{k} \left(\frac{1}{2}\right)^{19-a-b}.$$

7E-25 Let A_i be the event that number i does not show up in any of the 50 drawings. By the inclusion-exclusion formula, the desired probability is given by

$$P(A_1 \cup A_2 \cup \cdots \cup A_{10}) = \sum_{k=1}^{9} (-1)^{k+1} \binom{10}{k} \left(\frac{10-k}{10}\right)^{50} = 0.0509.$$

Note: The number of integers from $1, 2, \ldots, 10$ not showing up in 50 drawings is approximately Poisson distributed with expected value $\lambda = 10 \times (9/10)^{50} = 0.0515$. This claim is supported by the fact that the Poisson probability $1 - e^{-\lambda} = 0.0502$ is approximately equal to the probability 0.0509 that not all of the numbers $1, 2, \ldots, 10$ will show up.

7E-26 Number the seats as $1, 2, \ldots, 6$ and take as sample space the set all $6!$ possible ways the six people can be seated at the table. Let A_i be the event that the couple i sit together for $i = 1, 2, 3$. The desired probability is $1 - P(A_1 \cup A_2 \cup A_3)$. We have $P(A_1) = P(A_2) = P(A_3) = 6 \times 2! \times 4!/6!$, $P(A_1A_2) = P(A_1A_3) = P(A_2A_3) = 6 \times 3 \times 2! \times 2! \times 2!/6!$, and $P(A_1A_2A_3) = 6 \times 2 \times 1 \times 2! \times 2! \times 2!/6!$. By the inclusion-exclusion formula,

$$P(A_1 \cup A_2 \cup A_3) = 3P(A_1) - \binom{3}{2} P(A_1A_2) + P(A_1A_2A_3) = \frac{11}{15}.$$

Hence the desired probability $1 - \frac{11}{15} = \frac{4}{15}$.

7E-27 (a) Take as sample space the set of all 6^6 possible sequences of the integers $1, \ldots, 6$ to the length of 6. The probability that three of the six possible outcomes do not show up and each of the other three possible outcomes shows up two times is

$$\frac{\binom{6}{3}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{6^6} = 0.0386.$$

(b) Let A_i be the event that outcome i shows up *exactly* three times for $i = 1, \ldots, 6$. Then the probability that some outcome shows up at least three times is given by

$$P(A_1 \cup \cdots \cup A_6) + \sum_{k=4}^{6} P(\text{some outcome shows up exactly } k \text{ times}).$$

For any i, $P(A_i) = \binom{6}{3} \times 5^3/6^6$. Also, $P(A_iA_j) = \binom{6}{3}/6^6$ for any i, j with $i \neq j$. By the inclusion-exclusion formula,

$$P(A_1 \cup \cdots \cup A_6) = 6P(A_1) - \binom{6}{2} P(A_1A_2) = 0.3151.$$

The probability that any given outcome shows up exactly k times is equal to $\binom{6}{k}5^{6-k}/6^6$ for $4 \leq k \leq 6$ and so

$$P(\text{some outcome shows up exactly } k \text{ times}) = 6\sum_{k=4}^{6} \frac{\binom{6}{k}5^{6-k}}{6^6} = 0.0522.$$

Hence the desired probability is $0.3151 + 0.0522 = 0.3672$.

7E-28 Let A_i be the event that the ith boy becomes part of a couple. The desired probability is $1 - P(A_1 \cup \cdots \cup A_n)$. For any fixed i,

$$P(A_i) = \frac{n \times n^{2n-2}}{n^{2n}} = \frac{n}{n^2}.$$

For any fixed i and j with $i \neq j$,

$$P(A_iA_j) = \frac{n \times (n-1)n^{2n-4}}{n^{2n}} = \frac{n(n-1)}{n^4}.$$

Continuing in this way, we find

$$P(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{n(n-1)\cdots(n-k+1)}{n^{2k}}.$$

7E-29 Let A_i be the event that the ith team has a married couple. The desired probability is $1 - P(A_1 \cup A_2 \cup \cdots \cup A_8)$. For any i,

$$P(A_i) = \frac{12 \times 22}{\binom{24}{3}}.$$

Further, $P(A_iA_j) = \frac{12\times 11\times 20\times 19}{\binom{24}{3}\binom{21}{3}}$ for any $i < j$, $P(A_iA_jA_k) = \frac{12\times 11\times 10\times 18\times 17\times 16}{\binom{24}{3}\binom{21}{3}\binom{18}{3}}$ for any $i < j < k$, etc. Using the inclusion-exclusion formula, we calculate

$$P(A_1 \cup A_2 \cup \cdots \cup A_8) = 0.6553$$

and so the desired probability is $1 - 0.6553 = 0.3447$.

7E-30 Let A_i be the event that nobody gets off at the ith stop. The desired probability is $1 - P(A_1 \cup A_2 \cup \cdots \cup A_7)$. The probability that nobody gets off at k of the 7 stops is $\frac{(7-k)^{25}}{7^{25}}$. By the inclusion-exclusion formula,

$$P(A_1 \cup A_2 \cup \cdots \cup A_7) = \sum_{k=1}^{7} (-1)^{k+1} \binom{7}{k} \frac{(7-k)^{25}}{7^{25}} = 0.1438,$$

Hence the probability that somebody gets off at each stop is $1 - 0.1438 = 0.8562$.

7E-31 There are four paths from node n_1 to node n_4. These paths are the paths (l_1, l_5), (l_2, l_6), (l_1, l_3, l_6) and (l_2, l_4, l_5). Let A_j be the event that the jth path is functioning. The desired probability is given by $P(A_1 \cup A_2 \cup A_3 \cup A_4)$. By the inclusion-exclusion formula,

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3 \cup A_4) &= \sum_{j=1}^{4} P(A_j) - \sum_{j=1}^{3} \sum_{k=j+1}^{4} P(A_j A_k) \\ &\quad + \sum_{j=1}^{2} \sum_{k=j+1}^{3} \sum_{l=k+1}^{4} P(A_j A_k A_l) - P(A_1 A_2 A_3 A_4). \end{aligned}$$

Hence the probability that there is some functioning path from node n_1 to node n_4 is equal to

$$\begin{aligned} &p_1 p_5 + p_2 p_6 + p_1 p_3 p_6 + p_2 p_4 p_5 - p_1 p_2 p_5 p_6 - p_1 p_3 p_5 p_6 - p_1 p_2 p_4 p_5 \\ &- p_1 p_2 p_3 p_6 - p_2 p_4 p_5 p_6 - p_1 p_2 p_3 p_4 p_5 p_6 + p_1 p_2 p_3 p_5 p_6 + p_1 p_2 p_4 p_5 p_6 \\ &+ p_1 p_2 p_3 p_4 p_5 p_6 + p_1 p_2 p_3 p_4 p_5 p_6 - p_1 p_2 p_3 p_4 p_5 p_6. \end{aligned}$$

The four terms $p_1 p_2 p_3 p_4 p_5 p_6$ cancel out in this expression. For the special case that $p_i = p$ for all i, the above expression simplifies to

$$2p^2(1 + p + p^3) - 5p^4.$$

Chapter 8

8E-1 Take an ordered sample space whose outcomes are given by all possible 3-tuples (i_1, i_2, i_3), where i_k is the result of the roll of the kth die. Each of the 216 possible outcomes of the sample space gets assigned the same probability of $\frac{1}{216}$. Let A be the event that the sum of the results of the rolls of the three dice is 10 and B the event that the three dice show different results. The desired probability is $P(A \mid B) = P(AB)/P(B)$. The event AB consists of the 3! orderings of $\{1,3,6\}$, the 3! orderings of $\{1,4,5\}$ and the 3! orderings of $\{2,3,5\}$. This gives $P(AB) = \frac{3\times 3!}{216} = \frac{18}{216}$. Clearly, $P(B) = \frac{6\times 5\times 4}{216} = \frac{120}{216}$.

8E-2 Label the two red balls as R_1, R_2 and the blue and white balls as B, W. Take as sample space the collection of the subsets $\{R_1, R_2\}$, $\{R_1, B\}$, $\{R_1, W\}$, $\{R_2, B\}$, $\{R_2, W\}$, $\{B, W\}$ of two balls from the bag. All six unordered outcomes are equally likely. Let A be the event that both balls drawn out the bag are red and B be the event that there is a red ball among the two balls drawn out. The desired probability is given by $P(A \mid B)$ and

$$P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{1/6}{5/6} = \frac{1}{5}.$$

8E-3 Take an ordered sample space whose outcomes are given by all 2^n possible sequences of H's and T's to the length of n. Each outcome of the sample space gets assigned the same probability of $\frac{1}{2^n}$. Let A be the event that the first toss gives heads and let B the event that r heads are obtained in n tosses. The set B has $\binom{n}{r}$ outcomes and the set set AB has $\binom{n-1}{r-1}$outcomes. Hence $P(B) = \binom{n}{r}/2^n$ and $P(AB) = \binom{n-1}{r-1}/2^n$. This leads to the desired probability $P(AB)/P(B) = r/n$.

8E-4 The probability that the number of tens in the hand is the same as the number of aces in the hand is given by

$$\sum_{k=0}^{4} \binom{4}{k}\binom{4}{k}\binom{44}{13-2k} \Big/ \binom{52}{13} = 0.3162.$$

Hence, using a symmetry argument, the probability that the hand contains more aces than tens is $\frac{1}{2}(1-0.3162) = 0.3419$. By $P(A \mid B) = P(AB)/P(B)$, the other probability is equal to

$$\frac{0.341924}{\sum_{k=1}^{4} \binom{4}{k}\binom{48}{13-k}/\binom{52}{13}} = 0.4911.$$

8E-5 Let A be the event that a randomly chosen student takes Spanish and B be the event that the student takes French. The desired probability is $P(B \mid A) = P(AB)/P(A)$. It is given that $P(A) = 0.35$, $P(B) = 0.15$ and $P(A \cup B) = 0.40$. To find $P(AB)$, apply

$$P(A \cup B) = P(A) + P(B) - P(AB).$$

This gives $P(AB) = 0.10$ and so the desired probability is $\frac{0.10}{0.35} = \frac{2}{7}$.

8E-6 Since $A = AB \cup AB^c$ and the events AB and AB^c are mutually exclusive, it follows that

$$P(A) = P(AB) + P(AB^c) = P(A)P(B) + P(AB^c),$$

by the independence of A and B. Hence

$$P(AB^c) = P(A)[1 - P(B)] = P(A)P(B^c),$$

proving that A and B^c are independent. By letting B^c play the role of A and A the role of B, it follows next that B^c and A^c are independent.

8E-7 The number is randomly chosen from the matrix and so $P(A) = \frac{30}{50}$, $P(B) = \frac{25}{50}$ and $P(AB) = \frac{15}{50}$. Since $P(AB) = P(A)P(B)$, the events A and B are independent. This result can also be explained by noting that you obtain a random number from the matrix by choosing first a row at random and choosing next a column at random.

8E-8 (**a**) Imagining that the waiter first serves the two cups of coffee, let A_1 be the event that the first coffee is given to a person having ordered coffee and A_2 be the event that the second coffee is given to the other person having ordered coffee. The desired probability $P(A_1A_2)$ can be evaluated as $P(A_1)P(A_2 \mid A_1) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$.
(**b**) Let A_1 be the event that the first vacancy is not on the top floor and A_2 be the event that the second vacancy is not on the top floor. The desired probability is $1 - P(A_1A_2) = 1 - P(A_1)P(A_2 \mid A_1)$ and is thus equal to $1 - \frac{48}{56} \times \frac{47}{55} = 0.2675$.
(**c**) Let A_1 be the event that your first friend is in the same group as you are and A_2 the event that your second friend is in the same group as you are. The desired probability is $P(A_1A_2) = P(A_1)P(A_2 \mid A_1) = \frac{4}{9} \times \frac{3}{8} = \frac{1}{6}$.
(**d**) Let A_i be the event that the ith card you receive is a picture card that you have not received before. Then, by $P(A_1A_2A_3A_4) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1A_2)P(A_4 \mid A_1A_2A_3)$, the desired probability can also be computed as

$$P(A_1A_2A_3A_4) = \frac{16}{52} \times \frac{12}{51} \times \frac{8}{50} \times \frac{4}{49} = 9.46 \times 10^{-4}.$$ [8]

8E-9 Let A_i be the event that one red and one blue ball show up on the ith drawing for $i = 1, \ldots, 4$. The desired probability is given by

$$P(A_1A_2A_3A_4) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1A_2)P(A_4 \mid A_1A_2A_3).$$

[8] Problem 8E-7 illustrates again that conditioning is often easier than counting.

We have $P(A_1) = \frac{8\times 4}{8\times 7} = \frac{4}{7}$, $P(A_2 \mid A_1) = \frac{6\times 3}{6\times 5} = \frac{3}{5}$, $P(A_3 \mid A_1A_2) = \frac{4\times 2}{4\times 3} = \frac{2}{3}$, and $P(A_4 \mid A_1A_2A_3) = 1$ (alternatively, $P(A_1) = \binom{4}{1}\binom{4}{1}/\binom{8}{2}$, and so on). Hence the desired probability is given by

$$\frac{4}{7} \times \frac{3}{5} \times \frac{2}{3} \times 1 = \frac{8}{35}.$$

8E-10 Let A_i be the event that the ith English team is not paired with another English team. The desired probability is $P(A_1A_2A_3)$. This probability can be evaluated as

$$P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1A_2) = \frac{5}{7} \times \frac{4}{5} \times 1 = \frac{4}{7}.$$

8E-11 No more than five picks until all balls in the jar are black means three picks or five picks. The probability that you need three picks until all balls in the bar are black is given by $\frac{3}{5} \times \frac{2}{5} \times \frac{1}{5} = \frac{6}{125}$. The number of picks needed is five only if one of the first three picks gives a black ball and the other four picks give a white ball. Using conditional probabilities again, we find that the probability that five picks are needed until all balls in the jar are black is equal to

$$\frac{2}{5} \times \frac{4}{5} \times \frac{3}{5} \times \frac{2}{5} \times \frac{1}{5} + \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} \times \frac{1}{5} + \frac{3}{5} \times \frac{2}{5} \times \frac{4}{5} \times \frac{2}{5} \times \frac{1}{5} = \frac{6}{125}.$$

Hence the desired probability is $\frac{6}{125} + \frac{6}{125} = \frac{12}{125}$.[9]

8E-12 For fixed n, let A_n be the event that your name appears at the nth drawing and A_i be the event that your name does not appear at the ith drawing for $i = 1, \ldots, n-1$. Using the basic formula $P(A_1A_2\cdots A_n) = P(A_1)P(A_2 \mid A_1)\cdots P(A_n \mid A_1\ldots A_{n-1})$, it follows that

$$P(\text{you get assigned the } n\text{th placement}) = \prod_{k=1}^{n-1} \frac{N-k}{2+N-k} \times \frac{2}{2+N-n}.$$

8E-13 Let p_k be the probability that the kth announcement wins the bonus. Then, by $P(A_1A_2\cdots A_n) = P(A_1)P(A_2 \mid A_1)\cdots P(A_n \mid A_1\ldots A_{n-1})$, it follows that $p_1 = 0$, $p_2 = \frac{1}{100}$, $p_3 = \frac{99}{100} \times \frac{2}{100}$, and

$$p_k = \frac{99}{100} \times \frac{98}{100} \times \cdots \times \frac{99-k+3}{100} \times \frac{k-1}{100}, \quad k = 4, \ldots, 25.$$

[9]The complete probability mass function of the number of picks needed until all balls in the jar are black can be numerically obtained by using the method of absorbing Markov chains that is discussed in chapter 15 of my book *Understanding Probability*.

Numerical calculations show that p_k is maximal for $k = 11$ with $p_{11} = 0.0628$.

8E-14 Let p_k be the probability that the first ace appears at the kth card. Then, by $P(A_1 A_2 \cdots A_n) = P(A_1)P(A_2 \mid A_1) \cdots P(A_n \mid A_1 \ldots A_{n-1})$, it follows that $p_1 = \frac{4}{52}$, $p_2 = \frac{48}{52} \times \frac{4}{51}$, and

$$p_k = \frac{48}{52} \times \frac{47}{51} \times \cdots \times \frac{48-k+2}{52-k+2} \times \frac{4}{52-k+1}, \quad k = 3, \ldots, 49.$$

The three players do not have the same chance to become the dealer. For $P = A$, B, and C, let r_P be the probability that player P becomes the dealer. Then $r_A > r_B > r_C$, because the probability p_k is decreasing in k. The probabilities can be calculated as $r_A = \sum_{n=0}^{16} p_{1+3n} = 0.3600$, $r_B = \sum_{n=0}^{15} p_{2+3n} = 0.3328$, and $r_C = \sum_{n=0}^{15} p_{3+3n} = 0.3072$.

8E-15 Let A be the event that HAPPY HOUR appears again. To find $P(A)$, condition on the events B_1, B_2, and B_3, where B_1 is the event that the two letters H are removed, B_2 is the event that the two letters P are removed, and B_3 is the event that two different letters are removed. The formula $P(A) = \sum_{i=1}^{3} P(A \mid B_i)P(B_i)$ gives

$$P(A) = 1 \times \frac{1}{\binom{9}{2}} + 1 \times \frac{1}{\binom{9}{2}} + \frac{1}{2} \times \left(1 - \frac{1}{\binom{9}{2}} - \frac{1}{\binom{9}{2}}\right) = \frac{19}{36}.$$

8E-16 Let A be the event that the first question on the exam is marked as difficult. Let B_1 be the event that the exam is hard and B_2 be the event that the exam is easy. Applying the formula $P(A) = \sum_{i=1}^{2} P(A \mid B_i)P(B_i)$ gives

$$P(A) = 0.9 \times 0.8 + 0.15 \times 0.1 = 0.735.$$

The probability that the exam is hard given that the first question on the exam is marked as difficult is equal to

$$P(B_1 \mid A) = \frac{P(AB_1)}{P(A)} = \frac{P(A \mid B_1)P(B_1)}{P(A)} = \frac{0.9 \times 0.8}{0.735} = 0.9796.$$

8E-17 Let A be the event that Bill wins the match and B_i be the event that Bill loses i games of the first two games for $i = 0, 1$ and 2. By the law of conditional probability,

$$\begin{aligned} P(A) &= P(A \mid B_0)P(B_0) + P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) \\ &= 1 \times p^2 + P(A) \times 2pq + 0 \times q^2. \end{aligned}$$

This gives

$$P(A) = \frac{p^2}{1 - 2pq} = \frac{p^2}{p^2 + q^2},$$

where the last equality uses the fact that $p^2 + 2pq + q^2 = (p+q)^2 = 1$. This result can be also obtained from the gambler's ruin problem in which each player starts with two dollars. The gambler's ruin formula gives that the probability of Bill winning the match is equal to $[1-(q/p)^2]/[1-(q/p)^4] = p^2/(p^2+q^2)$.

8E-18 Let A be the event that the bowl contained five white balls and three red balls before the two red balls were added. Also, let B the event that all five balls picked out of the bowl are white. The desired probability $P(A \mid B$ is given by

$$P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{\binom{8}{5}(\frac{1}{2})^8 \times \binom{5}{5}/\binom{10}{5}}{P(B)}.$$

By the law of conditional probability,

$$P(B) = \sum_{i=5}^{8} \frac{\binom{8}{i}\left(\frac{1}{2}\right)^8 \times \binom{i}{5}}{\binom{10}{5}}.$$

This leads to the answer $P(A \mid B) = \frac{1}{8}$.

8E-19 It does not matter what question you ask. After hearing the answer of your friend, the probability of making a correct guess is $\frac{1}{26}$ for both questions. To see this, let A be the event that you guess correctly what the card is. For either question, let B_1 be the event that the answer of your friend to the question is yes and B_2 be the event that the answers is no. By the law of conditional probability, $P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2)$. For the question whether the card is red, this gives

$$P(A) = \frac{1}{26} \times \frac{1}{2} + \frac{1}{26} \times \frac{1}{2} = \frac{1}{26}.$$

For the second question whether the card is the ace of spades,

$$P(A) = 1 \times \frac{1}{52} + \frac{1}{51} \times \frac{51}{52} = \frac{1}{26}.$$

Hence it does not matter what question you pose to your friend.

8E-20 (a) Let A_k be the event that both players have obtained k heads after N tosses. The probability of a tie after N tosses is

$$\begin{aligned}\sum_{k=0}^{N} P(A_k) &= \sum_{k=0}^{N} \binom{N}{k}\left(\frac{1}{2}\right)^N \binom{N}{k}\left(\frac{1}{2}\right)^N \\ &= \left(\frac{1}{2}\right)^{2N} \sum_{k=0}^{N} \binom{N}{k}\binom{N}{N-k} = \left(\frac{1}{2}\right)^{2N}\binom{2N}{N}.\end{aligned}$$

Note that this probability is the same as the probability of getting N heads (and N tails) in $2N$ tosses of a fair coin.
(b) Let A be the event that player 1 wins the game. Then $P(A) = 0.5$, regardless of the value of N. The simplest way to see this is to define E_1 as the event that player 1 has more heads than player 2 after N tosses, E_2 as the event that player 1 has fewer heads than player 2 after N tosses, and E_3 as the event that player 1 has the same number of heads as player 2 after N tosses. Then $P(A) = \sum_{i=1}^{3} P(A \mid E_i)P(E_i)$, by the law of conditional probability. To evaluate this, it is not necessary to know the $P(E_i)$. Since $P(E_2) = P(E_1)$ and $P(E_3) = 1 - 2P(E_1)$, it follows that $P(A)$ is equal to

$$1 \times P(E_1) + 0 \times P(E_2) + \frac{1}{2} \times P(E_3) = P(E_1) + \frac{1}{2} \times \big(1 - 2P(E_1)\big) = 0.5.$$

8E-21 Let A be the event that all of the balls drawn are blue and B_i be the event that the number of points shown by the die is i for $i = 1, \ldots, 6$.
(a) By the law of conditional probability, the probability that all of the balls drawn are blue is given by

$$P(A) = \sum_{i=1}^{6} P(A \mid B_i)P(B_i) = \frac{1}{6} \sum_{i=1}^{5} \frac{\binom{5}{i}}{\binom{10}{i}} = \frac{5}{36}.$$

(b) The probability that the number of points shown by the die is r given that all of the balls drawn are blue is equal to

$$P(B_r \mid A) = \frac{P(B_r A)}{P(A)} = \frac{(1/6)\binom{5}{r}/\binom{10}{r}}{5/36}.$$

This probability has the values $\frac{3}{5}$, $\frac{4}{15}$, $\frac{1}{10}$, $\frac{1}{35}$, $\frac{1}{210}$ and 0 for $r = 1, \ldots, 6$.

8E-22 Let A be the event that John and Pete meet each other in the semi-finals. To find $P(A)$, let B_1 be the event that John and Pete are allocated to either group 1 or group 2 but not to the same group and B_2 be the event that John and Pete are allocated to either group 3 or group 4 but not to the same group. Then $P(B_1) = P(B_2) = \frac{1}{2} \times \frac{2}{7} = \frac{1}{7}$. By the law of conditional probability,

$$\begin{aligned} P(A) &= P(A \mid B_1) \times \frac{1}{7} + P(A \mid B_2) \times \frac{1}{7} \\ &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{7} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{7} = \frac{1}{14}. \end{aligned}$$

Let C be the event that John and Pete meet each other in the final. To find $P(C)$, let D_1 be the event that John is allocated to either group 1 or group

2 and Pete to either group 3 or group 4 and D_2 be the event that John is allocated to either group 3 or group 4 and Pete to either group 1 or group 2. Then $P(D_1) = P(D_2) = \frac{1}{2} \times \frac{4}{7} = \frac{2}{7}$. By the law of conditional probability,

$$\begin{aligned} P(C) &= P(C \mid D_1) \times \frac{2}{7} + P(C \mid D_2) \times \frac{2}{7} \\ &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{2}{7} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{2}{7} = \frac{1}{28}. \end{aligned}$$

The latter result can also be directly seen by a symmetry argument. The probability that any one pair contests the final is the same as that for any other pair. There are $\binom{8}{2}$ different pairs and so the probability that John and Pete meet each other in the final is $1/\binom{8}{2} = \frac{1}{28}$.

8E-23 Let $P(r,\, s)$ be the probability of Fermat being the overall winner given that Fermat has won so far r games and Pascal has won so far s games. Then, by conditioning on the result of the next game,

$$P(r,\, s) = \frac{1}{2} P(r+1,\, s) + \frac{1}{2} P(r,\, s+1)$$

with the boundary conditions $P(10,\, s) = 1$ for all $s < 10$ and $P(r,\, 10) = 0$ for all $r < 10$. By backward calculation, we get the desired probability $P(a,\, b)$.

8E-24 (a) For fixed n, let A_n be the event that the total number of heads after n tosses is even and let $P_n = P(A_n)$. Also, let B_0 be the event that the first toss results in tails and B_1 be the event that the first toss results in heads. Then, $P(A_n) = P(A_n \mid B_0)P(B_0) + P(A_n \mid B_1)P(B_1)$. This gives

$$P_n = (1-p)P_{n-1} + p(1 - P_{n-1}) \quad \text{for } n = 1, 2, \ldots,$$

where $P_0 = 1$. This recurrence equation has the explicit solution

$$P_n = \frac{1}{2} + \frac{1}{2}(1-2p)^n \quad \text{for } n = 1, 2, \ldots.$$

Note that, for any $0 < p < 1$, the probability P_n tends to $\frac{1}{2}$ as n gets large. For $p = \frac{1}{2}$, $P_n = \frac{1}{2}$ for all $n \geq 1$, as is obvious for reasons of symmetry.

(b) For fixed n, let A_n be the event that a sequence of n tosses does not show five or more consecutive heads and let $P_n = P(A_n)$. Also, let B_i be the event that the first $i-1$ tosses give heads and the ith toss gives tails for $i = 1, 2, \ldots, 5$ and B_6 be the event that the first five tosses result in heads. The events B_i are disjoint. Then $P(A_n) = \sum_{i=1}^{6} P(A_n \mid B_i)P(B_i)$ with $P(B_i) = p^{i-1}(1-p)$ for $1 \leq i \leq 5$ and $P(A_n \mid B_6) = 0$. This gives the recursion

$$P_n = \sum_{i=1}^{5} p^{i-1}(1-p)P_{n-i} \quad \text{for } n = 5, 6, \ldots,$$

where $P_0 = P_1 = P_2 = P_3 = P_4 = 1$.

8E-25 For fixed integer r, let A_r be the event that there are exactly r winning tickets among the one half million tickets sold. Let B_n be the event that there exactly n winning tickets among the one million tickets printed. Then, by the law of conditional probability,

$$P(A_r) = \sum_{n=0}^{\infty} P(A_r \mid B_n) P(B_n).$$

Obviously, $P(A_r \mid B_n) = 0$ for $n < r$. For all practical purposes the so-called Poisson probability $e^{-1}/n!$ can be taken for the probability of the event B_n for $n = 0, 1, \ldots$ (see Example 7.11 in the textbook). This gives

$$P(A_r) = \sum_{n=r}^{\infty} \binom{n}{r} \left(\frac{1}{2}\right)^r \frac{e^{-1}}{n!} = e^{-1} \frac{(1/2)^r}{r!} \sum_{j=0}^{\infty} \frac{1}{j!} = e^{-\frac{1}{2}} \frac{(1/2)^r}{r!}.$$

Hence the probability of exactly r winning tickets among the one half million tickets sold is given by the Poisson probability $e^{-\frac{1}{2}} \frac{(1/2)^r}{r!}$ for $r = 0, 1, \ldots$.

8E-26 Let the hypothesis H be the event that a 1 is sent and the evidence E be the event that a 1 is received. The desired posterior probability $P(H \mid E)$ satisfies

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{0.8}{0.2} \times \frac{0.95}{0.01} = 380.$$

Hence the posterior probability $P(H \mid E)$ that a 1 has been sent is $\frac{1}{1+380} = 0.9974$.

8E-27 Let the hypothesis H be the event that the inhabitant you overheard spoke truthfully and the evidence E be the event that the other inhabitant says that the inhabitant you overheard spoke the truth. The desired posterior probability $P(H \mid E)$ satisfies

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{1/3}{2/3} \times \frac{1/3}{2/3} = \frac{1}{4}.$$

Hence the posterior probability $P(H \mid E)$ that the inhabitant you overheard spoke the truth is $\frac{1/4}{1+1/4} = \frac{1}{5}$.

8E-28 Let the hypothesis H be the event that oil is present in the test area and the evidence E be the event that the test gives a positive signal. This leads to

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{0.4}{0.6} \times \frac{0.90}{0.15} = 4.$$

Hence the posterior probability $P(H \mid E)$ of oil being present in the test area is $\frac{4}{1+4} = 0.8$.

8E-29 Let the hypotheses H be the event that the family has two girls and the evidence E be the event that the family has a girl named Mary Ann. In the first problem, the prior probabilities $P(H)$ and $P(\overline{H})$ have the values $\frac{1}{4}$ and $\frac{3}{4}$, while $P(E \mid H)$ and $P(E \mid \overline{H})$ have the values 1 and $\frac{2}{3}$. This leads to

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{1/4}{3/4} \times \frac{1}{2/3} = \frac{1}{2}$$

and so the posterior probability $P(H \mid E)$ of having two girls has the value $\frac{1}{3}$ in the first problem. In the second problem, the prior probabilities $P(H)$ and $P(\overline{H})$ have again the values $\frac{1}{4}$ and $\frac{3}{4}$, but $P(E \mid H)$ and $P(E \mid \overline{H})$ now have the values $\frac{1}{10} + \frac{9}{10} \times \frac{1}{9} = \frac{2}{10}$ and $\frac{2}{3} \times \frac{1}{10}$. This gives

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{1/4}{3/4} \times \frac{2/10}{(2/3) \times (1/10)} = 1$$

and so the posterior probability $P(H \mid E)$ of having two girls has the value $\frac{1}{2}$ in the second problem. An intuitive explanation of why the second probability is larger lies in the fact that in the second problem it is more likely to have a girl named Mary Ann when there are two girls in the family rather than a single girl. In the first problem it makes no difference whether the number of girls in the family is one or two in order to have a girl named Mary Ann.

8E-30 Let the hypothesis H be the event that the family has one boy and two girls and the evidence E be the event that he family has a boy among the three children. Then $P(H) = \frac{3}{8}$, $P(\overline{H}) = \frac{5}{8}$, $P(E \mid H) = 1$, and $P(E \mid \overline{H}) = \frac{4}{5}$

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{3/8}{5/8} \times \frac{1}{4/5} = \frac{3}{4}.$$

Hence the desired probability $P(H \mid E)$ is equal to $\frac{3/4}{1+3/4} = \frac{3}{7}$.

8E-31 (a) Assume you have tossed the randomly chosen coin n times and all n tosses result in heads. Let the hypothesis H be the event that you have chosen the two-headed coin and the evidence E be the event that all n tosses result in heads. The desired posterior probability $P(H \mid E)$ satisfies

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{1/10{,}000}{9{,}999/10{,}000} \times \frac{1}{0.5^n}.$$

This gives

$$P(H \mid E) = \frac{2^n}{2^n + 9{,}999}.$$

The posterior probability $P(H \mid E)$ has the values 0.7662 and 0.9997 for $n = 15$ and $n = 25$.

(b) Let hypothesis H be the event that you choose coin s and the evidence E be the event that each of the first n tosses results in heads. By Bayes' rule in odds form,

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{1/(r+1)}{r/(r+1)} \times \frac{(s/r)^n}{(1/r)\sum_{j \neq s}(j/r)^n} = \frac{(s/r)^n}{\sum_{j \neq s}(j/r)^n}.$$

This gives

$$P(H \mid E) = \frac{(s/r)^n}{\sum_{j=0}^{r}(j/r)^n}.$$

8E-32 (a) Let the hypothesis H be the event that both numbers are even and the evidence E be the event that there is an even number among the two numbers. Then

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{25/100}{75/100} \times \frac{1}{(2 \times 5 \times 5)/100} = \frac{2}{3}.$$

Hence the posterior probability $P(H \mid E)$ that both numbers are even is $\frac{2/3}{1+2/3} = 0.4$.

(b) Let the hypothesis H be the event that both numbers are even and the evidence E be the event that there the number 2 is among the two numbers. Then

$$\frac{P(H \mid E)}{P(\overline{H} \mid E)} = \frac{25/100}{75/100} \times \frac{1-(4/5)^2}{(2 \times 5)/100} = \frac{6}{5}.$$

Hence the posterior probability $P(H \mid E)$ that both numbers are even is $\frac{6/5}{1+6/5} = \frac{6}{11}$.

8E-33 Let the random variable Θ represent the unknown probability that a single toss of the die results in the outcome 6. The prior distribution of Θ is given by $p_0(\theta) = 0.25$ for $\theta = 0.1$, 0.2, 0.3 and 0.4. The posterior probability $p(\theta \mid \text{data}) = P(\Theta = \theta \mid \text{data})$ is proportional to $L(\text{data} \mid \theta)p_0(\theta)$, where $L(\text{data} \mid \theta) = \binom{300}{75}\theta^{75}(1-\theta)^{225}$. Hence the posterior probability $p(\theta \mid \text{data})$ is given by

$$\begin{aligned} p(\theta \mid \text{data}) &= \frac{L(\text{data} \mid \theta)p_0(\theta)}{\sum_{k=1}^{4} L(\text{data} \mid k/10)p_0(k/10)} \\ &= \frac{\theta^{75}(1-\theta)^{225}}{\sum_{k=1}^{4}(k/10)^{75}(1-k/10)^{225}}, \quad \theta = 0.1, 0.2, 0.3, 0.4. \end{aligned}$$

The posterior probability $p(\theta \mid \text{data})$ has the values 3.5×10^{-12}, 0.4097, 0.5903, and 3.5×10^{-12} for $\theta = 0.1$, 0.2, 0.3, and 0.4.

8E-34 Let the random variable Θ represent the unknown probability that a free throw of your friend will be successful. The prior probability mass function $p_0(\theta) = P(\Theta = \theta)$ has the values 0.2, 0.6, and 0.2 for $\theta = 0.25$, 0.50, and 0.75. The posterior probability $p(\theta \mid \text{data}) = P(\Theta = \theta \mid \text{data})$ is proportional to $L(\text{data} \mid \theta)p_0(\theta)$, where $L(\text{data} \mid \theta) = \binom{10}{7}\theta^7(1-\theta)^3$. Hence the posterior probability $p(\theta \mid \text{data})$ is given by

$$\frac{\theta^7(1-\theta)^3 p_0(\theta)}{0.25^7 \times 0.75^3 \times 0.2 + 0.50^7 \times 0.50^3 \times 0.6 + 0.75^7 \times 0.25^3 \times 0.2}$$

for $\theta = 0.25$, 0.50, and 0.75. The possible values 0.25, 0.50 and 0.75 for the success probability of the free throws of your friend have the posterior probabilities 0.0051, 0.5812 and 0.4137.

Chapter 9

9E-1 This problem can be solved both by counting arguments and by conditional probabilities. The solution approach using counting arguments requires the specification of a sample space. Take as sample space the set of all possible 10! rankings of the scores of the ten people. The number of rankings for which the highest ranking achieved by a women equals 1 is given by $5 \times 9!$. Hence

$$P(X = 1) = \frac{5 \times 9!}{10!} = \frac{5}{10}.$$

The number of rankings for which the highest ranking achieved by a women equals 2 is given by $5 \times 4 \times 8!$. Hence

$$P(X = 2) = \frac{5 \times 4 \times 8!}{10!} = \frac{5}{18}.$$

Continuing in this way,

$$P(X = 3) = \frac{5 \times 4 \times 5 \times 7!}{10!} = \frac{5}{36},\ P(X = 4) = \frac{5 \times 4 \times 3 \times 5 \times 6!}{10!} = \frac{5}{84},$$
$$P(X = 5) = \frac{5 \times 4 \times 3 \times 2 \times 5 \times 5!}{10!} = \frac{5}{252},$$
$$P(X = 6) = \frac{5 \times 4 \times 3 \times 2 \times 1 \times 5 \times 4!}{10!} = \frac{1}{252}.$$

Alternatively, the problem can be solved by using conditional probabilities. To do so, note that the problem is one of the many problems that can be reformulated as "balls in a bag" problem. Imagine that there are five red balls and five blue balls in a bag. You pull them out one at a time. Let A_i be the event that the ith ball pulled out is the first blue ball you get. Then $P(X = i) = P(A_i)$ for $i = 1, \ldots, 6$. Using the relation

$$P(A_i) = P(A_1^c) \cdots P(A_{i-1}^c \mid A_1^c \cdots A_{i-2}^c) P(A_i \mid A_1^c \cdots A_{i-1}^c),$$

where A_i^c is the complement event of A_i, it follows that

$$\begin{aligned}
&P(X = 1) = \frac{5}{10},\ P(X = 2) = \frac{5}{10} \times \frac{5}{9},\ P(X = 3) = \frac{5}{10} \times \frac{4}{9} \times \frac{5}{8}, \\
&P(X = 4) = \frac{5}{10} \times \frac{4}{9} \times \frac{3}{8} \times \frac{5}{7},\ P(X = 5) = \frac{5}{10} \times \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} \times \frac{5}{6}, \\
&P(X = 6) = \frac{5}{10} \times \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} \times \frac{1}{6} \times \frac{5}{5}.
\end{aligned}$$

9E-2 To find the two depleted batteries, you need at least two tests but no more than four tests. Label the batteries as $1, 2, \ldots, 5$. Think of the order in which the batteries are placed for testing as a random permutation of the numbers $1, 2, \ldots, 5$. The sample space has 5! equally likely outcomes. You need two tests if the first two batteries tested are depleted. The number of outcomes for which the two depleted batteries are on the positions 1 and 2 is $2 \times 1 \times 3!$. Hence

$$P(X = 2) = \frac{2 \times 1 \times 3!}{5!} = \frac{1}{10}.$$

You need three tests if the first three batteries tested are not depleted or if a second depleted battery is found at the third test. This leads to

$$P(X = 3) = \frac{3 \times 2 \times 1 \times 2! + 2 \times 3 \times 1 \times 2! + 3 \times 2 \times 1 \times 2!}{5!} = \frac{3}{10}.$$

The probability $P(X = 4)$ follows from $P(X = 4) = 1 - P(X = 2) - P(X = 3) = \frac{6}{10}$. Alternatively, the probability mass function of X can be obtained by using conditional probabilities. This gives $P(X = 0) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$ and $P(X = 2) = \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} + \frac{2}{5} \times \frac{3}{4} \times \frac{1}{3} + \frac{3}{5} \times \frac{2}{4} \times \frac{1}{3} = \frac{3}{10}$.

9E-3 The random variable X is defined on a sample space consisting of the 36 equiprobable elements (i, j), where $i, j = 1, 2, \ldots, 6$. The random variable X takes on the value $i \times j$ for the realization (i, j). The random variable X takes on the 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24, 25, 30, and 36 with respective probabilities $\frac{1}{36}, \frac{2}{36}, \frac{2}{36}, \frac{4}{36}, \frac{2}{36}, \frac{4}{36}, \frac{2}{36}, \frac{1}{36}, \frac{2}{36}, \frac{4}{36}, \frac{2}{36}, \frac{1}{36}, \frac{2}{36}, \frac{2}{36}, \frac{2}{36}, \frac{1}{36}$, and $\frac{1}{36}$. The expected value of X is easiest computed as

$$E(X) = \frac{1}{36} \sum_{i=1}^{6} \sum_{j=1}^{6} i \times j = \frac{1}{6} \sum_{i=1}^{6} i \times \frac{1}{6} \sum_{j=1}^{6} j = \frac{7}{2} \times \frac{7}{2} = \frac{49}{4}.$$

Also,

$$E(X^2) = \frac{1}{36}\sum_{i=1}^{6}\sum_{j=1}^{6}(i \times j)^2 = \frac{1}{6}\sum_{i=1}^{6} i^2 \times \frac{1}{6}\sum_{j=1}^{6} j^2 = \frac{91}{6} \times \frac{91}{6} = \frac{8281}{36}.$$

Hence $E(X) = 12.25$ and $\sigma(X) = \sqrt{8281/36 - (49/4)^2} = 8.9423$.

9E-4 Let the random variable X be the payoff (expressed in dollars) on a lottery ticket. The random variable X takes on the values 0, 50, 500, and 5,000 with respective probabilities

$$P(X=0) = \frac{1 \times 9 \times 10 \times 10 + 9 \times 10 \times 10 \times 10}{10{,}000} = \frac{99}{100},$$
$$P(X=50) = \frac{1 \times 1 \times 9 \times 10}{10{,}000} = \frac{9}{1{,}000},$$
$$P(X=500) = \frac{1 \times 1 \times 1 \times 9}{10,000} = \frac{9}{10{,}000},\ P(X=5{,}000) = \frac{1}{10{,}000}.$$

This gives

$$E(X) = 0 \times \frac{99}{100} + 50 \times \frac{9}{1{,}000} + 500 \times \frac{9}{10{,}000} + 5{,}000 \times \frac{1}{10,,000} = 1.4.$$

The house edge of the lottery is $\left(\frac{2-1.4}{2}\right) \times 100\% = 30\%$.

9E-5 Let the random variable X be the payoff of the game. Then,

$$P(X=0) = \frac{6 \times 5 \times \times 5 \times 5}{6^4} = \frac{125}{216},$$
$$P(X=2) = \frac{6 \times \binom{3}{1} \times 1 \times 5 \times 4}{6^4} = \frac{60}{216},$$
$$P(X=2.5) = \frac{6 \times \binom{3}{1} \times 1 \times 5 \times 1}{6^4} = \frac{15}{216},$$
$$P(X=3) = \frac{6 \times \binom{3}{2} \times 1 \times 1 \times 5}{6^4} = \frac{15}{216},$$
$$P(X=4) = \frac{6 \times \times 1 \times 1 \times 1}{6^4} = \frac{1}{216}.$$

Alternatively, using conditional probabilities, the probability mass function of X can be calculated as $P(X=0) = (\frac{5}{6})^3$, $P(X=2) = \binom{3}{1}\frac{1}{6}(\frac{5}{6})^2 \times \frac{4}{5}$, $P(X=2.5) = \binom{3}{1}\frac{1}{6}(\frac{5}{6})^2 \times \frac{1}{5}$, $P(X=3) = \binom{3}{2}(\frac{1}{6})^2\frac{5}{6}$, and $P(X=4) = (\frac{1}{6})^3$. Hence

$$E(X) = 0 \times \frac{125}{216} + 2 \times \frac{60}{216} + 2.5 \times \frac{15}{216} + 3 \times \frac{15}{216} + 4 \times \frac{1}{216} = 0.956$$

Since $E(X) - 1 = -0.044 < 0$, the game is disadvantageous to you. The house edge is 4.4%.

9E-6 Let the random variable X be your payoff expressed in dollars. The random variable X can take the values 0.5, 5, 6, 7, 8, 9, and 10 with respective probabilities $\frac{4}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{10}$, and $\frac{1}{10}$. This gives

$$E(X) = 0.50 \times \frac{4}{10} + \sum_{j=5}^{10} j \times \frac{1}{10} = 4.7.$$

To calculate $\sigma(X)$, note that

$$E(X^2) = 0.50^2 \times \frac{4}{10} + \sum_{j=5}^{10} j^2 \times \frac{1}{10} = 35.6.$$

Hence $\sigma(X) = \sqrt{35.6 - 4.7^2} = 3.68$.

9E-7 Let the random variable X be the smallest number rolled. The easiest way to calculate $E(X)$ is to use $E(X) = \sum_{j=0}^{5} P(X > j)$. Noting that

$$P(X > j) = \left(\frac{6-j}{6}\right)^6 \qquad \text{for } j = 0, 1, \ldots, 5,$$

it follows that

$$E(X) = \sum_{j=0}^{5} \left(\frac{6-j}{6}\right)^6 = 1.4397.$$

To find $\sigma(X)$, note that $P(X = k) = P(X > k-1) - P(X > k)$ and so

$$E(X^2) = \sum_{k=1}^{6} k^2 \left[\left(\frac{6-k+1}{6}\right)^6 - \left(\frac{6-k}{6}\right)^6\right] = 2.5656.$$

This gives $\sigma(X) = \sqrt{E(X^2) - E^2(X)} = 0.702$.

9E-8 The idea is to use a one-stage-look-ahead rule. For the situation that you have collected so far a dollars and $k+1$ boxes are still closed including the box with the devil's penny, define the random variable $X_k(a)$ as the amount by which your capital would change when you would decide to open one other box. The one-stage-look-ahead-rule prescribes to stop if the expected value of $X_k(a)$ is smaller than or equal to zero and to continue otherwise. Let $A = a_1 + a_2 + \cdots + a_{10}$ be the original amount of dollars in the 10 boxes. To calculate $E\big(X_k(a)\big)$, you need not to know how the remaining amount of

$A-a$ dollars is distributed over the k remaining closed boxes not containing the devil's penny. To see this, imagine that the dollar amounts $a_{i_1}, \ldots, a_{i_k}$ are in these k closed boxes. Then,

$$E\big(X_k(a)\big) = \frac{1}{k+1}\sum_{j=1}^{k} a_{i_j} - \frac{1}{k+1}a = \frac{1}{k+1}(A-a) - \frac{1}{k+1}a.$$

Hence $E\big(X_k(a)\big) \leq 0$ only if $a \geq \frac{1}{2}A$, regardless of the value of k. A good strategy is to stop as soon you have collected $\frac{1}{2}A$ dollars or more and to continue otherwise.

9E-9 For any $k \geq 2$, let X_k be the amount you get at the kth game. Then the total amount you will get is $\sum_{k=2}^{s} X_k$. By the linearity of the expectation operator, the expected value of the total amount you will get is $\sum_{k=2}^{s} E(X_k)$. Since $P(X_k = 1) = \frac{1}{k(k-1)}$ and $P(X_k = 0) = 1 - P(X_k = 1)$, it follows that $E(X_k) = \frac{1}{k(k-1)}$ for $k = 2, \ldots, s$. Hence the expected value of the total amount you will get is equal to

$$\sum_{k=2}^{s} \frac{1}{k(k-1)} = \frac{s-1}{s}.$$

The fact that the sum equals $\frac{s-1}{s}$ is easily proved by induction.

9E-10 By conditioning, the probability of getting the same two outcomes when tossing the two coins is $\frac{1}{2} \times p + \frac{1}{2} \times (1-p) = \frac{1}{2}$. Hence the number of tosses until both coins simultaneously show the same outcome is geometrically distributed with parameter $\frac{1}{2}$ and thus has expected value 2 and standard deviation $\sqrt{2}$.

9E-11 For fixed k, let the random variable X_i be equal to 1 if box i contains exactly k apples and X_i be equal to 0 otherwise. Then,

$$P(X_i = 1) = \binom{25}{k}\left(\frac{1}{10}\right)^k \left(\frac{9}{10}\right)^{25-k}.$$

Hence the expected value of the number of boxes containing exactly k apples is given by

$$E(X_1 + \cdots + X_{10}) = 10 \times \binom{25}{k}\left(\frac{1}{10}\right)^k \left(\frac{9}{10}\right)^{25-k}, \quad k = 0, 1, \ldots, 25.$$

9E-12 Let us say that a success occurs each time an ace is drawn that you have not seen before. Denote by X_j be the number of cards drawn between the occurrences of the $(j-1)$st and jth success. The random variable X_j is geometrically distributed with success parameter $\frac{4-(j-1)}{52}$. Also, the random variables $X_1, \ldots, X_4$ are independent of each other. A geometrically distributed random variable with parameter p has expected value $1/p$ and variance $(1-p)/p^2$. Hence the expected value and the standard deviation of the number of times you have to draw a card until you have seen all four different aces are given by

$$E(X_1 + X_2 + X_3 + X_4) = \frac{52}{4} + \frac{52}{3} + \frac{52}{2} + \frac{52}{1} = 108.33$$

and

$$\sigma(X_1 + X_2 + X_3 + X_4) = \sqrt{\sum_{k=1}^{4} \frac{1 - k/52}{(k/52)^2}} = 61.16.$$

9E-13 It is easiest to compute the expected value and the standard deviation of the random variable X being the number of floors on which the elevator will not stop. Let the random variable $X_j = 1$ if the elevator does not stop on floor j and $X_j = 0$ otherwise. It holds that $P(X_j = 1) = \left(\frac{r-1}{r}\right)^m$ and so $E(X_j) = \left(\frac{r-1}{r}\right)^m$ for all for $j = 1, 2, \ldots, r$. Hence

$$E(X) = \sum_{j=1}^{r} E(X_j) = r\left(\frac{r-1}{r}\right)^m.$$

To find the variance of the random variable X, we use the relation

$$E(X^2) = \sum_{j=1}^{r} E(X_j^2) + 2\sum_{j=1}^{r}\sum_{k=j+1}^{r} E(X_j X_k).$$

Since X_j is a $0-1$ variable, we have $E(X_j^2) = E(X_j)$ for all for j. To find $E(X_j X_k)$ for $j \neq k$, note that

$$P(X_j = 1,\, X_k = 1) = \left(\frac{r-2}{r}\right)^m.$$

This gives

$$E(X^2) = r\left(\frac{r-1}{r}\right)^m + 2\binom{r}{2}\left(\frac{r-2}{r}\right)^m.$$

Next $\sigma(X)$ follows from $\sqrt{E(X^2) - E^2(X)}$. Finally, the expected value and the standard deviation of the number of stops of the elevator are given by $r - E(X)$ and $\sigma(X)$.

9E-14 (a) Let X_i be equal to 1 if the integer i does not show up in the 20 drawings and X_i be equal to 0 otherwise. Then the number of integers not showing up in the 20 drawings is $X_1 + X_2 + \cdots + X_{10}$. For all i, $E(X_i) = P(X_i = 1) = (9/10)^{20}$ and so

$$E(X_1 + X_2 + \cdots + X_{10}) = 10 \times \left(\frac{9}{10}\right)^{20} = 1.2158.$$

For any $i \neq j$, $E(X_i X_j) = P(X_i = 1, X_j = 1) = (8/10)^{20}$ and so

$$\begin{aligned} E(X_1 + X_2 + \cdots + X_{10})^2 &= 10 \times \left(\frac{9}{10}\right)^{20} + 2\binom{10}{2} \times \left(\frac{8}{10}\right)^{20} \\ &= 2.2534. \end{aligned}$$

This leads to $\sigma(X_1 + X_2 + \cdots + X_{10}) = 0.8805$.
(b) Let X_i be equal to 1 if the number i does not show up in the 15 lotto drawings and X_i be equal to 0 otherwise. The probability that a specified number does not show up in any given drawing is $\binom{44}{6}/\binom{45}{6} = 39/45$. Hence $E(X_i) = P(X_i = 1) = (39/45)^{15}$ and so

$$E(X_1 + X_2 + \cdots + X_{45}) = 45 \times \left(\frac{39}{45}\right)^{15} = 5.2601.$$

The probability that two specified numbers i and j with $i \neq j$ do not show up in any given drawing is $\binom{43}{6}/\binom{45}{6} = (39/45) \times (38/44)$. Hence $E(X_i X_j) = P(X_i = 1, X_j = 1) = [(39 \times 38)/(45 \times 44)]^{15}$ and so

$$\begin{aligned} E(X_1 + X_2 + \cdots + X_{45})^2 &= 45 \times \left(\frac{39}{45}\right)^{15} + 2\binom{45}{2} \times \left(\frac{39 \times 38}{45 \times 44}\right)^{15} \\ &= 30.9292. \end{aligned}$$

This leads to $\sigma(X_1 + X_2 + \cdots + X_{45}) = 1.8057$.

9E-15 For any $i \neq j$, let the random variable $X_{ij} = 1$ if the integers i and j are switched in the random permutation and $X_{ij} = 0$ otherwise. The total number of switches is $\sum_{i<j} X_{ij}$. Since $P(X_{ij} = 1) = \frac{(n-2)!}{n!}$, it follows that $E(X_{ij}) = \frac{1}{n(n-1)}$. Hence

$$E\left(\sum_{i<j} X_{ij}\right) = \sum_{i<j} E(X_{ij}) = \binom{n}{2} \frac{1}{n(n-1)} = \frac{1}{2},$$

irrespective of the value of n.

9E-16 Let the indicator variable I_k be equal to 1 if the kth team has a married couple and zero otherwise. Then $P(I_k = 1) = 12 \times 22/\binom{24}{3}$ for any k. Hence the expected number of teams having a married couple is equal to $\sum_{k=1}^{8} E(I_k) = (8 \times 12 \times 22)/\binom{24}{3} = 1.043$.

9E-17 Call your opponents East and West. The probability that East has two spades and West has three spades is

$$\frac{\binom{5}{2}\binom{21}{10}}{\binom{26}{13}} = \frac{39}{115}.$$

Hence the desired probability is $2 \times \frac{39}{115} = 0.6783$.

9E-18 Let the random variable X denote the number of times you will roll the die until you get an outcome that is larger than or equal to the randomly chosen integer from $1, 2, \ldots, 6$. Under the condition that the randomly chosen integer is j, the number of rolls of the die until you get an outcome that is larger than or equal to j is geometrically distributed with parameter $p_j = \frac{6-j+1}{6}$. By the law of conditional probability, it now follows that

$$P(X = k) = \sum_{j=1}^{6} \left(1 - \frac{6-j+1}{6}\right)^{k-1} \frac{6-j+1}{6} \times \frac{1}{6}, \quad k = 1, 2, \ldots.$$

Using the fact that the geometric distribution with parameter p has $1/p$ as expected value and $(1-p)/p^2$ as variance, it follows that

$$E(X) = \frac{1}{6} \sum_{k=1}^{\infty} k \sum_{j=1}^{6} \left(1 - \frac{6-j+1}{6}\right)^{k-1} \frac{6-j+1}{6} = \frac{1}{6} \sum_{j=1}^{6} \frac{6}{6-j+1},$$

yielding $E(X) = 2.45$. Also,

$$E(X^2) = \frac{1}{6} \sum_{k=1}^{\infty} k^2 \sum_{j=1}^{6} \left(1 - \frac{6-j+1}{6}\right)^{k-1} \frac{6-j+1}{6} = \sum_{j=1}^{6} \frac{6+j-1}{(6-j+1)^2},$$

yielding $E(X^2) = 15.4467$. This gives $\sigma(X) = \sqrt{15.4467 - 2.45^2} = 3.073$.

9E-19 Let the indicator variable $I_n = 1$ if the nth toss results in heads and $I_n = 0$ otherwise. To find $E(\sum_{n=1}^{r} I_n)$, let $a_n = P(I_n = 1)$ denote the probability that the nth toss will result in heads. By the law of conditional probabilities, $P(I_n = 1)$ is given by $P(I_n = 1 \mid I_{n-1} = 1)P(I_{n-1} = 1) + P(I_n = 1 \mid I_{n-1} = 0)P(I_{n-1} = 0)$ and so

$$a_n = 0.5a_{n-1} + p(1 - a_{n-1}) \quad \text{for } n = 0, 1, \ldots,$$

where $a_0 = 1$. The solution of this standard difference equation is given by

$$a_n = \alpha + \beta\gamma^n \quad \text{for } n = 1, 2, \ldots,$$

where $\alpha = \frac{p}{p+0.5}$, $\beta = \frac{1}{2p+1}$ and $\gamma = 0.5 - p$. This gives that the expected value of the number of heads in r tosses is given by $\sum_{n=1}^{r}(\alpha + \beta\gamma^n)$.

9E-20 Label the white balls as $1, \ldots, W$. Let the indicator variable I_k be equal to 1 if the white ball with label k remains in the bag when you stop and 0 otherwise. To find the probability $P(I_k = 1)$, you can discard the other white balls and consider the situation that you pick balls from a bowl with R red balls and one white ball. In this situation the probability of the white ball remaining as last ball in the bowl is $\frac{1}{R+1}$. Hence $E(I_k) = \frac{1}{R+1}$ for $k = 1, \ldots, W$. This gives that the expected number of white balls remaining in the bag when you stop is equal to $\sum_{k=1}^{R} E(I_k) = \frac{W}{R+1}$.

9E-21 For $x, y \in \{-1, 1\}$, $P(X = x, Y = y) = P(X = x \mid Y = y)P(Y = y)$ and $P(X = x \mid Y = y) = P(Z = x/y \mid Y = y)$. By the independence of Y and Z, $P(Z = x/y \mid Y = y) = P(Z = x/y) = 0.5$. This gives

$$P(X = x, Y = y) = 0.5 \times P(Y = y).$$

Also, by $P(X = 1) = P(Y = 1, Z = 1) + P(Y = -1, Z = -1)$ and the independence of Y and Z, it follows that $P(X = 1) = 0.5^2 + 0.5^2$. This shows that $P(X = x) = 0.5$ for $x = -1, 1$ and so, by $P(X = x, Y = y) = 0.5 \times P(Y = y)$,

$$P(X = x, Y = y) = P(X = x)P(Y = y) \quad \text{for } x, y \in \{-1, 1\},$$

proving that X and Y are independent. By the same reasoning, X and Z are independent. However, X is not independent of $Y + Z$. To see this, note that $P(X = 1, Y + Z = 0) = 0$ and $P(X = 1)P(Y + Z = 0) > 0$.

9E-22 (a) You can think of $n + m$ independent Bernoulli experiments with success probability p, where X is the number of successes in the first n experiments and Y is the number of successes in the last m experiments. This explains why $X + Y$ is binomially distributed with parameters $n + m$ and p. A formal proof goes as follows. Using the independence of X and Y, it follows that

$$\begin{aligned}
P(X+Y=k) &= \sum_{r=0}^{k} P(X=r,\, Y=k-r) = \sum_{r=0}^{k} P(X=r,\, Y=k-r) \\
&= \sum_{r=0}^{k} \binom{n}{r} p^r (1-p)^{n-r} \binom{m}{k-r} p^{k-r} (1-p)^{m-(k-r)} \\
&= p^k (1-p)^{n+m-k} \sum_{r=0}^{k} \binom{n}{r} \binom{m}{k-r}
\end{aligned}$$

for $k = 0, 1, \ldots, n+m$. Using the identity $\sum_{r=0}^{k} \binom{n}{r}\binom{m}{k-r} = \binom{n+m}{k}$, it follows that $X+Y$ has a binomial distribution with parameters $n+m$ and p.
(b) By $P(A \mid B) = \frac{P(AB)}{P(B)}$, it follows that

$$P(X=j \mid X+Y=k) = \frac{P(X=j,\, X+Y=k)}{P(X+Y=k)}.$$

Using the independence of X and Y,

$$\begin{aligned}
P(X=j,\, X+Y=k) &= P(X=j,\, Y=k-j) \\
&= \binom{n}{j} p^j (1-p)^{n-j} \binom{m}{k-j} p^{k-j} j (1-p)^{m-k+j}
\end{aligned}$$

and so

$$P(X=j,\, X+Y=k) = \binom{n}{j}\binom{m}{k-j} p^k (1-p)^{n+m-k}$$

for $0 \leq j \leq k$ and $0 \leq k \leq n+m$. Hence

$$P(X=j \mid X+Y=k) = \frac{\binom{n}{j}\binom{m}{k-j} p^k (1-p)^{n+m-k}}{\binom{n+m}{k} p^k (1-p)^{n+m-k}} = \frac{\binom{n}{j}\binom{m}{k-j}}{\binom{n+m}{k}}.$$

For fixed k, the probabilities $P(X=j \mid X+Y=k)$ for $j = 0, \ldots, k$ constitute a hypergeometric distribution.

9E-23 Let the random variable N be the number of particles emitted in the given time interval. Noting that $P(X=j,\, Y=k) = P(X=j,\, Y=k,\, N=j+k)$ and using the formula $P(AB) = P(A \mid B)P(B)$, it follows that

$$\begin{aligned}
P(X=j,\, Y=k) &= P(X=j,\, Y=k \mid N=j+k) P(N=j+k) \\
&= \binom{j+k}{j} p^j (1-p)^k e^{-\lambda} \frac{\lambda^{j+k}}{(j+k)!}.
\end{aligned}$$

Since $\binom{j+k}{j} = \frac{(j+k)!}{j!k!}$ and $e^{-\lambda} = e^{-\lambda p} e^{-\lambda(1-p)}$, this result can be restated as

$$P(X = j,\, Y = k) = e^{-\lambda p} \frac{(\lambda p)^j}{j!} \times e^{-\lambda(1-p)} \frac{(\lambda(1-p))^k}{k!} \quad \text{for all } j, k \geq 0.$$

Using the relations $P(X = j) = \sum_{k=0}^{\infty} P(X = j,\, Y = k)$ and $P(Y = k) = \sum_{j=0}^{\infty} P(X = j,\, Y = k)$, it next follows that

$$P(X = j) = e^{-\lambda p} \frac{(\lambda p)^j}{j!} \quad \text{and} \quad P(Y = k) = e^{-\lambda(1-p)} \frac{(\lambda(1-p))^k}{k!}$$

for all $j,\, k \geq 0$. Hence X and Y are Poisson distributed with expected values λp and $\lambda(1-p)$. Moreover,

$$P(X = j,\, Y = k) = P(X = j)P(Y = k) \quad \text{for all } j, k,$$

showing the remarkable that X and Y are independent.

9E-24 (a) By the substitution rule,

$$\begin{aligned}
&E[\lambda g(X+1) - Xg(X)] \\
&= \sum_{k=0}^{\infty} \lambda g(k+1) e^{-\lambda} \frac{\lambda^k}{k!} - \sum_{k=0}^{\infty} k g(k) e^{-\lambda} \frac{\lambda^k}{k!} \\
&= \sum_{k=0}^{\infty} \lambda g(k+1) e^{-\lambda} \frac{\lambda^k}{k!} - \lambda \sum_{l=0}^{\infty} g(l+1) e^{-\lambda} \frac{\lambda^l}{l!} = 0.
\end{aligned}$$

(b) Let $p_j = P(X = j)$ for $j = 0, 1, \ldots$. For fixed $i \geq 1$, define the indicator function $g(x)$ by $g(k) = 1$ for $k = i$ and $g(k) = 0$ for $k \neq i$. Then the relation $E[\lambda g(X+1) - Xg(X)] = 0$ reduces to

$$\lambda p_{i-1} - i p_i = 0.$$

This gives $p_i = \frac{\lambda}{i} p_{i-1}$ for $i \geq 0$. By repeated application of this equation, it next follows that $p_i = \frac{\lambda^i}{i!} p_0$ for $i \geq 0$. Using the fact that $\sum_{i=0}^{\infty} p_i = 1$, we get $p_0 = e^{-\lambda}$. This gives

$$P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!} \quad \text{for } i = 0, 1, \ldots,$$

proving the desired result.

9E-25 Let the random variable N be the outcome of the initial roll of the die. Using the law of conditional probability with the conditioning events $B_j = \{N = j\}$, it follows that

$$P(X = k) = \sum_{j=1}^{6} P(X = k \mid N = j) P(N = j), \quad k = 0, 1, \ldots, 7.$$

Under the condition that the first roll gives the outcome j, the probability of a total k sixes is equal to the binomial probability $\binom{j}{k}(1/6)^k(5/6)^{j-k}$ if $1 \leq j \leq 5$ and is equal to the binomial probability $\binom{6}{k-1}(1/6)^{k-1}(5/6)^{6-k+1}$ if $j = 6$, with the convention that $\binom{j}{k} = 0$ for $k > j$. This results in $P(X=0) = \frac{1}{6}\sum_{j=1}^{5}(5/6)^j$ and

$$\begin{aligned} P(X=k) &= \frac{1}{6}\sum_{j=k}^{5}\binom{j}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{j-k} \\ &+ \frac{1}{6}\binom{6}{k-1}\left(\frac{1}{6}\right)^{k-1}\left(\frac{5}{6}\right)^{6-k+1}, \quad k = 1, 2, \ldots, 7. \end{aligned}$$

The probability $P(X = k)$ has the numerical values 0.4984, 0.3190, 0.1293, 0.0422, 0.0096, 0.0014, and 3.57×10^{-6} for $k = 0$, 1, 2, 3, 4, 5, 6, and 7.

9E-26 Let the random variable X be your payoff in any play. The probability of drawing j gold-colored balls is

$$P(X = j) = \frac{\binom{4}{j}\binom{6}{3-j}}{\binom{10}{4}} \quad \text{for } j = 0, 1, 2, 3.$$

Alternatively, using conditional probabilities, $P(X = 2)$ and $P(X = 3)$ can be computed as $P(X=2) = \frac{4}{10} \times \frac{3}{9} \times \frac{6}{8} + \frac{6}{10} \times \frac{4}{9} \times \frac{3}{8} + \frac{4}{10} \times \frac{6}{9} \times \frac{3}{8} = \frac{3}{10}$ and $P(X=3) = \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{1}{30}$. Hence

$$E(X) = 1 \times P(X=2) + 11 \times P(X=3) = 1 \times \frac{3}{10} + 11 \times \frac{1}{30} = \frac{2}{3} \text{ dollars.}$$

Your expected loss in any play is $1 - \frac{2}{3} = \frac{1}{3}$ dollars and so the house edge is 33.3%.

9E-27 The problem can be translated into the urn model with 1,422 white balls and 1,405 black balls. The desired probability is equal to the probability the number of white balls remaining is smaller than or equal the number of black balls remaining when 101 randomly chosen balls are removed from the urn. Noting that $1{,}422 - m \leq 1{,}405 - (101 - m)$ only if $m \geq 59$, it follows that this probability is given by

$$\sum_{m=59}^{101} \frac{\binom{1,422}{m}\binom{1,405}{101-m}}{\binom{2,827}{101}} = 0.05917.$$

9E-28 Let E_0 be the event that you get $r-1$ red balls in the first $k-1$ draws and E_1 be the event of a red ball at the kth draw. The desired probability is $P(E_0E_1)$. It holds that

$$P(E_0) = \frac{\binom{n}{r-1}\binom{m}{k-1-(r-1)}}{\binom{n+m}{k-1}} \quad \text{and} \quad P(E_1 \mid E_0) = \frac{n-(r-1)}{n+m-(k-1)}$$

Hence, by $P(E_0E_1) = P(E_0)P(E_1 \mid E_0)$, the desired probability is given by

$$\frac{\binom{n}{r-1}\binom{m}{k-r}}{\binom{n+m}{k-1}} \times \frac{n-r+1}{n+m-k+1} \quad \text{for } k = r, \ldots, m+r.$$

9E-29 Assuming that the game is not stopped, let the random variable X_1 be the number of draws you need to get one of the numbers $1, \ldots, 4$ and X_2 be the number of draws your friend needs to get one of the numbers $5, \ldots, 10$. The probability that you will be the winner is $P(X_1 \le X_2)$. Since you and your friend draw the random numbers independently of each other with respective success probabilities $p_1 = \frac{4}{10}$ and $p_2 = \frac{6}{10}$, it follows that

$$\begin{aligned} P(X_1 \le X_2) &= \sum_{j=1}^{\infty} P(X_1 = j,\, X_2 \ge j) \\ &= \sum_{j=1}^{\infty} p_1(1-p_1)^{j-1}(1-p_2)^{j-1} = \frac{p_1}{p_1+p_2-p_1p_2}. \end{aligned}$$

Hence your probability of winning is 0.5263. It is a surprising result that your probability of winning is more than 50%, while p_1 is smaller than p_2. The length of the game is $\min(X_1, X_2)$. It holds that

$$\begin{aligned} P(\min(X_1, X_2) > l) &= P(X_1 > l,\, X_2 > l) \\ &= (1-p_1)^l(1-p_2)^l = (1-p)^l, \quad l = 0, 1, \ldots, \end{aligned}$$

where $p = p_1 + p_2 - p_1p_2$. Using the relation $P(\min(X_1, X_2) = l) = P(\min(X_1, X_2) > l-1) - P(\min(X_1, X_2) > l)$, it next follows that

$$P(\min(X_1, X_2) = l) = p(1-p)^{l-1} \quad \text{for } l = 1, 2, \ldots.$$

In other words, the length of the game is geometrically distributed with parameter $p_1 + p_2 - p_1p_2 = 0.76$.

9E-30 Let the random variable X be the total demand for spare parts in the remaining lifetime of the airplane. Using the law of conditional probabilities and noting that the sum of independent Poisson random variables is again Poisson distributed, it follows that

$$P(X = k) = \frac{1}{3}\sum_{l=3}^{5} e^{-l\lambda}\frac{(l\lambda)^k}{k!} \quad \text{for } k = 0, 1, \ldots.$$

The probability that there will be a shortage in the remaining lifetime of the plane is

$$\sum_{k=Q+1}^{\infty} P(X=k) = \frac{1}{3}\sum_{k=Q+1}^{\infty}\sum_{l=3}^{5} e^{-l\lambda}\frac{(l\lambda)^k}{k!} = \frac{1}{3}\sum_{l=3}^{5} e^{-l\lambda}\sum_{k=Q+1}^{\infty}\frac{(l\lambda)^k}{k!}$$

Using the fact that $\sum_{k=Q+1}^{\infty}\frac{(l\lambda)^k}{k!} = e^{l\lambda} - \sum_{k=0}^{Q}\frac{(l\lambda)^k}{k!}$, it follows that

$$P(\text{shortage}) = \frac{1}{3}\sum_{l=3}^{5}\Big[1 - \sum_{k=0}^{Q} e^{-l\lambda}\frac{(l\lambda)^k}{k!}\Big].$$

The number of spare parts left over at the end of the remaining lifetime of the plane is $\max(Q-X,0)$. By the substitution rule,

$$\begin{aligned} E(\text{number of spare parts left over}) &= \sum_{k=0}^{Q}(Q-k)P(X=k) \\ &= \frac{1}{3}\sum_{l=3}^{5} e^{-l\lambda}\sum_{k=0}^{Q}(Q-k)\frac{(l\lambda)^k}{k!}. \end{aligned}$$

The amount of demand that cannot be satisfied from the stock of Q units is $\max(X-Q,0)$. By the substitution rule,

$$\begin{aligned} E(\text{shortage}) &= \sum_{k=Q+1}^{\infty}(k-Q)P(X=k) \\ &= \sum_{k=0}^{\infty}(k-Q)P(X=k) - \sum_{k=0}^{Q}(k-Q)P(X=k) \\ &= \frac{1}{3}(3\lambda+4\lambda+5\lambda) - Q + \frac{1}{3}\sum_{l=3}^{5} e^{-l\lambda}\sum_{k=0}^{Q}(Q-k)\frac{(l\lambda)^k}{k!}. \end{aligned}$$

9E-31 Let E be the event that no more than one of your ten trees will not grow well and B_k be the event that there are k trees of tree-nurseryman A are among your ten trees. By the law of conditional probability, $P(E) = \sum_{k=0}^{10} P(E \mid B_k)P(B_k)$. For any $k = 0, 1, \ldots, 10$,

$$P(B_k) = \frac{\binom{50}{k}\binom{50}{10-k}}{\binom{100}{10}}.$$

The conditional probability $P(A \mid B_k)$ is given by

$$P(A \mid B_k) = (0.90)^k \times (0.95)^{10-k} + \binom{k}{1} 0.10\,(0.90)^{k-1} \times 0.95^{10-k}$$
$$+ \binom{10-k}{1} 0.05\,(0.95)^{9-k} \times 0.90^k \quad \text{for } k = 0, 1, \ldots, 10$$

with the convention $\binom{0}{1} = 0$. Substituting the expressions for $P(E \mid B_k)$ and $P(B_k)$ into $P(E) = \sum_{k=0}^{10} P(E \mid B_k)P(B_k)$ gives

$$P(E) = 0.8243.$$

9E-32 Let A_k be the event that the last drawing has k numbers in common with the second last drawing. Seeing the six numbers from the second last drawing as red balls and the other numbers as white balls, it follows that

$$P(A_k) = \frac{\binom{6}{k}\binom{43}{6-k}}{\binom{49}{6}} \quad \text{for } k = 0, 1, \ldots, 6.$$

Let E be the event that the next drawing will have no numbers common with the last two two drawings. Then, by the law of conditional probabilities, $P(E) = \sum_{k=0}^{6} P(E \mid A_k)P(A_k)$ and so

$$P(E) = \sum_{k=0}^{6} \frac{\binom{49-(6+6-k)}{6}}{\binom{49}{6}} \times \frac{\binom{6}{k}\binom{43}{6-k}}{\binom{49}{6}} = 0.1901.$$

9E-33 Let the indicator variable I_k be equal to 1 if Bill and Matt have both chosen the number k and 0 otherwise. Since $P(I_k = 1) = \frac{5}{100} \times \frac{5}{100} = 0.0025$ for all k, the expected number of common numbers in the choices of Bill and Matt is $\sum_{k=1}^{100} E(I_k) = 0.25$. Let the random variable X indicate how many numbers Bill and Matt have in common in their choices. Seeing the numbers chosen by Bill as red balls and the other numbers as white balls, it is directly seen that X has the hypergeometric distribution

$$P(X = k) = \frac{\binom{5}{k}\binom{95}{5-k}}{\binom{100}{5}} \quad \text{for } k = 0, 1, \ldots, 5.$$

In particular, the probability that the choices of Bill and Matt have a number in common is $1 - P(X = 0) = 0.2304$.

9E-34 Suppose your strategy is to stop as soon as you have picked a number larger than or equal to r. The number of trials until you have picked a number larger than or equal to r is geometrically distributed with success probability $p = \frac{25-r+1}{25}$. The dollar amount you get paid has the discrete uniform distribution on $r, \ldots, 25$. Hence

$$\begin{aligned} E(\text{net payoff}) &= \frac{1}{25-r+1}\sum_{k=r}^{25} k - \frac{25}{25-r+1} \\ &= \frac{1}{2}(25+r) - \frac{25}{25-r+1} \text{ dollars.} \end{aligned}$$

This expression takes on the maximal value \$18.4286 for $r = 19$.

Chapter 10

10E-1 The requirement $1 = c\int_0^1 (x + \sqrt{x})\,dx = \frac{7}{6}c$ gives $c = \frac{6}{7}$. To find the density function of $Y = \frac{1}{X}$, we determine $P(Y \leq y)$. Obviously, $P(Y \leq y) = 0$ for $y \leq 1$. For $y > 1$,

$$P(Y \leq y) = P\left(X \geq \frac{1}{y}\right) = 1 - P\left(X \leq \frac{1}{y}\right) = 1 - F\left(\frac{1}{y}\right),$$

where $F(x)$ is the probability distribution function of X. By differentiation, it follows that the density function $g(y)$ of Y is given by

$$g(y) = \frac{6}{7} f\left(\frac{1}{y}\right) \times \frac{1}{y^2} = \frac{6}{7}\left(\frac{1}{y^3} + \frac{1}{y^2\sqrt{y}}\right) \quad \text{for } y > 1$$

and $g(y) = 0$ otherwise.

10E-2 The area of the circle is $Y = \pi X^2$, where X is uniformly distributed on $(0, 1)$. The density function of X is given by $f(x) = 1$ for $0 < x < 1$ and $f(x) = 0$ otherwise. For any y with $0 \leq y \leq \pi$,

$$P(Y \leq y) = P\left(X \leq \sqrt{\frac{y}{\pi}}\right) = \int_0^{y/\pi} dx = \sqrt{\frac{y}{\pi}}.$$

Differentiation gives that the density function of the area of the circle is given by $g(y) = \frac{1}{2\sqrt{\pi y}}$ for $0 \leq y \leq \pi$ and $g(y) = 0$ otherwise.

10E-3 The range of the random variable X is the interval $[0, 1]$. Let A be the subset of points from the rectangle for which the distance to the closest side of the rectangle is larger than x, where $0 \leq x \leq 1$. Then A is a rectangle whose sides have the lengths $3 - 2x$ and $2 - 2x$ and so the area of A is $(3 - 2x)(2 - 2x)$. It now follows that

$$P(X \leq x) = \frac{6 - (3-2x)(2-2x)}{6} = \frac{5}{3}x - \frac{2}{3}x^2 \quad \text{for } 0 \leq x \leq 1.$$

The probability density $f(x)$ of X is given by $f(x) = \frac{5}{3} - \frac{4}{3}x$ for $0 < x < 1$ and $f(x) = 0$ otherwise. The expected value of X is $\int_0^1 x\left(\frac{5}{3} - \frac{4}{3}x\right)\, dx = \frac{7}{18}$. Noting that $E(X^2) = \int_0^1 x^2\left(\frac{5}{3} - \frac{4}{3}x\right)\, dx = \frac{2}{9}$, it follows that $\sigma(X) = \sqrt{2/9 - (7/18)^2} = \frac{1}{18}\sqrt{23}$.

10E-4 The probability distribution function of the weekly volume of waste in thousands of gallons is given by

$$\begin{aligned} F(x) &= 105\int_0^x y^4(1-y)^2\, dy = 105\int_0^x (y^4 - 2y^5 + y^6)\, dy \\ &= x^5(15x^2 - 35x + 21) \quad \text{for } 0 \leq x \leq 1. \end{aligned}$$

The solution of the equation $1 - F(x) = 0.05$ is $x = 0.8712$. Hence the capacity of the storage tank in thousands of gallons should be 0.8712.

10E-5 Let random variable X be the amount of waste (in thousands of gallons) produced during a week and Y be the total costs incurred during a week. Then the random variable Y can be represented as $Y = g(X)$, where the function $g(x)$ is given by

$$g(x) = \begin{cases} 1.25 + 0.5x & \text{for } 0 < x < 0.9, \\ 1.25 + 0.5 \times 0.9 + 5 + 10(x - 0.9) & \text{for } 0.9 < x < 1 \end{cases}$$

and $g(x) = 0$ otherwise. By the substitution rule, the expected value of the weekly costs is given by

$$105\int_0^1 g(x)x^4(1-x)^2\, dx = 1.69747.$$

To find the standard deviation of the weekly costs, we first calculate

$$E(Y^2) = \int_0^1 g^2(x)x^4(1-x)^2\, dx = 3.62044.$$

Hence the standard deviation of the weekly costs is equal to $\sqrt{3.62044 - 1.69747^2} = 0.8597$.

10E-6 A stockout occurs if the demand X is larger than Q and so

$$P(\text{stockout}) = \int_Q^\infty f(x)\, dx = 1 - \int_0^Q f(x)\, dx.$$

The amount of stock left over at the end of the period is $\max(Q - X, 0)$. By the substitution rule, we can compute the expected value of the random variable $Y = \max(Q - X, 0)$. This gives

$$E(\text{stock left over}) = \int_0^Q (Q - x)f(x)\, dx.$$

The amount that cannot be satisfied from stock is $\max(X - Q, 0)$. By the substitution rule,

$$E(\text{shortage}) = \int_Q^\infty (x - Q) f(x)\, dx = \mu - Q + \int_0^Q (Q - x) f(x)\, dx,$$

where $\mu = E(X)$.

10E-7 Denoting by the point O the center of the circle, let the random variable Θ be the angle between the line segments PO and QO. Then Θ is uniformly distributed between 0 and π. A little geometry shows that $\sin(\frac{1}{2}\Theta) = \frac{X/2}{r}$ and so $X = 2r\sin(\frac{1}{2}\Theta)$. Hence, by the substitution rule,

$$\begin{aligned} E(X) &= \int_0^\pi 2r\sin\left(\frac{1}{2}\theta\right)\frac{1}{\pi}\, d\theta = \frac{4r}{\pi}\int_0^{\pi/2} \sin(x)\, dx \\ &= \frac{-4r}{\pi}\cos(x)\,\Big|_0^{\pi/2} = \frac{4r}{\pi} \approx 1.273r. \end{aligned}$$

Noting that $\sin^2(x) + \cos^2(x) = 1$ and $\int_0^{\pi/2} \sin^2(x)\, dx = \int_0^{\pi/2} \cos^2(x)\, dx$, it follows that $\int_0^{\pi/2} \sin^2(x)\, dx = \frac{\pi}{4}$. Hence, by using again the substitution rule,

$$\begin{aligned} E(X^2) &= \int_0^\pi 4r^2\sin^2\left(\frac{1}{2}\theta\right)\frac{1}{\pi}\, d\theta = \frac{8r^2}{\pi}\int_0^{\pi/2} \sin^2(x)\, dx \\ &= \frac{8r^2}{\pi} \times \frac{\pi}{4} = 2r^2. \end{aligned}$$

This gives

$$\sigma(X) = \sqrt{2r^2 - \frac{16r^2}{\pi^2}} \approx 0.616r.$$

10E-8 Let the random variable X be the amount of damage. Then $Y = g(X)$, where the function $g(x)$ is given by $g(x) = 0$ for $x \leq 450$ and $g(x) = \min(500, x - 450)$ for $x > 450$. Using the substitution rule, it follows that

$$E(Y) = \int_{450}^{950} (x - 450)\frac{1}{1{,}000}\, dx + 500\int_{950}^{1{,}250} \frac{1}{1{,}000}\, dx = 275.$$

The probability distribution function of Y is given by

$$\begin{aligned} &P(Y = 0) = P(250 \leq X \leq 450) = 0.20, \\ &P(Y \leq y) = P(Y = 0) + \int_{450}^{450+y} \frac{1}{1{,}000} dx = 0.20 + \frac{y}{1{,}000} \text{ for } y < 500, \\ &P(Y = 500) = P(950 < X \leq 1{,}250) = 0.30. \end{aligned}$$

The random variable Y has a mixed distribution with a probability mass at each of the points 0 and 500 and a density on the interval $(0, 500)$.

10E-9 Let the $N(7, 4)$-distributed random variable X be the lifetime of the item. The expected value of the payment per insurance is

$$\begin{aligned}
&aP(X \leq 2) + \frac{1}{2}aP(2 < X \leq 4) \\
&= aP\left(\frac{X-7}{2} \leq \frac{2-7}{2}\right) + \frac{1}{2}aP\left(\frac{2-7}{2} < \frac{X-7}{2} \leq \frac{4-7}{2}\right) \\
&= a\Phi(-2.5) + \frac{1}{2}a[\Phi(-1.5) - \Phi(-2.5)] = a \times 0.03651.
\end{aligned}$$

Hence the value 1370 should be taken for a.

10E-10 The area of the triangle is given by $V = \frac{1}{2}\text{tg}(\Theta)$. The density $f(\theta)$ of Θ is given by $f(\theta) = \frac{4}{\pi}$ for $0 < \theta < \frac{\pi}{4}$ and $f(\theta) = 0$ otherwise. By the substitution rule, the first two moments of V are given by

$$E(V) = \frac{1}{2}\int_0^{\pi/4} \text{tg}(\theta)\frac{4}{\pi}d\theta = \frac{1}{\pi}\ln(2)$$

and

$$E(V^2) = \frac{1}{4}\int_0^{\pi/4} \text{tg}^2(\theta)\frac{4}{\pi}d\theta = \frac{1}{\pi}\left(1 - \frac{\pi}{4}\right).$$

10E-11 The quadratic distance is distributed as X^2+Y^2, where X and Y are independent random variables each having the standard normal density. The random variable $V = X^2+Y^2$ has the chi-square density with two degrees of freedom. This density is given by $f(v) = \frac{1}{2}e^{-\frac{1}{2}v}$ for $v > 0$. Denoting by $F(v)$ the probability distribution function of the χ_2^2 distributed random variable V, it follows that the probability distribution function of the distance from the center of the target to the point of impact is given by

$$P(\sqrt{X^2+Y^2} \leq r) = P(V \leq r^2) = F(r^2) \quad \text{for } r > 0.$$

Hence the probability density of the distance from the center of the target to the point of impact is $2rf(r^2) = re^{-\frac{1}{2}r^2}$ for $r > 0$. The expected value of the distance is $\int_0^\infty r^2 e^{-\frac{1}{2}r^2}\,dr = \frac{1}{2}\sqrt{2\pi}$, using the fact that $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty r^2 e^{-\frac{1}{2}r^2}\,dr = 1$ for the standard normal density. The mode of the distance follows by solving the equation $\int_0^x re^{-\frac{1}{2}r^2}\,dr = 0.5$. This gives the value $\sqrt{2\ln(2)}$ for the mode.

10E-12 The radius of the circle is given by the random variable $R = \sqrt{X^2+Y^2}$ and the area of the circle is $\pi(X^2+Y^2)$. Using the fact that X^2+Y^2 has the χ_2^2-density $f(x) = \frac{1}{2}e^{-\frac{1}{2}x}$ for $x > 0$, it follows that

$$P\big(\pi(X^2+Y^2) \leq v\big) = P\left(X^2+Y^2 \leq \frac{v}{\pi}\right) = \int_0^{v/\pi} \frac{1}{2}e^{-\frac{1}{2}w}dw, \quad v > 0.$$

Hence the density function of the area of the circle is $\frac{1}{2\pi}e^{-\frac{1}{2}v/\pi}$ for $v > 0$. This is the exponential density with parameter 2π. The expected value of the area of the circle is $\int_0^\infty v\frac{1}{2\pi}e^{-\frac{1}{2}v/\pi}dv = 2\pi$.

10E-13 Let the random variable L be length of the line segment joining the points (X, Y) and $(1, 0)$. Since $X^2 + Y^2 = 1$, $L = \sqrt{(X-1)^2+Y^2} = \sqrt{2-2X}$. Noting that X is distributed as $\cos(\Theta)$ with Θ being uniformly distributed on $(0, 2\pi)$, it follows from the substitution rule that that

$$E(L) = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{2-2\cos(\theta)}\, d\theta.$$

Using numerical integration to evaluate the integral, we get $E(L) = \frac{8}{2\pi}$.

10E-14 Let U be the randomly chosen point in the interval $(0, 1)$. Then the length L of the subinterval covering the given point s is given by $g(U)$, where the function $g(x)$ is defined by $g(x) = 1 - x$ if $x < s$ and $g(u) = x$ if $x \geq s$. The density function $f(u)$ of U is given by $f(u) = 1$ for $0 < u < 1$ and $f(u) = 0$ otherwise. Hence, by the substitution rule,

$$E(L) = \int_0^1 g(u)f(u)\, du = \int_0^s (1-u)\, du + \int_s^1 u\, du = s - s^2 + \frac{1}{2}.$$

10E-15 Let the random variable X denote your service time, For fixed $t > 0$, let A be the event that your service time is no more than t and B_i be the event that you are routed to server i for $i = 1, 2$. Then, by the law of conditional probability, $P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2)$. Using the fact that $P(V \leq t) = 1 - e^{-\lambda t}$ for an exponentially distributed random variable V with parameter λ, it now follows that

$$P(X \leq t) = p_1(1 - e^{-\mu_1 t}) + p_2(1 - e^{-\mu_2 t}).$$

Differentiation gives that the density function $f(t)$ of X is given by

$$f(t) = p_1\mu_1 e^{-\mu_1 t} + p_2\mu_2 e^{-\mu_2 t} \quad \text{for } t > 0.$$

This is the so-called hyperexponential density.

10E-16 (a) The probability distribution function of Y is given by

$$\begin{aligned}
P(Y \leq y) &= P(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sqrt{y}} e^{-\frac{1}{2}x^2}\, dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\sqrt{y}} e^{-\frac{1}{2}x^2}\, dx.
\end{aligned}$$

Differentiation of $P(Y \leq y)$ gives that the probability density function of Y is given by

$$\begin{aligned}
g(y) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} \frac{1}{2\sqrt{y}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y} \frac{1}{2\sqrt{y}} \\
&= \frac{1}{\sqrt{2\pi}} y^{\frac{1}{2}} e^{-\frac{1}{2}y} \quad \text{for } y \geq 0
\end{aligned}$$

and $g(y) = 0$ otherwise. This shows that Y has a gamma density with shape parameter $\frac{1}{2}$ and scale parameter $\frac{1}{2}$.
(b) Noting that $P(Y \leq y) = P(|X| \leq y^2)$ for $y \geq 0$, it follows that

$$P(Y \leq y) = P(-y^2 \leq X \leq y^2) = \Phi(y^2) - \Phi(-y^2) \quad \text{for } y \geq 0.$$

Put for abbreviation $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$. Differentiation of the distribution function $P(Y \leq y)$ gives that the density function of Y is equal to

$$2y\phi(y^2) + 2y\phi(-y^2) = \frac{4y}{\sqrt{2\pi}} e^{-\frac{1}{2}y^4} \quad \text{for } y > 0$$

and is equal to zero otherwise.

10E-17 The probability distribution function of Y is given by

$$\begin{aligned}
P(Y \leq y) &= P(2X - 1 \leq y) = P\left(X \leq \frac{1}{2}(y+1)\right) \\
&= \int_0^{\frac{1}{2}(y+1)} \frac{8}{\pi} \sqrt{x(1-x)}\, dx, \quad -1 \leq y \leq 1.
\end{aligned}$$

The derivative of $P(Y \leq y)$ is $(8/\pi)\sqrt{\frac{1}{2}(y+1)(1-\frac{1}{2}(y+1))} \times \frac{1}{2} = \frac{2}{\pi}\sqrt{1-y^2}$ for $-1 < y < 1$. Hence the probability density function of Y is given by $g(y) = \frac{2}{\pi}\sqrt{1-y^2)}$ for $-1 < y < 1$ and $g(y) = 0$ otherwise. This is the so-called semicircle density.

10E-18 The probability density function $g(y)$ of the random variable Y is given by $f(a(y))|a'(y)|$, where the inverse function $a(y) = \frac{1}{y}$. This gives

$$g(y) = \frac{1}{\pi(1 + (1/y)^2)} \left| \frac{-1}{y^2} \right| = \frac{1}{\pi(1+y^2)} \quad \text{for } -\infty < y < \infty.$$

In other words, the random variable $\frac{1}{X}$ has the same Cauchy distribution as X.

10E-19 Let us first define the conditional density function of X given that $X > a$. To do so, we determine $P(X \leq x \mid X \leq a)$ By $P(A \mid B) = P(AB)/P(B)$, we have for any $x < a$ that

$$P(X \leq x \mid X \leq a) = \frac{P(X \leq x)}{P(X \leq a)} = \frac{\int_{-\infty}^{x} f(y)\, dy}{\int_{-\infty}^{a} f(y)\, dy}.$$

Obviously, $P(X \leq x \mid X \leq a) = 1$ for $x \geq a$. This explains why conditional probability density $f_a(x)$ of X given that $X \leq a$ is defined by

$$f_a(x) = \frac{f(x)}{\int_{-\infty}^{a} f(y)\, dy} \qquad \text{for } x < a$$

and $f_a(x) = 0$ otherwise. The conditional expected value $E(X \mid X \leq a)$ is defined by

$$E(X \mid X \leq a) = \int_{-\infty}^{a} x f_a(x)\, dx.$$

For the case that x has the exponential density with parameter λ,

$$E(X \mid X \leq a) = \frac{\int_{-\infty}^{a} x\lambda e^{-\lambda x}\, dx}{\int_{-\infty}^{a} \lambda e^{-\lambda x}\, dx} = \frac{1 - e^{-\lambda a} - \lambda a e^{-\lambda a}}{\lambda(1 - e^{-\lambda a})}.$$

10E-20 You wish to cross a one-way traffic road on which cars drive at a constant speed and pass according to independent interarrival times having an exponential distribution with an expected value of $1/\lambda$ seconds. You can only cross the road when no car has come round the corner since c time seconds. What is the probability distribution of the number of passing cars before you can cross the road when you arrive at an arbitrary moment? What property of the exponential distribution do you use?

10E-20 The probability that the time between the passings of two consecutive cars is more than c seconds is given by

$$p = \int_{c}^{\infty} \lambda e^{-\lambda t}\, dt = e^{-\lambda c}.$$

By the lack of memory of the exponential distribution, the probability $p = e^{-\lambda c}$ gives also the probability that no car comes around the corner during the c seconds measured from the moment you arrive at the road. Denoting by the random variable N the number of passing cars before you can cross the road, it now follows that N has the shifted geometric distribution

$$P(X = k) = (1-p)^k p \quad \text{for } k = 0, 1, \ldots.$$

10E-21 By the lack of memory of the exponential distribution, the remaining washing time of the car being washed in the station has the same exponential density as a newly started washing time. Hence the probability that the car in the washing station will need no more than five other minutes is equal to

$$\int_0^5 \frac{1}{15} e^{-\frac{1}{15}t}\, dt = 1 - e^{-5/15} = 0.2835.$$

The probability that you have to wait more than 20 minutes before your car can be washed is equal to the probability of 0 service completions or 1 service completion in the coming 20 minutes. The latter probability is given by the Poisson probability (see Rule 10.2 in the book)

$$e^{-20/15} + \frac{20}{15} e^{-20/15} = 0.6151.$$

10E-22 The probability that the closest integer to the random observation is odd is equal to

$$\begin{aligned}
\sum_{k=0}^{\infty} P(2k + \frac{1}{2} < X < 2k + 1 + \frac{1}{2}) &= \sum_{k=0}^{\infty} \int_{2k+\frac{1}{2}}^{2k+1+\frac{1}{2}} e^{-x}\, dx \\
= \sum_{k=0}^{\infty} \left[e^{-(2k+\frac{1}{2})} - e^{-(2k+1+\frac{1}{2})} \right] &= e^{-\frac{1}{2}} \left(\frac{1 - e^{-1}}{1 - e^{-2}} \right) = \frac{e^{-\frac{1}{2}}}{1 + e^{-1}}.
\end{aligned}$$

The conditional probability that the closest integer to the random observation is odd given that it is larger than the even integer r is equal to

$$\begin{aligned}
&\sum_{k=0}^{\infty} P(2k + \frac{1}{2} < X < 2k + 1 + \frac{1}{2} \mid X > r) \\
&= \frac{1}{P(X > r)} \sum_{k=0}^{\infty} P(2k + \frac{1}{2} < X < 2k + 1 + \frac{1}{2},\, X > r) \\
&= \frac{1}{e^{-r}} \sum_{k=r/2}^{\infty} \int_{2k+\frac{1}{2}}^{2k+1+\frac{1}{2}} e^{-x}\, dx = \frac{1}{e^{-r}} \sum_{k=r/2}^{\infty} \left[e^{-(2k+\frac{1}{2})} - e^{-(2k+1+\frac{1}{2})} \right]
\end{aligned}$$

Since $\sum_{k=r/2}^{\infty} \left[e^{-(2k+\frac{1}{2})} - e^{-(2k+1+\frac{1}{2})} \right] = e^{-r} \sum_{l=0}^{\infty} \left[e^{-(2l+\frac{1}{2})} - e^{-(2l+1+\frac{1}{2})} \right]$, the conditional probability that the closest integer to the random observation is odd given that it is larger than r is equal to

$$\sum_{l=0}^{\infty}\left[e^{-(2l+\frac{1}{2})}-e^{-(2l+1+\frac{1}{2})}\right]=\frac{e^{-\frac{1}{2}}}{1+e^{-1}}.$$

The unconditional probability is the same as the unconditional probability that the closest integer to the random observation from the exponential density is odd. This result can also be explained from the memoryless property of the exponential distribution.

10E-23 Let the random variable X be the amount of time that the component functions is in the good state. The probability of having a replacement because of a system failure is given by

$$\sum_{n=0}^{\infty}P(nT<X\leq(n+1)T-a)=\sum_{n=0}^{\infty}\left(e^{-\mu nT}-e^{-\mu[(n+1)T-a]}\right)$$
$$=\left(1-e^{-\mu(T-a)}\right)\sum_{n=0}^{\infty}e^{-\mu nT}=\frac{1-e^{-\mu(T-a)}}{1-e^{-\mu T}}.$$

The time between two replacements is equal to nT with probability $P((n-1)T<X\leq nT)$ for $n=1,2,\ldots$. Hence

$$\begin{aligned}E(\text{time between two replacements}) &= \sum_{n=1}^{\infty}nT\left(e^{-\mu(n-1)T}-e^{-\mu nT}\right)\\ &= \frac{T}{1-e^{-\mu T}}.\end{aligned}$$

Remark. Denoting by p the probability that a replacement is because of a system failure and using the memoryless property of the exponential distribution, it follows that the expected time until the first system failure is given by $E(\text{time between two replacements})/p$ minus $T-E(X\mid X<T-a)$, where $E(X\mid X<T-a)$ can be evaluated as $[1-e^{-\mu(T-a)}-\mu(T-a)e^{-\mu(T-a)}]/[\mu(1-e^{-\mu(T-a)})]$.

10E-24 Your probability of winning is the probability of having exactly one signal in the time interval (s,T). By the memoryless property of the Poisson process, this probability is equal to $e^{-\lambda(T-s)}\lambda(T-s)$. Putting the derivative of this expression equal to zero, it follows that the optimal value of s is given by $T-\frac{1}{\lambda}$. The maximal win probability is $e^{-1}(\approx 0.3679)$, irrespective of the values of λ and T.

10E-25 Let p be the probability of no car passing through the village during the next half hour. Then, by the memoryless property of the Poisson process, the probability of no car passing through the village during one hour is p^2. Solving $p^2 = 1 - 0.64$ gives $p = 0.6$. Hence the desired probability is 0.4.

10E-26 Let the random variables X_1 and X_2 be the two measurement errors. The desired probability is $P(\frac{1}{2}|X_1+X_2| \leq 0.005l)$. Since the random variables X_1 and X_2 are independent, the random variable $\frac{1}{2}(X_1 + X_2)$ is normally distributed with mean 0 and standard deviation $\frac{1}{2}\sqrt{0.006^2 l^2 + 0.004^2 l^2} = \frac{l}{2{,}000}\sqrt{52}$. Letting Z be standard normally distributed, it now follows that

$$\begin{aligned} P\Big(\frac{1}{2}|X_1 + X_2| \leq 0.005l\Big) &= P\Big(\frac{1}{2}\frac{|X_1 + X_2 - 0|}{(l/2{,}000)\sqrt{52}} \leq \frac{1{,}000 \times 0.005}{\sqrt{52}}\Big) \\ &= P\left(\frac{1}{2}|Z| \leq \frac{5}{\sqrt{52}}\right) = P\left(-\frac{10}{\sqrt{52}} \leq Z \leq \frac{10}{\sqrt{52}}\right). \end{aligned}$$

Hence the desired probability is given by

$$\begin{aligned} P\left(\frac{1}{2}|X_1 + X_2| \leq 0.005l\right) &= \Phi\left(\frac{10}{\sqrt{52}}\right) - \Phi\left(-\frac{10}{\sqrt{52}}\right) \\ &= 2\Phi\left(\frac{10}{\sqrt{52}}\right) - 1 = 0.8345. \end{aligned}$$

10E-27 The desired probability is $P(|X_1 - X_2| \leq a)$. Since the random variables X_1 and X_2 are independent, the random variable $X_1 - X_2$ is normal distributed with expected value $\mu = \mu_1 - \mu_2$ and standard deviation $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$. It now follows that

$$\begin{aligned} P(|X_1 - X_2| \leq a) &= P(-a \leq X_1 - X_2 \leq a) \\ &= P\left(\frac{-a - \mu}{\sigma} \leq \frac{X_1 - X_2 - \mu}{\sigma} \leq \frac{a - \mu}{\sigma}\right). \end{aligned}$$

Hence, denoting by $\Phi(x)$ the standard normal distribution function,

$$P(|X_1 - X_2| \leq a) = \Phi\left(\frac{a - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right) - \Phi\left(\frac{-a - (\mu_1 - \mu_2)}{\sqrt{\sigma_1^2 + \sigma_2^2}}\right).$$

10E-28 The probability mass function of the number of copies of the appliance to be used when an infinite supply would be available is a Poisson distribution with expected value of $\frac{150}{2} = 75$. Suppose that Q copies of the appliance are stored in the space ship. Let the exponentially distributed random variable X_i be the lifetime (in days) of the ith copy used. Then the probability of a shortage during the space mission is $P(X_1 + \cdots + X_Q \leq 150)$. The random variables $X_1, \ldots, X_Q$ are independent and have an expected value of $\frac{1}{\lambda}$ days and a standard deviation of $\frac{1}{\lambda}$ days, where $\lambda = \frac{1}{2}$. By the central limit theorem,

$$\begin{aligned} P(X_1 + \cdots + X_Q \leq 150) &= P\left(\frac{X_1 + \cdots + X_Q - 2Q}{2\sqrt{Q}} \leq \frac{150 - 2Q}{2\sqrt{Q}}\right) \\ &\approx \Phi\left(\frac{150 - 2Q}{2\sqrt{Q}}\right). \end{aligned}$$

The 0.001 percentile of the standard normal distribution is -3.0902. Solving the equation $\frac{150-2Q}{2\sqrt{Q}} = -3.0902$ gives $Q = 106.96$ and so the normal approximation suggests to store 107 units. The exact value of the required stock follows by finding the smallest value of Q for which the Poisson probability$\sum_{k>Q} e^{-75}\frac{75^k}{k!}$ is smaller than or equal to 10^{-3}. This gives $Q = 103$.

10E-29 Let X_i be the amount (in dollars) the casino owner loses on the ith bet. Then $X_1, \ldots, X_{2{,}500}$ are independent random variables with $P(X_i = 10) = \frac{18}{37}$ and $P(X_i = -5) = \frac{19}{37}$ for $1 \leq i \leq 2{,}500$. The amount (in dollars) lost by the casino owner is $X_1 + \cdots + X_{2{,}500}$. For any i,

$$E(X_i) = \frac{85}{37} = 2.29730 \text{ and } \sigma(X_i) = \sqrt{\frac{2275}{37} - \left(\frac{85}{37}\right)^2} = 7.49726.$$

By the central limit theorem, $X_1 + \cdots + X_{2{,}500}$ is approximately $N(\mu, \sigma^2)$ distributed with $\mu = 2.297302 \times 2{,}500 = 5{,}743.250$ and $\sigma = 7.49726\sqrt{2{,}500} = 374.863$. Hence the probability that the casino owner will lose more than 6,500 dollars is approximately equal to $1 - \Phi\left(\frac{6{,}500-5{,}743.250}{374.863}\right) = 0.0217$.

10E-30 Let the random variable V_n be the bankroll (in dollars) of the gambler after the nth bet. Then $V_n = (1-\alpha)V_{n-1} + \alpha V_{n-1}R_n$ for $1 \leq n \leq 100$, where $\alpha = 0.05$, $V_0 = 1{,}000$ and $R_1, \ldots, R_{100}$ are independent random variables with $P(R_i = \frac{1}{4}) = \frac{19}{37}$ and $P(R_i = 2) = \frac{18}{37}$ for all i. Iterating this equality gives

$$V_n = (1 - \alpha + \alpha R_1) \times \cdots \times (1 - \alpha + \alpha R_n)V_0 \quad \text{for } n = 1, 2, \ldots, 100.$$

Taking logarithms, we get

$$\ln(V_n/V_0) = \ln(1 - \alpha + \alpha R_1) + \cdots + \ln(1 - \alpha + \alpha R_n) \quad \text{for all } n.$$

Let $X_i = \ln(1-\alpha+\alpha R_i)$. The random variables $X_1, \ldots, X_{100}$ are independent with

$$E(X_i) = \frac{19}{37}\ln(0.9625) + \frac{18}{37}\ln(1.05) = 0.0041086$$

and

$$\sigma(X_i) = \sqrt{\frac{19}{37}\ln^2(0.9625) + \frac{18}{37}\ln^2(1.05) - 0.0041086^2} = 0.0434898.$$

By the central limit theorem, $\ln(V_{100}/V_0)$ is approximately $N(\mu, \sigma^2)$ distributed, where

$$\mu = 0.0434898 \times 100 = 0.41086 \quad \text{and} \quad \sigma = 0.0434898 \times \sqrt{100} = 0.434898.$$

To approximate the probability that the gambler takes home more than d dollars for $d = 0$, 500, 1,000 and 2,500, we consider $P(V_n > aV_0)$ for $a = 1$, 1.5, 2 and 3.5. We have

$$\begin{aligned} P(V_n > aV_0) &= P(\ln(V_n/V_0) > \ln(a)) \\ &= P\left(\frac{\ln(V_n/V_0) - \mu}{\sigma} > \frac{\ln(a) - \mu}{\sigma}\right) \end{aligned}$$

and so

$$P(V_n > aV_0) \approx 1 - \Phi\left(\frac{\ln(a) - \mu}{\sigma}\right).$$

The probability that the gambler takes home more than d dollars has the values 0.8276, 0.5494, 0.2581, and 0.0264 for $d = 0$, 500, 1,000 and 2,500.

Remark. The random variable V_{100} being the gambler's bankroll after 100 bets is approximately lognormally distributed with an expected value of $e^{\mu+\frac{1}{2}\sigma^2} \times 1{,}000 = 1{,}000.51$ dollars and a standard deviation of $e^{\mu+\frac{1}{2}\sigma^2}\sqrt{e^{\sigma^2} - 1} \times 1{,}000 = 43.73$ dollars.

10E-31 Let the random variable X denote the original lifetime of the random variable X. To calculate the desired probability

$$P(X > s + t \mid X > s) = \frac{P(X > s + t)}{P(X > s)},$$

we need the probability distribution of X. Let A_i be the event that the battery comes from supplier i for $i = 1, 2$. Then, by the law of conditional probability,

$$\begin{aligned} P(X > x) &= P(X > x \mid A_1)P(A_1) + P(X > x \mid A_2)P(A_2) \\ &= p_1 e^{-\mu_1 x} + p_2 e^{-\mu_2 x} \quad \text{for } x \geq 0. \end{aligned}$$

This gives

$$P(X > s + t \mid X > s) = \frac{p_1 e^{-\mu_1(s+t)} + p_2 e^{-\mu_2(s+t)}}{p_1 e^{-\mu_1 s} + p_2 e^{-\mu_2 s}}.$$

10E-32 Taking the derivative of the failure rate function $r(x)$, it is a matter of simple algebra to verify that the failure rate function is increasing on $(0, x^*)$ and decreasing on (x^*, ∞), where x^* is the unique solution to the equation

$$\mu_2^2 e^{-\mu_1 x} + \mu_1^2 e^{-\mu_2 x} = (\mu_2 - \mu_1)^2.$$

The failure rate function determines uniquely the probability distribution function $F(x)$ of the lifetime of the vacuum tube. Using the basic relation $r(x) = \frac{f(x)}{1-F(x}$, it follows that

$$1 - F(x) = e^{-\mu_1 x} + e^{-\mu_2 x} - e^{-(\mu_1+\mu_2)x} \quad \text{for } x \geq 0.$$

Chapter 11

11E-1 (a) The joint probability mass function is given by $P(X = 0, Y = 0) = \frac{1}{8}$, $P(X = 0, Y = 1) = \frac{1}{8}$, $P(X = 1, Y = 0) = \frac{1}{8}$, $P(X = 1, Y = 1) = \frac{1}{4}$, $P(X = 1, Y = 2) = \frac{1}{8}$, $P(X = 2, Y = 1) = \frac{1}{8}$, and $P(X = 2, Y = 2) = \frac{1}{8}$. Using the relation $E(XY) = \sum_{x=0}^{2} \sum_{y=0}^{2})P(X = x, Y = y)$, it follows that $E(XY) = 1 \times \frac{1}{4} + 2 \times \frac{1}{8} + 2 \times \frac{1}{8} + 4 \times \frac{1}{8} = 1.25$.
(b) The joint probability mass function is given by

$$P(X = x, Y = y) = \binom{5}{x}\left(\frac{1}{2}\right)^5 \binom{x}{y}\left(\frac{1}{2}\right)^x \quad \text{for } 0 \leq x \leq 5,\ 0 \leq y \leq x.$$

Using the relation $E(X + Y) = \sum_{x=0}^{5} \sum_{y=0}^{x}(x + y)P(X = x, Y = y)$, it follows that

$$E(X + Y) = \sum_{x=0}^{5}(x + 0.5x)\binom{5}{x}\left(\frac{1}{2}\right)^5 = 2.5 + 1.25 = 3.75.$$

11E-2 The joint probability mass function is given by

$$P(X = 1, Y = k) = \binom{k-1}{3}(0.55)^4(0.45)^{k-4} \quad \text{for } k = 4, \ldots, 7.$$

and

$$P(X = 0, Y = k) = \binom{k-1}{3}(0.45)^4(0.55)^{k-4} \quad \text{for } k = 4, \ldots, 7.$$

11E-3 By the rule $P(A_1 A_2 \cdots A_n) = P(A_1)P(A_2 \mid A_1) \cdots P(A_n \mid A_1 \cdots A_{n-1})$ for conditional probabilities, it follows that the joint probability mass function of X_1 and X_2 is given by

$$\begin{aligned}
P(X_1 = i,\, X_2 = j) &= \frac{48}{52} \times \cdots \times \frac{48-(i-2)}{52-(i-2)} \times \frac{4}{52-(i-1)} \\
&\quad \times \frac{48-(i-1)}{52-i} \times \cdots \times \frac{48-(i-1)-(j-2)}{52-i-(j-2)} \\
&\quad \times \frac{3}{52-i-(j-1)}
\end{aligned}$$

for $i = 1, \ldots, 49$ and $j = i+1, \ldots, 50$. Of course, the probability distribution of X_1 is given by

$$P(X_1 = i) = \frac{48}{52} \times \cdots \times \frac{48-(i-2)}{52-(i-2)} \times \frac{4}{52-(i-1)},\ \ 1 \leq i \leq 49.$$

The marginal distribution of X_2 is computed from

$$P(X_2 = j) = \sum_{i=1}^{j-1} P(X_1 = i,\, X_2 = j),\ \ 2 \leq j \leq 50.$$

Remark. Let X_3 be the number of cards flipped over until the third ace appears and X_4 be the number of cards flipped over until the fourth ace appears. For reasons of symmetry, the probability distribution of $52 - X_4$ is the same as the probability distribution of $X_1 - 1$ and the probability distribution of $52 - X_3$ is the same as the probability distribution of $X_2 - 1$. An alternative solution to the problem of finding the marginal distributions of the X_i is provided by the solution to Problem 9E-25. It can be heuristically argued that the four aces divide the other 48 cards in five blocks of the same average length. This argument leads to $E(X_1) = 9.6 + 1 = 10.6$, $E(X_2) = 21.2$, $E(X_3) = 31.8$, and $E(X_4) = 42.4$.

11E-4 To find the joint probability mass function of X and Y, we first calculate

$$P(X = x,\, Y = y,\, N = n) = \frac{1}{6}\binom{n}{x}\left(\frac{1}{2}\right)^n \binom{n}{y}\left(\frac{1}{2}\right)^n$$

for $1 \leq n \leq 6$ and $0 \leq x, y \leq n$. It now follows that the joint probability mass function of X and Y is given by

$$P(X = x,\, Y = y) = \frac{1}{6}\sum_{n=1}^{6}\binom{n}{x}\binom{n}{y}\left(\frac{1}{2}\right)^{2n}, \quad 0 \leq x, y \leq 6,$$

with the convention $\binom{n}{k} = 0$ for $k > n$. To calculate the numerical value of $P(X = Y)$, note that

$$P(X = Y) = \frac{1}{6}\sum_{k=0}^{6}\sum_{n=1}^{6}\binom{n}{k}^2\left(\frac{1}{2}\right)^{2n} = \frac{1}{6}\sum_{n=1}^{6}\left(\frac{1}{2}\right)^{2n}\sum_{k=0}^{n}\binom{n}{k}^2.$$

The relation $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$ holds , as follows from $\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$. We can now conclude that

$$P(X = Y) = \frac{1}{6}\sum_{n=1}^{6}\binom{2n}{n}^2\left(\frac{1}{2}\right)^{2n} = 0.3221.$$

11E-5 The probability $P(X = i, Y = i)$ is equal the probability that two or more dice yield the highest score i and the other dice score less than i. Hence

$$\begin{aligned}
P(X = i, Y = i) &= \sum_{r=2}^{d}\binom{d}{r}\left(\frac{1}{6}\right)^r\left(\frac{i-1}{6}\right)^{d-r} \\
&= \sum_{r=0}^{d}\binom{d}{r}\left(\frac{1}{6}\right)^r\left(\frac{i-1}{6}\right)^{d-r} - \left(\frac{i-1}{6}\right)^d - d\left(\frac{i-1}{6}\right)^{d-1} \\
&= \frac{i^d - (i-1)^d - d(i-1)^{d-1}}{6^d}.
\end{aligned}$$

The probability $P(X = i, Y = j)$ for $i > j$ is equal the probability that exactly one die yields the highest score i, at least one die yields the second-highest score j and the other dice score less than j. Hence, for $i > j$,

$$\begin{aligned}
P(X = i, Y = j) &= \binom{d}{1}\frac{1}{6}\sum_{r=1}^{d-1}\binom{d-1}{r}\left(\frac{1}{6}\right)^r\left(\frac{j-1}{6}\right)^{d-1-r} \\
&= \binom{d}{1}\frac{1}{6}\left[\sum_{r=0}^{d-1}\binom{d-1}{r}\left(\frac{1}{6}\right)^r\left(\frac{j-1}{6}\right)^{d-1-r} - \left(\frac{j-1}{6}\right)^{d-1}\right] \\
&= \frac{d[j^{d-1} - (j-1)^{d-1}]}{6^d}.
\end{aligned}$$

11E-6 (a) The marginal distribution of X is given by

$$P(X = x) = \sum_{y=x}^{\infty}\frac{e^{-2}}{x!(y-x)!} = \frac{e^{-2}}{x!}\sum_{k=0}^{\infty}\frac{1}{k!} = \frac{e^{-1}}{x!} \quad \text{for } x = 0, 1, \ldots$$

and the marginal distribution of Y is given by

$$P(Y = y) = \sum_{x=0}^{y}\frac{e^{-2}}{x!(y-x)!} = \frac{e^{-2}}{y!}\sum_{x=0}^{y}\frac{y!}{x!(y-x)!} = \frac{e^{-2}2^y}{y!} \quad \text{for } y = 0, 1, \ldots.$$

(b) Let $V = Y - X$. The joint probability mass function of X and $Y - X$ is

$$P(X = x,\, Y - X = z) = P(X = x, Y = z + x) = \frac{e^{-2}}{x!z!} \quad \text{for } x, z = 0, 1, \ldots.$$

Since $P(Y - X = z) = \sum_{x=0}^{\infty} \frac{e^{-2}}{x!z!} = \frac{e^{-1}}{z!}$ for $z = 0, 1, \ldots$, it follows that

$$P(X = x,\, Y - X = z) = P(X = x)P(Y - X = z) \quad \text{for all } x, z$$

and so X and $Y - X$ are independent.
(c) $E(XY)$ can be computed from $E(XY) = \sum_{x=0}^{\infty}\sum_{y=x}^{\infty} xy \frac{e^{-2}}{x!(y-x)!}$, but it is easier to compute $E(XY)$ from $E(XY) = EX(Y - X + X)] = E(X)E(Y - X) + E(X^2)$, using the independence of X and Y. Since X and $Y - X$ are both Poisson distributed with expected value 1, it follows that $E(XY) = 1 + 2 = 3$. The expected value and the variance of the Poisson distributed variable X are given by $E(X) = 1$ and $\sigma^2(X) = 1$. The expected value and the variance of the random variable Y are easiest computes as $E(Y) = E(Y - X + X) = E(Y - X) + E(X) = 1 + 1 = 2$ and $\sigma^2(Y) = \sigma^2(Y - X) + \sigma^2(X) = 1 + 1 = 2$. This gives

$$\rho(X,\, Y) = \frac{3 - 1 \times 2}{1 \times \sqrt{2}} = \frac{1}{\sqrt{2}}.$$

11E-7 The joint probability mass function of X and Y is given by

$$P(X = x,\, Y = y) = \frac{\binom{6}{x}\binom{6-x}{y}4^{6-x-y}}{6^6} \quad \text{for } x, y \geq 0, x + y \leq 6.$$

The easiest way to compute $\rho(X,\, Y)$ is to use indicator variables. Let $I_j = 1$ if the jth roll gives the outcome 1 and $I_j = 0$ otherwise. Also, let $L_j = 1$ if the jth roll gives the outcome 6 and $L_j = 0$ otherwise. Then $X = \sum_{j=1}^{6} I_j$ and $Y = \sum_{j=1}^{6} L_j$. For each j, $E(I_j) = E(L_j) = \frac{1}{6}$ and $\sigma^2(I_j) = \sigma^2(L_j) = \frac{5}{36}$. This gives $E(X) = E(Y) = 1$ and $\sigma^2(X) = \sigma^2(Y) = \frac{5}{6}$, where the latter expression uses the fact that both $I_1, \ldots, I_6$ and $L_1, \ldots L_6$ are independent random variables. To find $E(XY)$, note that $I_j L_j = 0$ for all j and I_j is independent of L_k for $j \neq k$. This gives

$$E(XY) = E\left(\sum_{j=1}^{6} I_j \sum_{k=1}^{6} L_k\right) = \sum_{j=1}^{6}\sum_{k \neq j} E(I_j)E(L_k) = \frac{30}{36} = \frac{5}{6}.$$

We can now conclude that

$$\rho(X,\, Y) = \frac{5/6 - 1 \times 1}{\sqrt{5/6} \times \sqrt{5/6}} = -0.2.$$

11E-8 The constant c follows from the requirement $c\int_0^1 dx \int_0^x xy\,dy = 1$. This gives $c\int_0^1 \frac{1}{2}x^3\,dx = 1$ and so $c = 8$. The marginal density of X is given by

$$f_X(x) = \int_0^x 8xy\,dy = 4x^3 \quad \text{for } 0 < x < 1$$

and $f_X(x) = 0$ otherwise. The marginal density of Y is

$$f_Y(y) = \int_y^1 8xy\,dx = 4y(1-y^2) \quad \text{for } 0 < y < 1$$

and $f_Y(y) = 0$ otherwise.

11E-9 The radius of the circle is $\sqrt{X^2+Y^2}$ and so its circumference is given by $V = 2\pi\sqrt{X^2+Y^2}$. To find the desired probability, use the relation $P((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$. This gives

$$\begin{aligned} P(V \le 2\pi) &= P(X^2+Y^2 \le 1) = \int_0^1 dx \int_0^{\sqrt{1-x^2}} (x+y)dy \\ &= \int_0^1 \left(x\sqrt{1-x^2} + \frac{1}{2}(1-x^2) \right) dx = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

11E-10 Let X and Y be the two random points at which the stick is broken with X being the point that is closest to the left end point of the stick. Assume that the stick has length 1. Al three pieces are no longer than half the length of the stick only if $X \le 0.5$, $Y - X \le 0.5$ and $1 - Y \le 0.5$. That is (X,Y) should satisfy $0 \le X \le 0.5$ and $0.5 \le Y \le 0.5 + X$. The joint density function $f(x,y)$ of (X,Y) is given by $f(x,y) = 2$ for $0 < x < y < 1$ and 0 otherwise. To see this, note that $X = \min(U_1, U_2)$ and $Y = \max(U_1, U_2)$, where U_1 and U_2 are independent and uniformly distributed on $(0,1)$. For any $0 < x < y < 1$ and $dx > 0$, $dy > 0$ sufficiently small, $P(x \le X \le x + dx,\ y \le Y \le y + dy)$ is equal to the sum of $P(x \le U_1 \le x + dx,\ y \le U_2 \le y + dy)$ and $P(x \le U_2 \le x + dx,\ y \le U_1 \le y + dy)$. By the independence of U_1 and U_2, this gives $P(x \le X \le x + dx,\ y \le Y \le y + dy) = 2dxdy$, showing that $f(x,y) = 2$ for $0 < x < y < 1$. It now follows that

$$\begin{aligned} &P(\text{no piece is longer than half the length of the stick}) \\ &= \int_0^{0.5} dx \int_{0.5}^{0.5+x} 2dy = 2\int_0^{0.5} x\,dx = \frac{1}{4}. \end{aligned}$$

11E-11 Using the basic formula $P((X,Y) \in C) = \iint_C f(x,y)\,dx\,dy$, it follows that

$$
\begin{aligned}
P(X < Y) &= \frac{1}{10}\int_5^{10} dx \int_x^{\infty} e^{-\frac{1}{2}(y+3-x)}\, dy \\
&= \frac{2}{10}\int_5^{10} e^{-\frac{1}{2}(3-x)} e^{-\frac{1}{2}x} = e^{-\frac{3}{2}} = 0.2231.
\end{aligned}
$$

11E-12 Let $Z = XY$. Using the basic formula $P((X,Y) \in C) = \iint_C f(x,y)\, dx\, dy$, it follows that

$$
\begin{aligned}
P(Z \leq z) &= \int_0^z dx \int_0^{z/x} x e^{-x(y+1)}\, dy = \int_0^z e^{-x}(1 - e^{-x \times z/x})\, dx = \\
&= (1 - e^{-z})(1 - e^{-z}) \quad \text{for } z \geq 0.
\end{aligned}
$$

Hence the density function of $Z = XY$ is given by $f(z) = 2e^{-z}(1 - e^{-z})$ for $z > 0$ and $f(z) = 0$ otherwise.

11E-13 Let $Z = X+Y$. Using the basic formula $P((X,Y) \in C) = \iint_C f(x,y)\, dx\, dy$, it follows that

$$
\begin{aligned}
P(Z \leq z) &= \frac{1}{2}\int_0^z dx \int_0^{z-x} (x+y)e^{-(x+y)}\, dy = \frac{1}{2}\int_0^z dx \int_x^z u e^{-u}\, du \\
&= \frac{1}{2}\int_0^z (-ze^{-z} + xe^{-x} + e^{-x} - e^{-z})\, dx = 1 - e^{-z}(1 + z + \frac{1}{2}z^2)
\end{aligned}
$$

for $z \geq 0$. Hence the density function of $Z = X + Y$ is $f(z) = \frac{1}{2}z^2 e^{-z}$ for $z > 0$ and $f(z) = 0$ otherwise. This is the Erlang density with shape parameter 3 and scale parameter 1.

11E-14 (a) Let T be the time until neither of two components is still working. Then $T = \max(X, Y)$. To evaluate $P(T \leq t) = P(X \leq t, Y \leq t)$, we distinguish between $0 \leq t \leq 1$ and $1 \leq t \leq 2$. For $0 \leq t \leq 1$,

$$
\begin{aligned}
P(X \leq t, Y \leq t) &= \frac{1}{4}\int_0^t dx \int_0^t (2y + 2 - x)\, dy \\
&= \frac{1}{4}\int_0^t (t^2 + 2t - xt)\, dx = 0.25t^3 + 0.375t^2.
\end{aligned}
$$

For $1 \leq t \leq 2$,

$$
\begin{aligned}
P(X \leq t, Y \leq t) &= \frac{1}{4}\int_0^t dx \int_0^1 (2y + 2 - x)\, dy \\
&= \frac{1}{4}\int_0^t (3 - x)\, dx = 0.75t - 0.125t^2.
\end{aligned}
$$

The density function of T is given by $0.75(t^2+t)$ for $0 < t < 1$ and $0.75-0.25t$ for $1 \leq t < 2$.

(b) The amount of time that the lifetime X survives the lifetime Y is given by $V = \max(X,Y) - Y$. The random variable V is a mixed random variable with a probability mass at $v = 0$ and a density on $(0,2)$. The probability $P(V=0)$ is given by $P(Y \geq X)$ and so

$$\begin{aligned} P(V=0) &= \frac{1}{4}\int_0^1 dx \int_x^1 (2y+2-x)\,dy \\ &= \frac{1}{4}\int_0^1 (3-3x)\,dx = \frac{3}{8}. \end{aligned}$$

To find the density of V on the interval $(0,2)$, we distinguish between the cases $0 < v < 1$ and $1 \leq v < 2$. It is easiest to evaluate $P(V > v)$. For $0 < v < 1$,

$$\begin{aligned} P(V>v) &= \frac{1}{4}\int_0^1 dy \int_{y+v}^1 (2y+2-x)\,dx \\ &= \frac{1}{4}\int_0^1 (2y-1.5y^2-yv+2-2v+0.5v^2)\,dy = \frac{5}{8}-\frac{5}{8}v+\frac{1}{8}v^2. \end{aligned}$$

For $1 \leq v < 2$,

$$\begin{aligned} P(V>v) &= \frac{1}{4}\int_0^{2-v} dy \int_{y+v}^1 (2y+2-x)\,dx \\ &= \frac{1}{4}\int_0^{2-v} (2y-1.5y^2-yv+2-2v+0.5v^2)\,dy \\ &= \frac{1}{4}\big(-0.5v^3+3v^2-6v+4\big). \end{aligned}$$

Hence the density function of V on $(0,2)$ is given by $\frac{5}{8}-\frac{2}{8}v$ for $0 < v < 1$ and by $\frac{3}{8}v^2 - 1.5v + 1.5$ for $1 \leq v < 2$.

11E-15 Since $f(x,y)$ can be written of $f_X(x)f_Y(y)$ with $f_X(x) = e^{-x}$ and $f_Y(y) = e^{-y}$, the lifetimes X and Y are independent and have the same exponential density. Using the memoryless property of the exponential distribution, it follows that the probability of a system failure between two inspections is

$$\begin{aligned} P(X \leq T, Y \leq T) &= \int_0^T \int_0^T e^{-(x+y)}\,dxdy \\ &= \int_0^T e^{-x}\,dx \int_0^T e^{-y}\,dy = \big(1-e^{-T}\big)^2. \end{aligned}$$

Let the random variable D denote the amount of time the system is down between two inspections. Then $D = T - \max(X, Y)$ if $X, Y \leq T$ and $D = 0$ otherwise. This gives

$$\begin{aligned} E(D) &= \int_0^T \int_0^T \big(T - \max(x, y)\big) e^{-(x+y)}\, dx dy \\ &= 2\int_0^T e^{-x}\, dx \int_0^x (T - x)e^{-y}\, dy. \end{aligned}$$

This leads after some algebra to

$$E(D) = \int_0^T (T - x)e^{-x}(1 - e^{-x})\, dx = T - 1.5 - e^{-T} + 0.5e^{-2T}.$$

11E-16 The expected value of the time until the electronic device goes down is given by

$$\begin{aligned} E(X + Y) &= \int_1^\infty \int_1^\infty (x + y)\frac{24}{(x + y)^4} dx dy \\ &= \int_1^\infty dx \int_1^\infty \frac{24}{(x + y)^3} dy = \int_1^\infty \frac{12}{(x + 1)^2} dx = 6. \end{aligned}$$

To find the density function of $X + Y$, we calculate $P(X + Y > t)$ and distinguish between $0 \leq t \leq 2$ and $t > 2$. Obviously, $P(X + Y > t) = 1$ for $0 \leq t \leq 2$. For the case of $t > 2$,

$$\begin{aligned} P(X + Y > t) &= \int_1^{t-1} dx \int_{t-x}^\infty \frac{24}{(x + y)^4} dy + \int_{t-1}^\infty dx \int_1^\infty \frac{24}{(x + y)^4} dy \\ &= \int_1^{t-1} \frac{8}{t^3} dx + \int_{t-1}^\infty \frac{8}{(x + 1)^3} dx = \frac{8(t - 2)}{t^3} + \frac{4}{t^2}. \end{aligned}$$

By differentiation, the density function $g(t)$ of $X + Y$ is $g(t) = \frac{24(t-2)}{t^4}$ for $t > 2$ and $g(t) = 0$ otherwise.

11E-17 (a) Using the basic formula $P((X, Y) \in C) = \iint_C f(x, y)\, dx\, dy$, it follows that

$$P(B^2 \geq 4A) = \int_0^1 \int_0^1 \chi(a, b) f(a, b)\, da\, db,$$

where the function $\chi(a, b) = 1$ for $b^2 \geq 4a$ and $\chi(a, b) = 0$ otherwise. This leads to the desired probability

$$P(B^2 \geq 4A) = \int_0^1 db \int_0^{b^2/4} (a + b)\, da = \int_0^1 \Big(\frac{b^4}{32} + \frac{b^3}{4}\Big)\, db = 0.06875.$$

(b) Letting the function $\chi(a,b,c) = 1$ for $b^2 \geq 4ac$ and $\chi(a,b,c) = 0$ otherwise, it follows that

$$P(B^2 \geq 4AC) = \int_0^1 \int_0^1 \int_0^1 \chi(a,b,c) f(a,b,c)\, da\, db\, dc.$$

Any order of integration can be used to evaluate this three-dimensional integral. This leads to

$$\begin{aligned} P(B^2 \geq 4AC) &= \int_0^1 db \int_0^{b^2/4} da \int_0^1 \chi(a,b,c) f(a,b,c)\, dc \\ &\quad + \int_0^1 db \int_{b^2/4}^1 da \int_0^1 \chi(a,b,c) f(a,b,c)\, dc. \end{aligned}$$

Next it is readily seen that

$$\begin{aligned} P(B^2 \geq 4AC) &= \frac{2}{3} \int_0^1 db \int_0^{b^2/4} da \int_0^1 (a+b+c)\, dc \\ &\quad + \frac{2}{3} \int_0^1 db \int_{b^2/4}^1 da \int_0^{b^2/(4a)} (a+b+c)\, dc. \end{aligned}$$

We find

$$\int_0^1 db \int_0^{b^2/4} da \int_0^1 (a+b+c)\, dc = \int_0^1 \left(\frac{b^4}{32} + \frac{b^3}{4} + \frac{b^2}{8} \right) db = 0.110417$$

and

$$\begin{aligned} &\int_0^1 db \int_{b^2/4}^1 da \int_0^{b^2/(4a)} (a+b+c)\, dc \\ &= \int_0^1 \left[\left(1 - \frac{b^2}{4}\right)\left(\frac{b^2}{4} - \frac{b^4}{32}\right) - \frac{b^3}{4} \ln\left(\frac{b^2}{4}\right) \right] db. \end{aligned}$$

Using partial integration the integral $\int_0^1 \frac{b^3}{4} \ln\left(\frac{b^2}{4}\right) db$ can be evaluated as

$$\frac{1}{16} \int_0^1 \ln\left(\frac{b^2}{4}\right) db^4 = \frac{1}{16} \left(b^4 \ln\left(\frac{b^2}{4}\right)\Big|_0^1 - \int_0^1 b^4 \frac{4}{b^2} \frac{b}{2}\, db \right) = \frac{1}{16} \ln\left(\frac{1}{4}\right) - \frac{1}{24}.$$

This gives

$$\int_0^1 db \int_{b^2/4}^1 da \int_0^{b^2/(4a)} (a+b+c)\, dc = 0.183593.$$

Hence the desired probability that $Ax^2 + Bx + C = 0$ has two real roots is given by $P(B^2 \geq 4AC) = \frac{2}{3}(0.110417 + 0.183593) = 0.19601$.

11E-18 By the independence of the chosen random numbers, the joint density function of X_1, X_2 and X_3 is $1 \times 1 \times 1 = 1$ for $0 < x_1, x_2, x_3 < 1$ and 0 otherwise. Let $C = \{(x_1, x_2, x_3) : 0 < x_1, x_2, x_3 < 1,\, 0 < x_2 + x + 3 < x_1\}$. Then

$$P(X_1 > X_2 + X_3) = \int\int\int_C dx_1\, dx_2\, dx_3 = \int_0^1 dx_1 \int_0^{x_1} dx_2 \int_0^{x_1 - x_2} dx_3.$$

This gives

$$\begin{aligned} P(X_1 > X_2 + X_3) &= \int_0^1 dx_1 \int_0^{x_1} (x_1 - x_2)\, dx_2 = \int_0^1 \frac{1}{2} x_1^2\, dx_1 \\ &= \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}. \end{aligned}$$

Since the events $\{X_1 > X_2 + X_3\}$, $\{X_2 > X_1 + X_3\}$ and $\{X_3 > X_1 + X_2\}$ are mutually exclusive, the probability that the largest of the three random numbers is greater than the sum of the other two is $3 \times \frac{1}{6} = \frac{1}{2}$.
Remark. More generally, let $X_1, X_2, \ldots, X_n$ be independent random numbers chosen from $(0, 1)$, then $P(X_1 > X_2 + \cdots + X_n) = \frac{1}{n!}$ for any $n \geq 2$.

11E-19 The joint density of X_1 and X_2 is given by $f(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = 1$ for $0 < x_1, x_2 < 1$ and $f(x_1, x_2) = 0$ otherwise. Hence

$$P(X_1 + X_2 \leq 1) = \int_0^1 dx_1 \int_0^{1 - x_1} dx_2 = \frac{1}{2}.$$

In the same way,

$$P(X_1 + X_2 + X_3 \leq 1) = \int_0^1 dx_1 \int_0^{1 - x_1} dx_2 \int_0^{1 - x_1 - x_2} dx_3 = \frac{1}{6}.$$

Continuing in this way,

$$P(X_1 + X_2 + \cdots + X_n \leq 1) = \frac{1}{n!}.$$

11E-20 To find the joint density of V and W, we apply the transformation formula. The inverse functions $x = a(v, w)$ and $y = b(v, w)$ are given $a(v, w) = vw/(1 + w)$ and $b(v, w) = v/(1 + w)$. The Jacobian $J(v, w)$ is equal to $-v/(1 + w)^2$ and so the joint density of V and W is given by

$$f_{V,W}(v, w) = 1 \times 1 \times |J(v, w)| = \frac{v}{(1 + w)^2} \quad \text{for } 0 < v < 2 \text{ and } w > 0$$

and $f_{V,W}(v,w) = 0$ otherwise. The marginal density of V is

$$f_V(v) = \int_0^{\infty} \frac{v}{(1+w)^2}\, dw = \frac{1}{2}v \quad \text{for } 0 < v < 2$$

and $f_V(v) = 0$ otherwise. The marginal density of W is given by

$$f_W(w) = \int_0^{2} \frac{v}{(1+w)^2}\, dv = \frac{2}{(1+w)^2} \quad \text{for } w > 0$$

and $f_W(w) = 0$ otherwise. Since $f_{V,W}(v,w) = f_V(v)f_W(w)$ for all v, w, the random variables V and W are independent.

11E-21 It is not possible to obtain the joint density of V and W by applying directly the two-dimensional transformation rule. To apply this rule, we write V and W as $V = X + Y$ and $W = X - Y$, where $X = Z_1^2$ and $Y = Z_2^2$. The probability densities of X and Y are easily found. Note that

$$P(X \leq x) = P(-\sqrt{x} \leq Z_1 \leq \sqrt{x}) = \Phi(\sqrt{x}) - \Phi(\sqrt{x}) \quad \text{for } x \geq 0.$$

Denoting by $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ the standard normal density. it follows that the density $f_X(x)$ of X is

$$f_X(x) = \phi(\sqrt{x})\,\frac{1}{2}x^{-\frac{1}{2}} + \phi(-\sqrt{x})\,\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{\sqrt{2\pi}}x^{-\frac{1}{2}}e^{-\frac{1}{2}x} \quad \text{for } x > 0.$$

This is a gamma density with shape parameter $\frac{1}{2}$ and scale parameter $\frac{1}{2}$. The density function $f_Y(y)$ of Y is the same gamma density. Since Z_1 and Z_2 are independent, X and Y are also independent. Hence the joint density function $f_{X,Y}(x,y)$ of X and Y is given by

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi}(xy)^{-\frac{1}{2}}\, e^{-\frac{1}{2}(x+y)} \quad \text{for } x,\, y > 0.$$

The inverse functions $x = a(v,w)$ and $y = b(v,w)$ are $a(v,w) = \frac{1}{2}(v+w)$ and $b(v,w) = \frac{1}{2}(v-w)$. The Jacobian $J(v,w)$ is equal to $-\frac{1}{2}$. Next it follows that the joint density of V and W is given by

$$f_{V,W}(v,w) = \frac{1}{4\pi^2}(v^2 - w^2)^{-\frac{1}{2}}\, e^{-\frac{1}{2}v} \quad \text{for } v > 0 \text{ and } -\infty < w < \infty.$$

Since $f_{V,W}(v,w)$ is not the product of a function of only v and a function of only w, the random variables V and W are not independent.

11E-22 The inverse functions $x = a(v, w)$ and $y = b(v, w)$ are given $a(v, w) = vw$ and $b(v, w) = v(1 - w)$. The Jacobian $J(v, w)$ is equal to $-v$ and so the joint density of V and W is given by

$$f_{V,W}(v, w) = \mu e^{-\mu v w} \mu e^{-\mu v(1-w)} |-v| = \mu^2 v e^{-\mu v} \quad \text{for } v > 0 \text{ and } 0 < w < 1$$

and $f_{V,W}(v, w) = 0$ otherwise. The marginal density of V is

$$f_V(v) = \int_0^1 \mu^2 v e^{-\mu v}\, dw = \mu^2 v e^{-\mu v} \quad \text{for } v > 0$$

and $f_V(v) = 0$ otherwise. The marginal density of W is given by

$$f_W(w) = \int_0^\infty \mu^2 v e^{-\mu v}\, dv = 1 \quad \text{for } 0 < w < 1$$

and $f_W(w) = 0$ otherwise. In other words, V has the Erlang-2 density with scale parameter μ and W is uniformly distributed on $(0, 1)$. Since $f_{V,W}(v, w) = f_V(v) f_W(w)$ for all v, w, the random variables V and W are independent.

11E-23 (a) To find c, use the fact that $\frac{1}{a\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}y^2/a^2} dy = 1$ for any $a > 0$. This gives

$$\begin{aligned} 1 &= c \int_0^\infty dx \int_0^\infty x e^{-\frac{1}{2}x(1+y^2)}\, dy = \int_0^\infty c x e^{-\frac{1}{2}x}\, dx \int_0^\infty e^{-\frac{1}{2}y^2/(1/\sqrt{x})^2}\, dy \\ &= \int_0^\infty c x e^{-\frac{1}{2}x} (1/\sqrt{x}) \sqrt{2\pi}\, \frac{1}{2}\, dx = c \frac{1}{2} \sqrt{2\pi}\, \Gamma(1.5)/0.5^{1.5} = \pi. \end{aligned}$$

This gives

$$c = \frac{1}{\pi}.$$

The marginal densities of X and Y are given by

$$f_X(x) = \frac{1}{\pi} \int_0^\infty x e^{-\frac{1}{2}x(1+y^2)} dy = \frac{1}{\sqrt{2\pi}}\, x^{\frac{1}{2}} e^{-\frac{1}{2}x} \quad \text{for } x > 0$$

and

$$f_Y(y) = \frac{1}{\pi} \int_0^\infty x e^{-\frac{1}{2}x(1+y^2)} dx = \frac{4/\pi}{(1+y^2)^2} \quad \text{for } y > 0.$$

(b) Let $V = Y\sqrt{X}$ and $W = X$. To find the joint density of V and W, we apply the transformation formula. The inverse functions $x = a(v, w)$ and

$y = b(v, w)$ are given $a(v, w) = w$ and $b(v, w) = v/\sqrt{w}$. The Jacobian $J(v, w)$ is equal to $-1/\sqrt{w}$ and so the joint density of V and W is given by

$$f_{V,W}(v, w) = \frac{1}{\pi} w e^{-w(1+v^2/w)} \frac{1}{\sqrt{w}} = \frac{1}{\pi}\sqrt{w} e^{-\frac{1}{2}w} e^{-\frac{1}{2}v^2} \quad \text{for } v, w > 0$$

and $f_{V,W}(v, w) = 0$ otherwise. The densities $f_V(v) = \int_0^\infty f_{V,W}(v, w)dw$ and $f_W(w) = \int_0^\infty f_{V,W}(v, w)dv$ are given by

$$f_V(v) = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}v^2} \text{ for } v > 0, \quad f_W(w) = \frac{1}{\sqrt{2\pi}} w^{\frac{1}{2}} e^{-\frac{1}{2}w} \text{ for } w > 0.$$

The random variable V is distributed as $|Z|$ with Z having the standard normal distribution and W has a gamma distribution with shape parameter $\frac{3}{2}$ and shape parameter $\frac{1}{2}$. Since $f_{V,W}(v, w) = f_V(v) f_W(w)$ for all v, w, the random variables V and W are independent.

11E-24 The marginal density of X is given by

$$f_X(x) = 6\int_0^x (x - y)\, dy = 3x^2 \quad \text{for } 0 < x < 1$$

and $f_X(x) = 0$ otherwise. The marginal density of Y is given by

$$f_Y(y) = 6\int_y^1 (x - y)\, dx = 3y^2 - 6y + 3 \quad \text{for } 0 < y < 1$$

and $f_Y(y) = 0$ otherwise. Next we calculate

$$E(X) = \frac{3}{4}, \quad \sigma(X) = \frac{1}{4}\sqrt{3}, \quad E(Y) = \frac{1}{4}, \quad \sigma(Y) = \sqrt{3/80}$$

and

$$E(XY) = 6\int_0^1 dx \int_0^x xy(x - y)\, dy = \int_0^1 x^4\, dx = \frac{1}{5}.$$

This leads to

$$\rho(X, Y) = \frac{1/5 - (3/4) \times (1/4)}{(1/4)\sqrt{3} \times \sqrt{3/80}} = 0.1491.$$

11E-25 The linear least square estimate of D_1 given that $D_1 - D_2 = d$ is

$$E(D_1) + \rho(D_1 - D_2, D_1)\frac{\sigma(D_1)}{\sigma(D_1 - D_2)}[d - E(D_1 - D_2)]$$

Using the independence of D_1 and D_2, we have

$$E(D_1 - D_2) = \mu_1 - \mu_2, \quad \sigma(D_1 - D_2) = \sqrt{\sigma_1^2 + \sigma_2^2}$$

and

$$\operatorname{cov}(D_1 - D_2,\, D_1) = \operatorname{cov}(D_1,\, D_1) - \operatorname{cov}(D_1,\, D_2) = \sigma_1^2 - 0.$$

Hence the linear least square estimate of D_1 is

$$\mu_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}(d - \mu_1 + \mu_2).$$

Chapter 12

12E-1 Any linear combination of the random variables X and Y is of the form $aZ_1 + bZ_2$ for constants a and b. Since the normally distributed random variables Z_1 and Z_2 are independent, the random variable $aZ_1 + bZ_2$ has a normal distribution for any constants a and b. This is a basic result for the normal distribution (see Section 10.4.7 in the book). Hence any linear combination of the random variables X and Y is normally distributed. It now follows from Rule 12.3 in the book that the random vector $(X,\, Y)$ has a bivariate normal distribution. The parameters $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ of this distribution are easily found. We have $\mu_1 = E(X) = 0$, $\mu_2 = E(Y) = 0$, $\sigma_1^2 = \sigma^2(X) = 10$, $\sigma_2^2 = \sigma^2(Y) = 2$. Since $\operatorname{cov}(X, Y) = E(XY) = E(Z_1^2 + 4Z_1Z_2 + 4Z_2^2) = 1 + 4 = 5$, it follows that $\rho = 5/\sqrt{20} = \frac{1}{2}\sqrt{5}$.

12E-2 Since V and W are linear combinations of X and Y, the random vector (V, W) has a bivariate normal distribution. The random variables V and W are independent if and only if $\operatorname{cov}(V, W) = 0$. It holds that

$$\begin{aligned}\operatorname{cov}(V, W) &= \operatorname{cov}(aX, X) + \operatorname{cov}(aX, aY) + \operatorname{cov}(Y, X) + \operatorname{cov}(Y, aY)\\ &= a - 0.5a^2 - 0.5 + a = -0.5a^2 + 2a - 0.5.\end{aligned}$$

The solutions of the equation $-0.5a^2 + 2a - 0.5 = 0$ are given by $a = 2 + \sqrt{3}$ and $a = 2 - \sqrt{3}$. The random variables V and W are independent for these two values of a.

12E-3 Define the random variables V and W by $V = X$ and $W = \rho X + \sqrt{1 - \rho^2} Z$. Any linear combination of V and W is a linear combination of X and Z and is normally distributed, using the fact that X and Z are independent and normally distributed. Hence the random vector (V, W) has a bivariate normal distribution. The parameters of this bivariate normal distribution are given by $E(V) = 0$, $E(W) = 0$, $\sigma^2(V) = 1$, $\sigma^2(W) = \rho^2 + 1 - \rho^2 = 1$, and $\rho(V, W) = E(VW) = E\big(\rho X^2 + \sqrt{1 - \rho^2} ZX\big) = \rho$.

Hence the random vector (V, W) has the same standard bivariate normal distribution as (X, Y).

12E-4 Since $P(X>0)=\frac{1}{2}$, we have $P(Y>X \mid X>0)=2P(Y>X,\, X>0)$. Next we use the result that $\left(X, \rho X+\sqrt{1-\rho^2}Z\right)$ has the same bivariate normal distribution as (X, Y), where Z is an $N(0,1)$ distributed random variable that is independent of X, see Problem 12E-3. Hence

$$P(Y>X \mid X>0)=2P\left(Z>\frac{1-\rho}{\sqrt{1-\rho^2}}X, X>0\right).$$

This gives

$$P(Y>X \mid X>0)=\frac{2}{2\pi}\int_0^\infty e^{-\frac{1}{2}x^2}\,dx\int_{(1-\rho)x/\sqrt{1-\rho^2}}^\infty e^{-\frac{1}{2}z^2}\,dz.$$

Using polar coordinates, we find

$$\begin{aligned}\int_0^\infty e^{-\frac{1}{2}x^2}\,dx\int_{ax}^\infty e^{-\frac{1}{2}z^2}\,dz &= \int_{\mathrm{arctg}(a)}^{\frac{1}{2}\pi}\int_0^\infty e^{-\frac{1}{2}r^2}r\,dr\,d\phi\\ &= \frac{\pi}{2}-\mathrm{arctg}(a)\end{aligned}$$

for any constant $a \geq 0$. This gives

$$P(Y>X \mid X>0)=\frac{1}{2}-\frac{1}{\pi}\mathrm{arctg}\left(\frac{1-\rho}{\sqrt{1-\rho^2}}\right).$$

(b) Write $P(Y/X \leq 1)$ as

$$P\left(\frac{Y}{X}\leq 1\right)=P(Y\leq X, X>0)+P(Y\geq X, X<0).$$

Using the result of Problem 12E-3, it next follows that

$$\begin{aligned}P\left(\frac{Y}{X}\leq 1\right) &= P\left(Z\leq\frac{1-\rho}{\sqrt{1-\rho^2}}X,\, X>0\right)\\ &\quad +P\left(Z\geq\frac{1-\rho}{\sqrt{1-\rho^2}}X,\, X<0\right).\end{aligned}$$

Hence

$$\begin{aligned}P\left(\frac{Y}{X}\leq 1\right) &= \frac{1}{2\pi}\int_0^\infty e^{-\frac{1}{2}x^2}\,dx\int_{-\infty}^{(1-\rho)x/\sqrt{1-\rho^2}} e^{-\frac{1}{2}z^2}\,dz\\ &\quad +\frac{1}{2\pi}\int_{-\infty}^0 e^{-\frac{1}{2}x^2}\,dx\int_{(1-\rho)x/\sqrt{1-\rho^2}}^\infty e^{-\frac{1}{2}z^2}\,dz\end{aligned}$$

counts	observed	expected
0	57	54.3768
1	203	210.4604
2	383	407.2829
3	525	525.4491
4	532	508.4244
5	408	393.5610
6	273	253.8730
7	139	140.3699
8	45	67.9110
9	27	29.2046
10	10	11.3034
≥ 11	6	5.7831

Using the method of polar coordinates to evaluate the integrals, we next find

$$P\left(\frac{Y}{X} \leq 1\right) = \frac{1}{2} + \frac{1}{\pi}\text{arctg}\left(\frac{1-\rho}{\sqrt{1-\rho^2}}\right).$$

12E-5 By the definition of the multivariate distribution, the random variables $a_1X_1 + \cdots + a_nX_n$ and $b_1Y_1 + \cdots + b_mY_m$ are normally distributed for any constants $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$. Moreover, these two random variables are independent (any two functions f and g result into independent random variables $f(\mathbf{X})$ and $g(\mathbf{Y})$ if $\mathbf{X}$ and $\mathbf{Y}$ are independent random vectors). The sum of two independent normally distributed random variables is again normally distributed. Hence $a_1X_1 + \cdots + a_nX_n + b_1Y_1 + \cdots + b_mY_m$ is normally distributed for any constants $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$, showing that $(\mathbf{X}, \mathbf{Y})$ has a multivariate normal distribution.

12E-6 The value of the chi-square statistic is

$$\frac{(60.179 - 61,419.5)^2}{61,419.5} + \cdots + \frac{(61,334 - 61,419.5)^2}{61,419.5} = 642.46$$

The chi-square distribution with 12-1=11 degrees of freedom is used in the test. The probability χ^2_{11}- distributed random variable takes on a value larger than 642.46 is practically zero. This leaves no room at all for doubt about the fact that birth dates are not uniformly distributed over the year.

12E-7 The parameter of the hypothesized Poisson distribution describing the number of counts per time interval is estimated as $\frac{10,094}{2,608} = 3.8704$. We form

12 groups of time intervals by taking together the time intervals with 11 or more particles. In the table we give the expected Poisson frequencies. The value of the chi-square statistic is given by

$$\frac{(57-54.768)^2}{54.3768} + \frac{(203-210.4604)^2}{210.4604} + \cdots + \frac{(6-5.7831)^2}{5.7831} = 11.613.$$

Since the parameter of the hypothesized Poisson distribution has been estimated from the data, the chi-square distribution with $12-1-1=10$ degrees of freedom is used for the test. The probability $P(\chi^2_{10} \geq 11.613) = 0.3117$. The Poisson distribution gives a good fit.

12E-8 The parameter of the hypothesized Poisson distribution is estimated as

$$\lambda = \frac{37}{78}$$

vacancies per year. The data are divided in three groups: years with 0 vacancies, with 1 vacancy and with ≥ 2 vacancies. Letting $p_i = e^{-\lambda}\lambda^i/i!$, the expected number of years with 0 vacancies is $78p_0 = 48.5381$, with 1 vacancy is $78p_1 = 23.0245$ and with ≥ 2 vacancies is $78(1-p_0-p_1) = 6.4374$. The chi-square test statistic with $3-1-1=1$ degree of freedom has the value

$$\frac{(48-48.5381)^2}{48.5381} + \frac{(23-23.0245)^2}{23.0245} + \frac{(7-6.4374)^2}{6.4374} = 0.055.$$

The probability $P(\chi^2_1 > 0.055) = 0.8145$, showing that the Poisson distribution with expected value $\lambda = \frac{37}{78}$ gives an excellent fit for the probability distribution of the number of vacancies per year.

Chapter 13

13E-1 Let X be equal to 1 if the stronger team is the overall winner and let X be equal to 0 otherwise. The random variable Y is defined as the number of games the final will take. The joint probability mass function of X and Y satisfies

$$P(X=0,\, Y=k) = \binom{k-1}{3}(0.45)^4(0.55)^{k-4} \quad \text{for } k=4,\ldots,7$$

It now follows that the conditional probability mass function of Y given that $X=0$ is given by

$$\frac{P(Y=k,\, X=0)}{P(X=0)} = \frac{\binom{k-1}{3}(0.45)^4(0.55)^{k-4}}{\sum_{j=4}^{7}\binom{j-1}{3}(0.45)^4(0.55)^{j-4}}.$$

This conditional probability has the numerical values 0.1047, 0.2303, 0.3167, and 0.3483 for $k = 4$, 5, 6, and 7.

13E-2 First the marginal distribution of X must be determined. Using the binomium of Newton, it follows that

$$P(X = n) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{6}\right)^k \left(\frac{1}{3}\right)^{n-k} = \left(\frac{1}{3} + \frac{1}{6}\right)^n, \ n = 1, 2, \ldots .$$

That is, the random variable X is geometrically distributed with parameter $\frac{1}{2}$. Hence the conditional distribution of Y given that $X = n$ is given by

$$P(Y = k \mid X = n) = \frac{\binom{n}{k}\left(\frac{1}{6}\right)^k \left(\frac{1}{3}\right)^{n-k}}{\left(\frac{1}{2}\right)^n} = \binom{n}{k}\left(\frac{1}{3}\right)^n 2^k, \ k = 0, 1, \ldots, n.$$

Remark. Using the identity $\sum_{n=k}^{\infty} \binom{n}{k} a^{n-k} = (1-a)^{-k-1}$ for $|a| < 1$, it is readily verified that the marginal distribution of Y is given by $P(Y = 0) = \frac{1}{2}$ and $P(Y = k) = \frac{3}{2}\left(\frac{1}{4}\right)^k$ for $k = 1, 2, \ldots$. For $k = 0$ the conditional distribution of X given that $Y = k$ is

$$P(X = n \mid Y = 0) = 2\left(\frac{1}{3}\right)^n, \quad n \geq 1,$$

while for $k \geq 1$ the conditional distribution of X given that $Y = k$ is

$$P(X = n \mid Y = k) = \binom{n}{k}\left(\frac{2}{3}\right)^{k+1}\left(\frac{1}{3}\right)^{n-k}, \quad n \geq k.$$

13E-3 The conditional expected value $E(X \mid Y = 2)$ is computed from

$$E(X \mid Y = 2) = 1 \times P(X = 1 \mid Y = 2) + \sum_{i=3}^{\infty} i\, P(X = i \mid Y = 2).$$

It holds that $P(X = 1, Y = 2) = \frac{1}{6} \times \frac{1}{6}$ and

$$P(X = i, Y = 2) = \frac{4}{6} \times \frac{1}{6} \times \left(\frac{5}{6}\right)^{i-3} \times \frac{1}{6} \quad \text{for } i \geq 3$$

Also, $P(Y = 2) = \frac{5}{6} \times \frac{1}{6}$. Hence, by $P(X = i \mid Y = 2) = \frac{P(X=i, Y=2)}{P(Y=2)}$,

$$P(X = 1 \mid Y = 2) = \frac{1}{5}, \quad P(X = i \mid Y = 2) = \frac{4}{30}\left(\frac{5}{6}\right)^{i-3} \quad \text{for } i \geq 3.$$

This leads to

$$E(X \mid Y=2) = \frac{1}{5} + \sum_{i=3}^{\infty} i\,\frac{4}{30}\left(\frac{5}{6}\right)^{i-3} = 6.6.$$

In a similar way $E(X \mid Y = 20)$ is computed. Using the expressions

$$P(X=i,\,Y=20) = \left(\frac{4}{6}\right)^{i-1}\frac{1}{6}\left(\frac{5}{6}\right)^{19-i}\frac{1}{6} \quad \text{for } 1 \le i \le 19$$
$$P(X=i,\,Y=20) = \left(\frac{4}{6}\right)^{19}\frac{1}{6}\left(\frac{5}{6}\right)^{i-21}\frac{1}{6} \quad \text{for } i \ge 21,$$

we find that $E(X \mid Y = 20) = 5.029$.

13E-4 By $f(x,\,y) = f_X(x)f_Y(y \mid x)$, it follows that

$$f(x,\,y) = 2 \quad \text{for } 0 < x < 1,\, 0 < y < x$$

and $f(x,\,y) = 0$ otherwise. This leads to the marginal density $f_Y(y) = \int_y^1 f(x,\,y)\,dx = 2(1-y)$ for $0 < y < 1$. By $f(x,\,y) = f_Y(y)f_X(x \mid y)$, we next find for any y with $0 < y < 1$ that

$$f_X(x \mid y) = \frac{1}{1-y} \quad \text{for } y < x < 1$$

and $f_X(x \mid y) = 0$ otherwise. This leads to

$$E(X \mid Y=y) = \int_y^1 \frac{x}{1-y}\,dx = \frac{1}{2(1-y)}(1-y^2) = \frac{1}{2}(1+y), \quad 0 < y < 1.$$

13E-5 The approach is to use the relation $f(x,y) = f_X(x)f_Y(y \mid x)$ and to simulate first a random observation for x from $f_X(x)$ and to simulate next a random observation for y from $f_Y(y \mid x)$. It follows from $f_X(x) = \int_x^\infty f(x,\,y)dy$ that

$$f_X(x) = e^{-x} \quad \text{for } x > 0.$$

Next, by $f_Y(y \mid x) = f(x,\,y)/f_X(x)$, it holds for any x that

$$f_Y(y \mid x) = e^{-(y-x)} \quad \text{for } y > x.$$

Hence the marginal density $f_X(x)$ is the exponential density with parameter 1, while, for fixed x, the conditional density $f_Y(y \mid x)$ is the exponential density with parameter 1 shifted to the point x. A random observation from the exponential density can be obtained by the inverse-transformation method.

Hence a random observation from $f(x, y)$ can be simulated as follows:

Step 1. Generate two random numbers u_1 and u_2.
Step 2. Output $x = -\ln(u_1)$ and $y = x - \ln(u_2)$.

13E-6 The marginal density of the lifetime X of the first circuit is given by

$$f_X(x) = \int_1^\infty \frac{24}{(x+y)^4}\,dy = \frac{8}{(x+1)^3} \quad \text{for } x > 1$$

and $f_X(x) = 0$ otherwise. Hence the conditional density of the lifetime of the second circuit given that the first circuit has failed after s time units is given by

$$f_Y(y \mid s) = \frac{f(s,y)}{f_X(s)} = \frac{3(s+1)^3}{(s+y)^4} \quad \text{for } y > 1$$

and $f_Y(y \mid s) = 0$ otherwise. Hence the expected value of the lifetime of the second circuit given that the first circuit has failed after s time units is given by

$$\begin{aligned} E(Y \mid X = s) &= \int_1^\infty y\frac{3(s+1)^3}{(s+y)^4}\,dy = -(s+1)^3\int_1^\infty y\,d(s+y)^{-3} \\ &= 1 + (s+1)^3\int_1^\infty (s+y)^{-3}\,dy = 0.5s + 1.5. \end{aligned}$$

The probability that the second circuit will work more than v time units given that the first circuit has failed after s time units is equal to

$$\int_v^\infty f_Y(y \mid s)\,dy = \frac{(s+1)^3}{(s+v)^3} \quad \text{for } v \geq 1.$$

13E-7 Suppose you mark m boxes. Let the random variable G be the gain of the game. The random variable G can be represented as

$$G = X_1 + \cdots + X_m,$$

where X_k is the gain obtained from the ith of the m marked boxes. The random variables $X_1, \ldots, X_m$ are identically distributed but are in general not independent. To find $E(G) = \sum_{k=1}^m E(X_k)$, we condition upon the random variable Y which is defined to be 1 if the box with the devil's penny is not among the m marked boxes and is 0 otherwise. By the law of conditional expectation, $E(X_k) = E(X_k \mid Y = 0)P(Y = 0) + E(X_k \mid Y = 1)P(Y = 1)$. Since $E(X_k \mid Y = 0) = 0$ and $P(Y = 1) = 1 - \frac{m}{11}$, it follows that

$$E(X_k) = E(X_k \mid Y = 1)\left(1 - \frac{m}{11}\right)$$

The conditional distribution of the random variable X_k given that $Y = 1$ is the discrete uniform distribution on $a_1, \ldots, a_{10}$. Hence, for all k,

$$E(X_k \mid Y = 1) = \frac{1}{10}\sum_{i=1}^{10} a_i \quad \text{and} \quad E(X_k) = \Big(1 - \frac{m}{11}\Big)\frac{1}{10}\sum_{i=1}^{10} a_i.$$

Inserting the expression for $E(X_k)$ into $E(G) = \sum_{k=1}^{m} E(X_k)$ gives

$$E(G) = m\Big(1 - \frac{m}{11}\Big)\frac{1}{10}\sum_{i=1}^{10} a_i.$$

The function $x(1 - \frac{x}{11})$ of the continuous variable x is maximal for $x = 5.5$. The maximal value of $E(G)$ as function of m is attained for both $m = 5$ and $m = 6$. Hence the optimal decision is to mark 5 or 6 boxes in order to maximize the expected gain of the game. The choice $m = 6$ is more risky. Taking $m = 5$ or $m = 6$ in the expression for $E(G)$, we find

$$\text{maximal value of the expected gain} = \frac{3}{11}\sum_{i=1}^{10} a_i \quad \text{dollars.}$$

Remark. Using the reasoning as in the derivation of $E(G)$, we find that

$$E(G^2) = \Big(1 - \frac{m}{11}\Big)\Big[\frac{m}{10}\sum_{k=1}^{10} a_k^2 + \frac{m(m-1)}{10 \times 9}\sum_{k,\,l=1,\,l \neq k}^{10} a_k a_l\Big].$$

The standard deviation of the gain next follows from $\sigma(G) = \sqrt{E(G^2) - E^2(G)}$.

13E-8 (a) Let the random variable Y be the number of heads showing up in the first five tosses of the coin. Then,

$$E(X \mid Y = y) = y + \frac{1}{2}y \quad \text{for } 0 \leq y \leq 5.$$

Hence, by $E(X) = \sum_{y=0}^{5} E(X \mid Y = y)P(Y = y)$, we have $E(X) = E(Y) + \frac{1}{2}E(Y)$. Noting that $E(Y) = \frac{5}{2}$, it follows that $E(X) = \frac{15}{4} = 3.75$.
(b) Let the random variable Y be the outcome of the first roll of the die. Then,

$$E(X \mid Y = y) = \delta(6 - y) + \frac{1}{6}y \quad \text{for } 0 \leq y \leq 6,$$

where $\delta(0) = 1$ and $\delta(k) = 0$ for $k \neq 0$. Hence, by $E(X) = \sum_{y=0}^{6} E(X \mid Y = y)P(Y = y)$, we get $E(X) = \frac{1}{6}E(Y) + \frac{1}{6} \times 1$. Since $E(Y) = 3.5$, we find $E(X) = \frac{4.5}{6} = 0.75$.

13E-9 Using the formulas for the expected value and variance of the uniform distribution, it follows that

$$E(X \mid Y = y) = \frac{1 - y + 1 + y}{2} = 1,$$

$$E(X^2 \mid Y = y) = \frac{\big(1 + y - (1 - y)\big)^2}{12} + 1^2 = \frac{y^2}{3} + 1.$$

Denoting by $f_Y(y)$ the density of Y, it follows from the law of conditional expectation that

$$E(X) = \int_y E(X \mid Y = y) f_Y(y)\, dy = 1,$$

$$E(X^2) = \int_y E(X^2 \mid Y = y) f_Y(y)\, dy = \frac{E(Y^2)}{3} + 1 = \frac{3}{3} + 1.$$

Hence the expected value and the variance of X are both equal to 1.

13E-10 By the convolution formula,

$$P(X_1 + X_2 = r) = \sum_{k=1}^{r-1} p(1-p)^{k-1} p(1-p)^{r-k-1}.$$

This gives

$$\begin{aligned} P(X_1 = j \mid X_1 + X_2 = r) &= \frac{P(X_1 = j,\, X_1 + X_2 = r)}{P(X_1 + X_2 = r)} \\ &= \frac{p(1-p)^{j-1} p(1-p)^{r-j-1}}{\sum_{k=1}^{r-1} p(1-p)^{k-1} p(1-p)^{r-k-1}} = \frac{1}{r-1}. \end{aligned}$$

In other words, the conditional distribution of X_1 given that $X_1 + X_2 = r$ is the discrete uniform distribution on $1, \ldots, r-1$.

13E-11 The number p is a random observation from a random variable U that is uniformly distributed on $(0, 1)$. Let the random variable X denote the number of times that heads will appear in n tosses of the coin. Then, by the law of conditional probability,

$$P(X = k) = \int_0^1 P(X = k \mid U = p)\, dp = \int_0^1 \binom{n}{k} p^k (1-p)^{n-k}\, dp$$

for $k = 0, 1, \ldots, n$. Using the result for the beta integral, we next obtain

$$P(X = k) = \binom{n}{k} \frac{k!(n-k)!}{(n+1)!} = \frac{1}{n+1} \qquad \text{for } k = 0, 1, \ldots, n.$$

13E-12 Let $V = X_1$ and $W = X_1 + X_2$. To find the joint density function of V and W, we apply the transformation rule from Section 11.4. The inverse functions $a(v,w)$ and $b(v,w)$ are given by $a(v,w) = v$ and $b(v,w) = w - v$. The Jacobian $J(v,w) = 1$. Hence the joint density function of V and W is given by

$$f_{V,W}(v,w) = \mu e^{-\mu v} \mu e^{-\mu(w-v)} = \mu^2 e^{-\mu w} \quad \text{for } 0 < v < w$$

and $f_{V,W}(v,w) = 0$ otherwise. The density function of W is given by $\int_0^w \mu^2 e^{-\mu w} dv = \mu^2 w e^{-\mu w}$ for $w > 0$ (the Erlang-2 density with scale parameter μ). Hence the conditional density function of V given that $W = s$ is equal to

$$f_V(v \mid s) = \frac{\mu^2 e^{-\mu s}}{\mu^2 s e^{-\mu s}} = \frac{1}{s} \quad \text{for } 0 < v < s$$

and $f_V(v \mid s) = 0$ otherwise. In other words, the conditional probability density function of X_1 given that $X_1 + X_2 = s$ is the uniform density on $(0, s)$.

13E-13 The joint density function $f(\theta, r)$ of Θ and R is given by $f(\theta, r) = \frac{1}{2\pi} r e^{-\frac{1}{2}r^2}$ for $-\pi < \theta < \pi$ and $r > 0$. To find the joint density function of V and W, we apply the transformation rule from Section 11.4. The inverse functions $\theta = a(v,w)$ and $r = b(v,w)$ are given by $a(v,w) = \operatorname{arctg}(w/v)$ and $b(v,w) = \sqrt{v^2 + w^2}$. Using the fact that the derivative of $\operatorname{arctg}(x)$ is equal to $1/(1+x^2)$, we find that the Jacobian $J(v,w) = -\sqrt{v^2+w^2}$. Hence the joint density function of V and W is equal to

$$f_{V,W}(v,w) = \frac{1}{2\pi}(v^2 + w^2) e^{-\frac{1}{2}(v^2+w^2)} \quad \text{for } -\infty < v, w < \infty.$$

Using the fact that $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}v^2} dv = 1$ and $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v^2 e^{-\frac{1}{2}v^2} dv = 1$, it follows that the marginal density function of W is given by

$$f_W(w) = \frac{1}{\sqrt{2\pi}}(1 + w^2) e^{-\frac{1}{2}w^2} \quad \text{for } -\infty < w < \infty.$$

The random variable V has the same marginal density as W. The conditional density function of V given that $W = w$ is equal to

$$f_V(v \mid w) = \frac{1}{\sqrt{2\pi}} \frac{v^2 + w^2}{1 + w^2} e^{-\frac{1}{2}v^2} \quad \text{for } -\infty < v < \infty.$$

Since $\int_{-\infty}^{\infty} v^k e^{-\frac{1}{2}v^2}\, dv = 0$ for both $k = 1$ and $k = 3$, it follows that

$$E(V \mid W = w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{v^3 + vw^2}{1 + w^2} e^{-\frac{1}{2}v^2}\, dv = 0 \quad \text{for all } w.$$

13E-14 By the law of conditional probability,

$$P(N=k)=\int_0^1 P(N=k \mid X_1=u)\,du \quad \text{for } k=2,3,\ldots.$$

The conditional probability distribution of N given that $X_1=u$ is a shifted geometric distribution with success parameter $p=1-u$. Hence

$$P(N=k)=\int_0^1 u^{k-2}(1-u)du=\frac{1}{k(k-1)} \quad \text{for } k=2,3,\ldots.$$

The expected value of N is given by $E(N)=\infty$.

13E-15 By the law of conditional probability and the independence of X, Y and Z,

$$\begin{aligned}
P(V=v,\,W=w) &= \sum_{x=0}^{\infty} P(X+Y=v,\,X+Z=w \mid X=x)P(X=x)\\
&= \sum_{x=0}^{\min(v,w)} P(Y=v-x,\,Z=w-x)P(X=x)\\
&= \sum_{x=0}^{\min(v,w)} e^{-\lambda(v-x)}\frac{\lambda^{v-x}}{(v-x)!}e^{-\lambda(w-x)}\frac{\lambda^{w-x}}{(w-x)!}e^{-\lambda x}\frac{\lambda^{x}}{x!}
\end{aligned}$$

This gives

$$P(V=v,\,W=w)=e^{-3\lambda}\,\lambda^{v+w}\sum_{x=0}^{\min(v,w)}\frac{\lambda^{-x}}{(v-x)!(w-x)!x!}, \quad v,w=0,1,\ldots.$$

13E-16 For the case of n cars, let the random variable X_n be the number of clumps of cars that will be formed and the random variable Y be the position of the slowest car on the highway. Under the condition that $Y=i$ the conditional distribution of X_n is the same as the unconditional distribution of $1+X_{n-i}$ for $i=1,\ldots n$, where $X_0=0$. By the law of conditional expectation, we find

$$E(X_n)=\sum_{i=1}^{n}\frac{1}{n}\left[1+E(X_{n-i})\right].$$

The solution of this recursion is given by

$$E(X_n)=\sum_{k=1}^{n}\frac{1}{k} \quad \text{for } n\geq 1.$$

13E-17 Let c be the constant for which $E(X \mid Y = y) = c$ for all y. Then $E(X) = c$. To see this, denote by $f_Y(y)$ the marginal density function of Y and use the law of conditional expectation to obtain

$$E(X) = \int_y E(X \mid Y = y) f_Y(y)\, dy = c \int_y f_Y(y)\, dy = c.$$

Also, by the law of conditional expectation,

$$\begin{aligned} E(XY) &= \int_y E(XY \mid Y = y) f_Y(y)\, dy = \int_y y E(X \mid Y = y) f_Y(y)\, dy \\ &= c \int_y y f_Y(y)\, dy = E(X)E(Y). \end{aligned}$$

This shows that $\text{cov}(X, Y) = E(XY) - E(X)E(Y) = 0$.

13E-18 For any $0 \leq x \leq 1$,

$$P(U_1 \leq x \mid U_1 > U_2) = \frac{P(U_1 \leq x,\, U_1 > U_2)}{P(U_1 > U_2)} = \frac{\int_0^x du_1 \int_0^{u_1} du_2}{1/2} = x^2$$

and

$$P(U_2 \leq x \mid U_1 > U_2) = \frac{P(U_2 \leq x,\, U_1 > U_2)}{P(U_1 > U_2)} = \frac{\int_0^x du_2 \int_{u_2}^1 du_1}{1/2} = 2\left(x - \frac{1}{2}x^2\right).$$

Thus the conditional densities of U_1 and U_2 given that $U_1 > U_2$ are defined by $2x$ and $2(1-x)$ for $0 < x < 1$ and are zero otherwise. This gives

$$E(U_1 \mid U_1 > U_2) = \int_0^1 x\, 2x\, dx = \frac{2}{3}, \quad E(U_2 \mid U_1 > U_2) = \int_0^1 x\, 2(1-x)\, dx = \frac{1}{3}.$$

13E-19 The density functions of X and Y are $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$ and $f_Y(y) = \lambda e^{-\lambda}$ for $y > 0$. By $P(A \mid B) = P(AB)/P(B)$,

$$P(X \leq x \mid X > Y) = \frac{P(X \leq x,\, X > Y)}{P(X > Y)} = 2\int_0^x f_X(u) du \int_0^u f_Y(y)\, dy,\ x \geq 0$$

and

$$P(Y \leq y \mid X > Y) = \frac{P(Y \leq y,\, X > Y)}{P(X > Y)} = 2\int_0^y f_Y(u) du \int_u^\infty f_X(x)\, dx,\ y \geq 0.$$

Differentation shows that the conditional density of X given that $X > Y$ is $2f_X(x)\int_0^x f_Y(y)\, dy = 2\lambda e^{-\lambda x}(1 - e^{-\lambda x})$ for $x > 0$, while the conditional density of Y given that $X > Y$ is $2f_Y(y)\int_y^\infty f_X(x)\,dx = 2\lambda e^{-\lambda y}e^{-\lambda y}$ for $y > 0$. This gives

$$E(X \mid X > Y) = \int_0^\infty x\,2\lambda e^{-\lambda x}(1 - e^{-\lambda x})\,dx = \frac{3}{2\lambda},$$
$$E(Y \mid X > Y) = \int_0^\infty y\,2\lambda e^{-\lambda y}e^{-\lambda y}\,dy = \frac{1}{2\lambda}.$$

13E-20 The two roots of the equation $x^2 + 2Bx + 1 = 0$ are real only if $|B| \geq 1$. Since $P(|B| \geq 1) = P(B \leq -1) + P(B \geq 1)$, the probability of two real roots is equal to

$$P(|B| \geq 1) = \Phi(-1) + 1 - \Phi(1) = 2[1 - \Phi(1)] = 0.3174.$$

The sum of the two roots is $-2B$. To determine the conditional density function of $-2B$ given that $|B| \geq 1$, note that $P(-2B \leq x \big| |B| \geq 1)$ is given by

$$P(-2B \leq x \big| |B| \geq 1) = P(B \geq -\frac{1}{2}x \big| |B| \geq 1) = \frac{P(B \geq -\frac{1}{2}x,\, |B| \geq 1)}{P(|B| \geq 1)}.$$

We have $P(B \geq -\frac{1}{2}x,\, |B| \geq 1) = P(B \geq 1)$ for $-2 < x < 2$. Further

$$P(B \geq -\frac{1}{2}x,\, |B| \geq 1) = \begin{cases} P(B \geq -\frac{1}{2}x) = 1 - \Phi(-\frac{1}{2}x), & x \leq -2 \\ P(-\frac{1}{2}x \leq B \leq -1) = \Phi(-1) - \Phi(-\frac{1}{2}x), & x \geq 2. \end{cases}$$

Denoting by $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ the standard normal density, it now follows that the conditional probability density function of $-2B$ given that $|B| \geq 1$ is equal to $\frac{1}{4}\phi(-\frac{1}{2}x)/[1 - \Phi(1)]$ for both $x < -2$ and $x > 2$ and is equal to 0 for $-2 \leq x \leq 2$.

13E-21 Let the random variable T be the transmission time of a message and N be the number of newly arriving messages during the time T. Under the condition that $T = n$, the random variable N has a binomial distribution with parameters n and p. Hence, by the law of conditional expectation,

$$E(N) = \sum_{n=1}^{\infty} E(N \mid T = n)a(1-a)^{n-1}$$

and

$$E(N^2) = \sum_{n=1}^{\infty} E(N^2 \mid T = n)a(1-a)^{n-1}.$$

The first two moments of a binomial distribution with parameters n and p are given by np and $np(1-p)+n^2p^2$, while the first two moments of a geometric distribution with parameter a are given by $\frac{1}{a}$ and $\frac{1-a}{a^2}+\frac{1}{a^2}$. This gives

$$E(N)=\sum_{n=1}^{\infty} npa(1-a)^{n-1}=\frac{p}{a}$$

and

$$\begin{aligned} E(N^2) &= \sum_{n=1}^{\infty}\big(np(1-p)+n^2p^2\big)a(1-a)^{n-1} \\ &= \frac{p(1-p)}{a}+\frac{p^2(2-a)}{a^2}. \end{aligned}$$

Hence the expected value and the standard deviation of the number of newly arriving messages during the transmission time of a message are given by p/a and $\sqrt{p[a(1-p)+p(1-a)]}/a$.

13E-22 Let the random variable Y denote the number of messages waiting in the buffer. Then $P(Y=y)=p(1-p)^{y-1}$ for $y\geq 1$. By the law of conditional probability,

$$P(X=x)=\sum_{y=x+1}^{\infty}P(X=x\mid Y=y)p(1-p)^{y-1}=\sum_{y=x+1}^{\infty}\frac{1}{y}p(1-p)^{y-1}$$

for $x=0,1,\ldots$. Using the formula $\sum_{n=1}^{\infty}\frac{u^n}{n!}=-\ln(1-u)$ for $|u|<1$, the expression for $P(X=x)$ can be written as

$$P(X=x)=-\frac{p}{1-p}\ln(p)-\sum_{y=1}^{x}\frac{1}{y}p(1-p)^{y-1}.$$

Using the fact that the discrete uniform distribution on $0,1,\ldots,y-1$ has expected value $\frac{1}{2}(y-1)$, the expected value of X is calculated from

$$E(X)=\sum_{y=1}^{\infty}E(X\mid Y=y)p(1-p)^{y-1}=\sum_{y=1}^{\infty}\frac{1}{2}(y-1)\,p(1-p)^{y-1}=\frac{1}{2}\Big(\frac{1}{p}-1\Big).$$

13E-23 Given that the carnival master tells you that the ball picked from the red beaker has value r, let $L(r)$ be your expected payoff when you guess a larger value and let $S(r)$ your expected payoff when you guess a smaller value. Then

$$L(r)=\frac{1}{10}\sum_{k=r+1}^{10}k+\frac{r/2}{10}=\frac{1}{20}(10-r)(r+11)+\frac{r/2}{10}=\frac{110-r^2}{20}$$

and

$$S(r)=\frac{1}{10}\sum_{k=1}^{r-1}k+\frac{r/2}{10}=\frac{1}{20}(r-1)r+\frac{r/2}{10}=\frac{r^2}{20}.$$

It holds that $L(r)>S(r)$ for $1\leq r\leq 7$ and $L(r)<S(r)$ for $8\leq r\leq 10$, as follows by noting that $110-x^2=x^2$ has $x^*=\sqrt{55}\approx 7.4$ as solution. Thus, given that the carnival master tells you that the ball picked from the red beaker has value r, your expected payoff is maximal by guessing a larger value if $r\leq 7$ and guessing a smaller value otherwise. Applying the law of conditional expectation. it now follows that your expected payoff is

$$\sum_{k=1}^{7}\frac{110-r^2}{20}\times\frac{1}{10}+\sum_{k=8}^{10}\frac{r^2}{20}\times\frac{1}{10}=4.375\text{ dollars}$$

if you use the decision rule with critical level 7. The game is not fair, but the odds are only slightly in favor of the carnival master if you play optimally. Then the house edge is 2.8% (for critical levels 5 and 6 the house edge has the values 8.3% and 4.1%).

13E-24 For the case that you start with n strings, let X_n be the number of loops you get and let $Y_n=1$ if the first two loose ends you choose are part of the same string and $Y_n=0$ otherwise. By the law of conditional expectation, $E(X_n)=E(X_n\mid Y_n=1)P(Y_n=1)+E(X_n\mid Y_n=0)P(Y_n=0)$ and so

$$\begin{aligned}E(X_n)&=\big[1+E(X_{n-1})\big]\frac{1}{2n-1}+E(X_{n-1})\left(1-\frac{1}{2n-1}\right)\\&=\frac{1}{2n-1}+E(X_{n-1})\quad\text{for }n=1,2,\ldots,N,\end{aligned}$$

where $E(X_0)=0$. This gives $E(X_n)=\sum_{i=1}^{2n-1}\frac{1}{i}$ for $n=1,\ldots,N$.

13E-25 (a) Denote by N the Poisson distributed random variable you sample from. Let X be the number of heads you will obtain. By the law of conditional probability, for any $k=0,1,\ldots,$

$$\begin{aligned}P(X=k)&=\sum_{n=k}^{\infty}P(X=k\mid N=n)P(N=n)=\sum_{n=k}^{\infty}\binom{n}{k}\left(\frac{1}{2}\right)^n\frac{e^{-1}}{n!}\\&=\frac{e^{-1}(1/2)^k}{k!}\sum_{n=k}^{\infty}\frac{(1/2)^{n-k}}{(n-k)!}=\frac{e^{-1}(1/2)^k}{k!}e^{\frac{1}{2}}\\&=\frac{e^{-\frac{1}{2}}(1/2)^k}{k!}.\end{aligned}$$

In other words, the number of heads you will obtain is Poisson distributed with an expected value of 0.5.
(b) Let the random variable U be uniformly distributed on $(0,1)$. Then, by conditioning on U,

$$P(X = k) = \int_0^1 P(X = k \mid U = u)\, du = \int_0^1 e^{-u} \frac{u^k}{k!}\, du.$$

This integral can be interpreted as the probability that an Erlang-distributed random variable with shape parameter $k+1$ and scale parameter 1 will take on a value smaller than or equal to 1. This shows that

$$P(X = k) = \sum_{j=k+1}^{\infty} e^{-1} \frac{1}{j!} \quad \text{for } k = 0, 1, \ldots.$$

13E-26 By the law of conditional probability,

$$\begin{aligned} P(N = k) &= \int_0^\infty e^{-x} \frac{x^k}{k!}\, x^{r-1} \frac{[(1-p)/p]^r}{\Gamma(r)} e^{-(1-p)x/p} dx \\ &= \frac{(1-p)^r p^{-r}}{k!\,\Gamma(r)} \int_0^\infty x^{r+k-1} e^{-x/p} dx. \end{aligned}$$

Since the gamma density $x^{r+k-1}(1/p)^{r+k} e^{-x/p}/\Gamma(r+k)$ integrates to 1 over $(0,\infty)$, it next follows that

$$P(N = k) = \frac{\Gamma(r+k)}{k!\,\Gamma(r)} p^k (1-p)^r \quad \text{for } k = 0, 1, \ldots.$$

In other words, the random variable has a negative binomial distribution, where the scale parameter r is not necessarily integer-valued.

13E-27 The random variable Z has the probability density function $f(x) = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}x^2}$ for $x > 0$. The constant c such that $f(x) \leq cg(x)$ for all x is easily found. Since $f(x)/g(x) = \sqrt{2/\pi}\, e^{x - \frac{1}{2}x^2}$, it follows by differentiation that $f(x)/g(x)$ is maximal at $x = 1$ and has $\sqrt{2e/\pi}$ as its maximal value. Hence $c = \sqrt{2e/\pi}$ (≈ 1.32). Note that $f(x)/cg(x) = e^{-\frac{1}{2}(x-1)^2}$. The acceptance-rejection method proceeds as follows:

Step 1. Generate two random numbers u_1 and u_2. Let $y = -\ln(u_1)$ be the random observation from the exponential density with parameter 1.
Step 2. If $u_2 \leq e^{-\frac{1}{2}(y-1)^2}$, then $v = -\ln(u_1)$ is a random observation for the

random variable $|Z|$. Otherwise, return to Step 1.

Suppose that v is a random observation for the random variable $|Z|$. Then a random observation z from the standard normal density follows by generating a random number u and taking $z = v$ if $u \leq 0.5$ and $z = -v$ otherwise.

13E-28 Let the random variable Θ represent the unknown probability that a single toss of the die results in the outcome 6. The prior density of Θ is given by $f_0\theta) = \frac{10}{3}$ for $\theta \in (0, 1, 0.4)$. The posterior density $f(\theta \mid \text{data})$ is proportional to $L(\text{data} \mid \theta) f_0(\theta)$, where $L(\text{data} \mid \theta) = \binom{300}{75} \theta^{75} (1-\theta)^{225}$. Hence the posterior density is given by

$$f(\theta \mid \text{data} = \frac{\theta^{75}(1-\theta)^{225}}{\int_{0.1}^{0.4} x^{75}(1-x)^{225}\, dx} \qquad \text{for } 0.1 < \theta < 0.4.$$

and $f(\theta \mid \text{data} = 0$ otherwise. The posterior density is maximal at $\theta = \frac{75}{75+225} = 0.25$. The solution of $\int_0^x f(\theta \mid \text{data})\, d\theta = \alpha$ is given by 0.2044 for $\alpha = 0.025$ and by 0.3020 for $\alpha = 0.975$. Hence a 95% Bayesian confidence interval for θ is given by $(0.2044, 0.3020)$.

13E-29 Let the random variable Θ represent the unknown probability that a free throw of your friend will be successful. The posterior density function $f(\theta \mid \text{data})$ is proportional to $L(\text{data} \mid \theta) f_0(\theta)$, where $L(\text{data} \mid \theta) = \binom{10}{7} \theta^7 (1-\theta)^3$. Hence the posterior density is given by

$$f(\theta \mid \text{data}) = \begin{cases} c^{-1}\theta^7(1-\theta)^3 16(\theta - 0.25) & \text{for } 0.25 < \theta < 0.50, \\ \\ c^{-1}\theta^7(1-\theta)^3 16(0.75 - \theta) & \text{for } 0.50 < \theta < 0.75, \end{cases}$$

where

$$c = 16 \int_{0.25}^{0.50} x^7(1-x)^3(x-0.25)dx + 16 \int_{0.50}^{0.75} x^7(1-x)^3(0.75-x)dx.$$

The posterior density $f(\theta \mid \text{data})$ is zero outside the interval $(0.25, 0.75)$. Numerical calculations show that $f(\theta \mid \text{data})$ is maximal at $\theta = 0.5663$. Solving $\int_{0.25}^x f(\theta \mid \text{data}) d\theta = \alpha$ for $\alpha = 0.025$ and 0.975 leads to the 95% Bayesian confidence interval $(0.3883, 0.7113)$.

13E-30 The likelihood $L(\text{data} \mid \theta)$ is given by

$$L(\text{data} \mid \theta) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}(t_1-\theta)^2/\sigma_1^2},$$

where $t_1 = 123$ is the test score. Hence the posterior density $f(\theta \mid \text{data})$ is proportional to

$$e^{-\frac{1}{2}(t_1-\theta)^2/\sigma_1^2} \times e^{-\frac{1}{2}(\theta-\mu_0)^2/\sigma_0^2}.$$

Next a little algebra shows that $f(\theta \mid \text{data})$ is proportional to $e^{-\frac{1}{2}(\theta-\mu)^2/\sigma^2}$, where

$$\mu = \frac{\sigma_0^2 t_1 + \sigma_1^2 \mu_0}{\sigma_0^2 + \sigma_1^2} = 118.4 \quad \text{and} \quad \sigma^2 = \frac{\sigma_0^2 \sigma_1^2}{\sigma_0^2 + \sigma_1^2} = 45.$$

In other words, the posterior density is a normal density with an expected value of 118.4 and a standard deviation of 6.708. The posterior density is maximal at $\theta = 118.4$. Using the 0.025 and 0.975 percentiles -1.960 and 1.960 of the standard normal distribution, a Bayesian 95% confidence interval for the true value of the IQ is calculated as $(\mu - 1.960\sigma,\, \mu + 1.960\sigma) = (105.3, 131.5)$.

Chapter 14

14E-1 Put for abbreviation $p_n = P(X = n)$. By the definition of $G_X(z)$, we have $G_X(-1) = \sum_{n=0}^{\infty} p_{2n} - \sum_{n=0}^{\infty} p_{2n+1}$. Also, $\sum_{n=0}^{\infty} p_{2n} + \sum_{n=0}^{\infty} p_{2n+1} = 1$. Hence $G_X(-1) + 1 = 2\sum_{n=0}^{\infty} p_{2n}$, showing the desired result.

14E-2 Let the random variable X be the outcome of the first roll of the die. By conditioning on X,

$$E(z^S) = \sum_{j=1}^{6} \frac{1}{6} E(z^S \mid X = j) = \frac{1}{6} \sum_{j=1}^{6} E(z^{j+X_1+\cdots+X_j}),$$

where $X_1, \ldots, X_j$ are independent random variables each having the discrete uniform distribution on $\{1, 2, \ldots, 6\}$. This gives

$$\begin{aligned} E(z^S) &= \frac{1}{6} \sum_{j=1}^{6} z^j \left(\frac{1}{6} z + \frac{1}{6} z^2 + \cdots + \frac{1}{6} z^6 \right)^j \\ &= \frac{1}{6} \sum_{j=1}^{6} \left(\frac{1}{6} z^2 + \frac{1}{6} z^3 + \cdots + \frac{1}{6} z^7 \right)^j. \end{aligned}$$

Using the first two derivatives of $E(z^S)$, we find $E(S) = 15.75$ and $E(S^2) = 301.583$. Hence the expected value and the standard deviation of the sum S are given by 15.75 and $\sqrt{301.583 - 15.75^2}$ =7.32.

14E-3 The random variable X is distributed as $1 + X_1$ with probability p and distributed as $1 + X_2$ with probability $1 - p$. By conditioning on the outcome of the first trial, it follows that $E(z^X) = pE(z^{1+X_1}) + (1-p)E(z^{1+X_2})$. Hence

$$E(z^X) = pzE(z^{X_1}) + (1-p)zE(z^{X_2}).$$

The random variable X_1 is equal to 2 with probability p^2, is distributed as $1 + X_2$ with probability $1 - p$ and is distributed as $2 + X_2$ with probability $p(1 - p)$. This gives

$$E(z^{X_1}) = p^2 z^2 + (1-p)zE(z^{X_2}) + p(1-p)E(z^{X_2})$$

By the same argument

$$E(z^{X_2}) = (1-p)^2 z^2 + pzE(z^{X_1}) + (1-p)pE(z^{X_1}).$$

Solving the two linear equations in $E(z^{X_1})$ and $E(z^{X_2})$ gives

$$E(z^{X_1}) = \frac{p^2 z^2 + (1-p)^3 z^3 + p(1-p)^3 z^4}{1 - [(1-p)pz^2 + p(1-p)z^3 + p^2(1-p)^2 z^4]}.$$

By interchanging the roles of p and $1 - p$ in the right-hand side of this expression, the formula for $E(z^{X_2})$ follows. We can now conclude that the generating function of X is given by

$$\begin{aligned} E(z^X) \;&=\; \frac{p^3 z^3 + p(1-p)^3 z^4 + p^2(1-p)^3 z^5}{1 - [(1-p)pz^2 + p(1-p)z^3 + p^2(1-p)^2 z^4]} \\ &\quad + \frac{(1-p)^3 z^3 + (1-p)p^3 z^4 + (1-p)^2 p^3 z^5}{1 - [p(1-p)z^2 + (1-p)pz^3 + (1-p)^2 p^2 z^4]}. \end{aligned}$$

For the special case of $p = \frac{1}{2}$ (fair coin), this expression can be simplified to

$$E(z^X) = \frac{(1/4)z^3}{1 - z/2 - z^2/4}.$$

Remark. By differentiating $E(z^X)$ with respect to z and putting $z = 1$, we find after tedious algebra that

$$E(X) = A(p) + A(1-p),$$

where

$$A(x) = \frac{x^9 - 5x^8 + 6x^7 + 5x^6 - 16x^5 + 11x^4 + 7x^3 - 10x^2 + 4x}{\left(1 - 2x(1-x) - (1-x)^2 x^2\right)^2}.$$

As a sanity check, $E(X) = 7$ for the special case of $p = \frac{1}{2}$. It is always a good plan to check your answers, if possible, by reference to simple cases where you know what the answers should be.

14E-4 The moment-generating function

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} \frac{e^x}{(1+e^x)^2}\, dx$$

is defined only for $-1 < t < 1$. Using the change of variable $u = 1/(1+e^x)$ with $\frac{du}{dx} = -e^x/(1+e^x)^2$, it follows that

$$M_X(t) = \int_0^1 \left(\frac{1-u}{u}\right)^t du \quad \text{for } -1 < t < 1.$$

This integral is the well-known beta-integral $\int_0^1 u^{-t}(1-u)^t\, du$ that can be evaluated as $\Gamma(1+t)\Gamma(1-t)$, where $\Gamma(x)$ is the gamma function. Using the fact that the derivative of the function a^x is $\ln(a)a^x$ for any constant $a > 0$, it is easily verified that

$$M'_X(0) = \int_0^1 \ln\left(\frac{1-u}{u}\right) du \text{ and } M''_X(0) = \int_0^1 \ln^2\left(\frac{1-u}{u}\right) du.$$

The two integrals can be evaluated as

$$\int_0^1 [\ln(1-u) - \ln(u)] du = 0, \quad \int_0^1 [2\ln^2(u) - 2\ln(u)\ln(1-u)] du = \frac{\pi^2}{3},$$

showing that $E(X) = 0$ and $\sigma^2(X) = \frac{\pi^2}{3}$.

14E-5 The relation $M_X(t) = e^t M_X(-t)$ can be written as $E(e^{tX}) = e^t E(e^{-tX}) = E(e^{t(1-X)})$ and is thus equivalent with

$$M_X(t) = M_{1-X}(t) \quad \text{for all } t.$$

Since the moment-generating function determines uniquely the probability distribution, it follows that the random variable X has the same distribution as the random variable $1 - X$. Hence $E(X) = E(1-X)$ and so $E(X) = \frac{1}{2}$. The relation $M_X(t) = e^t M_X(-t)$ is not sufficient to determine the density of X. The property is satisfied for both the uniform density on (0, 1) and the beta density $6x(1-x)$ on (0, 1).

14E-6 (a) Chebyshev's inequality states that

$$P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2} \qquad \text{for any constant } c > 0.$$

Taking $c = k\sigma$, it follows that the random variable X falls in the interval $(\mu - k\sigma, \mu + k\sigma)$ with a probability of at least $1 - 1/k^2$. The inequality

$1 - 1/k^2 \geq p$ leads to the choice $k = 1/\sqrt{1-p}$.
(b) For any constant $a > 0$,

$$P(X > \mu+a) \leq P(|X-\mu| \geq a) \leq P(|X-\mu| \geq a+\sigma) \leq \frac{\sigma^2}{(a+\sigma)^2} \leq \frac{\sigma^2}{a^2+\sigma^2},$$

where the third inequality uses Chebyshev's inequality.

14E-7 (a) The moment-generating function of X is given by

$$M_X(t) = \int_{-1}^{1} e^{tx} \frac{1}{2}\, dx = t^{-1}\left(\frac{e^t - e^{-t}}{2}\right) \qquad \text{for } -\infty < t < \infty.$$

The function $\frac{1}{2}(e^t - e^{-t})$ is also known as $\sinh(t)$ and has the power series representation $t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots$.
(b) Put for abbreviation $\overline{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$. Using the assumption that the X_i are independent, it follows that the generating function of $\overline{X}_n$ satisfies

$$M_{\overline{X}_n}(t) = e^{\frac{t}{n}(X_1+\cdots+X_n)} = e^{\frac{t}{n}X_1} \cdots e^{\frac{t}{n}X_n}.$$

Using the result in (a), we get

$$M_{\overline{X}_n}(t) = \left(\frac{t}{n}\right)^{-n}\left(\frac{e^{t/n} - e^{-t/n}}{2}\right)^n.$$

This gives

$$P(\overline{X}_n \geq c) = \min_{t>0} e^{-ct}\left(\frac{t}{n}\right)^{-n}\left(\frac{e^{t/n} - e^{-t/n}}{2}\right)^n.$$

Next we use the inequality

$$\frac{1}{2}(e^t - e^{-t}) = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots \leq te^{t^2/6} \qquad \text{for } t > 0.$$

Hence

$$P(\overline{X}_n \geq c) = \min_{t>0} e^{-ct}e^{t^2/6n}.$$

The function $e^{-(ct - t^2/6n)}$ takes on its minimal value for $t = 3cn$. This gives the desired bound $P(\overline{X}_n \geq c) \leq e^{-\frac{3}{2}c^2 n}$ for $c > 0$.

14E-8 The moment generating function $M_X(t) = E(e^{tX})$ is given by

$$M_X(t) = \sum_{n=0}^{\infty} e^{tn}\, e^{-\lambda}\frac{\lambda^n}{n!} = e^{-\lambda(1-e^t)}, \quad -\infty < t < \infty.$$

To work out $P(X \geq c) \leq \min_{t>0} e^{-ct} M_X(t)$, consider the function $a(t) = ct + \lambda(1 - e^t)$ for $t \geq 0$. Since $c > \lambda$, this function is increasing at $t = 0$ with $a(0) = 0$. Further $a(t)$ tends to $-\infty$ as t gets large. Putting the derivative of $a(t)$ equal to zero, we find $t = \ln(c/\lambda)$. We can now conclude that the function $e^{-ct} M_X(t)$ on $(0, \infty)$ takes on its absolute minimum at the point $t = \ln(c/\lambda) > 0$ provided that $c > \lambda$. This gives the Chernoff bound

$$P(X \geq c) \leq \left(\frac{\lambda}{c}\right)^c e^{c-\lambda}.$$

14E-9 The extinction probability is the smallest root of the equation $u = P(u)$, where the generating function $P(u)$ is given by

$$P(u) = \sum_{k=0}^{\infty} p(1-p)^k \, u^k = \frac{p}{1-(1-p)u} \quad \text{for } |u| \leq 1.$$

Writing the equation $u = \frac{p}{1-(1-p)u}$ as $(1-p)u^2 - u + p = 0$, it follows that the equation $u = P(u)$ has the two roots $u = \frac{p}{1-p}$ and $u = 1$. Hence the extinction probability is $\frac{p}{1-p}$ if $p < \frac{1}{2}$ and is 1 otherwise.

14E-10 Let $X(t)$ be the number of orders present at time t. The continuous-time process $\{X(t)\}$ regenerates itself each time a production run is started. We take as cycle the time between two successive production runs. The expected length of one cycle is the expected amount of time needed to collect N orders and so

$$\text{the expected length of one cycle} = \mu + \cdots + \mu = N\mu.$$

The ih arriving order in a cycle is kept in stock until $N - i$ additional orders have been arrived and so the expected holding cost incurred for this order is $h \times (N - i)\mu$. This gives

$$\begin{aligned} \text{the expected costs incurred in one cycle} &= K + h(N-1)\mu + \cdots + h\mu \\ &= K + \frac{1}{2} hN(N-1)\mu. \end{aligned}$$

By dividing the expected costs incurred in one cycle by the expected length of one cycle, it follows from the renewal-reward theorem that

$$\text{the long-run average cost per unit time} = \frac{K}{N\mu} + \frac{1}{2} h(N-1)$$

with probability 1. The function $g(x) = K/(x\mu) + \frac{1}{2}hx$ is convex in $x > 0$ and has an absolute minimum in $x^* = \sqrt{2K/(h\mu)}$. Hence the optimal value of N is one of the integers nearest to x^*.

14E-11 Let $X(t)$ be the number of messages in the buffer at time t. The stochastic process $\{X(t), t \geq 0\}$ is regenerative. By the memoryless property of the Poisson arrival process, the epochs at which the buffer is emptied are regeneration epochs. Let a cycle be the time between two successive epochs at which the buffer is emptied. Obviously, the length of a cycle is equal to T. Let the random variable τ_n denote the arrival time of the nth message. Define the cost function $g(x)$ by $g(x) = h \times (T - x)$ for $0 \leq x \leq T$ and $g(x) = 0$ otherwise. Then, the expected value of the total holding cost in the first cycle is

$$E\Big[\sum_{n=1}^{\infty} g(\tau_n)\Big] = h\sum_{n=1}^{\infty} E[g(\tau_n)] = h\sum_{n=1}^{\infty}\int_0^T (T-x)f_n(x)\,dx,$$

where $f_n(x)$ is the probability density function of the arrival epoch τ_n. The interchange of the order of expectation and summation is justified by the non-negativity of the function $g(x)$. Since the interarrival times of messages are independent random variables having an exponential distribution with parameter λ, the density $f_n(x)$ is the Erlang density $\lambda^n x^{n-1}e^{-\lambda x}/(n-1)!$. Interchanging the order of summation and integration in the above expression for the expected holding cost and noting that $\sum_{n=1}^{\infty} f_n(x) = \lambda$, it follows that the expected value of the total holding cost in the first cycle is equal to

$$h\int_0^T (T-x)\sum_{n=1}^{\infty} f_n(x)\,dx = h\int_0^T (T-x)\lambda\,dx = \frac{1}{2}h\lambda T^2.$$

Hence the total expected cost in one cycle is $K + h\frac{1}{2}T^2$ and so, by the renewal reward theorem,

$$\text{the long-run average cost per unit time} = \frac{K + \frac{1}{2}h\lambda T^2}{T} = \frac{K}{T} + \frac{1}{2}h\lambda T$$

with probability 1. Putting the derivative of this convex cost function equal to zero, it follows that the optimal value of T is given by

$$T^* = \sqrt{\frac{2K}{h\lambda}}.$$

14E-12 Let $X(t)$ be the number of passengers waiting for departure of the boat at time t. The stochastic process $\{X(t), t \geq 0\}$ is regenerative. The regeneration epochs are the departure epochs of the boat. Consider the first cycle $(0, T)$. By conditioning on the arrival time of the nth potential customer and using the law of conditional probability, we have that

$$P(\text{the } n\text{th arrival joins the first trip}) = \int_0^T e^{-\mu(T-t)}\frac{\lambda^n t^{n-1}e^{-\lambda t}}{(n-1)!}\,dt.$$

It now follows that

$$E(\text{number of passengers on the first trip}) = \sum_{n=1}^{\infty} \int_0^T e^{-\mu(T-t)} \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}\, dt.$$

Interchanging the order of summation and integration and using the fact that $\sum_{n=1}^{\infty} \lambda^n t^{n-1} e^{-\lambda t}/(n-1)! = \lambda$, we find

$$E(\text{number of passengers on the first trip}) = \int_0^T e^{-\mu(T-t)} \lambda\, dt = \frac{\lambda}{\mu}(1 - e^{-\mu T}).$$

Hence

$$E(\text{the expected net reward in one cycle}) = \frac{R\lambda}{\mu}(1 - e^{-\mu T}) - K.$$

The expected length of one cycle is T. Hence, by the renewal-reward theorem,

$$\text{the long-average net reward per unit time} = \frac{(R\lambda/\mu)(1 - e^{-\mu T}) - K}{T}$$

with probability 1. Assuming that $R\lambda/\mu > K$, differentiation gives that the long-run average net reward is maximal for the unique solution of the equation

$$e^{-\mu T}(R\lambda T + R\lambda/\mu) = R\lambda/\mu - K.$$

14E-13 Define the following random variables. For any $t > 0$, define the indicator variable $I(t)$ by

$$I(t) = \begin{cases} 1 & \text{if the system is out of stock at time } t \\ 0 & \text{otherwise} \end{cases}$$

Also, for any $n = 1, 2, \ldots$, define the indicator variable I_n by

$$I_n = \begin{cases} 1 & \text{if the system is out of stock when the } n\text{th demand occurs} \\ 0 & \text{otherwise.} \end{cases}$$

The continuous-time stochastic process $\{I(t)\}$ and the discrete-time stochastic process $\{I_n\}$ are both regenerative. The regeneration epochs are the demand epochs at which the stock on hand drops to zero, by the memoryless property of the Poisson process. A cycle starts each time the stock on hand drops to zero. The system is out of stock during the time elapsed from the beginning of a cycle until the next inventory replenishment. This amount of time is exponentially distributed with mean $1/\mu$. The expected amount of time it takes to go from stock level Q to 0 equals Q/λ. Hence, by the renewal-reward theorem,

$$\text{the long-run fraction of time the system is out of stock} = \frac{1/\mu}{1/\mu + Q/\lambda}.$$

To find the fraction of demand that is lost, note that the expected amount of demand lost in one cycle equals $\lambda \times E$(amount of time the system is out of stock during one cycle) $= \lambda/\mu$. Hence, by the renewal-reward theorem,

$$\text{the long-run fraction of demand that is lost} = \frac{\lambda/\mu}{\lambda/\mu + Q}.$$

The above two results lead to the remarkable finding:

$$\begin{aligned}&\text{the long-run fraction of customers finding the system out of stock}\\ &= \text{the long-run fraction of time the system is out of stock.}\end{aligned}$$

This finding is a particular instance of the property "Poisson arrivals see time averages", see also Chapter 16 of the textbook. Roughly stated, this property expresses that in statistical equilibrium the distribution of the state of the system *just prior* to an arrival epoch is the same as the distribution of the state of the system at an *arbitrary* epoch when arrivals occur according to a Poisson process. This result requires only that the arrival process $\{N(t), t \geq 0\}$ can be seen as an exogenous factor to the system and is not affected by the system itself. An intuitive explanation of the property "Poisson arrivals see time averages" is that Poisson arrivals occur completely randomly in time.

14E-14 Define the following random variables. For any $t \geq 0$, let

$$I(t) = \begin{cases} 1 & \text{if the work station is busy at time } t. \\ 0 & \text{otherwise.} \end{cases}$$

Also, for any $n = 1, 2, \ldots$, let

$$I_n = \begin{cases} 1 & \text{if the work station is busy just prior to the } n\text{th arrival} \\ 0 & \text{otherwise.} \end{cases}$$

The continuous-time process $\{I(t)\}$ and the discrete-time process $\{I_n\}$ are both regenerative. The arrival epochs occurring when the working station is idle are regeneration epochs for the two processes Let us say that a cycle starts each time an arriving job finds the working station idle. The long-run fraction of time the working station is busy is equal to the expected amount of time the working station is busy during one cycle divided by the expected length of one cycle. The expected length of the busy period during one cycle equals β. The Poisson arrival process is memoryless and so the time elapsed from the completion of a job until a next job arrives has the same distributions as the interarrival times between the jobs. Hence the expected length of the idle period during one cycle equals the mean interarrival time $1/\lambda$. Hence, by the renewal-reward theorem,

$$\begin{aligned}&\text{the long-run fraction of time the work station is busy}\\&=\frac{\beta}{\beta+1/\lambda}.\end{aligned}$$

The long-run fraction of jobs that are lost equals the expected number of jobs lost during one cycle divided by the expected number of jobs arriving during one cycle. The arrival process is a Poisson process and so the expected number of arrivals in any time interval of the length x is λx. This gives that the expected number of lost arrivals during the busy period in one cycle equals $\lambda \times E(\text{processing time of a job}) = \lambda\beta$. Hence, by the renewal-reward theorem,

$$\begin{aligned}&\text{the long-run fraction of jobs that are lost}\\&=\frac{\lambda\beta}{1+\lambda\beta}.\end{aligned}$$

The above two results show that

$$\begin{aligned}&\text{the long-run fraction of arrivals finding the work station busy}\\&=\text{the long-run fraction of time the work station is busy.}\end{aligned}$$

Again we find that Poisson arrivals see time averages. Also, it is interesting to observe that in this particular problem the long-run fraction of lost jobs is insensitive to the form of the distribution function of the processing time but needs only the first moment of this distribution. Problem 14E-15 is a special case of Erlang's loss model that has been discussed in Chapter 16.