Lecture 3
Conditional Probability
Text: A Course in Probability by Weiss 4.1

STAT 225 Introduction to Probability Models
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Whitney Huang
Purdue University

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Notes

Agenda

1. Conditional Probability
2. General Multiplication Rule

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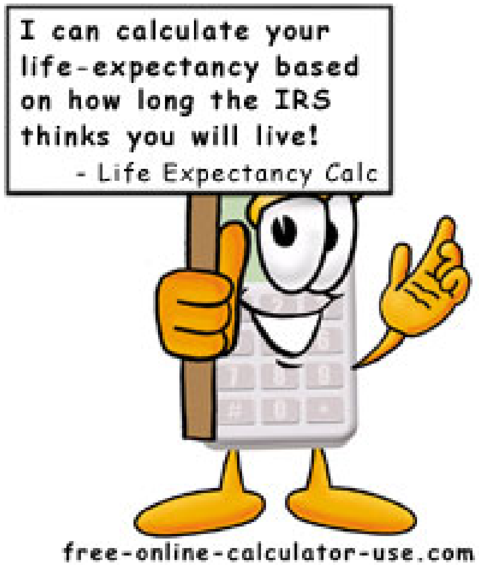
Notes

Motivating Example

In a certain population, the probability a person lives to be 80 is 80% while the probability a person lives to be 90 is 68%. Given that a person lives to be 80, what is the probability that she/he will live to be 90?



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Conditional Probability

Let A and B be events. The probability that event A occurs given (knowing) that event B occurs is called a conditional probability. It is denoted as $\mathbb{P}(A|B)$. The formula of conditional probability is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

The above formula works so long as $\mathbb{P}(B) > 0$. Under the equally likely framework the formula above can be written as

$$\mathbb{P}(A|B) = \frac{\#(A \cap B)}{\#(B)}$$

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Motivating Example

In a certain population, the probability a person lives to be 80 is 80% while the probability a person lives to be 90 is 68%. Given that a person lives to be 80, what is the probability that she/he will live to be 90?

Solution.

- Event A: a person lives to be 90
- Event B: a person lives to be 80

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(\text{a person lives to be 80 AND a person lives to be 90})}{\mathbb{P}(\text{a person lives to be 80})} = \frac{\mathbb{P}(\text{a person lives to be 90})}{\mathbb{P}(\text{a person lives to be 80})} = \frac{0.68}{0.80} = 0.85$$

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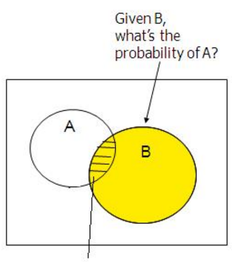
Venn Diagram Illustration of Conditional Probability



In a conditional probability problem, the sample space is "reduced" to the "space" of the given outcome (e.g. if given B, we now just care about the probability of A occurring "inside" of B)

Notes

General Multiplication Rule

Suppose we know the conditional probability $\mathbb{P}(A|B)$ and the marginal probability i.e. the probability of the given event $\mathbb{P}(B)$. Then the formula of conditional probability provides a way to compute the joint probability $\mathbb{P}(A \cap B)$

- 2 events:

$$\mathbb{P}(A \cap B) = \mathbb{P}(B) \times \mathbb{P}(A|B)$$

- More than 2 events:

$$\mathbb{P}(\cap_{i=1}^{n} A_i) = \mathbb{P}(A_1) \times \mathbb{P}(A_2|A_1) \times \mathbb{P}(A_3|A_1 \cap A_2) \times \cdots \times \mathbb{P}(A_n|A_{n-1} \cap \cdots \cap A_1)$$

Notes

Example 11

A Morgan Stanley Consumer Research Survey sampled men and women and asked each whether they preferred to drink plain bottled water or a sports drink such as Gatorade or Propel Fitness water *(The Atlanta Journal-Constitution, December 28, 2005)*. Suppose 200 men and 200 women participated in the study, and 280 reported they preferred plain bottled water. Of the group preferring a sports drink, 80 were men and 40 were women. Let

- M : the event the consumer is a man
- W : the event the consumer is a woman
- B : the event the consumer preferred plain bottled water
- S : the event the consumer preferred a sports drink

Notes

Example 11 (cont'd)

Answer the following:

1. What is the probability a person in the study preferred plain bottled water?
2. What is the probability a person in the study preferred a sports drink?
3. What are the conditional probabilities $\mathbb{P}(M|S)$ and $\mathbb{P}(W|S)$?
4. What are the joint probabilities $\mathbb{P}(M \cap S)$ and $\mathbb{P}(W \cap S)$?
5. Given a consumer is a man, what is the probability he will prefer a sports drink?

Notes

Example 11

Solution.

1. $\mathbb{P}(B) = \frac{280}{400} = 0.7$
2. $\mathbb{P}(S) = \frac{120}{400} = 0.3$
3. $\mathbb{P}(M|S) = \frac{\mathbb{P}(M \cap S)}{\mathbb{P}(S)} = \frac{\frac{80}{400}}{\frac{120}{400}} = \frac{2}{3}$, $\mathbb{P}(W|S) = \frac{\mathbb{P}(W \cap S)}{\mathbb{P}(S)} = \frac{\frac{40}{400}}{\frac{120}{400}} = \frac{1}{3}$
4. $\mathbb{P}(M \cap S) = \mathbb{P}(S) \times \mathbb{P}(M|S) = 0.3 \times \frac{2}{3} = 0.2$, $\mathbb{P}(W \cap S) = \mathbb{P}(S) \times \mathbb{P}(W|S) = 0.3 \times \frac{1}{3} = 0.1$
5. $\mathbb{P}(S|M) = \frac{\mathbb{P}(S \cap M)}{\mathbb{P}(M)} = \frac{\frac{80}{400}}{\frac{200}{400}} = 0.4$

Conditional Probability

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Conditional Probability
General Multiplication Rule

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Example 12 (Example 10 revisit)

Using the Venn Diagram summarizing the distribution of operating systems previously described, calculate the following:

1. The probability that a randomly chosen student uses all three operating systems, given the student uses Windows
2. The probability that a randomly chosen student uses all three operating systems, given the student does not use Windows
3. The probability that a randomly chosen student uses Windows, given the student uses Mac OS
4. The probability that a randomly chosen student does not use any of the operating systems, given the student does not use Windows

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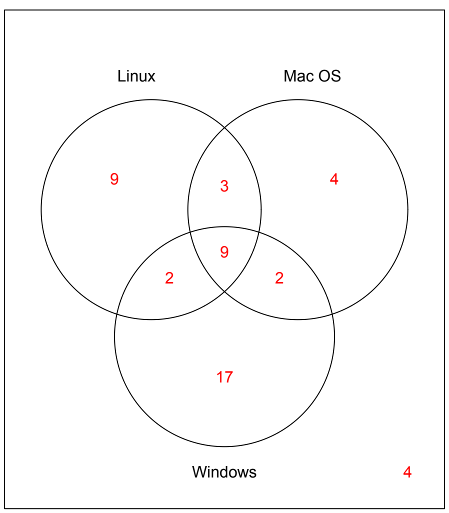
Example 12 Venn Diagram



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Example 12

Solution.

1. $\mathbb{P}(W \cap M \cap L | W) = \frac{\mathbb{P}((W \cap M \cap L) \cap W)}{\mathbb{P}(W)} = \frac{\mathbb{P}(W \cap M \cap L)}{\mathbb{P}(W)} = \frac{\frac{9}{50}}{\frac{30}{50}} = 0.3$
2. $\mathbb{P}(W \cap M \cap L | W^c) = \frac{\mathbb{P}((W \cap M \cap L) \cap W^c)}{\mathbb{P}(W^c)} = \frac{\mathbb{P}(\emptyset)}{\mathbb{P}(W^c)} = \frac{\frac{0}{50}}{\frac{20}{50}} = 0$
3. $\mathbb{P}(W | M) = \frac{\mathbb{P}(W \cap M)}{\mathbb{P}(M)} = \frac{\frac{11}{50}}{\frac{18}{50}} = \frac{11}{18}$
4. $\mathbb{P}((W \cup M \cup L)^c | W^c) = \frac{\mathbb{P}((W \cup M \cup L)^c \cap W^c)}{\mathbb{P}(W^c)} = \frac{\mathbb{P}((W \cup M \cup L)^c)}{\mathbb{P}(W^c)} = \frac{1 - \frac{46}{50}}{\frac{20}{50}} = 0.2$

Notes

Summary

In this lecture, we learned

- Conditional probability: definition, formula, venn diagram representation
- General multiplication rule

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