Probability Rules

with

Venn & Tree Diagram

What are some basic **Probability Rules**?

There are three basic **Probability Rules**:

- Complement Rule
- Addition Rule
- Multiplication Rule

What is the **Complement Rule**?

Given the event A, the **Complement Rule** states

$$P(\bar{A}) = 1 - P(A)$$

Example:

1. Given: $P(A) = 0.65$, find $P(\bar{A})$.
2. If you draw a random card from a full deck of playing cards, what is the probability of not getting a face card?

Solution:

1. Using the complement rule, we get
$$\begin{aligned} P(\bar{A}) &= 1 - P(A) \\ &= 1 - 0.65 = 0.35 \end{aligned}$$
2. There are 52 cards in a full-deck of playing cards with 12 face cards, we get
$$\begin{aligned} P(\overline{\text{Face Card}}) &= 1 - P(\text{Face Card}) \\ &= 1 - \frac{12}{52} = \frac{40}{52} = \frac{10}{13} \approx 0.769 \end{aligned}$$

What is the **Addition Rule**?

Given two different events A and B, the **Addition Rule** states

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Where $P(A \text{ and } B)$ implies the probability that both events occur at the same time as an outcome.

Example:

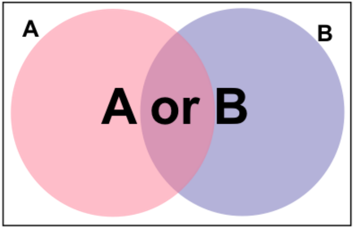
Given: $P(A) = 0.65, P(B) = 0.45,$ and $P(A \text{ and } B) = 0.35,$ find $P(A \text{ or } B)$.

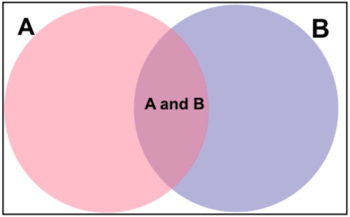
Solution:

Using the addition rule, we get

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= 0.65 + 0.45 - 0.35 = 0.75 \end{aligned}$$

Here is a visual interpretation of **A or B** and **A and B** events using Venn Diagram.

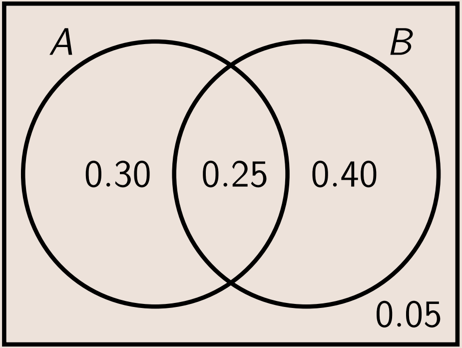




Example:

Given: $P(A) = 0.55, P(B) = 0.65,$ and $P(A \text{ and } B) = 0.25,$
Construct the Venn Diagram using the given information.

Solution:



What are **Mutually Exclusive Events**?

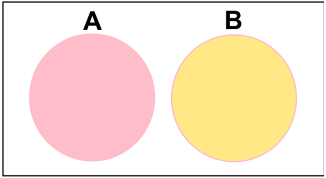
Two different events A and B, are called **Mutually Exclusive Events** or **Disjoint Events** if they cannot happen at the same time.

What is the property of two **Disjoint Events**?

For any two disjoint (mutually exclusive) events A and B,

$$P(A \text{ and } B) = 0$$

Here is a visual interpretation of two **Disjoint Events** using Venn Diagram.



Example:

Given: $P(A) = 0.55, P(B) = 0.25$, and A and B are mutually exclusive events, find $P(A \text{ or } B)$.

Solution:

Using the addition rule, we get

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= 0.55 + 0.25 - 0 = 0.8 \end{aligned}$$

Example:

A survey was conducted on 100 randomly selected adults to determine on what day of the week they did not report to work. The result of the survey is given below.

	Mon	Tues	Wed	Thurs	Fri
Females	15	5	3	7	15
Males	19	6	3	5	22

If one person is randomly selected from this group, find the probability that this person

1. is a male.
2. missed work on Friday.
3. is a male and missed work on Friday.
4. is a male or missed work on Friday.

Solution:

Let's begin by finding the total for each categories.

	Mon	Tues	Wed	Thurs	Fri	Total
Females	15	5	3	7	15	45
Males	19	6	3	5	22	55
Total	34	11	6	12	37	100

1. Male $\Rightarrow P(Male) = \frac{55}{100} = \frac{11}{20}$
2. Friday $\Rightarrow P(Friday) = \frac{37}{100}$
3. Male and Friday $\Rightarrow P(Male \text{ and } Friday) = \frac{22}{100} = \frac{11}{50}$
4. Male or Friday $\Rightarrow P(Male \text{ or } Friday) = \frac{55}{100} + \frac{37}{100} - \frac{22}{100} = \frac{70}{100} = \frac{7}{10}$

What is the **Multiplication Rule**?

Given two different events A and B, the **Multiplication Rule** states

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

Where $P(A \text{ and } B)$ implies that event A happens first, then event B is to happen.
And $P(B|A)$ implies that event A has taken place already, therefore the probability of event B has to be adjusted accordingly.

Example:

Given: $P(A) = 0.6, P(B|A) = 0.4$, find $P(A \text{ and } B)$.

Solution:

Using the multiplication rule, we get

$$P(A \text{ and } B) = P(A) \cdot P(B|A) = 0.6 \cdot 0.4 = 0.24$$

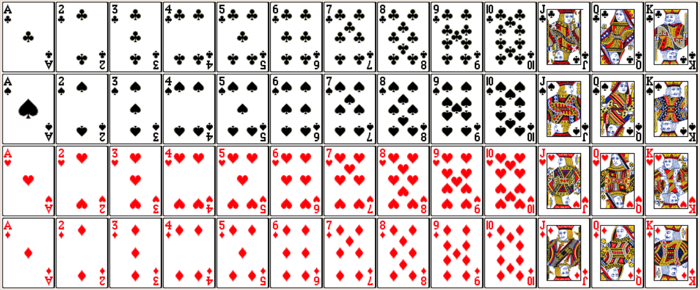
Example:

What is the probability of drawing two face cards from an ordinary full-deck of playing cards?

- With replacement
- Without replacement

Solution:

An ordinary full-deck of playing cards has 52 cards and 12 of them are face cards as shown below.



Let F_1 be the event that the first card is a face card, and F_2 be the event that the second card is a face card, using the multiplication rule, we need to eveluate the following formula.

$$P(F_1 \text{ and } F_2) = P(F_1) \cdot P(F_2|F_1)$$

Continued Solution:

Working with a full-deck of playing cards, $P(F_1) = \dfrac{12}{52}$, however when calculating $P(F_1|F_2)$, we need to know what happens to the first card.

- With replacement $\Rightarrow P(F_2|F_1) = \dfrac{12}{52}$
- Without replacement $\Rightarrow P(F_2|F_1) = \dfrac{11}{51}$

so

- With replacement
  $\Rightarrow P(F_1 \text{ and } F_2) = P(F_1) \cdot P(F_2|F_1) = \dfrac{12}{52} \cdot \dfrac{12}{52} = \dfrac{9}{169}$
- Without replacement
  $\Rightarrow P(F_1 \text{ and } F_2) = P(F_1) \cdot P(F_2|F_1) = \dfrac{12}{52} \cdot \dfrac{11}{51} = \dfrac{11}{221}$

What are **Independent Events**?

Two different events A and B, are called **Independent Events** means $P(B|A) = P(B)$.

What is the property of **Independent Events**?

For any two independent events A and B,

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example:

Given: $P(A) = 0.6, P(B) = 0.5$, and A and B are independent events, find $P(A \text{ and } B)$ and $P(A \text{ or } B)$.

Solution:

Since A and B are independent events, we get

$$\begin{aligned} P(A \text{ and } B) &= P(A) \cdot P(B) \\ &= 0.6 \cdot 0.5 = 0.3 \end{aligned}$$

Using the addition rule, we get

$$\begin{aligned} P(A \text{ or } B) &= P(A) + P(B) - P(A \text{ and } B) \\ &= 0.6 + 0.5 - 0.3 = 0.8 \end{aligned}$$

What are **Dependent Events**?

Two different events A and B, are called **Dependent Events** means that the result of one event is affected by the result of another event.

What is the property of **Dependent Events**?

For any two dependent events A and B,

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Example:

It has been reported that in Europe, 88% of all households have a TV. 55% of all households have a TV and a DVR. What is the probability that a randomly selected household in Europe has a DVR given that it has a TV?

Solution:

Let T be the event that a randomly selected household has a TV and D be the event that a randomly selected household has a DVR. So $P(T) = 0.88$ and $P(T \text{ and } D) = 0.55$, now we can use the conditional probability formula to find $P(D|T)$.

$$\begin{aligned} P(D|T) &= \frac{P(T \text{ and } D)}{P(T)} \\ &= \frac{0.55}{0.88} = 0.625 \end{aligned}$$

Example:

The probability that a freshman taking a math class is 0.75. The probability of taking a math class and an English class is 0.4. What is the probability that a randomly selected freshman taking an English class given that he or she is taking a math class?

Solution:

Let M be the event that a randomly selected freshman is taking a math class and E be the event that a randomly selected freshman is taking an English class. So $P(M) = 0.75$ and $P(M \text{ and } E) = 0.4$, now we can use the conditional probability formula to find $P(E|M)$.

$$\begin{aligned} P(E|M) &= \frac{P(M \text{ and } E)}{P(M)} \\ &= \frac{0.4}{0.75} = 0.533 \end{aligned}$$

What is a **Tree Diagram**?

Tree diagrams are a helpful tool for calculating probabilities when there are several events involved.

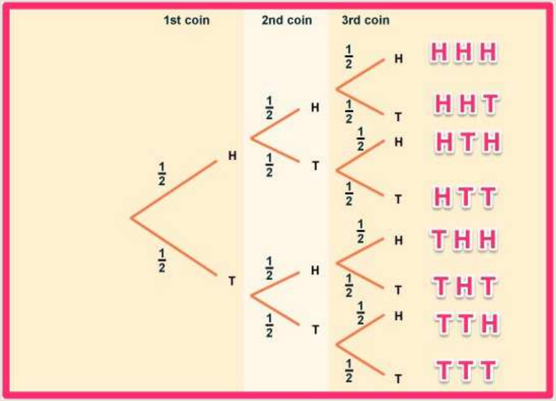
How do we use the **Tree Diagram**?

- The probability of each branch is written on the branch.
- The outcome is written at the end of the branch.
- Multiply probabilities along the branches.
- Add probabilities of those branches that satisfy the desired event.

Example:

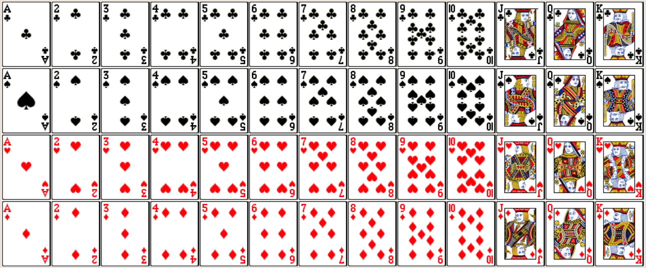
Use tree diagram to display all outcomes with indicated probabilities for each branch when you toss three fair coins.

Solution:



Example:

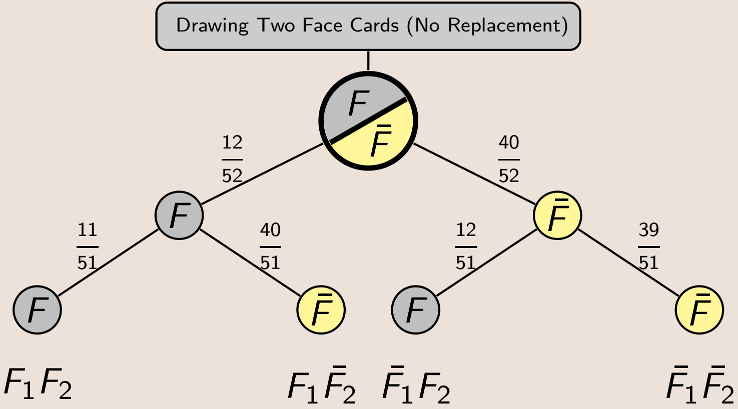
An ordinary full-deck of playing cards has 52 cards and 12 of them are face cards as shown below.



Use the tree diagram to find the probability of all outcomes of selecting two face cards without replacement.

Solution:

Let F_1 be the event that the first card is a face card, and F_2 be the event that the second card is a face card,



Continued Solution:

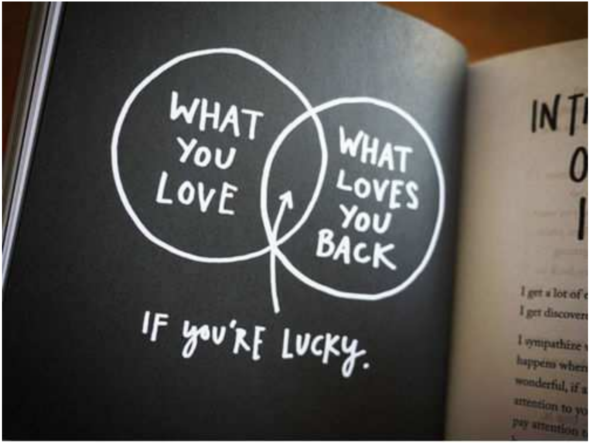
$$P(F_1 \text{ and } F_2) = \frac{12}{52} \cdot \frac{11}{51} = \frac{11}{221}$$

$$P(F_1 \text{ and } \bar{F}_2) = \frac{12}{52} \cdot \frac{40}{51} = \frac{40}{221}$$

$$P(\bar{F}_1 \text{ and } F_2) = \frac{40}{52} \cdot \frac{12}{51} = \frac{40}{221}$$

$$P(\bar{F}_1 \text{ and } \bar{F}_2) = \frac{40}{52} \cdot \frac{39}{51} = \frac{130}{221}$$

It is worth mentioning that if we add all these probabilities, we do get 1 as expected.


WHAT YOU LOVE
WHAT LOVES YOU BACK
IF YOU'RE LUCKY.