Probing Quantum Hall states with Flux Tubes and Cones

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work in preparation with:
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Outline

1. Write Laughlin wave function for arbitrary smooth R (scalar curvature and B (magnetic field).
   - density and generating functional using Ward identity
2. Generalization to singular configurations of curvature (cones) and B field (flux tubes)
   - density and generating functional
   - use density to motivate a regularization scheme for generating functional
3. Berry phase from generating functional

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Quantum Hall states

The N - electron wave function in the lowest Landau level takes the form

$$\Psi(z_1, ..., z_N) = F(z_1, ...z_N) \exp\left[\frac{1}{2}\sum_{i=1}^{N} Q(z_i, \bar{z}_i)\right] \qquad z = x + iy$$

- F is a holomorphic polynomial of 2D electron coordinates on a genus-0 surface.
- The exponential factor is related to the magnetic field via $\Delta_g Q = -2B(\mathbf{r})$
- Total magnetic flux through the surface $\frac{1}{2\pi}\int B\sqrt{g}d^2z = N_\phi$
- Scalar curvature on genus-0 (by Gauss-Bonnet) $\frac{1}{4\pi}\int R\sqrt{g}d^2z = \chi(S) = 2$
- Gauge fixing given by $\nabla \cdot \mathbf{A} = 0$, $\nabla \times \mathbf{A} = B(\mathbf{r})$ $\qquad ds^2 = \sqrt{g}dzd\bar{z}$, $R = -\Delta_g \log\sqrt{g}$
- Wf is zero mode of the anti-holomorphic kinetic momentum operator $\left(-i\frac{\partial}{\partial \bar{z}_i} - A_{\bar{z}}(\mathbf{r}_i)\right)\Psi(\{\mathbf{r}_i\}) = 0 \qquad \partial_{\bar{z}}Q = 2iA_{\bar{z}}$

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For FQH model wave functions, we use the holomorphic polynomial known from the planar models. E.g. for the Laughlin state we use

$$F(z_1, ..., z_N) = \prod_{i<j}(z_i - z_j)^\beta \qquad \nu = \frac{1}{\beta}$$

Laughlin PRL 1983

The “magnetic potential” also has a familiar form for constant curvature (sphere) and constant magnetic field…

$$Q = -N_\phi \log\left(1 + \pi|z|^2/V\right)$$

Haldane PRL 1983

…which degenerates to the appropriate Gaussian factor on the plane (infinite volume limit)

$$Q = -\pi\frac{1}{2\pi l^2}|z|^2$$

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Objects of Interest

The N - particle Laughlin state on compact genus-0 surface

$$\Psi(z_1, ..., z_N) = \mathcal{Z}^{-1/2} \prod_{i<j} (z_i - z_j)^{\beta} \exp\left[\frac{1}{2}\sum_{i=1}^{N} Q(z_i, \bar{z}_i)\right] \quad z_i \in \mathbb{C} \quad \Delta_g Q = -2B(\mathbf{r})$$

- Fixing the relation: $N_\phi = \beta(N-1)$ $\quad Q(z)_{z\to\infty} = -N_\phi \log|z|^2$

(ensures convergence) $N \to \infty$ $\quad$ Shift $\quad \sqrt{g(z)}_{z\to\infty} = |z|^{-4}$

Wen & Zee PRL 1992

Density

$$\langle \rho(z) \rangle = \int \prod_{i=1}^{N} \sqrt{g} d^2 z_i \; \rho(z) |\Psi|^2 \qquad \rho(z) = \sum_{i=1}^{N} \delta(z - z_i)$$

Generating functional (Normalization factor)

$$\mathcal{Z}[W] = \int_{\mathbb{C}^N} |\Delta(\{z_i\})|^{2\beta} \prod_{i=1}^{N} e^{W(z_i, \bar{z}_i)} d^2 z_i \qquad \frac{\delta \log \mathcal{Z}}{\delta W(\mathbf{r})} = \langle \sqrt{g} \rho(\mathbf{r}) \rangle$$

$$W = Q + \log\sqrt{g}$$

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$$\Psi(z_1,...,z_N) = \mathcal{Z}^{-1/2} F(z_1,...z_N) \exp\left[\frac{1}{2}\sum_{i=1}^{N} Q(z_i,\bar{z}_i)\right] \quad z_i \in \mathbb{C} \quad \Delta_g Q = -2B(\mathbf{r})$$

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This guy's usually easier to compute in a more general setting (e.g. Pfaffian state) - see next few talks

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Statement of Problem

Compute density in Laughlin state as an asymptotic expansion

$$\langle\rho(z)\rangle = \int \rho(z)|\Delta(\{z_i\})|^{2\beta} \prod_{i=1}^{N} e^{Q(z_i,\bar{z}_i)} \sqrt{g} d^2 z_i \qquad \rho(z) = \sum_{i=1}^{N} \delta(z - z_i)$$

1. Constant magnetic field, arbitrary (non-singular) curvature, as an expansion in magnetic field (for fixed area, this is equal to magnetic flux).

$$\langle\rho\rangle = B_0 \rho_1 + \rho_0 + \frac{1}{B_0}\rho_{-1} + \ldots$$

IQH: Douglas & Klevtsov 2010, Klevtsov 2014
FQH: TC, Laskin, Wiegmann 2014

2. Magnetic field with a delta function, zero curvature

$$B = B_0 - 2\pi a \delta(z) \qquad Q = -\frac{1}{2l^2}|z|^2 + a \log|z|^2 \quad B_0 = l^{-2}$$

TC, Laskin, Wiegmann Annals of Physics 2015, Appendix E

3. Constant magnetic field on a flat cone

$$R = 4\pi\alpha\delta(z) \qquad \sqrt{g} = |z|^{-2\alpha} \qquad Q = -\frac{1}{2l^2}\frac{|z|^{2-2\alpha}}{(1-\alpha)^2}$$

4. Use these results to find generating functional.

Statement of Problem

Compute density in Laughlin state as an asymptotic expansion

The density in **2 & 3** will be highly singular, and it makes more sense to treat it as a distribution by computing **moments**.

$$M_n = \int r^{2n}\left(\langle\rho\rangle - \frac{\nu}{2\pi l^2}\right) d^2 r$$

1. Constant magnetic field, arbitrary (non-singular) curvature, as an expansion in ma... al to magnetic flux).

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Results

Once we have a wave function, we can compute expectation values in the ground state. Particularly meaningful is the particle density for the Laughlin state. For weakly varying curvature and magnetic field, we found

Particle density

$$\langle \rho \rangle = \frac{\nu}{2\pi l^2} + \frac{1}{8\pi} R + \frac{1}{8\pi} \left(\frac{1}{12} + \frac{2 - \nu^{-1}}{4} \right) l^2 \Delta R + O(l^4)$$

For IQH, reproduces leading order of **Bergman kernel** expansion. Such asymptotic expansions are derived using exact Ward identity related to the BBGKY hierarchy from classical stat. mech.

From **Ward Identity** (aka loop equation) Zabrodin, Weigmann '06

$$\int \prod_j d^2 z_j \sum_i \partial_i \left(\frac{1}{z - z_i} \prod_{k<l} |z_k - z_l|^{2\beta} e^{\sum_p W(z_p)} \right) = 0 \qquad W = Q + \log \sqrt{g}$$

…plus a regularization for the two-point function.

Ward Identity for Laughlin States

- **Ward Identity** for Laughlin state: Generating functional invariant under coordinate transformation.

$$\int \prod_j d^2 z_j \sum_i \partial_i \left(\frac{1}{z - z_i} \prod_{k<l} |z_k - z_l|^{2\beta} e^{\sum_p W(z_p)} \right) = 0$$

$$-2\beta \int \frac{\partial W(\xi)}{z - \xi} \langle \rho(\xi) \rangle \sqrt{g} d^2 \xi = (2 - \beta) \langle \partial^2 \varphi(z) \rangle + \langle (\partial \varphi(z))^2 \rangle$$

Zabrodin, Weigmann '06

$$\varphi(z) = -\beta \sum_i \log |z - z_i|^2 \qquad -\Delta_g \varphi(z) = 4\pi \beta \rho(z)$$

$$\langle (\partial \varphi(z))^2 \rangle = \lim_{z' \to z} \langle \partial \varphi(z) \partial \varphi(z') \rangle$$

Exact equation connecting one- and two- point correlation functions

Applying two derivatives and rearranging, get useful form

$$\langle \rho \rangle = \frac{1}{2\pi \beta l^2} + \frac{1}{8\pi} R + \frac{(2 - \beta)}{8\pi \beta} \Delta \log \langle \rho \rangle - \frac{1}{2\pi^2 \beta^2} \frac{1}{\sqrt{g}} \bar{\partial} \left(\frac{\bar{\partial} \langle (\partial \varphi)^2 \rangle_c}{\sqrt{g} \langle \rho \rangle} \right)$$

Asymptotic Expansion and Short Distance Regularization

- Asymptotic expansion: $l \to 0 \quad N_\phi \to \infty \quad 2\pi l^2 N_\phi = A$ constant

Green function of Laplace-Beltrami

$$\langle \varphi(z)\varphi(z')\rangle_c = \beta G(z,z') \to -\beta \log |z-z'|^2$$

Leading order in l, valid at large separation

- UV Regularization: covariant regularization of Green function:

$$\langle \varphi(z)\varphi(z')\rangle_c = \beta \left(G(z,z') + 2\log d(z,z')\right)$$

Missing ingredient for Ward identity

$$\langle (\partial\varphi)^2\rangle_c = \frac{\beta}{6}\left[\partial^2 \log\sqrt{g} - \frac{1}{2}(\partial \log\sqrt{g})^2\right]$$

$$\frac{1}{\sqrt{g}}\bar\partial\langle(\partial\varphi)^2\rangle_c = -\frac{\beta}{24}\partial R$$

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Particle density

$$\langle \rho \rangle = \frac{\nu}{2\pi l^2} + \frac{1}{8\pi} R + \frac{1}{8\pi}\left(\frac{1}{12} + \frac{2-\nu^{-1}}{4}\right) l^2 \Delta R + O(l^4)$$

Origin of this is in the UV regularization of the 2-pt function.

For non-constant magnetic field, the asymptotic expansion is a gradient expansion in magnetic field

$$\langle \rho \rangle = \frac{\nu B}{2\pi} + \frac{1}{8\pi} R + \frac{(2\nu - 1)}{8\pi} \Delta \log \mathcal{B} + \frac{1}{8\pi}\left(\frac{1}{12} - \frac{(2\nu-1)^2}{4\nu}\right) \Delta \left(\frac{1}{\mathcal{B}}\left(R - \Delta \log \mathcal{B}\right)\right)$$

where $\mathcal{B} = B + \frac{1}{2} R$

Generating Functional

Generating Functional

$$\mathcal{Z}[W] = \int |F(\{z_i\})|^2 \prod_{i=1}^{N} e^{W(z_i,\bar{z}_i)} d^2 z_i \qquad \frac{\delta \log \mathcal{Z}}{\delta W(\mathbf{r})} = \langle \sqrt{g} \rho(\mathbf{r}) \rangle$$

Short-cut to get generating functional for arbitrary magnetic field. Start with constant B

$$W = -\frac{K}{2l^2} + \log \sqrt{g} \qquad \partial_z \partial_{\bar{z}} K = \sqrt{g} \qquad \langle \rho \rangle = \frac{\nu}{2\pi l^2} + \frac{1}{8\pi} R + \frac{1}{8\pi}\left(\frac{1}{12} + \frac{2-\nu^{-1}}{4}\right) l^2 \Delta R + O(l^4)$$

Note that variations wrt metric will hit both K and \sqrt{g}, such that

$$\delta \log \mathcal{Z} = \langle \sqrt{g} \rho \rangle \left(-\frac{1}{2l^2} \delta K + \frac{1}{\sqrt{g}} \partial \bar{\partial} \delta K \right)$$

$$\log \mathcal{Z} = -\frac{B^2}{8\pi\beta} \int K dV + \frac{B}{4\pi} \int \log \sqrt{g} dV - \frac{1}{96\pi}(1-3\beta) \int R \log \sqrt{g} dV$$

Now absorb factors of magnetic field into integrands and rewrite as

$$\log \mathcal{Z} = -\frac{1}{2\pi\beta} \int\int \left(B + \frac{\beta}{4} R\right) \Delta^{-1} \left(B + \frac{\beta}{4} R\right) + \frac{1}{96\pi} \int\int R \Delta^{-1} R$$

What we miss are extra terms which are functionals of local densities $+\mathcal{F}[B,R]$

Results

Choosing a gauge then allows us to write the generating functional in a very suggestive form

Generating Functional

$$\log \mathcal{Z}[A,\omega] = \frac{2\nu}{\pi}\int \left|\left(A_z + \frac{\beta}{2}\omega_z\right)\right|^2 dz d\bar{z} - \frac{1}{6\pi}\int |\omega_z|^2 dz d\bar{z}$$

$$+\mathcal{F}[B,R]$$

…plus a functional of local densities of the external fields

$$\mathcal{Z}[W] = \int |F(\{z_i\})|^2 \prod_{i=1}^{N} e^{W(z_i,\bar{z}_i)} d^2 z_i \qquad \frac{\delta \log \mathcal{Z}}{\delta W(\mathbf{r})} = \langle \sqrt{g}\rho(\mathbf{r})\rangle$$

$$W = Q + \log\sqrt{g} \qquad \frac{i}{2}\partial_z W = A_z + \omega_z \qquad \partial_{\bar{z}}\omega_z - \partial_z\omega_{\bar{z}} = -\frac{i}{4}R\sqrt{g}$$

Generalized Generating Functional

Applying the lessons learned from Bradlyn & Read 2015, Klevtsov & Ferrari 2014, Klevtsov & Wiegmann 2015, Klevtsov, Marinescu, Ma, and Wiegmann 2015

Generating Functional

For other (conformal block) FQH states, expect this to generalize

$$\log \mathcal{Z}[A,\omega] = \frac{2\nu}{\pi}\int |A_z + \bar{s}\omega_z|^2 dz d\bar{z} - \frac{c}{6\pi}\int |\omega_z|^2 dz d\bar{z}$$

$$+\mathcal{F}[B,R]$$

…plus a functional of local densities of the external fields

$\bar{s} = \mathcal{S}/2$ "orbital spin density", aka "shift"

c gravitational anomaly, aka "chiral central charge"

Generalized Density

Now we can take this hypothesized generalization, and take the functional derivative to find the density.

Particle density

$$\langle\rho\rangle = \frac{\nu}{2\pi l^2} + \frac{1}{8\pi}R + \frac{1}{8\pi}\left(\frac{1}{12} + \frac{2-\nu^{-1}}{4}\right) l^2 \Delta R + O(l^4)$$

Appealing to anomaly structure of generating functional, we can conjecture

$$\langle\rho\rangle = \frac{\nu}{2\pi l^2} + \frac{\nu\mathcal{S}}{8\pi}R + \frac{1}{8\pi}\left(\frac{c}{12} + \frac{\nu\mathcal{S}(2-\mathcal{S})}{4}\right) l^2 \Delta R$$

$\bar{s} = \mathcal{S}/2$ “orbital spin density”, related to “shift” and odd viscosity

c gravitational anomaly, also the “chiral central charge”, related to thermal Hall conductance

Also follows from demanding local galilean invariance in effective action

Gromov & Abanov PRL 113, 266802, 2014

Density around Flux Tubes

Now I add a flux tube on the plane. For the magnetic field and the potential, I will use

$$B = B_0 - 2\pi a\,\delta(z) \qquad Q = -\frac{|z|^2}{2l^2} + a\log|z|^2 \qquad B_0 = \frac{1}{l^2}$$

Computing the density from the Ward identity, I can approach it in two ways. The simplest approach that still yields the correct result (which we know by more sound approaches) assumes the ansatz

$$\langle\rho\rangle = \frac{\nu}{2\pi}\left(B_0 - 2\pi a\delta(z)\right) + \frac{M_2}{4B_0}\Delta\delta(z) + O(B_0^{-2})$$

Plug this into the **integrated** Ward identity, and solve order by order.

$$-2\beta\int_C dz\, z\int\frac{1}{z-\xi}\left(-\frac{\bar{\xi}}{2l^2}+\frac{a}{\xi}\right)\langle\rho\rangle d^2\xi = \int_C dz\, z\left[(2-\beta)\langle\partial^2\varphi\rangle + \langle(\partial\varphi)^2\rangle_c + (\langle\partial\varphi\rangle)^2\right]$$

$$\langle\varphi\rangle = -\frac{B_0}{2}|z|^2 - a\log|z|^2 + O(l^2)$$

We mainly need the *angular* integral. It is basically there to control singular terms in expansion.

Density around Flux Tubes

Density and moments

After following this procedure, we get the asymptotic expansion

$$\langle\rho\rangle = \frac{\nu}{2\pi l^2} - a\nu\delta(z) + \frac{1}{2}\left(h_a - \nu a\right) l^2 \Delta\delta(z) + O(l^4)$$

which only makes sense in the context of moments

$$\int \left(\langle\rho\rangle - \frac{\nu}{2\pi l^2}\right) d^2 z = -a\nu$$

1st Moment - Charge

$$\int \left(\frac{r^2}{2l^2} - 1\right)\left(\langle\rho\rangle - \frac{\nu}{2\pi l^2}\right) d^2 z = h_a$$

2nd Moment - Dimension (Spin)

$$h_a = \frac{a}{2} - \frac{\nu a^2}{2}$$

Generating Functional: One Flux Tube

Since we know the density, we can write the generating functional that follows

$$\langle \rho \rangle = \frac{\nu}{2\pi l^2} - a\nu\delta(z) + \frac{1}{2}(h_a - \nu a)\, l^2 \Delta\delta(z) + O(l^4)$$

the singular terms come from the generating functional

$$\log \mathcal{Z}[w, \bar{w}] = \frac{\nu a}{2l^2} K(w, \bar{w}) + \left(\frac{\nu a^2}{2} - \frac{a}{2} \right) \log \sqrt{g(w, \bar{w})}$$

this tells us how to properly regularize the generating functional for delta functional B field. Take the result from before, and plug in B for a flux tube, you will find a term like

$$\int \int B(x) \Delta^{-1}(x, x') B(x') dV dV' \sim 4\pi^2 a_i^2 \Delta^{-1}(w_i, w_i)$$

$$\Delta^{-1} = \frac{1}{4\pi} \log |x - x'|^2$$

$$\Delta^{-1}(x, x) = -\frac{1}{4\pi} \log \sqrt{g}$$

correct generating functional comes with this **UV regularization**

Generating Functional: Many Flux Tubes

Generating Functional

$$B = B(\mathbf{r}) - 2\pi \sum_{i=1}^{n} a_i \delta(z - w_i)$$

Can be computed in a large N expansion of the correlation function, represented as a string of vertex operators. We find the result $\log \mathcal{Z}[\{w_i, \bar{w}_i\}] = \log \mathcal{Z}[A, \omega] +$

$$-\sum_{i=1}^{n} \nu a_i Q(w_i) + \sum_{i=1}^{n} \left(\frac{\nu a_i^2}{2} - \frac{a_i}{2} \right) \log \sqrt{g(w_i)} - \sum_{i<j} \frac{\nu a_i a_j}{2} \log |w_i - w_j|^2$$

if we use the hypothesized generalization of the generating functional

$$\log \mathcal{Z} = -\frac{1}{2\pi\beta} \int\int \left(B + \frac{\mathcal{S}}{4} R \right) \Delta^{-1} \left(B + \frac{\mathcal{S}}{4} R \right) + \frac{c}{96\pi} \int\int R \Delta^{-1} R$$

this becomes

$$-\sum_{i=1}^{n} \nu a_i Q_0(w_i) + \sum_{i=1}^{n} \left(\frac{\nu a_i^2}{2} - \frac{a_i \nu \mathcal{S}}{2} \right) \log \sqrt{g(w_i)} - \sum_{i<j} \frac{\nu a_i a_j}{2} \log |w_i - w_j|^2$$

Summary for Flux Tubes

Ward Identity for density around single flux tube

Showed us how to regularize generating functional for single flux tube

Which let us confidently generalize the generating functional for *many* flux tubes

and every for other FQH states

Coherent State

Coherent State - Quasi-hole

Is there an **intrinsic** interpretation of these defects in the external field? For flux tubes, this is very well known, and I'm sure many of you assumed from the beginning this is what I was doing. Indeed, the moduli space of these wave functions is just the space of all w_i, which we assume is $\mathbb{C}^n/(w_1, \dots, w_n)$

and the FQH wave function is basically being put on a punctured sphere. Thus, we have n complex moduli. We many now **postulate** a wave function which depends on the moduli w_i in the following way

$$\Psi = \mathcal{Z}^{-1/2}[\{w_i, \bar{w}_i\}]\Psi(\{z_i, \bar{z}_i\}; w_1, \dots, w_n)$$

Then we impose some consistency conditions on this. Of course, we don't need to postulate this because we know the holomorphic wave function, and consistency (in this case, single-valuedness) requires

$$\Psi(\{z_i, \bar{z}_i); w_i) = \prod_{i,j}(z_i - w_j)^{a_j}\Psi_0 \qquad a_j = 1, 2, \dots$$

Coherent State

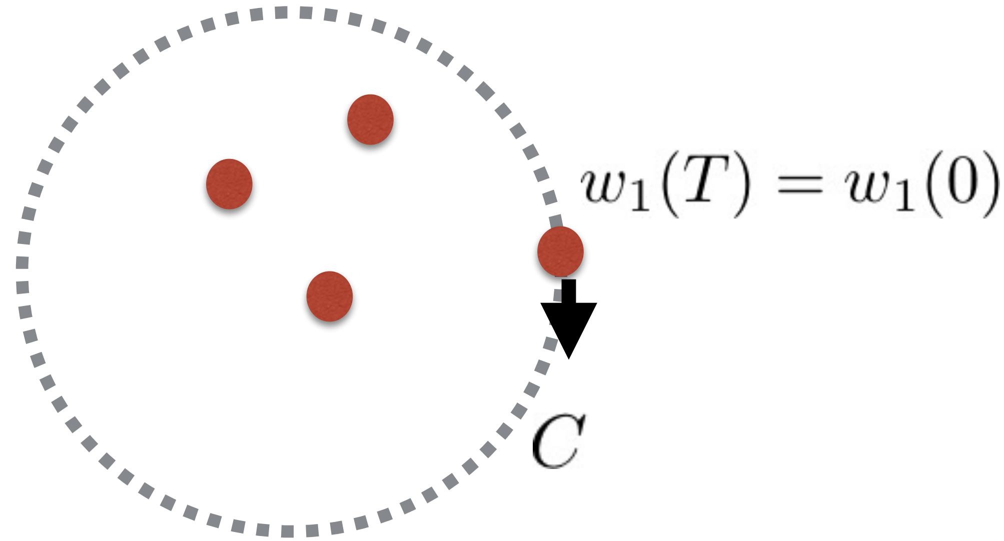
Coherent State - Quasi-hole transport and Berry Phase

In this coherent state representation of flux tubes, we may study adiabatic transport. In this case, the generating functional is the Kahler potential for the Berry curvature 2-form, specifically,

Berry curvature

$$F_{w_i,\bar{w}_i} = -i\frac{\partial}{\partial w_i}\frac{\partial}{\partial \bar{w}_i}\log \mathcal{Z}[\{w_i,\bar{w}_i\}]$$

Lévay, P. Journal Math. Phys. 1995
Avron, Seiler, Zograf PRL 1995



Berry Phase

$$\gamma = \int F_{w_i,\bar{w}_i} dw_i \wedge d\bar{w}_i$$

$$\gamma = -2\pi a_1 \nu \left(\frac{1}{2\pi}\int_C B dV\right) - h_{a_1}\frac{1}{2}\int_C R dV + 2\pi\nu a_1 \sum_{w_k \in C} a_k$$

Charge **Spin** **Mutual Statistics**

Coherent State

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Many previous workers to cite:
Charge: known to **Laughlin PRL 1983**
Statistics: computed in a similar way in **Arovas, Schrieffer, Wilczek PRL 1984**
Spin: computed on sphere w/ one quasihole in **D. Li Phys. Lett. A 1992**; apparently agrees with effective field theory calculation of **Wen & Zee PRL 1992**. and a special case of a generalized CFT results of **Kvorning PRB 2013**.
But a different interpretation is given in **N. Read arXiv:0807.3107**

$$\gamma = -2\pi a_1 \nu \left(\frac{1}{2\pi} \int_C B dV \right) - h_{a_1} \frac{1}{2} \int_C R dV + 2\pi\nu a_1 \sum_{w_k \in C} a_k$$

Charge **Spin** **Mutual Statistics**

Curvature Singularities

Starting point: flat cone, the curvature zero everywhere except the origin.

$$R = 4\pi\alpha\delta(z) \qquad ds^2 = |z|^{-2\alpha}|dz|^2 = |dw|^2 \qquad w = \frac{z^{1-\alpha}}{(1-\alpha)}$$

We can write the wave function in different coordinates:

$$\prod_{i<j}(z_i - z_j)^\beta \prod_i \exp\left(-\frac{1}{4l^2(1-\alpha)^2}|z_i|^{2-2\alpha}\right) \qquad z \in \mathbb{C}$$

In these coordinates, the **metric** completely accounts for the cone singularity through its scaling behavior near the cone point.

$$\prod_{i<j}\left(w_i^{\frac{1}{1-\alpha}} - w_j^{\frac{1}{1-\alpha}}\right)^\beta \prod_i \exp\left(-\frac{1}{4l^2}|w_i|^2\right) \qquad \begin{array}{c} w = re^{i\theta} \\ 0 \le \theta < 2\pi(1-\alpha) \end{array}$$

In these coordinates, the metric is flat, but the domain is restricted and **boundary conditions** are imposed which reproduce the cone singularity at the origin.

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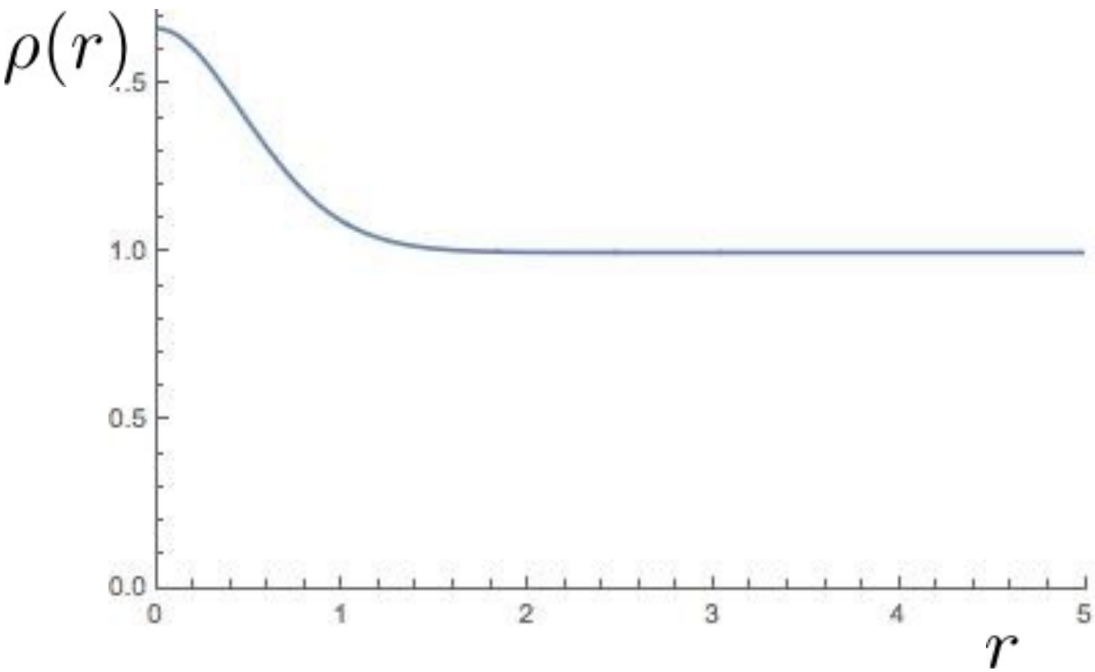
Curvature Singularities

$$R = 4\pi\alpha\delta(z) \qquad ds^2 = |z|^{-2\alpha}|dz|^2 = |dw|^2$$

Asymptotic expansion in curvature technically breaks down. More sensible to think about moments around a background.



$$\rho(r) \to \frac{\nu}{2\pi l^2}$$

$$\int f(r)\left(\rho(r) - \frac{\nu}{2\pi l^2}\right) d^2z$$

I will now do precisely what I did for the flux tube. Namely, I will assume an ansatz and solve an integrated version of the Ward identity.

Density around Cone

I will adopt the coordinates in which the metric explicitly has a conical singularity at the origin. Then I will assume that the density has the asymptotic expansion

$$R = 4\pi\alpha\delta(z) \qquad\qquad W = -\frac{|z|^{2-2\alpha}}{2l^2(1-\alpha)^2} - \alpha\log|z|^2$$

Computing the density from the Ward identity, I can approach it in two ways. The simplest approach that still yields the correct result (which we know by more sound approaches) assumes the ansatz

$$\langle\rho\rangle = \frac{\nu}{2\pi l^2} + \frac{1}{8\pi}\left(4\pi\alpha\delta(z)\right) + \frac{M_2}{4}\Delta_g\delta_g(z)$$

with a subtlety that the “second” moment is now defined with a fractional power

$$M_2 = \int \frac{|z|^{2-2\alpha}}{(1-\alpha)^2}\left(\langle\rho\rangle - \frac{\nu}{2\pi l^2}\right)\sqrt{g}d^2z$$

$$M_2 = \int_0^{2\pi(1-\alpha)} |w|^2\left(\langle\rho\rangle - \frac{\nu}{2\pi l^2}\right)d^2z$$

Density around Cone

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$$\langle\rho\rangle = \frac{\nu}{2\pi l^2} + \frac{1}{8\pi}(4\pi\alpha\delta(z)) + \frac{M_2}{4}\Delta_g\delta_g(z)$$

$$-2\beta\int_C dz\, z \int \frac{1}{z-\xi}\left(-\frac{|\xi|^{2-2\alpha}}{2l^2(1-\alpha)\xi} - \frac{\alpha}{\xi}\right)\langle\rho\rangle d^2\xi = \int_C dz\, z\left[(2-\beta)\langle\partial^2\varphi\rangle + \langle(\partial\varphi)^2\rangle_c + (\langle\partial\varphi\rangle)^2\right]$$

appeal of the cone is the this term (which give the gravitational anomaly) does not vanish!

$$\langle(\partial\varphi)^2\rangle_c = \frac{\beta}{6}\left(\partial^2\log\sqrt{g} - \frac{1}{2}(\partial\log\sqrt{g})^2\right) = \frac{\beta}{12}\frac{(2\alpha-\alpha^2)}{z^2}$$

Curvature Singularities

Density

Proceeding identically to the flux tube, we find the asymptotic expansion

$$\langle\rho\rangle_{cone} = \frac{\nu}{2\pi l^2} + \frac{\alpha}{2}\delta(z) + \frac{l^2\alpha}{2}\left(\frac{1}{2} + \frac{1-3\beta}{24}\frac{(2-\alpha)}{(1-\alpha)}\right)\Delta\delta(z)$$

Non-linear dependence on degree of singularity indicates that we could not take asymptotic expansion for smooth R, and simply substitute a delta function.

1st Moment - Charge

$$\int \left(\langle\rho\rangle - \frac{\nu}{2\pi l^2}\right) d^2r = \frac{\alpha}{2}$$

2nd Moment - Dimension (Spin)

$$\int r^2 \left(\langle\rho\rangle - \frac{\nu}{2\pi l^2}\right) d^2r = \frac{(1-3\beta)}{24}\frac{\alpha(2-\alpha)}{(1-\alpha)}$$

Curvature Singularities

Density

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$$\langle \rho \rangle_{cone} = \frac{\nu}{2\pi l^2} + \frac{\alpha}{2}\delta(z) + \frac{l^2\alpha}{2}\left(\frac{1}{2} + \frac{1-3\beta}{24}\frac{(2-\alpha)}{(1-\alpha)}\right)\Delta\delta(z)$$

Non-linear dependence on degree of singularity indicates that we could not take asymptotic expansion for smooth R, and simply substitute a delta function.

Generating Functional

$$\log \mathcal{Z} = \alpha\frac{K(0)}{2l^2} + \frac{(1-3\beta)}{24}\frac{(\alpha^2-2\alpha)}{(1-\alpha)}\log\sqrt{g(0)}$$

This is schematic, as it is rather divergent. We should rather consider the difference.

$$\log \mathcal{Z}[e^{2\sigma}\sqrt{g}] - \log \mathcal{Z}[\sqrt{g}]$$

Regularizing the Generating Functional

Structure of Generating Functional

The generating functional will mainly suffer from pathologies coming from the Polyakov Weyl anomaly term

$$\log \mathcal{Z} \propto \frac{(1-3\beta)}{96\pi} S_P[g] \qquad S_P[g] = \int\int R\Delta^{-1}R \sim (4\pi\alpha)^2 \Delta^{-1}(x,x)$$

The result for the density above tells us that we must find the difference given by

$$S_P[e^{2\sigma}g] - S_P[g] = \frac{4\pi(\alpha^2 - 2\alpha)}{(1-\alpha)} 2\sigma(0)$$

This also follows if we define this action using its relation to the determinant of the laplacian. Use this scheme in the following generalized form for other FQH states

$$\log \mathcal{Z} = -\frac{\nu}{2\pi} \int\int \left(B\Delta^{-1}B + \frac{\mathcal{S}}{2} R\Delta^{-1}B \right) + \frac{c - 3\nu\mathcal{S}^2}{96\pi} S_P[g]$$

Curvature Singularities

Density

Using the hypothesized generalization of the generating functional to other FQH states, and applying the regularization scheme we just learned about, we get for the moments

$$\langle \rho \rangle_{cone} = \frac{\nu}{2\pi l^2} + \frac{\alpha \nu \mathcal{S}}{2} \delta(z) + \frac{l^2 \alpha}{2} \left(\frac{\nu \mathcal{S}}{2} + \frac{c - 3\nu \mathcal{S}^2}{24} \frac{(2-\alpha)}{(1-\alpha)} \right) \Delta \delta(z)$$

1st Moment - Charge

$$\int \left(\langle \rho \rangle - \frac{\nu}{2\pi l^2} \right) d^2 z = \frac{\alpha \nu \mathcal{S}}{2}$$

2nd Moment - Dimension (Spin)

$$\int \left(\frac{r^2}{2l^2} - 1 \right) \left(\langle \rho \rangle - \frac{\nu}{2\pi l^2} \right) d^2 z = \frac{(c - 3\nu \mathcal{S}^2)}{24} \frac{\alpha(2-\alpha)}{(1-\alpha)}$$

Multiple Cones: Conjecture

Generating Functional

Since we now know how to regularize the generating functional, we can do two things at once: we can generalize it for other states, and write it for many cones. The simplest metric in which to write this would be the flat metric with conical singularities.

$$R = 4\pi \sum_i \alpha_i \delta(z-\zeta_i) \qquad \chi(S) - \sum_i \alpha_i = 0$$

$$ds^2 = |w'(z)|^2 |dz|^2 \qquad w'(z) = \prod_i (z-\zeta_i)^{-\alpha_i}$$

And now simply plugging this into the generating functional with the appropriate regularization, we find

$$-\sum_i \alpha_i \nu \mathcal{S} Q(\zeta_i) + \sum_i \frac{(c-3\nu\mathcal{S}^2)}{24}\frac{(\alpha_i^2-2\alpha_i)}{(1-\alpha_i)} \log\sqrt{g(\zeta_i)} - \frac{(c-3\nu\mathcal{S}^2)}{12}\sum_{i<j}\alpha_i\alpha_j \log|\zeta_i-\zeta_j|^2$$

Dimension $$h_\alpha = \frac{(c-3\nu\mathcal{S}^2)}{24}\frac{\alpha^2-2\alpha}{1-\alpha}$$

Coherent State

Coherent State

Is there an **intrinsic** interpretation of conical singularities? As with the analysis of flux tubes, I want to think about adiabatic transport in the space of cone singularities

$$\mathbb{C}^n/(\zeta_1, ..., \zeta_n)$$

This gives us n complex moduli to move around. Once again we **postulate** a coherent state wave function which depends on the cone moduli in the following way

$$\Psi = \frac{1}{\sqrt{\mathcal{Z}[\{\zeta_i, \bar{\zeta}_i\}]}} \Psi(\{w_i, \bar{w}_i\}; \zeta_1, ..., \zeta_n)$$

Consistency conditions are much trickier now. We may appeal to the single cone problem. If the w coordinate is allowed to roam through the entire plane, then the wave function is only consistent

$$\frac{1}{1-\alpha_i} = 1, 2, ...$$

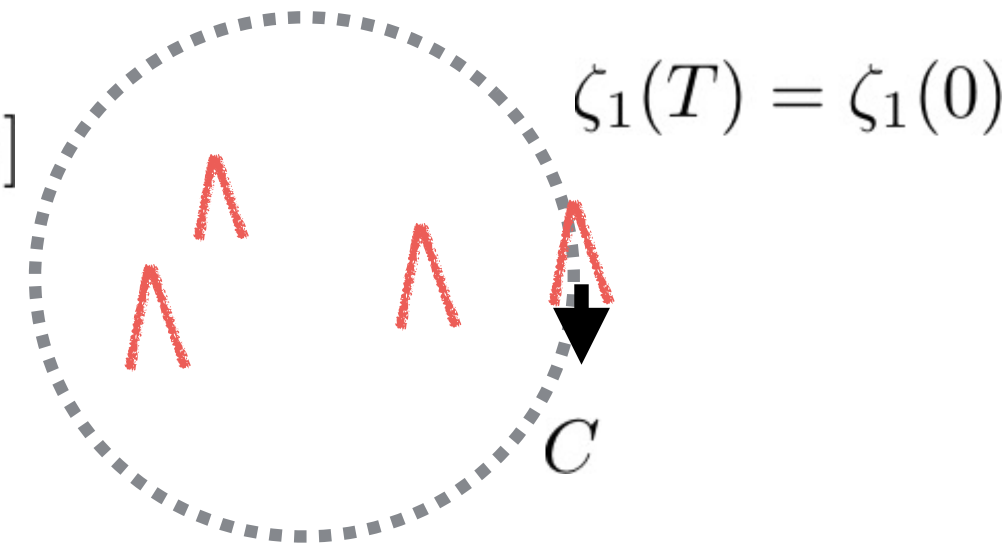
Coherent State

Coherent State - Cone transport and Berry Phase

In this coherent state representation of cones, we may study adiabatic transport. In this case, the generating functional is the Kahler potential for the Berry curvature 2-form, specifically,

Berry curvature

$$F_{\zeta_i,\bar{\zeta}_i} = -i\frac{\partial}{\partial\zeta_i}\frac{\partial}{\partial\bar{\zeta}_i}\log\mathcal{Z}[\{\zeta_i,\bar{\zeta}_i\}]$$



Berry Phase

$$\gamma = \int F_{\zeta_i,\bar{\zeta}_i}\, d\zeta_i \wedge d\bar{\zeta}_i$$

$$\gamma = -2\pi\nu\mathcal{S}\alpha_1\left(\frac{1}{2\pi}\int_C B\,dV\right) - h_{\alpha_1}\frac{1}{2}\int_C R\,dV + 2\pi\frac{(c-3\nu\mathcal{S}^2)}{6}\alpha_1\sum_{\zeta_k\in C}\alpha_k$$

Charge **Spin** **Mutual Statistics**