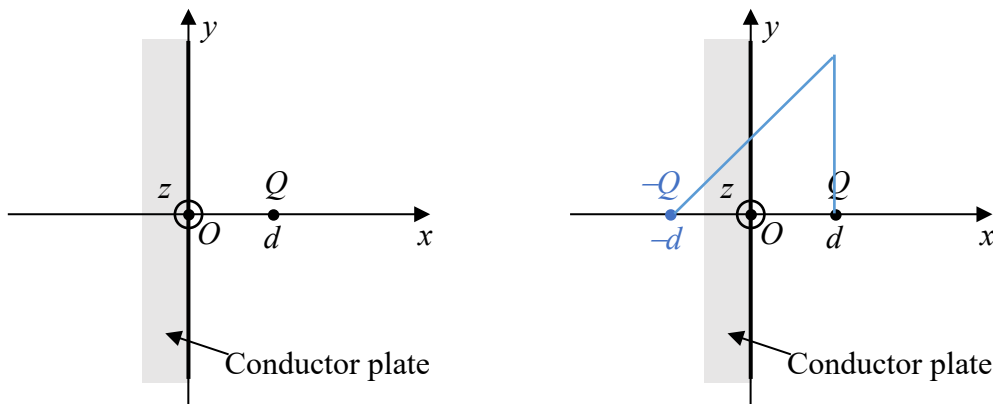


**Problem 1. Electrstatics: Coulomb's law & image method (24 points total)**

One surface of an infinitely large ideal conductor plate is at the plane  $x = 0$  of the Cartesian coordinate system, with the  $x$ - $y$  plane being the plane of the paper and the  $z$  axis represented by the dot, as shown in the figure. The conductor plate is grounded (i.e. at a potential 0). A positive point charge  $Q$  is located at  $(d, 0, 0)$ . Assuming free space (i.e. vacuum, with permittivity  $\epsilon_0$ ), find the following:

- (1) The electric field and potential anywhere with  $x < 0$ , and
- (2) The electric field and potential at point  $(d, 2d, 0)$ . **Note:** The field is a vector. You may express the field in the form of  $E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$ , or express its magnitude **and** indicate its direction verbally or graphically in the figure.



**Solutions**

(1) For  $x < 0$ ,  $\mathbf{E} = 0$  and  $V = 0$ .

(8 points: 4 for field and 4 for potential)

(2) Image charge  $-Q$  at  $(-d, 0, 0)$ .

2

The field due to charge  $Q$  is  $\vec{E}_1 = \hat{y} E_{1y}$

$$E_{1y} = \frac{1}{4\pi\epsilon_0} \frac{Q}{(2d)^2} = \frac{1}{4\pi\epsilon_0} \frac{Q}{4d^2}$$

2

Let the field due to the image charge be  $\mathbf{E}_2$ .

$$|\vec{E}_2| = E_2 = \frac{1}{4\pi\epsilon_0} \frac{Q}{(2\sqrt{2}d)^2} = \frac{1}{4\pi\epsilon_0} \frac{Q}{8d^2}$$

2

$$E_{2x} = -\frac{1}{\sqrt{2}} E_2 = -\frac{1}{4\pi\epsilon_0} \frac{Q}{8\sqrt{2}d^2}$$

1

$$E_{2y} = -\frac{1}{\sqrt{2}} E_2 = -\frac{1}{4\pi\epsilon_0} \frac{Q}{8\sqrt{2}d^2}$$

1

The total field  $\mathbf{E} = E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}}$

$$E_x = E_{2x} = \frac{-Q}{32\pi\epsilon_0 d^2} \cdot \frac{1}{\sqrt{2}} \quad 2$$

$$E_y = E_{1y} + E_{2y} = \frac{Q}{4\pi\epsilon_0 d^2} \left( \frac{1}{4} - \frac{1}{8\sqrt{2}} \right) = \frac{Q}{32\pi\epsilon_0 d^2} \left( 2 - \frac{1}{\sqrt{2}} \right) \quad 2$$

$$\therefore \vec{\mathbf{E}} = \frac{Q}{32\pi\epsilon_0 d^2} \left[ -\hat{\mathbf{x}} \frac{1}{\sqrt{2}} + \hat{\mathbf{y}} \left( 2 - \frac{1}{\sqrt{2}} \right) \right] \quad 2$$

The potential  $V$  at  $(d, 2d, 0)$  is the sum of the potentials due to  $Q$  and its image:

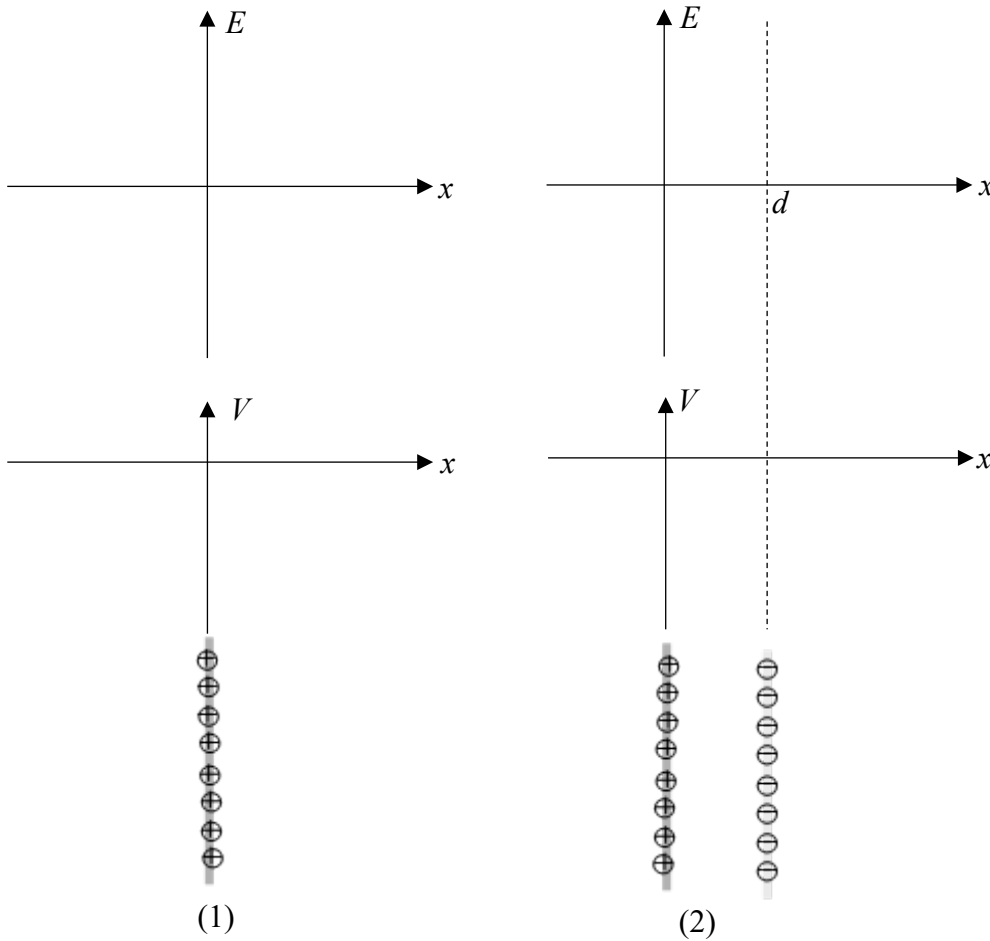
$$V = \frac{Q}{4\pi\epsilon_0} \cdot \frac{1}{2d} - \frac{Q}{4\pi\epsilon_0} \frac{1}{2\sqrt{2}d} \quad 2$$

Part (2) weighs **16** points. Listed in the right column are partial credit to be given if student does not get the final answer (for  $\mathbf{E}$ ) correctly. A student who gets the final answer correctly gets all 14 points for  $\mathbf{E}$ ; it is not required to follow the above procedure. A student who gets to a certain intermediate answer above correctly receives the partial credit for answers that are prerequisite, even if not explicitly writing out those. When grading, please keep in mind that there can be numerous equivalent correct expressions.

**Problem 2. Electrostatics: Gauss's law, potential, and capacitors (34 points total)**

(1) A uniform, two-dimensional (meaning zero thickness), infinitely large charge sheet is at the plane  $x = 0$ . The surface charge density is  $\rho_S$ . Express the field  $\mathbf{E}(x)$  and potential  $V(x)$  for all  $x$ . Select  $V(0) = 0$  as the reference potential. Assuming  $\rho_S$  is positive, schematically plot  $E(x)$  and  $V(x)$ . **Note:** Express your results in the notations defined here. Copying equations from elsewhere without adapting to the notations defined here will not earn points. Same for below.

(2) Now, another uniform, two-dimensional, infinitely large charge sheet is added at  $x = d$ , with a surface charge density  $-\rho_S$ . Express the field  $\mathbf{E}(x)$  and potential  $V(x)$  for all  $x$ . Again, take  $V(0) = 0$ . Assuming  $\rho_S$  is positive, schematically plot  $E(x)$  and  $V(x)$ .



**Solutions**

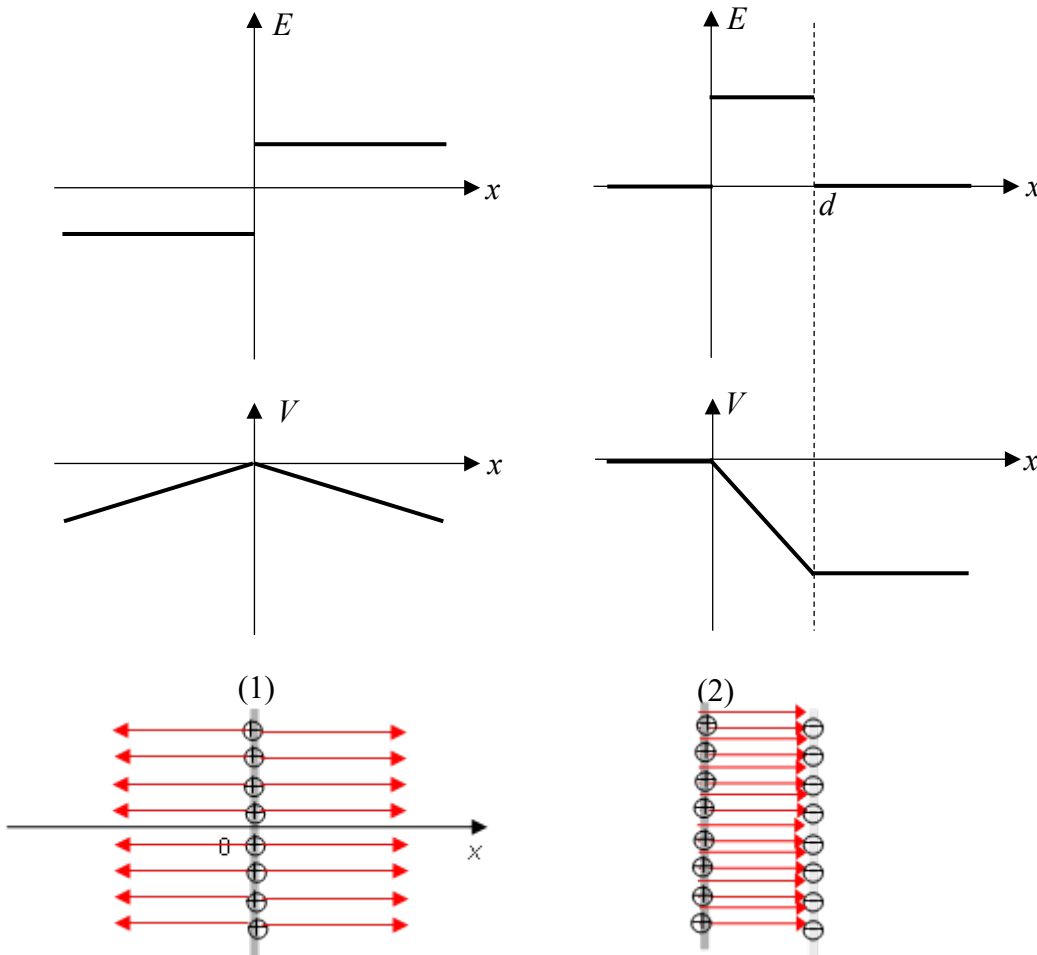
(1) 10 points for part (1).

$$\mathbf{E}(x) = \frac{\rho_S}{2\epsilon_0} \text{sgn}(x) \hat{\mathbf{x}}$$

3 points for **E**. These 3 points are for getting the correct  $E$  value. Evaluate the answer along with the sketches. A student who fail to mathematically express the correct answer but nonetheless sketches correctly is to receive the full three points. A student who understands that  $|E| = \frac{\rho_S}{2\epsilon_0}$  and  $E$  is in opposite directions on the two sides of the sheet also receives the full 3 points; mistakes in directions will be penalized in the sketch. Notice that there are equivalent correct answers, e.g., in a piecewise form instead of the sgn function.

$$V(x) = -\frac{\rho_S|x|}{2\epsilon_0}$$

3 points for  $V$ . Similar to the above for **E**, a student who gets the correct absolute value of the slopes of  $V(x)$  and understands that the slopes are opposite in sign on two sides receives the full 3 points; mistakes in other aspects are penalized in the sketches.



2 points for sketching  $E$ , and 2 for  $V$ . The field line sketch is not required, but a correct one earns 1 point if student does not get the  $E$  and  $V$  sketches correctly.

(2) For  $x < 0$ ,  $E(x) = 0$  and  $V(x) = 0$ .

For  $0 < x < d$ ,  $\mathbf{E}(x) = \frac{\rho S}{\epsilon_0} \hat{\mathbf{x}}$  and  $V(x) = -\frac{\rho S}{\epsilon_0} x$ .

For  $x > 0$ ,  $E(x) = 0$  and  $V(x) = -\frac{\rho S}{\epsilon_0} d$ .

10 points for part (2). Grading guideline same as (1).

**Problem 2. (continued)**

(3) While the two charge sheets in (2) approximately models the parallel-plate capacitor, a practical capacitor has a plate area  $A$ . The approximation is good when all lateral dimensions of the capacitor is significantly larger than the distance between the plates. In this approximation, we assume the field distribution at an edge of the capacitor is as shown in the left figure below, but the actual distribution is as shown in the right figure (the fringe effect). The approximate model on the left actually violates some fundamental principle of electrostatics. What fundamental principle does it violate? Explain. **Hint:** You may view the edge (side surface) of the capacitor as a “boundary” and see if the methods we used to arrive at the electric field boundary conditions help answer the question.

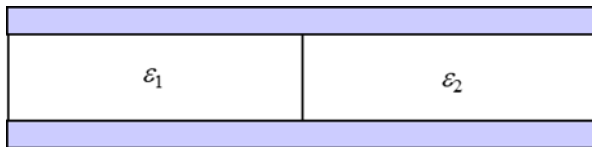


**Solutions**

(3) For a rectangle loop shown in Fig. (2) Left, with length  $l$  and width  $0$ ,  $\oint \mathbf{E} \cdot d\mathbf{l} = El \neq 0$ . The conservativity of the electrostatic field (a special case of Faraday’s law) is violated. (3 points)



(4) As shown in the figure below, a parallel-plate capacitor of plate distance  $d$  and area  $A$  is filled with two dielectrics, with permittivities  $\epsilon_1$  and  $\epsilon_2$ , each in half of the plate area. Find the fields  $E_1$  and  $E_2$  in the two dielectrics if the voltage between the plates is  $V$ , ignoring the fringe effect.



**Solutions**

(4)  $E_1 = E_2 = V/d$ . (3 points)

**Problem 2. (continued)**

(5) Find the charge  $Q_1$  and  $Q_2$  of the two parts of the capacitor.

**Solutions**

(5) The surface charge density of the two parts are

$$\rho_{s1} = \epsilon_1 E_1 = \epsilon_1 \frac{V}{d}, \quad \rho_{s2} = \epsilon_2 E_2 = \epsilon_2 \frac{V}{d}$$

Therefore,

$$Q_1 = \rho_{s1} \frac{A}{2} = \epsilon_1 \frac{VA}{2d}, \quad Q_2 = \rho_{s2} \frac{A}{2} = \epsilon_2 \frac{VA}{2d}.$$

(3 points. Student receives a partial credit of 2 points if getting charge density right but missing factor of  $\frac{1}{2}$  for the area)

(6) Find the total energy stored in this capacitor (including both parts) when charged to voltage  $V$ .

**Solutions**

(6) The total stored energy is

$$W = \frac{1}{2} C_1 V^2 + \frac{1}{2} C_2 V^2 = \frac{1}{2} \frac{\epsilon_1 + \epsilon_2}{2} \frac{A}{d} V^2 = \frac{\epsilon_1 + \epsilon_2}{4} \frac{A}{d} V^2$$

or

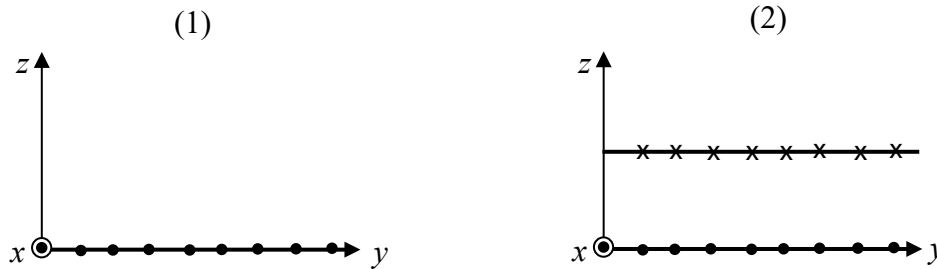
$$W = \frac{1}{2} (Q_1 + Q_2) V = (\epsilon_1 + \epsilon_2) \frac{A}{4d} V^2.$$

(3 points. Student receives a partial credit of 2 points if getting charge density right but missing factor of  $\frac{1}{2}$  for the area. Student receives a partial credit of 1 point if presenting the first half of either of above equations)

**Problem 3. Magnetostatics: Magnetic field due to dc current (30 points)**

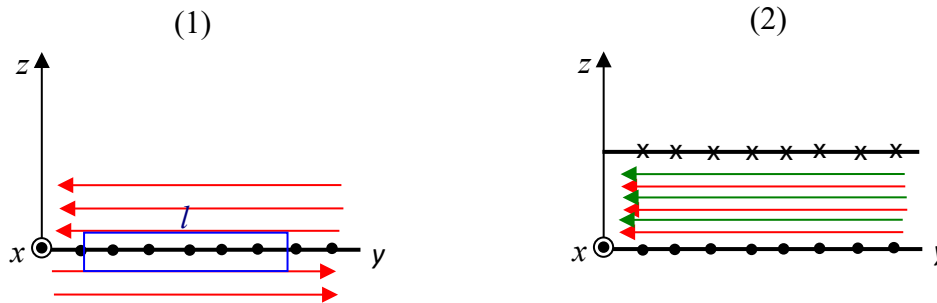
An infinitely large conductor sheet is at the  $z = 0$  plane, carrying a surface current density  $\mathbf{J} = J\hat{x}$ , as shown in Fig. (1).

- (1) What is the unit of  $J$ ? Find the magnetic field distribution  $\mathbf{B}(z)$  in the entire space (for  $z > 0$  and  $z < 0$ ). Assuming  $J > 0$ , draw the magnetic field lines (indicating directions with arrows) in Fig. (1).
- (2) Now, another conductor sheet, carrying a surface current density  $\mathbf{J} = -J\hat{x}$ , is added at  $z = d$ , as shown in Fig. (2). Find the magnetic field distribution  $\mathbf{B}(z)$  in the entire space. Assuming  $J > 0$ , draw the magnetic field lines (indicating directions with arrows) in Fig. (2).
- (3) Do the two sheets in Fig. (2) attract or repel each other? What is the force per unit area between the two sheets?



**Solutions**

- (1) The unit of  $J$  is A/m.  
 Draw a rectangular loop with length  $l$  and width  $w$  (which is infinitesimally small) as shown.  
 $2lB = \mu J l$ .  
 Therefore,  $B = \mu J/2$ . Directions shown in figure.



Part (1) weighs 10 points. 4 points for the unit of  $J$ , 3 for the  $B$  value, and 3 for the directions (0 out of the 3 points if depicting  $B$  in same direction on two sides of current sheet).

- (2) For  $z < 0$  and  $z > d$ ,  
 $B = 0$ .



For  $0 < z < d$ ,

$B = \mu J$ . Direction shown in Figure.

Part (2) weighs 10 points. 4 points for correctly getting  $B = 0$  for  $z < 0$  and  $z > d$ , 3 for  $B$  value between the sheets, and 3 for the directions.

(3) They repel each other.

The upper sheet is in the field  $B = \mu J/2$ , with the direction shown by red arrows, due to the lower sheet.

For a rectangle in the upper sheet, with a width (perpendicular to  $\mathbf{J}$ )  $w$ , and a length (along the direction of  $\mathbf{J}$ )  $l$ , the force is

$$F = IlB = JwlB = Jwl\mu J/2 = \mu J^2 wl/2.$$

Therefore, the force per area is

$$F = \mu J^2/2$$

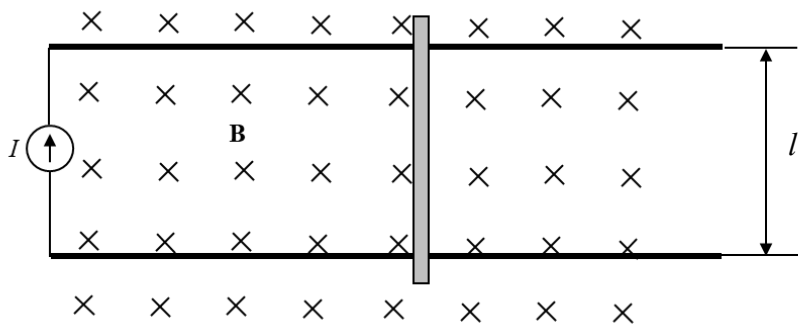
Part (3) weighs 10 points. 4 points for answering “repel”, 6 for getting the correct force.

Deduct 2 points for algebraic error if student demonstrates understanding of the concept (one sheet feels the force in the field of the other sheet, rather than the total field due to two sheets – this is the key concept being tested!)

**Problem 4. Magnetostatics: Rail motor (3 points + 10 bonus points)**

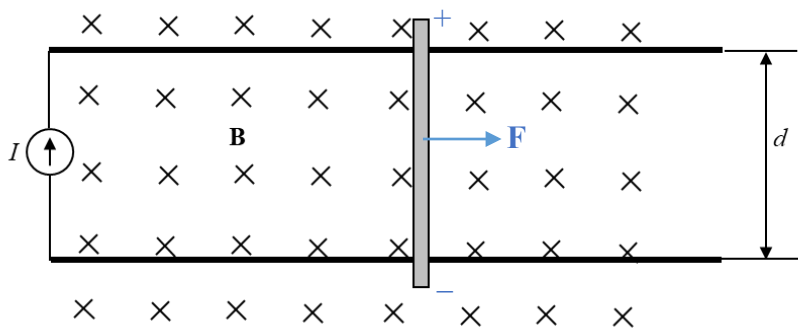
See the figure below. In a uniform magnetic field  $\mathbf{B}$  (shown by crosses, going into the paper), a metal bar (considered a perfect conductor) is placed on a pair of parallel metal rails (also perfect conductors) a distance  $l$  apart. An ideal current source  $I$  is connected to the rails.

- (1) Find the force  $\mathbf{F}$  exerted by the magnetic field on the bar in terms of  $\mathbf{B}$ ,  $I$ , and  $l$ . Note:  $\mathbf{F}$  is a vector. You need to specify its direction. You may indicate the direction on the figure.
- (2) **Optional, for bonus points.** Driven by  $\mathbf{F}$ , the bar slides, and it in turn drives a mechanical load. At the steady state, the bar moves at a constant velocity  $\mathbf{v}$ , in the same direction as  $\mathbf{F}$ . If we measure the voltage between the rails, what value will we get? Express your answers in terms of  $\mathbf{v}$ ,  $\mathbf{B}$ ,  $I$ , and  $l$ . Indicate the polarity of the voltage in the figure.
- (3) **Optional, for bonus points.** What is the electric power fed by the current source to the moving bar? Prove that energy is conserved and that the magnetic field does not do work.



**Solutions**

- (1)  $F = IBl$ . Direction of  $\mathbf{F}$  shown in figure. (3 points: 1.5 for value, 1.5 for direction)



- (2)  $V = \text{emf} = vBl$ . Polarity of  $V$  shown in figure. (3 bonus points: 1.5 for value, 1.5 for polarity)
- (3) The electric power is  $IV = IvBl$ .

With a finite  $\mathbf{B}$ , the bar moving at velocity  $\mathbf{v}$  pushes the load, doing work  $\mathbf{F} \cdot \mathbf{v} = Fv = IBlv$ .

Any mobile charge  $q$  in the bar, moving along with the bar at  $\mathbf{v}$ , experiences a force  $q\mathbf{v}\times\mathbf{B}$ , giving rise to the emf  $vBl$ . To counter this emf, the electric power  $IvBl$  must be done to push current  $I$  into the bar.

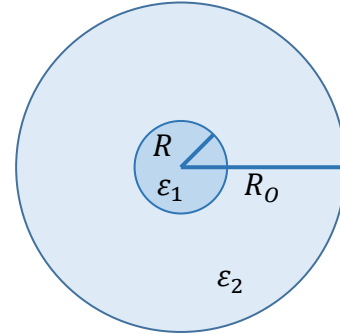
The electric power fed into the bar equals the mechanical power that the bar exerts to push the load. Therefore, energy is conserved and the net work done by magnetic forces is 0.

**(7 bonus points)**

**Problem 5. Electrstatics: Charge and Gauss's law (9 points + 40 bonus points)**

A sphere with radius  $R_0$  is charged with a spherically symmetric charge density  $\rho(r)$ , where  $r$  is the distance measured from the center of the sphere.

(1) Write an expression of the charge within a sphere of radius  $r$  and concentric to the charged sphere. Write an expression of the total charge of the charged sphere. **Note:** Since the form of  $\rho(r)$  is not provided here, there is no need to solve any integrals you may have in the expressions.



**Solutions**

(1) **Part (1) weighs 6 points.**

The charge inside a sphere of radius  $r$  is

$$\int \rho dV = \int_0^r \int_0^\pi \int_0^{2\pi} r' \sin \theta d\varphi r' d\theta \rho(r') dr' = 2\pi \int_0^r \int_0^\pi \sin \theta d\theta r'^2 \rho(r') dr'$$
$$= \int_0^r 4\pi r'^2 \rho(r') dr'$$

(3 points. No penalty for using  $r$  in both integrant and integral limit. No need to go thru these steps, which are written out here only for partial credits if final result not obtained. 1 point for writing out the first step; 2 points if either the 2<sup>nd</sup> or 3<sup>rd</sup> step expressions are obtained regardless whether the first step is present.)

The total charge inside the sphere of radius  $R_0$  is  $\int_0^{R_0} 4\pi r'^2 \rho(r') dr'$ . (3 points)

The charge sphere has a concentric spherical core with radius  $R$ . The core material has a permittivity  $\epsilon_1$  and the outer part material has a permittivity  $\epsilon_2$ . The charge density is

$$\rho(r) = K \left[ \frac{2R}{r} \left( 1 - e^{-\frac{3r}{R}} \right) + 3e^{-\frac{3r}{R}} \right],$$

where  $K$  is a proportional constant.

(2) What is the unit of  $K$ ?

**Solutions**

(2) The unit is  $C/m^3$ . (3 points. The answer “charge per volume”, although being the dimension rather than unit, is also acceptable and earns the full 3 points)

### Problem 5. (continued)

(3) **Optional, for bonus points.** Using the expression you derived in (1) to find the charge within a sphere of radius  $r$  and concentric to the charged sphere. Hint: If you find the following mathematical formulae (where  $C$  is an integral constant) useful, you are on the right track:

$$\int x^2 e^{-3x} dx = -\frac{1}{3} x^2 e^{-3x} + \frac{2}{3} \int x e^{-3x} dx$$
$$\int x e^{-3x} dx = -\frac{1+3x}{9} e^{-3x} + C$$

Typo found in original  
Test 2 posted in Canvas.  
Corrected here in red.

### Solutions

(3) From part (1), the charge inside a sphere of radius  $r$  is  $\int_0^r 4\pi r'^2 \rho(r') dr'$ . Let  $x = \frac{r'}{R}$ . Then the charge is

$$\int_0^{\frac{r}{R}} 4\pi R^2 x^2 K \left[ \frac{2}{x} (1 - e^{-3x}) + 3e^{-3x} \right] d(Rx) = 4\pi R^3 K \int_0^{\frac{r}{R}} x^2 \left[ \frac{2}{x} (1 - e^{-3x}) + 3e^{-3x} \right] dx$$
$$= 4\pi R^3 K \int_0^{\frac{r}{R}} (2x - 2xe^{-3x} + 3x^2 e^{-3x}) dx$$

Use the integral  $\int x^2 e^{-3x} dx$  given above, we have:

$$\int (2x - 2xe^{-3x} + 3x^2 e^{-3x}) dx = \int 2x dx - 2 \int x e^{-3x} dx - x^2 e^{-3x} + 2 \int x e^{-3x} dx$$
$$= x^2 (1 - e^{-3x})$$

(The second integral in the mathematical hint is not necessary as it cancels.)

Therefore, the charge inside a sphere of radius  $r$  is

$$4\pi R^3 K x^2 (1 - e^{-3x}) \Big|_0^{\frac{r}{R}} = 4\pi R^3 K \left( \frac{r}{R} \right)^2 (1 - e^{-\frac{3r}{R}})$$

**Note:**

The unfortunate typo (corrected in red font in this answer sheet) leads to the wrong integral:  $\int (2x - 2xe^{-3x} + 3x^2 e^{-3x}) dx = x^2 - e^{-3x}$ , and thus the wrong answer:

$$4\pi R^3 K (x^2 - e^{-3x}) \Big|_0^{\frac{r}{R}} = 4\pi R^3 K \left[ \left( \frac{r}{R} \right)^2 - e^{-\frac{3r}{R}} \right]$$

This “correct wrong answer” earns full points.

8 bonus points

### Problem 5. (continued)

(4) **Optional, for bonus points.** Find the electric displacement field  $D(r)$  at distance  $r$  from the center of the charged sphere. **Note:** Considering  $R_0 \gg R$ , you may assume  $R_0 \rightarrow \infty$  therefore no need to consider  $r > R_0$ . You **do need to consider both**  $r < R$  and  $r > R$ , however.

(5) **Optional, for bonus points.** Find the electric field strength  $E(r)$  at distance  $r$  from the center of the charged sphere. **Note:** Considering  $R_0 \gg R$ , you may assume  $R_0 \rightarrow \infty$  therefore no need to consider  $r > R_0$ . You **do need to consider both**  $r < R$  and  $r > R$ , however.

### Solutions

(4) By Gauss's law, we have for both  $r < R$  and  $r > R$ :

$$4\pi r^2 D(r) = 4\pi R^3 K \left(\frac{r}{R}\right)^2 \left(1 - e^{-\frac{3r}{R}}\right) = 4\pi r^2 RK \left(1 - e^{-\frac{3r}{R}}\right)$$

Therefore,

$$D(r) = RK \left(1 - e^{-\frac{3r}{R}}\right)$$

Sanity check by dimensions:  $K$  is of the dimension of charge density, i.e., charge per volume. Therefore,  $RK$  is of the dimension of charge per area, the same for  $D(r)$ .

**Note:**

The unfortunate typo (corrected in **red** font in this answer sheet) leads to the wrong answer:

$$D(r) = RK \left[1 - \frac{1}{\left(\frac{r}{R}\right)^2} e^{-\frac{3r}{R}}\right]$$

The wrong answer resembles the right one for  $r > R/2$ , with large deviation for small  $r$ . It is unphysical that  $D(r) \rightarrow -\infty$  as  $r \rightarrow 0$ . Fortunately, the value  $D(R)$  and the trend round  $r = R$ , as you sketch it in part (6), are not altered from the right answer.

**10 bonus points**

(5) For  $r < R$ ,

$$E(r) = \frac{D(r)}{\epsilon_1} = \frac{RK}{\epsilon_1} \left(1 - e^{-\frac{3r}{R}}\right)$$

For  $r > R$ ,

$$E(r) = \frac{D(r)}{\epsilon_2} = \frac{RK}{\epsilon_2} \left(1 - e^{-\frac{3r}{R}}\right)$$

8 bonus points

Note:

The unfortunate typo (corrected in red font in this answer sheet) leads to wrong answers:

For  $r < R$ ,

$$E(r) = \frac{D(r)}{\varepsilon_1} = \frac{RK}{\varepsilon_1} \left[ 1 - \frac{1}{\left(\frac{r}{R}\right)^2} e^{-\frac{3r}{R}} \right]$$

For  $r > R$ ,

$$E(r) = \frac{D(r)}{\varepsilon_2} = \frac{RK}{\varepsilon_2} \left[ 1 - \frac{1}{\left(\frac{r}{R}\right)^2} e^{-\frac{3r}{R}} \right]$$

The wrong answer resembles the right one for  $r > R/2$ , with large deviation for small  $r$ . It is unphysical that  $D(r) \rightarrow -\infty$  as  $r \rightarrow 0$ . Fortunately, the value  $D(R)$  and the trend round  $r = R$ , as you sketch it in part (6), are not altered from the right answer.

Again, any “correct wrong answers” to Parts (4) and (5) using and consistent with the “correct wrong answer” to Part (3) earn full points.

Incorrect answers based on and consistent with a student’s own result of Part (3) earn partial credit.

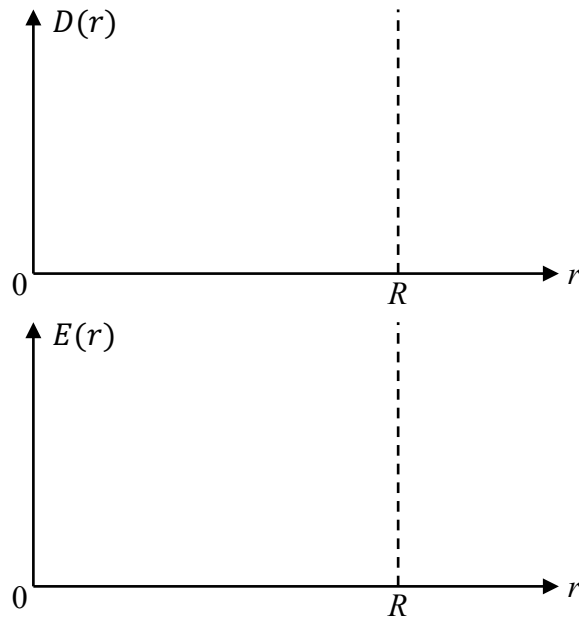
**Problem 5. (continued)**

(6) **Optional, for bonus points.** Sketch  $D(r)$  and  $E(r)$  in the figures to the right.

6 bonus points

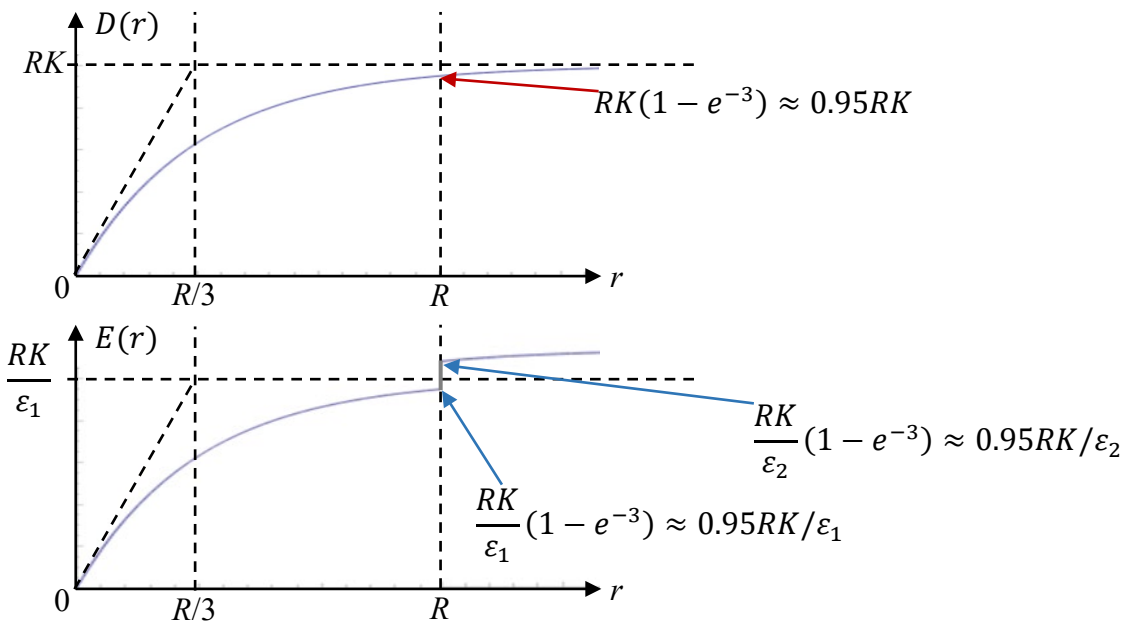
(7) **Optional, for bonus points.** Between  $D(r)$  and  $E(r)$ , which one is discontinuous at  $r = R$ ? What is the discontinuity due to?

8 bonus points



**Solutions**

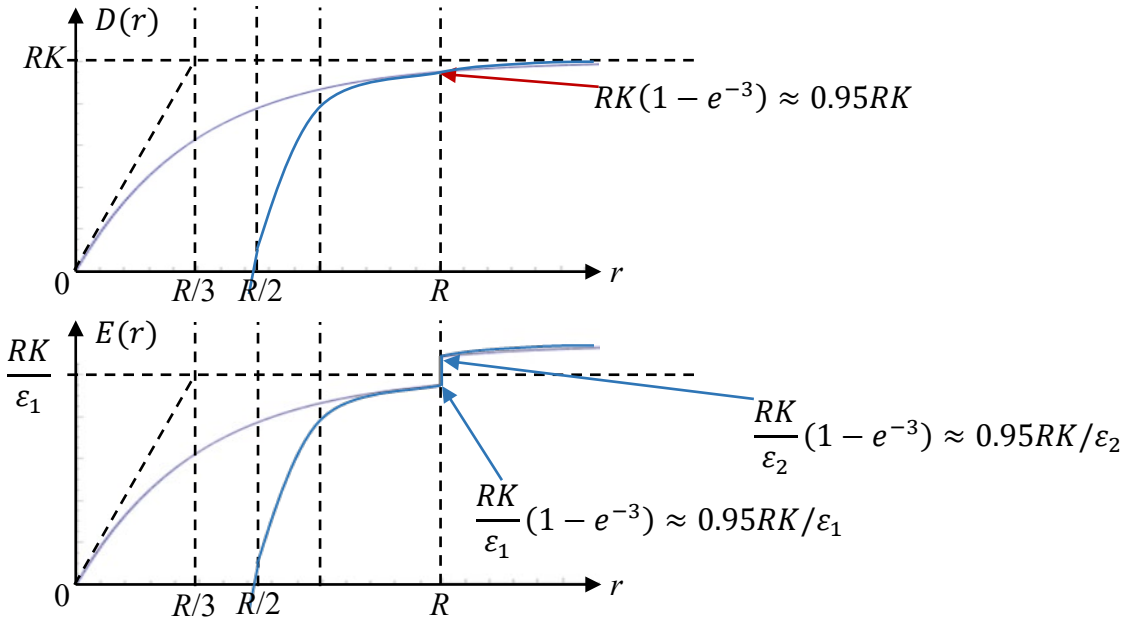
(6) Assuming  $\epsilon_2 < \epsilon_1$ ,



**Note:**

The unfortunate typo (corrected in **red** font in this answer sheet) leads to wrong answer for  $D(r)$  and  $E(r)$ , which are sketched below with **blue curves**, along with the right answers for comparison. You can see that the wrong answers resemble the right ones closely for  $r > 2R/3$  and that the discontinuity at  $r = R$  is unaltered by the mistake. The discontinuity is robust, unaltered by the mathematical mistake, because of the robust physical mechanism behind it, namely the interface polarization charge.





(7)  $E(r)$  is discontinuous at  $r = R$ , due to the surface polarization charge density on the sphere of radius  $R$ , i.e., the interface between the two materials of different dielectric constants.

Additions details not required to earn points (you are **strongly encouraged** to read and think about the following and then make sure you are able to apply the principles to much simpler charge distributions such as a point charge or a uniformly charged sphere):

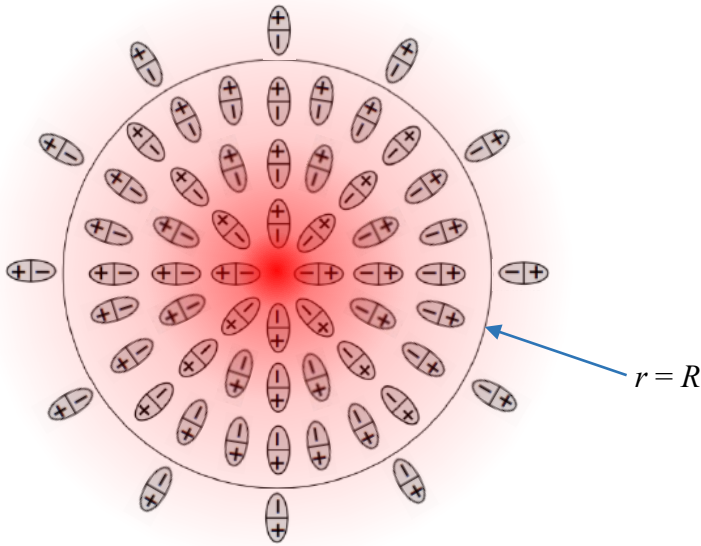
Unlike the simple parallel-plate geometry we discussed in detail in class, there is a volume polarization charge density distribution due to polarization  $\mathbf{P}(r) = P(r)\hat{\mathbf{r}}$ . For both  $r < R$  and  $r > R$ ,  $\mathbf{P}(r)$  is continuous and the volume polarization charge density  $\rho_p = -\nabla \cdot \mathbf{P}$ . At the interface where  $r = R$ , however,  $\mathbf{P}$  is discontinuous.

Define  $P_1 = P(r = R^-)$  and  $P_2 = P(r = R^+)$ .

$$P_1 = \chi_1 \epsilon_0 E(R^-) = \chi_1 \epsilon_0 \frac{D(R)}{\epsilon_1} = \frac{\epsilon_{r1} - 1}{\epsilon_{r1}} D(R) = \left(1 - \frac{1}{\epsilon_{r1}}\right) D(R)$$

$$P_2 = \chi_2 \epsilon_0 E(R^+) = \chi_2 \epsilon_0 \frac{D(R)}{\epsilon_2} = \frac{\epsilon_{r2} - 1}{\epsilon_{r2}} D(R) = \left(1 - \frac{1}{\epsilon_{r2}}\right) D(R)$$

Assuming  $\epsilon_2 < \epsilon_1$ , we have  $P_1 > P_2$ , resulting in a surface polarization charge density  $\rho_{sP} = \rho_{sP1} - \rho_{sP2} = P_1 - P_2$ , which can be visualized by the figure below.



$$\rho_{SP} = P_1 - P_2 = \left( \frac{1}{\epsilon_{r2}} - \frac{1}{\epsilon_{r1}} \right) D(R)$$

Notice that the total field  $E(r)$  is contributed by the external charge density  $\rho(r)$ , the volume polarization charge density that can be calculated by  $\rho_p = -\nabla \cdot \mathbf{P}$ , and the interface polarization charge described by surface density  $\rho_{SP}$ .

The first part is  $D(R)/\epsilon_0$ . The second part can be calculated from  $\rho_{SP}$  by using Gauss's law (**caution**: the permittivity of free space,  $\epsilon_0$ , is to be used).

Now, we use Gauss's law to calculate the third part, namely the field  $E_{SP}$  due to the polarization charge shell:  $4\pi R^2 \rho_{SP} = 4\pi r^2 \epsilon_0 E_{SP}$ , for  $r > R$ . Therefore,

$$E_{SP}(r) = \left( \frac{R}{r} \right)^2 \frac{1}{\epsilon_0} \rho_{SP} = \left( \frac{R}{r} \right)^2 \frac{1}{\epsilon_0} \left( \frac{1}{\epsilon_{r2}} - \frac{1}{\epsilon_{r1}} \right) D(R) = \left( \frac{R}{r} \right)^2 \left( \frac{1}{\epsilon_2} - \frac{1}{\epsilon_1} \right) D(R)$$

At the interface, where  $r = R$ , we have

$$E_{SP}(R) = \left( \frac{1}{\epsilon_2} - \frac{1}{\epsilon_1} \right) D(R)$$

Recall that  $E(R^-) = D(R)/\epsilon_1$  and  $E(R^+) = D(R)/\epsilon_2$ . You then immediately see that the jump in the E field at  $r = R$  is due to the shell of polarization charge:

$$E(R^+) - E(R^-) = \left( \frac{1}{\epsilon_2} - \frac{1}{\epsilon_1} \right) D(R) = E_{SP}(R)$$