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Problem Posing in Mathematics Classrooms

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"Everybody knows what a problem is."

-Student S, Ms. Gold's 5th-grade class, August 30, 2013

The aim of problem posing in a mathematics class is to provide students with experiences where they may question the constraints and assumptions provided in a given scenario, identify problematic features, carefully formulate a problem, and begin searching for potential solution methods or solutions. The generation of new problems and the reformulation of existing problems is an important component of mathematical proficiency. In the seminal publication, *Adding It Up*, the authors defined strategic competence as "the ability to formulate mathematical problems, represent them, and solve them" (Kilpatrick, Swafford, & Findell, 2001, p. 124); we view problem posing as one of the means by which students develop strategic competence. Ideally, school mathematics should provide fertile ground for students to engage in problem-posing activities.

Organizations such as the National Council of Teachers of Mathematics [NCTM] in the US, the Ministry of Education of Italy (2007), the Ministry of Education of the Peoples' Republic of China (NCSM, 2001), and *The National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991, p. 39) recognize the importance of problem posing

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stating that students should experience the opportunity to "formulate interesting problems based on a wide variety of situations, both within and outside mathematics" (NCTM, 2000, p. 258). Although there are numerous advocates in favor of problem posing, we wondered about the educational benefits of problem posing. In a well-known quote, Einstein and Infeld (1938) claimed that identifying a new problem or asking the right question is a key feature of scientific thought as new problems are the mechanisms by which science is propelled forward:

> The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skills. To raise new questions, new possibilities, to regard old questions from a new angle, requires creative imagination and marks real advance in science. (p. 92)

Silver (1994) argued that engaging in problem posing activities had the potential to improve students' problem solving abilities, improve students' dispositions towards mathematics, and reveal students' mathematical understandings (see also Kilpatrick, 1987). Kilpatrick, Swafford, and Findell (2001) also identified problem posing as a feature of the strategic competence strand of mathematical proficiency. They defined strategic competence as the "ability to *formulate* (emphasis added), represent, and solve mathematical problems" (p. 5).

What We Know From the Literature About Problem Posing

Despite the potential benefits, the literature base concerning problem posing in mathematics is relatively small. Singer, Ellerton and Cai (2013a) observed, "the topic of posing problems has largely remained outside the vision and interest of the mathematics education community" (p. 2). And according to Olson and Knott (2013), "Literature on problem-posing episodes is limited" (p. 27). Most of the literature that does exist is more than 20 years old, the exception being a special issue of *Educational Studies in Mathematics* (2013b) and the book

Mathematical Problem Posing: From Research to Effective Practice (Singer, Ellerton, & Cai, 2015).

In general, the literature on problem posing falls into three categories: studies of the interaction between problem posing and problem solving, studies of the relationship between problem posing and students' creativity, and studies of problem posing as a pedagogical tool for inquiry-oriented instruction. Studies of the interaction between problem posing and problem solving typically aimed to determine how the two might be linked. A subset of these studies examined the types of problems posed by students in order to see what, if anything, might be inferred about students' problem solving abilities based on the problems they posed. Silver (2013) observed that progress investigating this relationship "…has been stymied by the lack of an explicit, theoretically based explanation of the relationship between problem posing and problem solving that is consistent with existing evidence and that could be tested in new investigations" (p. 160). A second category of research on problem posing is comprised of studies concerning the interaction between creativity and problem posing. This literature contains both theoretical and empirical studies that argue problem-posing activities can stimulate creative mathematics thinking (e.g., Bonotto, 2013). Some of the older literature in this category suggest a relationship between creative thought and problem posing or problem finding in disciplines outside of mathematics (Einstein & Infeld, 1939; Getzels & Csikszentmihalyi, 1976; Silver, 1994). And, finally, studies from the third category focus on the use of problem posing as a form of instruction or assessment. For example, Tichá and Hošpesová (2013) found problem posing is an appropriate way to introduce prospective elementary school teachers to the teaching of mathematics. In another example, Contreras (2009) identified problem posing as a viable differentiation technique; that is, he identified problem posing as a means to vary instruction to meet the needs of students with diverse learning needs.

Within mathematics education, there is a need to better understand the kinds of problem posing experiences students have in order to determine in what ways these practices benefit students and to identify instructional practices that encourage student problem posing. The literature is largely silent in this area. One exception is a recent paper by Da Ponte and Henriques (2013) who observed problem posing by mathematics students in a university numerical analysis course. They observed that "…few studies have examined the cognitive processes involved when students generate their own problems in the course of their problem solving activity or about instructional strategies that can effectively promote problem posing" (p. 147) Thus, the present study is important because it helps to paint a picture of the extent to which problem posing occurs in certain K-12 contexts, and it reveals environments or situations that may engender problem posing.

Problem Posing Exemplified and Defined

We defined problem posing as the generation of new problems or reformulation of existing problems, following both Silver's (1994) and Duncker's (1945) earlier work. Because we believe problem-posing experiences are beneficial for students and their teachers, we were particularly interested in the nature and types of experiences *students* have to engage in problem posing in middle grades mathematics classrooms. To better explain our definition of the term problem posing, consider the following middle grades problem-posing activity and three examples of student responses and solutions.

Write a real-life problem for the expression $\frac{1}{2} - \frac{1}{3}$.

Example 1: Johnny has ½ of a pizza. Brett ate $\frac{1}{3}$ of Johnny's portion of pizza. How much pizza does Johnny have left? Student answer: $\frac{2}{3}$ pizza left

Example 2: Ms. B has half of a whole cookie. She eats one-third of the whole cookie. How much cookie does Ms. B have left? Student answer: $\frac{1}{6}$ of the whole cookie

Example 3: Amy and Emily each have their own pie. Amy ate one-half of her pie and Emily ate one-third of her pie. How much more pie did Amy eat than Emily? Student answer: $\frac{1}{6}$

Many students, when posing problems for the expression $\frac{1}{2} - \frac{1}{3}$, wrote a story similar to Example 1. But the real-life context in this example is more appropriately modeled with the expression $\frac{1}{2} - \left(\frac{1}{3} \cdot \frac{1}{2}\right)$. Responses like the "pizza problem" and the resultant answer of "two-thirds of a pizza" revealed that some students were struggling to define fractional amounts in comparison to the appropriate whole or unit. Johnny and Brett's amounts of pizza have different referents, whereas one-third and one-half in the original expression refer to the same unit. In contrast consider examples 2 and 3 which both identify a common referent or unit for the fractional amounts (the whole cookie and a whole pie). A critical understanding for children to develop with respect to fractions is that a fractional amount is defined in comparison to the whole or unit, and that the size or magnitude of a fraction is relative to the size of the unit (Barnett-Clarke, Fisher, Marks, & Ross, 2010). Students who generated problems like Example 1 still have work to do to develop this understanding.

As students engaged in this problem-posing activity, they had opportunities to interpret fractional amounts, to conceptualize the operation of subtraction, and to exemplify and connect fractions and operations to contexts as well as written symbolic notation. We believe these are rich learning opportunities for students as well as teachers. Moreover, this type of mathematical activity is consistent with the admonition from NCTM's publication *Principles to Actions* (2014) that encourages teachers ask students to create situations that can be modeled with various expressions, such as $6 \div \frac{3}{4}$. Such prompts may provide insight "…to assess students' conceptual understanding and reasoning" (NCTM, 2014, p. 93).

We wanted to develop a framework to identify key characteristics of problem posing in the examples just discussed. Our problem-posing framework contains three related components: the type of mathematical problem, the problem-posing structure with reference to the catalyst, and the problem-posing type (i.e., problem reformulation or new problem). Though a detailed discussion of the construct of problem is beyond the scope of this paper, we use Schoenfeld's (1992) three categories of problems: routine exercises, traditional problems, and problems that are problematic to characterize the kind of problems with which students are engaged during problem posing episodes (pp. 338–340). *Routine exercises* occur when students *practice* some specific mathematical skill, technique, or algorithm that they have already been shown or have developed expertise with. *Traditional problems* are tasks that students perform as a means to a focused end. Traditional problems are often the "word problems" found in traditional mathematics textbooks. Such problems are often ones for which the problem solver does not have a pre-determined solution path. P*roblems that are problematic* are "problems of the perplexing kind" (p. 338). School mathematics does not often expose students to problem of this type, perhaps because the teacher may not know the solution to this type of problem. The following are examples of problems that are problematic: "What mathematics is involved in determining how to tackle the BP Oil Spill in the Gulf of Mexico?," or, "How might we use mathematics to try to locate the missing Malaysia Airlines flight MH370?" Such problems may also be more mathematical-theoretical, such as the “four-color problem” (now known as the four-color theorem). We realize that others might not consider the categories of routine problem and traditional problems as bona fide problems. Because we suspected the category of *problems that are problematic* to occur rarely in school mathematics, we included all three categories as ‘problems’. For this study, we used Schoenfeld’s three categories of problems as a means to classify the kinds of mathematics problems that students are asked to pose in mathematics classes.

Additionally, we used Stoyanova and Ellerton’s (1996) classifications of problem posing situations—structured, semi-structured, and free—to describe the nature of the constraints in the problem-posing episodes. Stoyanova and Ellerton’s classifications have been used in studies in which problem posing is an explicit goal or method of instruction. Through the course of observing lessons we encountered instances of problem posing that occurred organically; that is, they were initiated by students rather than teachers. Because Stoyanova and Ellerton’s classifications grew out of an expectation to see problem posing, we determined that their classifications are most useful when the teacher is the catalyst for problem posing episodes but are difficult to apply in situations where students initiate problem posing during the course of a mathematical task or discussion. As a result, we modified their classification to identify the catalyst of the problem-posing episode. Any episode initiated by a student was coded as “student.” Instances of problem posing initiated by the teacher were coded using Stoyanova and Ellerton’s classifications of structured, semi-structured, or free. We refer to this category of four codes as problem-posing structure with reference to catalyst.

In the course of the study we also developed a new descriptive category, problem-posing type (our third and final component in the framework). This category distinguishes between problem reformulations and the creation of new problems. We describe the problem-posing type category in more detail in the methods section. One of the goals of this study is to describe the frequency, quality, and kinds of opportunities students have to engage in the mathematical practice of problem posing in school settings using these categories. In the following section, we describe the participants, data, and analysis methods.

Methods

In this observational study we analyzed mathematics instruction in six middle grades classrooms to identify key aspects of mathematical problem posing. The research questions guiding our study were: How often does problem posing occur in six middle grades mathematics classrooms and what types of problem posing occur? In this paper, we report the frequency and types of problem posing in which these teachers engaged and share potential implications for instruction.

Participants and Data

Data include video recordings and transcripts of 88 mathematics lessons taught during the 2013-2014 school year. We worked with six teachers in grades 5–7, filming lessons they identified as providing good opportunities for problem solving and student discussion. Table 1 provides information about the teachers, grade levels taught, and the number of lessons we observed in their classrooms. Because this study is part of a larger NSF-funded research project investigating differences in mathematical discourse across classrooms, the potential for problem posing was not one of the criteria for lesson selection. In addition, teachers were selected without reference to their experience (or lack of experience) with problem posing. Thus, observations in these classrooms provided a unique opportunity to determine how often problem posing occurs without outside researcher influence. Lessons focused on three different content areas—fractions, algebraic reasoning, and integer operations.

Table 1: *Counts of Lessons Observed by Teacher and Topic*

Grade	Teacher	Location	Algebra (# of lessons)	Integers (# of lessons)	Fractions (# of lessons)	Total # of Lessons
5	Mr. Blue	Western US	5	0	8	13
5	Ms. Gold	Southern US	7	0	7	14
5	Ms. Violet	Southern US	4	0	10	14
6	Ms. Green	Southern US	6	7	5	18
6	Ms. White	Western US	5	3	4	12
7	Ms. Lavender	Southern US	6	5	6	17

Data Analysis

Analysis began by identifying all problem-posing episodes in the data corpus, which required multiple passes through the data. Problem-posing episodes are characterized by student generation of a mathematical problem either at the teacher's request or as a result of a student's curiosity or questioning. Most teacher-initiated instances of problem posing were clearly identifiable because the teacher asked students to create problems. However, student-initiated instances of problem posing took a variety of forms; at times, students questioned the constraints of the given problem suggesting a modification and, at other times, students generated mathematical conjectures. (Note that we did not consider student questions to be instances of problem posing.) In order to account for these student-initiated instances, as mentioned earlier we added a category to Stoyanova and Ellerton's (1996) framework. Using Schoenfeld's (1992) problem framework and our modification of Stoyanova and Ellerton's problem-posing framework, each problem-posing episode was *initially* characterized by (a) problem type (routine exercise, traditional problem, and problems that are problematic), (b) problem-posing structure with reference to catalyst (student-initiated, teacher-initiated free, teacher-initiated semi-structured, and teacher-initiated structured), and (c) the mathematical topic addressed. Multiple revisions to the coding framework were made over time as we independently considered instances of problem posing and resolved disagreements about their coding. Additionally, because of the importance in the literature of problem reformulation[1] as a key form of problem posing, we decided to report the distinction between *reformulation of given problems* and the creation of *new problems* in each of the problem-posing episodes. This is reflected in our problem-posing framework with the third category of problem-posing type. Final coding involved assigning four codes to each problem-posing episode—3 codes specific to problem-posing and one code that identify the mathematical topic. The first author completed all final coding.

Results

In this section we discuss findings from our analysis of problem posing for the six participating teachers. We first share two brief transcript excerpts to illustrate what typical problem-posing episodes looked like "in action" and then share key characteristics and the frequency of problem posing in our sample of teachers.

Problem-Posing Examples

The first except is from a sixth-grade classroom during a unit introducing integers.

> Ms. Green: Alright, so what I want you to do on your paper, okay? You are gonna come up with a real-life problem that we might see that has to, that deals with integers. We

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[1] Problem reformulation refers to changing a given in the problem and identifying the resultant consequences. A common example is asking "what-if-not" (Brown & Walter, 2005) questions in given problems. Brown and Walter coined the phrase "what-if-not" to refer to challenging the givens in a posed problem. Any modification of given information in a problem constitutes the reformulation of a previous problem by asking "what-if-not?" For example, applying the "what-if-not" strategy to Euclid's Parallel Postulate formed non-Euclidean geometries. Additional examples of this strategy may be found throughout Brown and Walter's book, *The Art of Problem Posing*.

could, it could deal with integers, it could deal with absolute values, um, it could deal with opposites; anything like that, I want you to think about and you think about that in your group and come up with an idea to, to test your fellow classmates…

In this problem-posing episode, students were asked to create a context and corresponding problem within which an integer-related problem could be posed to a classmate. We categorized this as a semi-structured episode involving a traditional problem with the teacher as the catalyst. This instance was not problem reformulation, but involved the creation of a 'new' problem for the students. One interesting aspect of the above example is that one group of students created a problem requiring integer subtraction—a topic that was scheduled later in the unit. In a follow-up interview, the teacher specifically identified this problem her students created as a "set up" or a "reference" problem later when 'officially' discussing integer subtraction.

The second excerpt is from Ms. Gold's fifth-grade classroom where students had been working on fraction division with unit fractions *before* being shown an algorithm. Students had spent most of the class period solving and discussing solutions to the measurement division problem, $5 \div \frac{1}{3}$, situated within the following context: Mac has 5 cups of dog food left. If he feeds his dog, Nick, 1/3 cup a day, how many days will it be when Nick runs out of food? One student observed that the solution of 15 could be obtained if you "switch it [the one-third] around" and multiply it by 5. Another student, Jason, responded to her observation:

Jason: But what if it was like two thirds. What would you do?...

Ms. Gold: If what was two thirds?

Jason: Um – if – if that were to be two thirds.

Ms. Gold: If that. What's that?

Jason: If the one third were – were like two thirds what would you do? Like how would you do that?

Ms. Gold: Alright. So I have a question (for the whole class). What if the servings was two thirds? … So now we don't have – we have five cups. But, Jason wants to change this to two thirds. Alright. How many two thirds are there in five?

Jason: [overtalk] Would it be thirty?

Ms. Gold: [overtalk] are there in five?

Jason: Would it be thirty? Would that be thirty?

Ms. Gold: Look, I don't know. You see. Can you prove it? What would that be if two-thirds was the serving?

Ms. Gold initially posed a scenario asking for the number of groups of size 1/3 that are in 5 wholes. During the discussion, Jason asked a what-if-not question (Brown & Walter, 2005), to consider what would happen if the divisor was changed from 1/3 to 2/3. Jason's question was a problem reformulation that Ms. Gold capitalized on by asking the entire class to solve the problem Jason posed: Mac has 5 cups of dog food left. If he feeds his dog, Nick, 2/3 cup a day, how many days will it be when Nick runs out of food? It is possible that Ms. Gold intended to eventually modify the divisor later, but, in this instance, she leveraged Jason's question to redirect the class's activity.

In this example, it is noteworthy that the student initiated the problem-posing episode. Though not all of the students in Ms. Gold's class engaged in problem posing, one student reformulated the given problem and all students were then invited to consider how changing the initial constraints might change the solution. We categorized this excerpt as a student-generated, traditional problem involving a problem reformulation. This example illustrates a student-initiated episode of problem posing based on changing a constraint (problem reformulation) in the original problem, which was a traditional problem. It also illustrates how the teacher was responsive to the student's mathematical idea and used it as the basis for the rest of the lesson.

In the next section we discuss descriptive data about the frequency and categories of problem posing occurring within each classroom.

Description of Problem-Posing Episodes

Across the 6 teachers, we filmed 88 mathematics lessons and observed 24 distinct instances of problem posing. Table 2 summarizes key characteristics of the 24 instances of problem posing across all teachers. Problem-posing episodes are characterized by (a) problem-posing type (reformulation of a given problem or new/novel problem) (b) problem type (routine exercise, traditional problem, or problem that is problematic), and (c) problem-posing structure with reference to catalyst (student-initiated, structured, semi-structured, or free). Table 2 also includes the total number of problem posing episodes by teacher.

Table 2: *Characteristics of Problem-Posing Episodes Across Teachers*

Episode	PP Type		Problem Type			PP Structure with Reference to Catalyst				Number of PP Episodes
	Reformu-lation	New	Rout	Trad	Problematic	Student	Teacher			
						Student	Struct	Semi-St.	Free	
Ms. Green	2	9	6	5	0	3	2	6	0	11
Ms. Gold	2	1	2	1	0	3	0	0	0	3
Ms. Violet	0	1	0	1	0	0	0	1	0	1
Ms. Lavender	0	2	2	0	0	0	0	2	0	2
Ms. Blue	1	2	3	0	0	1	2	0	0	3
Ms. White	1	3	2	2	0	1	1	2	0	4
Total Across Teachers	6	18	15	9	0	8	5	11	0	24
Mean Across Teachers	1.00	3.00	2.50	1.50	0.00	1.33	0.83	1.83	0.00	4.0

Std. Dev. Across Teachers	0.89	3.03	1.97	1.87	0.00	1.37	0.98	2.23	0.00	3.58

There was large variation across teachers in our sample in terms of the frequency of problem posing (1, 3 and 11 instances and a standard deviation of 3.58 around a mean of 4). Problem posing occurred most often in Ms. Green's classroom both in terms of the number of instances (11 episodes) and in terms of her problem-posing rate (one instance every 1.6 lessons, or 11 instances in 18 lessons). At the other end of the spectrum, problem posing occurred only once in Ms. Violet's classroom for rate of one instance every 14 lessons. On average, problem posing occurred about once out of every four lessons in this data set.

Type of Problems Posed in Episodes

Across the 24 problem-posing instances in this data we did not find any situations in the data where students posed problems that were problematic. In fact, the only types of problems the students posed were routine exercises or traditional problems, with over 60% of the problems classified as routine exercises. For the routine exercises, students either created an exercise similar to a previous exercise, or they created contexts to model a *given*, routine exercise. Nine of the 15 routine exercises episodes occurred as a result of occasions where a student or group of students created a context for a routine exercise. For example, Ms. Lavender asked her students to "give me a story" for the addition of the two numbers, 5 + -8. We considered computing the sum of positive five and negative eight as a routine exercise because it was designed to "provide practice on a particular mathematical technique" (Schoenfeld, 1992, p. 337) that was recently demonstrated. In response to Ms. Lavender's request, Beth created a story involving borrowing and paying off debt:

Beth Um Jim walked um [Jim had] [um] eight dollars and he gave them- Tim gave um like uh no- Jim gave Mike eight dollars and so, Mike was in debt with eight dollars. And then he paid back five dollars to Jim so, he had three dollars left to pay back.

In contrast to the creation of a routine exercise, consider the earlier examples of traditional problems from Ms. Gold and Ms. Green's classrooms. In particular, the traditional problems from Ms. Green's class were new in that the students created them *in the absence of a given numerical expression* such as the 5 + -8 above.

Type of Problem-Posing Episodes

As Table 2 shows, teachers initiated two-thirds of the problem-posing episodes (see the structured, semi-structured, and free columns under Problem Posing Type). In these instances, the teacher specifically asked the students to engage in creating a mathematical problem. We observed both structured and semi-structured episodes of problem posing in the teacher-initiated episodes, but no instances of free problem posing. Five of the episodes were structured episodes, and 11 were semi-structured. And seven out of the 11 semi-structured problem-posing episodes comprised situations in which the teacher asked the students to contextualize (create a story or context for) a routine exercise. The example from Ms. Lavender's class in the previous section is an instance of a structured problem-posing episode whereas the first example in this section from Ms. Green's class is an instance of a semi-structured episode. Both involve the contextualization of real-life situations and appropriately representing that context symbolically; however, the difference between these episodes lies in the freedom students have. In the structured episodes, a numerical expression, algebraic expression, or algebraic equation was given and students were asked to provide contexts for the given expressions. In semi-structured episodes, students were not given any expressions but were instructed to create problems in a slightly larger context.

The eight student-initiated, problem-posing episodes appeared spontaneously within the flow of classroom instruction, and it did not appear as though the teacher had planned for them. These instances of problem posing occurred when students posed questions or questioned the constraints of the task in ways that the teacher did not appear to anticipate. Sometimes the teacher took up these unexpected ideas and questions, and other times the teacher did not. If the teacher did not pursue the question or idea we did *not* count the episode as a problem-posing episode. The identification of the problem posing catalyst is useful data for two reasons. First, as reported above, the most widely used framework to study problem posing (Stoyanova & Ellerton, 1996) does not account for such instances. Second, student-initiated problem posing episodes may help us understand how teachers respond to students' questions and intuitions about the mathematics. This type of analysis can help us gain insight into how willing a teacher is to capitalize on students' mathematical ideas.

In general, the teacher was more often the catalyst for problem posing opportunities, but we find it particularly noteworthy when students initiate problem posing episodes (as in the second example from Ms. Gold's classroom above) as these occurrences are indicators of students who engage in mathematical wondering of their own volition.

Creation of New Problems and Reformulation of Existing Problems

Because problem posing refers to the reformulation of previously given problems or the creation of new problems, we distinguished between these two types of instances of problem-posing scenarios in our codes. One out of four instances of problem posing (6 episodes) were examples of reformulations of previously given problems. Interestingly, five of the 6 problem reformulation episodes involved the student as a catalyst. Consider the measurement division example above from Ms. Gold's classroom. In this instance the student reformulated the initial problem $5 \div \frac{1}{3}$ by asking how the problem would change if the divisor were 2/3 instead of 1/3. Ms. Gold then redirected the students to consider the reformulated problem as part of their discussion.

Problem posing via reformulation is important because it can provide insight into student perceptions of the mathematics. For example, Justin hypothesized (incorrectly) that the 2/3 modification of the divisor might double the quotient. Justin's reformulation helped to reveal a misconception about the roles of the numerator and denominator when dividing with fractions.

Reformulation of problems can also provide opportunities for students to extend, refine, or deepen their current understanding about mathematics. An additional example of problem reformulation occurs when individuals "…reverse givens and unknowns in a problem situation" (NCTM, 2014, p. 92). For example, a common exercise found in introductory, secondary algebra courses is to find the unknown and compute the value of a logarithm, such as $log_2 16 = x$. Students may then use various strategies to conclude $x = 4$ and quickly move on to complete another similar exercise. A possible reformulation of the problem might look like the following: Find several logarithmic expressions of the form $log_x y$ such that $log_x y = 4$. Such a reformulation (regardless of the catalyst) may provide opportunities for students to develop more insight into the relationship between the argument of a logarithmic function and its base. This reformulation also reversed the givens and the unknowns in the original exercise. In addition, reformulation tasks may help reveal mathematical misconceptions that were not previously visible to a teacher.

The majority of the problem-posing episodes (75%) involved the creation of new problems. The most common type of new problem students generated was the contextualization of a more general mathematical idea or concept (signed numbers, fraction division) or the contextualization of a specific computation or equation. The first example from Ms. Green's classroom is an example of the creation of a new problem involving contextualization. As mentioned earlier, prompts such as the one Ms. Green posed can provide opportunities to assess students' conceptual understanding and reasoning of mathematical content (NCTM, 2014, pp. 92–93). But not all new problems posed by students involved contextualizations. Some new problems were based on student's mathematical wonderings, observations, and conjectures.

In the following example, two fifth-grade students engaged in problem posing by jointly generating a mathematical conjecture. Prior to this transcript excerpt, a student has posited that one can add denominators when adding fractions. The teacher then asked the whole class to decide in their small groups if they agreed or disagreed and why. In this exchange, one student is reasoning about a specific case which is then extended by a classmate to a more general claim about what happens to a fractional sum if one (incorrectly) adds denominators.

Ms. Gold: Jacob, you believe you add the denominators? Alright explain. …

Jacob: You could add the denominators, but the number – but the number is practically staying the same except you just wouldn't reach a whole number.

Ms. Gold: So then can you add your denominators?

Jacob: Yes.

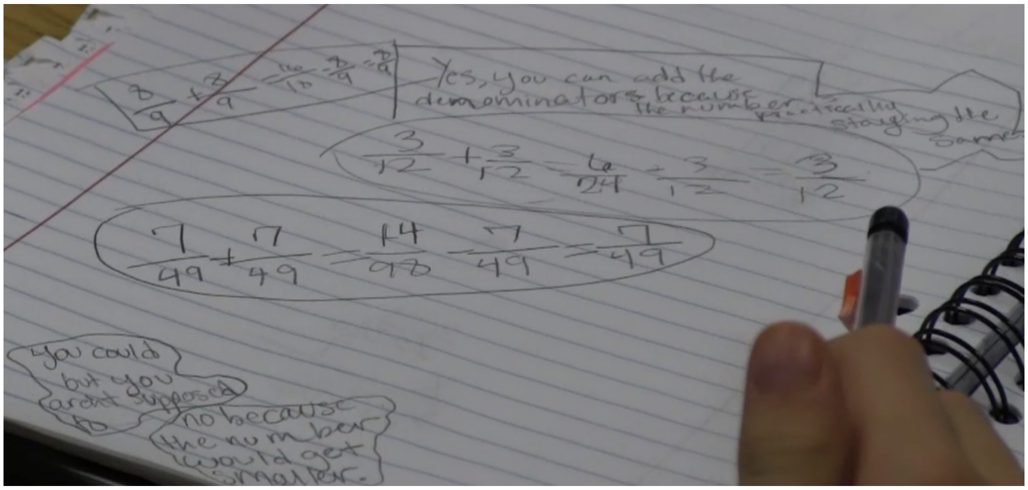
Justin: You just said why you don't add denominators.

Jacob: Well.

Justin: You would never get the whole. [overalk]

Jacob: Well you wouldn't get a whole number. [overtalk]

Figure 2. Jacob's Work



Ms. Gold:	Alright wait a minute. … Let's have 'professor Jack' [referring to Justin] over here explain what, what issues do you have with what he said?
Justin:	Well, you would always get a smaller number and you would never get a whole number.
Ms. Gold:	So as you are adding those denominators, you're getting a smaller and smaller amount right? So—and you would never reach what?
Justin:	You would never reach a whole number.

First, Jacob uses a specific case, adding 7/49 and 7/49 by adding the denominators, and noting that his resultant sum remains 7/49 (after reducing the fraction 14/98). But then Justin and the teacher refine Jacob's statement, focusing on Jacob's second observation that not only is the sum "practically the same" but that it does not approach a "whole number" like one would expect when adding fractional amounts of the same size. Justin, Jacob, and the teacher co-construct a related conjecture about the behavior of a fractional sum, ignoring specific values of the numerator: As the denominator increases the fractional amount is "smaller and smaller". Two students—Jacob and Justin—are the catalyst for this mathematical conjecture.

An instance like this is noteworthy for at least two reasons. First, this is the only example in our data set in which we observed the emergence of a conjecture, essentially from two students. Jacob and Justin's conjecture is similar to Polya's "problems to prove" (1962/1981, pp. 120–121). The question of the appropriateness of adding denominators when adding fractions required some level of proof or a counterexample. In this case, they formulate and then disprove the original conjecture—Is it true that you add denominators when adding fractions—using a proof by contradiction and creating a new conjecture of their own in the process. One might also argue that Justin utilized adaptive reasoning in his statement, "Well, you would always get a smaller number and you would never get a whole number." In other words, it is not appropriate to add denominators when adding fractions because the addition of positive fractions should result in a sum that is larger than either fraction. Furthermore, repeated addition of two such fractions should eventually yield a sum that is at least one.

In summary, many studies suggest problem posing needs additional attention, but they do not describe how much attention is required. We have also found no studies that report the prevalence of problem posing from a given sample of teachers. Researchers, teachers, and teacher educators need to know how often problem posing occurs as well as how it occurs. Such knowledge will help advance the discussion of problem posing in the field. Thus, our study makes an important contribution to the problem-posing literature.

To recap, we observed instances of problem posing in approximately one out of every four lessons. Most of the episodes were semi-structured (teacher-initiated), and most of the problems posed were routine exercises. We also found that in the majority of these semi-structured episodes, students created a story or context to model a given numerical expression, algebraic expression, or algebraic equation. Students did not have the opportunity to pose problems that were problematic, nor did they have the opportunity to engage in free problem posing (Stoyanova & Ellerton, 1996). Of the problems posed, 25% (six) were problem reformulations and 75% (18) were creations of new problems. Of the six problem reformulations, students initiated five. The remaining instance of reformulation was a (teacher-initiated) structured problem-posing task. In general, we have found that problem posing does happen in classrooms, but that there is room to expand the kinds of tasks and the amount of structure students are given.

Discussion

Our primary goal in this paper was to describe the frequency and types of problem posing activities in which middle grades mathematics students engaged. Although our focus was not to describe the characteristics of instruction that led to problem posing, we did learn about the environments in these six classrooms that appeared to support problem-posing activities.

All teachers in our study engaged in problem posing, but the frequency of problem posing varied across the six classrooms. In general, these teachers provided an expectation for students to engage in mathematical discussions during instruction. This expectation was most often evidenced by the fact that students were in pairs or small groups during lessons. It was also common for the students to present their work to the class and for other students to comment and question one another. Each of the 24 problem-posing episodes we observed occurred in a lesson that contained some type of group-work dynamic. Classrooms where mathematical discussions occur regularly may provide an environment to promote mathematical problem posing because students have opportunities to discuss their ideas with each other and question and wonder about ideas they may not yet understand.

Most of the problem-posing episodes in this study involved teacher-initiated, semi-structured, routine problems. However, the students in this study *did* initiate problem-posing episodes even when their teacher did not ask them to pose problems. It seems that each of these instances involved a student's desire to better understand the content or to satisfy his or her own mathematical curiosity. It is these instances of student-initiated problem posing that we find most noteworthy because they are instances of where students engage in doing mathematics out of their own motivation and initiative. Of the eight instances of student-initiated problem posing, three instances occur in Ms. Green's classroom and three in Ms. Gold's classroom.

The instances of problem posing in Ms. Gold's classes were unique because none of her 14 lessons contained any teacher prompts for students to create their own problems. Her students were the catalyst for *all* problem-posing episodes in her classroom, which were among the richest examples of problem posing in our data set (e.g., students making mathematical conjectures and questioning constraints of the problem). We believe that each of the three problem-posing episodes from Ms. Gold's class occurred, at least in part, because of Ms. Gold's classroom structure and expectations. She regularly modeled the questioning of constraints and she asked students to think deeply about mathematical ideas. A consistent feature of Ms. Gold's instruction was to 'unpack' problems by discussing what the problems meant and were asking. For example, after posing this problem, "If eight children shared five hamburgers equally, how many would each child get?" Ms. Gold asked the class a series of questions including the following: What is this problem about?, Is there anything unusual about it?, How are we sharing?, Does it say dividing in the problem?, What are we dividing?, What does it mean to share equally? Ms. Gold demonstrated a pattern of asking these kinds of questions about problems she posed for the students. To put it another way, she regularly questioned the constraints of problems that she presented. It is possible that her persistent questioning of the constraints of problems encouraged her students to question constraints and develop a sense of mathematical wonder and curiosity. In Ms. Gold's case, problem posing seemed to arise spontaneously from her students out of a mathematical curious environment that she had deliberately created, but she did not necessarily plan for problem posing activities.

Ms. Green, too, had three student-initiated instances of problem posing in her classroom, but that information is less surprising in light of the fact that she engaged in the most problem posing episodes overall (11 episodes out of the 24 total). Ms. Green deliberately planned for students to engage in problem posing and believed that asking students to generate examples and problems supported their mathematical learning. In her case, problem posing was designed, deliberate, and a regular classroom activity. Because of her normative practice of problem posing, it is plausible that some students appropriated this mathematical behavior and began to pose problems for themselves.

Cai et al. (2015) identified the need to carefully analyze classroom practices where problem posing occurs observing that, "few researchers have tried to carefully describe the dynamics of classroom instruction where students are engaged in problem-posing activities" (p. 34). Though our work has not fully answered this call, we have begun to paint a partial picture of the types, frequency, and circumstances surrounding problem posing in the middle grades classrooms in our study. Because our study is an observational study of only six classrooms, we are limited in the conclusions we can draw about the prevalence of problem posing beyond these classrooms or classroom that have a commitment to mathematical discussion. This is a limitation of our study, but one that we hope will encourage others to further investigate problem posing in K-12 settings with broader samples.

Implications

In closing we share two implications from this work. First, problem posing is a valuable pedagogical tool mathematics teachers can use to improve student motivation and mathematical thinking. Teachers can use problem posing to link abstract mathematics to real world phenomena and increase the relevance of mathematics to students' lived experiences. They can also ask students to create new problems or reformulate previously given problems as a means to both assess and promote conceptual understanding, procedural fluency, and, of course, strategic competence. Problem posing provides insight into how students think about mathematics by identifying important information about students' background knowledge and what students do and do not understand. As a result, it is a valuable means of formative assessment for teachers.

Our second and final implication concerns appropriate entry points for teachers who are open to problem posing but are unsure of how to incorporate it into existing pedagogical practices. Based on our findings, the use of contextualization problems and the reformulation of problems are good starting points. As mentioned above, the contextualization of routine exercises was the most common form of problem posing in this study. Contextualization episodes seemed to help students make sense of important mathematical concepts. For example, in one class, students initially (and erroneously) thought 3 divided by 4 was not equivalent to three-fourths. One student corrected her error after she assigned a context to the routine division exercise. This example is also consistent with Stoyanova and Ellerton's (1996) observation that such instances "help students learn how to generalise, as well as [to make] mathematics more meaningful to them" (p. 520). Additionally, NCTM's *Principles to Actions* (2014, p. 93) identifies the contextualization of routine exercises as a useful tool to promote conceptual understanding and reasoning. In the spirit of Brown and Walter's (2005) 'what-if-not' questions, we also encourage teachers to experiment with questioning and changing the constraints for a given problem. We believe that these are particularly useful kinds of problem-posing activities because of the potential for students themselves to initiate the questioning of constraints. Mathematics teachers can create classroom environments that encourage students to question constraints of given problems and begin to develop a sense of wonder about mathematics. Again, this suggestion is consistent with the *Principles to Actions* document that describes the reversal of givens and unknowns (NCTM, 2014).

We began this study with the aim to learn more about problem posing in mathematics classrooms. We observed problem posing in action and cataloged each instance in order to try to better understand the circumstances that might support problem posing as well as the ways in which problem posing might be useful. For teachers who have not engaged with problem posing, we hope they will give problem posing strong consideration. As Kilpatrick (1987) observed, "How to design instruction that will help students learn to formulate mathematical problems is itself a problem in need of a more complete formulation" (p. 139). Even so, teachers can begin to take incremental steps towards using problem posing more and more in their instruction of mathematics. It is our conviction that problem posing will help students engage with mathematics and develop a deeper understanding of mathematical concepts.

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