Vectors have both *magnitude* and *direction*
displacement, velocity, acceleration
Scalars have *magnitude* only
distance, speed, time, mass

Unit vectors
Specify direction only.
Used to represent a vector in terms of components.
$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$

Multiplication of Vector by Scalar
Physics application
Momentum: $\mathbf{p} = m\mathbf{v}$
Electric force: $\mathbf{F} = q\mathbf{E}$
Result
A *vector* with the same direction, a different magnitude and perhaps different units.

Multiplication of Vector by Vector (Dot Product)
$C = AB\cos\theta$
$C = A_xB_x + A_yB_y + A_zB_z$
Physics application
Work: $W = \mathbf{F} \bullet \mathbf{d}$
Result
A *scalar* with magnitude and no direction.

Multiplication of Vector by Vector (Cross Product)
$\mathbf{C} = \mathbf{A} \times \mathbf{B}$
$C = AB\sin\theta$ (magnitude)
Physics application
Work $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$
Magnetic force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$
Result
A *vector* with magnitude and a direction perpendicular to the plane established by the other two vectors.

Kinematic Equations (in 3 dimensions)
$\mathbf{v} = \mathbf{v}_o + \mathbf{a}t$
$\mathbf{r} = \mathbf{r}_o + \mathbf{v}_o t + \frac{1}{2}\mathbf{a}t^2$
$\mathbf{v} \bullet \mathbf{v} = \mathbf{v}_o \bullet \mathbf{v}_o + 2\mathbf{a} \bullet \Delta\mathbf{r}$

Projectile Motion
Horizontal velocity is constant.
$x = v_{xo}t$
Vertical velocity is accelerated at -g.
$v_y = v_o - gt$
$y = y_o + v_{yo}t - {}^1/_2 gt^2$
$v_y^2 = v_{yo}^2 - 2g(y - y_o)$
The trajectory is defined mathematically by a parabola.

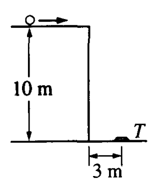
Problem: Projectile (CM-1998)

2. The velocity of a projectile at launch has a horizontal component v_h and a vertical component v_v. Air resistance is negligible. When the projectile is at the highest point of its trajectory, which of the following show the vertical and horizontal components of its velocity and the vertical component of its acceleration?

	Vertical Velocity	*Horizontal Velocity*	*Vertical Acceleration*
(A)	v_v	v_h	0
(B)	v_v	0	0
(C)	0	v_h	0
(D)	0	0	g
(E)	**0**	**v_h**	**g**

Justify your answer:
The vertical velocity, v_y, is zero because the projectile is neither rising nor falling at that moment. The horizontal velocity, v_h, doesn't change throughout the trajectory. Gravity, g, is always on and doesn't change throughout the trajectory.

Problem: Projectile (CM-1998)



26. A target *T* lies flat on the ground 3 m from the side of a building that is 10 m tall, as shown above. A student rolls a ball off the horizontal roof of the building in the direction of the target. Air resistance is negligible. The horizontal speed with which the ball must leave the roof if it is to strike the target is most nearly

(A) 3/10 m/s (B) $\sqrt{2}$ m/s **(C) $\frac{3}{\sqrt{2}}$ m/s**

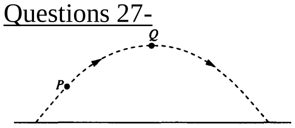
(D) 3 m/s (E) $10\sqrt{\frac{5}{3}}$ m/s

Show your work:
The time it takes to cover the 10 meters vertically is given by
$y = -\frac{1}{2}gt^2 \rightarrow -10 = -\frac{1}{2}(10\text{ m/s})t^2 \rightarrow -10 = -5t^2$
$\rightarrow 2 = t^2 \rightarrow \mathbf{\sqrt{2} = t}$

How fast horizontally does the ball need to move to cover 3 meters? $x = v_h t \rightarrow 3 = v_h\sqrt{2} \rightarrow 3 = v_h\sqrt{2} \rightarrow \mathbf{3/\sqrt{2} = v_h}$

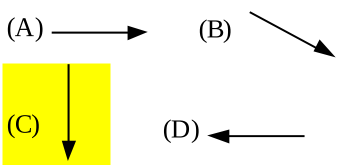
Problem: Projectile (CM-1993)

Questions 27-



A ball is thrown and follows a parabolic path, as shown above. Air friction is negligible. Point Q is the highest point on the path.

27. Which of the following best indicates the direction of the acceleration, if any, of the ball at point Q ?



(E) There is no acceleration of the ball at point Q.

Justify your answer:

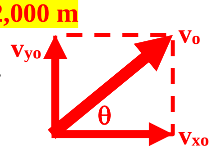
Gravity is accelerating the object downwards, is always on, and acts towards the center of mass of the earth.

Problem: Projectile (CM-1988)

10. A projectile is fired from the surface of the Earth with a speed of 200 meters per second at an angle of 30° above the horizontal. If the ground is level, what is the maximum height reached by the projectile?

(A) 5 m (B) 10 m (C) 500 m
(D) 1,000 m **(E) 2,000 m**

Show your work:



The initial **vertical velocity**, v_y, is given by:

$v_{yo} = v_o \sin\theta = (200\text{ m/s})\sin 30^o = \mathbf{200}(½)$

$v_y = 100$ m/s

At the top of the projectile's flight, its **vertical velocity** is zero.

$v_y = v_{yo} - gt$

$0 = 200\text{ m/s} - 10t$

This happens after t = 20 s:

$-200 = -10t \rightarrow 20 = t$

The **vertical height** is given by

$y = v_{yo}t - ½gt^2$

$y = 200\text{ m/s}(\mathbf{20\ s}) - ½(10)(\mathbf{20})^2$

$y = 2000$ m

Relative Motion

Usually requires vector addition.

You may make any observer the "stationary" observer.

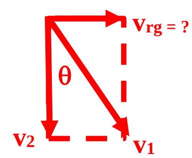
Problem: Relative Motion (CM-1993)

3. At a particular instant, a stationary observer on the ground sees a package falling with speed v_1 at an angle to the vertical. To a pilot flying horizontally at constant speed relative to the ground, the package appears to be falling vertically with a speed v_2 at that instant. What is the speed of the pilot relative to the ground?

(A) $v_1 + v_2$ (B) $v_1 - v_2$ (C) $v_2 - v_1$

(D) $\sqrt{v_1^2 - v_2^2}$ (E) $\sqrt{v_1^2 + v_2^2}$

Show your work:



The velocity of the pilot relative to the ground is given by the **Pythagorean Theorem.**

$a^2 + b^2 = c^2$

$v_{rg}^2 + v_2^2 = v_1^2$

$v_{rg}^2 = v_1^2 - v_2^2$

$\mathbf{v_{rg} = \sqrt{v_1^2 - v_2^2}}$

Problem: Relative Motion (CM-1988)

6. Two people are in a boat that is capable of a maximum speed of 5 kilometers per hour in still water, and wish to cross a river 1 kilometer wide to a point directly across from their starting point. If the speed of the water in the river is 5 kilometers per hour, how much time is required for the crossing?

(A) 0.05 hr (B) 0.1 hr (C) 1 hr (D) 10 hr

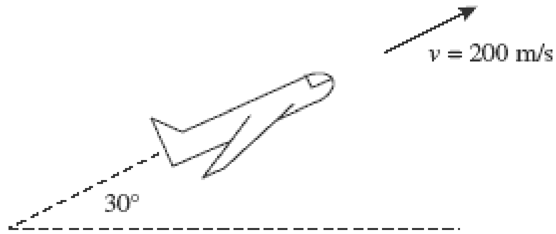
(E) The point directly across from the starting point cannot be reached under these conditions.

Show your work:

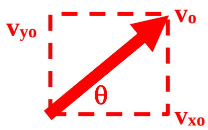
The point cannot be reached because if the river is moving at 5 km/hr and the maximum speed of the boat is 5 km/hr, then even if he went straight across, he would end up down river of his starting point.

FREE RESPONSE 1

An airplane attempts to drop a bomb on a target. When the bomb is released, the plane is flying upward at an angle of 30° above the horizontal at a speed of 200 m/s, as shown below. At the point of release, the plane's altitude is 2.0 km. The bomb hits the target.



a. Determine the magnitude and direction of the vertical component of the bomb's velocity at the point of release.

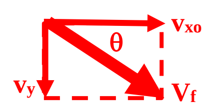


Ans. The **vertical** velocity $v_{yo} = v_o \sin\theta$ = 200 m/s sin 30° = 200(½) = **100 m/s**
This comes from **sin θ =** $\frac{v_{yo}}{v_o}$**.**

b. Determine the magnitude and direction of the horizontal component of the bomb's velocity at the point when the bomb contacts the target.
Ans. The **horizontal** component of the bomb's velocity will not change through the trajectory in the absence of air resistance. It will always be $v_{xo} = v_o \cos\theta$ = 200 m/s cos 30° = 200(√3/2) = **100√3 m/s**

c. Determine how much time it takes for the bomb to hit the target after it is released.
Ans. The formula for the height of the bomb at any time *t*, after it is released is given by $y = y_o + v_{yo}t - \frac{1}{2}gt^2$
$y = 2000\text{ m} + 100t - \frac{1}{2}(10)t^2 = 2000 + 10t - 5t^2$
Setting y = 0 to find and solving (using a graphing calculator) for *t* to find when the bomb hits the target.
$0 = 2000 + 10t - 5t^2$
t = 21.02 s

d. At the point of release, what angle below the horizontal does the pilot have to look in order to see the target?
Ans. This is a fancy, convoluted way of saying, what is the angle *to the horizontal* at which the bomb hits the target. Since we have v_{xo} (doesn't change) and we can find v_y, we can find θ, the angle the bomb makes when it hits the ground.

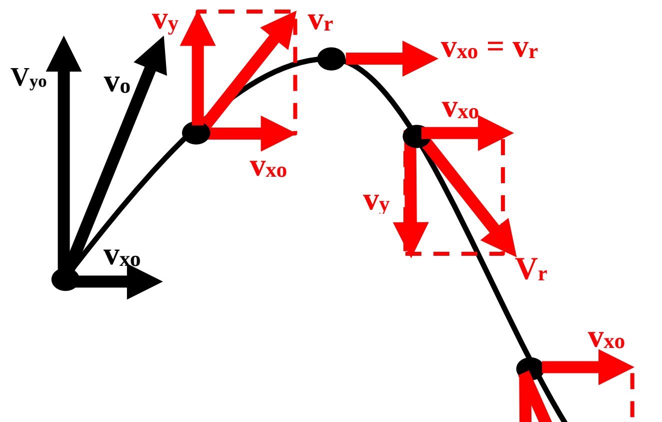


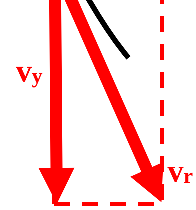
$v_y = v_{yo} - gt$
$v_y = 100 - 10(21.02\text{ s}) =$ **–110.2 m/s**
To find **θ,** we can use **tan θ =** $\frac{v_y}{v_{xo}} \rightarrow \theta = \tan^{-1}\left(\frac{v_y}{v_{xo}}\right) \rightarrow \theta = \tan^{-1}\left(\frac{100\ m/s}{100\sqrt{3}\ m/s}\right)$ = **30°**

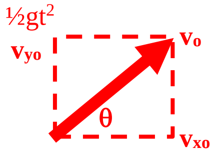
FREE RESPONSE 2

A projectile is launched from the top of a cliff. The cliff is 30 m high, and the projectile is launched from the cliff in the direction of the level plane below. At launch, the projectile has a velocity of 35m/s at an angle 30° above the horizontal. Air resistance is negligible.

a. Draw a representation of the trajectory of the projectile.







b. Calculate the total time from launch until the projectile hits the ground.

The **vertical distance** travelled by the ball is given by by $y = y_o + v_{yo}t - \frac{1}{2}gt^2$

$y = 30 \text{ m} + v_{yo}t - \frac{1}{2}(10)t^2 = 30 + v_{yo}t - 5t^2$

The **vertical** velocity $v_{yo} = v_o \sin\theta = 35 \text{ m/s} \sin 30^\circ = 35(\frac{1}{2}) =$ **17 m/s**

This comes from **sin θ =** $\frac{v_{yo}}{v_o}$.

Setting y = 0 to find and solving (using a graphing calculator) for *t* to find when the bomb hits the target.

$0 = 30 + 17t - 5t^2 \quad \rightarrow$ **t = 4.68 s**

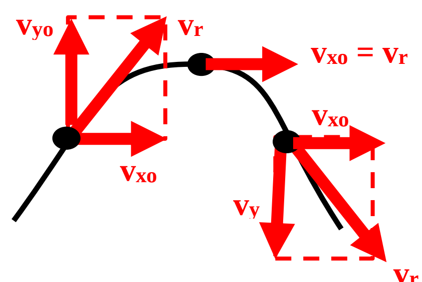
c. Calculate the horizontal distance that the projectile travels before it hits the ground

Ans. The **horizontal** component of the bomb's velocity will not change through the trajectory in the absence of air resistance. It will always be $v_{xo} = v_o \cos\theta = 30 \text{ m/s} \cos 30^\circ = 30(\sqrt{3}/2) =$ **15√3 m/s**

The **horizontal distance** $x = v_{xo}t = (15\sqrt{3})(4.68 \text{ s}) =$ **121.6 m**

d. Calculate the speed at points A, B, and C, where A is maximum height, B is point at which projectile returns to its original height, and C is just before impact.

POINT A: At the **maximum height**, the velocity is all **horizontal** and, as can be seen in the graphic above, $v_r = v_{xo} =$ **15√3 m/s**

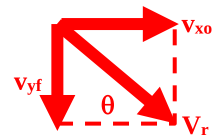


POINT B: On the way down, at the **original height**, the velocity will be the same as the initial velocity except pointing downwards. But, since we are finding **speed** (a scalar), the direction doesn't matter and so $v_B = v_o =$ **35 m/s**

POINT C: The **speed** before impact would be **the hypotenuse** of the triangle formed by the initial horizontal velocity, v_{xo} and the final vertical velocity, v_{yf}.

We know the horizontal velocity is $v_{xo} =$ **15√3 m/s.** The **vertical velocity** at when the projectile hits the ground can be obtained from

$\mathbf{v_y = v_{yo} - gt = 17 - 10(4.68 \text{ s}) = -29.8 \text{ m/s}}$

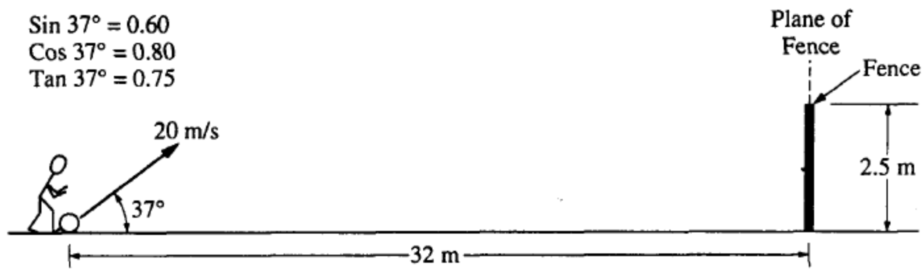


To find v_o, the final velocity, we get $v_o = \sqrt{v_{xo}^2 + v_{yf}^2} = \sqrt{\left(15\sqrt{3}\right)^2 + 29.8^2}$

$V_r =$ **30.5 m/s**

FREE RESPONSE 3. 1994 AP Test.

A ball of mass 0.5 kilogram, initially at rest, is kicked directly toward a fence from a point 32 meters away, as shown below. The velocity of the ball as it leaves the kicker's foot is 20 meters per second at an angle of 37° above the horizontal. The top of the fence is 2.5 meters high. The kicker's foot is in contact with the ball for 0.05 second. The ball hits nothing while in flight and air resistance is negligible.

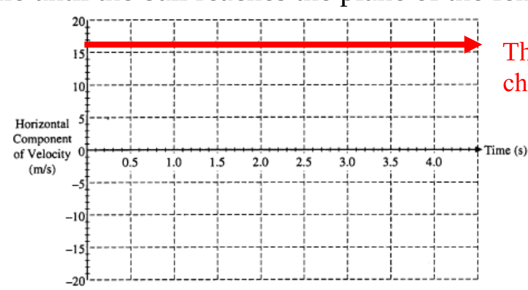


Note: Diagram not drawn to scale.

a. Determine the time it takes for the ball to reach the plane of the fence.
The **horizontal distance** can be related to the **time of travel** and the **horizontal velocity** with the formula $\mathbf{x = v_{xo} \cdot t}$. The **horizontal distance** is 32 m. The **horizontal component of the velocity** of the ball is given by $v_{xo} = v_o \cos\theta$ = **20 m/s** cos **37°** = 20(**0.8**) = **16 m/s**
The time of travel is then $t = \frac{x}{v_{xo}} = \frac{32\ m}{16\ m/s} =$ **2 s**

b. Will the ball hit the fence? If so, how far below the top of the fence will it hit? If not, how far above the top of the fence will it pass?
After 2 s, the ball will be at a **height** given by: $y = v_{yo}t - \frac{1}{2}gt^2$ where v_{yo} is given by $v_o \sin\theta$.
$v_{yo} = v_o \sin\theta$ = **20 m/s** sin **37°** = 20(**0.6**) = **12 m/s**
Substituting: y = (**12 m/s**)(**2 s**) – ½(**10 m/s²**)(**2 s**)² = **4 m**
The ball will pass **4 m – 2.5 m = 1.5 m** above the fence.

c. On the set of axes below, sketch the horizontal and vertical components of the velocity of the ball as functions of time until the ball reaches the plane of the fence.



The **horizontal velocity** does not change throughout the ball's flight.

The **vertical velocity** decreases by 10 m/s per second, the value of gravity, g.