Finance 30210
Solutions to Problem Set #4: Production and Cost Analysis

1) Consider the following output table:

Labor	**Output**	***Marginal Product***	***Average Product***	***Elasticity of Production***
1	2	*2*	*2*	*1*
2	6	*4*	*3*	*1.3*
3	16	*10*	*5.3*	*1.9*
4	29	*13*	*7.3*	*1.8*
5	43	*14*	*8.6*	*1.7*
6	55	*12*	*9.2*	*1.3*
7	58	*3*	*8.3*	*.36*
8	60	*2*	*7.5*	*.27*
9	59	*-1*	*6.6*	*-.15*
10	56	*-3*	*5.6*	*-.53*

a) Calculate Marginal product, Average Product, Elasticity of Production.

See above chart

b) Within what ranges do we see increasing returns, decreasing returns and negative returns?

Increasing returns: 1-5
Decreasing returns: 6-8
Negative returns: 9-10

2) Consider the following short run production function:

$$Q = 6L^2 - .4L^3$$

a) Find the value of L that maximizes output.

First, find marginal product:

$$MP = 12L - 1.2L^2$$

Now, set marginal product equal to zero and solve for L:

$$MP = 12L - 1.2L^2 = 0$$
$$12L - 1.2L^2 = 0$$
$$L(12 - 1.2L) = 0$$
$$L = 10$$

b) Find the value of L that maximizes marginal product.

$$MP = 12L - 1.2L^2$$

Take the derivative and set it equal to zero.

$$12 - 2.4L = 0$$
$$L = 5$$

c) Find the value of L that maximizes average product

$$AP = 6L - .4L^2$$

Take the derivative and set it equal to zero:

$$6 - .8L = 0$$
$$L = 7.5$$

Note: Alternatively, we could find where the elasticity of production equals 1 (or, where MP = AP)

$$\varepsilon = \frac{MP}{AP} = \frac{12L - 1.2L^2}{6L - .4L^2} = 1$$

$$12L - 1.2L^2 = 6L - .4L^2$$
$$6L = .8L^2$$
$$6L - .8L^2 = 0$$
$$L(6 - .8L) = 0$$
$$L = 7.5$$

3) Consider the following production function for bus transportation in a particular city:

$$Q = \alpha L^{\beta_1} F^{B_2} K^{\beta_3}$$

Where $L =$ Fuel input in gallons
$K =$ Capital input in number of busses
$L =$ Labor input in worker hours
$Q =$ Output in millions of bus miles

We estimate the various parameters as follows using historical data:

$$\alpha = .0012 \qquad \beta_1 = .45 \qquad \beta_2 = .20 \qquad \beta_3 = .30$$

a) Determine output elasticities for Labor, Fuel and Capital.

Elasticity of output with respect to labor is as follow:

$$\varepsilon_L = \frac{MP_L}{AP_L}$$

Lets calculate average product and marginal product:

$$MP = \alpha\beta_1 L^{\beta_1 - 1} F^{B_2} K^{\beta_3}$$
$$AP = \alpha L^{\beta_1 - 1} F^{\beta_2} K^{\beta_3}$$

Now, take the ratio…if we did this right, everything should cancel out!

$$\varepsilon_L = \frac{MP_L}{AP_L} = \frac{\alpha\beta_1 L^{\beta_1 - 1} F^{B_2} K^{\beta_3}}{\alpha L^{\beta_1 - 1} F^{B_2} K^{\beta_3}} = \beta_1 = .45$$

Similarly,

$$\varepsilon_F = \frac{MP_F}{AP_F} = \beta_2 = .2$$

$$\varepsilon_K = \frac{MP_K}{AP_K} = \beta_3 = .3$$

b) Suppose that labor hours increase by 10%. By what percentage will output increase?

We know that $\varepsilon_L = \frac{\%\Delta Q}{\%\Delta L} = .45$

Therefore, $\%\Delta Q = .45(\%\Delta L) = .45(10) = 4.5\%$

c) Suppose that every year, 3% of the busses are taken out of service? What effect will this have on output?

Similar to (b)…. $\%\Delta Q = .3(\%\Delta K) = .3(-3) = -.9\%$

d) Suppose that we increase all inputs by 10%, what will happen to output?

$\%\Delta Q = .45(\%\Delta L) + .3(\%\Delta K) + .2(\%\Delta F) = .95(10) = 9.5\%$

4) Suppose that you have the following production function:

$$y = k^{\frac{2}{3}} l^{\frac{1}{3}}$$

Where k represents the units of capital employed at your production facility, l is the number of labor hours employed and y is your total production. You face wage rate equal to \$15 per hour and a cost of capital equal to \$1,920 per unit. You have 1 unit of capital and that can't be changed.

a) You are currently employing 8 hours of labor. Calculate your Total costs, Average cost and marginal cost.

$TC = \$1920(1) + \$15(8) = \$2,040$

To get average cost, we need to calculate total production.

$$y = (1)^{\frac{2}{3}}(8)^{\frac{1}{3}} = 2$$

$$AC = \frac{TC}{Q} = \frac{\$2,040}{2} = \$1,020$$

For marginal cost, we need to calculate the marginal product of labor.

$$MPL = \left(\frac{1}{3}\right)k^{\frac{2}{3}} l^{\frac{-2}{3}} = \left(\frac{1}{3}\right)(1)^{\frac{2}{3}}(8)^{\frac{-2}{3}} = \frac{1}{12}$$

Now,

$$MC = \frac{w}{MPL} = \frac{15}{(1/12)} = \$180$$

b) Calculate your technical rate of substitution. Are you behaving optimally? Explain.

$$TRS = \frac{MPK}{MPL} = \frac{(2/3)k^{\frac{-1}{3}}l^{\frac{2}{3}}}{(1/3)k^{\frac{2}{3}}l^{\frac{-1}{3}}} = 2\left(\frac{l}{k}\right)$$

If we are acting optimally, we should have the technical rate of substitution equal to relative prices.

$$TRS = \frac{P_k}{w}$$

However, with the price of capital equal to $1,920, and the price of labor equal to $15

$$\frac{P_k}{w} = \frac{1920}{15} = 128 \neq 16 = 2\left(\frac{8}{1}\right) = 2\left(\frac{l}{k}\right) = TRS$$

No, we are not acting optimally. We should be hiring a lot more labor. (the productivity of capital is 16 times that of labor, but costs 128 times more. Labor is the better bargain.

Alternatively, we can look at is this way. Given the production function,

$$y = k^{\frac{2}{3}}l^{\frac{1}{3}}$$

The exponents give is the expenditure shares when we are acting optimally.

2/3 = percentage of TC going to capital (67%)
1/3 = percentage of TC going to labor (33%)

But if we calculate our actual cost shares

$$TC = \$1920(1) + \$15(8) = \$2,040$$

$$\frac{P_k k}{TC} = \frac{1,920}{2,040} = .94 \qquad \frac{wl}{TC} = \frac{120}{2,040} = .06$$

We have way too much capital.

c) Given your fixed capital stock of 1, if you were acting optimally, what should your labor be?

We want the technical rate of substitution to equal the relative price

$$TRS = \frac{P_k}{w} \Rightarrow 2\left(\frac{l}{k}\right) = \frac{1920}{15} = 128$$

$$\left(\frac{l}{k}\right) = 64$$

$$l = 64$$

d) Calculate your TC, AC, and Marginal Cost at the optimal choice of labor.

$$TC = \$1920(1) + \$15(64) = \$2,880$$

Note:

$$\frac{P_k k}{TC} = \frac{1,920}{2,880} = .67 \qquad \frac{wl}{TC} = \frac{960}{2,880} = .33$$

To get average cost, we need to calculate total production.

$$y = (1)^{\frac{2}{3}}(64)^{\frac{1}{3}} = 4$$

$$AC = \frac{TC}{Q} = \frac{\$2,880}{4} = \$720$$

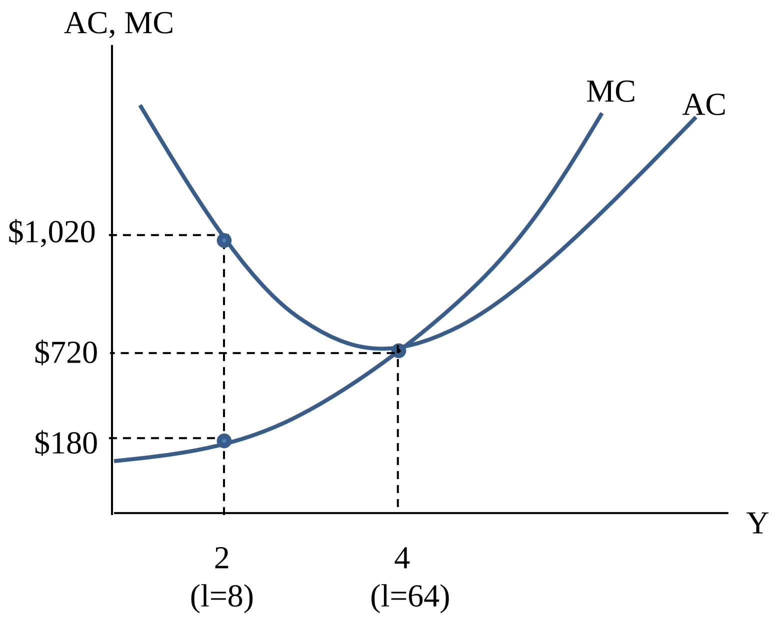
For marginal cost, we need to calculate the marginal product of labor.

$$MPL = \left(\frac{1}{3}\right)k^{\frac{2}{3}}l^{\frac{-2}{3}} = \left(\frac{1}{3}\right)(1)^{\frac{2}{3}}(64)^{\frac{-2}{3}} = \frac{1}{48}$$

Now,

$$MC = \frac{w}{MPL} = \frac{15}{(1/48)} = \$720$$

Side note: Let's picture this….



Optimal behavior minimizes total costs – which also minimizes average costs.

5) Suppose that you are a firm that produces xylophones. You have a production technology to produce xylophones that can be written as:

$$y = k^{\frac{1}{2}} l^{\frac{1}{2}}$$

Where k represents the units of capital employed at your production facility, l is the number of labor hours employed and y is your total production of xylophones. Assume that labor costs \$10 per hour and that capital costs \$250 per unit.

a) Suppose that you are currently employing 100 units of capital. If you have expected sales equal to 1,000. Calculate your optimal choice of labor.

We know that you must produce 1,000 units of output. Therefore, we know that

$$1000 = (100)^{\frac{1}{2}} l^{\frac{1}{2}}$$

We can solve this for labor.

$$1000 = 10 l^{\frac{1}{2}}$$

$$100 = l^{\frac{1}{2}}$$

$$l^{\frac{1}{2}} = (100)^2 = 10{,}000$$

b) Given your answer to (a), calculate your marginal and average cost of production.

Therefore, we know that in the short run, you will hire 10,000 hours of labor to go along with your 100 units of capital. To calculate total costs, we must add up expenditures on capital and labor.

$$TC = \$10L + \$250K = \$10(10{,}000) + \$250(100) = \$125{,}000$$

Average costs (unit costs) are equal to total costs divided by output (which we know is 1,000).

$$AC = \frac{\$125{,}000}{1000} = \$125$$

Marginal costs represent the additional costs incurred by producing a little bit more output. This equals

$$MC = \frac{\$10}{MP_L} = \frac{\$10}{\frac{1}{2} k^{\frac{1}{2}} l^{-\frac{1}{2}}} = \frac{\$10}{\frac{1}{2}(100)^{\frac{1}{2}}(10{,}000)^{-\frac{1}{2}}} = \$200$$

Let's think about this number for a minute. Recall that $F_l(k,l)$ *represents the marginal product of labor (MPL). That is, the additional output that each hour of labor can produce. Here, we have*

$$MP_L = \frac{1}{2}k^{\frac{1}{2}}l^{-\frac{1}{2}} = \frac{1}{2}(100)^{\frac{1}{2}}(10{,}000)^{-\frac{1}{2}} = .05$$

Therefore, each extra hour of labor can produce .05 units of output. Therefore, to produce 1 unit of output would require 20 hours of labor which would cost $200. (Capital costs aren't included because capital is fixed).

c) Now, assume that you can adjust your capital as well as labor. Calculate your optimal capital/labor choice.

In the long run, you are free to choose both capital and labor. Therefore, we need to set up the maximization problem

$$\min_{k,l}\left\{\$10L + \$250K\right\}$$

Subject to the constraint that output is at least 1,000 units. ($k^{\frac{1}{2}}l^{\frac{1}{2}} \geq 1{,}000$)

Setting up the problem:

$$\ell = \$10L + \$250K - \lambda\left(k^{\frac{1}{2}}l^{\frac{1}{2}} - 1000\right)$$

Take the derivatives with respect to K and L and set them equal to zero

$$\$10 - \lambda\left(\frac{1}{2}\right)k^{\frac{1}{2}}l^{-\frac{1}{2}} = 0$$

$$\$250 - \lambda\left(\frac{1}{2}\right)k^{-\frac{1}{2}}l^{\frac{1}{2}} = 0$$

Now, solve each for lambda.

$$\frac{\$10}{\left(\frac{1}{2}\right)k^{\frac{1}{2}}l^{-\frac{1}{2}}} = \lambda = \frac{\$250}{\left(\frac{1}{2}\right)k^{-\frac{1}{2}}l^{\frac{1}{2}}}$$

Rearranging, we get

$$\frac{\left(\frac{1}{2}\right)k^{-\frac{1}{2}}l^{\frac{1}{2}}}{\left(\frac{1}{2}\right)k^{\frac{1}{2}}l^{-\frac{1}{2}}}=\lambda=\frac{\$250}{\$10}$$

This simplifies to

$$L=25K$$

The cost minimizing combination of capital and labor is 25 hours of labor for every unit of capital. Now, use the output constraint

$$1000=k^{\frac{1}{2}}l^{\frac{1}{2}}=k^{\frac{1}{2}}(25k)^{\frac{1}{2}}$$

Solve for k (and l)

$$k=\frac{1000}{5}=200 \qquad l=25k=5{,}000$$

d) Calculate your long run average cost and marginal cost.

In the long run, your optimal scale is 200 units of capital and 5,000 hours of labor. Now, if we recalculate your costs

$$TC=\$10L+\$250K=\$10(5{,}000)+\$250(200)=\$100{,}000$$

$$AC=\frac{\$100{,}000}{1000}=\$100$$

$$MC=\frac{\$10}{MPL}=\frac{\$10}{\frac{1}{2}k^{\frac{1}{2}}l^{-\frac{1}{2}}}=\frac{\$10}{\frac{1}{2}(200)^{\frac{1}{2}}(5{,}000)^{-\frac{1}{2}}}=\$100$$

6) Suppose that you have the following production function:

$$Q=.5LK-.1L^2-.05K^2$$

The process per unit of L and K are \$20 and \$25 respectively. Suppose you are interested in maximizing output given a budget constraint of \$500.

a) Write down the lagrangian associated with this optimization.

$$\ell = .5LK - .1L^2 - .05K^2 + \lambda\left(500 - 20L - 25K\right)$$

b) *Solve for the optimal values for K and L.*

First, take derivitives with respect to K and L and set equal to 0.

$$.5K - .2L - 20\lambda = 0$$
$$.5L - .1K - 25\lambda = 0$$

Now, solve for lambda in each…

$$\lambda = .025K - .01L$$
$$\lambda = .02L - .004K$$

Now, set them equal…

$$.025K - .01L = .02L - .004K$$
$$.029K = .03L$$
$$K = \left(\frac{.03}{.029}\right)L = 1.03L$$

Now, use the constraint…

$$20L + 25K = 500$$
$$20L + 25(1.03L) = 500$$
$$45.86L = 500$$
$$L = 10.90$$
$$K = 11.22$$

c) Assuming that your budget was increased by 10%, calculate (approximately) the increase in output given the new budget.

If we calculate lambda…

$$\lambda = .025K - .01L = .025(11.22) - .01(10.90) = .1715$$

A 10% increase ($50) increases output by 50(.1715) = 8.575*

7) Suppose that you have estimated the following production function.

$$\ln Q = 3.2 + .2\ln k + .7\ln l$$

Where l represents unskilled labor (currently being paid minimum wage) and k represents capital equipment.

So, this production function looks like this…

$$Q = \left(e^{3.2}\right)k^{.2}l^{.7} = 24.5k^{.2}l^{.7}$$

a) Will short run marginal costs be increasing or decreasing in Q? Explain.

Marginal costs are based on the marginal product of labor.

$$MC = \frac{w}{MPL} = \frac{15}{(1/48)} = \$720$$
$$MPL = 24.5(.7)k^{.2}l^{-.3} = 17.15k^{.2}l^{-.3}$$

Here, MPL is declining in L, which means that MC is increasing. With this functional form, MC is increasing as long as the exponent on l is less than 1. IF it is equal to 1, MC is constant, and if it is greater than 1, MC is decreasing.

b) Will long run average costs be increasing or decreasing in Q? Explain.

Imagine doubling both inputs,,,

$$24.5(2k)^{.2}(2l)^{.7} = \underbrace{(2)^{.9}}_{\boxed{\text{Less than 2}}}24.5k^{.2}l^{.7}$$

So, when both inputs are doubled (this will double costs), output less than doubles. Therefore,

$AC = \dfrac{TC}{Q}$ *increases. This will be true whenever the exponents sum to less than 1. If the exponents sum to 1, AC is constant. IF they sum to greater than 1, AC is decreasing.*

c) What will this firm's cost structure look like in the long run (i.e. what percentage of this firms costs will be labor costs? What percentage will be capital costs?)

When the production function is of the form

$Q = Ak^{\alpha}l^{\beta}$ (Here, alpha = .2 and beta = .7)

$$\frac{P_k k}{TC} = \frac{\alpha}{\alpha+\beta} = \frac{.2}{.2+.7} = .22 \qquad \frac{wl}{TC} = \frac{\beta}{\alpha+\beta} = \frac{.7}{.2+.7} = .78$$

8) Continuing with the above example, we have the following estimated production function.

$$\ln Q = 3.2 + .2\ln k + .7\ln l$$

Where l represents unskilled labor (currently being paid minimum wage) and k represents capital equipment.

a) What will this firm's elasticity of demand for labor with respect to the wage be in the short term? How about the long term?

For a fixed production target, the short term elasticity is zero. In the long term, for a production function of the form

$Q = Ak^{\alpha}l^{\beta}$ (Here, alpha = .2 and beta = .7)

$$\varepsilon_{lw} = -\left(\frac{\alpha}{\alpha+\beta}\right) = -\left(\frac{.2}{.2+.7}\right) = -.22$$

b) Suppose that Hillary Clinton is elected President and manages to pass an increase in the minimum wage to $15 per hour. What will happen to employment at this firm in the short run? How about the long run? Explain.

(Again, this is assuming a constant production target)

In the short run, with the elasticity of labor equaling zero, nothing happens. In the long run, with labor being more expensive, the firm will substitute out of labor and into capital. We have that the elasticity of labor demand with respect to the wage is -.22

$$\frac{\%\Delta l}{\%\Delta w} = -.22$$

An increase in the minimum wage from its current $7.25 to $15 would be a

$$\left(\frac{15-7.25}{7.25}\right)100 = 107\% \qquad \textit{Increase.}$$

So, employment falls by 107(-.22) = -23.5%

As a side note, the elasticity of capital with respect to the wage is

$$\varepsilon_{kw} = \left(\frac{\beta}{\alpha+\beta}\right) = \left(\frac{.7}{.2+.7}\right) = .78$$

So, capital demand rises by 107(.78) = 83.5%*

Another side note….the percentage change in $\left(\frac{l}{k}\right)$ *is -107%...as the relative price of labor rises by 107%, the use of labor relative to capital falls by 107%. This is a consequence of the elasticity of substitution equaling 1.*

c) How will a $15 per hour wage affect marginal cost in the short run? How about the long run? Explain.

In the short term (labor demand is unaffected), so MC increases by 107%. In the long term, when firm's substitute out of labor, the elasticity of MC with respect to wages is

$$\varepsilon_{MC,w} = \left(\frac{\beta}{\alpha+\beta}\right) = \left(\frac{.7}{.2+.7}\right) = .78$$

So, MC only rises by 78% (That is, business can avoid some of the wage increase by hiring less labor)

9) Suppose that you have following production function:

$$Q = \left(.5k^{.5} + .5l^{.5}\right)^2$$

a) Calculate the TRS for this production function.

$$MPK = 2\left(.5k^{.5} + .5l^{.5}\right)\left(.25k^{-.5}\right)$$
$$MPL = 2\left(.5k^{.5} + .5l^{.5}\right)\left(.25l^{-.5}\right)$$

$$TRS = \frac{MPK}{MPL} = \left(\frac{l}{k}\right)^{.5}$$

b) If the wage rate is \$20 and the cost of capital is \$100, calculate the minimum cost of producing 54 units of production.

$$TRS = \left(\frac{l}{k}\right)^{.5} = \frac{100}{20} = \frac{P_k}{w}$$

$$\left(\frac{l}{k}\right)^{.5} = 5 \Rightarrow \left(\frac{l}{k}\right) = 25 \Rightarrow l = 25k$$

$$54 = \left(.5k^{.5} + .5\left(25k\right)^{.5}\right)^2$$
$$18 = \left(3k^{.5}\right)^2$$
$$k = 2$$
$$l = 25k = 50$$

c) Calculate the elasticity of substitution.

(this is a little tricky…)

$$TRS = \left(\frac{l}{k}\right)^{.5}$$

So, the elasticity of substitution is

$$\varepsilon = \frac{\%\Delta\left(\frac{l}{k}\right)}{\%\Delta TRS} = \left(\frac{\Delta\left(\frac{l}{k}\right)}{\Delta TRS}\right)\frac{TRS}{\left(\frac{l}{k}\right)}$$

So, first, take the derivative of TRS with respect to (l/k)

$$TRS = \left(\frac{l}{k}\right)^{.5}$$

$$\frac{\Delta TRS}{\Delta\left(\frac{l}{k}\right)} = .5\left(\frac{l}{k}\right)^{-.5}$$

Now, I want the reciprocal….

$$\frac{\Delta\left(\frac{l}{k}\right)}{\Delta TRS} = \frac{1}{.5\left(\frac{l}{k}\right)^{-.5}}$$

Almost there….now, multiply by TRS and divide by (l/k)

$$\frac{\Delta\left(\frac{l}{k}\right)TRS}{\Delta TRS\left(\frac{l}{k}\right)} = \frac{\left(\frac{l}{k}\right)^{.5}}{.5\left(\frac{l}{k}\right)^{.5}} = \frac{1}{.5} = 2$$

Or, you can just remember that for a production function of the form

$$Q = \left(ak^{\theta} + (1-a)l^{\theta}\right)^{\frac{1}{\theta}}$$ (here, theta is .5)

$$\varepsilon = \frac{1}{1-\theta} = \frac{1}{.5} - 2$$

d) Suppose that the price of capital increases to $120. By how much does the ratio of capital to labor utilized change?

A rise in the price of capital from \$100 to \$120 is a 20% increase. With an elasticity of substitution equal to 2, if the relative price of capital goes up by 20%, the ration of capital to labor goes down by 40%.

10) Suppose that you have the following production function:

$$Q = 15k + 7l$$

Where l represents unskilled labor and k represents capital equipment. You face a wage rate of \$8 per hour and a price of capital equal to \$20 per unit.

a) Calculate the technical rate of substitution.

$$TRS = \frac{MPK}{MPL} = \frac{15}{7} = 2.14$$

b) Calculate the elasticity of substitution.

$$\varepsilon = \frac{\%\Delta\left(\frac{l}{k}\right)}{\%\Delta TRS} = \left(\frac{\Delta\left(\frac{l}{k}\right)}{\Delta TRS}\right)\frac{TRS}{\left(\frac{l}{k}\right)}$$

So, first, take the derivative of TRS with respect to (l/k)

$$TRS = 2.14$$

$$\frac{\Delta TRS}{\Delta\left(\frac{l}{k}\right)} = 0$$

Now, I want the reciprocal….

$$\frac{\Delta\left(\frac{l}{k}\right)}{\Delta TRS} = \frac{1}{0} = \infty$$

Infinite elasticity of substitution!

c) Calculate the optimal choice for capital and labor given a production goal of 140 units of output.

$$TRS = \frac{MPK}{MPL} = \frac{15}{7} = 2.14$$

$$\frac{P_k}{w} = \frac{20}{8} = 2.5$$

Capital is 2.14 times as productive as labor (always) while capital costs 2.5 times as much as labor. Labor is always the better deal, so you use only labor.

d) What is the elasticity of demand in the short run? How about the long run?

The elasticity of labor demand is 0 for the short run and long run…unless the relative price of labor gets high enough that you switch into capital exclusively.

e) Plot out your labor demand curves.