Ellina Grigorieva

Methods of Solving Complex Geometry Problems

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To my beautiful daughter,
        Sasha
And to my wonderful parents,
        Natali *and* ***Valery Grigoriev***
Your encouragement made this book possible

And to my university mentor and scientific advisor academician,
        Stepanov Nikolay Fedorovich
Without your help and brilliant mind my career as a scientist would not have been successful!

Preface

Many of us fear mathematics. Sometimes we live with this fear all our lives. We assume that math is not for us and believe that our next-door neighbor is a genius just because he or she is a math major. When I was young, I trained professionally as a musician. One day in the sixth grade after missing 2 weeks of school and coming back from a national violin competition, I did not do well on a plane geometry test because I could not understand how to do proofs. I received a B− on the test—a great disappointment. I asked the teacher why. My math teacher told me, in front of the class, that I had no talent for the subject and that I should continue practicing the violin. Now, being a teacher myself, I understand that this approach was not the best pedagogical "trick" that can be applied in a classroom. However, it worked in my case and it motivated me. I went home and worked until I could solve all the problems in the book. It is important to mention that before I was publicly criticized by my math teacher I did not realize that my understanding of the subject was not clear. I remember sitting in the back of the class and watching how other students went to the board to perform proofs. I had no idea how they did it.

I was always good in algebra but geometry was different. Geometry was not just the calculation of something or the application of a formula. In geometry class we had to do proofs using Euclidean postulates and axioms. I had never done such things before. It was even more shocking for me since I missed the first two weeks of the subject. I did not know how to even start a proof. I hated geometry as strongly as I hated playing violin in the first grade when nothing was good enough for my teacher. That teacher did not want to work with me anymore because she also believed that I did not have talent for the violin and sent me home crying. I was seven years old and did not like the violin until I got a new violin teacher, Mr. London. He was able to give me hope and developed my skills so I became the best in my violin class. I did not like geometry in the same way. I did not understand it and I did not like it. But, when I made an effort and solved every problem in the book I started to love the subject. We always like things that we are good at or the things that come easy. We love something if we have visible, continuous success. Such success comes only with hard work. I later went on to win various Math Olympiads and graduate from

Lomonosov Moscow State University (MGU) *summa cum laude*, defend a Ph.D. in Mathematical and Physical Sciences, and publish over 60 papers in the field of differential equations, game theory, economics, and optimal control theory.

At Moscow State, at the end of each semester the students had to pass four oral exams given by renowned professors. The professors could ask any tricky questions on the topic of the examination ticket. When I was preparing for such exams and in order to have an "A," I did not try to memorize all the definitions and proofs, but rather I tried to develop a "global" understanding of the subject. I thought of possible questions that an examiner could ask me and tried to predict the type of a problem that I could be asked to solve. I developed my own way of learning and I want to share it with you. If you are struggling with math, this book is for you.

For over 30 years whenever I spotted an especially interesting or tricky problem, I added it to my notebooks along with my original solutions. I've accumulated thousands of these problems. I use them every day in my teaching and included many of them in this book. Please look at one of my notebook pages from the ninth grade in the Appendix. You will see how carefully and neatly each problem was presented and solved. Please notice the importance of making an accurate sketch. Many of the problems were created when I was an assistant professor and also worked as a tutor to prepare high school students for the MGU entrance exams. Before accepting an academic position at a university, I worked as a teacher at Ursuline Academy of Dallas and used my problem-solving techniques in my students' college preparation. I was pleased to receive appreciation letters from MIT and Harvard where some of my students were admitted.

If students see an elegant solution but do not apply the approach to other problems, they will not remember it just as nobody remembers phone numbers these days. However, if a teacher uses and reuses the same approach throughout the entire curriculum, students will remember it and learn to value the beauty of the method. Most math books start from theoretical facts, give one or two examples, and then a set of problems. In this book almost every statement is followed by problems. You are not just memorizing a theorem; you apply the knowledge immediately. Upon seeing a similar problem in the homework section you will be able to recognize and solve it.

Although each section of this book can be studied independently, this book is constructed to reinforce patterns developed at stages throughout this book. This helps you to see how math topics are connected. This book can be helpful for self-education, for people who want to do well in math classes or to prepare for competitions. This book is also meant for math teachers and college professors who would like to use it as an extra resource in their classroom.

What Is This Book About?

Geometry is a very important field of mathematics with applications in science, architecture, and art. It is difficult to find a field of human activity where geometry is not applied. People build houses and office buildings, roads, and airplanes. Geometry is everywhere. Even if you are not going to become a scientist or engineer and wish only to graduate from a university, without knowledge of geometry you may not pass the SAT.

Geometry is probably the oldest part of mathematics. Ancient people wanted to know how big their property was, its capacity for growing crops, and how much fence was needed. Around that time simple geometric figures were introduced such as triangles, squares, and circles. Problems solved by ancient Greeks and Egyptians are very unusual and challenging and their different versions appear on some exams and math contests. Reviewing overlooked problems and their solutions that I offer in this book can prepare you for solving Math Olympiad problems. Few teachers cover such problems. Some teachers do not have time to explain a strategy for solving these problems; others do not have the background. This book is not a textbook. This book does not cover every topic in geometry, but it will provide you with a brief course in plane geometry and it will help you to develop problem-solving skills. It will help you to improve your mathematical abilities.

This book is briefly divided into four chapters: Triangle, Quadrilaterals and Polygons, Circles, and Problems on Construction. Each chapter has its own home-work. However, there are overlaps between chapters, because, for example, a problem on a triangle and a circle inscribed into it cannot be solved without knowledge of the properties of each. Sometimes for the same reason, in a solution or proof we will use a property (lemma or theorem) proven in a later section. So if you know that property you can follow along right away and, if not, then you may find it in the following sections or in the suggested references.

It is obvious that two distinct points form a unique line. How do you find out whether or not three given points are on the same line (collinear)? It is well known (but you will learn how to prove it) that three medians and three altitudes concur. What if we take some points M, N, and K on the sides of a triangle and connect them with the corresponding opposite vertex? Under what condition will such segments (cevians) intersect at one point (concur)? The answer to these and many other questions will be found in this book. This book contains important and sometimes overlooked topics on triangles, quadrilaterals, and circles such as Menelaus-Ceva theorems, Simson line, theorems of the three altitudes and medians, and Heron's formula along with their proofs. You will be able to dissect a segment in the Golden ratio, construct an angle of $36°$, and visualize Fibonacci numbers. You will prove the inequality between geometric, arithmetic, and harmonic means in a purely geometric way: the way it was done 1000 years ago by ancient Greeks, and prove why not every angle can be trisected using modern methods. You will learn which quadrilateral with diagonals d_1 and d_2 has the maximum area and solve unusual

problems involving cyclic quadrilaterals and problems on location of circles and polygons with respect to each other.

Problems in geometry can be roughly divided into three groups:

1. Problems on construction (inscribing a pentagon into a circle, trisecting an angle, or construct a segment of certain length),
2. Problems on evaluation (numerical problems on finding something like perimeters, lengths, or areas), and
3. Problems on proof where you must prove certain statements.

Many problems in geometry are a combination of all three types and involve proof, construction, and evaluation together. So I do not separate problems by their type, but rather put problems together because you need to learn how to recognize and attack them.

Usually when you take geometry in high school, the problems are straightforward and clear, but would not prepare you well for solving Olympiad-type problems. Many textbooks have a picture that goes together with the condition of the problem, so much of the necessary thought is already done for you and your space for creativity is limited. Therefore, now and forever, you will always read a problem and then, without looking at anything this book provides, you will draw your own picture. Sometimes you will have to go through several sketches until you find a solution. My figures are prepared with software packages such as MAPLE and Geometer's Sketchpad. However, following the new rules of any mathematics contest or Olympiad, I suggest that you prepare all sketches by hand with the use of ruler and compass only. I also assume that you will not use any calculator or computer when solving the problems.

How to Use This Book

Here are my suggestions about how to use this book. Read the corresponding section and try to solve the problem without looking at my solution. If a problem is not easy, then sometimes it is important to draw an auxiliary element that is not a part of the problem's condition, but that will help you to find a solution to the problem in a couple of steps. In this book I will show simple and challenging problems and will point out ideas we used in the auxiliary constructions so that you can develop your own experience in such "business" and hopefully become an expert soon. If you find any question or section too difficult, skip it and go to another one. Later you may come back and try to understand it. Different people respond differently to the same question and this response sometimes is not related to intelligence or education. Return to difficult sections later and then solve all the problems. Read my solution when you have found your own solution or when you think you are just absolutely stuck. Think about similar problems that you would solve using the same or similar approach. Find a similar problem from the homework section. Create your own problem and write it down

alone with your original solution. Now it is your powerful method. You will use it when it is needed.

I promise that this book will make you successful in problem solving. If you do not understand how a problem was solved or if you feel that you do not understand my approach, please remember that there are always other ways to do the same problem. Maybe your method is better than that one proposed in this book. If a problem requires knowledge of trigonometry or number theory or another field of mathematics that you have not learned yet, then skip it and do other problems that you are able to understand and solve. This will give you a positive record of success in problem solving and will help you to attack that "hard" problem later. Do not ever give up! The great American inventor Thomas Edison once said, "Genius is one percent inspiration and 99% perspiration." Accordingly, a "genius" is often merely a talented person who has done all of his or her homework. Remember that it is never late to become an expert in any field. Archimedes, himself, became a mathematician only at the age of 54.

I hope that upon finishing this book you will love math and its language as I do. Good luck and my best wishes to you!

Denton, TX, USA   Ellina Grigorieva

Acknowledgements

Contents

Chapter 1
Problems Involving Triangles

If three points of a plane do not lie on the same line, the segments that connect these points form a **triangle**. From school we know that there are different types of triangles such as the right triangle (one angle is 90°), equilateral triangle (all sides are equal), isosceles triangle (two sides of the same length), and scalene (a triangle of a general type). Triangles are either **acute** if all angles are less than 90° or **obtuse** if one of the angles is greater than 90°. Some facts listed below probably are known to almost everyone:

A **right triangle** is a triangle with a 90° angle. The side that is opposite to the right angle is called a hypotenuse.

A **perimeter of a triangle** is the sum of all sides of the triangle, $P = a + b + c$.

Here and below a, b, and c are the lengths of sides of a triangle, and A, B, and C are the corresponding opposite angles.

The **area of a triangle** is half of the product of its base and height.

To solve problems with triangles, we should know the following formulas and theorems.

1.1 Conditions of the Existence of a Triangle

If three segments with lengths a, b, and c are given, then the triangle with sides a, b, and c exists if and only if (iff) $a + b > c$, $a + c > b$, and $b + c > a$. We call these relations the **triangle existence inequalities**. Conversely, if a triangle with sides a, b, and c exists, then each side is less than the sum of the other two sides.

Another fact is that the longest side is opposite the largest angle, i.e., if a triangle has sides a, b, and c, then $a \geq b \geq c$ iff $\angle A \geq \angle B \geq \angle C$.

Problem 1. One side of some triangle is 5.3, and the other is 0.7. Find the third side of the triangle, if its length is a natural number.

Solution. Let us make a sketch. In geometry, an accurate picture is often 50 % of work required towards the solution (Fig. 1.1).
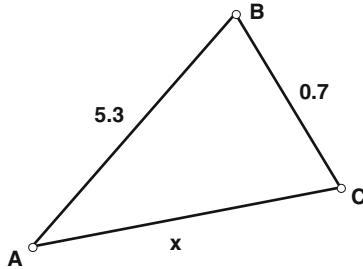
Fig. 1.1 Sketch for Problem 1

Looking at the picture, we can write three inequalities:

$$\begin{cases} x < 5.3 + 0.7 \\ 5.3 < x + 0.7 \\ 0.7 < x + 5.3 \end{cases} \Leftrightarrow \begin{cases} x < 6 \\ x > 4.6 \\ x > -4.6 \end{cases} \Leftrightarrow \quad x \in \ (4.6, 6)$$

There is only one natural number, 5, satisfying the conditions on the interval (4.6, 6).

Answer. The length of the third side is 5.

The following problem was proposed by Professor Gregory Galperin. Despite its apparent simplicity, many have found this problem difficult.

Problem 2. A point D is inside of the triangle ABC. Prove that $AB + BC > AD + DC$

Solution. Let us draw a picture of a scalene triangle with point D inside. Connect A to D and connect D to C. The triangle may be obtuse but our solution cannot depend on the shape of the triangle (Fig. 1.2).
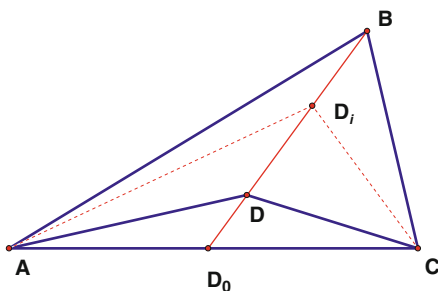
Fig. 1.2 Point D is connected to vertices

If we connect point D to the vertex B and continue the line until it intersects the opposite side at point D_0, then it is obvious that $AB + BC > AD_0 + D_0C = AC$ by the triangle sides inequality. Since by the triangle property $AD + DC > AC$, then it is also clear that $AD + DC > AD_0 + D_0C = AC$ (by the same reason).

Can we state that the relationship is true for any point D between points B and D_0?

Let D be on the height dropped from B to the opposite side as shown in Fig. 1.3. The height subdivides the triangle ABC into two right triangles that share a common side. By the Pythagorean Theorem it is obvious that the greater the height of these triangles, the greater the length of each hypotenuse. Since $(BD_0 > DD_0)$, it is true that $AB > AD$ and $BC > DC$. Therefore, it follows that $AB + BC + AD > DC$.
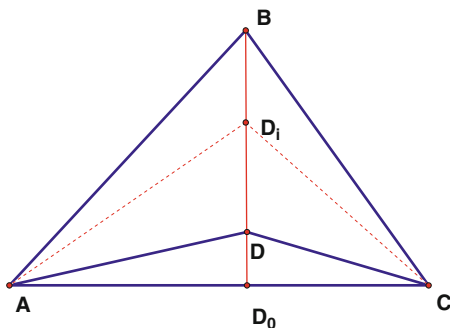
Fig. 1.3 Point D is on the height from B

Unfortunately, if D is not on the height, then AD is not always shorter that AB. See the picture below (Fig. 1.4).

m $\overline{AB}$ = 7.05 cm
m $\overline{AD}$ = 7.45 cm
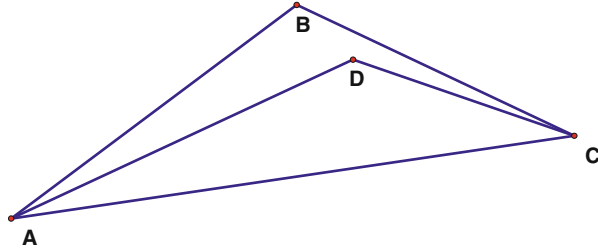
Fig. 1.4 D is not on the height

You can see that $AD > AB$ and that $DC < BC$? How can we be sure that $AB + BC$ is always greater than $AD + DC$?

Maybe an additional (**auxiliary**) construction is needed here.

Let us continue the segment AD until it intersects BC at E (Fig. 1.5).
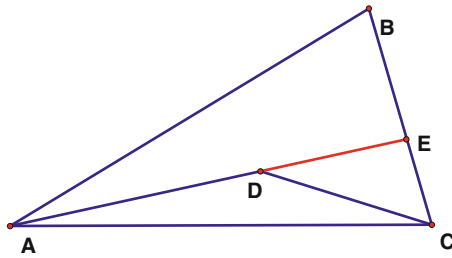
Fig. 1.5 Auxiliary construction for Problem 2

Using obvious inequalities for corresponding triangles we obtain:

$$
\begin{aligned}
AB + BC &= AB + BE + EC \\
&= (AB + BE) + EC > AE + EC \\
AE + EC &= (AD + DE) + EC \\
&= AD + (DE + EC) + DC
\end{aligned}
$$

The proof is complete. Moreover, the following statement is valid.

Lemma 1. Given a triangle ABC and an arbitrary point D inside the triangle. The following is valid (a) perimeter of the triangle ADC and (b) its area is less than the perimeter of triangle ABC and its area, respectively.

Note. Above we actually proved part (a) of the Lemma 1. So,

$$P(ABC) = AB + BC + AC > AD + DC + AC > P(ADC)$$

Here P denotes the perimeter. Try to prove part (b) yourself. The proof will be given later in the corresponding section.

The Triangle Interior Angle Sum Theorem. The sum of the measures of the interior angles of a triangle is $180°$. If A, B, and C are angles of a triangle, then $m\angle A + m\angle B + m\angle C = 180°$.

Proof. The proof of this fact is easy. Just consider a triangle ABC and mark the angles as $\angle A$, $\angle B$, $\angle C$. Draw, for example, a line parallel to side AC through point B. (This is our auxiliary construction!) Find angles that are congruent to $\angle A$ and $\angle C$. All three angles make $180°$ (Fig. 1.6).
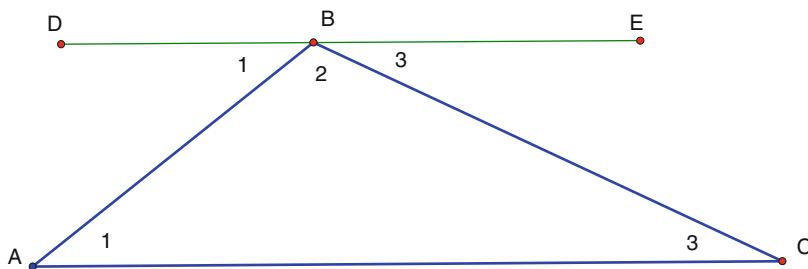
Fig. 1.6 The Triangle interior Angle Sum Theorem

Problem 3. In a triangle ABC the median AM is half of side BC. The angle between AM and height of the triangle, AH is $40°$. Find all angles of triangle ABC.

Solution. This problem can be solved by the **angle chasing method**. First, we will draw a triangle ABC and drop from vertex A the median (AM) with height (AH), and mark angle MAH as $40°$. Since the median, $AM = BC/2$, it is clear that $CM = MA = MR = R$, the radius of the **circumscribed** circle (shown in red) with CB as its diameter. Obviously, ABC is a right triangle and CAB is a $90°$ angle (see Sect. 3.1.1.1 for a detailed coverage of the topic). Moreover, triangles CMA and AMB are isosceles triangles, so their base angles shown by one and two arcs, respectively, are

equal. Since triangle *AHM* is a right triangle, angle *AMH* $= 50°$. Hence the base angle in triangle *CMA* is 25°and the base angle, *ABM* is 65° (Fig. 1.7).
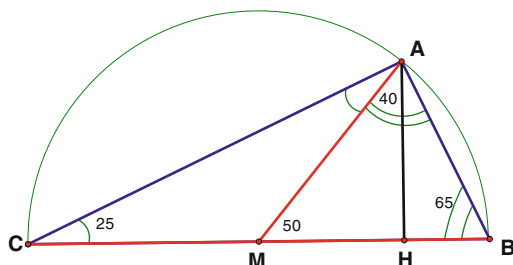
Fig. 1.7 Sketch for Problem 3

Answer. 90, 25, and 65° are the angles of the triangle.

What was the key part of our investigation that allowed us to solve this problem quickly?

Probably, it was our conclusion that *CB* is the diameter of a circumscribed circle and that the triangle *ABC* has angle *CAB* as the 90° angle. Additionally, it was very important to learn that triangles *AMC* and *AMB* are isosceles and so each of them have equal base angles. You will learn more about these topics in the followings sections.

An **exterior angle** of a triangle is an angle formed between the extension of one of the sides of the triangle and the outside of the triangle.

Exterior Angle Theorem. The measure of an exterior angle of a triangle equals the sum of the two nonadjacent opposite interior angles.

The proof of this theorem is easy and you should do it yourself.

Now here is a problem for your consideration. The problem was stated and solved by the Greek geometer **Archimedes** in his manuscript "Lemmas." To many modern students it has a very confusing condition.

Problem 4. From a point *A* outside of a circle, construct two secants such that one goes through the center and the other is cut by the circle into two segments. If the length of the exterior segment equals the radius of the circle, show that the angle between the secants is one third the larger arc's measure concluded between sides of the angle.

Proof. Let us draw a picture first.
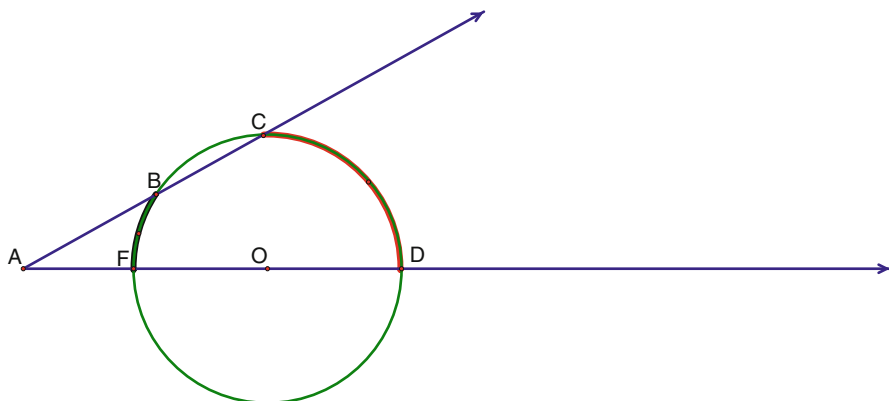
Fig. 1.8 Problem 4, first view

In Fig. 1.8, *AC* and *AD* are the two secants. *AD* goes through the center *O*. The other, *AC*, is cut by the circle into two segments. Based on the condition of the problem, $AB = FO = BO = r$, the radius of the circle. Next, there is some information about the arcs that are concluded between the sides of the angle *CAD*. In our sketch we have a small arc *BF* and big arc *CD* (shown in red). We have to prove that the angle *CAD* is 1/3 of the measure of the arc *CD* (or 1/3 of the central angle *COD*).

Since now we understand the condition of the problem, we can add something to our sketch so that our proof will be easier. For example, we can connect points *B* and *C* with the center of the circle *O* (Fig. 1.9). Further we can rephrase "antique" problems into modern language.

Rephrased Problem. Two secants are dropped to a circle from point *A* outside the circle. As it is shown on the picture provided, $AB = OB = r$, the radius of the circle. The angle between two secants equals α. Evaluate the measure of angle *COD* and prove that $COD = 3\alpha$.
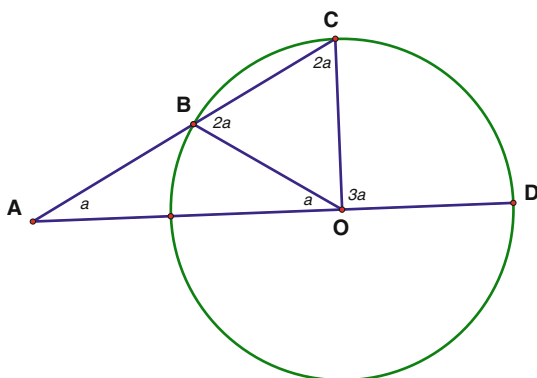
Fig. 1.9 Problem 4, second view

Why is this problem on an application of the Exterior Angle Theorem?

Since $AB = BO = r$, then triangle ABO is an isosceles and $\angle BOA = \alpha$. If $\angle CBO$ is an exterior angle of triangle ABO, then $\angle CBO = 2\alpha = \angle BCO$ since $BO = OC = r$. Finally, if $\angle CBO$ is an exterior angle of the triangle AOC, then it is equal to the sum of $\angle BOA$ and $\angle ACO$, i.e., $\angle COD = \alpha + 2\alpha = 3\alpha$.

Note. It is interesting that this problem was proposed by Archimedes for trisecting an angle. More about angle trisection and why Archimedes' method has obvious drawbacks will be discussed in Chapter 4 of the book.

1.2   Properties of Right Triangles

1.2.1   Pythagorean Theorem

For a right triangle the following theorem is valid.

> **Pythagorean Theorem.**   The square of the hypotenuse of a right triangle is equal to the sum of the squares of the two legs.

This theorem was known to the Babylonians more than 1,000 years earlier, but the first general proof of this theorem was given by **Pythagoras**. We believe that Pythagoras gave a proof similar to one illustrated in the figure below (Fig. 1.10).
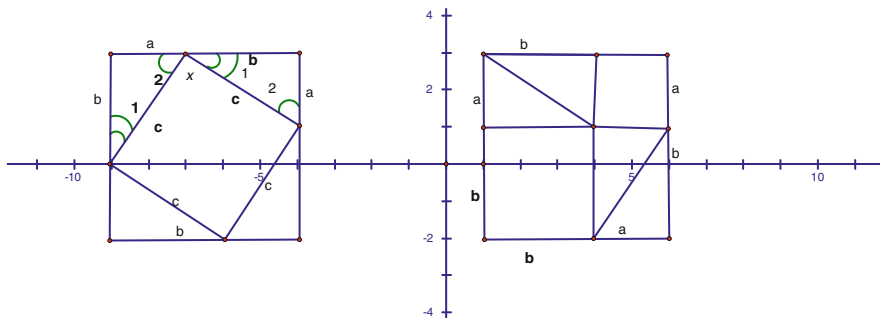
Fig. 1.10 Pythagorean Theorem

Proof 1. Let a, b, and c be the legs and the hypotenuse of the right triangle, respectively. Consider two squares, each with sides $(a + b)$. The second square is divided into two squares with sides a, b and two congruent triangles with legs a, b. The first square is divided into four congruent triangles and a square between them

with side c. You can easily see that after such dissection, the figure between triangles must be a square. Since angles 1 and 2 add up to 90°, angle x (the angle formed by the sides of the quadrilateral) must be a 90° angle as well. The area of each big square is the same and can be written in two different ways as

$$4\frac{ab}{2} + c^2 = 4\frac{ab}{2} + a^2 + b^2$$

Subtracting $2ab$ from each side, we obtain the required formula, $c^2 = a^2 + b^2$.

Proof 2. This proof is based on relationship between *similar triangles* formed by the height dropped from the vertex of the right angle in the right triangle as the auxiliary construction. The proof will be given in the corresponding section.

Let us apply the Pythagorean Theorem by solving Problem 5.

Problem 5. Medians AD and BE of triangle ABC form a 90° angle. It is known that $AC = 3$ and $BC = 4$. Find AB.

Solution. First, draw a picture by hand and include all known information. Because BE and AD are medians, D is the midpoint of BC (i.e., $BD = 2$) and E is the midpoint of AC (i.e., $AE = 3/2 = 1.5$). In the picture below, O is the point of intersection of medians (Fig. 1.11).
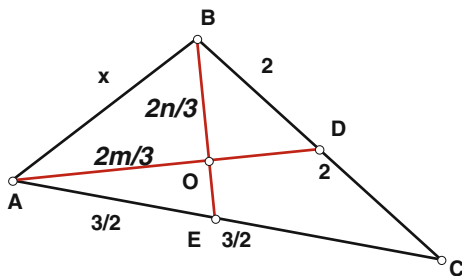
Fig. 1.11   Sketch for Problem 5

By the Theorem of the Three Medians (see Sect. 5.2), $BO{:}OE = 2{:}1$ and $AO{:}OD = 2{:}1$. Let us introduce three variables: the lengths of medians, $n = BE$ and $m = AD$, and the unknown side $x = AB$. Now we can express the lengths of all segments in terms of n and m: $BE = 2n/3$, $OE = n/3$, $AO = 2m/3$, and $OD = m/3$. By the condition of the problem, $BE \perp AD$ the triangles AOE, AOB, and BOD are right triangles. Using the Pythagorean Theorem for each triangle, we obtain the following system:

$$\begin{cases} \left(\dfrac{2n}{3}\right)^2 + \left(\dfrac{m}{3}\right)^2 = 2^2 \\[2mm] \left(\dfrac{n}{3}\right)^2 + \left(\dfrac{2m}{3}\right)^2 = \left(\dfrac{3}{2}\right)^2 \\[2mm] \left(\dfrac{2n}{3}\right)^2 + \left(\dfrac{2m}{3}\right)^2 = x^2 \end{cases} \Rightarrow \begin{cases} 4n^2 + m^2 = 36 \\[2mm] 4n^2 + 16m^2 = 81 \\[2mm] \dfrac{4n^2}{9} + \dfrac{4m^2}{9} = x^2 \end{cases}$$

Subtracting the first equation from the second, we have $15m^2 = 45$, $m^2 = 3$. Replacing it back into the first, we obtain, $n^2 = \frac{33}{4}$. Using the last two results we can evaluate x from the third equation of the system

$$x^2 = \frac{4}{9} \cdot \frac{33}{4} + \frac{4 \cdot 3}{9}$$
$$x^2 = 5$$
$$x = \sqrt{5}$$

Answer. $x = AB = \sqrt{5}$.

Note. When they see this problem, some students rush to give me a solution without actually solving it. They say that $3^2 + 4^2 = 5^2$ so the answer is 5. However, such a solution is wrong, and it would be true only if the triangle ABC was the right triangle. You can apply the Pythagorean Theorem only in the case of right triangles.

Moreover, in general there are infinitely many possibilities for the length of side AB if we apply the triangle existence inequality relationship between the sides. Let x be the length of side AB then we would have the following system.

$$\begin{cases} x < 4 + 3 \\ 4 < x + 3 \\ 3 < x + 4 \end{cases} \Leftrightarrow \begin{cases} x < 7 \\ x > 1 \\ x > -1 \end{cases} \Leftrightarrow \quad x \in (1, 7)$$

If we did not have an additional constraint on the type of the triangle given by the condition about its medians, then the length of side AB would be given by an open interval, $1 < x < 7$, which would satisfy infinitely many cases including one of a right triangle with hypotenuse of length five. In fact, our correct solution, $x = \sqrt{5}$, also satisfies the inequality because $1 < \sqrt{5} < 7$.

Problem 6. In a right triangle, all sides are relatively prime numbers. Prove that if the length of the hypotenuse is an odd number and that the length of one leg is odd, then the other is an even number.

Solution. Let m and k be the legs of the triangle and let n be its hypotenuse. Numbers m, k, n are relatively primes and satisfy the Pythagorean Theorem:

$$n^2 = m^2 + k^2$$

Now suppose that the hypotenuse is an even number so that $n = 2l$, then

$$m^2 + k^2 = 4l^2$$

Since the right side is an even number, m and n are either both odd or even. The latter case is not applicable because two even numbers cannot be relatively prime. Therefore, m and n must be both odd and

$$m = 2l_1 + 1, \quad \text{and} \quad k = 2l_2 + 1, \text{ so}$$
$$4l^2 = (2l_1 + 1)^2 + (2l_2 + 1)^2 = 4l_1^2 + 4l_2^2 + 4l_1 + 4l_2 + 2$$

It follows from the last formula that the left side is a multiple of 4, but the right side is not (it is only divisible by 2) then such a relationship is not valid. Therefore, the length of the hypotenuse must be only an odd number. Moreover, if the hypotenuse is an odd number, then the length of one leg must be odd and the length of the other leg must be even.

1.2.2 Trigonometric Relationships in a Right Triangle

Consider a right triangle ABC, such that $AB = c$ (hypotenuse), $AC = b$, and $CB = a$ with opposing angles C (right angle), B and A, respectively.
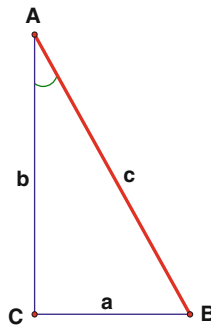
Fig. 1.12 Right triangle

Then the following statement is valid.

A length of a leg of a right triangle equals the length of the hypotenuse times the sine of the opposite angle or the length of the hypotenuse times the cosine of the adjacent angle.

Therefore, the relationships for the right triangle are:

$$a = c \cdot \sin(\angle A) = c \cdot \cos(\angle B)$$
$$b = c \cdot \sin(\angle B) = c \cdot \cos(\angle A)$$

(1.1)

This statement can be easily proven. If we square the left and right sides of both equations and then add them, we will obtain

$$a^2 = c^2 \cdot \sin^2(\angle A) = c^2 \cdot \cos^2(\angle B)$$
$$b^2 = c^2 \cdot \sin^2(\angle B) = c^2 \cdot \cos^2(\angle A)$$

Angles A and B are **complimentary** since $\angle A + \angle B = 90°$ (i.e., $\sin(\angle A) = \cos(\angle B)$ and $\sin(\angle B) = \cos(\angle A)$). Using the trigonometric identity, $\sin^2 x + \cos^2 x = 1$ we obtain the following correct statement:

$$a^2 + b^2 = c^2$$

In the USA, students learn this formula in a different form so that the sine of an angle is the ratio of the opposite side and a hypotenuse. Formulas (1.1), in my opinion, have more sense and are easier to memorize. Additionally this form makes clear that sine or cosine of any angle is always bounded by one. Otherwise, the legs of a right triangle would be longer than hypotenuse! Second, if a student sketches a right triangle like one in the Fig. 1.12 above, he or she can simply write the relationships remembering that a leg of a triangle is a product of a hypotenuse and the sine of the opposite angle. Clearly, if an opposite angle is 30°, then such a leg is half of the hypotenuse, $a = c/2$.

The only trigonometric formula involving sides of a right triangle that I believe can be presented to a student as a ratio is the formula involving the tangent. This is because as the student learns calculus, the derivative of a function at a point will be defined as a limit of the slope of a secant line based on the ratio formula. Therefore, the tangent of an opposite angle equals the ratio of the opposite leg to the adjacent leg and the following is valid:

$$\tan(\angle A) = \frac{a}{b}$$

$$\tan(\angle B) = \frac{b}{a}$$

(1.2)

Consider one of the formulas (1.2), for example, $a/b = \tan(\angle A)$. If the legs of a triangle are equal, then $a = b$ and consequently $a/b = 1$. So if $\tan(\angle A) = \tan(\angle B) = 1$, then $A = B = 45°$. We just gave a trigonometric proof to the well-known fact: isosceles right triangles have base angles of 45°. You can easily prove it by angle chasing and by using the Triangle Angles Theorem and the property of the isosceles triangle.

Problem 7. In a triangle ABC with the right angle C, side BC is divided by points D and E into three equal parts. Find the sum of angles AEC, ADC, and ABC if it is known that $BC = 3AC$.

Solution 1. Consider a right triangle ABC and denote $|AC| = b$, $\angle AEC = \alpha$, $\angle ABC = \beta$. Then $CD = DE = EB = b$, and $CB = 3b$ (Fig. 1.13).
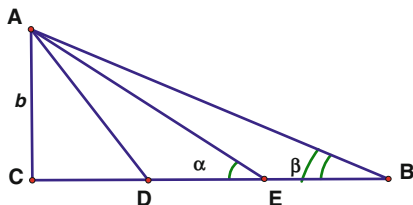
Fig. 1.13 Problem 7, Solution 1

From the right triangles ADC, ACE, and ACB we have

$$\tan(\angle ADC) = \frac{|CD|}{|AC|} = \frac{b}{b} = 1$$

$$\tan \alpha = \frac{|CE|}{|AC|} = \frac{b}{2b} = \frac{1}{2}$$

$$\tan \beta = \frac{|CB|}{|AC|} = \frac{b}{3b} = \frac{1}{3}$$

The sum of the angles is

$$45° + \left(\arctan \frac{1}{2} + \arctan \frac{1}{3} \right)$$

This could be the answer but it does not look pretty. Let us simplify the expression inside the parentheses and denote it by γ, then

$$\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \gamma$$

or $\alpha + \beta = \gamma$. Take the tangent of both sides so that $\tan(\alpha + \beta) = \tan \gamma$.

Applying to the left side the formula of tangent of the sum of two angles we obtain

$$\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta} = \tan \gamma$$

$$\frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} = 1 = \tan \gamma$$

$$\alpha + \beta = 45° = \gamma$$

Therefore, the sum of the three angles is $90°$.

Would it be nice to solve this problem without any trigonometry, if possible?

One of the "pure geometric" solutions is given for your consideration. Be sure to understand the importance of the auxiliary construction.

Solution 2. Construct point G such that the vectors AC and DG are equal. Let $\alpha = \angle AEC$, $\beta = \angle ABC$. We will show next that AGB is an isosceles right triangle. Indeed, $\angle DGA = \alpha$ since triangles ADG and ADE are congruent. Angle $\angle BGD = 90° - \alpha$ since triangles BDG and ECA are congruent. Thus, $\angle BGA = \angle BGD + \angle DGA = 90°$ (Fig. 1.14).
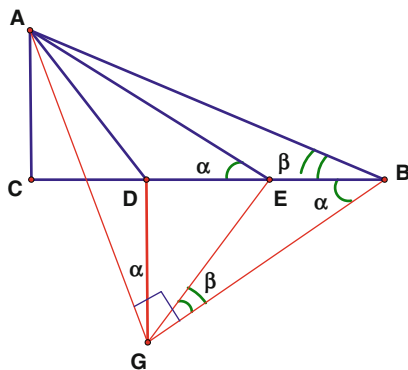
Fig. 1.14 Problem 7, Solution 2

Finally, $BG = AG$ since triangles ADG and GEB are congruent. Angle $\angle DBG = \alpha$ (BDG and ECA are congruent). So $\angle ABG = \alpha + \beta = 45°$.

Answer. $\angle ADC + \angle AEC + \angle ABC = 45° + \alpha + \beta = 90°$.

1.3   Law of Cosines and Law of Sines

The **Law of Cosines** is the relationship between the sides and the angles of a triangle: $a^2 = b^2 + c^2 - 2bc \cdot \cos(\angle A)$, where $\angle A$ is opposite of side a. Students usually learn this formula at the end of a geometry course but don't understand how to apply it until taking trigonometry. Nevertheless, this formula is very powerful. In addition, the Pythagorean Theorem is a particular case of the Law of Cosines. Let us now solve Problem 8:

Problem 8.   A triangle with sides of length 5, 12, and 13 is given. Find the angle that is opposite the biggest side.

Solution.   First, we will draw an accurate picture and put all known information on it so it will help us to calculate the measure of angle C (Fig. 1.15).
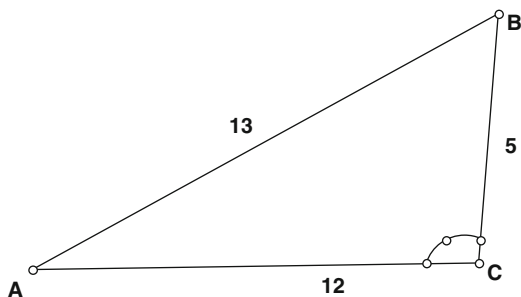
Fig. 1.15   Sketch for Problem 8

Let us apply the Law of Cosines to the triangle:

$$13^2 = 5^2 + 12^2 - 2 \cdot 5 \cdot 12 \cdot \cos(\angle C)$$
$$0 = \cos(\angle C), \quad \text{then} \quad m\angle C = 90°$$

We found that triangle ABC is a right triangle.

Let us prove that the Pythagorean Theorem is a particular case of the Law of Cosines. For this purpose we will write the Law of Cosines for the cosine of angle C using the same picture:

$$c^2 = a^2 + b^2 - 2ab\cos(\angle C)$$
$$\cos(\angle C) = \frac{a^2 + b^2 - c^2}{2ab} \tag{1.3}$$

The last formula can be used to evaluate angles in triangles, but because $\cos(\angle C)$ is a fraction, it equals 0 iff the numerator of the fraction equals 0, i.e., $a^2 + b^2 - c^2 = 0$ or $a^2 + b^2 = c^2$. Of course, we recognize the Pythagorean Theorem.

Remark. Formula (1.3) can be used to evaluate angles in triangles and even to predict the type of triangles. Suppose that the sides of a triangle satisfy the inequality $c > b > a$. The following is true:

1. If the square of the biggest side is greater than the sum of the squares of two other sides, i.e., $c^2 > b^2 + a^2$, then the triangle is **obtuse**. (The proof is simple: if $\angle C > 90° \Rightarrow \cos \angle C < 0 \Rightarrow a^2 + b^2 - 2ab \cos \angle C > a^2 + b^2$.)
2. If the square of the biggest side is less than the sum of the squares of two other sides, i.e., $c^2 < b^2 + a^2$, then the triangle is **acute** (all triangle angles are less than 90°.)

Problem 9. Consider a scalene triangle, ABC, with sides $BC = 6$, $AC = 5$, and angle ABC of 30°. Find the length of side AB, if the distance from the vertex A to the line BC is less than $\frac{1}{\sqrt{2}}$.

Solution. Let us construct triangle ABC satisfying the given conditions (Fig. 1.16):

1. Draw segment CB of length 6.
2. Draw a ray from vertex B that makes angle 30° with BC.
3. Draw a circle with center C and radius 5.
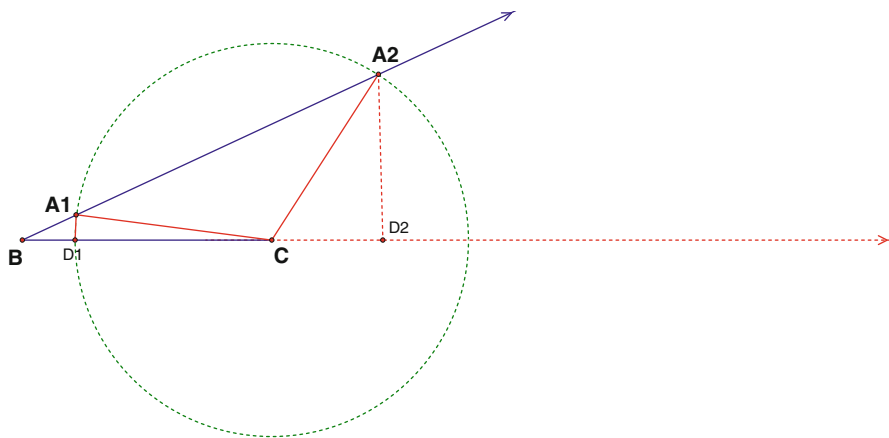
Fig. 1.16 Sketch for Problem 9

Such a circle will intersect the ray at two points: A_1 and A_2 such that $|CA_1| = |CA_2| = 5$. Two different triangles can be constructed before the last condition is satisfied.

Apply to it the Law of Cosines to find the length of side AB:

$$5^2 = |AB|^2 + |BC|^2 - 2|AB| \cdot |BC| \cos 30°$$

$$25 = |AB|^2 + 36 - \frac{2 \cdot \sqrt{3}}{2} \cdot |AB|$$

$$|AB|^2 - 6\sqrt{3}|AB| + 11 = 0$$

$$|AB| = 3\sqrt{3} \pm 4$$

After solving the quadratic equation we obtained two answers, hence either the length of $BA_1 = 3\sqrt{3} - 4$ or the length of $BA_2 = 3\sqrt{3} + 4$.

In order to select one out of two possible triangles, we will use the **distance condition**.

Do you remember that the distance between a point and a line is the length of the perpendicular dropped from the point to the line?

By dropping perpendiculars A_1D_1, A_2D_2 to the line BC from points A_1A_2 we will obtain two right triangles, ΔBA_1D_1, ΔBA_2D_2 with hypotenuses CA_1, CA_2, respectively, and angle $ABC = 30°$. For such right triangles the leg opposite to $30°$ angle (distance to the line BC) equals half of the hypotenuse. Thus, we finally have

$$A_2D_2 = A_2B \sin 30° = \frac{3\sqrt{3} + 4}{2} > \frac{1}{\sqrt{2}}$$

$$A_1D_1 = A_1B \sin 30° = \frac{3\sqrt{3} - 4}{2} < \frac{1}{\sqrt{2}}$$

Since only the second inequality satisfies the condition of the problem, it follows that $|BA| = 3\sqrt{3} - 4$.

The **Law of Sines** is a relation between the length a side and the opposite angles of a triangle. For any triangle ABC, the ratios between sides and sines of opposite angles are the same and equal to double the radius of the circumscribing circle:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \tag{1.4}$$

Please remember that for any triangle the center of the circumscribing circle is at the intersection of its perpendicular bisectors. Connecting the center of the circle with each vertex of a triangle, we get three radii, R, of the circle as shown in Fig. 1.17.
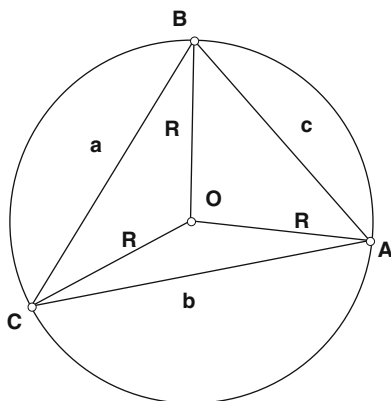
Fig. 1.17   Circumradius and sine law

Now it is easy to prove the statement. Each of the triangles, *AOC*, *COB* and *BOA* is an isosceles since $CO = OB = OA = R$. Think of dropping a perpendicular from *O* to side *CB*; then its foot, say *D*, will be also the midpoint of the segment *CB*. From the right triangle *BDO* we find that

$$\frac{a}{2} = BD = R \ \sin(\angle BOD) = R \ \sin\left(\frac{\angle BOC}{2}\right) = R \ \sin(\angle BAC) = R \ \sin A$$

From which we obtain that $a = 2R\sin A$, etc.

Here we used the relationship between *central* (*COB*) and *inscribed* (*CAB*) angles. You will see more on this topic in Chap. 3.

Remark. Suppose that the sides of a triangle satisfy $a \le b \le c$, then without loss of generality it follows from (1.4) that $\sin A \le \sin B \le \sin C \Rightarrow A \le B \le C$.

Please practice in the Law of Sine by solving Problem 10.

Problem 10. Consider the right triangle *KLM* with $\angle KML = 90°$. Point *D* is on the hypotenuse *KL* such that $|DL| = 1$, $|DM| = \sqrt{2}$, $|DK| = 2$. Evaluate the angle *KMD*.

Solution. First, we will draw a right triangle *KML* and place point *D* on segment *KL* (Fig. 1.18).
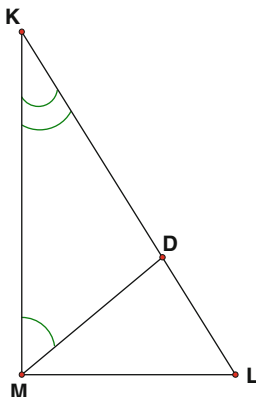
Fig. 1.18 Sketch for Problem 10

Let the angle marked by one arc be $\angle KMD = \alpha$ and the angle marked by two arcs be $\angle MKD = \beta$. Next, apply the Law of Sines to two triangles:

$$\Delta KMD : \frac{MD}{\sin \beta} = \frac{KD}{\sin \alpha} \quad \text{or} \quad \frac{\sqrt{2}}{\sin \beta} = \frac{2}{\sin \alpha},$$

$$\Delta MDL : \frac{DL}{\sin(\angle DML)} = \frac{MD}{\sin(\angle MLD)} \quad \text{or} \quad \frac{1}{\sin(90° - \alpha)} = \frac{\sqrt{2}}{\sin(90° - \beta)}$$

Using the relation between the sine and cosine of complimentary angles and simplifying the proportions above, we have that

$$\sin \beta = \frac{\sin \alpha}{\sqrt{2}}$$
$$\cos \beta = \sqrt{2} \cos \alpha$$

Squaring the left and the right sides of both relations and applying a trigonometric identity we will obtain an equation in terms of $\cos \alpha$ from which the value of α follows:

$$1 = 3\cos^2 \alpha$$
$$\cos \alpha = \frac{1}{\sqrt{3}}$$
$$\alpha = \arccos\left(\frac{1}{\sqrt{3}}\right)$$

Note. We selected only such cosines that may satisfy the given condition so that the angle is less than 90°.

1.4 Similar Triangles

Two triangles are called **similar** if their corresponding angles are equal and the ratio of corresponding sides is the same. Similar triangles or, in general, any similar figures have a similar shape. You probably remember that two triangles are similar to each other by two angles (*AA*), by two sides and the included angle (*SAS*), and by three sides (*SSS*). Especially important is the fact that in similar triangles the ratio of corresponding sides, medians, heights, and bisectors equals k, the **coefficient of similitude**. The ratio of the areas of similar triangles equals k^2, the square of the coefficient of similitude.

The following picture will illustrate one way of constructing similar triangles by hand, with $k > 1$ (magnification) and $k < 1$ (making the image smaller). This is based on **Homothetic Transformation** or such transformation at which any point of the original (pre-image), its image and the center of homothety lie on the same line. Using homothety, different similar figures can be constructed, so all corresponding angles are equal.
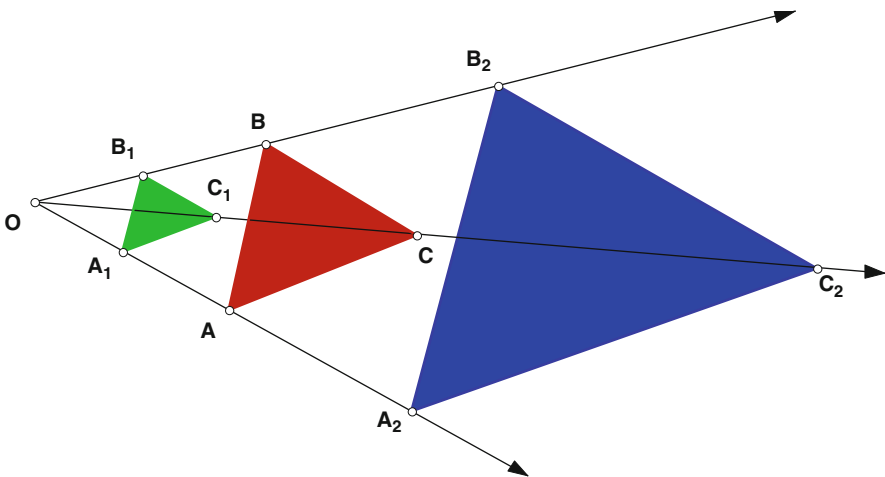
Fig. 1.19 Triangle homothetic transformation

In the figure above (Fig. 1.19), all triangles are similar and $\Delta ABC \sim \Delta A_1B_1C_1 \sim \Delta A_2B_2C_2$. Let us consider the first situation where the red triangle, *ABC*, is the pre-image (we will make it bigger or smaller), O is the center of transformation, and the green triangle, $A_1B_1C_1$ is the image of *ABC*. Then $\frac{|OA|_1}{|OA|} = \frac{|OB_1|}{|OB|} = \frac{|OC_1|}{|OC|} = \frac{1}{2}$ by our construction with $k = \frac{1}{2}$,

$$\frac{|A_1B_1|}{|AB|} = \frac{|B_1C_1|}{|BC|} = \frac{|C_1A_1|}{|CA|} = k = \frac{1}{2}$$

so

$$\frac{\text{Area of } \Delta A_1 B_1 C_1}{\text{Area of } \Delta ABC} = k^2 = \frac{1}{4}, \ k = \frac{1}{2} < 1$$

and we can see that the image is smaller than the pre-image. For the second situation, again the red triangle is the pre-image, but the blue triangle is the image.

In our diagram if

$$\frac{|OA_2|}{|OA|} = \frac{|OB_2|}{|OB|} = \frac{|OC_2|}{|OC|} = 2$$

then $k = 2$ and

$$\frac{|A_2 B_2|}{|AB|} = \frac{|B_2 C_2|}{|BC|} = \frac{|C_2 A_2|}{|CA|} = k = 2 \quad \text{and} \quad \frac{\text{Area of } \Delta A_2 B_2 C_2}{\text{Area of } \Delta ABC} = k^2 = 4$$

Here $k = 2 > 1$ and we notice that triangle $A_2 B_2 C_2$ is larger that it's pre-image, triangle ABC.

Homothetic transformations have many applications, for example, in animation or computer game development. Thus, you can think of taking the image of your favorite cartoon character and using this technique to magnify it or make it smaller. For your consideration in Chap. 4 I will offer you a problem on inscribing a polygon into a given shape using homothety.

Below is a problem on similar triangles.

Problem 11. Given a right triangle ABC, let CD be the height dropped from the $90°$ angle. (1) Prove that the triangles ACD, CBD, and ABC are similar. (2) Find the ratio of the corresponding sides in triangles ACD and BCD.

Proof. Since a good picture usually is 50 % of the successful solution, I advise you to rotate a typical right triangle ABC so its hypotenuse would become its base. In such setting you can find relationships between similar triangles better. The next step is to mark angles of the same measure (Fig. 1.20).
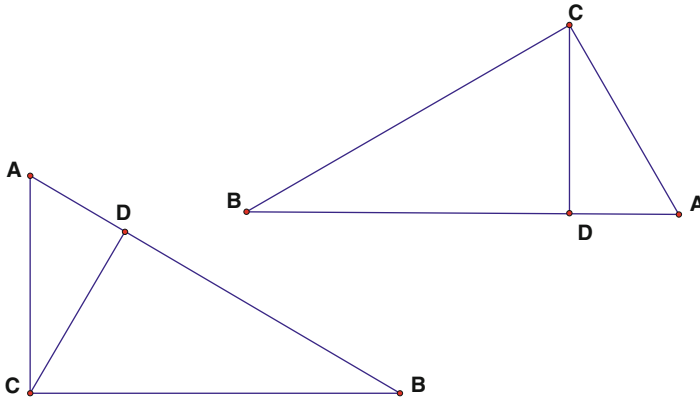
Fig. 1.20 First sketch for Problem 11
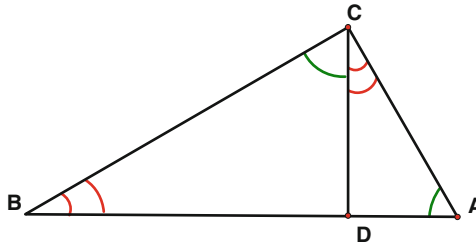
Fig. 1.21 Second sketch for Problem 11

Let us have $\angle BAC = \alpha$, $\angle CBA = \beta$. Then $\angle BCD = \alpha$, $\angle ACD = \beta$. Therefore, $\Delta ACD \sim \Delta DCB \sim \Delta ABC$ by the two angles (AA) property. In Fig. 1.21, angle α is shown by one arc and angle β by two arcs. Therefore, triangles DCB and CDB are similar and the ratio of the corresponding sides is the same. Many students have difficulties in writing these ratios. Remember that <u>we need to consider the ratio of only such sides that both are opposite to an angle of the same measure.</u>

Thus, in triangle BDC, side CD is opposite to two arcs. In triangle CDA, side DA is opposite to the same angle (marked by two arcs). The ratio between the sides is $|CD|/|DA|$. On the other hand, in triangle BDC side BD is opposite to angle marked by one arc as is side CD in triangle CDA. The ratio between these sides can be written as BD/CD.

Since two ratios of the lengths of the sides are equal,

$$\frac{|CD|}{|DA|} = \frac{|BD|}{|CD|} \quad \text{or} \quad |CD|^2 = |BD| \cdot |DA|$$

From which we can conclude that *CD* is the **geometric mean** (the square root of the product of the components) of the sides *DA* and *BD*. See more on this topic in Sect. 4.1.

1.4.1   Thales' Theorem

Similar triangles and their properties were known to Egyptians and Babylonians. However, we need to give credit to the famous Greek geometer **Thales** who first formulated the properties of similar triangles formed by two parallel lines and two intersecting transversals. He is also the one who measured the height of the Great Pyramid using shadows.

> **Thales' Theorem.**   If two parallel transversals cut two equal segments on one side of the angle, then on the other side of the angle they form two other segments of equal length.

We can say that the Midline Segment Theorem follows from Thales' Theorem. The Midline Segment Theorem states that the segment containing two midpoints of adjacent sides is both parallel and half the length of the respective side. Figure 1.22 illustrates the theorem.

AB = 1.21 inches
BC = 1.21 inches
$m\angle BB_1A = 45°$
$m\angle CC_1A = 45°$
$AB_1 = 1.61$ inches
$B_1C_1 = 1.61$ inches
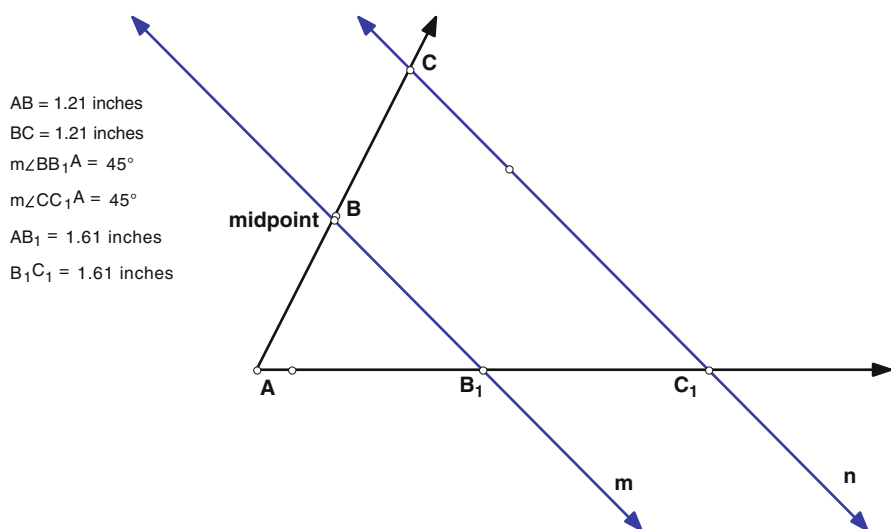
Fig. 1.22   Illustration for Thales' Theorem

We can see that B is the midpoint of AC because $|AB| = |BC| = 1.21$. Lines m and n are parallel to each other $(m||n)$ because they form the same angles (in our case of $45°$) with side AC_1. After this construction we obtain segments of equal length, $|AB_1| = |B_1C_1| = 1.61$ on the second side.

Proof. Since B is the midpoint of AC and $BB_1||CC_1$, then B_1 is a midpoint of segment AC_1. The proof is completed.

Generalized Thales' Theorem (The Split Converse Theorem). If on one side of an angle we construct segments of length $n, l, m, p, q, \ldots$ etc., starting from the vertex of the angle and draw parallel lines through the end points of each segment until they intersect the other side of the angle, then segments of length n_1, l_1, m_1, p_1, q_1 are formed such that $n : l : m : p : q = n_1 : l_1 : m_1 : p_1 : q_1$.

The following example will illustrate how this theorem can be applied (Fig. 1.23).

AB = 0.21 inches

BC = 0.71 inches

CD = 1.12 inches

AB_1 = 0.25 inches

B_1C_1 = 0.86 inches

$\dfrac{AB}{BC} = 0.29$

$\dfrac{AB_1}{B_1C_1} = 0.29$

CD = 1.12 inches

C_1D_1 = 1.36 inches

$\dfrac{BC}{CD} = 0.63$

$\dfrac{B_1C_1}{C_1D_1} = 0.63$

$\dfrac{AB}{CD} = 0.19$

$\dfrac{AB_1}{C_1D_1} = 0.19$
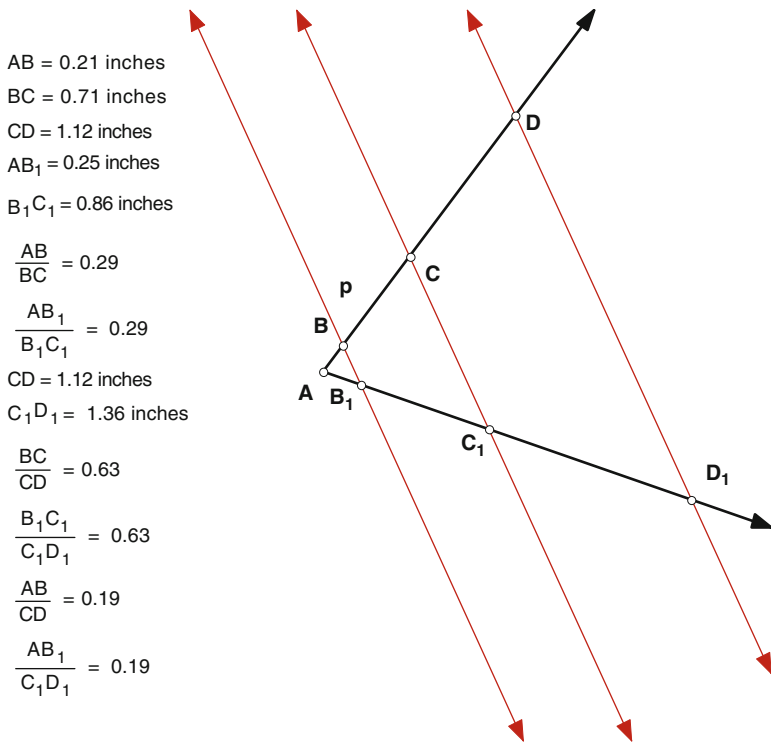
Fig. 1.23 Illustration for generalized Thales' Theorem

We measured the lengths of segments AB, BC, and CD on one side of the angle that were 0.21, 0.71, and 1.12 respectively. Then we constructed parallel lines through point B, C, and D, obtaining points of the intersection with the second side of the angle, B_1, C_1, and D_1. We measured distances $|AB_1|$, $|B_1C_1|$, and $|C_1D_1|$ that were 0.25, 0.86, and 1.36 respectively. The theorem states that $|AB| : |BC| : |CD| = |AB_1| : |B_1C_1| : |C_1D_1|$.

This notation means that any corresponding ratios are the same:

$$\frac{|AB|}{|BC|} = \frac{|A_1B_1|}{|B_1C_1|} = 0.29$$

$$\frac{|BC|}{|CD|} = \frac{|B_1C_1|}{|C_1D_1|} = 0.63$$

$$\frac{|AB|}{|CD|} = \frac{|A_1B_1|}{|C_1D_1|} = 0.19$$

that we can see from Sketchpad measurements.

Proof of the Generalized Thales' Theorem.   Consider two parallel transversals going through points B and C of the angle above. Assume that $|AB| : |BC| = k$. Let us show that $|AB_1| : |B_1C_1| = k$. Since $\Delta ABB_1 \sim \Delta ACC_1$ by the AA property, then the ratios of the lengths of the corresponding sides are the same (Fig. 1.24)
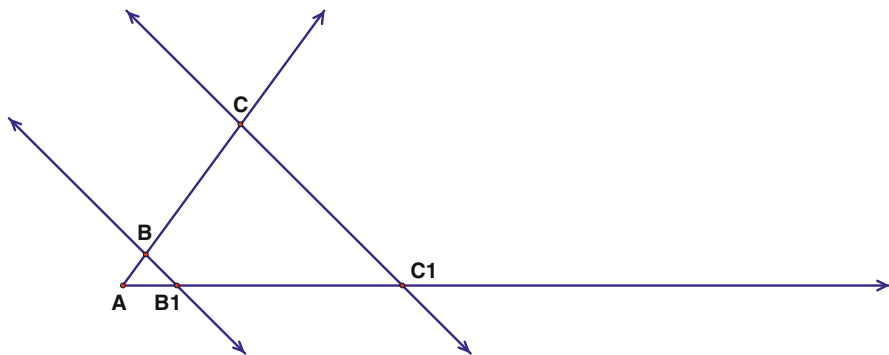
Fig. 1.24   Proof of Thales' Theorem

$$\frac{|AC|}{|AB|} = \frac{|AC_1|}{|AB_1|} = m$$

From this we can state that

$$|AC| = m \cdot |AB|$$

$$|AC_1| = m \cdot |AB_1|$$

$$|BC| = |AC| - |AB| = (m-1) \cdot |AB|$$

$$|BC_1| = |AC_1| - |AB_1| = (m-1) \cdot |AB_1|$$

$$\frac{|B_1C_1|}{|BC|} = \frac{|AB_1|}{|AB|} \Leftrightarrow \frac{|AB|}{|BC|} = \frac{|AB_1|}{|B_1C_1|} = k$$

Note. If you have difficulties following this proof, please see Appendix A on proportions and ratios for clarification. From this point forward, AB will be used in place of $|AB|$ when it is clear from the context that the length of AB is the object of the reference.

Corollary to Thales' Theorem. If lines containing segments AC and BD intersect at a point E and E is either:

(1) Interior to both segments AC and BD, or
(2) exterior to both segments AC and BD

and lines $AB||CD$, then the triangles AEB and CED are similar.

This is true because in case (1) the triangles share the angle $AEB =$ angle CED and else in case (2) these two angles are vertical angles. Therefore, $EC/EA = ED/EB$.

Case 1.
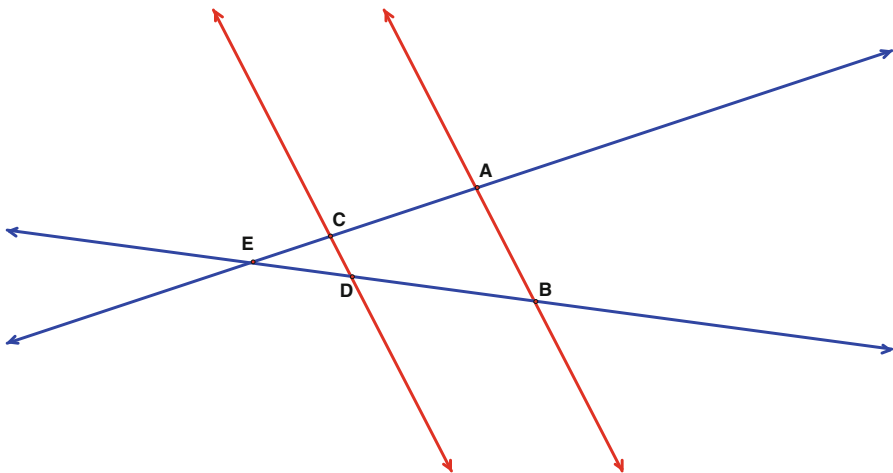
Fig. 1.25 Thales' Theorem, Case 1

Case 2.
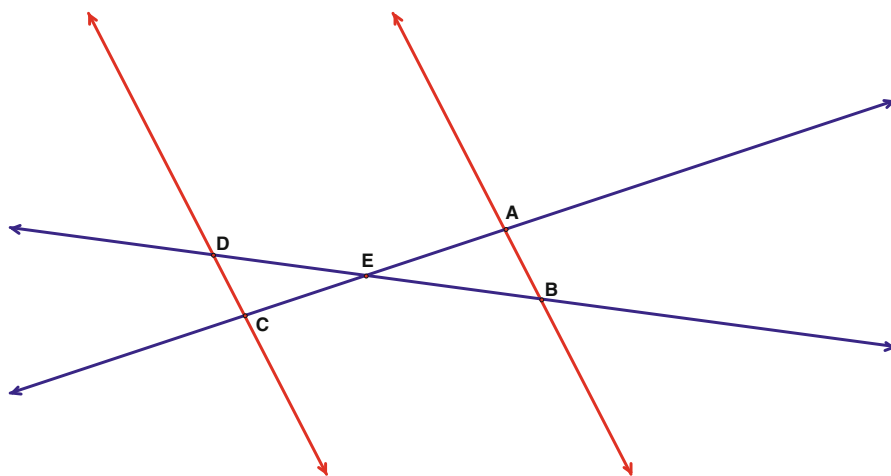
Fig. 1.26 Thales' Theorem, Case 2

Several beautiful problems on the application of Thales' Theorem will be presented in the "Construction" section.

Problem 12. Prove the Pythagorean Theorem using relations between similar triangles formed by the height dropped from the vertex of the right angle of the right triangle.

Proof. Actually this height dropped from a 90° angle is a very important auxiliary element! For example, as it is shown in Fig. 1.27, triangles *DCB* and *ABC* are similar, then the following is true
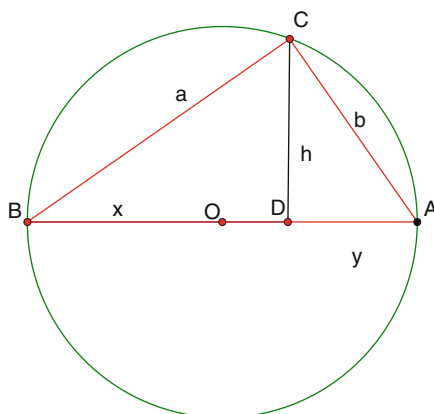
Fig. 1.27 Proof of Pythagorean Theorem using similar triangles

$$\frac{a}{x} = \frac{c}{a}, \quad \text{or} \quad a^2 = x, \ c = AB \tag{1.5}$$

And triangles DCA and ABC are similar, then the following is valid

$$\frac{b}{y} = \frac{c}{b}, \quad \text{or} \quad b^2 = yc, \ x + y = AB = c \tag{1.6}$$

Adding (1.5) and (1.6) we obtain: $a^2 + b^2 = xc + yc = (x + y)c = c^2$. The proof is completed.

In the most of the previous problems sides or angles of triangles are given by numbers. Try Problem 13 which at first glance has an unusual condition.

Problem 13.   Consider a scalene triangle ABC with area S. Point P is in the interior of ABC. Draw three lines through P parallel to each side of the triangle to form three triangles with areas S_1, S_2, and S_3. Find S.

Solution.   In order to start thinking of this problem I recommend drawing a picture (Fig. 1.28).
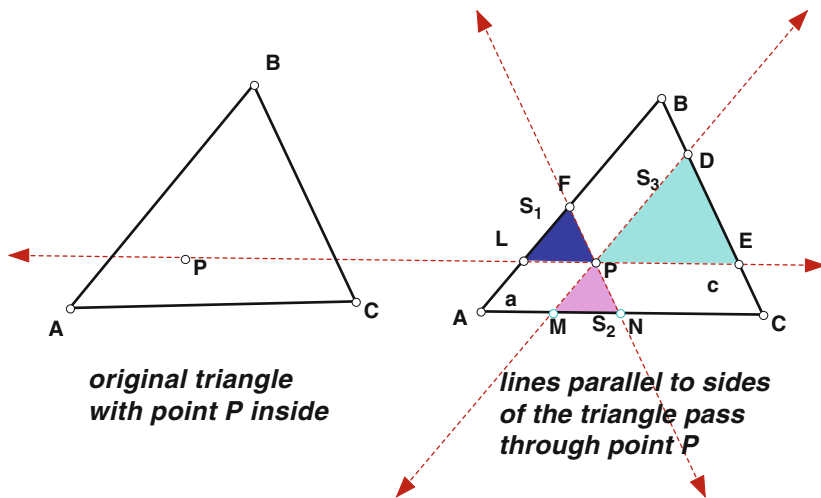
Fig. 1.28   Sketch for Problem 13

Having a picture, we notice that $LE \| AC$, $MD \| AB$, and $FN \| BC$, then we can conclude that $|LP| = |AM|$ and $|PE| = |NC|$. It seems to be reasonable to denote three variables: $a = |AM|$, $b = |MN|$, and $c = |NC|$. Besides, each of the small triangles is similar to triangle ABC.

How can we use this information?

Of course, you remember (if not then the proof will be provided in the following sections) that the ratio of areas of similar triangles equals to the square of the coefficient of similitude, k^2.

However, how can we find these three coefficients?

Again we know the answer to this question: The coefficient of similitude, k, is the ratio of the corresponding sides of similar triangles. Now we are ready to solve the problem.

$$\Delta FLP \sim \Delta ABC \Rightarrow \frac{S_1}{S} = \left(\frac{|LP|}{|AC|}\right)^2$$

$$\Delta MPN \sim \Delta ABC \Rightarrow \frac{S_2}{S} = \left(\frac{|MN|}{|AC|}\right)^2$$

$$\Delta PDE \sim \Delta ABC \Rightarrow \frac{S_3}{S} = \left(\frac{|PE|}{|AC|}\right)^2$$

Replacing $AC = a + b + c$, $LP = a$, $MN = b$, and $PE = c$ and taking square root from the left and the right sides of each equality, we have

$$\frac{\sqrt{S_1}}{\sqrt{S}} = \frac{a}{a+b+c}$$

$$\frac{\sqrt{S_2}}{\sqrt{S}} = \frac{b}{a+b+c}$$

$$\frac{\sqrt{S_3}}{\sqrt{S}} = \frac{c}{a+b+c}$$

Does it look too far from the answer?

Try to add the left and the right sides. Putting expressions over the common denominator, we obtain:

$$\frac{\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}}{\sqrt{S}} = \frac{a+b+c}{a+b+c} = 1$$

that is equivalent to $\sqrt{S} = \sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}$ and squaring gives S:

Answer. $S = \left(\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}\right)^2.$

A similar approach can be used for Problem 14.

Problem 14 (Lidsky). Let P be a point inside an equilateral triangle ABC. Drop three perpendiculars PD, PE and PF to the sides BC, CA, and AB, respectively. Show that $\frac{|PD|+|PE|+|PF|}{|BD|+|CE|+|AF|}$ is a constant and find it.

Solution. We will draw an equilateral triangle ABC with point P inside the triangle. Then draw three lines parallel to the side of the triangle through P. This auxiliary construction will make three other equilateral triangles (shown in Fig. 1.29 in grayscale) and the sum of their sides equals the length of sides of the original triangle. Let $AB = a$. Then the sum of their heights equals the height of triangle ABC, so that

$$|PD| + |PE| + |PF| = \frac{a\sqrt{3}}{2}$$

The other sum $|BD| + |CE| + |AF|$ equals the sum of the sides of the shaded triangles plus half of this sum. Thus, $|BD| + |CE| + |AF| = 3a/2$.
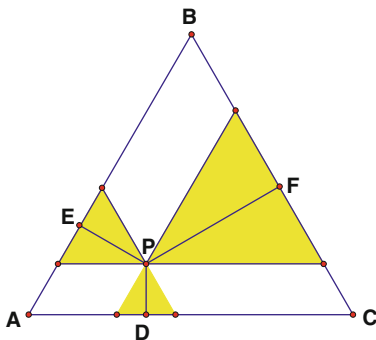
Fig. 1.29 Sketch for Problem 14

Finally,

$$\frac{|PD| + |PE| + |PF|}{|BD| + |CE| + |AF|} = \frac{a\sqrt{3}}{2} \div \frac{3a}{2} = \frac{\sqrt{3}}{3}$$

Note that this ratio is a constant and it does not depend on the length of the sides of the triangle.

The following problem was offered at the 2001 Entrance Exam to Moscow Lomonosov State University and it requires knowledge of similar triangles.

Problem 15. A triangle ABC is given. Point D is on side AC, such that $\angle CBD = 2\angle ABD$. Radii of the circles inscribed into triangles BDC and ADB equal 8 and 4 cm, respectively. The distance between tangent points of these circles with the line AC is $\sqrt{129}$. Find BD exactly.

Solution. Let O_1 be the center of the circle inscribed into triangle ADB; F and M are points of tangency with lines DA and BD. Let O_2 be the center of the circle inscribed into triangle BCD; E and N are points of tangency with lines CD and BD (see Fig. 1.30). Denote $m\angle O_2AD = \alpha$. Since BO_1 and BO_2 are bisectors of angles ABD and DBC, respectively, and using the condition of the problem we get, $m\angle O_2BD = 2\alpha$.

Consider triangles BO_2N and BO_1M. Using the double angle formula for $\tan 2\alpha$ we obtain:

$$BN = \frac{8}{\tan 2\alpha} = \frac{8(1 - \tan^2\alpha)}{2\tan^2\alpha} < \frac{4}{\tan\alpha} = BM$$

and $|DM| < |DN|$. Since $|DF| = |DM|$ and $|DN| = |DE|$ (property of a tangent to a circle), then $|DM| + |DN| = |DE| + |DF| = |EF| = \sqrt{129}$.
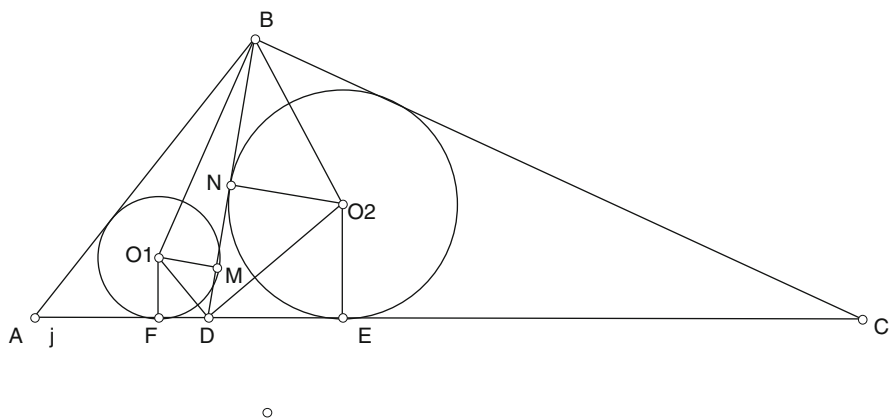
Fig. 1.30 Sketch for Problem 15

Since O_2D is the bisector of angle CDB, O_1D is the bisector of angle ADB, and angles ADM and NDE are supplementary, we have that $m\angle O_1DO_2 = 90°$.

$O_1D \perp O_2D$ and $m\angle O_1DM = m\angle DO_2N$, then $\triangle O_2ND$ and $\triangle DMO_1$ are similar triangles by the two angles (AA) property. Also,

$$\frac{|O_2N|}{|DM|} = \frac{|DN|}{|O_1M|} \Leftrightarrow |DM| \cdot |DN| = 4 \cdot 8 = 32$$

Since $|DM| < |DN|$, $|DM| + |DN| = \sqrt{129}$ and $|DM| \cdot |DN| = 32$, then we obtain

$$|DM| = \frac{\sqrt{129} - 1}{2} \quad \text{and} \quad |DN| = \frac{\sqrt{129} + 1}{2}$$

Let $|AN| = x$, then $|AM| = x + |MN| = x + 1$, then $\tan 2\alpha = 8/x$ add $\tan \alpha = 4/(x + 1)$. Using the formula for the tangent of a double angle, we obtain the following equation:

$$\frac{8}{x} = 2 \cdot \frac{4}{x+1} \div \left(1 - \frac{16}{(1+x)^2} \right)$$

for which $x = 15$. Therefore, $|BD| = x + \frac{\sqrt{129}+1}{2} = \frac{\sqrt{129}+31}{2}$.

What was the most important step in solving this problem?

It was probably the fact that triangles $\Delta O_2 ND$ and ΔDMO_1 are similar triangles by the two angles (*AA*) property.

Answer. $|BD| = \dfrac{\sqrt{129} + 31}{2}$.

1.4.2   Menelaus' Theorem

Menelaus of Alexandria (c. 70–140 AD) was a Greek mathematician who proved a theorem on the collinearity of points on the edges of a triangle. The theorem has his name and it is very useful in solving many geometry problems. For its proof, we will use the properties of similar triangles.

Menelaus' Theorem.   Let ABC (Fig. 1.31) be a triangle and let line l cut the sides of the triangle (extended if necessary) at points D, E, and F. The ratios lengths of segments are then $\frac{|AD|}{|BD|} \cdot \frac{|BF|}{|CF|} \cdot \frac{|CE|}{|AE|} = 1$.
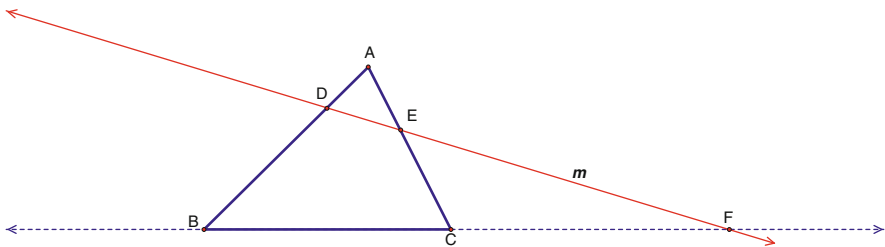

Fig. 1.31   Line intersecting sides of a triangle

Proof.   Let $D = AB \cap m$, $E = AC \cap m$, $F = BC \cap m$. From Fig. 1.31 we can see that intersecting lines m and BA form two equal vertical angles, one of which is

$\angle BDE$. Likewise, intersecting lines m and AC form another pair of vertical angles, $\angle AED = \angle CEF$.

Question. We have two pairs of vertical angles. What kind of construction will give us two pairs of similar triangles?

Hint. Let us draw a line parallel to side BC and passing through vertex A. What does it do for us? Figure 1.32 will help you get the answer and further ideas of solving this problem.
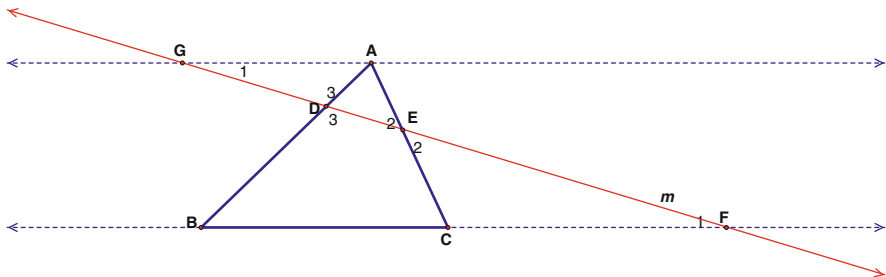
Fig. 1.32 Proof of Menelaus Theorem

Let GA be parallel to BC, then $\Delta AGD \sim \Delta GFB$. Now we have two pairs of similar triangles: $\Delta GDA \sim \Delta BDF$ and $\Delta GAE \sim \Delta ECF$. Considering the corresponding ratios of sides in the similar triangles we obtain:

$$\frac{|DA|}{|BD|} = \frac{|GA|}{|BF|} \Rightarrow |GA| = \frac{|AD| \cdot |BF|}{|BD|} \tag{1.7}$$

$$\frac{|AE|}{|CE|} = \frac{|GA|}{|CF|} \Rightarrow |GA| = \frac{|AE| \cdot |CF|}{|CE|} \tag{1.8}$$

Dividing (1.7) by (1.8) and regrouping fractions we have the following:

$$\frac{|AD| \cdot |BF|}{|BD|} \cdot \frac{|CE|}{|AE| \cdot |EF|} = \frac{|AD|}{|BD|} \cdot \frac{|BF|}{|CF|} \cdot \frac{|CE|}{|AE|} = 1$$

The proof is completed.

Problem 16. Given a triangle ABC and points K and M on sides AB and AC, respectively. It is known that $AK{:}KB = 2{:}3$ and $AM{:}MC = 4{:}5$. Find the ratio at which the point of the intersection of lines KC and BM divides segment BM.

Solution. First, we will draw a triangle ABC (Fig. 1.33).
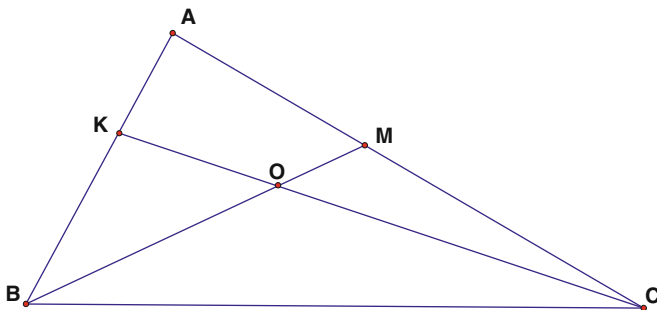
Fig. 1.33 Sketch for Problem 16

Let O be the intersection of lines KC and BM. Let $AK = 2x, KB = 3x, AM = 4y$, $MC = 5y$, so that $AB = 5x$ and $AC = 9y$. Applying Menelaus' Theorem to the triangle ABM and transversal KO, we have

$$\frac{|AK|}{|KB|} \cdot \frac{|BO|}{|OM|} \cdot \frac{|MC|}{|CA|} = 1 \Rightarrow \frac{|BO|}{|OM|} = \frac{9y}{5y} \cdot \frac{3x}{2x} = \frac{27}{10}$$

Answer. $|BO|:|OM| = 27:10$.

1.5   Cevians of a Triangle

A segment from one vertex of a triangle to a point on the opposite side is called a *cevian*. Thus, medians, heights, bisectors are cevians of a triangle. A cevian is named after **Giovanni Ceva**, an early seventeenth century Italian geometer.

1.5.1   Ceva's Theorem

If you draw a cevian from each vertex of a triangle, in general, all three cevians will not run through the same point, as for example in Fig. 1.34.
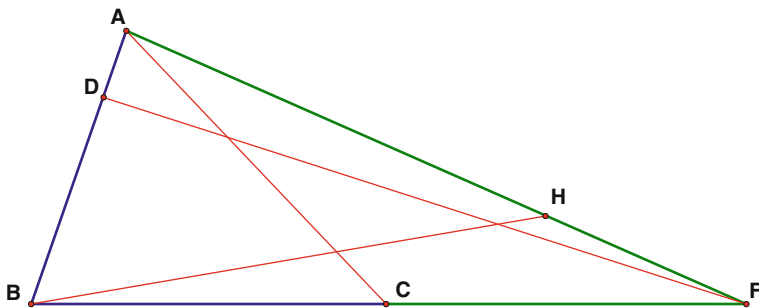
Fig. 1.34 Cevians do not concur

Ceva was very interested in finding the condition under which three lines would concur and stated his theorem in his paper titled "De lineis rectis se invicem secantibus, static constructione." It is interesting that his original proof was based on mechanical statics. The theorem was stated as:

Let point O be in the plane of a triangle ABC. If the lines connected each vertex with point O meet the sides AB, AC and BC in points, C_1, B_1, A_1, respectively and cut on two segments on each side, then the product of each three segments without common vertex are equal, (i.e., $|AC_1| \cdot |BA_1| \cdot |CB_1| = |AB_1| \cdot |CA_1| \cdot |BC_1|$).

I think that this theorem is easier to understand if we mark segments that do not have a common vertex by different colors. As shown in Fig. 1.35, AC_1, BA_1, and CB_1, are shown in red, and the other triple remains black.
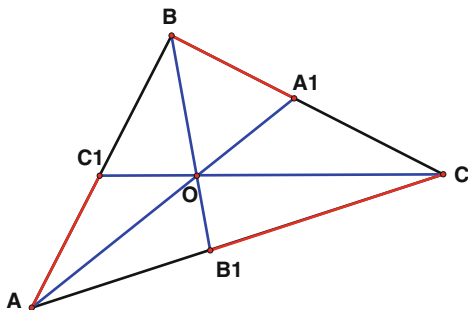
Fig. 1.35 Original Ceva's Thereom

Below is the proof of the original Ceva's Theorem (Fig. 1.36).

$\triangle A_1HB \sim \triangle DCA_1$

$$\frac{BH}{CD} = \frac{BA_1}{CA_1}$$

Fig. 1.36 Proof of original Ceva's Theorem

Consider two triangles AOB and AOC (shaded in our sketch) that share the same side AO. Then the ratio of their areas equals the ratio of their heights BH and CD, dropped from vertices B and C, respectively. This can be written as

$$\frac{[AOB]}{[AOC]} = \frac{|BH|}{|CD|} \tag{1.9}$$

On the other hand, we can recognize a pair of similar right triangles for which $\triangle A_1HB \sim \triangle DCA_1$, and

$$\frac{|BH|}{|CD|} = \frac{|BA_1|}{|CA_1|} \tag{1.10}$$

Combining (1.9) and (1.10) we can state

$$\frac{[AOB]}{[AOC]} = \frac{|BA_1|}{|CA_1|}$$

Without loss of generality we can state similar relationships for two other pairs of triangles $\{BOC, BOA\}$ and $\{COA, COB\}$. Thus,

$$\frac{[AOB]}{[AOC]} = \frac{|BA_1|}{|CA_1|}$$

$$\frac{[BOC]}{[BOA]} = \frac{|CB_1|}{|AB_1|} \tag{1.11}$$

$$\frac{[COA]}{[COB]} = \frac{|AC_1|}{|BC_1|}$$

Multiplying the left and right sides of (1.11) we obtain 1 on the left side:

$$1 = \frac{|BA_1| \cdot |CB_1| \cdot |AC_1|}{|CA_1| \cdot |AB_1| \cdot |BC_1|}$$

or the required formula,

$$|AC_1| \cdot |BA_1| \cdot |CB_1| = |AB_1| \cdot |CA_1| \cdot |BC_1|$$

Remark. Further in the text, in order to prove formulas (1.11), we instead of consideration of similar triangles use an algebraic approach based on properties of equal ratios.

1.5.1.1 Modern Formulation of Ceva's Theorem

> **Theorem.** Let ABF be any triangle and E be an interior point. If lines from the vertices passing through E meet at C, D, and H, then it is true that $\frac{|BC|}{|CF|} \cdot \frac{|FH|}{|HA|} \cdot \frac{|AD|}{|DB|} = 1.$

The theorem actually works in two directions: if the calculation holds, then the segments concur and if the segments concur, then the calculation holds. Therefore, three cevians of a triangle concur iff the ratios of the segment lengths they create satisfy the main equation above.

Remark. In order to use this theorem correctly, one way to remember it is to think about walking around the triangle. Pick any vertex to start (for example, B), and walk around in either direction. Insert each segment into the ratios as you come to it and then they'll be in the right place.

Proof of Ceva's Theorem Using Menelaus' Theorem. Considering Fig. 1.37, we will apply Menelaus' Theorem twice. First, consider triangle ABC and let line DEF be its transversal so that

$$\frac{|BF|}{|FC|} \cdot \frac{|CE|}{|EA|} \cdot \frac{|AD|}{|DB|} = 1$$

Next, applying the same theorem to triangle ACF and its transversal BEH, we have

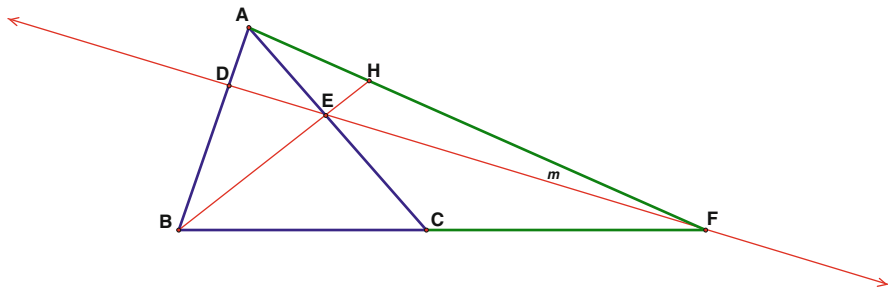$$\frac{|CE|}{|EA|} \cdot \frac{|AH|}{|HF|} \cdot \frac{|FB|}{|BC|} = 1$$

Fig. 1.37 Proof of Ceva's Theorem using Menelaus Theorem

Ceva's Theorem follows by dividing these two equations

$$\frac{|AD|}{|DB|} \cdot \frac{|BC|}{|CF|} \cdot \frac{|FH|}{|AH|} = 1$$

1.5.1.2   Connection to Menelaus' Theorem

Suppose we have a triangle *ABC* and line *m* is its transversal that crosses sides *AB*, *BC*, and *AC* at points *D*, *F* and *E*, respectively (Fig. 1.38).

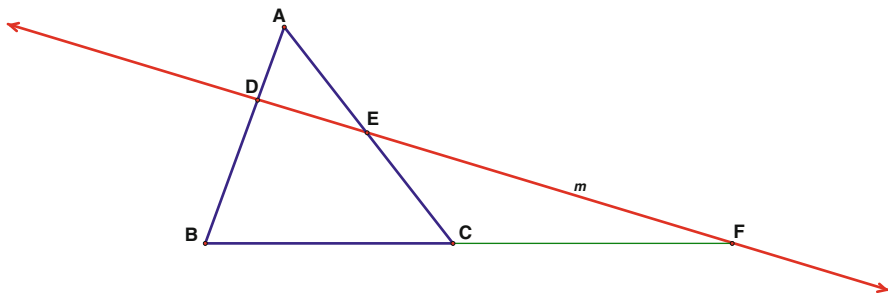**Fig. 1.38** Proof of Ceva's Theorem, first sketch

1. By Menelaus' Theorem for triangle *ABC* we obtain:

$$\frac{|BF|}{|FC|} \cdot \frac{|CE|}{|EA|} \cdot \frac{|AD|}{|DB|} = 1 \qquad\qquad (1.12)$$

2. We can connect *A* and *F* and form a triangle *ABF*. Continue *BE* until its intersection with *AF* at *H*. Then *E* is the point at which three cevians meet, *FD* ∩ *BH* ∩ *AC* = *E* (Fig. 1.39).
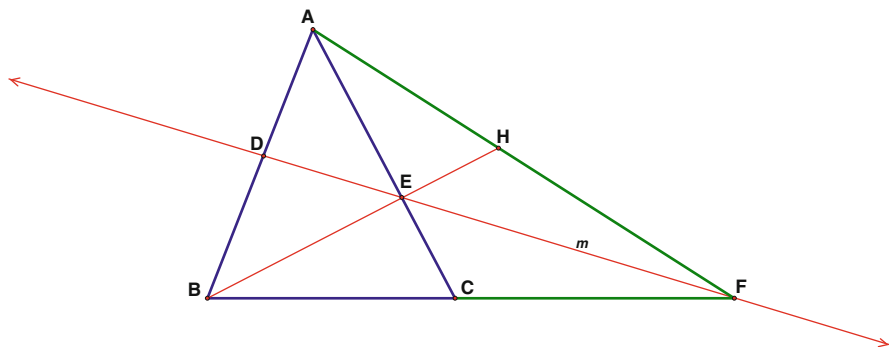
Fig. 1.39 Proof of Ceva's Thereom, auxiliary construction

Then from Ceva's Theorem the following must be also true:

$$\frac{|BC|}{|FC|} \cdot \frac{|AD|}{|DB|} \cdot \frac{|FH|}{|HA|} = 1 \qquad (1.13)$$

So if we take points D, C and H on sides AB, BC and FA respectively and (1.13) is valid then AC, BH and DF intersect at the same point E, concurrent. Moreover, (1.12) will hold since D, E and F are **collinear**.

Problem 17. Points M, N, K belong to sides AB, BC and CA, respectively. $|AM|$: $|MB| = 1 : 4 |BN| : |NC| = 2 : 3$, It is known that the cevians AN, BK and CM are concurrent. What is the ratio $|CK| : |KA|$?

Solution. First, let us sketch the triangle with intersecting cevians (Fig. 1.40).

Fig. 1.40 Sketch for Problem 17

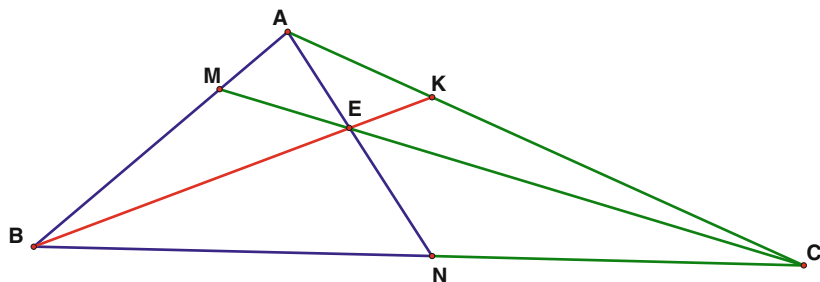Since the cevians are concurrent then by Ceva's Theorem the following is valid:

$$\frac{|AM|}{|MB|} \cdot \frac{|BN|}{|NC|} \cdot \frac{|CK|}{|KA|} = 1$$

Substituting here given ratios we have

$$\frac{1}{4} \cdot \frac{2}{3} \cdot \frac{|CK|}{|KA|} = 1$$

From which we obtain that $|CK|{:}|KA| = 6{:}1$. Otherwise, the cevians will not **concur** (intersect at one point).

Answer. 6:1.

1.5.2   The Median and its Properties

A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side. Three medians of a triangle concur.

Can you prove this now? If not, the answer follows.

Theorem of the Three Medians. (a) All medians of a triangle intersect at one point (concur), the center of gravity of the triangle. (b) This point divides each median in the ratio 2:1 starting from a vertex.

Proof. Part (a) can be easily proved using Ceva's Theorem. Try it yourself knowing that each median goes through a midpoint of a corresponding segment.

Proof of Part (b). Consider a triangle ABC. The two medians BD and AE have D and E as midpoints of AC and BC, respectively as shown in Fig. 1.41.
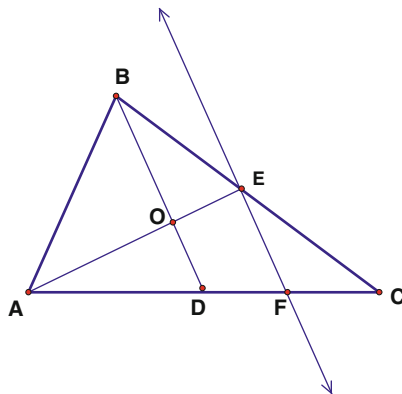

Fig. 1.41 Theorem of Three Medians

Let $AE \cap BD = O$. Draw a line through midpoint E of segment BC parallel to BD so that F is the midpoint of segment DC, i.e., $|DF| = 1/2\,|DC|$ and also $|DC| = |AD|$. Note that triangles AOD and AEF are similar. Therefore, it follows from Thales' Theorem that $|AO| : |OE| = |AD| : |DF| = 2 : 1$.

The Length of the Median Theorem. Let m_a, m_b, m_c be the medians dropped to sides a, b, and c, respectively of a triangle. Then (1.14) express the median lengths in terms of the lengths of the sides.

$$4m_a{}^2 = 2b^2 + 2c^2 - a^2$$
$$4m_b{}^2 = 2a^2 + 2c^2 - b^2 \qquad\qquad (1.14)$$
$$4m_c{}^2 = 2a^2 + 2b^2 - c^2$$

Proof. Consider Fig. 1.41 and let

$$AB = c$$
$$BC = a$$
$$AC = b$$
$$AD = \frac{b}{2}$$
$$\angle BDA = \alpha$$
$$\angle BDC = \pi - \alpha$$

Applying the Law of Cosines to triangles ABD and BDC we obtain

$$AB^2 = AD^2 + BD^2 - 2 \cdot AD \cdot BD \cdot \cos\alpha,$$
$$BC^2 = BD^2 + DC^2 - 2 \cdot BD \cdot DC \cdot \cos(\pi - \alpha)$$

or

$$c^2 = \frac{b^2}{4} + m_b^2 - b \cdot m_b \cdot \cos\alpha,$$
$$a^2 = \frac{b^2}{4} + m_b^2 - b \cdot m_b \cdot \cos(\pi - \alpha)$$

Adding the left and the right sides of the expressions and remembering that $\cos(\pi - \alpha) = -\cos\alpha$, we obtain the desired formula: $4m_b^2 = 2a^2 + 2c^2 - b^2$. The proof is completed.

Fact. Each median divides a triangle into two triangles of the same area.

This fact becomes almost obvious if we draw a picture and drop the median and height from the same vertex of a triangle. In Fig. 1.42, median BD divides triangle ABC into two triangles, ABD and BDC. If segment BE is perpendicular to AC, then BE is the height of triangles ABC, ABD, and BDC.
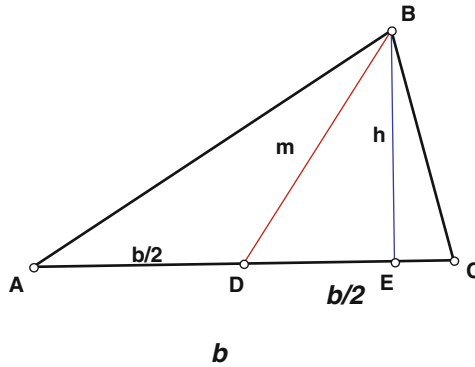

Fig. 1.42 Property of a median

Denoting $|BE| = h$, $|AC| = b$, and assuming that $|AD| = |DC| = b/2$ (D is a midpoint by the definition of a median), we can express the areas of all the triangles in terms of h and b. We will sometimes denote the area of triangle ABC as $[ABC]$ for simplicity:

$$[ABD] = \frac{1}{2} \cdot |AD| \cdot |BE| = \frac{1}{2} \cdot \frac{b}{2} \cdot h, \quad [BDC] = \frac{1}{2}|DC| \cdot |BE| = \frac{1}{2} \cdot \frac{b}{2} \cdot h$$

Knowing that the area of $[ABC] = \frac{1}{2} \cdot b \cdot h$, we can see that the area of each small triangle is exactly one-half of the area of the big triangle ABC. Moreover, we can state that areas of triangles with the same height are proportional to their bases.

1.5.3 Altitude of a Triangle

A segment from a vertex of a triangle perpendicular to the opposite side is an **altitude** (height) of the triangle. The three altitudes of a triangle concur. Can you prove this?

Theorem of the Three Perpendicular Bisectors of a Triangle. The perpendicular bisectors of the sides of a triangle intersect at a single point. This point is the center of a circumscribed circle. For a right triangle inscribed into a circle, the center of the circle is the midpoint of the hypotenuse. Thus, the radius of a circumscribed circle is one-half the hypotenuse and equals the length of the median dropped from the vertex of the right angle.

Proof. Consider a triangle EFD of Fig. 1.43. The perpendicular bisector of ED is the set of all points that are equidistant from both E and D. So for any point on the perpendicular bisector, e.g., point Q, we have $EQ = QD$. Similarly, since the perpendicular bisector of segment EF is the set of all points equidistant from E and F, then for any point on the bisector, e.g., point P, we have (the lengths of) $EP = PF$. Next, the point of the intersection of two perpendicular bisectors, O, is such that it is equidistant from points on D and E and also from points E and F so that (the lengths of) the segments $EO = OD = OF$. Finally, if O is equidistant from D and F, then it lies on the perpendicular bisector of DF. Therefore, the perpendicular bisectors concur.

Theorem of Three Altitudes of a Triangle. All altitudes of a triangle concur. Such a point is called the **orthocenter** of the triangle.

Proof. Let ABC be a given triangle and let us drop perpendiculars from each vertex to the opposite sides (Fig. 1.43). Next we will draw parallel lines to all sides of the triangle. Their intersection will form a new triangle DEF. Since $AB \parallel DF$, $AC \parallel EF$, and $BC \parallel ED$, then $ABFC$ is a parallelogram with $AC = BF$. On the other hand $EBAC$ is also parallelogram with sides $AC = EB$, so that B is a midpoint of EF and BG is perpendicular to EF. It also follows that BG is the perpendicular bisector of EF. Using similar arguments we can easily show that C is the midpoint of DF and that HC is the perpendicular bisector of side DF. Finally line AN is the perpendicular bisector of ED. However, we know that three perpendicular bisectors of the same triangle intersect at the same point, its **circumcenter**. Thus, point O in Fig. 1.43 is both the circumcenter of triangle EDF and orthocenter of triangle ABC. Therefore, the three altitudes of a triangle concur.
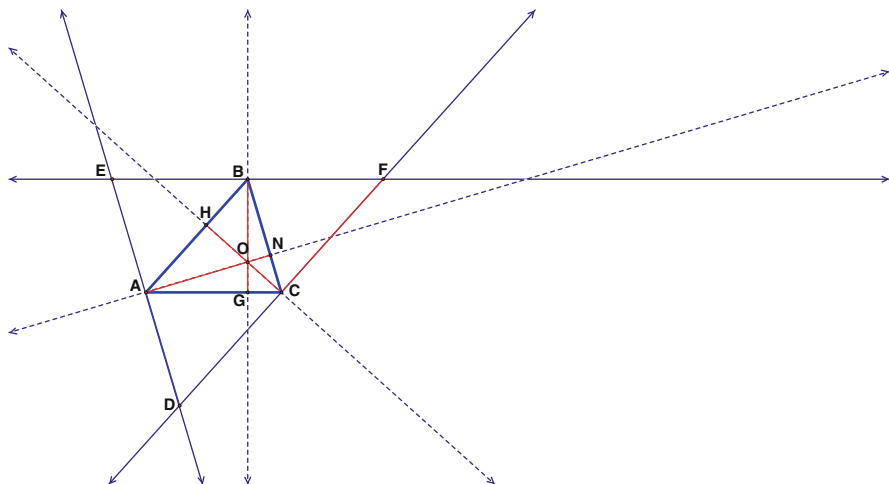
Fig. 1.43 Theorem of Three Altitudes

Problem 18. If the lengths of two altitudes of a triangle *ABC* are greater than 1, prove that the area of *ABC* must be greater than 1/2.

Proof. Let us draw a triangle and drop two altitudes in it (Fig. 1.44).
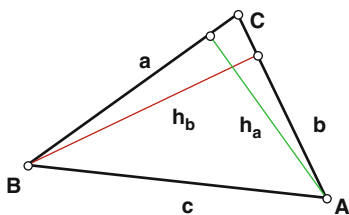
Fig. 1.44 Sketch for Problem 18

If $h_a > 1$ and $h_b > 1$ are perpendicular to sides *BC* and *CA*, respectively, then $c > h_a > 1$ and $a > h_b > 1$ since they are hypotenuses of right triangles.

Evaluating the area of triangle *ABC* yields

$$[ABC] = \frac{h_a \cdot a}{2} > \frac{1 \cdot a}{2} > \frac{1 \cdot 1}{2} = 1/2$$

We proved it with elegance.

Some interesting problems on construction including the orthocenter of a triangle are presented in Chapter 4 of the book.

1.5.4   Angle Bisector and its Properties

Theorem of the Three Angle Bisectors. In a triangle all angle bisectors intersect at one point. This point is the center of the circle inscribed into the triangle.
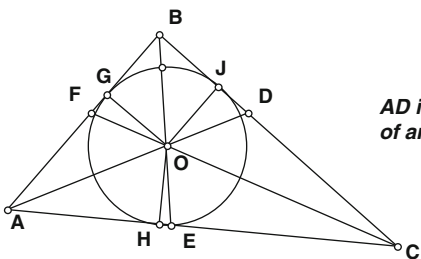

One of the proofs of this statement can be done easily with the use of Ceva's Theorem and knowledge of some properties of angle bisectors that are given below. However, just for demonstration, we sketch

Figure 1.45 with the use of Geometry Sketch Pad and obtain the following:

m∠ABE = 45°   m∠BCF = 18°

m∠EBC = 45°   m∠FCA = 18°

m∠BAD = 27°   m∠DAC = 27°

AD is the bisector of angle BAC, BE is the bisector of angle ABC, and CF is the bisector of BCA.

m∠BAC = 54°

m∠ACB = 36°

m∠BJO = 90°   m∠OHA = 90°   m∠AGO = 90°

OJ = 0.62 inches

OH = 0.62 inches

GO = 0.62 inches

r=OJ=OH=OG=0.62 is the radius of the inscribed circle

Fig. 1.45 Three angle bisectors

Triangle Angle Bisector Theorem. A bisector of an angle of a triangle divides the opposite side of the triangle into segments, which are proportional to the adjacent sides of the triangle.

The following formulas are valid.

$$\frac{b}{c} = \frac{a_b}{a_c}, \quad \frac{a}{b} = \frac{c_a}{c_b}, \quad \frac{c}{a} = \frac{b_c}{b_a} \tag{1.15}$$

For example, in Fig. 1.46, the bisector of $\angle BAC$ of triangle ABC divides the opposite side into segments a_b and a_c adjacent to sides b and c, respectively, so that $\dfrac{a_b}{a_c} = \dfrac{b}{c}$.
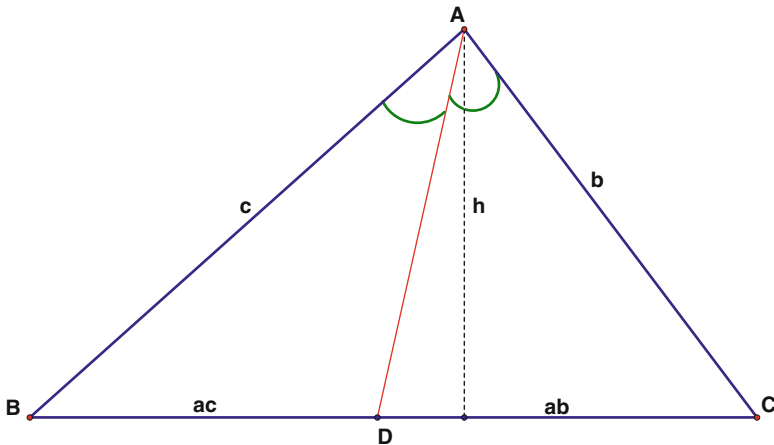

Fig. 1.46   Angle bisector

Proof. Since AD is the bisector of $\angle BAC$, then $\angle BAD = \angle DAC = \alpha$. Next, we will write the areas of triangles BAD and DAC in two different ways: (1) using the half base and height formula and (2) using a half product of two sides and the angle between them. Then we have

$$\frac{[DAC]}{[BAD]} = \frac{0.5h \cdot a_b}{0.5h \cdot a_c} = \frac{0.5b \cdot AD \cdot \sin\alpha}{0.5c \cdot AD \cdot \sin\alpha}$$

Canceling common factors, we obtain the required relation.

Problem 19. Given that the bisector of a right angle of a triangle, ABC, divides the opposite side into segments of lengths 1 and 2, find $[ABC]$.

Solution. Denote by x and y the legs of the triangle as shown in Fig. 1.47 where the right angle is $\angle BAC$. From the angle bisector theorem, it follows that if $x/y = 1/2$, then $y = 2x$ and $y > x$.
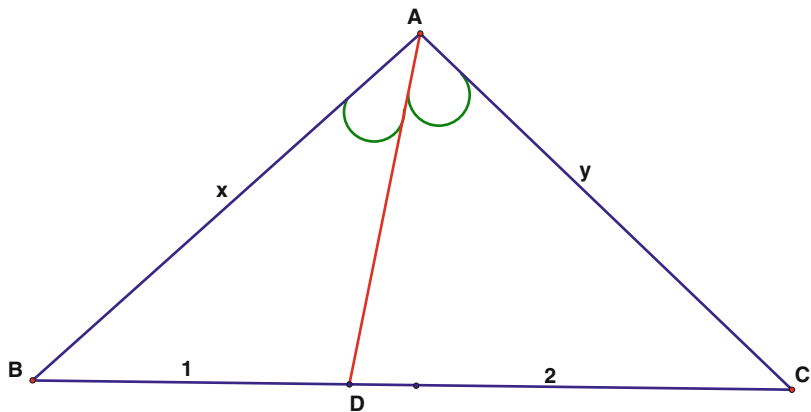
Fig. 1.47 Sketch for Problem 19

From the Pythagorean Theorem, $x^2 + y^2 = (1 + 2)^2 = 9$. So $5x^2 = 9$.
At last, the area of the right triangle is $\frac{1}{2}xy = \frac{x \cdot 2x}{2} = x^2 = \frac{9}{5} = 1.8$.

Answer. 1.8.

1.5.4.1   Median, Bisector and Height from a Vertex

Here are formulas for the bisector, l_a, median, m_a, and height, h_a dropped from the same vertex to the opposite side, a, of triangle ABC (Fig. 1.48):

$$l_a = \frac{2bc \cdot \cos\left(\frac{\angle A}{2}\right)}{b + c}$$

$$l_a{}^2 = bc - a_b \cdot a_c$$

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

$$h_a = \frac{bc}{a} \cdot \sin(\angle A)$$

Note. By analogy, you can rewrite these with respect to other sides.

Fig. 1.48 Median, bisector and height dropped from the same vertex
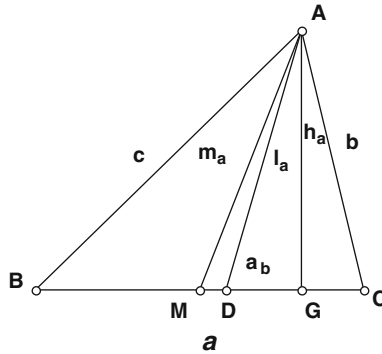
Consider further the triangle ABC of Fig. 1.48 where $AB = c$, $CB = a$, and $AC = b$. M is the midpoint of BC and $m_a = AM$ is a median. If $m\angle BAD = m\angle DAC$, then $l_a = AD$ is the bisector of $\angle BAC$. Let $BD = a_c$, $DC = a_b$, $AG \perp BC$ and $h_a = AG$ is the height of triangle ABC.

The length of the angle bisector can be evaluated in two different ways:

1. The length of the angle bisector dropped to a side of a triangle equals twice the product of the two other sides of the triangle and the cosine of the half angle between them divided by the sum of these sides.

$$l_a = \frac{2bc \cdot \cos\frac{A}{2}}{b + c}; \quad l_b = \frac{2ac \cdot \cos\frac{B}{2}}{a + c}; \quad l_c = \frac{2ab \cdot \cos\frac{C}{2}}{a + b} \qquad (1.16)$$

Proof. In order to prove these formulas we will find the areas of triangles ABC, AMC and BMC (see Fig. 1.49).

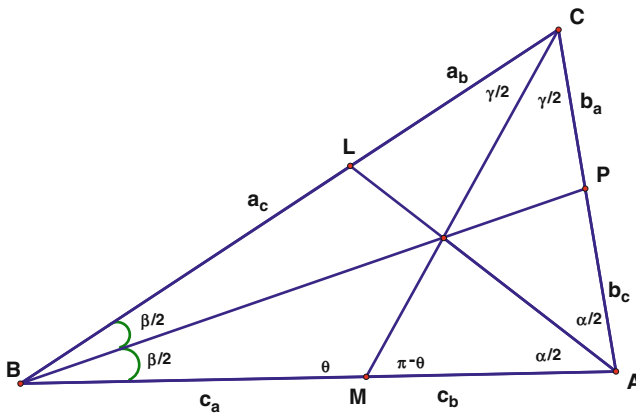**Fig. 1.49** Length of angle bisector

$$\begin{cases} [ABC] = \dfrac{1}{2}|AC| \cdot |BC| \cdot \sin(\angle ACB) = \dfrac{1}{2}ab\sin\gamma \\[2mm] [AMC] = \dfrac{1}{2}|AC| \cdot |MC| \cdot \sin(\angle ACM) = \dfrac{1}{2}al_c\sin\dfrac{\gamma}{2} \\[2mm] [BMC] = \dfrac{1}{2}|BC| \cdot |MC| \cdot \sin(\angle BCM) = \dfrac{1}{2}bl_c\sin\dfrac{\gamma}{2} \end{cases}$$

It is clear that if $[ABC] = [AMC] + [BMC]$, then

$$\frac{1}{2}ab\sin\gamma = \frac{1}{2}al_c\sin\frac{\gamma}{2} + \frac{1}{2}bl_c\sin\frac{\gamma}{2}$$

Cancelling common factors and using the formula for the sine of a double angle, $\sin\gamma = 2\sin\frac{\gamma}{2} \cdot \cos\frac{\gamma}{2}$, we obtain the equation:

$$l_c = \frac{2ab \cdot \cos\frac{\gamma}{2}}{a+b} = \frac{2ab \cdot \cos\frac{C}{2}}{a+b}$$

2. The square of the angle bisector dropped to a side of a triangle equals the difference of the product of the lengths of the two other and the product of the lengths of the segments on which the bisector cuts the side of the triangle, i.e.,

$$l_a{}^2 = bc - a_b a_c; \quad l_b{}^2 = ac - b_a b_c; \quad l_c{}^2 = ab - c_a c_b \qquad (1.17)$$

Let us use Fig. 1.49 and prove this formula for the length of bisector CN.

Proof. Consider the Cosine Theorem for triangles AMC and BMC.

$$\begin{cases} |AC|^2 = |AM|^2 + |CM|^2 - 2|AM| \cdot |CM| \cdot \cos(\angle AMC) \\ |BC|^2 = |BM|^2 + |CM|^2 - 2|BM| \cdot |CM| \cdot \cos(\angle BMC) \end{cases}$$

Angles AMC and BMC are supplementary so $\cos(\pi - \theta) = -\cos\theta$ where $\theta = \angle BMC$. Using the notation of Fig. 1.49, we can rewrite the system as

$$\begin{cases} b^2 = c_b^2 + l_c^2 + 2c_b l_c \cos\theta \\ a^2 = c_a^2 + l_c^2 - 2c_a l_c \cos\theta \end{cases}$$

In order to solve this system we will multiply the top equation by c_a and the bottom by c_b and add in order to cancel the $2c_b c_a l_c \cdot \cos\theta$ term. Now we have

$$a^2 c_b + b^2 c_a = c_a{}^2 c_b + l_c{}^2 c_b + c_b{}^2 c_a + l_c{}^2 c_a \qquad (1.18)$$

Is this too far from the formula that we are trying to prove?

Not really if we remember another property of the angle bisector, that is

$$\frac{a}{c_a} = \frac{b}{c_b}$$
$$ac_b = bc_a \qquad (1.19)$$

We will now rewrite each term in (1.18) in a different form and substitute (1.19) into it. First, for the left side:

$$a^2 c_b + b^2 c_b = a \cdot ac_b + b \cdot bc_a = a \cdot bc_a + b \cdot ac_b = ab(c_a + c_b) \qquad (1.20)$$

Now for the right hand side:

$$c_a c_a c_b + c_b c_b c_a + l_c^2 \cdot (c_b + c_a) = (c_a c_b + l_c^2) \cdot (c_b + c_a) \qquad (1.21)$$

By equating (1.20) to (1.21) and dividing both sides by $(c_b + c_a)$, we obtain the relations for the length of the bisector CM

$$ab = l_c{}^2 + c_a \cdot c_b$$
$$l_c{}^2 = ab - c_a \cdot c_b$$

The proof is completed. Using arguments similar to these leads us to the proof of the other two formulas.

Problem 20. Consider triangle ABC with length of $AB = 21$, BD a bisector of angle ABC, $BD = 8\sqrt{7}$, and $DC = 5$. Evaluate the perimeter of the triangle ABC.

Solution. Let us draw a triangle and mark all known information on it (Fig. 1.50). In order to evaluate the perimeter we need to find the lengths of sides AD and BC. Let $AD = x$ and $BC = y$.
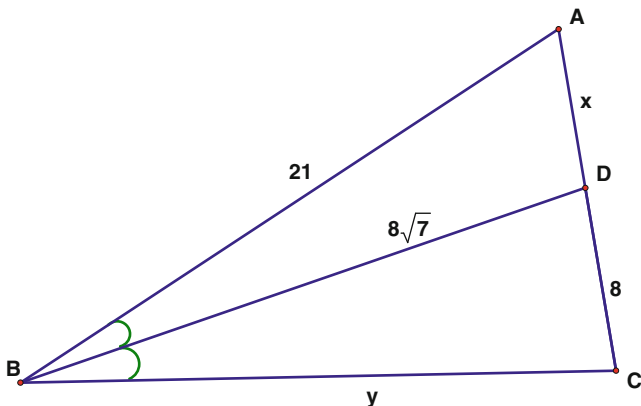
Fig. 1.50 Sketch for Problem 20

Using the property of the angle bisector (1.15) and the second formula for its length (1.17), we obtain the following

$$\frac{AB}{BC} = \frac{AD}{DC} \Rightarrow \frac{21}{y} = \frac{x}{8}$$

$$xy = 168 \qquad\qquad (1.22)$$

$$BD^2 = AB \cdot BC - AD \cdot DC$$

$$\left(8\sqrt{7}\right)^2 = 21y - 8x$$

$$448 = 21y - 8x \qquad\qquad (1.23)$$

Next, we will solve the system of (1.22) and (1.23). Remembering the inequalities $x > 0$ and $y > 0$, we obtain the answer

$$\begin{cases} xy = 168 \\ 21y = 8x + 448 \end{cases} \Rightarrow \begin{cases} x(8x + 448) = 168 \cdot 21 \\ 21y = 8x + 448 \end{cases} \Rightarrow x = 7, y = 24$$

Finally, the perimeter is the sum of sides AB, AD, DC and BC, i.e., $P = 60$.

Answer. $P = 60$.

Problem 21 brings together facts on bisectors, medians, and heights.

Problem 21. Consider a right non-isosceles triangle, CAN. A bisector, a median, and a height are dropped from the vertex of the right angle. Prove that this bisector is also bisects the angle between the median and height.

Solution. Let us draw a picture. Triangle *CAN* is a right triangle with $m\angle C = 90°$. Segments *CB*, *CM*, and *CH* are the bisector, median, and height, respectively dropped from the vertex *C* to side *AN*.
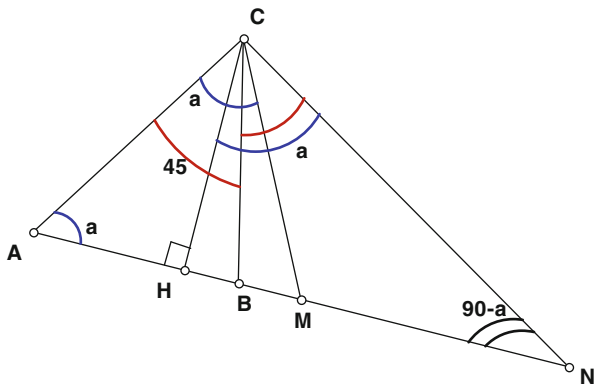

Fig. 1.51   Sketch for Problem 21

CB is a bisector, it divides the right angle, $\angle CAN$, into two equal angles, then $m\angle ACB = m\angle BCN = 45°$. We need to prove that $m\angle HCB = m\angle MCB$. Assuming that $m\angle A = \alpha$, we can express other angles in terms of α. In Fig. 1.51, $m\angle N = 90° - \alpha$. Because $CH \perp AN$, then we can conclude that $m\angle HCN = \alpha$. Now by the angle additional property we can evaluate the $m\angle HCB$ in terms of α,

$$m\angle HCB = \alpha - 45° \tag{1.24}$$

How can we show that $m\angle HCB = m\angle BCM$?

Let us recall some properties of right triangles. Let *CM* be the median dropped from the 90° angle.

What does this give us?

AM = MN. Yes, but what is else?

If *M* is the midpoint of the hypotenuse and the center of the circle circumscribed over the triangle *CAN*, then *MC = MA = MN*!

Fact. Any median dropped from the vertex of a right triangle divides such a triangle into two isosceles triangles.

Now we can find everything we need. In triangle *AMC*, if *AM = MC*, then $m\angle AMC = m\angle MCA = \alpha$. In the picture, we show all angles equal to α with blue arcs, angles of 45° with red arcs, and angles of $(90° - \alpha)$ with a double black arc. Now we can easily determine $m\angle BCM$,

$$m\angle BCM = m\angle NCH - m\angle NCB = \alpha - 45° \tag{1.25}$$

If the right sides of (1.24) and (1.25) are equal, then the left sides must be equal and $m\angle HCB = m\angle MCB$.

Fact. A bisector of the 90° angle of a right triangle also bisects the angle formed by the height and median dropped from the same vertex as the bisector.

1.6   Area of a Triangle

1.6.1   Five Important Formulas

There are five formulas for calculation of an area of a triangle that are worth being able to derive and remember:

$$S = \frac{1}{2}a \cdot h_a \tag{1.26}$$

$$S = \frac{1}{2}a \cdot b \cdot \sin(\angle C) \tag{1.27}$$

$$S = p \cdot r \tag{1.28}$$

$$S = \frac{abc}{4R} \tag{1.29}$$

$$S = \sqrt{p(p-a)(p-b)(p-c)} \tag{1.30}$$

Here and below a, b, and c are sides of a triangle; A, B, C are the corresponding opposite angles; h_a, h_b, and h_c are heights dropped to sides a, b, and c, respectively; p is a semiperimeter of a triangle; r is the radius of the circle inscribed into a triangle (**inradius**); R is the radius of the circle circumscribed about a triangle (**circumradius**). If the first two formulas are well known, the last three are not. Nevertheless, formulas (1.26)–(1.30) are very useful.

Let us show how each formula can be derived. The proofs can vary so I encourage you to find your own proof.

Problem 22. Prove that the area of a triangle equals half the product of its base and corresponding height: $S = \frac{1}{2}a \cdot h_a$.

Proof. Draw a triangle ABC and fill it up to the rectangle as it is shown in Fig. 1.52 below.
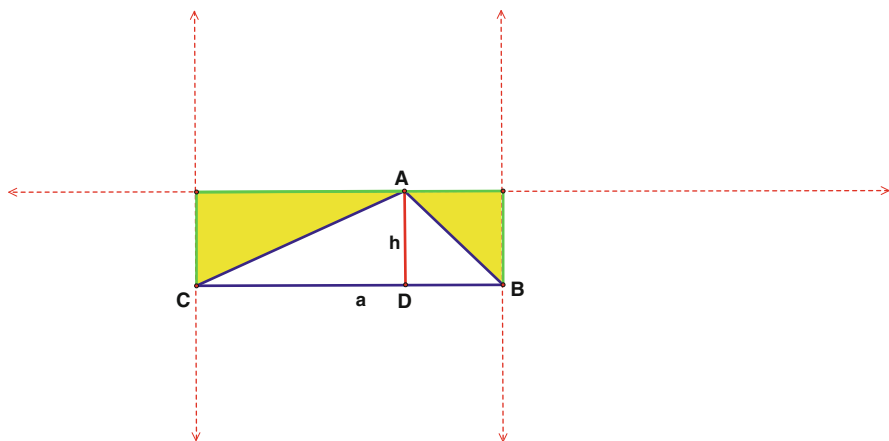

Fig. 1.52 Sketch for Problem 22 (proof of formula (1.26))

We can see that the area of triangle ABC is exactly the sum of the areas of two yellow triangles, that is half of the rectangle area, that is $S = [ABC] = \frac{1}{2} a \cdot h_a$.

Problem 23. Prove that the area of a triangle is half of the product of two sides and the sine of the angle between them, i.e., $S = \frac{1}{2} a \cdot b \cdot \sin(\angle C)$.

Proof. Let us sketch a scalene triangle ABC and mark its height and vertices (Fig. 1.53).
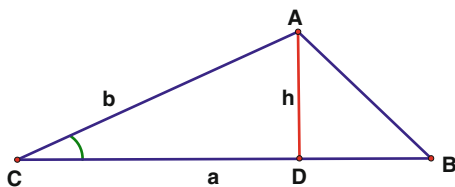

Fig. 1.53 Sketch for Problem 23 (proof of formula (1.27))

On one hand, its area is $S = [ABC] = \frac{1}{2} a \cdot h$. On the other hand, from the right triangle ADB, its leg ($AD = h$) can be written as a product of the hypotenuse ($AC = b$) and the sine of the opposite angle ($\angle C$). After substitution we have

$$S = [ABC] = \frac{1}{2}a \cdot h = \frac{1}{2}a \cdot b \sin(\angle C)$$

Problem 24. Find the formula for calculation of the area of a triangle if its sides a, b, c and the radius of the inscribed circle, r, are given.

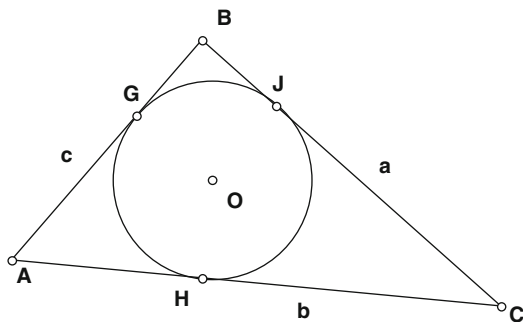**Solution.** Hence we need to prove that $S = p \cdot r$. First, we'll draw a nice picture (Fig. 1.54).

Fig. 1.54 Problem 24

In this sketch, the circle with center O is tangent to sides a, b, and c at points J, H, and G, respectively and has radius $r = OH = OJ = OG$.

How can we use these data in order to find the area?

Everyone remembers that the area of a triangle equals one-half base times height. But we are not given a height and it would be difficult to find it.

Remember that if we cut some figure into parts, the sum of the areas of all parts equals the area of the original figure. Much like assembling a puzzle, we put the pieces together. Let us divide the triangle into pieces with areas we can calculate. The inscribed circle is tangent to each side of the triangle. What does it mean? It means that the radius of the circle is perpendicular to each side of the triangle.
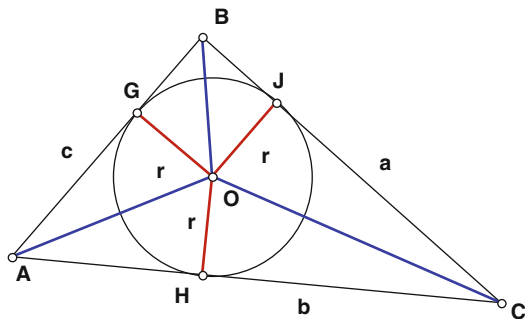
Fig. 1.55   Axillary construction for Problem 24 (proof of formula (1.28))

Let us add the radius to our sketch (Fig. 1.55). Three radii are shown in red connecting the center of the circle, O, with each vertex of the triangle, A, B, and C. We have divided the big triangle ABC into three small triangles: AOB, BOC, and AOC. They have the same height, r, and known bases, a, b, and c. We can find the area of each small triangle and in turn the area of the big triangle, S.

$$S_{\triangle ABC} = S_{\triangle AOB} + S_{\triangle BOC} + S_{\triangle AOC}$$
$$= \frac{1}{2}c \cdot r + \frac{1}{2}b \cdot r + \frac{1}{2}a \cdot r$$
$$= \frac{1}{2}r(c + b + a) = r \cdot p$$

Constructing an auxiliary element and a good, accurate sketch allowed us not only to solve this problem in one step but simultaneously to prove formula (1.28). I personally never tried to memorize formula (1.28), but once I proved it I always could derive it.

Now I want to offer the following problem.

Problem 25.   A right triangle with legs of length 3 and 4 in. is given. Find the radius of the inscribed circle.

Solution.   We have to calculate the radius of the inscribed circle so we can again use formula (1.28) or the idea of how we proved it. Let us draw a picture.
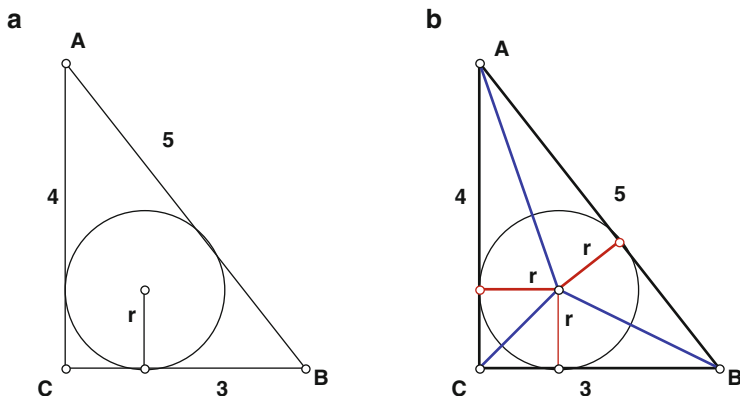
Fig. 1.56 Sketch for Problem 25

Knowing two legs, we always can find the hypotenuse using the Pythagorean Theorem. It is 5 in. Dividing the triangle into three triangles (Fig. 1.56b), and assuming that r is the radius of the circle, we can express the area as

$$S_{\triangle ABC} = \frac{1}{2} \cdot 4 \cdot r + \frac{1}{2} \cdot 3 \cdot r + \frac{1}{2} \cdot 5 \cdot r = \frac{1}{2} r \cdot 12 = 6r \qquad (1.31)$$

On the other hand, because ABC is a right triangle, its area can be found differently as

$$S_{\triangle ABC} = \frac{1}{2} AC \cdot BC = \frac{1}{2} \cdot 4 \cdot 3 = 6 \qquad (1.32)$$

Since expressions (1.31) and (1.32) have the same left sides, their right sides must be equal as well so that $6r = 6$ and $r = 1$.

Answer. The radius of the inscribed circle is 1 in.

Problem 26. Prove that the area of a triangle can be evaluated as $S = \frac{abc}{4R}$, where a, b, and c are the sides of a triangle and R is the radius of the circumscribed circle.

Proof. Sometimes students are afraid to prove this formula thinking that it is too difficult. However, those who try always prove it right away. We will sketch a triangle inscribed into a circle of radius R (Fig. 1.57).
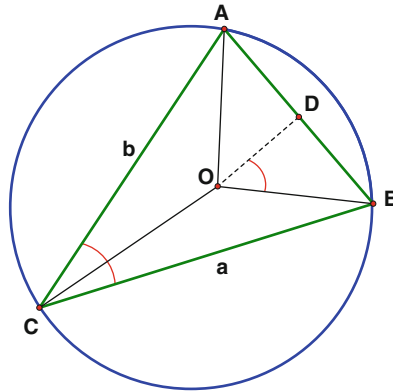
OA=OB=OC=R
AB=c

Fig. 1.57 Sketch for Problem 26 (proof of formula (1.29))

Next, we will use the formula for its area proven previously

$$S = \frac{1}{2}a \cdot b \cdot \sin(\angle C)$$

Angle C is marked on our sketch by a red arc and is shorthand for the angle ACB. Since triangle ABC is inscribed in the circle, the following relation between its inscribed (ACB) and central (AOB) angles is true:

$$\angle C = \angle ACB = \frac{1}{2}\angle AOB$$

On the other hand, triangle AOB is an isosceles triangle with vertex angle AOB. If D is the midpoint of AB, then

$$\angle DOB = \frac{1}{2}\angle AOB = \angle C$$

From the right triangle ODB we have

$$DB = OB \cdot \sin(\angle ACB)$$
$$\frac{c}{2} = R\sin(\angle C)$$

The last formula can be solved as

$$\sin(\angle C) = \frac{c}{2R}$$

After substituting this formula into the original area formula we finally get

$$S = [ABC] = \frac{1}{2}a \cdot b \cdot \frac{c}{2R} = \frac{abc}{4R}$$

The proof is completed.

Problem 27. Given a triangle ABC with circumradius $R = \sqrt{3} - 1$ and side $AB = \sqrt{2}$. It is given that points E and D are on sides AB and BC, respectively such that $\angle BAD = 2\angle DAC$, $\angle BCE = 2\angle ECA$, and $AB \cdot CE = BC \cdot AD$. Find the area of triangle ABC.

Solution. Let $\angle DAC = \alpha, \angle ECA = \beta \Rightarrow \angle BAD = 2\alpha, \angle BCE = 2\beta$ (Fig. 1.58).
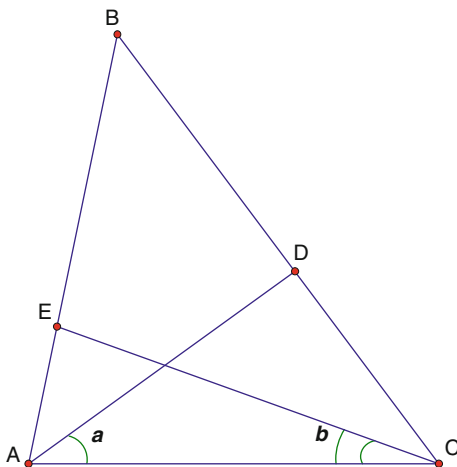

Fig. 1.58 Sketch for Problem 27

Using the Triangle Angle Theorem we now know that $\angle B = 180° - 3(\alpha + \beta)$, $\angle ADC = 180° - (\alpha + 3\beta) \Rightarrow \angle ADB = \alpha + 3\beta$.

Applying the Law of Sines for triangles ABD and CEB, respectively, we obtain

$$\frac{AB}{AD} = \frac{\sin \angle ADB}{\sin \angle ABD} = \frac{\sin(\alpha + 3\beta)}{\sin(3(\alpha + \beta))}$$

$$\frac{BC}{CE} = \frac{\sin \angle BEC}{\sin \angle CBE} = \frac{\sin(3\alpha + \beta)}{\sin(3(\alpha + \beta))}$$

We simplified our formulas above with $\sin(180° - \gamma) = \sin \gamma$.

Since we can rewrite the condition of the problem $AB \cdot CE = BC \cdot AD$ in a different form as

$$\frac{AB}{AD} = \frac{BC}{CE}$$

then from the previous equalities it follows that

$$\sin(\alpha + 3\beta) = \sin(3\alpha + \beta)$$

Many students at this point make a mistake by equating the arguments of the sines!

Instead we need to apply standard trigonometric formulas for the difference of sine functions:

$$\sin x - \sin y = 2 \sin\frac{x - y}{2} \cdot \cos\frac{x + y}{2}$$

and obtain

$$\sin(3\alpha + \beta) - \sin(\alpha + 3\beta) = 0$$
$$2 \sin(\alpha - \beta) \cdot \cos(2(\alpha + \beta)) = 0$$

By the zero product property, we equate each factor individually to zero.

Case 1. $\sin(\alpha - \beta) = 0$

Case 2. $\cos(2\alpha + 2\beta) = 0$

Since for any triangle the sum of any two angles is less than $180°$, from the two cases we know that $3\alpha + 3\beta < 180° \Leftrightarrow \alpha + \beta < 60°$ or $\alpha + \beta < \frac{\pi}{3}$.

Do you remember how to solve trigonometric equations?

If you have completely forgotten trigonometry, let's recall the definitions of sine and cosine using the unit circle idea (Fig. 1.59).
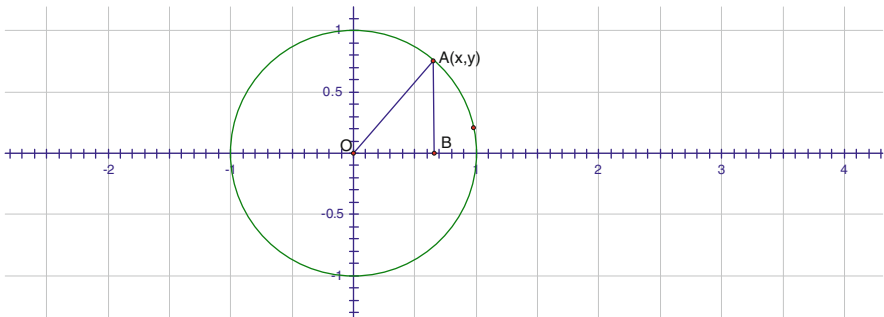

Fig. 1.59 Point A on the unit circle

Sine is the **y-coordinate** of a point on the unit circle (of radius 1).

Cosine is the **x-coordinate** of a point on the unit circle.

Thus, any point on the unit circle can have coordinates only within the interval $[-1, 1]$. Let us assume that the point (say A in the graph above) corresponds to the angle t. ($\angle BOA$). We can conclude that functions **sine** and **cosine** have the same range $[-1, 1]$ and can be written as

$$-1 \leq \sin t \leq 1 \ \text{ and } -1 \leq \cos t \leq 1 \tag{1.33}$$

From the same picture we can see that $\sin t = 0$ would correspond to two points on the horizontal diameter of the unit circle for angle $t = 0$ and $t = \pi$ within each revolution, then in general $t = \pi \cdot n, n = 0, \pm 1, \pm 2 \ldots$

On the other hand, $\cos t = 0$ (the first coordinate equals zero) corresponds to two points on the vertical diameter of the unit circle for the angle $t = \frac{\pi}{2}; \frac{3\pi}{2}$, which in general can be written as $t = \frac{\pi}{2} + \pi n, n = 0, \pm 1, \pm 2 \ldots$

Next, we can solve our equations for cases 1 and 2:

Case 1	**Case 2**

Case 1

$$\sin(\alpha - \beta) = 0, \ \alpha + \beta < \frac{\pi}{3}$$

$$\alpha - \beta = \pi \cdot n, n \in Z$$

$$\alpha = \beta$$

Case 2

$$\cos(2\alpha + 2\beta) = 0, \alpha + \beta < \frac{\pi}{3}$$

$$2\alpha + 2\beta = \frac{\pi}{2} + \pi \cdot n$$

$$\alpha + \beta = \frac{\pi}{4} + \frac{\pi}{2} \cdot n$$

$$\alpha + \beta = \frac{\pi}{4}$$

We can see that our problem will have different answers determined by the two cases:

Case1. $\alpha = \beta$.

The triangle ABC is an isosceles and $AB = BC = \sqrt{2}$. Its area can be found as half the product of the sides and the sine of the included angle.

$[ABC] = \frac{1}{2} \cdot AB \cdot BC \cdot \sin(\angle B) = \sin(\angle B) = \sin(180° - (3\alpha + 3\beta)) = \sin 6\alpha$. Hence, we need to evaluate $\sin 6\alpha$ and this will complete Case 1.

How can we find it? What do we have for that? What condition of the problem has not been used?

It is the radius of the circumscribed circle!

Using this formula we can write

$$\sin(\angle BAC) = \sin 3\alpha = \frac{BD}{2R} = \frac{\sqrt{2}}{2(\sqrt{3} - 1)} = \frac{\sqrt{2}(\sqrt{3} + 1)}{4}.$$

Can we evaluate $\sin 6\alpha$ using the value of $\sin 3\alpha$?

Yes we can if we recall the formula of the sine of double angle
$\sin 6\alpha = 2 \sin 3\alpha \cdot \cos 3\alpha$ and then replace

$$\cos 3\alpha = \sqrt{1 - \sin^2 3\alpha} = \sqrt{1 - \frac{(\sqrt{3}+1)^2}{8}} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Therefore, in Case 1, the area of triangle ABC,

$$[ABC] = 2 \sin 3\alpha \cos 3\alpha = \frac{(1+\sqrt{3})(1-\sqrt{3})}{4} = \frac{1}{2}.$$

Case 2. $\alpha + \beta = \frac{\pi}{4}$

By the Law of Sines we have,

$$\frac{BC}{\sin 3\alpha} = \frac{AB}{\sin 3\beta}$$

$$BC = \frac{AB \sin 3\alpha}{\sin 3\beta} = \frac{\sqrt{2} \cdot \sin\left(\frac{3\pi}{4} - 3\beta\right)}{\sin 3\beta} = \cot 3\beta + 1$$

You probably noticed that our plane geometry problem became mostly a trigonometric problem. Without knowledge of trigonometry, many relations would not be found. The problem can serve as a short review of trig. For example, let's see how the last formula can be obtained:

$$\sin(x - y) = \sin x \cos y - \sin y \cos x$$

$$\sin\left(\frac{3\pi}{4} - 3\beta\right) = \sin\frac{3\pi}{4} \cos 3\beta - \sin 3\beta \cos\frac{3\pi}{4}$$

$$= \frac{\sqrt{2}}{2} \cos 3\beta - \left(-\frac{\sqrt{2}}{2}\right) \sin 3\beta$$

$$= \frac{\sqrt{2}}{2}(\cos 3\beta + \sin 3\beta)$$

Multiplying by $\frac{\sqrt{2}}{\sin 3\beta}$ and after simplification we get. $BC = \cot 3\beta + 1$.

We want to evaluate $\cot 3\beta$ using the value of $\sin 3\beta$. If we again use the unit circle idea then we will see that for any value of $\sin 3\beta$ there will be two possible values of $\cot 3\beta$ (one positive and the other negative). We will consider two cases again:

A. If $\cot 3\beta > 0$, then $BC = \sqrt{\frac{1}{\sin^2 3\beta} - 1} + 1 = 3 - \sqrt{3}$.

B. If $\cot 3\beta < 0$, then $BC = -\sqrt{\frac{1}{\sin^2 3\beta} - 1} + 1 = \sqrt{3} - 1$.

The area may be one of two other values depending on Cases A and B.

Case A. $\cot 3\beta > 0$

$$
\begin{aligned}
[ABC] &= \frac{1}{2} \cdot AB \cdot BC \cdot \sin(\angle B) \\
&= \frac{1}{2} \cdot \sqrt{2} \cdot \left(3 - \sqrt{3}\right) \sin(180° - (3\alpha + 3\beta)) \\
&= \frac{\left(3 - \sqrt{3}\right)}{\sqrt{2}} \cdot \sin(3(\alpha + \beta)) \\
&= \frac{\left(3 - \sqrt{3}\right)}{\sqrt{2}} \sin\frac{3\pi}{4} \\
&= \frac{\left(3 - \sqrt{3}\right)}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{\left(3 - \sqrt{3}\right)}{2}.
\end{aligned}
$$

Case B. $\cot 3\beta < 0$

$$
\begin{aligned}
[ABC] &= \frac{1}{2} \cdot AB \cdot BC \cdot \sin \angle B \\
&= \frac{1}{2} \cdot \sqrt{2} \cdot \left(\sqrt{3} - 1\right) \sin(3(\alpha + \beta)) \\
&= \frac{\left(\sqrt{3} - 1\right)}{\sqrt{2}} \sin\frac{3\pi}{4} = \frac{\left(\sqrt{3} - 1\right)}{2}.
\end{aligned}
$$

Answer.   Depending on the considered cases, there are <u>three</u> total possible values of the area of triangle ABC: $\frac{1}{2}$, $\frac{\left(3 - \sqrt{3}\right)}{2}$, and $\frac{\left(\sqrt{3} - 1\right)}{2}$.

Note.   When I gave this problem to my graduate students in a History of Math class as homework, all of them tried to solve it but nobody got all three cases. Remember that in order to do well in problem solving, we always have to consider the most general case.

1.6.1.1   Heron's Formula

In this section we will show some proofs of Heron's Formula:

$$
S = \sqrt{p(p - a)(p - b)(p - c)}
$$

Heron of Alexandria was a mathematician who lived from 10 to 75 AD, taught at the Museum of Alexandria, and wrote his famous proofs in notebooks. Although the authorship of some of the works attributed to Heron are disputed, it is believed that

Heron derived his famous formula himself by considering cyclic quadrilaterals. Later many famous mathematicians tried to prove the formula because it looks quite unusual. For example, Euler proved it using relations between similar triangles. Some modern proofs include the use of cosine and sine formulae for a triangle (e.g., see Gardner's book by Cambridge University Press). It is interesting that mathematicians are still trying to find new proofs for this formula. In fact, in January 2012, a review of different proofs was presented by Professor Dunham at his plenary talk of the annual AMS meeting.

By Heron's formula, we can evaluate an area of any triangle if we are given the lengths of all sides. Here I will go over two proofs of the formula: one algebraic and one geometric. Heron's and Euler's Proofs can be found in (Dunham 1991).

Raifaizen's Algebraic Proof. Claude Raifaizen created a proof that primarily relied on algebra. The first step was to construct $\triangle ABC$. Denote CP, is the height and AB as the base (Fig. 1.60).
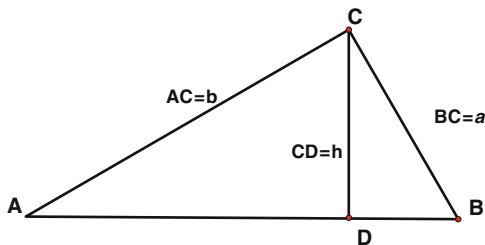

Fig. 1.60 Raifaizen Proof of Heron's Formula

Let $CD = h$, then by the Pythagorean formula,

$$AD = \sqrt{b^2 - h^2}$$

$$DB = \sqrt{a^2 - h^2}$$

$$\therefore c = AB = AD + DB = \sqrt{b^2 - h^2} + \sqrt{a^2 - h^2}$$

Observing that, $c \cdot h = 2A$ where A is the area, we have

$$\sqrt{a^2 - h^2} = c - \sqrt{b^2 - h^2}$$

$$a^2 - h^2 = c^2 - 2c\sqrt{b^2 - h^2} + b^2 - h^2$$

$$\left(a^2 - b^2 - c^2\right)^2 = 4c^2\left(b^2 - h^2\right) = 4b^2c^2 - 4c^2h^2$$

$$\left(a^2 - b^2 - c^2\right)^2 = 4c^2\left(b^2 - h^2\right) = 4b^2c^2 - 4c^2h^2$$

$$\left(a^2 - b^2 - c^2\right)^2 = 4b^2c^2 - 16A^2$$

then

$$16A^2 = 4b^2c^2 - \left(a^2 - b^2 - c^2\right)^2$$
$$= \left(2bc - a^2 + b^2 + c^2\right)\left(2bc + a^2 - b^2 - c^2\right)$$
$$= \left[\left(b^2 + 2bc + c^2\right) - a^2\right]\left[a^2 - (b-c)^2\right]$$
$$= \left[(b+c)^2 - a^2\right]\left[a^2 - (b-c)^2\right]$$
$$= (b+c+a)(b+c-a)(a-b+c)(a+b-c)$$
$$= (a+b+c)(a+b+c-2a)(a+b+c-2b)(a+b+c-2c)$$

Letting $a + b + c = 2p$, we get

$$16A^2 = (2p)(2p - 2a)(2p - 2b)(2p - 2c)$$
$$A^2 = p(p-a)(p-b)(p-c)$$
$$A = \sqrt{p(p-a)(p-b)(p-c)}$$

Raifaizen's algebraic approach with its single construction stands in contrast to the geometric proofs of Heron and Euler with their multitude of constructions (Raifaizen 1971). The Pythagorean Theorem is used at the beginning of this theorem to define the side lengths which is incorporated into the area formula. After several steps of algebraic simplification, the formula is solved for area. This is an excellent proof for those who know only basic algebra and substitution, or who have trouble with geometric theorems. However, it also a very abstract proof with little to aid visualization.

Nelsen's Geometric Proof. Roger Nelsen stated two lemmas (we actually proved one of them earlier in the chapter) as well as constructed several figures to prove Heron's formula in his "proof without words." Consider $\triangle ABC$ with side lengths of a, b, c and bisected angles that locate the center of the incircle. Extending the **inradius** to each side of the triangle creates six smaller triangles, as seen in Fig. 1.61.
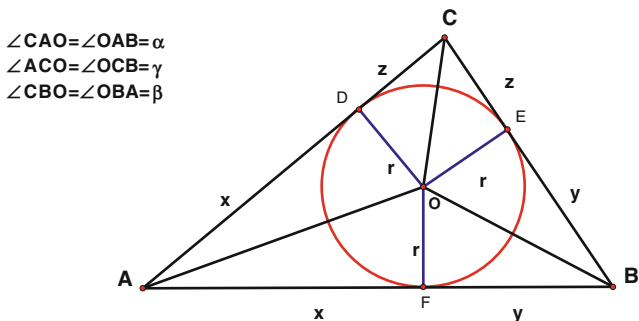

Fig. 1.61 Nelsen's Geometric Proof

Here we note that the **semiperimeter**, p, can be defined as one-half the perimeter,

$$p = x + y + z = x + a = y + b = z + c$$

So far, Nelsen has done little different from Heron, however the next step is to prove two lemmas from which "Heron's formula readily follows."

Lemma 2. The area A of a triangle is equal to the product of its inradius and semiperimeter.

Proof. Figure 1.62 shows ΔABC with two of the inside triangles shaded and the height of the triangles labeled as r.
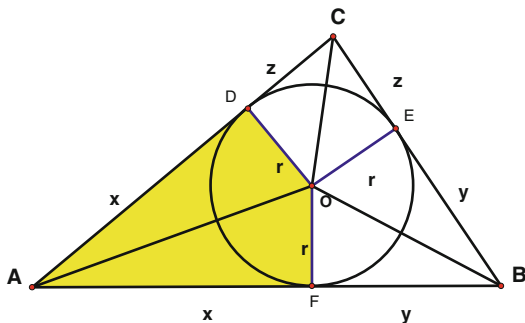

Fig. 1.62 Evaluating the area of a triangle

Then the area of the triangle can be written as

$$A = r(x + y + z) = rp$$

Lemma 3. If α, β, and γ are positive angles such that $\alpha + \beta + \gamma = \frac{\pi}{2}$, then $\tan\alpha \cdot \tan\beta + \tan\beta \cdot \tan\gamma + \tan\gamma \cdot \tan\alpha = 1$ (Nelsen 2001).

Proof. Knowing that α, β, and γ are half of the sum of the angles of the original triangle, it follows that their sum is half of 180° or $\pi/2$ in radians. Figure 1.63 proves this relation.
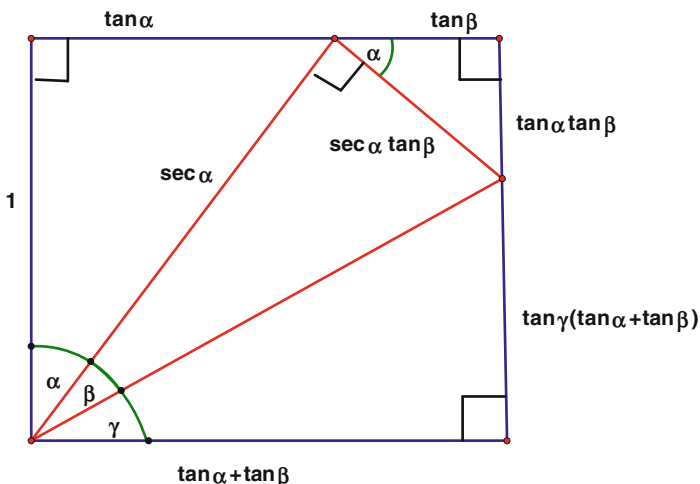
Fig. 1.63 Illustration of Lemma 3

Let us explain his construction step by step. First, the angles were combined in the bottom left of a rectangle with the side length of 1. With this we are able to discern that the opposite side of the large triangle with angle α is going to equal $\tan \alpha$ and the hypotenuse would equal $\sec \alpha$. Secondly, looking at the triangle with angle β and side length of $\sec \alpha$, we find that the other side length is $\sec \alpha \tan \beta$. Thirdly, we look at the small triangle with angle α and hypotenuse of $\sec \alpha \tan \beta$. Observe that the adjacent side is $\tan \beta$ and the opposite side is $\tan \alpha \tan \beta$. Finally, from the last triangle with angle γ, we observe that the adjacent length is equal to the top length of the rectangle which is $\tan \alpha + \tan \beta$. We can therefore find the length of the side opposite of angle γ as $\tan \gamma (\tan \alpha + \tan \beta)$. This side length can be expanded to $\tan \alpha \tan \gamma + \tan \beta \tan \gamma$ which is added to the remaining portion of the rectangle's side length of $\tan \alpha \tan \beta$, giving us $\tan \alpha \tan \beta + \tan \alpha \tan \gamma + \tan \beta \tan \gamma$. This is the opposite side of the rectangle, 1. Therefore the lemma has been proved.

Using Lemma 2 and Lemma 3 stated by Nelsen we can finish the proof of the Heron's formula. Consider Lemma 3,

$$1 = \tan \alpha \tan \beta + \tan \beta \tan \gamma + \tan \gamma \tan \alpha$$

and replace the tangents of all angles by the ratios from the original picture of the triangle ABD:

$$1 = \frac{r}{x} \cdot \frac{r}{y} + \frac{r}{y} \cdot \frac{r}{z} + \frac{r}{z} \cdot \frac{r}{x} = \frac{r^2(x + y + z)}{xyz} = \frac{r^2 p}{xyz} = \frac{r^2 p^2}{pxyz} = \frac{A^2}{pxyz}$$

Now, we will apply Lemma 2, $A = r(x + y + z)$. After algebraic manipulations we obtain

$$A^2 = \frac{(pxyz)^2}{pxyz} = pxyz = p(p-a)(p-b)(p-c)$$

Taking the square root of both sides we conclude the proof (Nelsen 2001).

$$A = \sqrt{p(p-a)(p-b)(p-c)}$$

Remark. Of course, this proof is not purely geometric. Nelsen used trigonometry and algebraic manipulations as well as descriptions of the steps of the proof. There is no proof without words.

The following problems will demonstrate how important it is to know Heron's formula.

Problem 28. Peter has drawn a triangle with sides 11, 13 and 6. Can you help him to evaluate the area of the triangle?

Solution. Using Heron's Formula, we can help Peter right away. First, we will calculate half the perimeter, $p = \frac{1}{2}(11 + 13 + 6) = 15$. Replacing p, a, b, and c into (1.30) we obtain the area of the triangle:

$$\begin{aligned} S &= \sqrt{p(p-a)(p-b)(p-c)} \\ &= \sqrt{15(15-11)(15-13)(15-6)} \\ &= \sqrt{15 \cdot 4 \cdot 2 \cdot 9} \\ &= 6\sqrt{30} \approx 32.86 \text{ in.}^2 \end{aligned}$$

Answer. Area of triangle is $6\sqrt{30}$ in.^{2}.

Are you ready for another proof?

Problem 29. Prove that for any triangle, the inequality $h_a \leq \sqrt{p(p-a)}$ is valid. Here h_a is the height dropped to side a and p is the semiperimeter.

Proof. We will equate the area of a triangle written in terms of h_a with that of Heron's Formula,

$$S = \frac{ah_a}{2} = \sqrt{p(p-a)(p-b)(p-c)}$$

and obtain formula for h_a:

$$h_a = \sqrt{p(p-a)} \cdot \frac{2 \cdot \sqrt{(p-b)(p-c)}}{a}$$

Let us prove that

$$\frac{2 \cdot \sqrt{(p-b)(p-c)}}{a} < 1, \quad \text{i.e., that } \sqrt{(p-b)(p-c)} \leq \frac{a}{2}$$

First, we substitute the formula for the semiperimeter, $p = \frac{a+b+c}{2}$ and use the relationship between the arithmetic and geometric mean $\left(\sqrt{xy} \leq \frac{x+y}{2}, \ x > 0, \ y > 0 \right)$ to give

$$\sqrt{(p-b)(p-c)} = \sqrt{\frac{a+c-b}{2} \cdot \frac{a+b-c}{2}} \leq \frac{\frac{a+c-b}{2} + \frac{a+b-c}{2}}{2} = \frac{a}{2}$$

Since

$$\sqrt{(p-b)(p-c)} \leq \frac{a}{2} \Rightarrow \frac{2 \cdot \sqrt{(p-b)(p-c)}}{a} \leq \frac{2}{a} \cdot \frac{a}{2} = 1$$

we have that

$$h_a \leq \sqrt{p(p-a)}$$

Remark. Formulas similar to Heron's Formula can be used for finding the areas of other geometric figures, such as the cyclic quadrilaterals which are discussed in Chapter 3.

1.6.2   Important Lemmas on the Area of a Triangle

Now we will see how algebra can help us obtain some interesting relationships between the sides of a triangle and its heights. For this purpose, we are going to use only the well-known formula (1.26) written in three different ways:

$$S = \frac{1}{2} a \cdot h_a \tag{1.34}$$

$$S = \frac{1}{2} b \cdot h_b \tag{1.35}$$

$$S = \frac{1}{2}c \cdot h_c \tag{1.36}$$

Of course, you understand that because any triangle has three sides, we can drop three different heights to those sides and calculate the area of the triangle three different ways. Dividing the left hand sides and the right hand sides of (1.34) and (1.35) by each other we obtain:

$$1 = \frac{a \cdot h_a}{b \cdot h_b}$$

or in a different form:

$$\frac{a}{b} = \frac{h_b}{h_a}$$

Applying the same procedure with (1.34) and (1.36) as well as with (1.35) and (1.36), we can get the following three equations:

$$\frac{a}{b} = \frac{h_b}{h_a}, \quad \frac{a}{c} = \frac{h_c}{h_a}, \quad \frac{c}{b} = \frac{h_b}{h_c} \tag{1.37}$$

Formulas (1.37) give us new information which we state in Lemma 4.

Lemma 4.   The lengths of the sides of a triangle are inversely proportional to the corresponding heights.

Problem 30.   Consider a scalene triangle in which two sides equal 10 and 15 in. A height dropped to the side of 15 in. has length 6 in. Find the height to side of 10 in.

Solution. Because sides are inversely proportional to their heights, we obtain $\frac{10}{15} = \frac{6}{x}$, where x the unknown second height. Solving the proportion we have $x = 9$ in. You do not have to memorize all formulas. What is more important is to remember some ideas we used to prove such formulas. Solving the problem, you can think of what relations connect the lengths of the sides to the heights of a triangle. It is a formula of an area! Writing this formula in two different ways as

$$S = \frac{1}{2} \cdot 15 \cdot 6$$

$$S = \frac{1}{2} \cdot 10 \cdot x$$

we can see that if the left sides are equal, then the right sides are equal. So $90 = 10x$.

Answer. 9 in.

Lemma 5. If two triangles are similar with the coefficient of similitude k then the ratio of their areas equals k^2.

Prove this statement by writing the areas of each triangle in terms of half of the product of the adjacent sides and the concluded angle.

Hint. The corresponding angles of similar triangles are the same and that the ratio of the corresponding sides equals k.

What if the heights of two triangles are the same, but the lengths of their bases is not? What can we say about the relationship between their areas?

Lemma 6. If two triangles have a common vertex and their sides that are opposite to this vertex lie on the same line, then the ratio of the areas of the triangles equals the ratio of the corresponding bases.
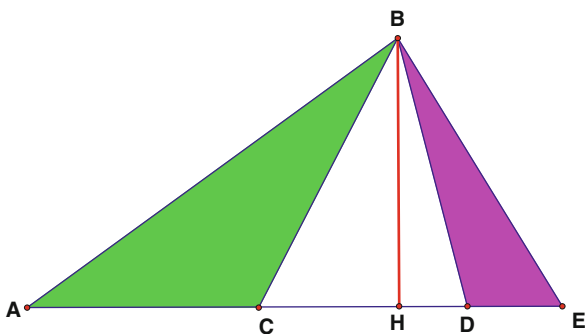

Fig. 1.64 Lemma 6

Proof. Triangles ABC and BDE have common vertex B and because their bases, AC and DE lie on the same line (AE) then they also have the same height (BH).

The height is the length of the perpendicular from vertex B to AE. Expressing the area of each triangle in terms of that height and corresponding base and taking the ratio of the areas we have that (Fig. 1.64):

$$\frac{[ABC]}{[DBE]} = \frac{\frac{1}{2} \cdot |AC| \cdot |BH|}{\frac{1}{2} \cdot |DE| \cdot |BH|} = \frac{|AC|}{|DE|} \tag{1.38}$$

What if two triangles have a common side?

In that case we have the following relations which we state as a Corollary to Lemma 6.

Corollary. When a cevian divides a triangle, the areas of the two resulting triangles are in the same ratio as the two parts of the side to which the cevian is drawn.

Prove this Corollary by solving Problem 31.

Problem 31. A cevian BD dropped from the vertex B divides a triangle ABC into two triangles, ABD and BDC. Prove that the following is true: $\frac{[ABD]}{[BDC]} = \frac{|AD|}{|DC|}$.

Hint. Write down both areas in terms of the half product of the corresponding height and base. Both triangles have the same height.
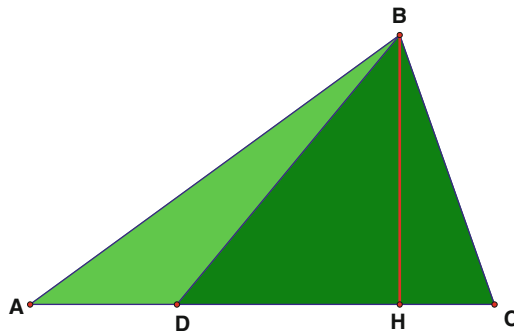

Fig. 1.65   Sketch for Problem 31

If the heights of triangles are equivalent, the areas are related by the proportions of the lengths of their bases. Lemma 7 gives us a way to visualize this algebraic relationship (Fig. 1.65).

Lemma 7. Consider two parallel lines, n and m, such that line n is above m and place points A and C on line m and the set of points $B_1, B_2, B_3, \ldots B_i, \ldots$ on line n. Then all triangles with base AC and a vertex on line n will have the same area. $[AB_1C] = [AB_2C] = \cdots = [AB_iC] = S$.
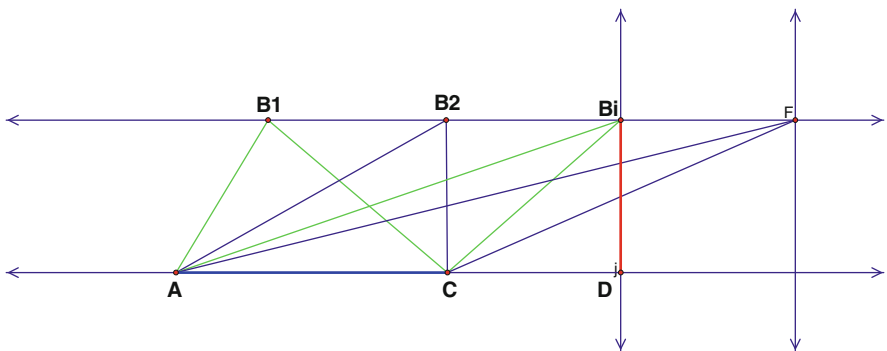

Fig. 1.66 Lemma 7

Alternatively, we can hold the lengths of the bases as a constant while we vary the proportion of the heights (Fig. 1.66).

Lemma 8. If two triangles have a side of the same length, then the ratio of their areas equal the ratio of their altitudes dropped to that side.

Proof. Consider two triangles ACB and ADB with the same base AB and altitudes $CE = h_1$ and $DF = h_2$ respectively. If we write the area of each triangle with respect to the base AB and evaluate the ratio, the proof will be complete.

$$\frac{[ABC]}{[ABD]} = \frac{\frac{1}{2}|AC| \cdot |CE|}{\frac{1}{2}|AC| \cdot |DF|} = \frac{|CE|}{|DF|} = h_1 : h_2$$

Let's test our understanding of the invariance of areas of triangles by solving Problems 32 and 33 (Fig. 1.67).
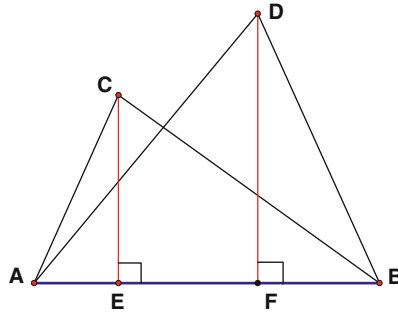
Fig. 1.67 Lemma 8

Problem 32. If h_a, h_b, and h_c are the corresponding altitudes of the triangle ABC and r is the radius of its inscribed circle, prove that $\frac{r}{h_a} + \frac{r}{h_b} + \frac{r}{h_c} = 1$.

Proof. Consider a triangle ABC (Fig. 1.68) with an inscribed circle of radius $r = OF = OE = OD$.
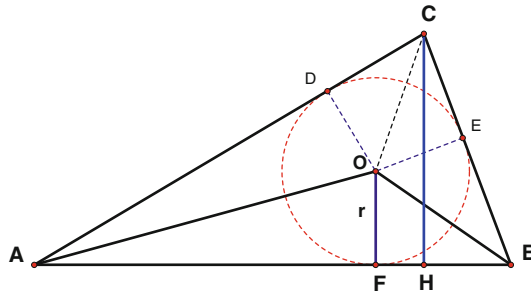

Fig. 1.68 Sketch for Problem 32

Note that the triangles AOB and ACB have a common base, AB, so that by Lemma 8,

$$\frac{[AOB]}{[ACB]} = \frac{|OF|}{|CH|} = \frac{r}{h_c}$$

Similarly, for pairs of the triangles (AOC, ABC) and (COB, ABC) we have $\frac{[AOC]}{[ACB]}$ $= \frac{r}{h_b}$. If we add all three formulas and use the fact that the small triangles compose triangle ABC, we will obtain the desired equality:

$$\frac{r}{h_a} + \frac{r}{h_b} + \frac{r}{h_c} = \frac{[COB]}{[ACB]} + \frac{[COA]}{[ACB]} + \frac{[AOB]}{[ACB]}$$
$$= \frac{[COB] + [COA] + [AOB]}{[ACB]}$$
$$= \frac{[ACB]}{[ACB]} = 1$$

Problem 33. Points N and M of the cevian BX are located so that the area of triangle MNC is 2/7 of the area of triangle ABC. Find the area of triangle ANM if $|AX|:|XC| = 3:5$ and $[ABC] = S$.

Solution. Using Fig. 1.69 we can write the following relationships for the areas of the shaded triangles
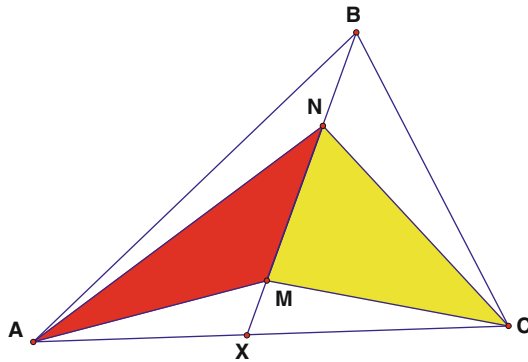
Fig. 1.69 Problem 33

$$\frac{[AMX]}{[XMC]} = \frac{3}{5} = k = \frac{[ANX]}{[XNC]} = \frac{[ANX] - [AMX]}{[XNC] - [XMC]} = \frac{[ANM]}{[NMC]}$$

Here we used the following property about ratios: (See more about properties of ratios in Appendix A)

$$k = \frac{a}{b} = \frac{c}{d} \Rightarrow \frac{c - a}{d - b} = k \text{ (Since } c = d \cdot k, a = b \cdot k, c - a = k(d - b))$$

so $[ANM] = \frac{3}{5} \cdot \frac{2}{7} S = \frac{6S}{35}$.

1.6.3   Proof of Ceva's Theorem by Ratio of the Areas

Assume that all three cevians intersect at a single point P as shown in Fig. 1.70. We need to prove that

$$\frac{|AZ|}{|ZB|} \cdot \frac{|BX|}{|XC|} \cdot \frac{|CY|}{|YA|} = 1$$

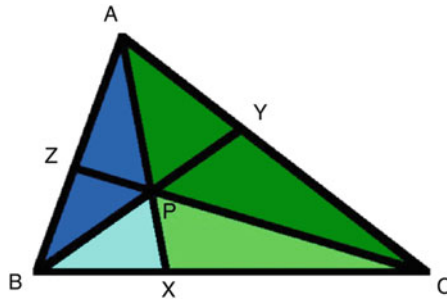The idea is to find each ratio separately and then multiply them together.

Fig. 1.70   Ratio of the areas

Notice that segments BX and XC are bases for the pairs of triangles (BAX, BPX) and (XAC, XPC), respectively. Thus,

$$\frac{|BX|}{|XC|} = \frac{[BXP]}{[CXP]} = \frac{[BXA]}{[CXA]}$$

These ratios can be rewritten in a different form as in Problem 33.
Therefore,

$$\frac{|BX|}{|XC|} = \frac{[BXA] - [BXP]}{[CXA] - [CXP]} = \frac{[BPA]}{[APB]} \tag{1.39}$$

By exactly the same reasoning, with the use of triangle-pairs (PAY, BAY) and (YPC, BYC) we get similar ratios:

$$\frac{|CY|}{|YA|} = \frac{[PYC]}{[APY]} = \frac{[BYC]}{[ABY]} = \frac{[BPC]}{[BAP]} \tag{1.40}$$

$$\frac{|AZ|}{|ZB|} = \frac{[ACP]}{[BPC]} \tag{1.41}$$

Multiplying the ratios (1.39)–(1.41) together we prove the forward implication of Ceva's Theorem.

$$\frac{|AZ|}{|ZB|} \cdot \frac{|BX|}{|XC|} \cdot \frac{|CY|}{|YA|} = 1$$

We must prove the reverse implication of the equivalence in order to prove the theorem completely.

Assume that the given segments BY and CZ in Fig. 1.70 intersect at point P and that a cevian from A intersects segment BC at point M. Suppose also that the product using that cevian is also 1, but that the fact that the cevian passes through P is unknown. That is, suppose

$$\frac{|AZ|}{|ZB|} \cdot \frac{|BM|}{|MC|} \cdot \frac{|CY|}{|YA|} = 1$$

Since the cevian that goes through P clearly ends at X, by the first part of Ceva's Theorem, we can compare the two equations and obtain that the following ratios are equal: $|BX|/|XC|$ and $|BM|/|MC|$. However, this occurs if and only if points X and M are the same points, i.e., segment AM equals AX and the cevian from A passes through P. Therefore, if the product of the corresponding ratios equals 1, then all three cevians concur. That's what we wanted to prove for the backwards implications so the proof is complete.

1.6.4   Vector Form of Menelaus' Theorem

Theorem. Given a triangle ABC let points C_1, B_1, and A_1 be on sides AB, AC, and BC, respectively. Points C_1, B_1, and A_1 are collinear (lie on the same line) iff

$$\frac{\overrightarrow{AC_1}}{\overrightarrow{C_1B}} \cdot \frac{\overrightarrow{BA_1}}{\overrightarrow{A_1C}} \cdot \frac{\overrightarrow{CB_1}}{\overrightarrow{B_1A}} = -1$$

For the collinear vectors $\overrightarrow{AC_1}$ and $\overrightarrow{C_1B}$, we have that $\dfrac{\overrightarrow{AC_1}}{\overrightarrow{C_1B}} = \pm\dfrac{AC_1}{C_1B}$.

The sign depends on whether the two vectors have the same (+) or opposite (−) direction. This form of Menelaus' Theorem is useful when the position of the points with respect to each other can change (see Appendix A for the proof of this statement).

Please try to solve Problem 34 offered at the USAMO 2012 as the fifth problem by yourself before looking at the solutions given.

Problem 34. Let P be a point in the plane of triangle ABC, and m is a line passing through P. Let A_1, B_1, and C_1 be the points where the reflections of lines PA, PB, PC with respect to m intersect the lines BC, AC, AB, respectively. Prove that A_1, B_1, and C_1 are collinear (Fig. 1.71).
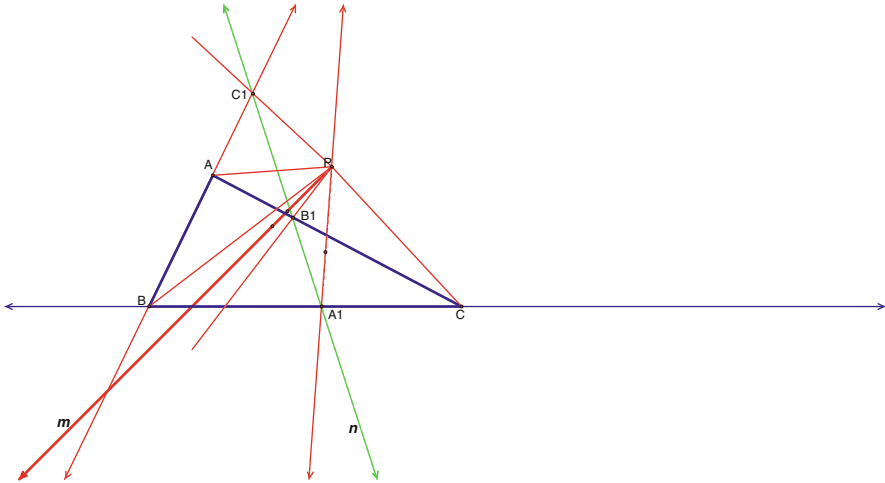

Fig. 1.71 Sketch 1 for Problem 34

Solution. Let us first draw a triangle and make the required constructions for the problem. Remember that it usually takes three or four sketches in order to make a good one. Clearly when a problem is understood, some lines can be erased in order not to work with a messy picture.

After constructing all reflections about line m and marking their intersections with the corresponding sides, we can draw a dashed line through points A_1, B_1 and C_1 (if you used a ruler then you probably would have three points on that line). Next, let us assume that three points were collinear so that we can rename the line that passes through as n. Since the line usually will not be parallel to any side of the triangle (particular cases can be considered separately) then such a line must divide sides of the triangle ABC in the ratios satisfying Menelaus' Theorem.

Without loss of generality can we prove that $\dfrac{\overrightarrow{BA_1}}{\overrightarrow{A_1C}} \cdot \dfrac{\overrightarrow{CB_1}}{\overrightarrow{B_1A}} \cdot \dfrac{\overrightarrow{AC_1}}{\overrightarrow{C_1B}} = -1$ and then conclude that A_1, B_1, and C_1 are on the same line, i.e., collinear?
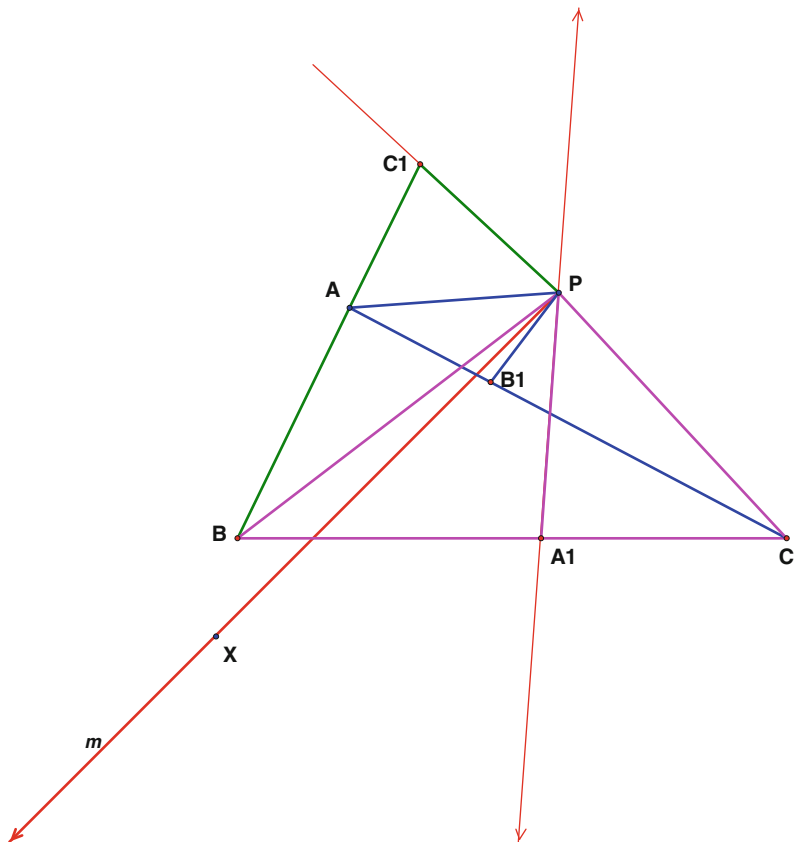
Fig. 1.72 Sketch 2 for Problem 35

We will emphasize only such elements that we will use in our proof and make invisible everything that makes our drawing messy. First, consider triangles BPA_1 and A_1PC (we show their sides in purple in Fig. 1.72). They have a common vertex P. The sides opposite to P both lie on the same line (BC). Therefore, the triangles have the same height! It means that the ratio of their areas equals the ratio of their bases. On the other hand, each area can be written using the formulas on two sides and the sine of the angle between them (Fig. 1.72).

Next, we can write down the relation:

$$\frac{\overrightarrow{BA_1}}{\overrightarrow{A_1C}} = \frac{[BPA_1]}{[A_1PC]} = \frac{\frac{1}{2} \cdot PB \cdot PA_1 \sin \angle BPA_1}{\frac{1}{2} \cdot PC \cdot PA_1 \sin \angle A_1PC} = \frac{PB}{PC} \frac{\sin \angle BPA_1}{\sin \angle A_1PC} \qquad (1.42)$$

Likewise, for the triangles shown partially in blue, APB_1 and PB_1C we obtain:

$$\frac{\overrightarrow{CB_1}}{\overrightarrow{B_1A}} = \frac{[CPB_1]}{[A_1PA]} = \frac{PC}{PA}\frac{\sin\angle CPB_1}{\sin\angle B_1PA} \tag{1.43}$$

Considering triangles BC_1P and AC_1P, we have

$$\frac{\overrightarrow{AC_1}}{\overrightarrow{C_1B}} = \frac{[APC_1]}{[C_1PB]} = \frac{PA}{PB}\frac{\sin\angle APC_1}{\sin\angle C_1PB} \tag{1.44}$$

Let X be a point of line m. Then by reflection properties and construction it is obvious that

$$\angle BPX = -\angle B_1PX, \quad \angle APX = -\angle A_1PX, \quad \angle CPX = -\angle C_1PX \tag{1.45}$$

Additionally,

$$\angle BPA_1 = \angle BPX + \angle XPA_1 = -(\angle B_1PX + \angle XPA) = -\angle B_1PA \tag{1.46}$$

Similarly,

$$\angle CPB_1 = -\angle C_1PB, \quad \angle APC_1 = -\angle A_1PC \tag{1.47}$$

Multiplying (1.42), (1.43), and (1.44) while using (1.45)–(1.47) as well as the property of the Sine function that $\sin(-x) = -\sin x$, we have the following chain of the equations:

$$\begin{aligned}
\frac{\overrightarrow{BA_1}}{\overrightarrow{A_1C}}\frac{\overrightarrow{CB_1}}{\overrightarrow{B_1A}}\frac{\overrightarrow{AC_1}}{\overrightarrow{C_1B}} &= \frac{PB}{PC}\frac{\sin\angle BPA_1}{\sin\angle A_1PC}\frac{PC}{PA}\frac{\sin\angle CPB_1}{\sin\angle B_1PA}\frac{PA}{PB}\frac{\sin\angle APC_1}{\sin\angle C_1PB} \\
&= \frac{\sin\angle BPA_1}{\sin\angle A_1PC}\frac{\sin\angle CPB_1}{\sin\angle B_1PA}\frac{\sin\angle APC_1}{\sin\angle C_1PB} \\
&= \frac{\sin\angle BPA_1}{\sin\angle B_1PA}\cdot\frac{\sin\angle CPB_1}{\sin\angle C_1PB}\cdot\frac{\sin\angle APC_1}{\sin\angle A_1PC} \\
&= (-1)(-1)(-1) = -1
\end{aligned}$$

This proves that points A_1, B_1, and C_1 are on the same line.

1.7   Homework on Triangles

1. Find an acute angle between the medians of an isosceles right triangle dropped from vertices of its acute angles.

   Answer. arccos(4/5).

Hint. Use the property of the medians and the Law of Cosines.

Solution. Consider a right triangle ABC with the right angle C and legs of length $a = BC = AC$ and the medians AD and BE (Fig. 1.73).

Applying the Pythagorean Theorem to the right triangle ACD we obtain that $AD = \sqrt{a^2 + \left(\frac{a}{2}\right)^2} = \frac{a\sqrt{5}}{2} = BE$.
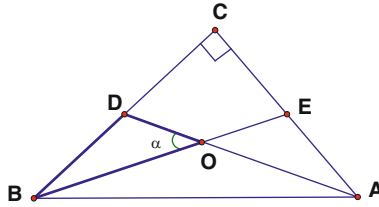

Fig. 1.73 Acute angle between the medians

The medians are cut by the point of intersection O in the ratio 2:1 starting from the vertex. Thus, $OD = \frac{1}{3}AD = \frac{a\sqrt{5}}{6}, BO = \frac{2}{3}BE = \frac{a\sqrt{5}}{3}, BD = \frac{a}{2}$

Denote angle DOB by α and consider the Law of Cosines for triangle BDO, in which we know the length of all sides:

$$\left(\frac{a}{2}\right)^2 = \left(\frac{a\sqrt{5}}{6}\right)^2 + \left(\frac{a\sqrt{5}}{3}\right)^2 - 2 \cdot \left(\frac{a\sqrt{5}}{6}\right)\left(\frac{a\sqrt{5}}{3}\right)\cos\alpha$$

$$\cos\alpha = \frac{4}{5}, \quad \alpha = \arccos\left(\frac{4}{5}\right)$$

2. Find the angles of a right triangle such that its hypotenuse is twice longer than one of its legs.

Answer. 30, 60, and 90°.

3. The sides of a right triangle are integers less than or equal to 10 and form an arithmetic progression. Find the perimeter of such triangle.

Answer. 12 or 24.

Solution. Let a, b, and c be sides of a triangle, such that $a < b < c$. Using that fact that a, b, and c are terms of arithmetic progression and assuming that its common difference equals d, we can write Pythagorean theorem for the right triangle:

$$(a + d)^2 + a^2 = (a + 2d)^2$$
$$a^2 - 2ad - 3d^2 = 0$$

Factoring the last quadratic expression yields

$$(a + d)(a - 3d) = 0,$$

that gives us two options for a: $a = -d$ and $a = 3d$. We choose $a = 3d$ because $a > 0$ and natural number. Now we can evaluate possible values of d. Number $c = a + 2d = 5d \leq 10$, then $d \leq 2$, $d = 1$ or $d = 2$ only.

(a) If $d = 1$, then $a = 3$, $b = 3 + 1 = 4$, $c = 4 + 1 = 5$. In a triple $(3, 4, 5)$ all numbers less than 10. The perimeter of such a triangle, P, is given as $P = 3 + 4 + 5 = 12$.

(b) If $d = 2$, then a $= 6$, $b = 6 + 2 = 8$, and $c = 8 + 2 = 10$. In a triple $(6, 8, 10)$ all sides again less than or equal to 10. Perimeter of the second triangle equals $P = 6 + 8 + 10 = 24$.

4. The base angles of some isosceles triangle are $80°$. Point D is on segment AC and point E is on AB such that $m\angle DBC = 60°$ and $m\angle ECB = 50°$. Find the angle EDB.

   Answer. $30°$.

5. Point D is in the interior of triangle ABC. Lines AD, BD, and CD intersect the sides of the triangle at points E, F, and G, respectively. Find the ratio $CF : FA$ if it is known that $\frac{AG}{GB} = p$ and $\frac{BE}{EC} = q$.

   Answer. $\dfrac{1}{pq}$.

   Hint. Draw a triangle ABC and three cevians AE, BF and CG though point D in the interior of the triangle. Apply Ceva's Theorem, $\frac{CF}{FA} \cdot \frac{AG}{GB} \cdot \frac{BE}{EC} = 1$ and solve for $\frac{CF}{FA}$

6. $AC = b$ and $AB = c$ are two sides of a scalene triangle ABC. Angle A is twice angle B. Find the length of side a.

   Answer. $a = \sqrt{b^2 + bc}$.

   Hint. Use trigonometric relations of a right triangle; the Law of Sines, and the trigonometric formulas for sine and cosine of a double angle.

   Solution. In the triangle ABC of Fig. 1.74, $AC = b$, $CB = a$ (it needs to be found), $AB = c$. Denote $\alpha = \angle CBA$, $2\alpha = \angle CAB$. Let CD be the height of the triangle.
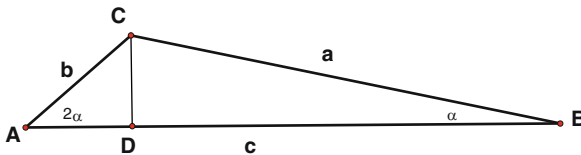

Fig. 1.74 Finding the length of side a

(a) $AB = AD + DB = c = b\cos 2\alpha + a\cos\alpha$. Using the trigonometric identity $\cos 2\alpha = 2\cos^2\alpha - 1$ and solving the first equation for a: $a = \frac{c - b(2\cos^2\alpha - 1)}{\cos\alpha}$ that is

$$a\cos\alpha = c - 2b\cos^2\alpha + b \tag{1.48}$$

(b) Applying the Law of Sines to triangle ABC and the formula for the sine of the double angle ($\sin 2x = 2\sin x \cdot \cos x$):

$$\frac{a}{\sin 2\alpha} = \frac{b}{\sin\alpha}$$
$$\frac{a}{2\cos\alpha} = b$$

so

$$a = 2b\cos\alpha \tag{1.49}$$

(c) Multiplying (1.48) by $2b$ and substituting into (1.49) we finally have

$$a \cdot 2b \cdot \cos\alpha = 2bc - 4b^2\cos^2\alpha + 2b^2$$
$$a^2 = 2bc - a^2 + 2b^2$$
$$a = \sqrt{b^2 + bc} = \sqrt{b(b + c)}$$

7. Triangle ABC is inscribed in a circle. A median AM is drawn so that it intersects the circle at point D. It is known that $AB = 1$ and $BD = 1$. Find BC.

Answer. $\sqrt{2}$

8. Find the area of a triangle with sides of lengths 6, $\sqrt{2}$, and $\sqrt{50}$.

Answer. 3.

Hint. Use Heron's formula.

Solution. If p is the semiperimeter, then $p = \frac{6 + \sqrt{2} + 5\sqrt{2}}{2} = 3 + 3\sqrt{2}$.

$$\begin{aligned}
\text{Area} &= \sqrt{(3 + 3\sqrt{2})(3\sqrt{2} - 3)(3 + 2\sqrt{2})(3 - 2\sqrt{2})} \\
&= \sqrt{(9 \cdot 2 - 9)(9 - 4 \cdot 2)} \\
&= \sqrt{9} = 3
\end{aligned}$$

9. Find all angles of a triangle in which the centers of inscribed and circumscribed circles are symmetric with respect to one side.

Answer. 36, 36, and 108°.

10. A height and a bisector of a right triangle dropped from the vertex of the right angle equals 3 and 4 respectively. Find the area of a triangle.

   Answer. 72.

11. An acute triangle has two sides of lengths 4 and 6. The area of the triangle is $6\sqrt{3}$. Find the length of the third side.

   Answer. $2\sqrt{7}$.

12. Is there a triangle such that

   (a) All its heights are less than 1, but its area is greater than 100?
   (b) Two heights are greater than 100, but its area is less than 1?

   Answer. (a) Such a triangle exists. (b) No.

   Hint. See Problem 18.

   Solution to (b). If $h_a > 100$ and $h_b > 100$ are perpendicular to sides BC and CA, respectively, then $c > h_a > 100$ and $a > h_b > 100$ since they are hypotenuses of right triangles.
      Evaluating the area of triangle ABC yields $[ABC] = \frac{h_a \cdot a}{2} > \frac{100 \cdot 100}{2} > 5,000$, which cannot be less than 1.

13. The area of an isosceles triangle equals 1/3 of the area of the square shown in Fig. 1.75. The leg of the triangle is 1 in. shorter than its base. Find the lengths of the sides of the triangle and its altitude.

   Answer. $AB = 6$, $BC = 5$, and $AH = 4$.
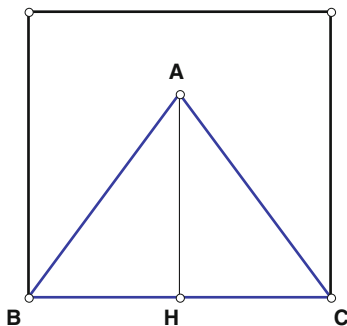
Fig. 1.75 Triangle inside a square

   Solution.

   (a) $BC^2 = 3 \cdot \frac{1}{3} BC \cdot AH \Rightarrow AH = \frac{2}{3} BC$.
   (b) $AH^2 = AB^2 - (0.5BC)^2$, this yields $\frac{4}{9} BC^2 = AB^2 - \frac{1}{4} BC^2$, or $AB^2 = \frac{25}{36} BC^2$ so $AB = \frac{5}{6} BC$.
   (c) Noticing that BC is 1 in. longer than AB, we obtain $AB = \frac{5}{6}(AB + 1) \Rightarrow$ $AB = 5$, $BC = 6$, and $AH = 4$.

14. Height CH is dropped from the vertex of the 90° angle of a right triangle ABC. Where on the segment BH do we need to place a point M such that segments AH, AM, and CM would form a right triangle?

Solution. Denote $AH = d$, $CH = h$, and $HM = x$. Then $AM^2 = (d+x)^2$ and $CM^2 = h^2 + x^2$. We have to find x. By the Pythagorean Theorem applied to triangle formed by sides AH, AM, and CM, we have that

$$AM^2 = AH^2 + CM^2$$
$$(d+x)^2 = d^2 + h^2 + x^2$$

$$h^2 = 2dx \qquad\qquad (1.50)$$

On the other hand, CH is the geometric mean of AH and HB, then

$$h^2 = d \cdot HB \qquad\qquad (1.51)$$

Combining (1.50) and (1.51) we will find that $x = HB/2$. So M must be the midpoint of segment HB.

15. Prove that for any triangle that the sum of the medians is

   (a) Less than P (perimeter)
   (b) Greater than $(3/4)P$.

Proof of (a). Consider a triangle ABC and $CD = m_c$, its median. Draw line AC' parallel to side CB and line BC' parallel to AC. Assume that the two lines intersect at point C'. Therefore $ACBC'$ is a parallelogram such that $AC = C'B$, $AC' = CB$, $CC' = 2m_c$. Next, we can use the triangle existence inequality for triangle ACC' : $CC' < AC' + AC$
   or

$$2m_c < a + b \qquad\qquad (1.52)$$

Using two other medians m_a, m_b and making similar parallelogram construction for each median, we will obtain additionally the following

$$2m_a < b + c \qquad\qquad (1.53)$$

$$2m_b < a + c \qquad\qquad (1.54)$$

Adding the left the right sides of all inequalities we obtain

$$2m_a + 2m_b + 2m_c < b + c + a + c + a + b = 2P$$

Dividing the both sides by 2, we finally can state the desired fact:

$$m_a + m_b + m_c < P.$$

Proof of (b). Again consider triangle ABC with sides a, b, and c such that its medians m_a, m_b, and m_c concur at point O. Using the triangle inequality we find that for triangle ACO

$$AC < CO + AO.$$

Because the medians are divided by the point of their intersection in the ration 2:1 then the inequality above can be rewritten as

$$b < \frac{2}{3}(m_c + m_a).$$

Similarly for triangles AOB and COB, respectively we can write

$$c < \frac{2}{3}(m_b + m_a)$$

$$a < \frac{2}{3}(m_c + m_b)$$

If we add the inequalities for a, b, and c above, we have that the perimeter $(P = a + b + c)$ of the triangle satisfies,

$$P = a + b + c < \frac{4}{3}(m_a + m_b + m_c).$$

Finally, $m_a + m_b + m_c > \frac{3}{4}P$. The proof is complete.

16. Consider triangle ABC with sides a and b. Prove that its area satisfies:

$$S = [ABC] \leq \frac{a^2 + b^2}{4}$$

Under what condition does this inequality become an equality?

Proof. First, use (1.27) for the area of a triangle, ABC. Secondly, by the application of the boundedness of sine by 1 together with the inequality between the arithmetic and geometric means, we have that

$$S = [ABC] = \frac{1}{2} \cdot ab \sin(\angle C) \leq \frac{ab}{2} \leq \frac{a^2 + b^2}{4}$$

Under what condition does this inequality becomes an equality? The geometric and arithmetic means above satisfy the inequality: $\frac{a^2+b^2}{4} = \frac{1}{2}\frac{(a^2+b^2)}{2} \geq \frac{1}{2}\sqrt{a^2 \cdot b^2} = \frac{ab}{2}$. The equal sign occurs iff $a = b$. Moreover, such a triangle would be an isosceles and right triangle.

17. A triangle has heights 3, 4, and 5. What is the type of the triangle? (Scalene, obtuse, equilateral, etc.).

Hint. Express the sides of the triangle in terms of its area and then compare the square of one side with the sum of the squares of two other sides.

Solution. Rewrite formula (1.26) in three different ways as $S = [ABC] = \frac{ah_a}{2} = \frac{bh_b}{2} = \frac{ch_c}{2}$

Substituting the values of the heights $h_a = 3, h_b = 2, h_c = 5$ into this formula we have the length of each side in terms of S: $a = \frac{2S}{3}; b = \frac{S}{2}; c = \frac{2S}{5}$. The largest side here is a. Let us compare the square of this side with the sum of the squares of two other sides.

$$a^2 = \frac{4S^2}{9}; b^2 = \frac{S^2}{4}; c = \frac{4S^2}{25}$$

$$\frac{4S^2}{9} > \frac{S^2}{4} + \frac{4S^2}{25} = \frac{41S^2}{100} \Leftrightarrow a^2 > b^2 + c^2$$

Therefore the triangle is obtuse.

18. Find the type of a triangle with the following length of the medians: 3, 4, and 5.

Hint. Evaluate the sides of the triangle using formulas (1.14) for the length of a median. Then compare the square of the longest side with the sum of the squares of two other sides.

Answer. All angles are less than $90°$. (Acute triangle).

19. In a triangle ABC there are two points P and Q on side AC, such that $AP > AQ$. The lines BP and BQ divide the median AM into three equal parts. It is known that $PQ = 3$. Find the length of side AC.

Solution. Consider a triangle ABC. Since AM is the median and BQ and BP divides it into three equal parts that is $AR = RS = SM$, then

$$|AS|:|SM| = 2:1.$$

By the property of the three medians, point S is the point at which all three medians concur. Therefore, BP is a median of triangle ABC and $AP = PC$ (Fig. 1.76).
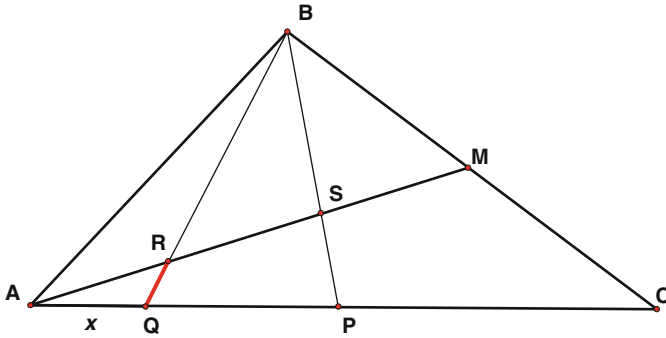
Fig. 1.76 The median divided into three equal parts

Denote the length of AQ as x. Since P is the midpoint of AC and since $QP = 3$, by the condition of the problem we have that

$$PC = AP = AQ + QP = x + 3 \text{ and } AC = 2x + 6.$$

We need to find x.

Consider a triangle AMC with RQ as a transversal. From the Menelaus' Theorem we have

$$\frac{|MR|}{|RA|} \cdot \frac{|AQ|}{|QC|} \cdot \frac{|CB|}{|BM|} = 1$$

Each ratio of the formula above can be found separately. Thus,

$$\frac{|MR|}{|RA|} = \frac{2}{1}$$

$$\frac{|AQ|}{|AC|} = \frac{x}{x+3+3} = \frac{x}{x+6}$$

$$\frac{|CB|}{|BM|} = \frac{2}{1}$$

Multiplying the left and right hand sides each of the above three equations together we obtain

$$\frac{4x}{x+6} = 1$$

$$x = 2$$

$$|AC| = 2 \cdot 2 + 6 = 10$$

Answer. $|AC| = 10$.

20. (Lidsky) The medians dropped to sides a and b of a triangle are perpendicular. Find the third side of a triangle. Under what condition such a triangle exists?

Solution Part 1. We need to find the length of the unknown side.

Recall what we remember about medians: (a) a median cuts the opposite side into two equal parts. (b) Three medians concur, and (c) each median is divided by the point of their intersection in the ratio 2:1 starting from the vertex.

Denote $DO = x$ and $OF = y$. From Fig. 1.77 we have the that $CB = a$, $DB = \frac{a}{2}$, $AC = b$, and $FA = \frac{b}{2}$
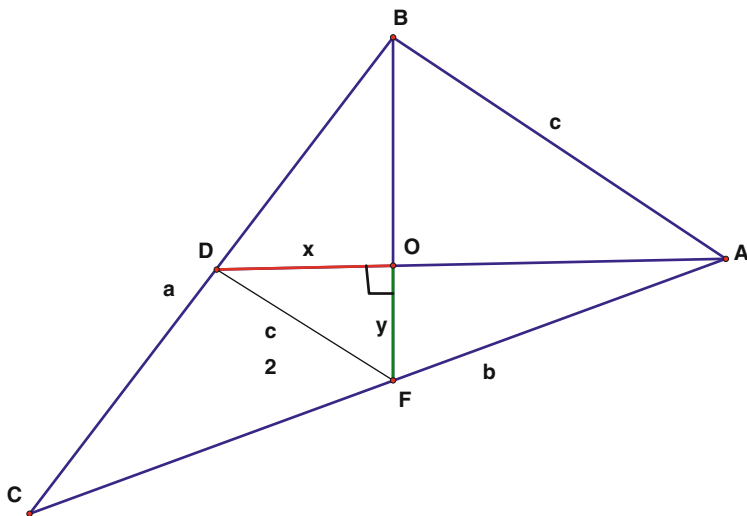
Fig. 1.77 Problem on two perpendicular medians

From the right triangles DOB, FOA, and DOF we have the system

$$
\begin{cases}
x^2 + (2y)^2 = \left(\dfrac{a}{2}\right)^2 \\[2mm]
y^2 + (2x)^2 = \left(\dfrac{b}{2}\right)^2 \\[2mm]
x^2 + y^2 = \left(\dfrac{c}{2}\right)^2
\end{cases}
$$

If we add the first two equations together and substitute the result into the third one as it is shown below, we obtain

$$20(x^2 + y^2) = 5 \cdot 4(x^2 + y^2) = a^2 + b^2$$
$$4(x^2 + y^2) = c^2$$
$$5c^2 = a^2 + b^2$$
$$c^2 = \frac{a^2 + b^2}{5}$$

$$AB = c = \sqrt{\frac{a^2 + b^2}{5}}$$

Part 2. Under what condition does a triangle with sides $a, b, c = \sqrt{\frac{a^2+b^2}{5}}$ exist?

We have to understand that the answer is valid only when we define the value of the parameters a and b so that we can make a triangle with the given sides. For example, assume that $a = 1$, $b = 2$, then $c = 1$ (calculated by formula). However, such a triangle does not exist because it does not satisfy the side triangle inequality condition, i.e., each side must be shorter than the sum of two other sides ($2 = 1 + 1$).

Consider all three triangle existence inequalities

$$\begin{cases} c < a + b \\ a < b + c \\ b < a + c \end{cases}$$

We can also to rewrite it as

$$\begin{cases} c < a + b \\ a - b < c \\ b - a < c \end{cases} \Leftrightarrow \begin{cases} c^2 < (a + b)^2 \\ c^2 > (a - b)^2 \end{cases}$$

Next, we will substitute the value of c^2 into each inequality and solve it.

$$c^2 = \frac{a^2 + b^2}{5} < (a + b)^2 \text{ is always true}$$

$$c^2 = \frac{a^2 + b^2}{5} > (a + b)^2. \text{ This will need to solved}$$

$$5(a - b)^2 < a^2 + b^2$$
$$5a^2 - 10ab + 5b^2 - a^2 - b^2 < 0$$
$$4a^2 - 10ab + 4b^2 < 0$$

In order to simplify this inequality we will divide both sides by $2b^2$ and then consider a quadratic inequality for new variable $z = a/b$ (Fig.):

$$2\left(\frac{a}{b}\right)^2 - 5\frac{a}{b} + 2 < 0, \; z = \frac{a}{b}$$
$$2z^2 - 5z + 2 < 0$$
$$z_1 = \frac{5 + \sqrt{5^2 - 4 \cdot 2 \cdot 2}}{2 \cdot 2} = 2$$
$$z_2 = \frac{1}{2}$$

g(x) = (2·x2-5·x)+2
q(x) = g(x)
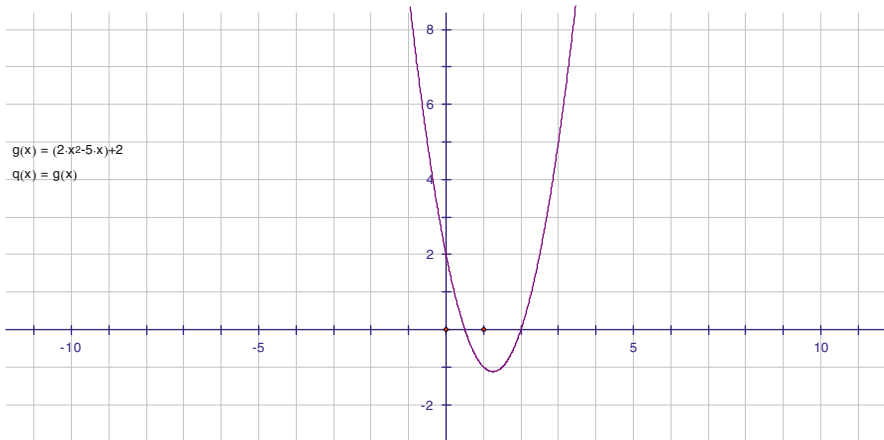
Fig. 1.78 Parabola

Since a parabola with a positive leading coefficient takes a negative value only between its two real zeros, the answer would be $\frac{1}{2} < z < 2$, which can be rewritten as

$$\frac{1}{2} < \frac{a}{b} < 2$$

This is why such a triangle can have sides $a = b = 1$. The inequality is satisfied $\frac{1}{2} < 1 < 2$ but cannot have sides $a = 1$, $b = 2$ or $\left(\frac{a}{b} = \frac{1}{2}\right)$ because then $\frac{1}{2} < \frac{1}{2} < 2$ is false.

Chapter 2
Quadrilaterals and Other Polygons

2.1 Convex and Nonconvex Polygons

A **convex polygon** is one in which no part of a line segment connecting any two parts of the polygon is outside the polygon. So certainly the polygon below is nonconvex because it does not obey the definition of convexity (part of the line segment *FG* is outside of the polygon's interior) (Fig. 2.1).
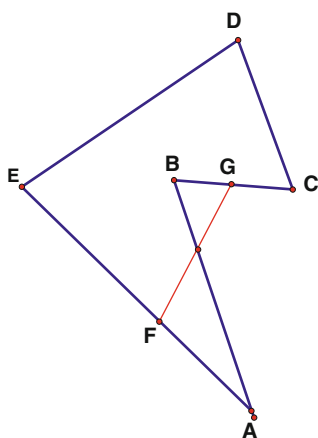
Fig. 2.1 Nonconvex polygon

2.1.1   Properties of Convex Polygons

Sum of the Interior Angles. For any convex polygon the sum of all interior angles is $(n-2) \cdot 180°$, where n is the number of sides of the polygon. Thus, for a quadrilateral, $n = 4$, and the sum of all interior angles is $360°$. As we know, the sum of all interior angles in a triangle is $180°$.

How can we use this fact?

Maybe we can divide a given convex polygon into nonoverlapping triangles by connecting some of its vertices. For example, let us draw a pentagon and connect its vertices in such a way that only nonoverlapping triangles will be formed as shown in Fig. 2.2.
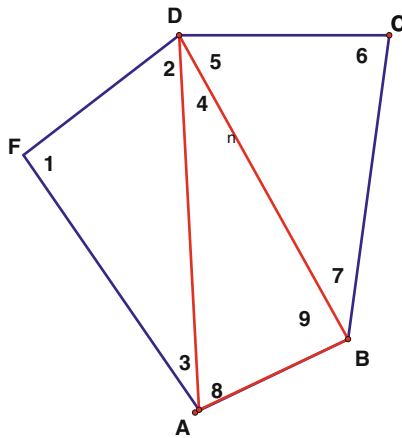
Fig. 2.2 Sum of the interior angles

We will get the sum of all interior angles of the pentagon if we add together angles 1, 2, 3, 4, 5, 6, 7, 8, and 9. However, each triple of angles (1–2–3), (4–8–9), and (5–6–7) will give us $180°$ for a total of $180° \cdot 3 = 540°$. Thus, for any n-gon, there are exactly $(n-2)$ nonoverlapping triangles, each of which have angles that sum to $180°$. Therefore, the sum of all interior angles in a convex polygon is $(n-2) \cdot 180°$.

An **exterior angle of a polygon** is an angle between the extension of one of the sides of the polygon and the outside of the polygon. Each such angle is a supplement of an adjacent interior angle, so the interior plus exterior angle equals $180°$ (Fig. 2.3).

m∠DFA = 121.63°

m∠HFA = 58.37°
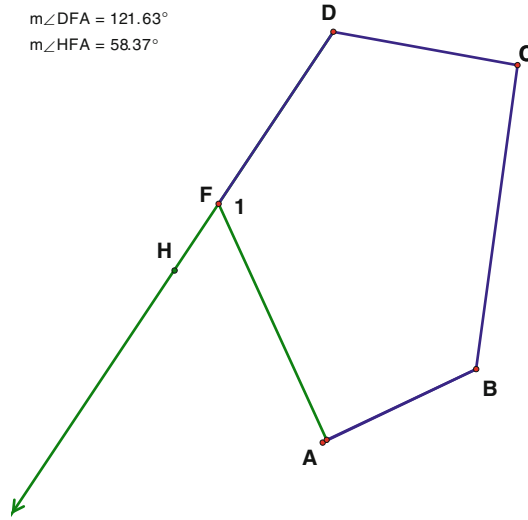
Fig. 2.3 Exterior angles of a polygon

For example, for the interior angle 1 (*DFA*) there is exterior angle *HFA*. Together they make a supplemental couple.

Exterior Angles Polygon Sum Theorem. The sum of the measures of the exterior angles, one at each vertex of any convex polygon is $360°$.

Any convex polygon with four sides is a **convex quadrilateral**. Squares, rectangles, parallelograms, rhombuses, trapezoids, and kites are examples of convex quadrilaterals and most problems given in this book are related to them.

2.2   Parallelograms

- Opposite sides of any **parallelogram** are equal, opposite angles are equal, and a sum of adjacent angles equals $180°$, or π.
- If opposite sides of a quadrilateral are equal or two opposite sides are equal and parallel to each other, then such a quadrilateral is a parallelogram.
- A quadrilateral is a parallelogram if and only if its diagonals bisect each other. (It means that the point of intersection of the diagonals is the midpoint of each diagonal.)
- There is an equality for any parallelogram: $d_1^2 + d_2^2 = 2(a^2 + b^2)$, where a and b are sides and d_1 and d_2 are the diagonals.

- A **rhombus** is a particular case of a parallelogram such that its diagonals bisect corresponding angles, bisect each other, and are perpendicular to each other.
- A **rectangle** is a parallelogram such that adjacent sides are perpendicular to each other. The diagonals of each rectangle are equal.
- The diagonals of a **square** are perpendicular to each other and equal. Of course, each square is a particular case of a rectangle.

2.2.1   Varignon's Theorem

Varignon's Theorem is a statement in Euclidean geometry by **Pierre Varignon** that was first published in 1731. It deals with the construction of a particular parallelogram (**Varignon parallelogram**) from an arbitrary quadrangle (another name for a quadrilateral).

> The midpoints of the sides of an arbitrary quadrangle form a parallelogram. If the quadrangle is convex, then the area of the parallelogram is half as large as the area of the quadrangle.

You will prove the first part of the theorem by solving Problem 35

Problem 35.   Prove that if consecutive midpoints of all sides of a quadrilateral are connected, that they form a parallelogram.

Solution. I have watched many students trying to solve this problem. Many of them draw a nice, accurate picture of the problem. From the picture, they make a conclusion: "It is always a parallelogram." Yes, an accurate picture is 50 % of a successful solution in geometry, but it is not the solution itself.
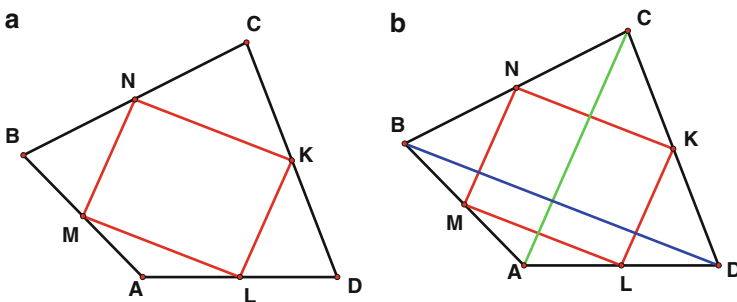
Fig. 2.4   Varignon' Theorem

My students showed me a picture like Fig. 2.4a. *ABCD* is a general quadrilateral and *M*, *N*, *K*, and *L* are midpoints. We connect them and get *MNKL* shown in red that looks like a rectangle. We must show that the opposite sides of *MNKL* are parallel.

Do you remember that in any triangle a midpoint line is parallel to the base of a triangle? In the original sketch we can see four triangles but none of them contains its midsegment.

Can we construct such a triangle?

<u>Yes</u>. First we will connect vertices *A* and *C*. To distinguish our additional construction from the original we will show this segment, *AC*, in green. Now let us look at the triangle *ABC*. You can see that *MN* is parallel to *AC*. Further, in triangle *ACD*, *KL* is a midpoint segment and *KL* is parallel to *AC*. We know that if two different lines are parallel to the third line, then they are parallel to each other. Moreover, since triangles *ABC* and *ACD* have the same base, *AC*, it follows that their midpoint segments are equal. This can be written as

$$\left. \begin{array}{l} MN \parallel AC \\ KL \parallel AC \\ MN = \dfrac{1}{2}AC \\ KL = \dfrac{1}{2}AC \end{array} \right\} \Rightarrow MN \parallel KL, MN = KL \qquad (2.1)$$

Now construct the second segment, *BD* shown in blue in Fig. 2.4b. Considering triangles *BDC* and *ABD* we conclude that

$$\left. \begin{array}{l} ML \parallel BD \\ NK \parallel BD \\ ML = \dfrac{1}{2}BD \\ NK = \dfrac{1}{2}BD \end{array} \right\} \Rightarrow ML \parallel NK, ML = NK \qquad (2.2)$$

Combining (2.1) and (2.2) we obtained a sufficient property of a parallelogram. Yes, *MNKL* is always a parallelogram.

Problem 36. Prove the second part of the Varignon's Theorem that "If the quadrangle is convex, then the area of the parallelogram is half as big as the area of the quadrangle."

Proof. Consider the second picture of the previous problem. Let the area of the original convex quadrilateral be S. We can see that the area of the parallelogram *MNKL* can be written as

$$\text{Area}(MNKL) = S - ([AML] + [NCK] + [BNM] + [LKD])$$

Using relations between similar triangles we obtain

$$[AML] = \frac{1}{4}[ABD], \quad \Delta AML \sim \Delta ABD$$

$$[NCK] = \frac{1}{4}[BCD], \quad \Delta NCK \sim \Delta BCD$$

$$[BNM] = \frac{1}{4}[ABC], \quad \Delta BNM \sim \Delta ABC$$

$$[LKD] = \frac{1}{4}[ACD], \quad \Delta LKD \sim \Delta ACD$$

If we add the left and the right sides of these relationships we obtain the following:

$$[AML] + [NCK] + [BNM] + [LKD] = \frac{1}{4}([ABD] + [BCD] + [ABC] + [ACD])$$

On the other hand, pairs of the triangles (ABD, BCD) and (ABC, ACD) make the original quadrilateral that can be written in terms of areas as

$$[ABD] + [BCD] = [ABC] + [ACD] = S$$

then

$$[ABD] + [BCD] + [ABC] + [ACD] = 2S$$

Therefore,

$$\text{Area}(MNKL) = S - \frac{1}{4} \cdot 2S = \frac{S}{2}$$

The proof is completed.

After solving Problem 36, Problem 37 will look familiar to you.

Problem 37. A quadrilateral with diagonals d_1 and d_2 is given. Prove that the parallelogram obtained by connecting the midpoints of the original quadrilateral has a perimeter of $d_1 + d_2$.

Proof. Let us again use Fig. 2.4b from the previous problem. The perimeter of *MNKL* can be calculated as *MN* + *NK* + *KL* + *LM*. We have that *MN* = *LK* = ½*AC* and *NK* = *LM* = ½*BD*, so with *AC* = d_1 and *BD* = d_2, we have

$$P = \frac{1}{2}d_1 + \frac{1}{2}d_2 + \frac{1}{2}d_1 + \frac{1}{2}d_2 = d_1 + d_2$$

This completed our proof.

Use a ruler for Problem 38 so the constructions will be as neat as required.

Problem 38 (Lidsky). A parallelogram with area *S* is given. Each vertex of the parallelogram is connected by a line segment to the midpoints of the two opposite sides. Find the area of the polygon that is formed by the intersection of the all line segments.

Solution. Construct a parallelogram and connect each of its vertices with the midpoints of two opposite sides. As a result we obtain a polygon with eight sides as shown in Fig. 2.5.
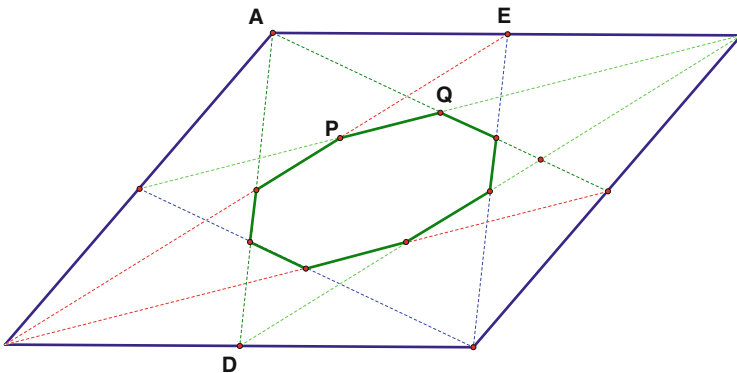
Fig. 2.5 First sketch for Problem 38

In order to evaluate its area, we will need an additional (auxiliary) construction, i.e., we will need to connect the midpoints of the adjacent sides.

Connect the midpoints of opposite sides (e.g., *E* to *D* and *G* to *H*) as shown in Fig. 2.6. Additionally, connect points *A* and *M* of the parallelogram. From the figure, we can see that the entire parallelogram consists of eight equal area triangles (e.g., triangle *AOE*). On the other hand, the 8-gon also consists of eight triangles with areas equal the area of triangle *POQ*.

We have to find the relation between the area of the small triangle, *POQ*, shown in red and the area of the large triangle, *AOE*. Since *P* is the point of intersections of

the medians in triangle DAE, then $PO = 1/3\ OA$ and $OQ = 1/2\ OE$, then $[POQ]$ $= \frac{1}{6} \cdot [AOE]$. Therefore, the area of the green 8-gon is $S/6$.
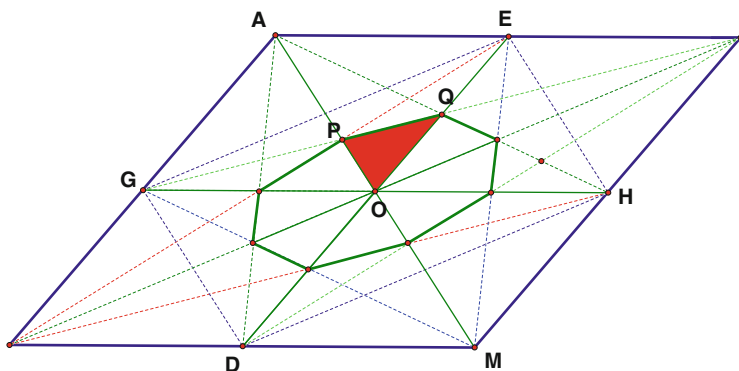
Fig. 2.6 Second sketch for Problem 38

Students may think that a square is the simplest geometric figure imaginable; nonetheless there are many good problems concerning squares.

Problem 39. The areas of a square and a circle are equal. Which is bigger: the circumference of the circle or the perimeter of the square?

Solution. Let us assume that the square has side a and the circle has radius R. Using the condition of the problem, we can write the equation: $\pi R^2 = a^2$ and $a = \sqrt{\pi}R$. Let us express the circumference and perimeter in terms of R and then compare the expressions:

$$P \text{ of square} = 4\sqrt{\pi}R$$
$$P \text{ of circle} = 2\pi R$$

In order to compare the formulas, we can the divide the perimeter of the circle by that of the square giving $\frac{\sqrt{\pi}}{2}$. Now compare $\frac{\sqrt{\pi}}{2}$ to 1. Pretending that we don't have a calculator, let us do it mentally. What is $\sqrt{\pi}$? It is some irrational number. But how big is it? $\pi \approx 3.14 < 4$, then $\sqrt{\pi} < \sqrt{4}$, and $\frac{\sqrt{\pi}}{2} < \frac{\sqrt{4}}{2} = \frac{2}{2} = 1$. Because the ratio is less than 1, the circumference is less than the perimeter of the square.

Problem 40. A side of a square is a. The midpoints of its sides are joined to form an inscribed square. This process is continued as shown in the diagram. Find the sum of the perimeters of the squares if the process is continued without end (Fig. 2.7).
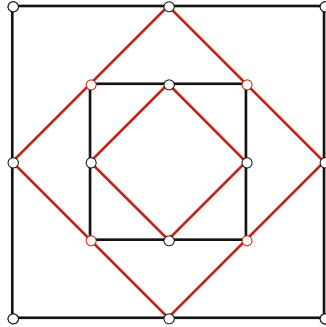
Fig. 2.7 Sketch for Problem 40

Solution. From the diagram we can see that sides of the black squares form a geometric progression with the first term of a and common ratio $\frac{1}{2}$:

$$a, \frac{a}{2}, \frac{a}{4}, \frac{a}{8}, \dots, \frac{a}{2^{n-1}}$$

All red squares, in turn, form a geometric progression with the same common ratio but the first term $\frac{a\sqrt{2}}{2}$ (half of the diagonal of the original square):

$$\frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{4}, \frac{a\sqrt{2}}{8}, \dots$$

Because the perimeter of a square with side b is $4b$, we obtain the following expression for the sum of the perimeters of black and red squares:

$$P = 4\left(a + \frac{a\sqrt{2}}{2} + a/2 + \frac{a\sqrt{2}}{4} + a/4 + \frac{a\sqrt{2}}{8} + a/8 + \dots + \frac{a}{2^{n-1}} + \frac{a\sqrt{2}}{2^{n-1}} + \dots\right)$$

$$= 4a(1 + 1/2 + 1/4 + \dots) + 4a \cdot \frac{\sqrt{2}}{2}(1 + 1/2 + 1/4 + \dots)$$

$$= 4a\left(2 + \sqrt{2}\right)\left(1 - \frac{1}{2^n}\right) = 4a\left(2 + \sqrt{2}\right)$$

Answer. $P = 4a(2 + \sqrt{2})$.

2.3 Trapezoids

A **trapezoid** is a convex quadrilateral with at least one pair of parallel sides. Any pair of angles adjacent to a parallel side sums to 180° since they are supplementary. The parallel sides are bases of the trapezoid. A midline segment (**midsegment**) of a trapezoid connects the midpoints of nonparallel sides and is parallel to the bases.

Trapezoid Midsegment Length Theorem. The length of a midsegment is equal to $\frac{1}{2}(a+b)$, where a and b are the bases of a trapezoid.

So, for a trapezoid with bases 4 and 10, irrespective of the lengths of other sides, the midline segment equals $(4+10)/2 = 7$.

We will give the proof of the Trapezoid Midsegment Length Theorem in Problem 41.

Problem 41. A trapezoid $ABCD$ with bases $AD = a$ and $BC = b$ is given. M is the midpoint of AB, N is the midpoint of CD. Prove that $MN = (a+b)/2$.

Proof. First, we will sketch a trapezoid and with all known information.
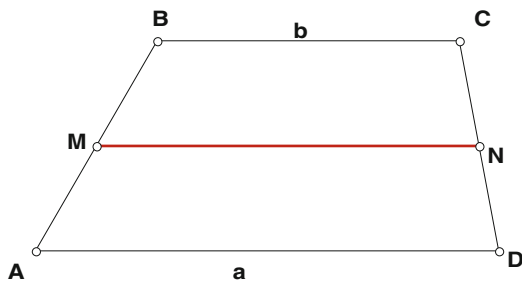
Fig. 2.8 First sketch for Problem 41

From Fig. 2.8 we cannot find MN right away, but drawing diagonals of $ABCD$ (Fig. 2.9) gets us closer. In our picture, segments AC and MN intersect at point K, diagonal BD intersects with MN at point L and the two diagonals intersect at O. Segment MN can be thought of as the sum of ML and LN. Our purpose is to find ML and LN. Let us translate all this into mathematical language. Here and below, symbol $\cap$ means intersection.
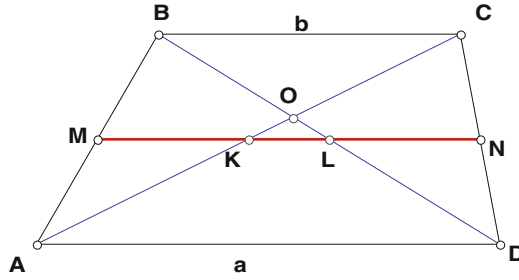
Fig. 2.9 Second sketch for Problem 41

$$AC \cap MN = K$$
$$BD \cap MN = L$$
$$BD \cap AC = O$$
$$MN = ML + LN$$

In a triangle *ABD*, if *ML* is a midline segment, then

$$ML = \frac{1}{2}AD = \frac{a}{2} \qquad (2.3)$$

In a triangle *BCD*, *LN* is a midline segment, then

$$LN = \frac{1}{2}BC = \frac{b}{2} \qquad (2.4)$$

Combining (2.3) and (2.4) we find the unknown *MN*:

$$MN = \frac{a}{2} + \frac{b}{2} = \frac{(a+b)}{2}$$

It follows from this formula that the length of a midsegment of a trapezoid is the **arithmetic mean** of its bases.

The reason I encourage you to do proofs is that you have to remember the ideas behind the proofs in order to gain experience.

Problem 42. Trapezoid *ABCD* with bases $AD = a$ and $BC = b$ is given. A line segment, *KL*, is drawn parallel to the bases and through the point of intersection of the diagonals. Find the length of *KL*.
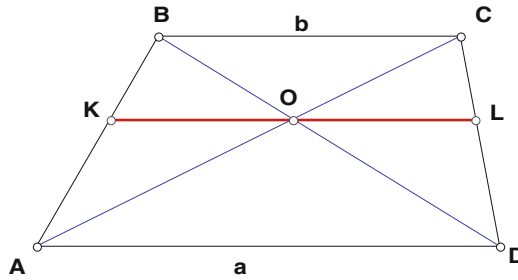
Fig. 2.10 Sketch for Problem 42

Solution. Again we start from a picture (Fig. 2.10).

$AC \cap BD = O$ and $KL \parallel BC \parallel AD$. We must find the length of KL. This picture looks very similar to a picture from Problem 41: three parallel segments, intersecting diagonals, but K and L are not midpoints of segments AB and CD so we have to come up with a new idea.

$$\left.\begin{array}{l} BC \parallel AD \text{ (given)} \\ m\angle BOC = m\angle DOA \text{ (as vertical angles)} \end{array}\right\} \Rightarrow \Delta AOD \sim \Delta COB \text{ (AA)}$$

For similar triangles we can write ratios of corresponding sides:

$$\begin{aligned} \frac{AD}{BC} &= \frac{AO}{OC} = \frac{OD}{OB} = \frac{a}{b} \Rightarrow \\ \frac{AO}{AC} &= \frac{a}{a+b} \quad \text{and} \quad \frac{OD}{BD} = \frac{a}{a+b} \end{aligned} \tag{2.5}$$

Likewise, triangles AKO and ABC are similar too:

$$\Delta AKO \sim \Delta ABC \Rightarrow \frac{AO}{AC} = \frac{KO}{BC}$$

Replacing BC by b and using ratio (2.5) we obtain:

$$\frac{a}{a+b} = \frac{KO}{b} \tag{2.6}$$

From (2.6) we find KO,

$$KO = \frac{ab}{a+b} \tag{2.7}$$

Our unknown, KL, is the sum of KO and OL. We know KO. Let us find OL using the similarity of triangles OLD and BCD:

$$\frac{OL}{BC} = \frac{OD}{BD}$$

Replacing BC by b and using ratio (2.5) we can find OL:

$$OL = \frac{ba}{a+b} \qquad (2.8)$$

Adding (2.7) and (2.8), we calculate KL:

$$KL = \frac{2ab}{a+b}$$

Moreover, the length of segment KL is the **harmonic mean** of the trapezoid bases.

Answer. $2ab/(a+b)$.

If you enjoyed solving Problem 42, I think you will like the following one even better.

Problem 43. Numbers a and b are bases of a trapezoid. Find the length of a segment connecting the midpoints of its diagonals.

Solution. Again, let us draw a trapezoid showing all known information. Please do not forget to use a ruler (Fig. 2.11).
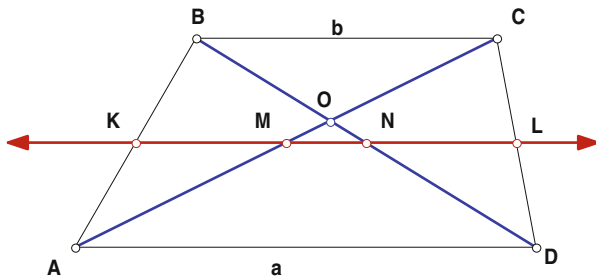
Fig. 2.11 Sketch for Problem 43

$$BD \cap AC = O$$
$$AM = MC$$
$$BN = ND$$
$$KL \parallel BC \parallel AD$$

So we have to find the length of MN. We have that $MN = KN - KM$, but how about KN and KM? From triangle ABD, KN is a midline segment, then

$$KN = \frac{1}{2}AD = \frac{a}{2} \qquad\qquad (2.9)$$

From triangle BCA, KM is a midline segment, then

$$KM = \frac{1}{2}BC = \frac{b}{2} \qquad\qquad (2.10)$$

Subtracting (2.10) from (2.9), we find our unknown MN:

$$MN = \frac{a - b}{2}$$

Answer. $(a - b)/2$.

2.4   Formulas for the Areas of Quadrilaterals

Area of any convex quadrilateral:

$$S = \frac{1}{2}d_1 \cdot d_2 \cdot \sin \phi \qquad\qquad (2.11)$$

where d_1 and d_2 are the diagonals of a quadrilateral and ϕ is the angle between them. Of course, this formula will work for any quadrilateral. I ask you to prove this as a homework exercise.

Area of a Parallelogram: There are two common formulas for evaluating areas of parallelograms:

1. The area of a parallelogram is the product of a base and height dropped to this base and can be written as

$$S = a \cdot h_a = b \cdot h_b \qquad\qquad (2.12)$$

2. The area of a parallelogram is the product of its sides and the sine of the included angle:

$$S = a \cdot b \cdot \sin \gamma \qquad\qquad (2.13)$$

Note. These are not independent formulas, i.e., $h_a = b \cdot \sin \gamma$ and $h_b = a \cdot \sin \gamma$.

Sometimes it is not enough to know these formulas by memorization. Experience will help you to visualize the idea of the formula in its application to a particular problem. Let us try Problem 44.

Problem 44. *ABCD* is a parallelogram. $AB = 2$ and $BC = 3$. Find the area of *ABCD*, if its diagonal *AC* is perpendicular to segment *BE* which connects vertex *B* with the midpoint of segment *AD*.

Solution. As usual, we are going to draw a parallelogram and put in the figure all known information. *E* is the midpoint of AD, $BE \perp AC$, and $AC \cap BE = M$ (segment *AC* intersects *BE* at point *M*). If in Fig. 2.12, *BM* is perpendicular to *AC*, then *BM* is the height of triangle *ABC* dropped to side *AC*. Having experience solving different problems, you might want to find the area of *ABCD* as a sum of the areas of the two triangles, *ABC* and *ACD*. Let us drop a perpendicular to side *AC* from vertex *D*. *DN* is the height of triangle *ACD* and *BM* ∥ *ND* as they are two perpendiculars to the same line.

Let us introduce two variables: $AC = x$ and $BE = y$. Triangle *AME* is similar to triangle *CMB* by the *AA* property because $m\angle AME = m\angle CMB$ (since they are vertical angles) and $m\angle MAD = m\angle BCM$ (since they are alternate interior angles (AIA)).

Note. If two angles of similar triangles are equal, then the third is equal as well, i.e., $m\angle AEM = m\angle MBC$ since they also are AIA.
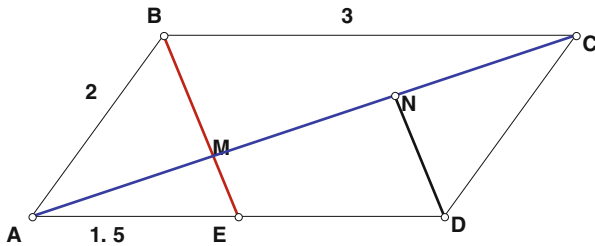
Fig. 2.12 Sketch for Problem 44

We are going to write a ratio of corresponding sides and express some important segments in terms of *x* and *y*.

$$\Delta BMC \sim \Delta EMA \Rightarrow \frac{BM}{ME} = \frac{CM}{MA} = \frac{3}{1.5} = \frac{2}{1}$$

The ratio tells us that point M divides segment BE in ratio of 2:1, but divides segment AC in a ratio 1:2. This gives us the following relations:

$$BM = \frac{2}{3}y, \quad ME = \frac{1}{3}y$$

$$AM = \frac{1}{3}x, \quad MC = \frac{2}{3}x$$

In order to find x and y we can apply the Pythagorean Theorem to the right triangles AMB and BMC. Now we must solve the system:

$$\begin{cases} \left(\frac{2y}{3}\right)^2 + \left(\frac{x}{3}\right)^2 = 4 \\ \left(\frac{2y}{3}\right)^2 + \left(\frac{2x}{3}\right)^2 = 9 \end{cases} \Leftrightarrow \begin{cases} 4y^2 + x^2 = 36 \\ 4y^2 + 4x^2 = 81 \end{cases}$$

Solving for x by subtraction of the equations on the right, we have that $x^2 = 15$, $x = \sqrt{15}$ which can be substituted to give $y^2 = \frac{21}{4}$, $y = \frac{\sqrt{21}}{2}$.

Knowing x and y we can easily evaluate the area of triangle ABC. There is only one thing we don't know: the length of segment ND. This will be our next step.

We see that $\triangle AME \sim \triangle AND$ (AA) and E is the midpoint of AD. Then, we have

$$ND = 2ME = \frac{2}{3}BE = BM$$

Finally,

$$\begin{aligned}
\text{Area of } ABCD &= \text{Area of } ABC + \text{Area of } ACD \\
&= \frac{1}{2}BM \cdot AC + \frac{1}{2}BM \cdot AC \\
&= BM \cdot AC \\
&= \frac{2}{3}y \cdot x \\
&= \frac{2}{3} \cdot \frac{\sqrt{21}}{2} \cdot \sqrt{15} \\
&= \frac{\sqrt{3 \cdot 7} \cdot \sqrt{3 \cdot 5}}{3} = \sqrt{35}
\end{aligned}$$

Answer. The area of parallelogram equals $\sqrt{35}$.

Having solved the problem above, you have proven again that there is no one recipe or standard approach of solving a problem. Thus, we could not apply the

formula for the area of a parallelogram right away because we didn't know the height. However, by dividing the parallelogram into two triangles and using algebra, we were able to get the right answer.

Area of a Trapezoid: For any trapezoid with bases a and b and height h, the area can be calculated as

$$S = \frac{1}{2}(a + b) \cdot h \tag{2.14}$$

Problem 45. Which quadrilateral with diagonals d_1, d_2 has the maximum area?

Solution. Let us imagine a couple quadrilaterals with diagonals d_1 and d_2. In Fig. 2.13, even diagonals of the same length form absolutely different quadrilaterals and of course they will have different areas.
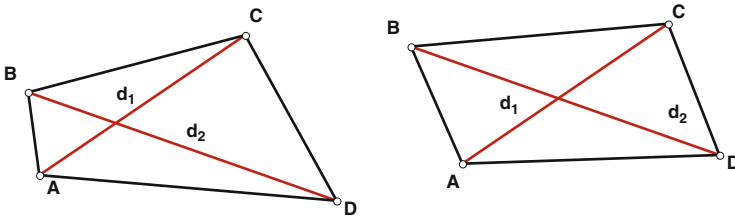
Fig. 2.13   Sketch for Problem 45

Recall the formula of the area of a quadrilateral. Looking at (2.11) again we understand that S is maximal if $\sin \gamma = 1$, i.e., the angle between the diagonals is 90°.

Answer. Among all quadrilaterals with diagonals of the same length, the quadrilateral with two perpendicular diagonals has the maximum area.

Recalling the previous problem, we can think of quadrilaterals with perpendicular diagonals: **kites**, rhombuses, and squares. A square has diagonals of the same length and is a particular case of a rhombus. A kite is a quadrilateral, such that its diagonals are perpendicular to each other and one of them is also the line of symmetry. Kites have perpendicular diagonals. The diagonals of a rhombus are not only perpendicular, but also bisect each other making the rhombus a parallelogram. There are many quadrilaterals with perpendicular diagonals and with no symmetry at all. For example, in the Fig. 1.77 you can see that quadrilateral *DBAF* has perpendicular diagonals, but is not symmetric along either diagonal.

Let us solve a problem on the properties of a rhombus.

Problem 46. The perimeter of a rhombus is 20 and the sum of its diagonals is 14. Find its area.

Solution. We start from drawing a picture (Fig. 2.14).
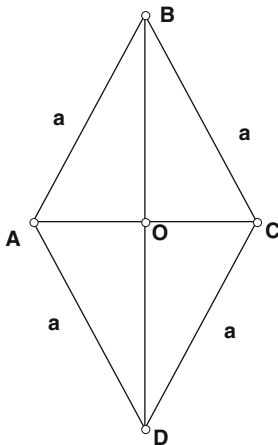
Fig. 2.14 Sketch for Problem 46

Let us introduce three variables: $a = AB = BC = CD = AD$ is the length of the sides of the rhombus, $d_1 = AC$ and $d_2 = BD$ are the diagonals. At this point we are analyzing the problem and trying to get the idea of the solution. Translating our conditions into mathematical language we obtain:

$$\begin{cases} 4a = 20 \ (\text{perimeter of a rhombus}) \\ d_1 + d_2 = 14 \end{cases} \Leftrightarrow \begin{cases} a = 5 \\ d_1 + d_2 = 14 \end{cases} \tag{2.15}$$

The area of the rhombus is our unknown. We can try to find it as

$$S = \frac{1}{2} d_1 \cdot d_2 \tag{2.16}$$

From (2.16) we notice that if we know the product of diagonals, $d_1 \cdot d_2$, we know the area. Squaring both sides of the second equation of system (2.15) and expanding the left side we have

$$(d_1 + d_2)^2 = 14^2 \quad \text{or} \quad d_1^2 + d_2^2 + 2 \cdot \underbrace{d_1 d_2} = 196 \tag{2.17}$$

from which we can find the product of diagonals if we are able to evaluate $d_1^2 + d_2^2$.

Are we getting closer?

I encourage you to look at the rhombus again and especially at the right triangle *AOB*. Applying the Pythagorean Theorem to it we can relate the sum of squares of diagonals and the side, a, of the rhombus:

$$\left(\frac{d_1}{2}\right)^2 + \left(\frac{d_2}{2}\right)^2 = a^2$$
$$d_1^2 + d_2^2 = 4 \cdot 25 = 100 \tag{2.18}$$

Replacing (2.18) into (2.17) we obtain

$$100 + 2d_1 d_2 = 196$$
$$d_1 d_2 = 48$$
$$S = \frac{1}{2} \cdot 48 = 24$$

Answer. The area of the rhombus equals 24 units^2.

Problem 46 shows us how algebra can help us to solve geometric problems. Do not be afraid to introduce variables. As you may recall, we originally introduced three (!) variables. By nicely manipulating them with algebraic expressions, we never needed to find the particular values of d_1 and d_2 to successfully find their product!

Problem 47 will help us to understand more about the rhombus.

Problem 47. Find the angles of the rhombus if its area equals 8 and the area of the inscribed circle equals π.

Solution. First, we will draw a rhombus and inscribe into it a circle of radius, r (Fig. 2.15).
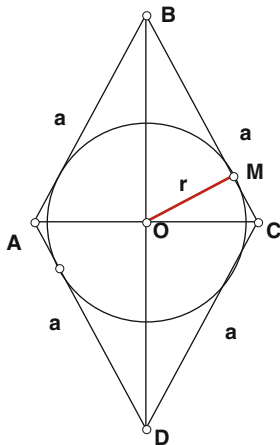
Fig. 2.15 Sketch for Problem 47

Let us introduce three variables: a side of the rhombus, $a = BC = CD = DA = AB$, the radius of the inscribed circle, $r = OM$ (OM is tangent to BC), and the angle OBM, α. Because the area of the circle is given, we can find its radius right away as $\pi r^2 = \pi$. Dividing both sides by π, we have $r = 1$. (Naturally, r cannot be negative.)

Now we have to find a way of using the second part of the condition: that the area equals 8. From Fig. 2.15 we can see that the diagonals divide the rhombus into four equal right triangles, so the area of the rhombus equals 4 times the area of triangle BOC. The area of the triangle can be found in different ways, but it seems reasonable to find it as half of a product of r (the height) and a (the base). From the condition of the problem, the area of the rhombus equals 8. This we can write as $[ABCD] = 4 \cdot \frac{1}{2} \cdot r \cdot a = 2a = 8 \Rightarrow a = 4$.

Now it is the time to find α. For this purpose we introduce additional variables x and y such that $x = BM$ and $y = MC$. From Fig. 2.15 we notice that $BM + MC = BC$ and

$$x + y = 4 \tag{2.19}$$

Values of y and x we can find from triangles OMC and OBM respectively.

$$\Delta OBM : \ x \cdot \tan \alpha = OM, \quad OM = 1, \ \text{then } x = \frac{1}{\tan \alpha} \tag{2.20}$$

$$\Delta OMC : \ y = OM \cdot \tan \alpha, \quad y = \tan \alpha \tag{2.21}$$

Now, substituting (2.20) and (2.21) into equation (2.19) we come up with a trigonometric equation of a quadratic type:

$$\tan \alpha + \frac{1}{\tan \alpha} = 4$$
$$\tan^2 \alpha - 4 \tan \alpha + 1 = 0$$
$$\tan \alpha = 2 \pm \sqrt{3}$$
$$m\angle OBC = \alpha = 15°$$

Knowing angle OBC we can find its complementary angle, $OCB = 75°$. The angles of the rhombus are twice the angles, $15°$ and $75°$, respectively.

2.5   Regular Polygons

A **regular polygon** is a polygon that is **equiangular** (all angles are equal in measure) and **equilateral** (all sides have the same length). Regular polygons may be convex or **star** (nonconvex). In the limit, a sequence of regular polygons with an increasing number of sides becomes a circle. In this text we will consider problems regarding convex regular polygons such as equilateral triangle, square, pentagon, hexagon, etc. Constructions of a regular polygon will be discussed in detail in Sect. 4.4.2. Some images of well-known regular polygons are given in Fig. 2.16.
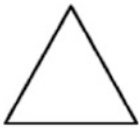
Triangle   Square   Pentagon
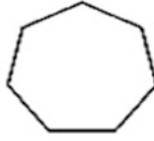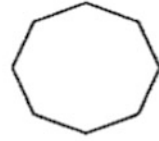
Hexagon   Heptagon   Octagon
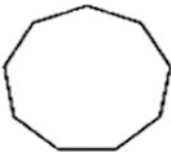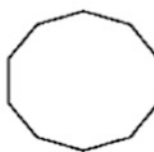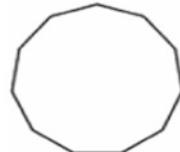
Nonagon   Decagon   Hendecagon

Fig. 2.16 Images of some regular polygons

2.5.1 Properties of Regular Polygons

Every regular polygon is **cyclic**, i.e., it can be inscribed into a circle. In order to construct a regular n-polygon, we need to construct a circle of radius R and a **central angle** of the circle of $\frac{360°}{n}$. Thus, for construction of a square we can create a circle with two perpendicular diameters. In order to construct a regular hexagon, we can use the fact that the side equals the radius of the circle and then make the entire construction with compass as shown in Fig. 2.17. Here $OB = OA = R$, $\angle AOB = \frac{360°}{6} = 60°$.
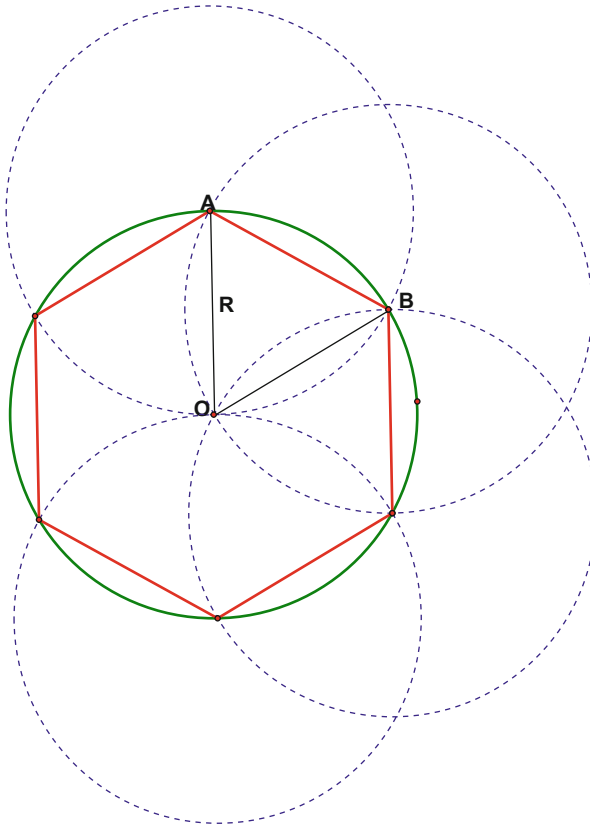
Fig. 2.17 Regular hexagon

Let's consider some important definitions and properties of a regular polygon using the constructed hexagon as an example.

The center of a regular polygon is a point on its interior that is equidistant from its vertices. Moreover, the center of a regular polygon is the common center of its

inscribed and circumscribed circles as shown Fig. 2.18 in blue and green color, respectively.

The **apothem** of a regular polygon is a segment extended from its center perpendicular to one of its sides. The apothem is also the radius of an inscribed circle. It bisects the sides to which it is drawn. Thus, $OC = r$ is the apothem of the hexagon.

Segments $OA = OB = R$, the radii of the circumscribed circle.
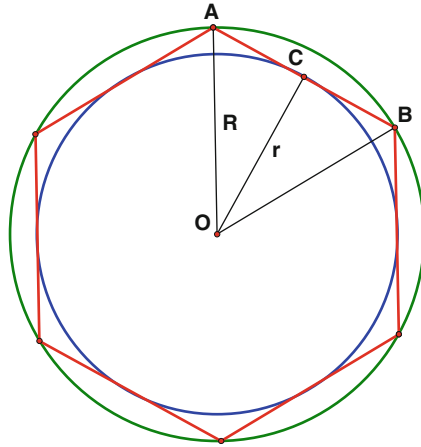
Fig. 2.18 Inscribed and circumscribed regular polygon

The central angle for a regular polygon is an angle between two radii drawn to two consecutive vertices. In our hexagon, the central angle is AOB. There are precisely n central angles, in each regular polygon, then its measure is $\frac{360°}{n}$ or $\frac{2\pi}{n}$, and for a hexagon it is $60°$. In Fig. 2.18 we can see that for any regular polygon, a triangle AOB is isosceles with AC as the height, the median, and the bisector dropped from O to the base AB. Therefore, the side of a regular polygon, $a = AB$, can be found from the right triangle AOC with angle $\angle AOC = \frac{\angle AOB}{2} = \frac{2\pi}{2n} = \frac{\pi}{n}$ and written in terms of either the circumscribed radius, $R = AO$ (hypotenuse) or the inscribed radius, $r = OC$ (leg) as follows

$$a = 2R \cdot \sin\left(\frac{\pi}{n}\right) = 2r \cdot \tan\left(\frac{\pi}{n}\right) \qquad (2.22)$$

Since we can divide any arc into two equal parts (by dividing into two parts the corresponding chord), then it is easy to construct a regular 8-gon, 16-gon or 12-gon, etc.

Therefore, construction of regular polygons with n sides with the use of compass and ruler is closely deals with the construction of angles with given measure because such construction can be considered as construction of the central angle of a circle with radius R with the degree measure $\frac{360°}{n}$. For example, as you have

seen, if we able to construct angle of $60°$ then we can easily construct a regular hexagon, and if we have an angle of $36°$ then we can construct a regular 10-gon, and if we can construct an angle of $72°$ then we can construct a regular pentagon, etc. Problems on construction of such angles and polygons are given in the corresponding section of the book.

This way we can construct regular n-gons with the following sides: $n = m \cdot 2^{k-1}$ and $m = 3, 4, 5$; $k \in N$.

Not all regular polygons can be constructed with compass and straightedge. For example, the heptagon ($n = 7$) and nonagon ($n = 9$) are impossible to construct by compass and straightedge. However, a 17th-gon can be constructed! Interestingly the solution of this problem is beyond the scope of a course in elementary geometry and deals with number theory and the properties of algebraic equations and expression of their solutions in terms of square roots. **Karl Friedrich Gauss** solved this problem at the age of 17. He proved that it is possible to construct a regular n-gon if and only if $n = 2^k p_1 p_2 p_3 \dots p_m$, $k \in N$, where $p_i = 2^{2^s} + 1, s = 0$ or $s \in N$, the relatively prime Fermat numbers. For example, the first Fermat numbers are 3, 5, 17, 257, etc. It is known that solving this problem motivated Gauss to become a mathematician and that he even asked to be buried under a stone in the shape of a regular 17th gon.

2.5.2 Area and Perimeter of Regular Polygons

Since the perimeter of any polygon is the sum of the lengths of the sides, the perimeter of a regular n-gon is n times the length of one.

The area of a regular polygon can be found as one half the product of the perimeter and the apothem of the polygon.
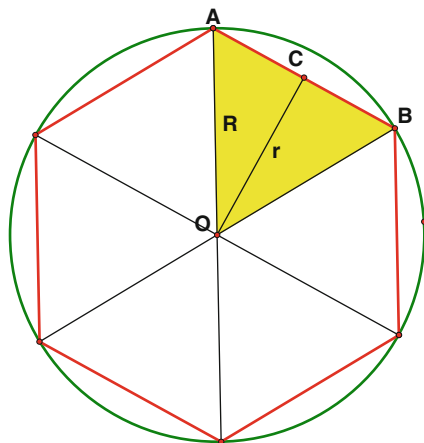
Fig. 2.19 Finding area of a regular polygon

The proof of this statement is simple. If we connect the center of a regular polygon with each vertex as shown in Fig. 2.19, we divide the polygon into n-pieces: n equal isosceles triangles with a vertex angle as the central angle. The area of each triangle (for example $[AOB]$ in the figure above) is equal to half the product of the height (apothem) OC and the base (side of the polygon) AB. Thus,

$$\text{Area of } n\text{-gon} = n \cdot \frac{1}{2} OC \cdot a = \frac{1}{2} h \cdot P \text{ (perimeter) where } P = n \cdot a$$

Problem 48 will help you to practice in several concepts.

Problem 48 (Lidsky). Given a regular n-gon with side a. There are n circles inscribed into the polygon so that each circle is tangent to two adjacent sides of the polygon and two other circles. What is the area of "star" formed in the center of the polygon?

Solution. In order to understand the problem, let us use Fig. 2.20 which shows four circles inscribed into a regular quadrilateral (square) of side length a. (There will be n circles inscribed this way into a regular n-gon). If we connect the centers of the circles we will obtain a polygon similar to that given, with the center at the center of the original polygon and with the sides parallel to the original polygon.

Let r be the radii of the circles, then the side of the small polygon (shown in dashed line) will be $2r$ and its area will be 4 times the area of the triangle with the central angle $GEO = \frac{2\pi}{n} = \frac{2\pi}{4} = \frac{\pi}{2}$. The area of GEO can be found as two areas of the right triangle HEO, where angle HEO is half of the central angle, i.e., $\frac{2\pi}{2n} = \frac{\pi}{n} = \frac{\pi}{4}$. Thus, the area of a green star is $S_* = (2r)^2 - 4 \cdot \frac{\pi r^2}{4} = r^2(4 - \pi)$. In general, for a regular n-gon, the area of the right triangle is half the product of its legs: $A_{HEO} = \frac{1}{2} r \cdot r \cdot \cot\left(\frac{\pi}{n}\right) = \frac{1}{2} r^2 \cot\left(\frac{\pi}{n}\right)$, then the area of triangle

$$[GEO] = 2[HEO] = r^2 \cot\left(\frac{\pi}{n}\right)$$
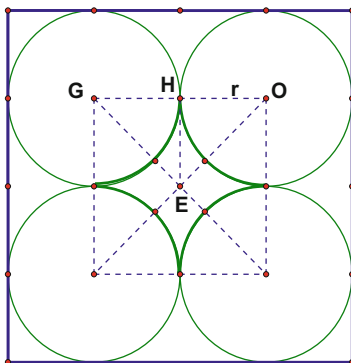
Fig. 2.20 First sketch for Problem 48

If there are exactly n right triangles to make a "small" regular n-gon, then its area is

$$S = n \cdot r^2 \cot\left(\frac{\pi}{n}\right)$$

$$S = n \cdot 2 \cdot \frac{1}{2} r \cdot r \cot\left(\frac{\pi}{n}\right) = nr^2 \cot\left(\frac{\pi}{n}\right)$$

Let the interior angle of the original polygon be $\angle FGH = \alpha = \frac{\pi(n-2)}{n}$ (it is marked by two arcs on the sketch). Next, in order to evaluate the area of the star in the middle, we need to subtract the area above and the total area of n sectors with a central angle α. Each such sector will have area

$$\frac{\alpha \cdot r^2}{2} = \frac{\pi(n-2)r^2}{2n}$$

So the area of the central "star" will be evaluated as (Fig. 2.21)

$$S_* = n \cdot r^2 \cot\left(\frac{\pi}{n}\right) - \frac{\pi(n-2)r^2}{2n} = r^2\left(n\cot\left(\frac{\pi}{n}\right) - \frac{\pi(n-2)}{2n}\right)$$
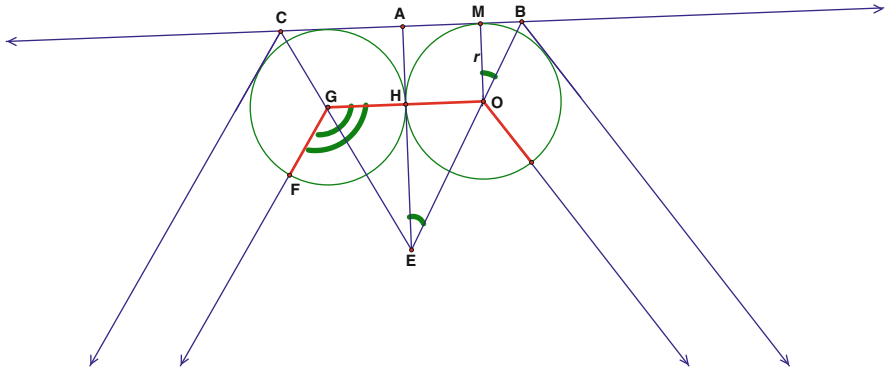
Fig. 2.21 Second sketch for Problem 48

Next, we need to find r.

We know that $\angle MOB = \angle HEO = \frac{\pi}{n}$ (half of the central angle marked by one arc) and $AB = \frac{a}{2}$. We can see that $r = HO = AB - MB$, where $AB = a/2$. Expressing the leg MB of the right triangle MOB in terms of tangent of the opposite angle we obtain

$$\frac{a}{2} - r = r \tan \frac{\pi}{n}$$

Solving this formula for r gives us

$$r = \frac{a}{2\left(1 + \tan \frac{\pi}{n}\right)}$$

Hence, the area of the "star" is $S_* = \dfrac{a^2 \left(n \cot \frac{\pi}{n} - (n-2)\frac{\pi}{2}\right)}{4\left(1 + \tan \frac{\pi}{n}\right)^2}$.

2.6 Homework on Polygons

1. A side of an equilateral triangle is 10 cm. The midpoints of its sides are joined to form an inscribed equilateral triangle and the process is continued. Find the sum of the perimeters of the triangles if the process is continued without end.

   Hint. See Problem 40.

   Answer. 80 cm.

2. Consider a rhombus $ABCD$ where angle $A = 60°$. Point N divides side AB in the ratio $|AN|:|BN| = 2:1$. Find $\tan(\angle DNC)$.

   Answer. $\dfrac{9\sqrt{3}}{11}$.

3. A trapezoid with bases 3 and 4 is given such that its diagonal has length 6 and bisects one of the trapezoid's angles. Can such a trapezoid be an isosceles?

   Answer. No.

4. Two opposite angles of a quadrilateral are right angles. The length of the diagonals passing through the two angles equals d. Find the range of the area of the quadrilateral.

   Answer. $\left(0, \dfrac{d^2}{2}\right]$.

5. Find the point inside a quadrilateral such that the sum of distances from the point to all vertices of a quadrilateral is minimal.

   Answer. It is the point of intersection of the diagonals.

6. A square with sides, a, is rotated about its center in $45°$. Find the area of the common part of the "old" and "new" squares.

   Answer. $2(\sqrt{2} - 1)a^2$.

7. Two diagonals of a trapezoid divide its area so that the areas of two triangles adjacent to the base equals S_1 and S_2 respectively. Find the area of the trapezoid.

   Answer. $\left(\sqrt{S_1} + \sqrt{S_2}\right)^2$.

8. The diagonals of a quadrilateral form four triangles. The areas of three of them are S_1, S_2, and S_3, respectively. Find the area of the fourth triangle.

   Answer. $\dfrac{S_1 \cdot S_2}{S_3}$.

   Solution. Let us draw quadrilateral $ABCD$ such that $AC \cap BD = O$ (Fig. 2.22). We denote five variables: $BO = a$, $OD = b$, $AO = c$, $OC = d$, and $m\angle BOC = \phi$.
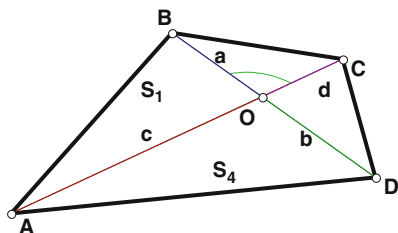
Fig. 2.22 Quadrilateral as a union of four triangles

Now we can write the area of each of four triangles as

$$\frac{a \cdot c \cdot \sin(180° - \phi)}{2} = S_1$$

$$\frac{ad \cdot \sin \phi}{2} = S_2$$

$$\frac{db \cdot \sin \phi}{2} = S_3$$

$$\frac{cb \cdot \sin(180° - \phi)}{2} = S_4$$

Using the fact that $\sin(180° - x) = \sin x$ and dividing S_1 by S_2 and S_4 by S_3 we obtain

$$\frac{c}{d} = \frac{S_1}{S_2}$$

$$\frac{c}{d} = \frac{S_4}{S_3}$$

This yields $S_4 = \dfrac{S_1 \cdot S_2}{S_3}$.

9. Let the lengths of sides AB, BC, CD, DA of a quadrilateral $ABCD$ be a, b, c, and d, respectively and let S denote the area of $ABCD$. Show that $S \le \frac{ad+bc}{2}$ and equality holds iff angles A and C are right angles.

   Hint. Connect points B and D. The area of $ABC = S = [ABD] + [CDB]$. Using formula (1.27) for the area of each triangle we obtain that $[ABD] = \frac{a \cdot d \sin(\angle BAD)}{2}$ $\le \frac{ad}{2}$ (because sine is bounded by 1). Similarly, for the area of triangle CBD we have $[CBD] = \frac{b \cdot c \cdot \sin(\angle BCD)}{2} \le \frac{bc}{2}$. Adding both areas, the required relationship will be obtained.

10. Using Problem 9 show that $S \le \frac{(a+c)(b+d)}{4}$ with equality iff $ABCD$ is a rectangle.

   Hint. Since $S \le \frac{ad+bc}{2}$ is true, then $S \le \frac{ab+dc}{2}$ is also true (we can connect two other vertices, A and C) then adding the two inequalities we obtain $2S \le \frac{ad+bc+ab+dc}{2} = \frac{(a+c)(b+d)}{2}$.

11. According to Euler, in order to inscribe a regular pentagon in a circle (Fig. 2.23), you can draw a circle with center O and construct two perpendicular diameters through its center. Let OA be one of the diameters and M be the midpoint of the other diameter. Connect M with A and bisect the angle AMO. Next place point N on OA and draw segment NP perpendicular to OA. Denote the point of intersection with the circle by P. Using a compass with the radius equal to the length of segment AP, you will find all vertices of the regular pentagon. Prove that AP is the side of the regular pentagon.
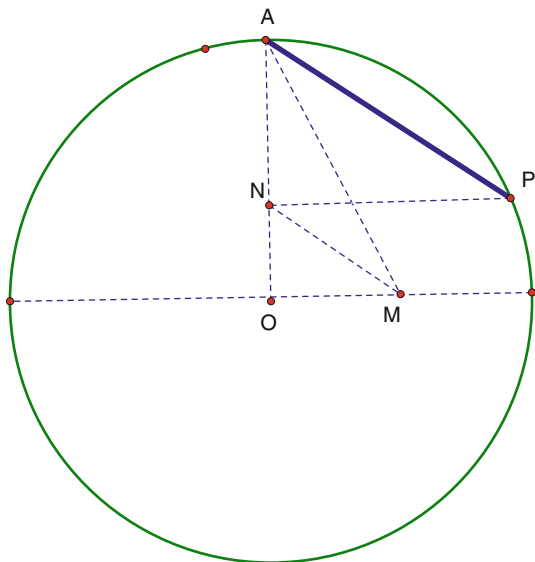
Fig. 2.23 Euler's method of constructing a regular pentagon

Proof. Denote $AO = OP = R$, $AN = x$, $AP = a$. Then $ON = R - x$, and from the right triangle AOM, we can find $AM = \frac{R\sqrt{5}}{2}$. Further, if MN is the bisector of angle AMO, then by the Angle Bisector Theorem applied to triangle AMO, we obtain the following ratios:

$$\frac{AM}{AN} = \frac{OM}{NO}$$

$$\frac{R\sqrt{5}}{2x} = \frac{R}{2(R - x)}$$

$$x = \frac{R\sqrt{5}}{1 + \sqrt{5}}$$

Next, applying the Pythagorean Theorem to triangles ANP and PNO, we obtain the following:

$$AP^2 = AN^2 + NP^2 = AN^2 + OP^2 - ON^2$$

$$AP^2 = x^2 + R^2 - (R - x)^2$$

$$AP^2 = 2Rx$$

Finally, substituting x from the previous formula and replacing AP by a, we obtain

$$a^2 = \frac{2R^2\sqrt{5}}{1 + \sqrt{5}}$$

If we rationalize the denominator and take the square root from both sides, then we obtain the required formula for the side of a regular pentagon inscribed into a circle with radius R:

$$a = R\sqrt{\frac{5 - \sqrt{5}}{2}}$$

You can check it yourself. By formula (2.22) the side of a regular pentagon can be written as

$$a = 2R\sin 36° = 4R\sin 18 \ \cos 18° = R\sqrt{\frac{5 - \sqrt{5}}{2}}$$

The proof is completed. (See more about how to find the sine of $18°$ in Chap. 4 of the book.)

12. In this chapter we proved that if for any quadrilateral, the consecutive midpoints of all sides are connected, then they form a parallelogram. Can this parallelogram be a rhombus? If yes, then under what condition? Under what condition can the parallelogram be a square?

    Answer. For a quadrilateral $ABCD$, assume that K, F, M, and N are midpoints of sides AB, BC, CD, and AD, respectively. Since KF is the midsegment of triangle ABC and MN is the midsegment of triangle ADC, we have that $KF\|$ MN and $KF = MN$ and that $KFMN$ is a parallelogram. If the diagonals of $ABCD$ are equal ($AC = BD$) then the parallelogram $KFMN$ is a rhombus. If the diagonals are equal and perpendicular to each other, then $KFMN$ is a square.

13. Consider a convex quadrilateral $ABCD$ with diagonals AC and BD in which the heights AE and DF are dropped to sides CD and AB, respectively. It is known that $AE \geq BD, DF \geq AC, AD = 2AB$. Find all angles of $ABCD$.

    Solution. Consider a quadrilateral $ABCD$. Because $AD = 2AB$, triangle ABD is a right triangle inscribed into a circle of radius $R = AB$ and diameter AD. Let O be the midpoint of AD. Obviously the height DF dropped to the side AB is actually the diagonal, BD.

    The inequality of the problem implies that $AE \geq DF = BF \geq AC$ from which we obtain that $AE \geq AC$. However, if AE is perpendicular to side CD then the triangle AEC is the right triangle in which AC is the hypotenuse. Since leg AC cannot be greater than the hypotenuse, we conclude that AC is

perpendicular to CD and $AE = AC$. Therefore, the only quadrilateral satisfying the given conditions is an isosceles trapezoid inscribed into a circle such that its base AD is also the diameter of the circle. Moreover, $\angle A = \angle C = 60°$ and $\angle B = \angle D = 120°$.

14. Given a convex quadrilateral $ABCD$ such that $AB + BD \le AC + CD$. Compare the length of segments AB and AC.

    Answer. $AB < AC$.

    Solution. Let us use Fig. 2.22 to solve this problem. By the triangle existence inequality for triangles ABO and COD we can write the following:

    $$\left. \begin{array}{l} AB < AO + BO \\ CD < CO + DO \end{array} \right\} \Leftrightarrow AB + CD < AO + CO + BO + DO = AC + BD$$

    On the other hand, by the condition of the problem we know that $AB + BD \le AC + CD$.

    Let us add both inequalities together and collect like terms. We will obtain

    $$AB + CD + AB + BD < AC + BD + AC + CD$$
    $$2AB < 2AC$$
    $$AB < AC$$

15. Prove that if each diagonal of a convex quadrilateral divides it into equal triangles then such a quadrilateral is a parallelogram.

    Proof. Using again Fig. 2.22 as our basic figure, let us make some additional constructions in it by dropping a perpendicular from B to the diagonal AC (height BH) and a perpendicular from D to AC (height DK). Since each diagonal of the quadrilateral divides it into two equal triangles then we can write that

    $$[ABC] = [ACD]$$
    $$\frac{1}{2}BH \cdot AC = \frac{1}{2}DK \cdot AC$$
    $$BH = DK$$

    Since a side of the triangles BHO and DKO is equal as well as the angles adjacent to them, we have that $\triangle BHO \cong \triangle DKO$. All sides of congruent triangles are equal, so O is the midpoint of the diagonal BD.

    By an analogues argument, we can show that O is the midpoint of AC. Therefore, $ABCD$ must be a parallelogram.

16. Find an interior point of a convex quadrilateral such that the sum of the distances from the point to all of the vertices is minimal.

    Hint. Think about the triangle existence inequality.

    Solution. Consider a quadrilateral $ABCD$ with diagonals that intersect at point O. The distance from O to all vertices equals

    $AO + BO + CO + DO = AC + BD$ (the sum of the diagonals) (Fig. 2.24)
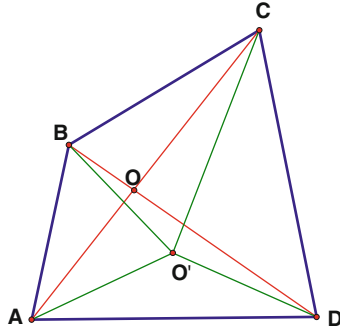
Fig. 2.24 Problem on an interior point of a quadrilateral

Let O' be another point inside the quadrilateral. Using the triangle existence inequality we have

$$AO_1 + O_1C \geq AC, \quad BO_1 + O_1D \geq BD$$

Hence, $AO_1 + O_1C + BO_1 + O_1D \geq AC + BD$.

Therefore, the point that minimizes the sum of the distances is the point of intersection of diagonals.

17. Prove that the area of any quadrilateral can be evaluated with the use of its diagonals and the sine of the angle between them as $S = \frac{1}{2} \cdot d_1 d_2 \sin \varphi$.

    Proof. Consider a quadrilateral with diagonals $AC \cap BD = O$. Let $AO = z$, $OC = t, BO = x, OD = y, \angle AOD = \varphi$. We have that $[ABCD] = [AOD] + [AOB] + [BOC] + [COD]$. Applying formula (1.27) to each triangle and using the fact that angles $\angle BOC = \angle AOD = \varphi$ are vertical and $\angle AOB = \pi - \varphi$, we finally complete the proof.

18. Consider a convex quadrilateral $ABCD$ such that the area of triangle ABC is half the area of triangle ACD. Adjacent sides AB and BC are equal and side AD is twice as long as side AB. Find the angle ADC of angle $BAC = 30°$.

    Solution. Let h_a, h_c be the length of the perpendiculars dropped to sides BC from vertex A and to AD from vertex C, respectively. Let $AB = BC = a$, then $AD = 2a$ (by the condition of the problem).

From the relationship between the areas of triangle ABC and ACD we have the following:

$$2[ABC] = [ACD] \text{ or}$$

$$2\frac{1}{2}ah_a = \frac{1}{2}2ah_c$$

after cancelation of common factors we have $h_a = h_c$ and then the sides of the quadrilateral BC and AD must be parallel (Fig. 2.25).
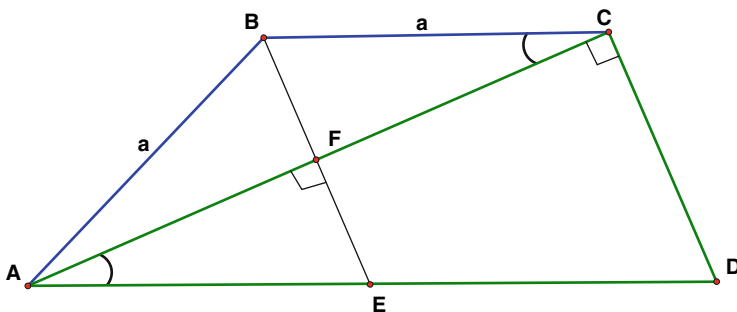
Fig. 2.25 Finding an angle of a quadrilateral

Therefore, our quadrilateral is a *trapezoid*. Moreover, $\angle CAD = \angle ACB = 30°$ (since triangle ABC is an isosceles, i.e., $AB = BC$).

Let us draw line BE parallel to CD so that $ED = BC = a$. Then it is true that $AE = 2a - a = a = AB$. Also note that triangle ABE is an isosceles and its bisector (AF) is its height so that $AC \perp BE \Rightarrow \angle ACD = 90°$. Finally, from triangle ACD we find that $\angle ADC = 60°$.

Answer. $\angle ADC = 60°$.

19. A convex pentagon $ABCDE$ is inscribed into a circle such that the segment AD contains the center of the circle with diameter d. The sides of the pentagon are $AB = a, BC = b, CD = c, DE = f, AE = e$. Evaluate the area of the pentagon.

    Solution. If AD contains the center of the circle, then AD is its diameter and $AD = d$. Moreover, $ABCD$ is a cyclic quadrilateral and triangle AED is the right triangle. The area of $ABCDE$ can be found as the sum of the areas of the quadrilateral and the triangle. The area of the quadrilateral can be found with the use of Brahmagupta's Formula for the area of a cyclic quadrilateral. Denoting p as the semiperimeter,

$$[ABCDE] = [ABCD] + [AED]$$

$$[ABCDE] = \sqrt{(p-a)(p-b)(p-c)(p-d)} + \frac{ef}{2}$$

$$[ABCDE] = \frac{\sqrt{(b+c+d-a)(a+c+d-b)(a+b+d-c)(a+b+c-d)} + ef}{2}$$

20. Given three regular polygons. Two of them, with perimeters a and b, respectively, are circumscribed about the circle and the third one is inscribed into the circle. It is known that the second and third polygon each has twice as many sides as the first polygon. Find the perimeter of the third regular polygon.

Answer. $P = b\sqrt{\dfrac{a}{2a - b}}$.

Solution. If a polygon is circumscribed about a circle, then the circle is inscribed into the polygon. Let R be the radius of the inscribed circle and assume that the first polygon has n sides and that the second and third polygons have $2n$ sides. Using Fig. 2.18 and formula (2.22) we can express the sides of the polygons in terms of the inradius, R. Then for the perimeters of the first and second polygons with n and $2n$ sides, respectively, we obtain

$$a = 2nR \tan \frac{\pi}{n} = 4nR \frac{\tan \frac{\pi}{2n}}{1 - \tan^2 \frac{\pi}{2n}}$$

$$b = 4nR \tan \frac{\pi}{2n}$$

In order to obtain the second part of the first expression above, we used the formula for the tangent of double angle.

Assume that the third polygon has perimeter P. Suppose that the side of the inscribed polygon is AB as it is shown in the Fig. 2.19. Since the third polygon has $2n$ sides, then its central angle is $\frac{2\pi}{2n} = \frac{\pi}{n}$. In order to express the side of the polygon in terms of the circumradius of the circle R, we need to use the relation between the leg AC and the hypotenuse, $AO = R$ in the right triangle AOC. Note that the angle AOC is half the central angle, $\left(\angle AOC = \frac{\pi}{2n}\right)$. Applying trigonometric identity to the sine function, the perimeter of the third polygon can be written as

$$P = 4nR \sin \frac{\pi}{2n} = 4nR \frac{\tan \frac{\pi}{2n}}{\sqrt{1 + \tan^2 \frac{\pi}{2n}}}$$

From the first two formulas we can see that

$$1 - \tan^2 \frac{\pi}{2n} = \frac{b}{a}$$

Therefore, $1 + \tan^2 \frac{\pi}{2n} = \frac{2a-b}{a}$.

Substituting this into formula for P, we finally obtain

$$P = b\sqrt{\frac{a}{2a - b}}$$

Chapter 3
Problems Involving Circles

3.1 Circles and Their Properties

A **circle** is the set of all points in a plane that are equidistant from a given point in the plane known as the **center** of the circle.

A **radius** is a segment from the center of the circle to a point on the circle.

A **chord** is a segment whose endpoints line on a circle.

A **diameter** is a chord that contains the center of a circle.

A **secant** to a circle is a line that intersects the circle at two points.

A **tangent** is a line in the plane of the circle that intersects the circle at exactly one point, which is known as the point of tangency.

An **arc** is an unbroken part of a circle. Any two points on a circle divide the circle into two arcs.

Let us list most important facts we have to remember in order to successfully solve problems involving circles.

A **central angle of a circle** is an angle of a circle whose vertex is the center of the circle. An arc whose endpoints lie on the sides of the angle and whose other parts lay in the interior of the angle is the **intercepted arc** (**supported arc**) of the central angle.

An **inscribed angle** is an angle whose vertex lies on a circle and whose sides are chords of the circle. Knowing the measure of the inscribed angle we can determine the measure of its intercepted arc.

3.1.1   The Inscribed Angle of a Circle

The Inscribed Angle Theorem.   The measure of an inscribed angle in a circle is equal to half the measure of the intercepted arc.

   In Fig. 3.1, $\angle ABC$ is inscribed in circle with center O, $\angle ABC$ intercepts arc AC, $\angle AOC$ is the central angle, $\angle AOC$ intercepts arc AC, and $m\angle AOC = 2m\angle ABC$.
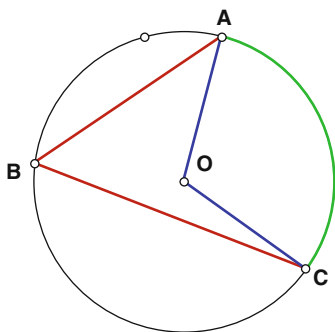
Fig. 3.1   Inscribed and central angles

Proof. Let us consider three possible cases associated with the corresponding picture.

Case 1. One side of the inscribed angle goes through the diameter of the circle (Fig. 3.2).
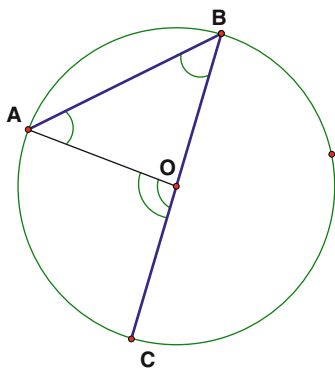
Fig. 3.2   An Inscribed angle, Case 1

Let angle $ABC = \alpha$, so that the isosceles triangle AOB has angle $BAO = \alpha$. The central angle AOC is an exterior angle of triangle AOB so $\angle AOC = 2\alpha \Rightarrow \alpha = \frac{1}{2} \angle AOC$.

Case 2. The center of the circle is inside the interior of the inscribed angle ABC (Fig. 3.3).
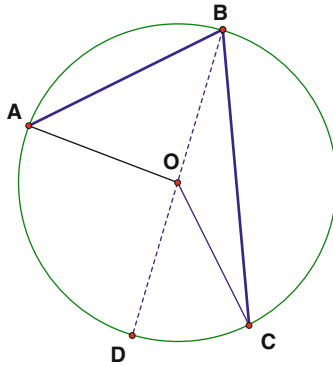
Fig. 3.3 An Inscribed angle, Case 2

In this case, we will need an auxiliary construction: we will draw segment BD through the vertex of the angle and the center of the circle. Now we can use what we proved in Case 1.

$$\angle ABC = \angle ABD + \angle BDC$$

$$\angle ABC = \frac{1}{2}\angle AOD + \frac{1}{2}\angle COD = \frac{1}{2}(\angle AOD + \angle COD) = \frac{1}{2}\angle AOC$$

Case 3. The interior of the inscribed angle does not include center O as shown in Fig. 3.4.
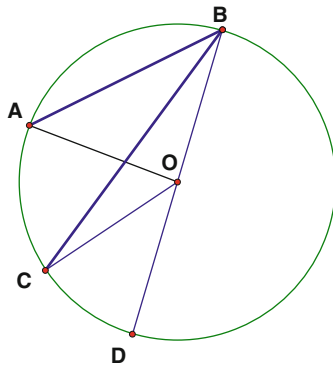
Fig. 3.4 An Inscribed angle, Case 3

Now, the inscribed angle *ABC* is the algebraic difference of angles *ABD* and *CBD*. Please finish this proof by analogy with the other cases.

3.1.1.1 Angle Inscribed in a Semicircle Theorem

This is another theorem that was stated and proved by Thales (Fig. 3.5).

Thales' Semicircle Inscribed Angle Theorem. Any angle inscribed in a semicircle is a right angle.
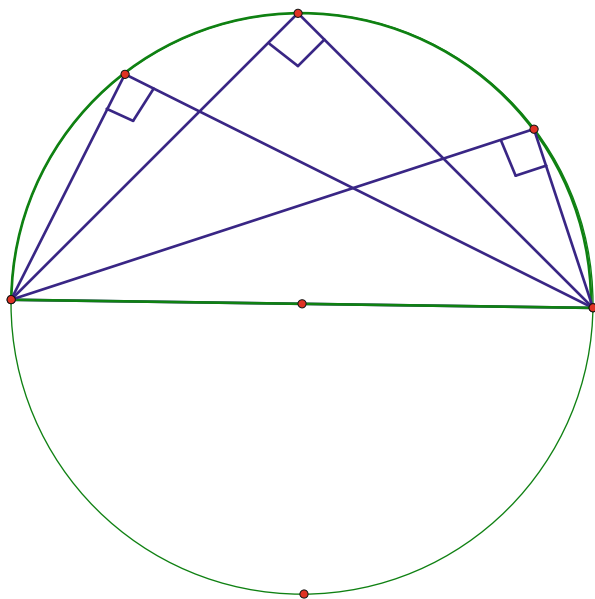
Fig. 3.5 Thales' semicircle inscribed angle theorem

Proof. Since a semicircle's measure is 180°, the validity of Thales' Theorem follows from the theorem on the inscribed angle proven above. If you want to have another proof, then the following is an option.

Let the diameter of a circle with its center at *O* be *AB*. Let *C* be another point on the circle. Since *A*, *B*, and *C* are on the circle, $AO = OC = OB$ (as radii) and the triangles *AOC* and *COB* are isosceles (Fig. 3.6).
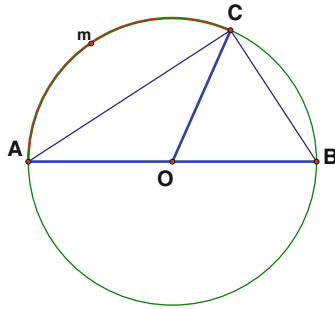
Fig. 3.6 Illustration of Thales's Theorem

Let $\angle CAO = \angle ACO = \alpha$ and $\angle BCO = \angle CBO = \beta$. Next, we have the relation for the inscribed and central angles supporting arc AMC as $2\angle ABC = \angle AOC$.

On the other hand, using the triangle angle theorem for AOC, we can rewrite $2\angle ABC = \angle AOC$ as

$$2\beta = \angle AOC = 180° - 2\alpha$$
$$2\beta + 2\alpha = 180°$$
$$\alpha + \beta = 90°$$

Therefore, triangle ABC is the right triangle and angle C is a 90° angle. Our proof is complete.

3.1.1.2 Arc-Intercept Corollary

Arc-Intercept Corollary. If two inscribed angles intercept the same arc, then they have the same measure.

As seen in Fig. 3.7, $m\angle ABC = m\angle ADC$.
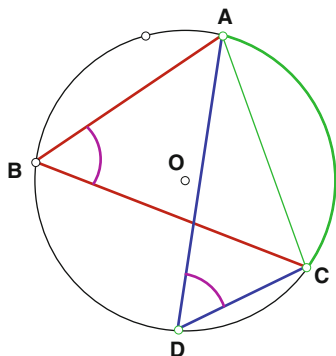
Fig. 3.7 Arc-Intercept corollary

3.1.2 Chords of a Circle

3.1.2.1 Intersecting Chords

The Intersecting Chords Theorem. The product of the lengths of the segments of each of two intersected chords is the same (Fig. 3.8).
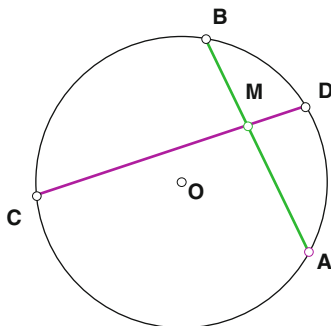
Fig. 3.8 Two intersecting chords

Proof. Let us make an auxiliary construction and connect points *C–B* and *B–A* as it is shown in Fig. 3.9.
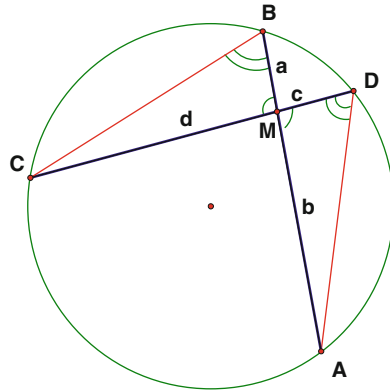
Fig. 3.9 Auxiliary construction for the theorem on two intersecting chords

Otherwise, if two chords of a circle meet, cutting each other in segments of lengths $a = BM$, $b = MA$, $c = MD$, $d = MC$, then $ab = cd$. If AB and CD are two intersected chords ($AB \cap CD = M$) of a circle with center O, then angles BMC and DMA are equal because they are **vertical angles**. Angles CBA and ADA are equal because they support the same arc CA. Thus, triangles CMB and DMA are similar and $CM/MA = BM/MD$; $BM \cdot MA = CM \cdot MD$.

3.1.2.2 The Angle Between Two Intersecting Chords

Angle Between Two Intersecting Chords Theorem. Any vertical angle between two intersecting chords equals half the sum of the angular measure of the arcs intercepted by it.
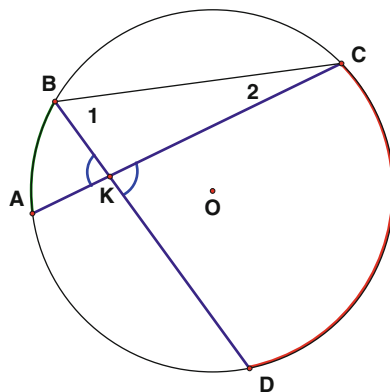
Fig. 3.10 Angle between two intersecting chords

As seen in Fig. 3.10 $\angle AKB = \frac{1}{2}(\cup AB + \cup CD) = \angle ACB + \angle CBD$.

Proof. Let two cords, AC and BC, intersect at point K inside the circle. Then we want to evaluate angle AKB. We will connect points B and C and make a triangle BKC for which angle AKB is exterior angle and

$$\angle AKB = \angle KBC + \angle ACB = \angle 1 + \angle 2$$

On the other hand, by the property of inscribed angles we can write that

$$\angle KBC = \frac{1}{2}\cup CD, \quad \angle ACB = \frac{1}{2}\cup AB$$

Therefore, $\angle AKB = \frac{1}{2}\cup CD + \frac{1}{2}\cup AB = \frac{1}{2}(\cup CD + \cup AB)$.

Remark. If two chords are perpendicular, then obviously the sum of the angle measured by the supporting arcs is 180°.

3.1.3   Tangents and Secants of a Circle

3.1.3.1   Tangents from the Same Point

> **Tangents from the Same Point Theorem.** Segments of tangents to a circle drawn from a point exterior to the circle have the same length. These tangents form congruent angles with the line passing through the point and the center of the circle (Fig. 3.11).
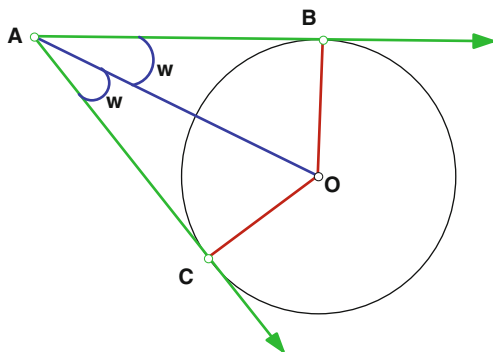
Fig. 3.11 Tangent from the same exterior point

By this theorem, B and C are tangent points ($AB \perp OB$ and $AC \perp OC$), then $AB = AC$ and $m\angle CAO = m\angle BAO = w$.

3.1.3.2   Secant–Tangent Segments

Secant–Tangent Segment Theorem. If a secant and a tangent are drawn to a circle from an exterior point, then the square of the length of the tangent segment equals the product of the entire length of the secant and the length of the exterior segment of that secant (Fig. 3.12).
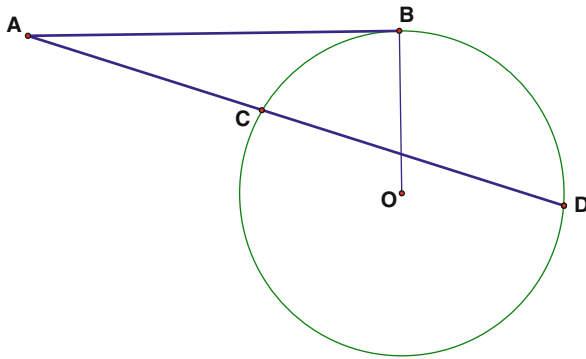
Fig. 3.12 Secant-tangent segments

On the picture above, $AB^2 = AD \cdot AC$ (i.e., tangent squared = "whole" times "outside").

Proof. Let us connect C–B and B–D, and C–O as shown in Fig. 3.13. This auxiliary construction will allow us to demonstrate that the angles ABC and ADB are equal. In fact, $\angle CDB = \alpha = \frac{1}{2}\angle COB$ as the inscribed and central angles support the same arc, CB. If the isosceles triangle COB has vertex angle of 2α, then the base angles are

$$\frac{180° - 2\alpha}{2} = 90° - \alpha = \angle CBO$$

AB is tangent so angle ABO is a right angle and ABC and CBO are complimentary angles, i.e., $\angle ABC = \alpha = \angle BDA$. Since the angle at A is common to triangles ABC and ADB, the triangles are similar (AA). Hence the corresponding sides are proportional,
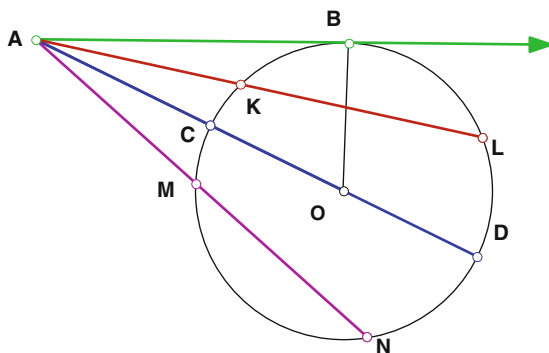
Proof. This corollary is illustrated in Fig. 3.14.

From point A we drew a tangent AB and three secants: AL, AD, and AN. The theorem states that $AL \cdot AK = AD \cdot AC = AN \cdot AM$. This can be easily proven using the previous theorem. All shown secants relate to the same tangent by

$$AB^2 = AK \cdot AL$$
$$AB^2 = AC \cdot AD$$
$$AB^2 = AM \cdot AN$$

The left sides are the same, so the right sides must be equal.

3.1.3.3   The Angle Between Two Secants

Theorem of the Angle Between Two Secants. Two secants drawn from a point outside the circle form an angle which measures one-half the positive difference of the measures of the intercepted arcs.
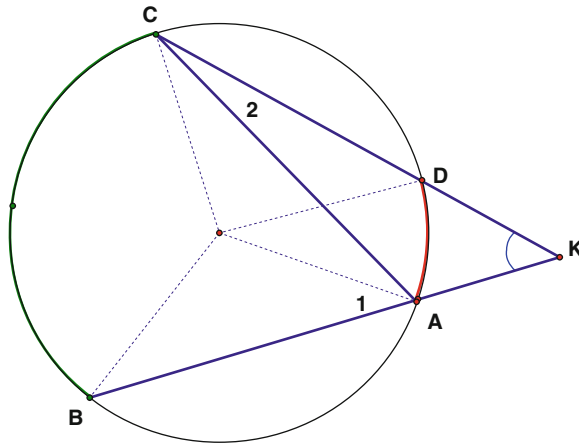
Fig. 3.15 The angle between two secants (must be replaced)

Proof. Let us draw a picture illustrating this theorem (see Fig. 3.15). Regarding our sketch we have to prove that

$$\angle AKC = \frac{1}{2}(\cup BC - \cup AD) \tag{3.1}$$

where $\cup BC$ is the angular measure of arc BC.

It follows from the inscribed angle theorem that (3.1) is true if

$$\angle AKC = \angle BAC - \angle DCA$$

Angle BAC is an exterior angle of triangle CKA so its measure equals the sum of the measures of angles AKC and ACK. Now we have

$$\angle BAC = \frac{1}{2}\cup BC, \quad \angle AKC + \angle ACD = \angle BAC, \quad \angle ACD = \frac{1}{2}\cup AD$$

Subtracting $\angle ACD$ from $\angle BAC$ and factoring out ½ we complete the proof.

3.1.3.4   Two Intersecting Tangents

Theorem of Two Intersecting Tangents. If two tangents are drawn to a circle from the same point outside the circle, then the measure of the angle formed is half the measure of the larger intercepted arc minus the measure of the smaller intercepted arc.
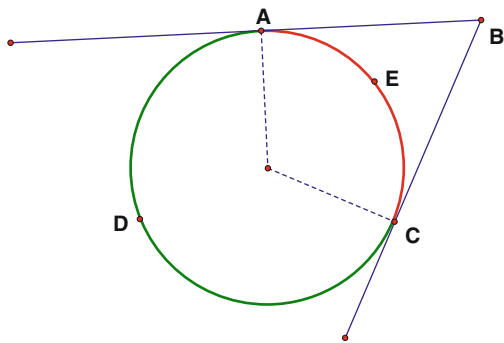
Fig. 3.16   Two intersecting tangents

As can be seen in Fig. 3.16, $\angle ABC = \frac{1}{2}(\cup ADC - \cup AEC)$.

3.1.3.5   Archimedes' Problems

> **Problem 49.** Two chords of a circle AB and CD intersect at point E at a $90°$
> angle. Prove that the sum of the squares of segments AE, BE, CE, and DE equals
> the square of the diameter of the circle.

Proof. We will show you here a solution given by the famous Greek mathemati-
cian **Archimedes** (287–212 BC) in his paper entitled "Lemmas."

Let us draw diameter AF and chords AC, AD, CF, and BD (Fig. 3.17).
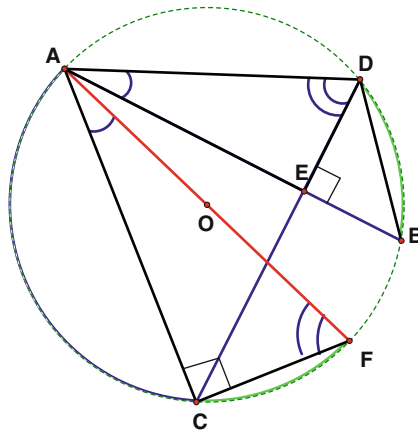
Fig. 3.17   Archimedes problem of two intersecting chords

Since angle $\angle AED = 90°$, it is equal to angle ACF by Thales' Theorem because C
is a point of a circle and AF is the diameter. Also, $\angle ADC = \angle AFC$ since they support
the same arc AC (shown in blue) and so the other angles of the respective right
triangles are equal, i.e., $\angle CFA = \angle DAE$. Hence, arc CF equals arc BD (marked by
green color) and the corresponding chords are also equal, $BD = CF$. We have that

$$DE^2 + BE^2 = CF^2$$
$$AE^2 + CE^2 = AC^2$$
$$AC^2 + CF^2 = AF^2$$

Substituting the values of CF^2 and AC^2 into the last formula, we obtain:

$$DE^2 + BE^2 + AE^2 + CE^2 = AF^2$$

Problem 50 can also be attributed to Archimedes.

Problem 50. (Popov) Let *AB* be a diameter of a circle and let *CD* be a chord. The chord is not perpendicular to the diameter. Prove that the perpendiculars *AE* and *BF* dropped from the end of the diameter to the chord cuts the chord *CD* into equal segments *CF* and *DE*.

Proof. Let us draw diameter *MN*∥*CD* (Fig. 3.18).
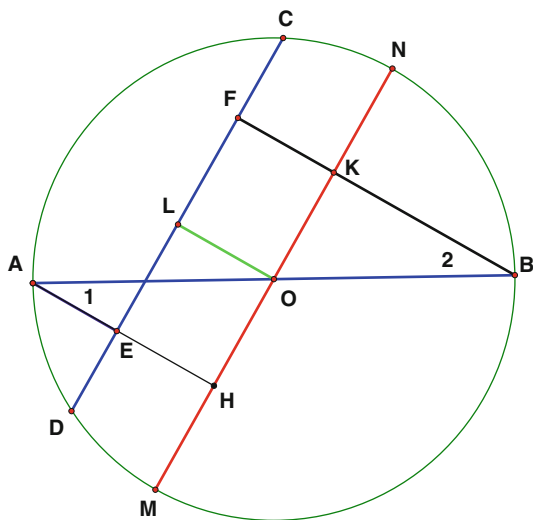
Fig. 3.18 Sketch for Problem 50

It is clear that $\angle 1 = \angle 2$ because *CD* is a transversal and *AE* and *BF* are two parallel lines (two perpendiculars to a line are parallel). Then $\angle AOH = \angle BOK$ $\Rightarrow HO = OK$. Let us drop a perpendicular *OL* from the center of the circle *O* to chord *CD*. Then $EL = HO$ and $LF = OK$. It follows that $EL = LF$ and therefore $DE = CF$.

3.2   Metrics of Circles, Sectors, and Segments

A **circumference** is the perimeter of a circle. For a circle of radius r, its circumference, C, can be evaluated from the radius as $C = 2\pi r$.

The **area of circle** can be evaluated as $A = \pi r^2$.

What if we want to find the length of an arc or the area of a sector? Suppose we have an arc with a central angle measure of θ (e.g., Fig. 3.20), then the length of the arc and area of the sector can be evaluated using **proportions**. (See Appendix A for more information on proportions.)

The property of proportion can be written as:

$$\begin{aligned} a &\to b \\ &\quad\text{then}\quad a \cdot d = c \cdot b \\ c &\to d \end{aligned}$$

Let us apply it to a circle with radius r, in order to find the length, x, of the arc with angle measure θ.

$$\begin{aligned} 2\pi &\to 2\pi r \\ &\quad\text{then}\quad 2\pi x = 2\pi r \theta \\ \theta &\to x \end{aligned}$$

Therefore, the length of the arc equals $x = r \cdot \theta$.
By the same idea we will find the area of a **sector**, y.

$$\begin{aligned} 2\pi &\to \pi r^2 \\ &\quad\text{then}\quad y \cdot 2\pi = \theta \cdot \pi r^2 \\ \theta &\to y \end{aligned}$$

Therefore, the area of a sector with central angle θ is $y = \theta \cdot r^2/2$.
Consider the **segment** of a circle that is shown in red in Fig. 3.19.
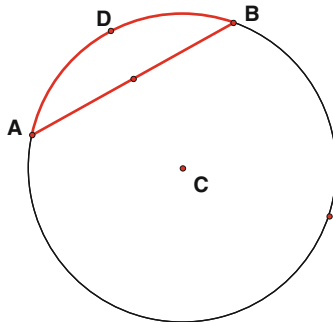
Fig. 3.19   Sector of a circle

How can we evaluate its area? If we connect the ends of the chord with the center of the circle, we can see that the area bounded in red equals the difference between sector $ADBC$ and the area of triangle ABC (see Fig. 3.20). If the central angle $ACB = \theta$ and the radius $AC = CB = r$, then by denoting the area of the segment ADB by Ω, we have

$$\Omega = \frac{\theta \cdot r^2}{2} - \frac{r^2 \sin \theta}{2} = \frac{r^2(\theta - \sin \theta)}{2}$$
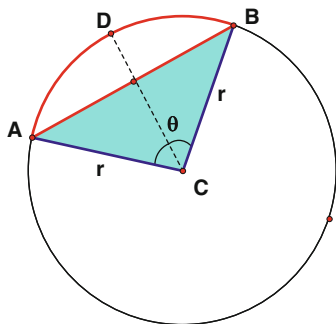
Fig. 3.20   Segment of a circle

Let us solve some problems on the material of this section.

Problem 51.   Each side, a, of a square is a diameter of a circle. Find the area bounded by the arcs of the four semicircle cut a within the interior of the square.

Solution. This problem sounds unusual. Let's sketch the situation as shown in Fig. 3.21. *ABCD* is the given square with side $a = AB$. Chord *AO* is related to similar chords through a 90° angle since the diagonals of a square are perpendicular. The area of one-half of a leaf equals

$$\frac{\pi \cdot a^2}{16} - \frac{a^2}{8} = \frac{a^2(\pi - 2)}{16}$$

Multiplying by 8, we get the area of the entire flower.

Answer.   Area $= \dfrac{a^2(\pi - 2)}{2}$.
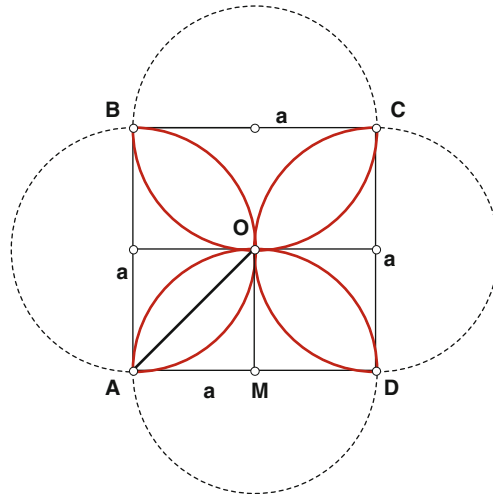
Fig. 3.21 Sketch for Problem 51

Problem 52. A trapezoid $ABCD$ with bases $AD = a$ and $BC = b$ is inscribed in a circle with radius r. The center of the circle is in the interior of the trapezoid. Two segments of the circle that are cut by the bases of the trapezoid are reflected with respect to the corresponding base. Find the area of the region formed by two sides of the trapezoid and the boundaries of the images of the reflected segments.

In Fig. 3.22 we need to find the area of the region within the red boundary.
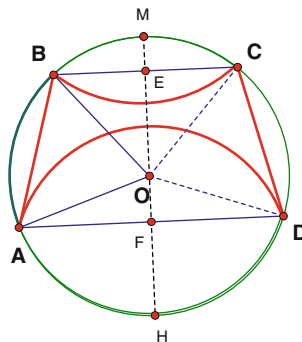
Fig. 3.22 Sketch for Problem 52

Since the trapezoid is inscribed in a circle, it is an isosceles trapezoid and $AB = CD$.

If the center of the circle is inside the trapezoid then the boundary of the segments' images will not overlap. The area of the figure equals the area of the trapezoid minus the area of two segments, *BMCE* and *AHDF*.

Denote $\angle BOC = \alpha$ and $\angle AOD = \beta$. Applying the Law of Cosines to triangles *BOC* and *AOD* we obtain

$$a^2 = 2r^2 - 2r^2 \cdot \cos\alpha = 2r^2(1 - \cos\alpha) = 4r^2 \sin^2\frac{\alpha}{2}$$
$$b^2 = 2r^2 - 2r^2 \cdot \cos\beta = 2r^2(1 - \cos\beta) = 4r^2 \sin^2\frac{\beta}{2}$$

(3.2)

from which we can find angles α and β:

$$\cos\alpha = 1 - \frac{a^2}{2r^2}, \quad \alpha = \arccos\left(1 - \frac{a^2}{2r^2}\right)$$
$$\cos\beta = 1 - \frac{b^2}{2r^2}, \quad \beta = \arccos\left(1 - \frac{b^2}{2r^2}\right)$$

(3.3)

Using formulas (3.3) for the angles, we can find the area within the red boundary. The area of the trapezoid is

$$\begin{aligned}
[ABCD] &= \frac{a+b}{2} \cdot h \\
&= \frac{a+b}{2} \cdot \left(r\cos\frac{\alpha}{2} + r\cos\frac{\beta}{2}\right) \\
&= \frac{r(a+b)\left(\cos\frac{\alpha}{2} + \cos\frac{\beta}{2}\right)}{2}
\end{aligned}$$

(3.4)

By (3.2), the area of segment

$$BMCE = \frac{r^2(\alpha - \sin\alpha)}{2}$$

(3.5)

The area of segment

$$AHDF = \frac{r^2(\beta - \sin\beta)}{2}$$

(3.6)

Subtracting (3.5) and (3.6) from (3.4) we finally have

$$\text{Area} = \frac{r(a+b)\left(\cos\frac{\alpha}{2} + \cos\frac{\beta}{2}\right)}{2} - \frac{r^2(\alpha - \sin\alpha)}{2} - \frac{r^2(\beta - \sin\beta)}{2}$$

Remark. This formula can be improved if we use the double angle formula with (3.2) and the Pythagorean identity to express sine in terms of cosine of the angle of a right triangle. You can try it as algebraic exercise.

3.3   Circles and Convex Polygons

3.3.1   Inscribed Triangle Theorem

Inscribed Triangle Theorem. Any triangle can be inscribed in a circle. The center of the circle will be the point of intersection of perpendicular bisectors of the sides of the triangle.

3.3.2   Circumscribed Triangle Theorem

Circumscribed Triangle Theorem. A circle can be circumscribed about a triangle if and only if its center is the point of intersection of the bisectors of angles of the triangle.

3.3.3   Circumscribed Quadrilateral Theorem

A **cyclic quadrilateral**, *ABCD*, is formed by joining segments between a set of four points, *A*, *B*, *C*, and *D*, arranged on a circle in that order. Segments *AC* and *BD* are the diagonals of the cyclic quadrilateral.

Circumscribed Quadrilateral Theorem. A circle can be circumscribed about a quadrilateral iff the sum of the opposite angles of the quadrilateral equals $180°$ (π).

Proof. First, we construct a circle and randomly put four distinct points on it, A, B, C, and D, in order to obtain a convex quadrilateral. A particular case of this cyclic quadrilateral is shown in Fig. 3.23.

The theorem claims that $m\angle 1 + m\angle 3 = m\angle 2 + m\angle 4 = 180°$. Let

$$\angle 1 = \angle BAD, \quad \angle 2 = \angle ABC, \quad \angle 3 = \angle BCD, \quad \text{and} \quad \angle 4 = \angle CDA$$

Consider angle BAD and an opposite angle BCD. Angle BAD supports arc BCD and angle BCD supports arc BAD, i.e., $\angle BAD = \frac{1}{2} \cup BCD$ and $\angle BCD = \frac{1}{2} \cup BAD$. Adding the left and the right sides of the two equalities we obtain the sum of two opposite angles in terms of the respective arcs. Since the arcs together form the entire circle, then the sum of their measures is $360°$. Half of this measure is $180°$.

This completes our proof.
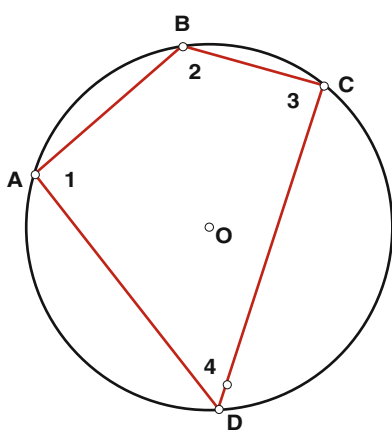
m∠DAB = 93°

m∠BCD = 87°

m∠DAB + m∠BCD = 180°

m∠ABC = 124°

m∠CDA = 56°

m∠ABC + m∠CDA = 180°

Fig. 3.23 Cyclic quadrilateral

3.3.3.1   Ptolemy's Theorem on Cyclic Quadrilaterals

Ptolemy (c. 90–168 AD) was a famous Greek astronomer and philosopher. His theorem on cyclic quadrilaterals is very important to solving complex geometry problems.

> **Ptolemy's Theorem on Cyclic Quadrilaterals.** If a quadrilateral is inscribable in a circle then the product of the measures of its diagonals is equal to the sum of the products of the measures of the pairs of opposite sides.

Moreover, the converse of Ptolemy's Theorem is also true.

Fact. If the sum of the products of its two pairs of opposite sides of a quadrilateral is equal to the product of its diagonals, then the quadrilateral can be inscribed in a circle.
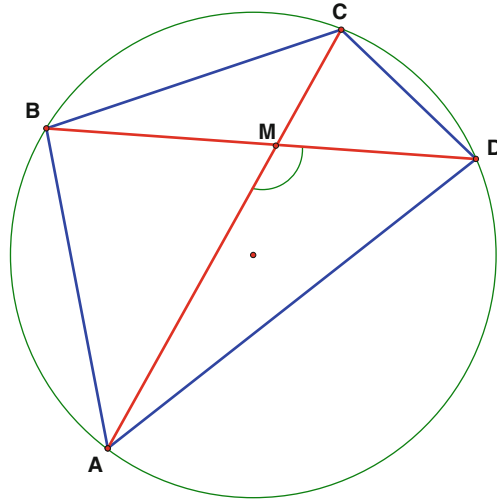
Fig. 3.24 First sketch for the proof of Ptolemy's Theorem

If the quadrilateral is given with vertices A, B, C, and D in that order as shown in Fig. 3.24, then the theorem states that

$$|AC| \cdot |BD| = |AB| \cdot |CD| + |BC| \cdot |AD|$$

Proof. Consider a cyclic quadrilateral $ABCD$ with diagonals AD and BD. Denote $AB = b, AD = a, BC = c, CD = d, AC = d_1$, and $BD = d_2$. We need to prove that $d_1 \cdot d_2 = ac + bd$ (Fig. 3.25).

m $\overline{AB}$ = 6.47 cm

m $\overline{DF}$ = 6.47 cm
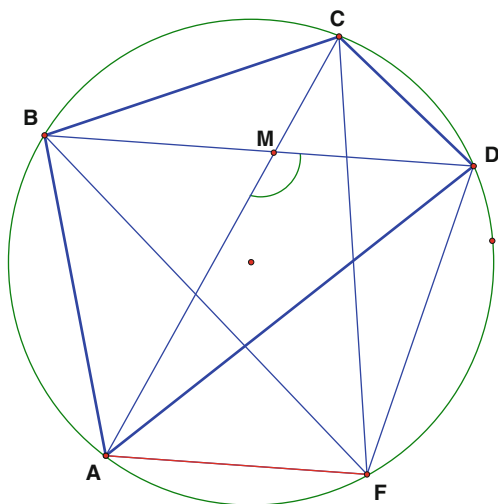
Fig. 3.25 Second sketch for the proof of Ptolemy's Theorem

Construct segment AF parallel to BD so that the quadrilateral $AFDB$ is a trapezoid with $FD = AB = b$ and $FB = AD = a$. Triangle BAD is congruent to BFD (three sides) so $[BFD] = [BAD]$. Since

$$[ABCD] = [BCD] + [BAD]$$
$$[FBCD] = [BCD] + [BFD]$$

the areas of the two quadrilaterals are the same.

Let angle $DMA = \varphi$, then on one hand $[ABCD] = \frac{1}{2} d_1 \cdot d_2 \sin\varphi$

On the other hand,

$$[ABCD] = [FBCD] = [FBC] + [FCD]$$
$$= \frac{1}{2} FB \cdot BC \cdot \sin(\angle FBC) + \frac{1}{2} FD \cdot CD \cdot \sin(\angle FDC)$$
$$= \frac{1}{2} a \cdot c \cdot \sin(\angle FBC) + \frac{1}{2} b \cdot d \cdot \sin(\angle FDC)$$

We can see that $\angle FBC + \angle FDC = \pi$ as they are opposite angles of the cyclic quadrilateral $FBCD$. Hence their sines are equal. In order to prove that $d_1 \cdot d_2 = ac + bd$, it would be enough to demonstrate that $\angle FDC = \varphi$. In order to see this, denote $\cup AB = \cup DF = \alpha, \cup BC = \beta$, and $\cup AF = \gamma$. By the property of the inscribed angles we have that

$$\angle FDC = \frac{\alpha + \beta + \gamma}{2}$$

and by an application of the Angle Between Two Intersecting Chords Theorem (see Sect. 1.2.2) we obtain

$$\angle AMD = \varphi = \frac{\beta + \gamma + \alpha}{2}$$

Therefore, it follows that $\angle FDC = \varphi$ from which it follows from the other hand that $[ABCD] = \frac{1}{2}(ac + bd) \cdot \sin\varphi$. Comparing the two hands, we find that $d_1 \cdot d_2 = ac + bd$ and so the proof is completed.

3.3.3.2   Area of a Cyclic Quadrilateral

Brahmagupta's Formula. Given a cyclic quadrilateral with sides a, b, c, and d, then its area, S, is given by the formula

$$S = \sqrt{(p-a)(p-b)(p-c)(p-d)}, \quad \text{where}$$
$$p = \frac{a+b+c+d}{2}$$

Brahmagupta was an Indian mathematician who is famous for his contributions to many fields of mathematics to include number theory and geometry. His formula on the area of cyclic quadrilaterals must look familiar to you. Let us prove this formula using (1.27) and the Laws of Cosines (Fig. 3.26).
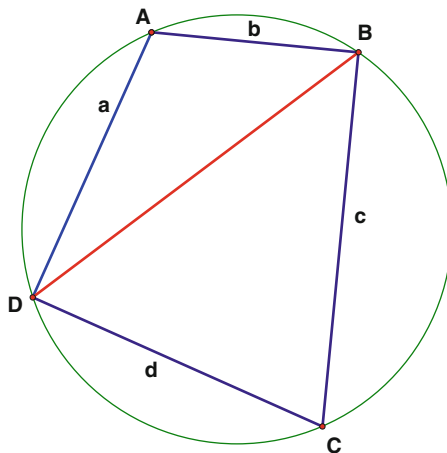
Fig. 3.26 Derivation of Brahmagupta's formula

Proof. Denote the area of $ABCD$ as S and draw a diagonal DB dividing the quadrilateral so that $S = [ABD] + [DBC]$. Applying formula (1.27) to both triangles and with $|AD| = a$, $|AB| = b$, $|BC| = c$, and $|DC| = d$, we have

$$S = \frac{1}{2} ab \cdot \sin(\angle A) + \frac{1}{2} dc \cdot \sin(\angle C)$$

Angles A and C are supplementary angles because $ABCD$ is cyclic. So $\angle A + \angle C = 180° \Rightarrow \sin(\angle C) = \sin(\angle A)$ and

$$S = \frac{1}{2}(ab + dc) \cdot \sin(\angle A)$$

Squaring both sides we obtain the following chain of transformations:

$$S^2 = \frac{1}{4}(ab + dc)^2 \cdot \sin^2(\angle A)$$
$$S^2 = \frac{1}{4}(ab + dc)^2 \cdot \left(1 - \cos^2(\angle A)\right) = \frac{1}{4}(ab + dc)^2 - \frac{1}{4}(ab + dc)^2 \cdot \cos^2(\angle A)$$

which can be written as

$$4S^2 = (ab + dc)^2 - (ab + dc)^2 \cdot \cos^2(\angle A) \tag{3.7}$$

Note that we replaced sin by cos using a trigonometric identity in order to apply the Law of Cosines to triangles ABD and BCD:

$$DB^2 = a^2 + b^2 - 2ab \cdot \cos(\angle A)$$
$$DB^2 = d^2 + c^2 - 2dc \cdot \cos(\angle C)$$

Equating the right sides of the equalities and using the fact that $\cos(\angle C) = -\cos(\angle A)$ as cosines of supplementary angles, we have

$$a^2 + b^2 - 2ab \cdot \cos(\angle A) = d^2 + c^2 + 2dc \cdot \cos(\angle A)$$

Collecting cosine terms on the right and eliminating the common factor:

$$a^2 + b^2 - d^2 - c^2 = 2(ab + dc) \cdot \cos(\angle A)$$

or

$$\frac{(a^2 + b^2 - d^2 - c^2)^2}{4} = (ab + dc)^2 \cdot \cos^2(\angle A) \tag{3.8}$$

Let us substitute (3.8) into (3.7) to arrive at an equation solely in terms of the lengths of the sides:

$$16S^2 = 4(ab+dc)^2 - (a^2+b^2-d^2-c^2)^2$$

Next, for the right hand side of the equality above, we will use a common algebraic factorization called the **difference of squares formula** $x^2 - y^2 = (x-y)(x+y)$.

$$(2(ab+dc))^2 - (a^2+b^2-d^2-c^2)^2$$
$$= (2(ab+dc) - a^2 - b^2 + d^2 + c^2) \cdot (2(ab+dc) + a^2 + b^2 - d^2 - c^2)$$

We can further simplify the equation by grouping terms inside parenthesis and by using the algebraic factorizations:

$$x^2 + 2xy + y^2 = (x+y)^2$$
$$x^2 - 2xy + y^2 = (x-y)^2$$

So

$$16S^2 = \left((d+c)^2 - (a-b)^2\right) \cdot \left((a+b)^2 - (d-c)^2\right)$$

Applying the difference of squares formula again to the right side, we obtain

$$16S^2 = (a+d+c-b)(d+c+b-a)(a+b+d-c)(a+b+c-d) \quad (3.9)$$

Now we will introduce a semiperimeter, $p = \dfrac{a+b+c+d}{2}$.
Replacing each factor in (3.9) similar to

$$a+d+c-b = (a+b+c+d) - 2b = 2p - 2b = 2(p-b)$$

we obtain

$$16S^2 = 16(p-b)(p-a)(p-c)(p-d)$$

Dividing by 16 and taking a square root, we finally arrive at the formula:

$$S = \sqrt{(p-b)(p-a)(p-c)(p-d)}$$

3.3.4   The Inscribed Circle Theorem

> **The Inscribed Circle Theorem.** A circle can be inscribed into a quadrilateral
> if the sums of its opposite sides are equal to each other.

Proof. As shown in Fig. 3.27, let *ABCD* be a quadrilateral circumscribed about a
circle with radius $r = OF$ and center at *O*. We must show that it is necessary that
the sum of opposing sides of *ABCD* is equal.

   Using the fact that segments of a tangent line to a circle dropped from the same
point are of equal lengths, let $|AF| = |AL| = x, |FD| = |DP| = y, |PC| = |CN| = z,$
and $|NB| = |LB| = t$, so

$$|AB| + |DC| = |AL| + |LB| + |DP| + |PC| = x + t + y + z$$
$$|AD| + |CB| = |AF| + |FD| + |CN| + |NB| = x + y + z + t$$

Therefore, $|AB| + |DC| = |AD| + |BC|$ (see Fig. 3.28).

m $\overline{AB}$ = 1.14 inches

m $\overline{DC}$ = 2.16 inches

m $\overline{AD}$ = 1.43 inches

m $\overline{BC}$ = 1.88 inches

(m $\overline{AB}$) + (m $\overline{DC}$) = 3.30 inches

(m $\overline{AD}$) + (m $\overline{BC}$) = 3.31 inches
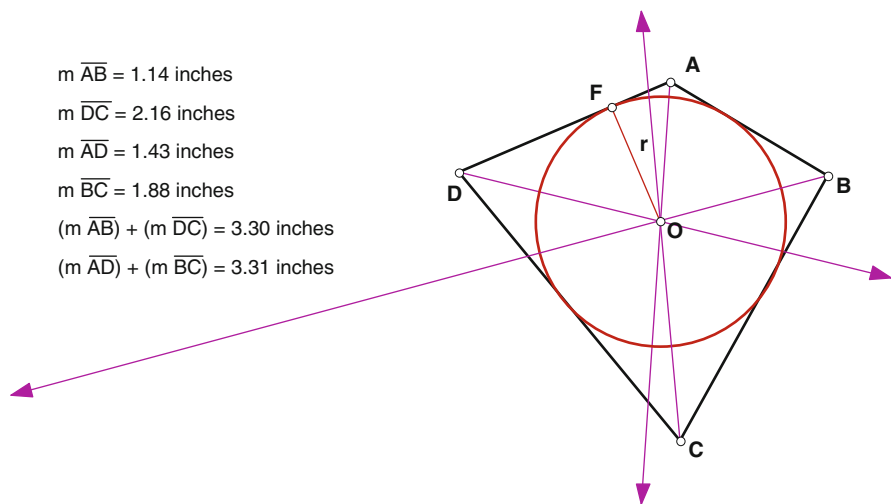
Fig. 3.27   Inscribed circle theorem

We can also prove that *O* is the intersection of the bisectors of all angles (Fig. 3.28).
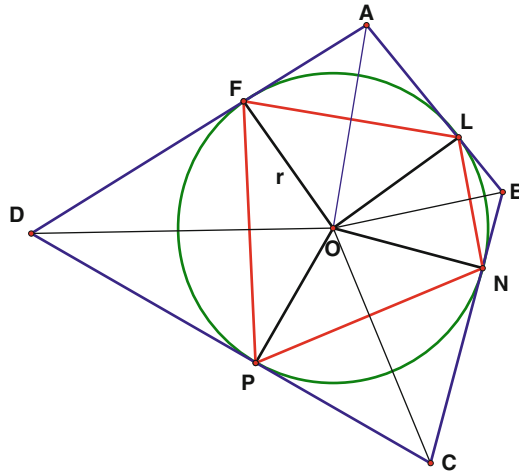
Fig. 3.28 Inscribed and circumscribed quadrilaterals

Proof. Let *FLNP* be inscribed into the same circle as that of Fig. 3.27. Then $FO = LO = ON = OP = r$ is the radius of the inscribing circle. *FALO* is a kite as there are three other quadrilaterals: *DFOP*, *PONC*, and *LBNO*. Diagonals of a kite are perpendicular. Moreover, *AO*, *BO*, *CO*, and *DO* are perpendicular bisectors of *FL*, *LN*, *NP*, and *PF*, respectively. Therefore, *O* is the intersection of the angle bisectors in a circumscribed quadrilateral *ABCD*.

3.3.4.1 Inscribed Circle of a Rhombus

> **Circumscribed Rhombus Theorem.** If a circle is inscribed in a rhombus, its center is the point of intersection of the diagonals.

Note that the diagonals of a rhombus bisect the respective angles.

3.3.5 Simson Line Theorem

> **Simson Line Theorem.** Let *M* be a point on the circumscribed circle of a triangle *ABC*. If A_1, B_1, and C_1 are the feet of the perpendiculars from *M* to the sides of the triangle, then A_1, B_1, and C_1 lay on the same line.

Robert Simson (1687–1768) was a British mathematician and professor at Glasgow University who worked primarily in the field of geometry. However, based on the book by Hartshorne (Hartshorne 2000), this result was first proven by **Wallace** in 1799. I would like to offer you a proof of the theorem based on the properties of cyclic quadrilaterals.

Proof. Consider a triangle ABC inscribed in a circle (Fig. 3.29). Let point M be a point on the circle. Let us drop perpendiculars MC_1, MB_1, and MA_1 from M to sides AB, AC, and BC, respectively. Connect points B_1–A_1, B_1–C_1, and M with vertices A and C. Since points M, A, B, and C are on the same circle, $MABC$ is a cyclic quadrilateral.

$$\angle MCB + \angle MAB = 180° \tag{3.10}$$

Denote angle $\angle CMA_1 = \angle 1$ and $\angle C_1MA = \angle 2$. Since ΔCA_1M is a right triangle and angle MAB is an exterior angle of the right triangle ΔMC_1A,

$$\angle MCB = 90° - \angle 1$$
$$\angle MAB = 90° + \angle 2 \tag{3.11}$$

Adding the left and right hand sides of (3.11) and equating to the right hand side of (3.10), $\angle MCB + \angle MAB = 180° = 180° + \angle 2 - \angle 1$ which implies $\angle 2 = \angle 1$.
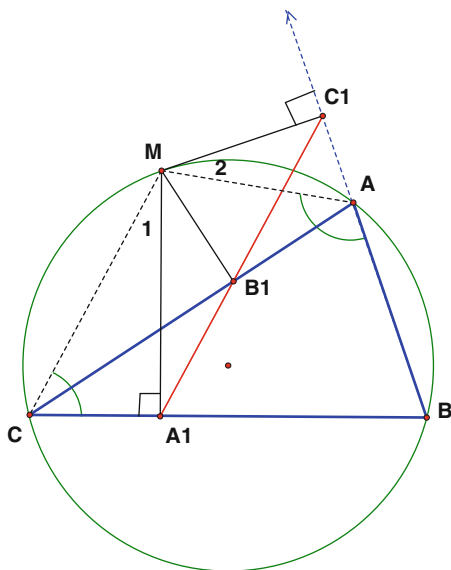
Fig. 3.29 Simson line theorem

Next, we note that since CMB_1A_1 is cyclic so $\angle 1 = \angle CB_1A_1$. On the other hand, quadrilateral C_1MB_1A is also cyclic so $\angle 2 = \angle AB_1C_1$. Finally, since $\angle CB_1A_1 = \angle AB_1C_1$, we conclude that segments B_1A_1 and B_1C_1 are on the same line. Therefore, the feet of the perpendiculars dropped from a point M on a circle to the sides of an inscribed triangle are collinear.

Remark. The application of cyclic quadrilaterals is maybe the least intuitive auxiliary construction for the proof of the Simson Line Theorem. However, the ability to do such constructions together with the ability to see their necessity is very important for solving many Olympiad-type problems. I hope that Problems 64 and 66 and their detailed solutions will help you to master the art of constructing proofs where intuition seems to be lacking.

3.4   Problems on Circles

3.4.1   Inscribed, Circumscribed, and Tangent Circles

I think we are now ready to solve some problems. We will mention theorems and properties we use. Before you start, I recommend that you review the entire chapter.

Problem 53. Prove that the sum of the lengths of the legs of a right triangle equals the sum of diameters of the inscribed and circumscribed circles.

Proof. Of course, we will start from a sketch (Fig. 3.30). The midpoint of the hypotenuse of the right triangle will be the center of the circumscribed circle and the intersection of all bisectors of the triangles will be the center of the inscribed circle.
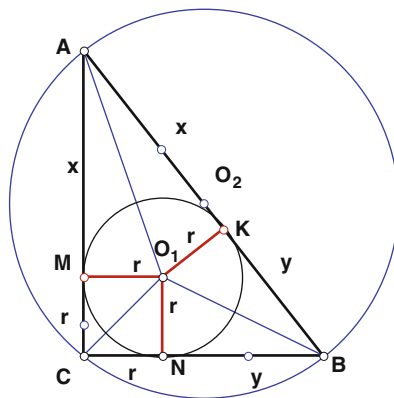
Fig. 3.30   Sketch for Problem 53

Let us introduce three variables x, y, and r, such that $AM = AK = x$, $BN = BK = y$, and $CM = CN = r$. By the Tangents from the Same Point Theorem, $CM = CN = O_1M = O_1N = r$, the radius of the inscribed circle. Now we can express the diameters of both circles in terms of the new variables. The diameter of the circumscribed circle,

$$D = 2 \cdot AO_2 = AB = x + y$$

The diameter of the inscribed circle $d = 2r$, so

$$D + d = x + y + 2r \tag{3.12}$$

While the sum of the two legs of the triangle is

$$AC + CB = x + r + r + y = x + y + 2r \tag{3.13}$$

The right sides of (3.12) and (3.13) are equal and the problem is solved.

Problem 54. The tangent point of a right triangle and an inscribed circle divide a leg into segments of length m and n such that $m < n$. Find the length of the second leg.

Solution. Let us draw a picture with all relevant information (Fig. 3.31).
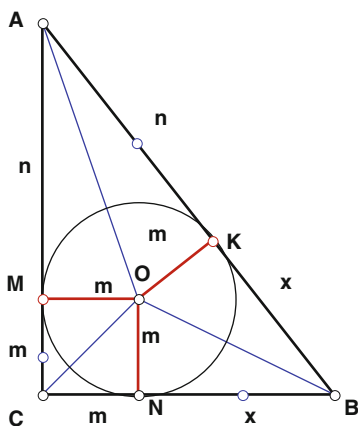
Fig. 3.31 Sketch for Problem 54

If *CA* and *CB* are two tangents dropped to the circle from vertex *C*, then $CM = CN = m$ by the Tangents from the Same Point Theorem. By the same property $AM = AK = n$. We must calculate the length of *CB*. It is obvious that $CB = CN + NB$, but we don't know the length of *NB*. Let us introduce a variable $x = |NB| = |KB|$. In order to find x and then *CB*, we will use the Pythagorean Theorem:

$$AC^2 + CB^2 = AB^2$$
$$(n + m)^2 + (m + x)^2 = (n + x)^2$$
$$m^2 + 2mn + n^2 + m^2 + 2mx + x^2 = n^2 + 2nx + x^2$$
$$m^2 + mn = x(n - m)$$
$$x = \frac{m^2 + mn}{n - m}$$

Using x we can find the second leg *CB*:

$$CB = m + \frac{m^2 + mn}{n - m} = \frac{2mn}{n - m}$$

Answer. $CB = \dfrac{2mn}{n - m}$.

In order to solve the problem below we need to demonstrate our cumulative knowledge. I would call this problem a problem-investigation where you have to apply many properties.

Problem 55. A circle is inscribed into a triangle with sides $AB = 8$, $BC = 6$, and $AC = 4$. Find the length of segment *DE* if points *D* and *E* are tangent to sides *AB* and *AC*, respectively.

Solution. Of course, we again will start from a picture (Fig. 3.32).

This nice picture can give us some ideas for finding *DE* from the quadrilateral *DOEA*. The quadrilateral consists of two right triangles (*ODA* and *OEA*) the areas of which we can find. This area is a half of a product of its diagonals. Continuing this idea, we need to know the radius of the circle and the length of *DA*.

As shown in Fig. 3.32, points *G*, *D*, and *E* are tangent to sides *BC*, *BA*, and *AC*, respectively. By the Tangents from the Same Point Theorem, $CG = CE$, $BG = BD$, and $AD = AE$. *O* is the center of the circle and the point of the intersection of the bisectors of triangle *ABC*.
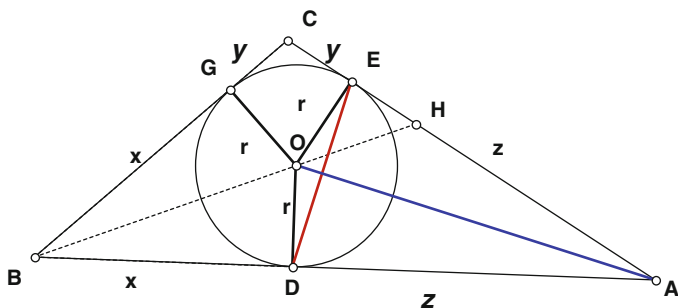
Fig. 3.32 Sketch for Problem 55

Let us introduce the variables $x = BG = BD$, $y = GC = CE$, $z = DA = AE$, and $r = OD = OC = OG$. We can find the radius, r, using the relationship between the area of the triangle ABC, the radius of the inscribed circle, and the half perimeter, $p = \frac{a+b+c}{2} = \frac{6+4+8}{2} = 9$. By Heron's Formula,

$$S = \sqrt{p(p-a)(p-b)(p-c)} = \sqrt{9(9-6)(9-8)(9-4)} = 3\sqrt{15}$$
$$S = p \cdot r = 9r$$

The left sides of the equations are the same, so we can equate the right sides and find the radius:

$$9r = 3\sqrt{15}$$
$$r = \frac{\sqrt{15}}{3}$$

Variables x, y, and z can found from the system:

$$\begin{cases} x + z = 8 \\ x + y = 6 \\ y + z = 4 \end{cases} \Leftrightarrow \begin{cases} z - y = 2 \\ z + y = 4 \\ z = 3 \end{cases} \Leftrightarrow \begin{cases} x = 5 \\ y = 1 \\ z = 3 \end{cases}$$

The last step of our investigation is to find DE. Let us find the area of $DEOA$ in two different ways:

$$\frac{DE \cdot OA}{2} = 2 \cdot \frac{r \cdot z}{2}$$
$$DE = 2 \cdot \frac{\sqrt{15} \cdot 3}{3 \cdot OA} = \frac{2\sqrt{15}}{OA}$$

In order to find the length of DE we need OA. This can be found by applying the Pythagorean Theorem to triangle ODE

$$OA^2 = \left(\frac{\sqrt{15}}{3}\right)^2 + 3^2 = \frac{32}{3}$$

$$OA = \frac{\sqrt{32}}{\sqrt{3}} = \frac{4\sqrt{6}}{3}$$

so that

$$DE = \frac{2\sqrt{15} \cdot 3}{4\sqrt{6}} = \frac{3\sqrt{10}}{4}$$

Answer. $DE = \dfrac{3\sqrt{10}}{4}$.

Problem 56. An isosceles triangle, ABC, with sides of length 2 and a 120° vertex angle is given. Find the radius of the circle that can be inscribed in triangle ABC.

Solution. One thought that could come into your mind is to use a very familiar formula connecting a half perimeter of a polygon, a radius of the inscribed circle, and an area of the polygon.

$$S = p \cdot r \tag{3.14}$$

This formula has always helped us with problems of such type. It should work now. After drawing Fig. 3.33 with all relevant information labeled, we can think of introducing some appropriate variables. Triangle ABC is isosceles so $AB = BC = 2$. A circle with radius $r = OD$ is inscribed into ABC. We must find r.
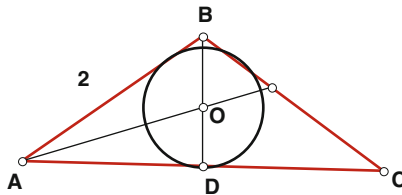
Fig. 3.33 Sketch for Problem 56

Construct segment *BD* as the height of bisector of the triangle. Since *ABC* is isosceles, *BD* bisects angle *ABC* and so $m\angle ABD = m\angle DBC = 60°$ from the condition of the problem (and $m\angle BAC = 30°$). Triangle *ABD* is a right triangle. Knowing the hypotenuse and angles, we find that $AD = \sqrt{3}$ and $BD = 1$. Because *D* is the midpoint of *AC*, $AC = 2AD = 2\sqrt{3}$. Knowing the height and base of triangle *ABC* we can find its area, *S*, and half perimeter, *p*.

$$S = \frac{1}{2}2\sqrt{3} \cdot 1 = \sqrt{3}$$

$$p = \frac{1}{2}(2 + 2\sqrt{3} + 2) = 2 + \sqrt{3}$$

The inscribed radius of formula (3.14) is then

$$r = \frac{S}{p} = \frac{\sqrt{3}}{2 + \sqrt{3}} \tag{3.15}$$

If we want to rationalize the denominator, we should multiply the numerator and the denominator of (3.15) by $\left(2 - \sqrt{3}\right)$, that is the conjugate of $\left(2 + \sqrt{3}\right)$,

$$r = \frac{\sqrt{3}}{\left(2 + \sqrt{3}\right)} \cdot \frac{\left(2 - \sqrt{3}\right)}{\left(2 - \sqrt{3}\right)} = \frac{2\sqrt{3} - 3}{2^2 - \left(\sqrt{3}\right)^2} = 2\sqrt{3} - 3$$

Answer. $r = 2\sqrt{3} - 3$.

Now we can try solving problems on two or more circles.

Problem 57. Two circles are tangent to each other. Through the center of the second is drawn a tangent line to the first. The distance from the point of tangency to the center of the second circle equals 3 times its radius. What is the ratio of radii of the first circle to the second?

Solution. Of course, we start with a picture (Fig. 3.34).
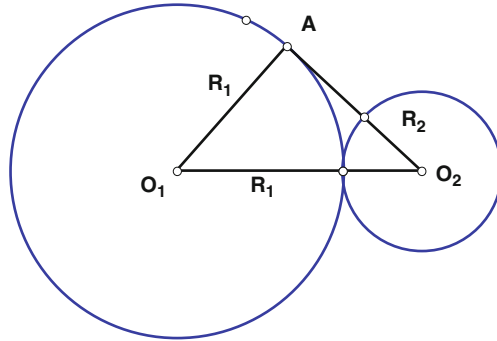
Fig. 3.34   Sketch for Problem 57 (two tangent circles)

As shown, O_1 and O_2 are the centers of the first and second circles, respectively and AO_2 is the tangent. The radius of the first circle is $O_1A = R_1$ and R_2 is the radius of the second circle. From the condition of the problem, $O_1O_2 = R_1 + R_2$ and $O_2A = 3R_2$. We need to find $\frac{R_1}{R_2}$.

Consider the right triangle $O_1AO_2(\angle A = 90°)$. By the Pythagorean Theorem $O_1O_2^2 = O_1A^2 + O_2A^2$ or $(R_1 + R_2)^2 = R_1^2 + (3R_2)^2$. Simplifying the expression we obtain $R_1 = 4R_2$ from which we have: $\frac{R_1}{R_2} = 4$.

Answer.   The ratio equals 4.

By solving the problem below you will gain practice in creating proofs.

Problem 58.   Prove that if an isosceles trapezoid can be circumscribed about a circle, then its height is the geometric mean of its bases.

Proof.   Let us assume that $ABCD$ is an isosceles trapezoid with bases $AD = a$ and $BC = b$, and sides $AB = CD = c$. If $ABCD$ is circumscribed by the circle, then by Secant–Tangent Segment Theorem, we have that $a + b = 2c$ from which we find (Fig. 3.35)

$$AB = \frac{a+b}{2} \qquad\qquad (3.16)$$
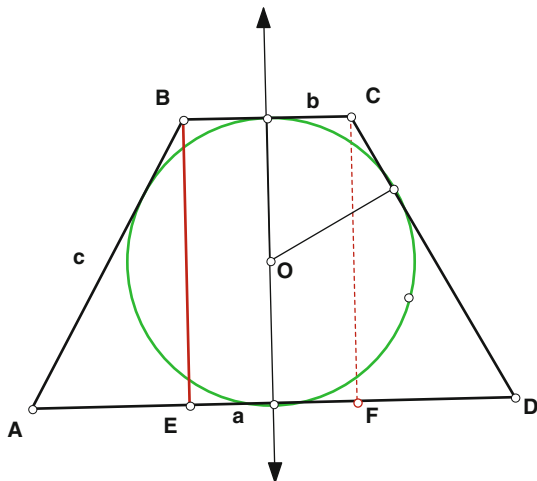
Fig. 3.35 Sketch for Problem 58 on circumscribed trapezoid

Because any isosceles trapezoid is symmetric about the line passing through the midpoints of the bases, *BE* is the height, *h*, of the trapezoid and

$$AE = \frac{AD - BC}{2} = \frac{a - b}{2} \tag{3.17}$$

Applying the Pythagorean Theorem to triangle *BEA*, $BE^2 = AB^2 - AE^2$. Replacing *AB* and *AE* by (3.16) and (3.17), respectively, we have that

$$h^2 = \left(\frac{a + b}{2}\right)^2 - \left(\frac{a - b}{2}\right)^2 = ab$$
$$h^2 = ab$$
$$h = \sqrt{ab}$$

The problem is solved.

Note. This is not the only solution to this problem. I included this problem in the homework and hope that you will find a pair of the right similar triangles and use the ratio of similar sides.

Problem 59. Five circles are inscribed in a 60° angle so that each consecutive circle starting after the first is tangent to the previous one. How much is the combined area of the first five circles greater than that of the first?

Solution. Let A be the vertex of the angle, O_i and r_i are the center and radius of the ith circle ($i = 1, 2, \ldots, 5$). Since $0.5 \cdot \angle A = 30°$, $AO_i = 2r_i$, $AO_{i-1} = 2r_{i-1}$, $AO_i = AO_{i-1} + r_{i-1} + r_i$, and $2r_i = 2r_{i-1} + r_{i-1} + r_i$. The radius of the ith circle can be written as $r_i = 3r_{i-1}$. The radii of consecutive circles form a geometric progression so that the combined area of the five circles is

$$S = \pi \cdot r_1^2 (1 + 9 + 9^2 + 9^3 + 9^4) = \frac{\pi \cdot r_1^2 (9^5 - 1)}{9 - 1} = 7{,}381 \pi r_1^2$$

The area of the first circle is $S_1 = \pi r_1^2$ so $\frac{S}{S_1} = 7{,}381$

Answer. 7,381 times.

Problem 60. (Lidsky) Infinitely many circles are inscribed in an acute angle so that each circle is tangent to its neighbors and to the sides of the angle. Prove that the radii of these circles form a geometric progression. Find this progression as a function of the measure of the angle.

Solution. Let O be the angle vertex, $m\angle O = \alpha$, O_n be the center of the nth circle, and O_{n-1} the center of the nth $-$ 1 circle (Fig. 3.36). It is clear that the line OO_n is the bisector of angle α. The radius of the nth circle can be evaluated as $r_n = OO_n \cdot \sin\frac{\alpha}{2}$. Denote by r_{n-1} the radius of the nth $-$ 1 circle. On one hand, $(OO_n - OO_{n-1}) \cdot \sin\frac{\alpha}{2} = r_n - r_{n-1}$. On the other hand, $(OO_n - OO_{n-1}) = r_n + r_{n-1}$.
Dividing these formulas by each other we obtain the following

$$\sin\frac{\alpha}{2} = \frac{r_n - r_{n-1}}{r_n + r_{n-1}}$$

$$r_{n-1} = r_n - r_n \sin\frac{\alpha}{2} - r_{n-1} \cdot \sin\frac{\alpha}{2}$$
$$r_{n-1}\left(1 + \sin\frac{\alpha}{2}\right) = r_n\left(1 - \sin\frac{\alpha}{2}\right)$$
$$\frac{r_n}{r_{n-1}} = \frac{1 + \sin\frac{\alpha}{2}}{1 - \sin\frac{\alpha}{2}} = r$$
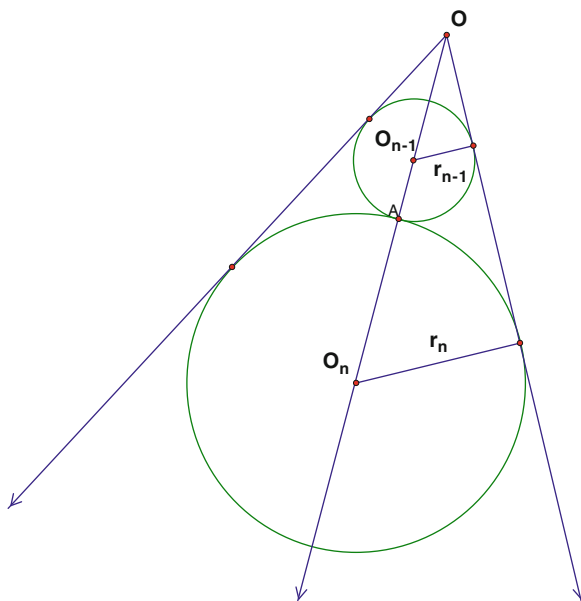
Fig. 3.36 Sketch for Problem 60

Therefore, the radii of the consecutive circles form a geometric progression with the ratio r given by the formula above. This ratio keeps the same value for the given angle α.

Problem 61. Can the area of a triangle be decreased if all sides of a triangle are increased?

Solution. Yes. It is possible. For example, consider an equilateral triangle with side $a = 1$ so that its area is $S_1 = \frac{\sqrt{3}}{4}$. Let us increase the sides of the triangle and allow it to become an isosceles triangle with sides $a = 2$, $b = c = 1.01$. Then the area of the new triangle with bigger sides will be much less. In fact, it will be $S_2 = \frac{2 \cdot \sqrt{1.01^2 - 1}}{2} = \sqrt{0.0201} \ll S_1 = \frac{\sqrt{3}}{4}$.

How can we explain this phenomenon? The explanation will surprise you. Do you remember the formula connecting the area of a triangle and a radius, R, of the subscribed circle? Here it is! $S = \frac{abc}{4R}$.

This formula is a key to the solution. Increasing a, b, and c and simultaneously letting R become much greater than abc, we can make S become very small and almost 0. How can we get such an effect? Let us imagine a circle of a very big radius, R (Fig. 3.37).
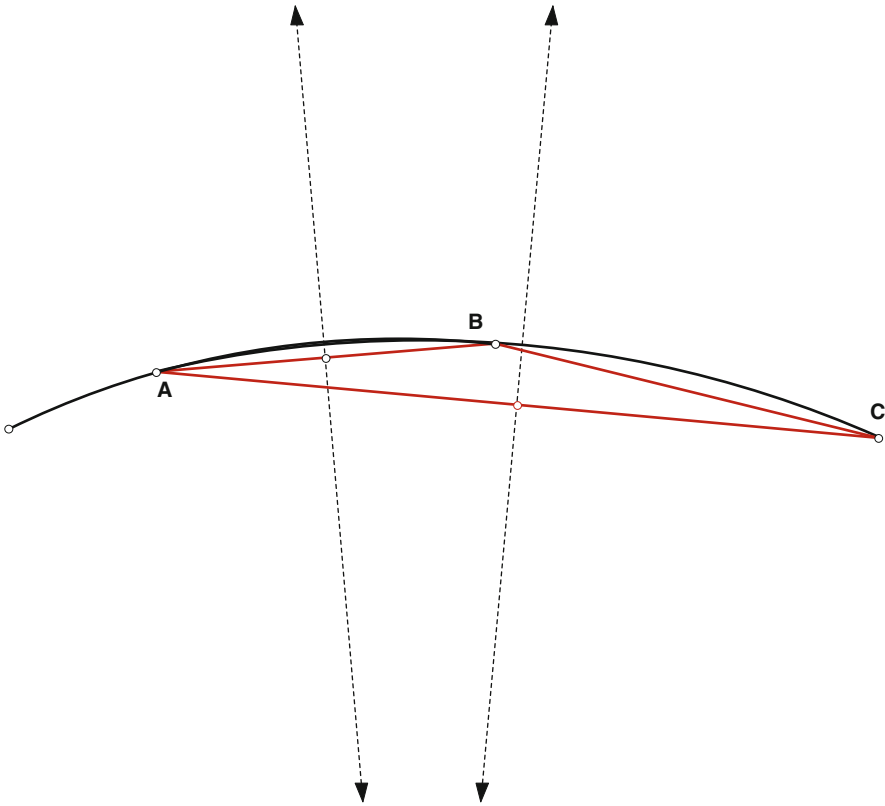
Fig. 3.37 Sketch for Problem 61

Problem 62. Consider two tangent circles such that one is within the other. A line passes through points B, C, and the center of the smaller circle and crosses the bigger one at points A and D. Find the ratio of radii of circles if $|AB|:|BC|:|CD| = 2:4:3$.

Solution. Let us sketch a picture (Fig. 3.38). Denote O_1 and O_2 as centers of the big and the small circles, respectively. Let M be a point of tangency of the two circles, so that a line drawn through points O_1, O_2, and M forms a chord MN of the big circle. By the properties of chords,

$$|MO_2| : |O_2N| = |AO_2| \cdot |O_2D| \tag{3.18}$$

We need to find these components in terms of r and R.

If the radius of the small circle is denoted as r, by the condition of the problem $|AB|:|BC| = 2:4$ or $\frac{|AB|}{2r} = \frac{2}{4}$, so $r = |AB|$.
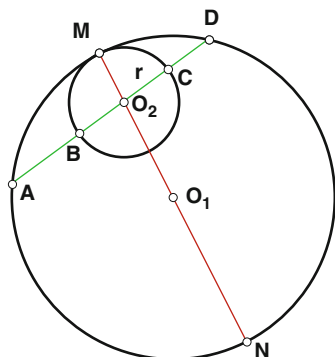
Fig. 3.38 Sketch for Problem 62

Also from the condition of the problem, $|BC|:|CD| = 4:3$; $\dfrac{|BC|}{|CD|} = \dfrac{4}{3}$ or in terms of r, $\dfrac{2r}{|CD|} = \dfrac{4}{3}$; $|CD| = \dfrac{3}{2}r$. The lengths of AO_2 and O_2D are

$$|AO_2| = |AB| + |BO_2| = 2r$$
$$|O_2D| = |O_2C| + |CD| = r + \frac{3}{2}r = \frac{5}{2}r$$

Denoting the radius of the big circle as R, we have that MO_2 and O_2N are

$$|MO_2| = r$$
$$|O_2N| = R + R - r = 2R - r$$

Substituting the lengths: $|MO_2|$, $|O_2N|$, $|AO_2|$, and $|O_2D|$ into (3.18),

$$r \cdot (2R - r) = 5r^2$$
$$R = 3r$$

Answer. 3:1.

3.4.2 Problems on Cyclic Quadrilaterals

I hope the following problem will be of interest to those who enjoy "tricky" problems. Problem 63 is not really tricky, but at first glance seems to have too many unknowns.

Problem 63. The angles between opposite sides of a quadrilateral inscribed in a circle are α and β. Find all angles of the quadrilateral.

Solution. First, we will draw a quadrilateral *ABCD* inscribed in a circle (Fig. 3.39). From the condition of the problem, we know the angles between opposite sides of the quadrilateral. The opposite sides themselves do not intersect, but the lines containing them do. Continue the opposite sides until they intersect at points *M* and *N* and form angles α and β that we show with one and two red arcs, respectively.
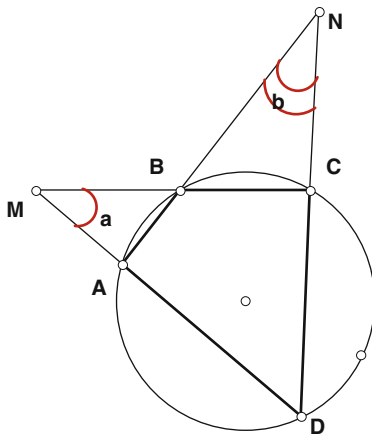
Fig. 3.39 Sketch for Problem 63

$$AD \cap BC = M, \quad m\angle DMC = \alpha$$
$$AB \cap CD = N, \quad m\angle AND = \beta$$

Let us assume that we are looking for the measures of angles *A*, *B*, *C*, and *D* of the quadrilateral, *ABCD*, that is inscribed in a circle. By the Circumscribed Quadrilateral Theorem, we can write the first two equations.

$$\angle B + \angle D = 180°$$
$$\angle A + \angle C = 180°$$

(3.19)

In order to find all four angles of the quadrilateral we need to create two more equations. We can notice that angles C and D are also the angles of triangle MCD and angles A and D are also angles of triangle AND. We know that for any triangle the sum of the interior angles equals $180°$. This can be written as

$$\angle A + \angle D + \beta = 180°$$
$$\angle C + \angle D + \alpha = 180° \tag{3.20}$$

Combining (3.19) and (3.20), we have

$$\begin{cases} \angle A + \angle D = 180° - \beta \\ \angle C + \angle D = 180° - \alpha \\ \angle A + \angle C = 180° \ldots \\ \angle B + \angle D = 180° \ldots \end{cases} \tag{3.21}$$

Subtracting the second equation of the system from the first we obtain

$$\angle A - \angle C = \alpha - \beta \tag{3.22}$$

Adding the third equation of system (3.21), we find $\angle A$,

$$\angle A = 90° + \frac{\alpha - \beta}{2} \tag{3.23}$$

Now from (3.22) and (3.23) we can find angle C,

$$\angle C = 90° - \frac{\alpha - \beta}{2}$$

We will find angle D by adding the left and the right sides of the first two equations of the system:

$$\angle A + \angle C + 2\angle D = 360° - (\alpha + \beta)$$

and replacing $\angle A + \angle C = 180°$, we have

$$180° + 2\angle D = 360° - (\alpha + \beta)$$

or

$$\angle D = 90° - \frac{\alpha + \beta}{2}, \quad \angle B = 90° + \frac{\alpha + \beta}{2}$$

Problem 64. Prove that if points K, L, M are the feet of the altitudes of an acute triangle ABC, then these altitudes are the angle bisectors of the *orthic* triangle KLM.

Proof. In order to solve this problem we will use the properties of cyclic quadrilaterals. First, we will draw a triangle *ABC* and show its three heights *AM*, *BL*, and *CK*. (See Fig. 3.40.)
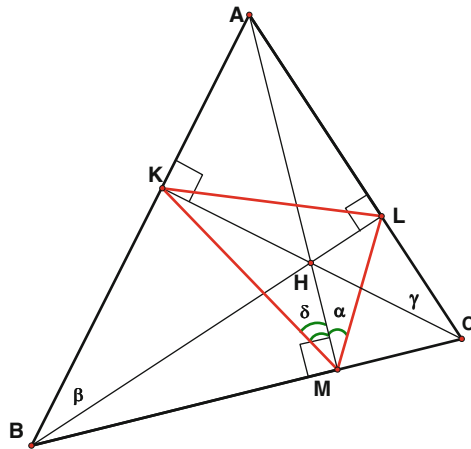
Fig. 3.40 First sketch for Problem 64

Denote $\angle AML = \alpha$, $\angle KCA = \gamma$, $\angle ABL = \beta$, and $\angle KMA = \delta$. If each height of triangle *ABC* is also a bisector of the **orthic triangle** *KLM* (shown in red as formed by the altitudes of *ABC*), then we need to prove that $\alpha = \delta$. However, Fig. 3.40 has too many lines and it is hard to work with. Let us temporarily erase segments *KL* and *KM* and connect only points *M* and *L* as shown in Fig. 3.41.
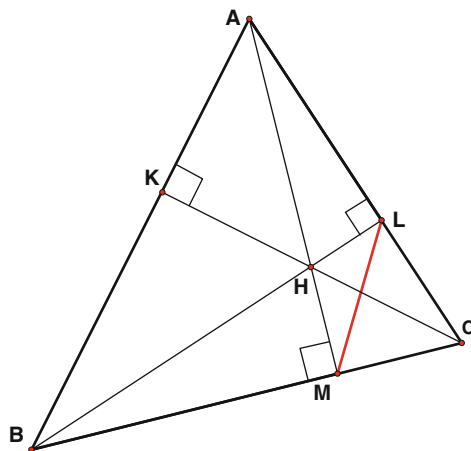
Fig. 3.41 Second sketch for Problem 64

We can state that the quadrilateral *LMAB* is cyclic because the angles at *L* (∠*ALB* = 90°) and at *M* (∠*AMB* = 90°) are right angles, so triangles *ALB* and *AMB* are inscribed in the same circle with diameter *AB*. Therefore, by Thales' Theorem, *AB* is a diameter of the circle in which the cyclic quadrilateral *ALMB* is inscribed. Hence, $\alpha = \angle AML = \beta = \angle ABL$ since they support the same arc, *AL*. The angles and the arc are shown in green in Fig. 3.42. Next, *KLCB* is also a cyclic quadrilateral because the angles *K* and *L* are right and *BC* is the diameter of the circle in which the quadrilateral *KLCB* is inscribed.
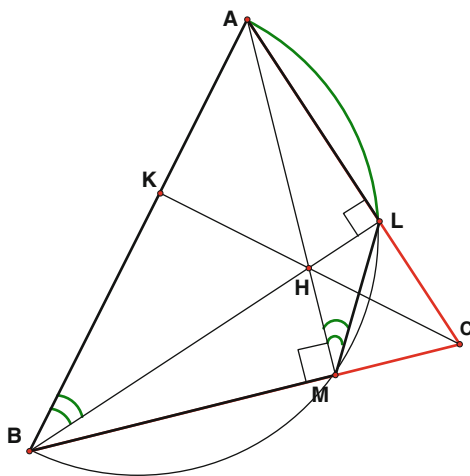
Fig. 3.42 Problem 64. View of cyclic quadrilateral BALM

Hence $\beta = \gamma = \angle KCL$ are angles supporting arc *KL*. Finally, *MKAC* is a cyclic quadrilateral (Fig. 3.43) because the angles *M* and *K* are right, so $\gamma = \delta = \angle AMK$. Thus, $\alpha = \delta$ and ∠*KMA* = ∠*AML*. Therefore, the altitude *AM* of *ABC* is the angle bisector of the angle *KML* of the orthic triangle. The same argument can be applied to the other two altitudes.
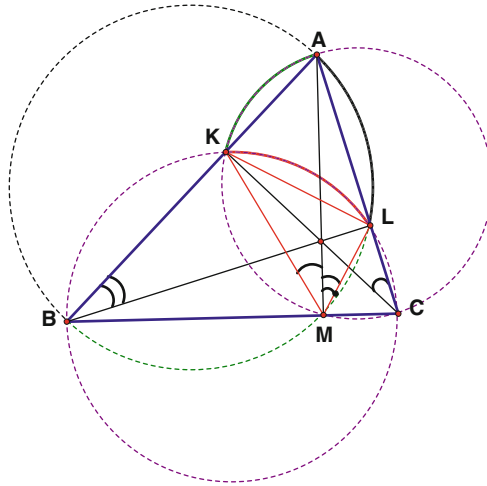
Fig. 3.43 Problem 64. View of cyclic quadrilaterals

Note. Since the angle bisectors of triangle *KLM* meet at a point, this gives us another proof that the altitudes of *ABC* meet in one point.

Problem 65. Let A, B, C, D be distinct points in the plane, with A, B both on the same side of the line *CD*. Prove that A, B, C, D lie on a circle if and only if the angles *DAC* and *DBC* are equal.

It was noted first by **Euclid** that if A, B, C, D lie on a circle (with A and B on the same side of *CD*), then the angles *CAD* and *DBC* are equal since they both support the same arc *DC*.
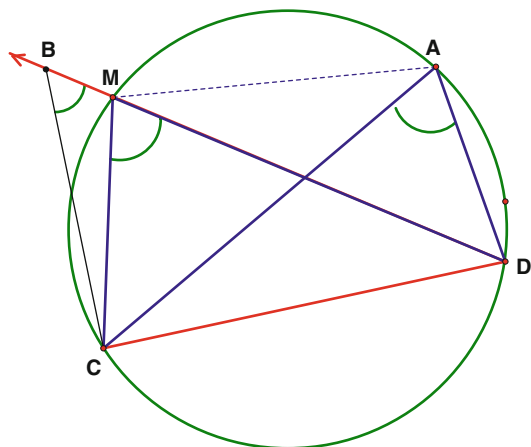
Fig. 3.44   Sketch for Problem 65

Proof. Suppose conversely that angles *CAD* and *DBC* are equal, but point *B* is not on the circle (Fig. 3.44). Let the circle through *A*, *D*, and C meet the line *BD* at *M*. In our figure *B* lies outside the circle, but the argument will be similar if *B* lies inside the circle. Then the angle at *M* is also equal to the angles at *A* and at *B*. We know that it cannot be true that $B \neq M$ because the angle *DMC* at *M* is an exterior angle to the triangle *BCM* and so it must be greater than the opposite interior angle at *B*. Hence $B = M$ and all four points lie on the same circle.

Problem 66. Let *AXYZB* be a convex pentagon inscribed in a semicircle of diameter *AB*. Denote by *P*, *Q*, *R*, *S* the feet of the perpendicular from *Y* onto lines *AX*, *BX*, *AZ*, *BZ*, respectively. Prove that the acute angle formed by lines *PQ* and *RS* is half of the angle *XOZ*, where *O* is the midpoint of segment *AB*.

Solution. Let us draw a semicircle and inscribe a pentagon *AXYZB* in it (Fig. 3.45). Additionally, we will construct points *P*, *Q*, *R*, and *S*, and continue lines *PQ* and *SR* to intersect at *T*.
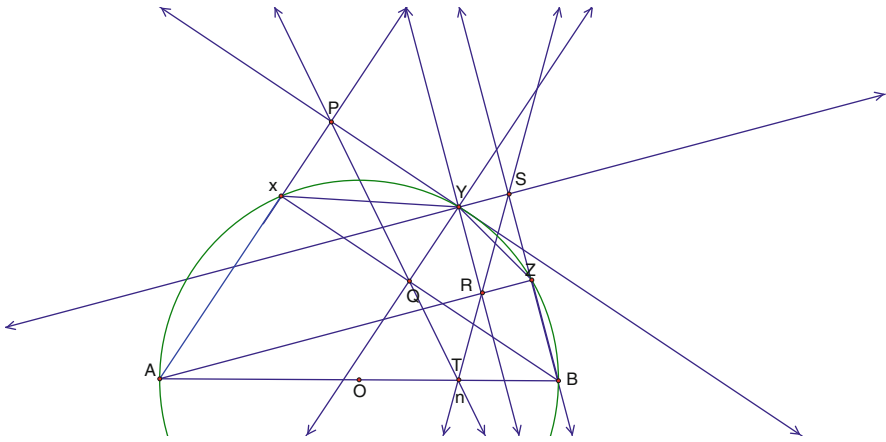
Fig. 3.45 First sketch for Problem 66

First, consider triangle *AXB*:

$$YP \perp AX$$
$$YQ \perp XB$$
$$YT \perp AB \Rightarrow \{P, Q, T\} \in \text{the same line (Simson)}$$

Secondly, consider triangle *ZAB*:

$$YR \perp AZ$$
$$YS \perp ZB$$
$$YT \perp AB \Rightarrow \{S, R, T\} \in \text{the same line (Simson)}$$

And note that

$$\angle PTS = \angle PTY + \angle YTS$$

First, we will evaluate angle *PTY*: Consider quadrilateral *APYT*. It is cyclic, because $\angle YPA = \angle YTA = 90°$ Also, $\angle PTY = \angle PAY = \frac{1}{2} \angle XOY$ (both angles support the same arc) as shown in Fig. 3.46. The angle $\angle YTS$ can be found from the cyclic

quadrilateral *YSBT* (see Fig. 3.47) since *YB* is the diameter of the circumscribed circle: We can state that $\angle YTS = \angle YBS$. Further, we can continue and rewrite the last equality as $\angle YTS = \angle YBS = \angle YBZ = \frac{1}{2} \angle YOZ$. In order to understand how the last part of the equality was obtained, we note that $\angle YBS = \angle YBZ$ because *Z* is on the line *BS* (see Fig. 3.45). Next $\angle YBZ = \frac{1}{2} \angle YOZ$ since they are the inscribed and central angles supporting the same arc.
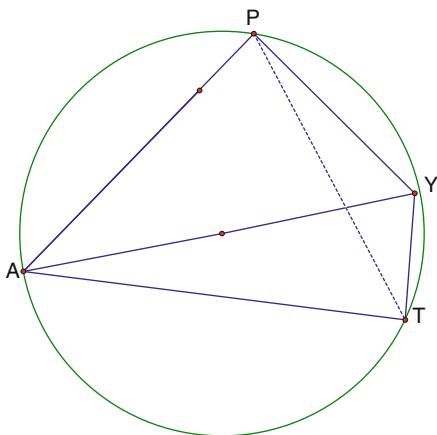
Fig. 3.46 Second sketch for Problem 66
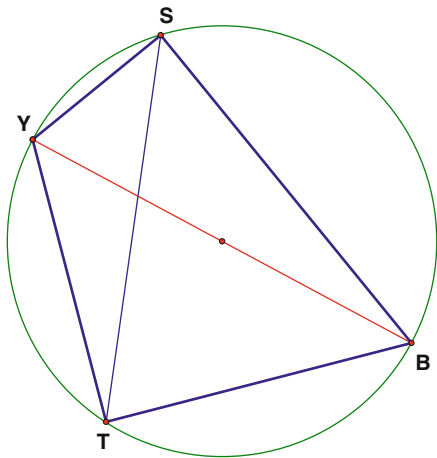
Fig. 3.47 Third sketch for Problem 66

Finally,

$$\angle PTS = \frac{1}{2} \angle XOY + \frac{1}{2} \angle YOZ = \frac{1}{2} \angle XOZ$$

The proof is completed.

3.5   Homework on Circles

1. Prove that if an isosceles trapezoid can be circumscribed about a circle, then its height is the geometric mean of its bases (Fig. 3.48).

   Solution.
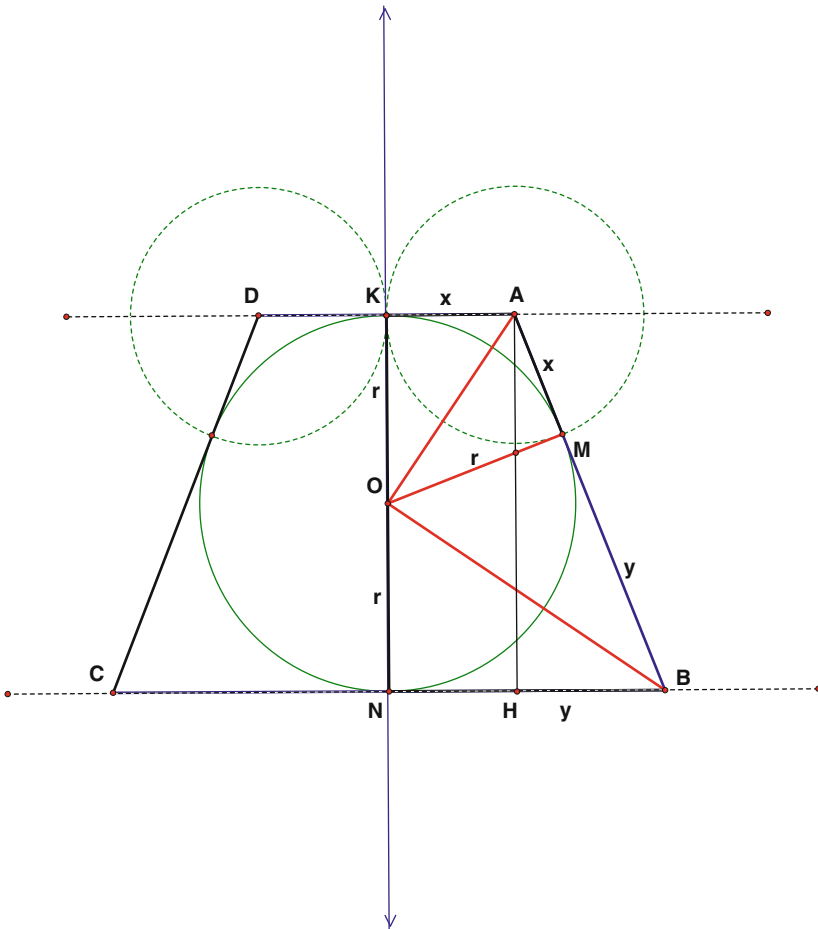
Fig. 3.48 Circumscribed trapezoid problem

Denote $|AD| = a$, $|BC| = b$, $|KA| = x$, then $|DA| = 2x$. Also let $|NB| = y$ so $|CB| = 2y$. $|KO| = r$ (the radius of the inscribed circle), then the height of the trapezoid $|AH| = |KN| = h = 2r$.

Drop a perpendicular OM from the center of the circle O to the side AB. Then triangles OMA and OMB are similar and the ratio of the corresponding sides is the same:

$$\frac{|OM|}{|MB|} = \frac{|AM|}{|OM|}$$
$$\frac{r}{y} = \frac{x}{r}$$
$$r^2 = xy$$

Since $h = 2r$ and $a = 2y$, and $b = 2x$, then multiplying both sides of the last relationship by 4 and after the substitution we obtain

$$4r^2 = 4xy$$
$$(2r)^2 = 2x \cdot 2y$$
$$h^2 = a \cdot b$$
$$h = \sqrt{ab}$$

We got the proof.

2. Given triangle ABC, such that its altitudes BL and CK intersect at O. Let line AO intersect the opposite side at point M. Prove that $AM \perp BC$. (Archimedes Problem.)

   Hint: Use cyclic quadrilaterals.

3. Two circles of the same radius R intersect each other so that the distance between their centers equals R as well. In a figure formed by the intersection of two circles a square is inscribed. Find the side of the square.

   Answer. $\frac{\sqrt{7} - 1}{2} \cdot R$.

4. A circle is inscribed into a triangle ABC, such that $AB = 4$, $BC = 2$, and $AC = 3$. Find the area of triangle AMN, where M and N are tangent points of the circle with sides AB and AC, respectively.

   Answer. $\frac{25\sqrt{15}}{64}$.

5. Four congruent circles are placed in a 4×4 square so that each is tangent to two sides of the square and to two sides of the other circles. A smaller fifth circle is drawn tangent to each of the four circles, as shown by Fig. 3.49. Find its radius.
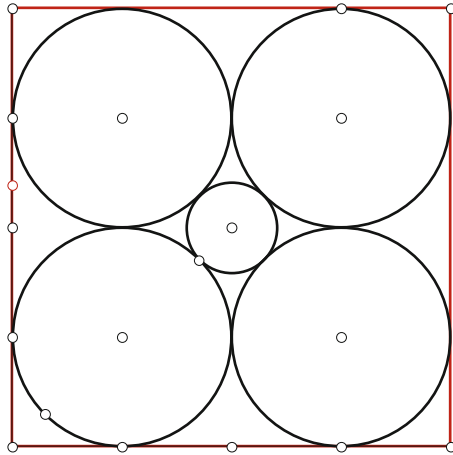
Fig. 3.49 Problem on five circles
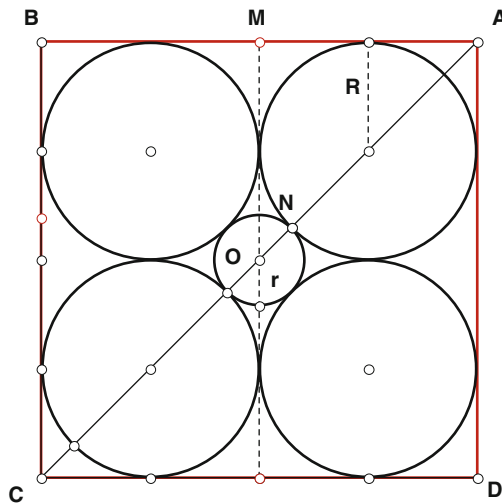
Fig. 3.50 Second sketch of five circles

Answer. $r = \sqrt{2} - 1$.

Solution. In Fig. 3.50, we have that $ABCD$ is a square, M is the midpoint of side AB, O is the center of the small circle, N is the tangent point of the small and the big circle, r is the radius of the small circle, and R is the radius of the big circle.

By the Pythagorean Theorem with $AB = 4$, then $AC = 4\sqrt{2}$ and radius $OA = \frac{1}{2}AC = 2\sqrt{2}$. Also, $4R = AC$ so $R = 1$. In terms of the radii of the circles, $OA = ON + NA = r + R + R\sqrt{2} = r + 1 + \sqrt{2}$. So OA, $r = \sqrt{2} - 1$.

6. Prove that the product of the diagonals of the quadrilateral inscribed in a circle equals the sum of the product of its sides

7. A circle of radius r is inscribed in the sector of a circle of radius R. The chord of the sector equals $2a$. Prove that $\dfrac{1}{r} = \dfrac{1}{R} + \dfrac{1}{a}$

8. Prove that internal bisectors of angles A, B, C, and D of quadrilateral $ABCD$ intersect at four points on a circle.

9. Prove that the distance from a point on a circle to a chord of the circle is a geometric mean of the distances between the end points of the chord and the tangent to the circle that goes through this point.

   Proof.

   A. If a chord is parallel to the tangent then the statement is true.
   B. Assume that chord is not parallel to the tangent. Then extension of the chord and tangent will intersect at point S. Let MN be the distance from the tangent point M to the chord and let AK and BF be the distances from the ends of the chord to the tangent. We need to prove that $MN = \sqrt{BF \cdot AK}$.

   Let us draw a picture of the situation (Fig. 3.51).
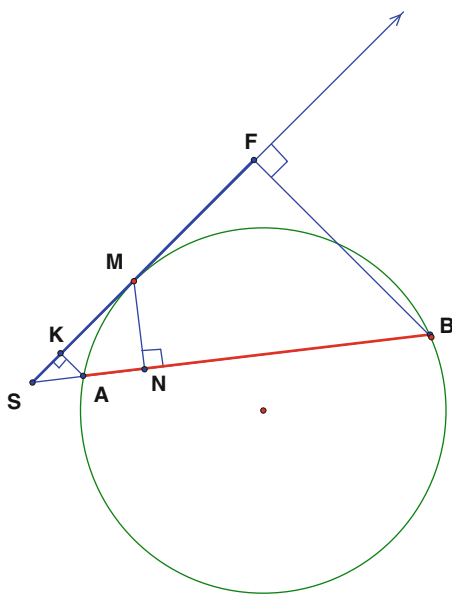
Fig. 3.51 Distance from a point on a circle to a chord of the circle

Triangles SKA and SFB are similar, so $\dfrac{AS}{SB} = \dfrac{AK}{BF}$.

Triangles SAK and SMN are similar so $\dfrac{AS}{SM} = \dfrac{AK}{MN}$ or $MN = \dfrac{SM}{AS} AK$.

By the property of the tangent and secant dropped to a circle from the same point we can write $SM = \sqrt{SA \cdot SB}$ from which we obtain that

$$MN = \frac{SM}{AS} \cdot AK = \sqrt{\frac{SB}{AK}} AK = \sqrt{BF \cdot AK}$$

10. Given a scalene triangle ABC, prove that the point of intersection of the bisector of angle A and the perpendicular dropped from the midpoint of side BC lies on the circumscribed circle of triangle ABC.

   Proof. Let point D be the intersection of the angle bisector (of angle BAC) of triangle ABC with the circle (Fig. 3.52).
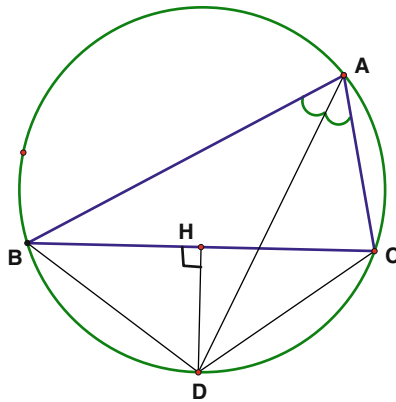
Fig. 3.52 A homework problem on an cyclic quadrilateral

   Since angles BAD and CAD are equal, $BD = CD$ and the triangle BDC is isosceles. Hence, the perpendicular dropped from point D to the segment BC divides this segment into two equal parts (bisects it).

11. Prove that the sum of the legs of a right triangle is equal to the sum of diameters of the inscribed and circumscribed circles.

12. Two unequal trapezoids with parallel corresponding sides are inscribed into the same circle. Prove that their diagonals are equal.

13. The tangent to a circle at point K is parallel to chord LM of the circle. It is known that $LM = 6$ and $KM = 5$. Find the radius of the circle.

   Answer. 25/8.

14. Find the area of a pentagon $ABCDE$ inscribed in a circle with radius 1 if $AB = \sqrt{2}$, $\angle ABE = \dfrac{\pi}{4}$, $\angle EBD = \dfrac{\pi}{6}$, and $BC = CD$.

Hint: Note that the triangle ABE is also inscribed in the same circle. Using formula (1.27) for its area we can find that $\sin \angle AEB = \dfrac{AB}{2R} = \dfrac{1}{\sqrt{2}}$ so $\angle AEB = \dfrac{\pi}{4}$. Then we conclude that triangle BAE is the right isosceles triangle with BE to be the diameter of the circle.

Answer. $1 + \dfrac{3}{4}\sqrt{3}$.

15. It is known that the area of a convex quadrilateral, $ABCD$, is equal to the arithmetic mean of the products $AB \cdot CD$ and $AD \cdot BC$. Find the length of segment CD if $|BC| = 4$, $\angle ADC = \dfrac{\pi}{3}$, and $\angle BAD = \dfrac{\pi}{2}$.

    Answer. $4\sqrt{3}$.

16. Consider a trapezoid $ABCD$ where $BC \parallel AD$ and side AB is perpendicular to the bases. A circle, for which $|AB|$ is the diameter, intersects side CD in two points that divide the side in ratio 2:1:3, starting from the vertex C. Find the acute angle of the trapezoid.

    Hint. Use theorem on tangent and secant.

    Answer. $\alpha = \arccos\left(\dfrac{\sqrt{2} - 1}{\sqrt{6}}\right)$.

17. Chords AP and CQ of the circle circumscribed about acute triangle ABC contain its heights dropped from vertices A and C. Find the radius of the circle circumscribed about triangle PLQ where L is the point of the intersection of chords AP and CQ, $\angle BAC = 65°$, $\angle ACB = 70°$, and $PQ = 2\sqrt{2}$.

    Solution. Let M and N be the feet of the heights of the triangle ABC dropped from vertex A and C, respectively. These feet belong to the sides of the triangle because the triangle is acute (see Fig. 3.53).
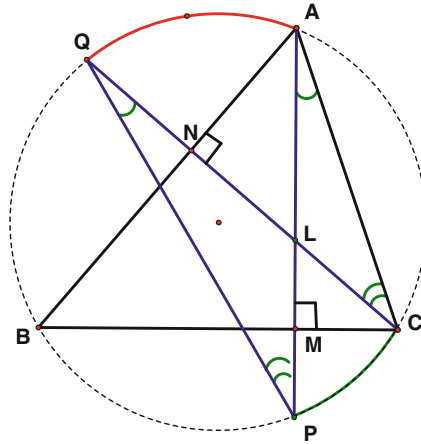
Fig. 3.53 Homework problem on an inscribed triangle

CAN and ACM are right triangles so $\angle ACQ = 25°$ and $\angle CAP = 20°$. From the inscribed angle theorem, it follows that

$$\angle APQ = ACQ = 25°, \quad \angle CQP = \angle CAP = 20° \Rightarrow$$
$$\angle QLP = 180° - 25° - 20° = 135°$$

Finally, applying the Law of Sines to the triangle PLQ, we can find the radius of its circumscribed circle

$$R = \frac{PQ}{2\sin \angle PLQ} = \frac{2\sqrt{2}}{2\sin 135°} = \frac{\sqrt{2}}{\sin 45°} = 2$$

Answer. $R = 2$.

18. Triangle ABC is inscribed in a circle. A tangent dropped from vertex C of the triangle intersects the extension of side AB from vertex B at point D. It is known that $\angle CDA + \angle ACB = 2\angle BAC$. Find the angle between tangent and chord CB.

Solution. Denote $\alpha = \angle BAC$ and $\beta = \angle ABC$ (Fig. 3.54). The angle between tangent CD and chord CB equals the inscribed angle BAC: $\angle BCD = \alpha$. Also, in triangle ACD we note that $\angle CAD = \alpha$ and $\angle ACD = 180° - \beta$ so $\angle CDA = \beta - \alpha$.
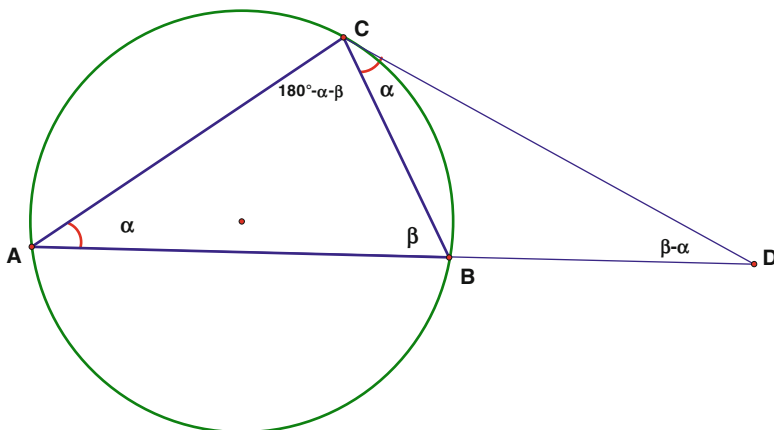
Fig. 3.54 Problem on finding the angle between a tangent and chord

Let us use the given relationship between angles: $\angle CDA + \angle ACB = 2\angle BAC$ it can be written as

$$(\beta - \alpha) + (180° - \alpha - \beta) = 2\alpha$$
$$\alpha = 45°$$

Answer. $\alpha = 45°$.

19. A quadrilateral *PQRS* is inscribed in a circle. Diagonals *PR* and *QS* are perpendicular to one another and intersect at a point *M*. It is known that $PS = 13$, $QM = 10$. And $QR = 26$. Evaluate the area of the quadrilateral *PQRS*.

Answer. 319.

Solution. Let us draw a quadrilateral with perpendicular diagonals that is inscribed into a circle (Fig. 3.55).
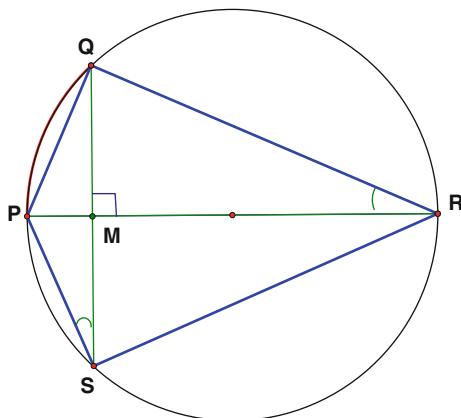
Fig. 3.55 Cyclic quadrilateral with perpendicular diagonals

From the right triangle QMR we obtain that $MR = \sqrt{26^2 - 10^2} = 24$. Since angles QSP and PRQ support the same arc, they are equal and the following is true.

$$\angle QSP = \angle PRQ$$
$$\triangle SPM \sim \triangle RQM$$
$$\frac{PM}{QM} = \frac{MS}{MR} = \frac{PS}{QR} = \frac{1}{2}$$
$$PM = 5, SM = 12$$
$$[PQRS] = \frac{1}{2}PR \cdot QM + \frac{1}{2}PR \cdot MS = 319$$

20. A trapezoid with base $a = \sqrt{8}$ and height $h = \sqrt{3} + \sqrt{2}\,h$ is inscribed in a circle with radius $R = \sqrt{5}$. The center of the circle is inside the trapezoid. Each of the four segments of the circle cut by the sides of the trapezoid is reflected with respect to the cutting side. Find the area of the geometric figure that is formed by the interior points of the trapezoid that do not belong to any of the cutting segments.

Hint. See Problem 52 and show that $KBCL$ of Fig. 3.56 is a square. The area of the figure equals the area of this square minus the four segment areas.

Answer. $S = 8 + 4\sqrt{6} - 10\arcsin\left(\frac{2\sqrt{6}}{5}\right)$.
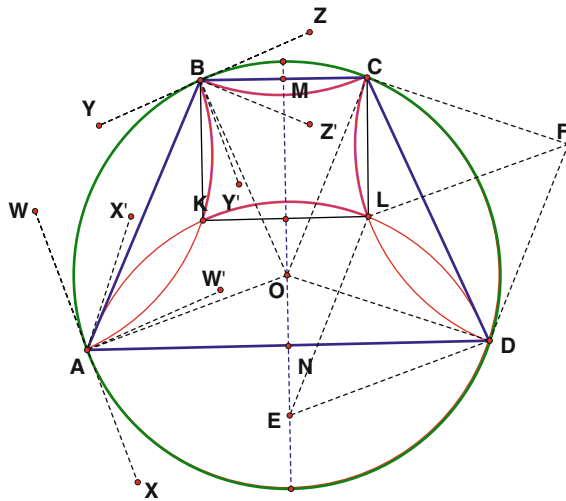
Fig. 3.56 Illustration for Homework Problem 20

Chapter 4
Problems on Construction

In this section we will make our constructions as it was done by the Greeks, using only an unmarked rulers and a compass. Our constructions will be based on two Euclidean postulates of plane geometry:

1. There is a unique line through two distinct points.
2. Given a point and a segment length, a circle can be contracted with the given point as a center and the given length as a radius.

Ancient Greeks were able to construct many things using only straightedges and compasses. However, they were unsuccessful in solving some problems such as trisecting an angle, constructing a regular heptagon or nonagon, and duplicating a cube. The answers to these problems that were unsolved for thousands of years came with the development of modern mathematics such as analytical geometry, algebra, and number theory. Mathematicians who greatly contributed to unsolved problems were Vieta, Fermat, Descartes, Euler, and Gauss. Ancient constructions are still the basis of Euclidean geometry. In this section we will do some of them and discuss why some construction problems are unsolvable and how modern development in the field explains this fact.

4.1 Angles

4.1.1 Construction of Angles

Problem 67. Can you construct an angle of 30°?

If yes, then do it right away. There are several ways to do it. I will show one of them here. Suppose you have a right triangle with angles 30°, 60°, and 90°. Then for this triangle, the side opposite to the 30° angle is half of the hypotenuse. This fact can

be explained by the Law of Sines for the right triangle. Remember that the length of a leg of a right triangle equals a product of its hypotenuse and the sine of the opposite angle. Thus, for the Fig. 4.1 we have that $a = c \sin(\angle CAB) = c \sin 30° = \frac{c}{2}$.
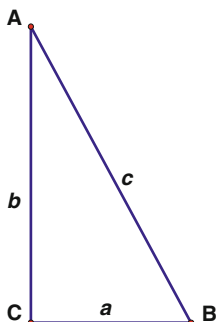
Fig. 4.1   Angle of 30°

Therefore, the construction of a 30° angle can be reduced to the construction of a right triangle with the length of one of the legs equal to half of that of the hypotenuse. The next step is very important: if AB is a diameter of a circle then any triangle with a vertex on the circle will be a right triangle (by Thales' Theorem). We need a unique one such that its side equals half of the hypotenuse. To do this we just intersect the circle by another circle of the same radius with its center at B (or A) (Fig. 4.2).
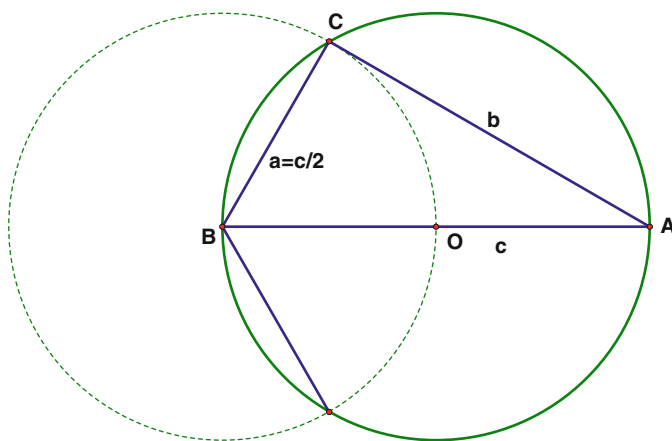
Fig. 4.2   Construction of 30° angle

One such problem is offered in this section for your consideration.

Problem 68. Construct an angle of 36°.

Solution. 36° is not 30° and we need to think of some fresh ideas.
Method 1: Combination of Geometric and Algebraic Approaches

Assume that a 36° angle is constructed and it is a vertex angle of isosceles triangle AOB (Fig. 4.3).
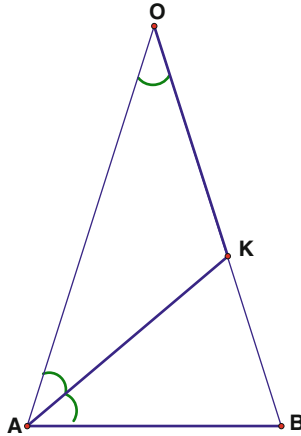
Fig. 4.3 Angle of 36°

Denote $|AO| = |OB| = R$, $|AB| = a$, and $\angle AOB = 36°$. Then the base angles of this triangle equal 72°. Let us drop a bisector AK; then $\angle OAK = \angle BAK = 36°$ and $\angle AKB = 72°$. So $|OK| = |AK| = |AB| = a$ (equal segments are shown in blue). By the angle bisector theorem and from the similarity of triangles AOB and BAK (by the AA property) we obtain the following relationship:

$$\frac{|OA|}{|AB|} = \frac{|OK|}{|BK|}$$
$$\frac{R}{a} = \frac{a}{R-a}$$
$$a^2 + Ra - R^2 = 0$$

$$\tag{4.1}$$

This quadratic equation has only one positive root, $a = \frac{R(\sqrt{5}-1)}{2}$.

Therefore, in order to solve this problem, we can select a segment of length R and construct an isosceles triangle with side R and base $a = \frac{R(\sqrt{5}-1)}{2}$. The vertex

angle will be 36°. Of course, if $R = 1$, then $a = \dfrac{(\sqrt{5} - 1)}{2}$, a **Fibonacci number**!

Hence, our construction can consist of two steps (Fig. 4.4):

1. Construct a segment of length $a = \dfrac{(\sqrt{5} - 1)}{2}$. One approach is to construct a right triangle with legs 1 and 2, so that the hypotenuse will be $\sqrt{5}$. We then draw an arc with center at point B of the hypotenuse and of radius 1. It will cut from the hypotenuse a segment AM of length $\sqrt{5} - 1$. Finally, we will find its midpoint, N. Therefore, $|AN| = |NM| = a = \dfrac{(\sqrt{5} - 1)}{2}$.

2. Consider segment AN as the base of the future isosceles triangle with a vertex angle of 36°. Using the same sketch (Fig. 4.4) draw semicircles of radius 1 from vertices at A and N. The point of the intersection, O, will complete our construction of $\angle AON = 36°$.

Method 2: Trigonometric Approach

This approach is based on the properties of complementary angles, e.g., $\cos \beta = \sin(90° - \beta)$ so $\cos 36° = \sin 54°$ can be rewritten as $\sin 3\alpha = \cos 2\alpha$ where $\alpha = 18°$.

I remember when I was in ninth grade, a similar problem appeared at the city math Olympiad. We had to evaluate $\sin 18°$ precisely. I first rewrote $\sin 3\alpha$ as the sine of the sum of two angles and replaced the right hand side in terms of a sine of single angle:

$$\sin(2\alpha + \alpha) = 1 - 2\sin^2\alpha$$
$$\sin \alpha \cdot \cos 2\alpha + \cos \alpha \cdot \sin 2\alpha = 1 - 2\sin^2\alpha$$
$$\sin \alpha(1 - 2\sin^2\alpha) + \cos \alpha \cdot 2 \sin \alpha \cos \alpha = 1 - 2\sin^2\alpha$$
$$\sin \alpha(1 - 2\sin^2\alpha) + 2 \sin \alpha(1 - \sin^2\alpha) = \sin \alpha(1 - 2\sin^2\alpha)$$
$$4 \sin^3 \alpha - 2\sin^2\alpha - 3 \sin \alpha + 1 = 0$$

This equation can be factored as $(\sin \alpha - 1)(4 \sin^2 \alpha + 2 \sin \alpha - 1) = 0$.
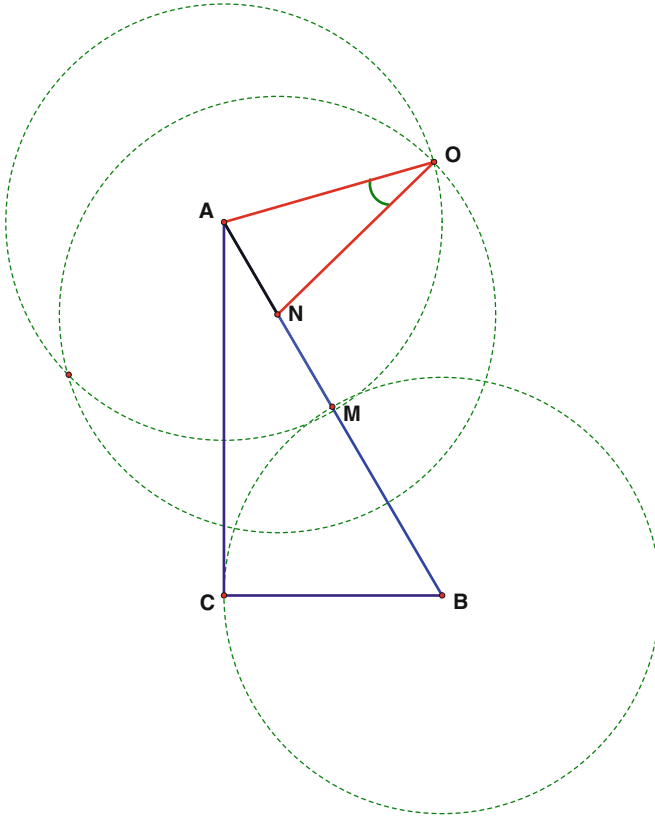
Fig. 4.4 Construction of 36° angle, Method 1

We are not interested in $\sin\alpha = 1$ $(\alpha = 90°)$. Our answer comes from solving the quadratic equation $4\sin^2\alpha + 2\sin\alpha - 1 = 0$ and selecting the root that is less than 1; $\sin\alpha = \sin 18° = \dfrac{\sqrt{5}-1}{4}$.

First we will construct 18° and then we will double it (Fig. 4.5):

1. Construct a right triangle with legs 1 and 2.

2. Construct $\sqrt{5} - 1$ and then $\dfrac{\sqrt{5}-1}{4}$.

3. Construct a right triangle AMO with leg $AM = \dfrac{\sqrt{5}-1}{4}$ opposite to angle of 18° with hypotenuse $OA = 1$.

4. Reflect this triangle over the line passing through OM to obtain the image OMB. Angle BOA is 36°.
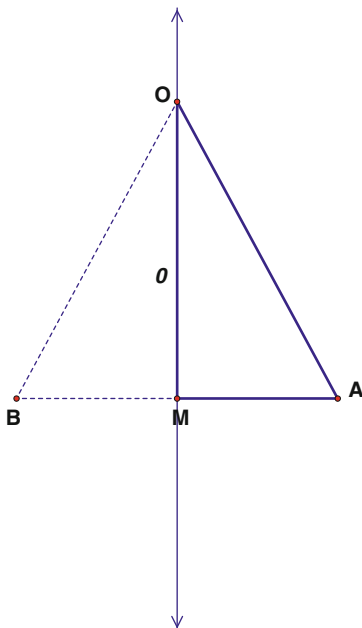
Fig. 4.5 Construction of 36° angle, Method 2

4.1.2 Angle Bisection

> **Problem 69.** Can we bisect any angle (divide it into two equal parts)?

Yes! For example, it can be done by constructing an arc with the center at the angle vertex *A* of any radius *AB*, then by finding the midpoint *D* of the base *BC* of the formed isosceles triangle *ABC*, and then by drawing the ray *AD* passing through the vertex of the angle and the midpoint (see constructions in Fig. 4.6).
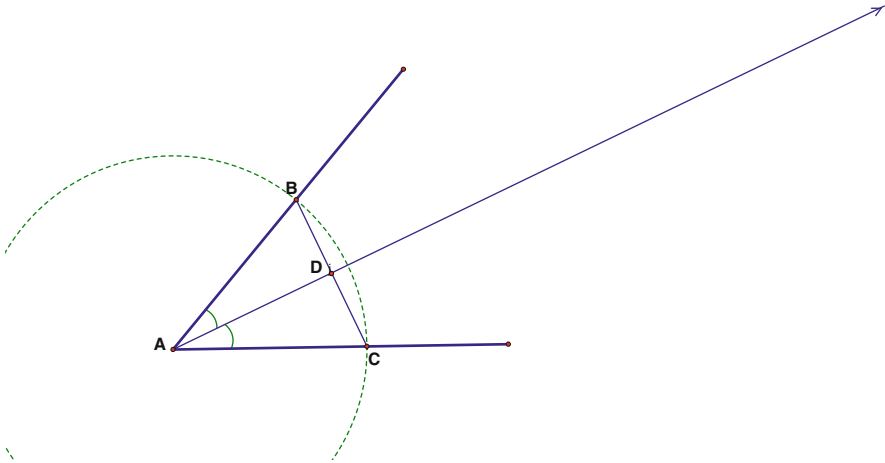
Fig. 4.6 Angle bisection

4.1.3 Angle Trisection

Angle trisection for any given angle is a famous ancient geometry problem and its explanation can be found in many books. As you have already seen in this book, Archimedes proposed the following method of trisection of an angle:

> Let *COD* be the given angle. Construct a circle with center *O* of any radius. Construct a line through *C* intersecting the diameter *FD* extended from *OD* so that |*AB*| is equal to the radius of the circle. Then $\angle CAD = \frac{1}{3} \angle COD$ (Fig. 4.7).
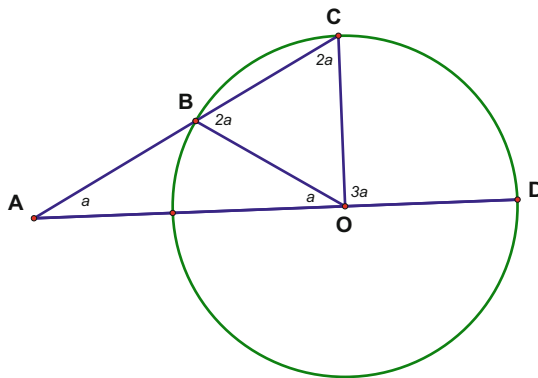
Fig. 4.7 Angle trisection idea by Archimedes

Although there is nothing wrong in this problem algebraically and we proved that by solving a corresponding problem in the beginning of the book, this method of an angle trisection is not valid from the point of view of construction with straightedge

and compass rules: In order to construct the line ABC so that $AB = r$ we would require placing two marks on the straightedge (which should be unmarked).

Continuing Bisecting is another ingenious procedure for trisecting an angle. The idea of it follows:

Let AOB be the given angle and OC be the bisector of angle AOB, OD the bisector of angle COB, etc., i.e., continue bisecting the angle formed by OB and the last angle bisector drawn. We have the following fractional parts of angle

$AOB : \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \cdots$

We can combine the parts starting with 1/4 with a pair of compasses and a straightedge. Eventually, after infinitely many operations we would get

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{b_1}{1-r} = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$$

Again, although there is nothing wrong with this idea algebraically and an infinite geometric series does converge to 1/3, from a construction point of view, this procedure would require bisection by an infinite number of operations. For good reason, only a finite number of operations with straightedge and compass are permitted.

Fact. There is no procedure using only an unmarked ruler and compass to trisect an arbitrary angle.

Why? Modern algebra gives us the answer to this question. For instance, if 3α is the angle to trisect and if we were to use the trigonometric identity, $\cos 3\alpha = 4\cos^3\alpha -3\cos\alpha$, we might substitute 3α into the left side of the equation and evaluate $\cos 3\alpha$. Letting $\cos 3\alpha = b$ and substituting $x = \cos\alpha$, we have a cubic equation, $4x^3 - 3x - b = 0$. This equation can be solved to give real solutions only for certain values of x (complex solutions have no meaning for $x = \cos\alpha$).

Trisection of an angle is possible if the cubic equation is "reducible." You can also say that an angle 3α can be trisected if $\cos\alpha$ can be constructed. For example, let $3\alpha = 90°$, then $\cos 90° = 0$, and $4x^3 - 3x = 0$. It is clear that $\alpha = 90°$ can be trisected.

Problem 70. Trisect a 90° angle without a protractor (i.e., divide a 90° angle into three equal angles).

A 90° angle can be trisected easily. For example, the following scenario can be offered (Fig. 4.8).

A right angle can be constructed as a perpendicular bisector. Label its vertex as C and take a point B anywhere on one of its sides (for example, on its "horizontal" side).

Next, construct a semicircle with center B and radius equal to double the length of side CB. The intersection with the other side of the angle is marked as A. Now triangle ABC is a special triangle (30°, 60°, and 90° triangle). By dropping a perpendicular from C to AB we will obtain another right triangle, such that angle DCB equals 30°. Thus, the ray CD divides the angle into angles of 30° and 60°. Finally, by bisecting angle ACD we will trisect the given angle ACB.
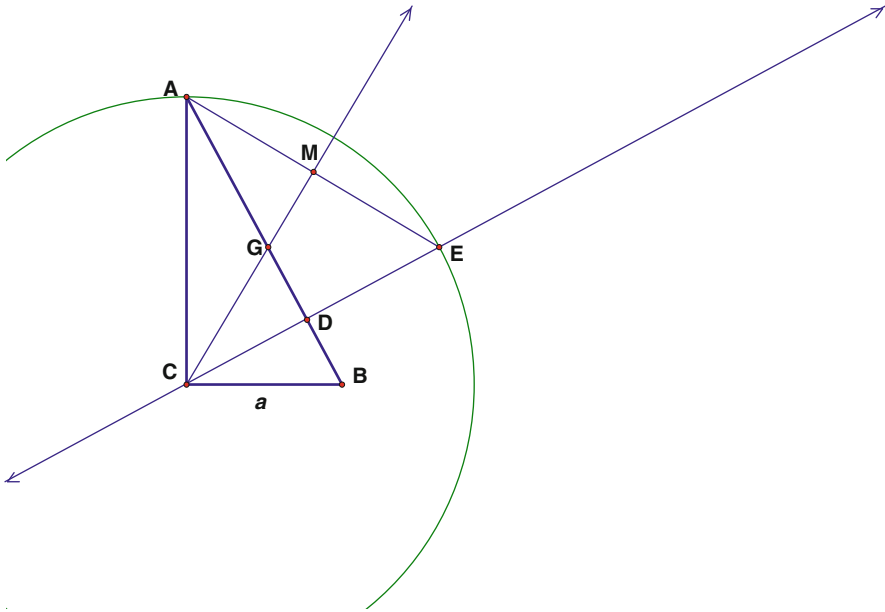

Fig. 4.8 Trisection of the right angle

The angle of Problem 71 can be also trisected. Please prove it in your homework exercise.

Problem 71. Divide an angle of 54° into three equal parts using only a compass.

Solution 1. First, we will construct an angle of $18° = 54°{:}3$. Since $18° = 36° : 2 = \dfrac{90° - 54°}{2}$, it is enough to construct a complementary angle of 36° and divide it by two. Finally, using the constructed angle we will divide the given one into three equal parts.

Solution 2. We will first construct an angle of measurement $18°$. Note that $18° = 180° - 3 \cdot 54°$. Next, we will construct a circle with center O at the vertex of the angle of measurement $54°$ that intersects the sides of the angle at points A and B. Starting from point A and using a compass to mark off the radius of the circle, we progress counterclockwise along the circumference of the circle, with each mark at $60°$ until point M is reached on the third mark. This point M will be central-symmetric to point A with respect to O. Consequently, we will construct arcs BC and CD equal to arc AB. Thus, arc DM is equal to $18°$.

4.1.4   Challenge: Line to an Unreachable Vertex

In ancient times people were interested in the following construction.

Problem 72.   Construct the line that goes through the interior point K and the vertex of the angle formed by two nonparallel lines n and m, such that the vertex of the angle cannot be reached (for example, it is located outside of the sheet of paper).

Solution 1. Assume that K is the orthocenter of a triangle with sides lying on lines n and m (Fig. 4.9). Such a triangle exists and is unique. The construction is similar to Problem 80 on the construction of a triangle by its two vertices and the orthocenter.

Draw two sides of the given angle with an unattainable vertex. Put point K somewhere between two line segments.
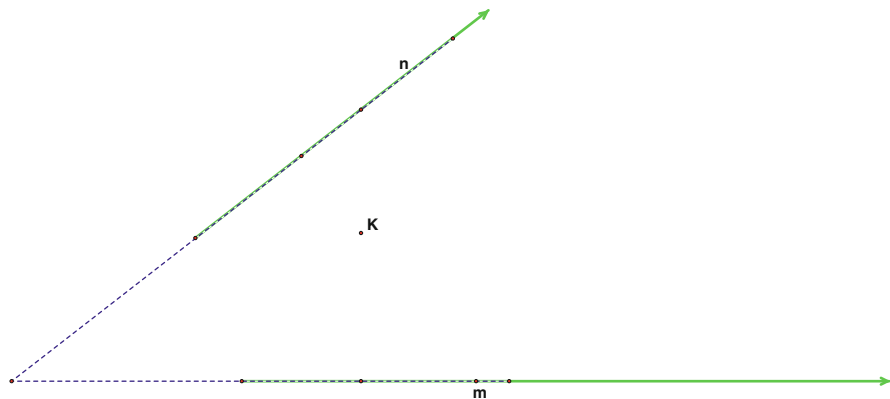
Fig. 4.9   A point between two nonparallel lines

If *K* is the orthocenter of some triangle with two sides lying on *n* and *m*, then we can find two of the vertices by drawing two lines through point *K* perpendicular to *n* and *m*. For example, the line that is perpendicular to line *m* will intersect the line *n* at point *B*. Likewise, the line that is perpendicular to line *n* will intersect line *m* at point *C* (Fig. 4.10).
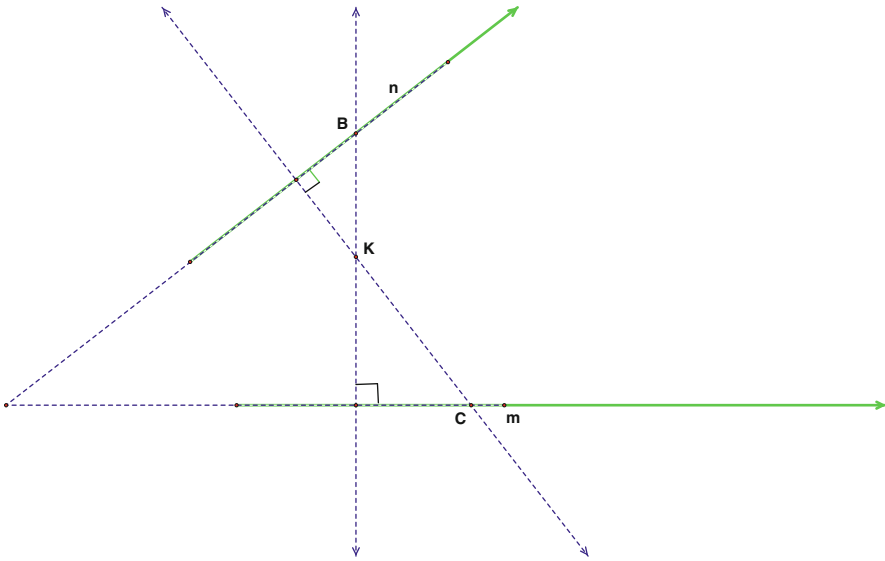
Fig. 4.10   Making a point the orthocenter of a triangle

Connect *B* and *C*. The red line *KL* dropped from *K* and perpendicular to *BC* is the line through vertex *A* that would reach point *A* (Fig. 4.11).
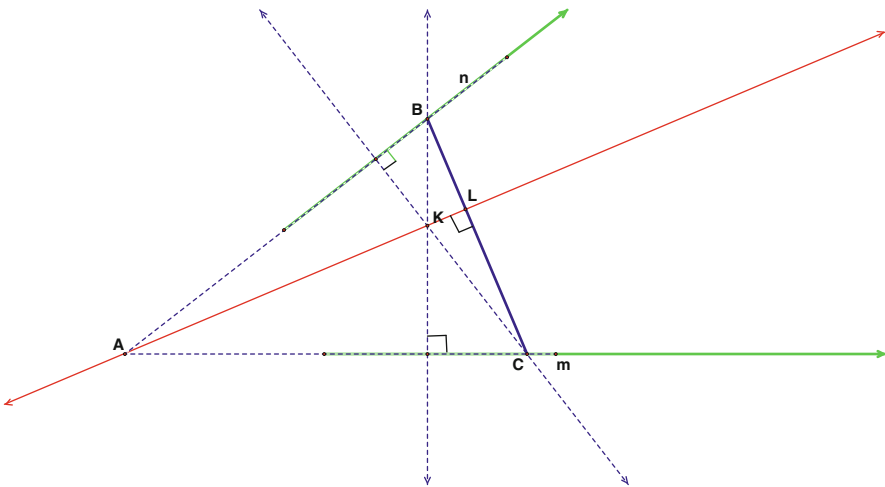
Fig. 4.11   Sketch for Solution 1 of Problem 72

Solution 2. We can use Menelaus' Theorem. Draw two segments *DM* and *EF* through point *K* such that {*D*, *E*} lie on *n* and points {*F*, *M*} on line *m* (Fig. 4.12). Suppose that lines *n* and *m* intersect at point *C* outside of the figure. Let *FD* ∩ *EM* = *Q* and let the line *QT* intersect *n* at *N* and line *m* at *T*. If *DT* ∩ *NF* = *L*, then *L* ∈ (*CK*). Here (*CK*) is the line that goes through points *C* and *K*.
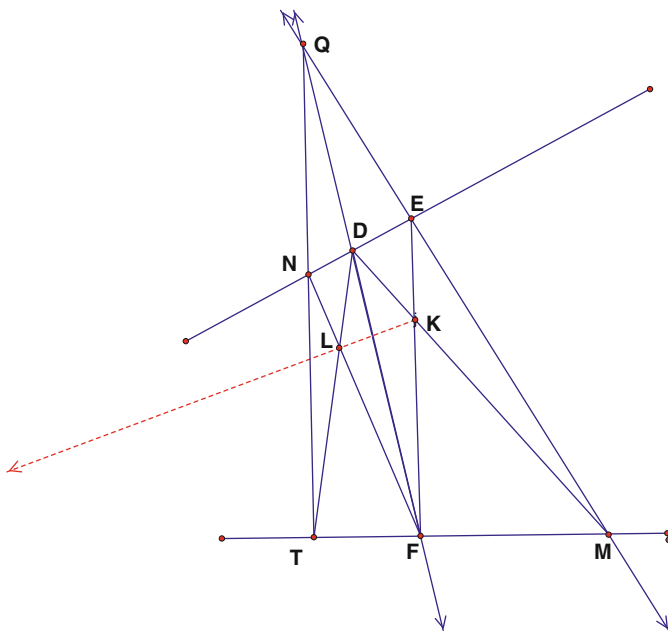
Fig. 4.12 Construction for Solution 2 of Problem 72

Let us prove that $L \in (CK)$

1. Apply Menelaus' Theorem to triangle *QDT*, where (*NF*) is a transversal line:

$$\frac{|QN|}{|NT|} \cdot \frac{|TL|}{|LD|} \cdot \frac{|DF|}{|FQ|} = 1$$

2. Apply Menelaus' Theorem to triangle *QDM*, where (*EF*) is a transversal line:

$$\frac{|QE|}{|EM|} \cdot \frac{|MK|}{|KD|} \cdot \frac{|DF|}{|FQ|} = 1$$

From these two relationships we have,

$$\frac{|QN|}{|NT|} \cdot \frac{|TL|}{|LD|} = \frac{|QE|}{|EM|} \cdot \frac{|MK|}{|KD|}$$

3. Apply Menelaus' Theorem to triangle TQM, where (EN) is a transversal line:

$$\frac{|QN|}{|NT|} \cdot \frac{|TC|}{|CM|} \cdot \frac{|ME|}{|EQ|} = 1 \Rightarrow \frac{|QN|}{|NT|} = \frac{|QE|}{|EM|} \cdot \frac{|MC|}{|CT|}$$

Substituting the second into third we have

$$\frac{|QE|}{|EM|} \cdot \frac{|MC|}{|CT|} \cdot \frac{|TL|}{|LD|} = \frac{|QE|}{|EM|} \cdot \frac{|MK|}{|KD|}$$

$$\frac{|TL|}{|LD|} \cdot \frac{|DK|}{|KM|} \cdot \frac{|MC|}{|CT|} = 1$$

Therefore, (LK) is transversal to triangle TDM and $C \in (LK)$ (Blinkov)

4.2   Segments: Ancient Mathematicians' Construction Problems

When I teach a course in history of mathematics, I always give my students geometry problems that regularly were offered at the Moscow State University oral entrance exams. Some of the problems are on constructions with the use of only ruler, compass, and pencil. For example, the following set is an example of such problems.

Problems 73–76.   If segments of length 1, a, and b are given, then construct the following segments:

73. $\sqrt{a^2 + 1}$, $\sqrt{a^2 + b^2}$, $\sqrt{b^2 + 1}$

74. $\sqrt{ab}$, $\sqrt{a}$, $\sqrt{b}$, $\dfrac{2ab}{a+b}$

75. $\dfrac{a}{b}$, $\dfrac{b}{a}$, $\dfrac{1}{a}$, $\dfrac{1}{b}$

76. ab

Solving Problem 73.   Many students offer the Pythagorean Theorem approach. For example, if we construct right triangles with sides 1 and a, a and b, and 1 and b, then they will have a hypotenuses of lengths $\sqrt{a^2 + 1}$, $\sqrt{a^2 + b^2}$, and $\sqrt{b^2 + 1}$, respectively.

Solving Problem 74. Considering just the first segment of length $\sqrt{ab}$, does it remind you anything, algebraically?

It must remind you a geometric mean of a and b, so the constructions of ancient Babylonians and Greeks can be used! We all know the following inequality between the arithmetic and geometric means

$$\frac{a+b}{2} \geq \sqrt{ab}, \text{ for } a > 0 \text{ and } b > 0.$$

The inequality becomes equality only when $a = b$. For any other a and b an arithmetic mean (AM) is always greater than the geometric mean (GM). This can be easily proved as follows:

$$(a-b)^2 \geq 0$$
$$a^2 - 2ab + b^2 \geq 0$$
$$a^2 + 2ab + b^2 \geq 4ab$$
$$(a+b)^2 \geq 4ab$$
$$a + b \geq 2\sqrt{ab}$$

In 1893 the Russian collector **Golenischev** purchased an Egyptian papyrus which was about 18 feet long and about 3 inches high. From a sample of problems from the papyrus it was clear that the inequality between arithmetic and geometric means was known to ancient Egyptians in 1850 BC. The Egyptians were impressively good at building pyramids and they used a geometric approach for establishing important relationships. They also introduced so-called harmonic mean (HM) and knew that

$$\text{HM} \leq \text{GM} \leq \text{AM} \tag{4.2}$$

Let us demonstrate that the inequality is true using plane geometry. Our approach will be similar to that for a geometric mean in the proof of the Pythagorean Theorem, but we will add to it other constructions.

We will construct a circle with diameter $|BD| + |DC|$ (Fig. 4.13). Let $a = |BD|$, $b = |DC|$ and let A be the point where the perpendicular to BC in D intersects the circle and let E be the foot of the perpendicular from D to the radius AO. Radius of the circle, $|OA| = \dfrac{a+b}{2}$, represents an **arithmetic mean** of lengths a and b. Let us denote $|AD| = h$, $|AE| = g$. Since $\triangle ABD$ and $\triangle CAD$ are similar right triangles,

$$\frac{h}{b} = \frac{a}{h}, \quad \text{then} \quad h = \sqrt{ab}$$

Therefore, $|AD| = h = \sqrt{ab}$, the geometric mean of lengths a and b, is now constructed. If $|BD| = a$ and $|DC| = 1$, then $|AD| = h = \sqrt{a}$.
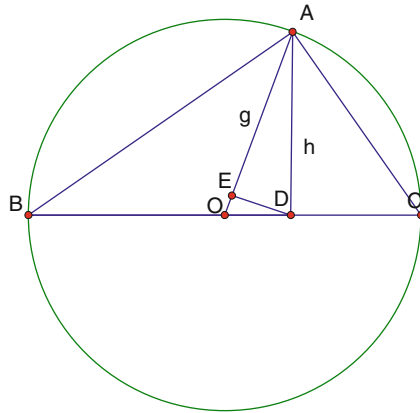
Fig. 4.13 Arithmetic and geometric means on a circle

Also, since *AOD* and *ADE* are similar right triangles, we have

$$\frac{g}{\sqrt{ab}} = \frac{\sqrt{ab}}{\dfrac{a+b}{2}}, \quad \text{then} \quad g = \frac{2ab}{a+b} = \frac{2}{\left(\dfrac{1}{a}+\dfrac{1}{b}\right)}$$

Therefore, segment length *AE* is the harmonic mean. Finally, from geometry we know that in a right triangle, the length of any leg is always smaller than the length of the hypotenuse. Hence, $g \le h \le \dfrac{a+b}{2}$, which can be rewritten as

$$\frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2}$$

Problems 74 and 75 are quite difficult at first, but can be easily solved with the use of Thales' Theorem and the similar triangles approach.

Solving Problem 74. Assume that three segments of length 1, *a*, and *b* are given. Let us show how a segment of length *a/b* can be constructed. First, we will create an angle of any measure. Starting from its vertex, *O*, on one side we will make segment of length *b* (*ON*) and 1 (*NK*). On the other side we will place point *M* so that *OM* = *a*. Next, we will connect points *N* and *M* and then draw a segment through point *K* parallel to *NM*. Therefore, segment *MP* has length *a/b*. Proof of this is simple. Triangles *ONM* and *OMP* are similar and let *MP* = *x* (Fig. 4.14).
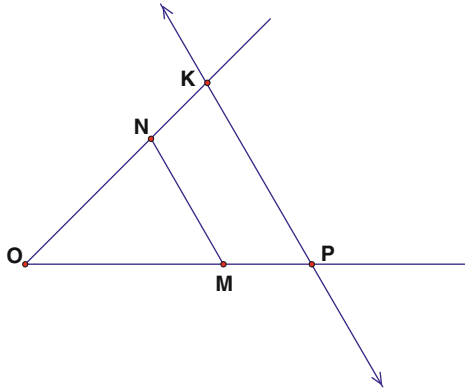
Fig. 4.14 Sketch for Problem 74

Then we have the following relationships:

$$\frac{b}{1+b} = \frac{a}{a+x}$$
$$b(a+x) = a(1+b)$$
$$x = \frac{a}{b}$$

The same can be obtained from the Split Converse Theorem:

$ON = b, NK = 1, OM = a, KP\|NM, MP = x$, then $\dfrac{b}{1} = \dfrac{a}{x}$, $x = \dfrac{a}{b}$

Solving Problem 75. Many of you are trying to solve it using constructions similar to the problem above and you are right! The difference here is only in the order of the segments. We will start on one side with segment of length 1 (ON) followed by segment NK of length a. On the other side of an acute angle we will place point M ($OM = b$). Next, by drawing line through K parallel to segment NM, we will make segment $KP = ab$. In more details: Triangles ONM and OKP are similar (Fig. 4.15):

$$\frac{1}{1+a} = \frac{b}{b+x}$$
$$b + x = b(1+a)$$
$$x = ab$$
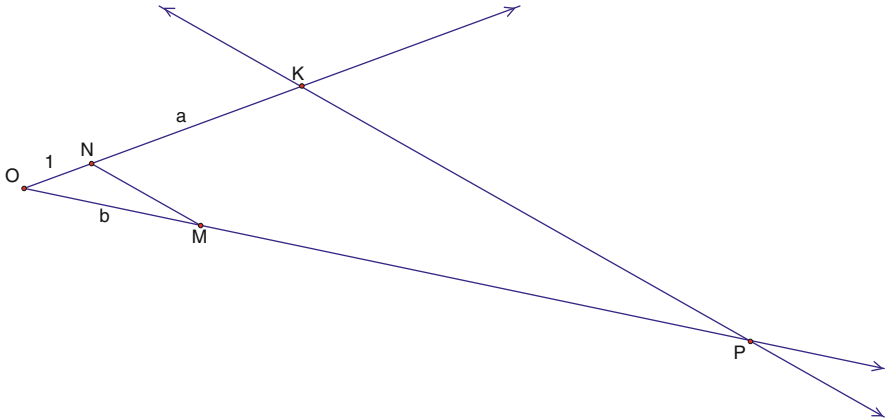
Fig. 4.15 Sketch for Problem 75

Remark. Of course, in order to make a segment of the given length we use compass and draw parallel line using a ruler.

4.3 Dissection, Golden Ratio, and Related Problems

Remark 1. It is interesting that construction of angle of $36°$ is simultaneously the construction of the **golden ratio** of the given segment. The golden ratio of the given segment divides a segment into two parts such that the ratio of the segment to its biggest part is the same as the ratio of the biggest part to the smallest part of the segment.

Consider the ratio (4.1): $\dfrac{R}{a} = \dfrac{a}{R-a}$ and rewrite it as $a^2 = R(R-a)$. The last relationship is equivalent to $a = \sqrt{R(R-a)}$. This equation can be recognized as a geometric mean. Construct the golden ratio of a segment means to place on the segment, AB, a dividing point, G, such that the length of the smaller segment is a geometric mean of the segment and its larger part. Such a point is unique (Fig. 4.16).

$$AB=R,$$
$$AG=R-a,$$
$$GB=a$$

Fig. 4.16 Dissection

The golden ratio is not a trivial problem that cannot be solved differently. We will mention here two ways to do it and want you to think of some other possible ways. In Problem 68 we constructed a 36° angle where point K divides segment OB in a golden ratio. Therefore, if we have an isosceles triangle AOB with vertex angle of 36°, then in order to construct the golden ratio of segment OB it is sufficient to construct the bisector AK of the base angle. The foot of the bisector will divide OB in the golden ratio!

An ancient Greek solution will be given in Problem 77.

Problem 77. Given a segment AB, cut the segment in the golden ratio.

Solution. First, having a square $ABDC$ with side $a = AB$, we will construct a segment BE (E is midpoint of the side AC) as shown in Fig. 4.17.

$$BE = \sqrt{a^2 + \frac{a^2}{4}} = \frac{a\sqrt{5}}{2}$$

Next, we will draw an arc of radius EB with center at E until it intersects AC at F.

$$BE = EF = \frac{a\sqrt{5}}{2}$$

Notice that $AF = AH = \dfrac{a\sqrt{5}}{2} - \dfrac{a}{2} = \dfrac{a(\sqrt{5} - 1)}{2}.$
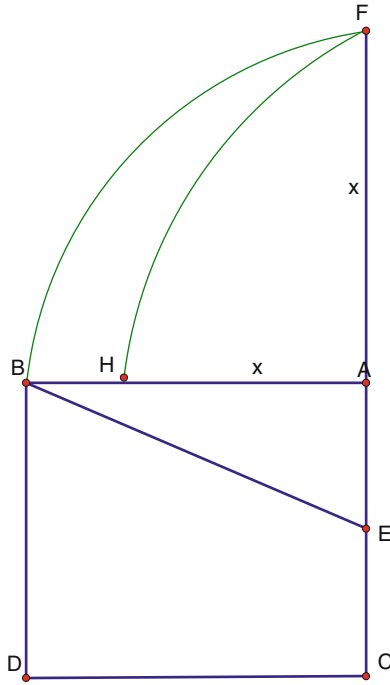
Fig. 4.17 Dividing a segment in the golden ratio

Finally, draw an arc of radius *AF* at center *A* until it intersects *BA* at point *H*. *H* divides segment *AB* in the golden ratio.

The following problem has a very unusual condition and came from ancient Greece.

Problem 78. (Popov) Consider a figure formed by a triangle and an arc as shown in Fig. 4.18. Dissect the figure into equal areas using one line.
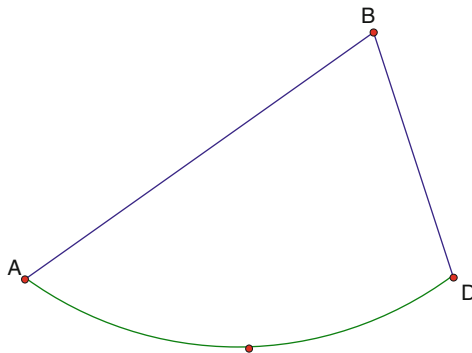
Fig. 4.18 Dissecting a figure into figures of equal areas. First Sketch for Problem 78

Solution. This nontrivial problem came from ancient Greeks. When I offer it to my graduate students in my history of mathematics class, they quickly propose a solution by making two cuts as shown in Fig. 4.19.
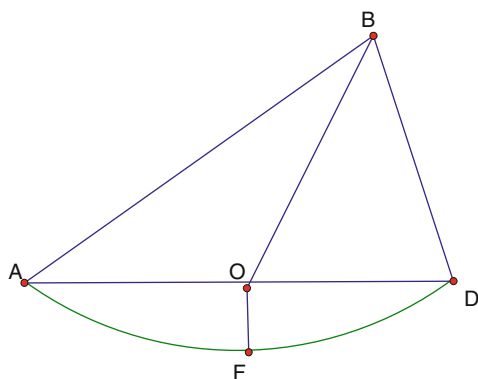
Fig. 4.19 Second sketch for Problem 78

Although this is an incorrect approach, we can use it and Fig. 4.19 to find the correct one. Obviously, if O is the midpoint of AD then the two segments OB and OF divide the figure into two equal ones: $ABOF$ and $BOFD$. Let us keep this in mind.
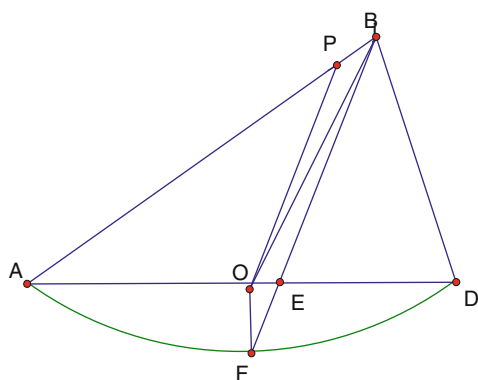
Fig. 4.20 Third sketch for Problem 78

Connect points F and B and then draw segment OP through point O and parallel to FB (Fig. 4.20). Now $FOPB$ is a trapezoid. Next, we will prove that the other diagonal PF is that one required cut!

Proof. Consider the figure *FOBD*; its area is half of the given figure and consists of the area of figure *FBD* and triangle *FOB*. However, the area of triangle *PFB* equals the area of triangle *FOB* (Fig. 4.21). We obtain the following relation:

Half of the given figure = [FBD] + [FOB] = [FBD] + [FPB].

Done!
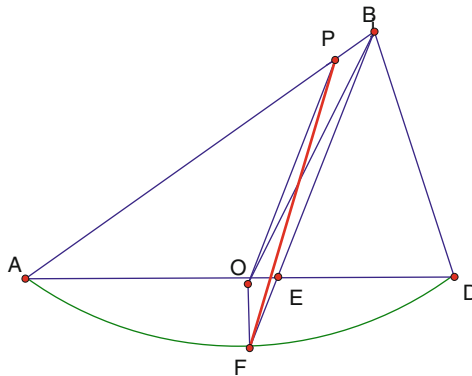
Fig. 4.21 Final sketch for Problem 78

Remark 2. Note that the construction of angles with a given measure closely deals with the construction of regular polygons with n sides with the use of compass and ruler, because such construction can be considered as construction of the central angle of a circle with radius R with the degree measure $\dfrac{360°}{n}$. For example, if we are able to construct an angle of $60°$, then we can easily construct a regular hexagon. If we have an angle of $36°$, then we can construct a regular decagon.

4.4 Polygons

4.4.1 Constructions Involving Polygons

A polygon is inscribed into a shape if all its vertices are on the boundary of the shape. For example, Fig. 4.22 shows a quadrilateral *DEFG* inscribed into triangle *ABC*.
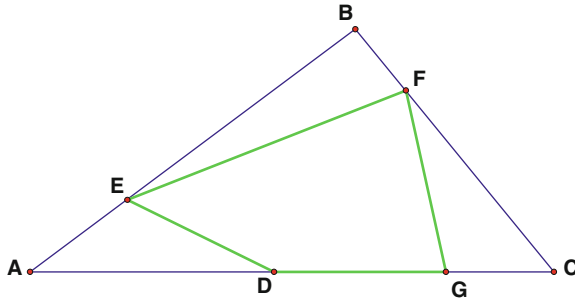
Fig. 4.22 Polygon inscribed into a triangle

Homothetic transformations can help us to solve the following problem.

Problem 79. Inscribe a square in a scalene triangle ABC so that two of its vertices are on its base, AC, and two others are on sides AB and BC.

Solution 1. Since two vertices of the square must belong to the base of the triangle, any square that has two vertices on this base and just one other vertex on side AB will be an image of the square that we have to construct. Here A is the center of homothety. Thus, first we will construct any square that satisfies the condition and then construct its image (Fig. 4.23).

1. Construct a scalene triangle ABC with base AC.
2. Take any point D' on side AB as a vertex of the image square and then drop a perpendicular to side AC. $D'G'$ is the side of the image square.
3. Construct square $D'G'F'E'$. First, draw an arc of radius $G'D'$ from center G' to obtain the third vertex of the square, F', on side AC. Second, obtain the fourth vertex E' of the square as the point of intersection of two lines, parallel $D'E'$ and perpendicular $F'E'$.
4. Find the image E of point E' under homothety at the point of intersection of lines AE' and the side of the triangle BC. One of vertices on the future square, E, is found.
5. Draw parallel and perpendicular lines to the base of the triangle through point E to points of intersection at D and F.
6. Draw a line parallel to EF through D. The point of the intersection, G, with side AC is the last vertex of the constructed square.
7. $DEFG$ is now inscribed in triangle ABC.
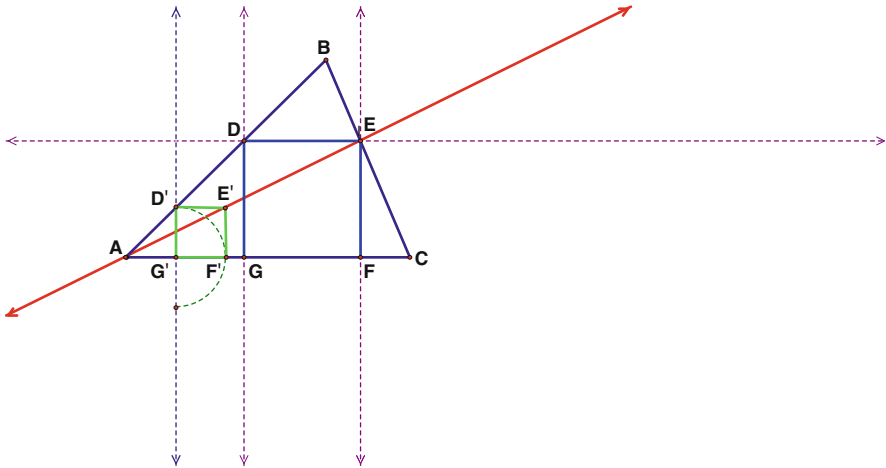
Fig. 4.23 Sketch for Problem 79. Solution 1

Solution 2. The solution was given by **Leonardo Fibonacci** in the *Practica Geometriae* in the case of equilateral triangles. Inscribe a square into an equilateral triangle so the side of the square is on the side of the triangle. This problem can be solved using an algebraic approach.
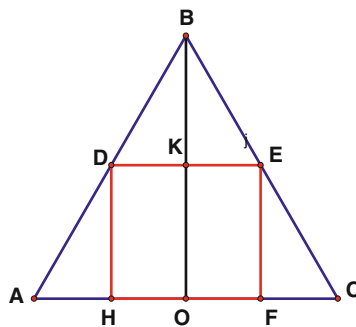
Fig. 4.24 Sketch for Problem 79. Solution 2

Although it seems to be easy, if you try to inscribe a square in a triangle, you most likely get a rectangle rather than a square. So assume first that we have already constructed the square. Given the side of the triangle, a, let us find the length of the side of the square, x. As shown in Fig. 4.24, $x = HF = DE = DH = EF$ and $a = AB = BC = AC$. Drop a perpendicular BO from vertex B to side AC and let K be the intersection of BO with side DE of the square. Triangles DBE and ABC are similar so,

$$\frac{DE}{AC} = \frac{BK}{BO}$$

$$\frac{x}{a} = \frac{\left(\frac{a\sqrt{3}}{2} - x\right)}{\frac{a\sqrt{3}}{2}}$$

$$x = a\left(2\sqrt{3} - 3\right)$$

The next step is to construct a square with this side length. I wish for you to do it yourself. If you need help, please practice by solving similar problems in Sect. 5.

Problem 80. Construct a triangle by its two vertices, A and B, and its orthocenter, H.

Solution. Connect points A and H to draw the line AH as shown in Fig. 4.25. Drop a perpendicular from vertex B to line AH. The point of their intersection, D, is the foot of the altitude from A. Next, connect B and H with the line BH. Drop a perpendicular to this line from A. The point of their intersection is E. Then BE is the second altitude of the triangle.
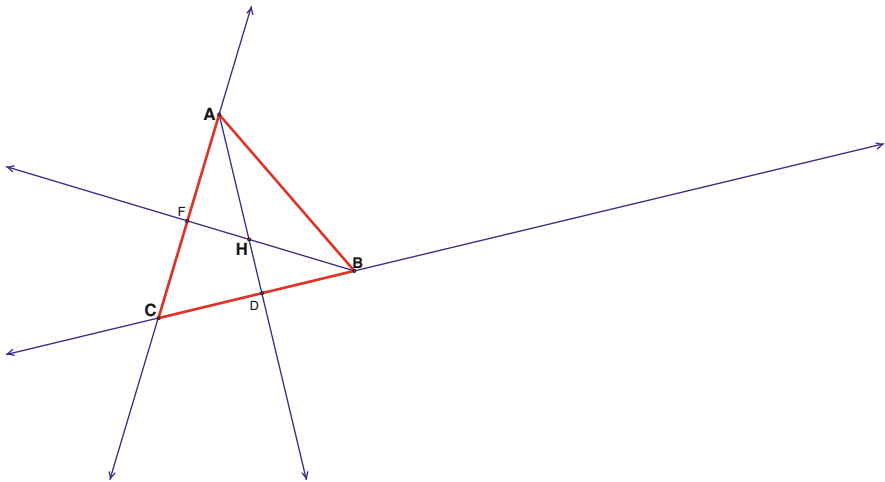
Fig. 4.25 Sketch for Problem 80

4.4.2 Construction of a Regular Polygon

Every regular polygon is cyclic, so it can be inscribed into a circle. In order to construct a regular n-polygon, we need to construct a circle of radius R and a central angle of the circle with measurement $\dfrac{360°}{n}$. Thus, to construct a square, we can create a circle and two perpendicular diameters of it. To construct a regular hexagon, we can use the fact that its side equals the radius of the circle and then make the entire construction with compass only as it is shown in Fig. 4.26. Here $OB = OA = R$, $\measuredangle\, AOB = \dfrac{360°}{6} = 60°$.
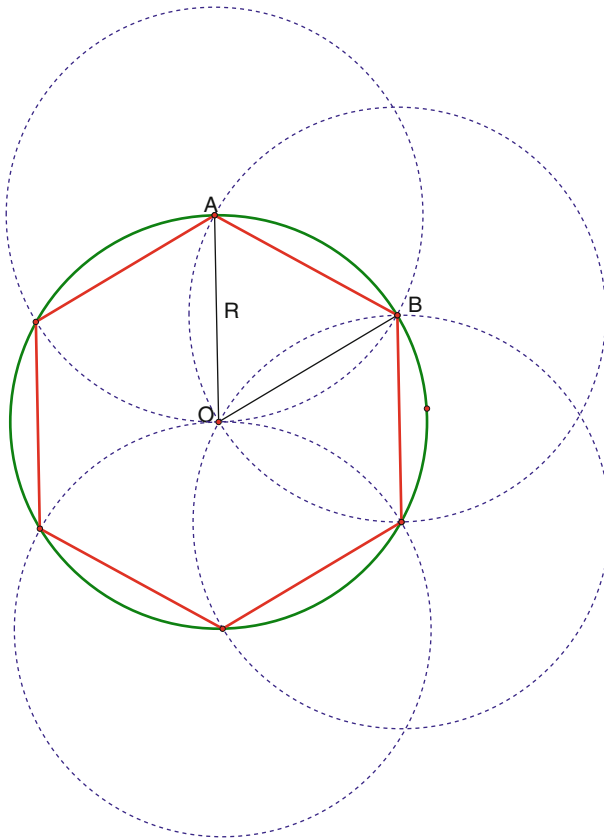
Fig. 4.26 Construction of regular hexagon

Next, let us recall that the side of the regular 10-gon in terms of the radius R of the circumscribed circle can be written as $a = \dfrac{R(\sqrt{5}-1)}{2}$ which means that we can construct a regular pentagon, regular 20-gon, etc.

4.4.2.1 Construction of a Regular Pentagon and Decagon

Problem 81. Construct a side of a regular pentagon and decagon using the same circle and the same sketch.
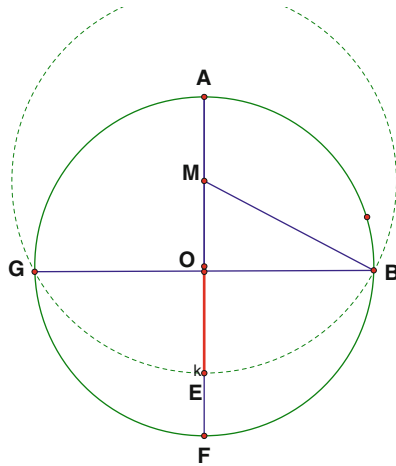
Fig. 4.27 Construction of a regular pentagon and decagon

Consider a circle of radius R and center O with two diameters AF and GB perpendicular to each other (Fig. 4.27). Divide segment AO at midpoint M into two equal parts, then $MB = \sqrt{R^2 + \left(\dfrac{R}{2}\right)^2} = \dfrac{R\sqrt{5}}{2}$. Then by constructing an arc with center M and radius MB until it intersects segment AF at point E, we obtain that $OE = ME - MO = MB - MO = \dfrac{R(\sqrt{5} - 1)}{2}$. Therefore, OE is the side of the regular decagon and BE is the side of the regular pentagon inscribed in the same circle of radius R (Fig. 4.28).

Let us prove that BE is the side of the regular pentagon inscribed into the same circle. We have that $a_5 = 2a_{10}\cos 18°$. From the isosceles triangle AOB below, we can easily obtain some trigonometric functions of $18°$ angles (Fig. 4.29).
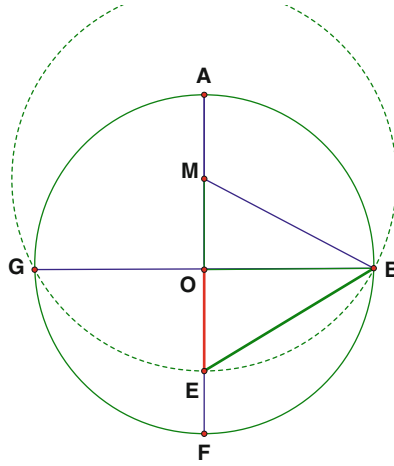
Fig. 4.28 Side of a regular pentagon and decagon

$$\sin 18° = \frac{0.5a}{R} = \frac{\sqrt{5}-1}{4}, \quad \cos 18° = \frac{\sqrt{2(5+\sqrt{5})}}{4} \quad \text{so} \quad a_5 = \frac{R\sqrt{2(5-\sqrt{5})}}{2}$$
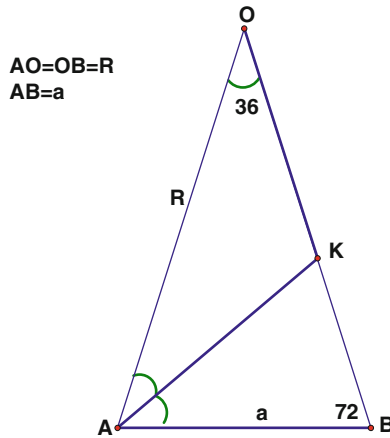
Fig. 4.29 Illustration for Problem 81

From the right triangle BOE we have $BE = \sqrt{OB^2 + OE^2} = \dfrac{R\sqrt{2(5-\sqrt{5})}}{2}$.

4.5   Homework on Construction

1. Prove that a 54° angle can be trisected using straightedge and compass.

   In Problems 2–10, given segments of unit length, segment of length a, and segment of length b, construct the following (solutions can vary):

2. $a + b$.
3. $a - b$.
4. $\sqrt{a}$.
5. a^2.
6. $\dfrac{a^2}{b}$.
7. $\sqrt[4]{a}$.

Hint. Use $\sqrt{a}$ and 1 as segments for diameter.

8. $\sqrt[8]{a}$.
9. $a^2 - b^2$
10. $(a + b)^2$
11. Place a point on segment AB such that it will divide the segment by the golden ratio (Fig. 4.30).

B   H   A

Fig. 4.30   Point on a segment

Solution. Let us start from the analysis. Assume that H is that point, then the following relationship must hold: $AH^2 = AB \cdot BH$. Solution of this problem is not obvious; it probably took some hours for Greeks to find it. Here it is. On side AB draw a square $ABCD$, such that $AB \parallel CD$ and $AC \parallel DB$. Then from the midpoint, E, of side AC draw an arc of radius EB with center E until it intersects the continuation of segment AC at point F. At last, draw an arc from point A as a center of radius AF until it intersects side AB at point H. Therefore, H is the required point (Fig. 4.31).
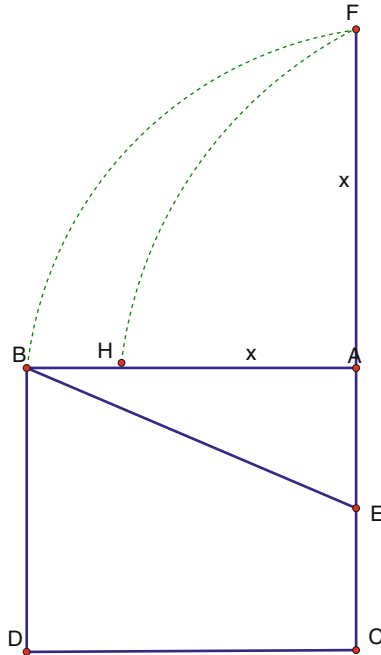
Fig. 4.31 Golden ration of segment AB

Next, the proof of this ancient problem will be done using a modern algebraic approach. Let $HA = x$, $BA = a$, $BH = a - x$, $ABCD$ be a square, and E be the midpoint of AC.

$$BE^2 = a^2 + \left(\frac{a}{2}\right)^2$$
$$FE = x + \frac{a}{2}$$
$$BE = FE$$
$$a^2 + \left(\frac{a}{2}\right)^2 = \left(x + \frac{a}{2}\right)^2$$
$$a^2 + \frac{a^2}{4} = x^2 + xa + \frac{a^2}{4}$$
$$x^2 = a(a - x)$$

Here a is the length of segment AB and $(a - x)$ is the length of the longest segment.

12. Given a circle and a point A outside the circle. Construct a tangent from A to the circle.
13. Given a triangle ABC. Inscribe a circle into the triangle.
14. Given a triangle ABC circumscribe a circle about the triangle.

15. Given a quadrilateral *ABCD* determine if a circle can be inscribed in it. Can a circle be circumscribed about it? Inscribe or circumscribe a circle (or both) if possible.

16. Inscribe a square into a sector of a circle.

    Solution. Let *O* be the center of the sector. Construct a square *ABCD*, two vertices of which (*A* and *D*) are on the radius of the sector at the equal distance from point *O*. Draw lines *OB* and *OC* which intersect the arc of the sector at points *B'* and *C'*. Further, we will construct $B'A' \perp B'C'$ and $C'D' \perp B'C'$. Quadrilaterals *ABCD* and *A'B'C'D'* are similar (one is obtained from the other one by means of **homothety** with center *O*). Hence, *A'B'C'D'* is a square, but it is also inscribed in the sector (Fig. 4.32).
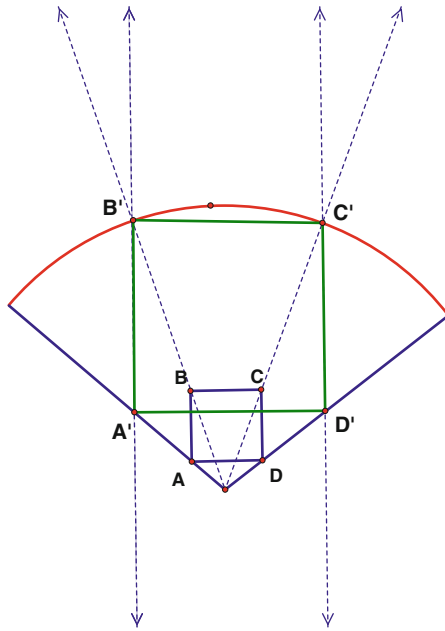
Fig. 4.32 Square inscribed into a sector

17. Construct a circle that is tangent to the given angle.

    Solution. Bisect the angle by placing a point on the angle bisector. It will be the circle center. Drop perpendiculars from the center to both sides of the angle. Connect the center of the circle with the feet of the perpendicular to obtain the radius of the circle. We can now draw a circle using a compass. Note: many inscribed circles can be constructed.

18. Construct a circle that is tangent to the given angle and that passes through a given point of the angle interior.

Solution. Start as in the previous problem and construct an auxiliary circle with center O' inscribed in the angle. Next, connect the vertex of the angle B with the given point A in the interior of the angle. Let point A' be the point of the intersection of line BA with the auxiliary circle. Draw a line parallel to segment $A'O'$ through point A so that it will intersect the ray BO' at point O, the center of the required circle (Fig. 4.33).
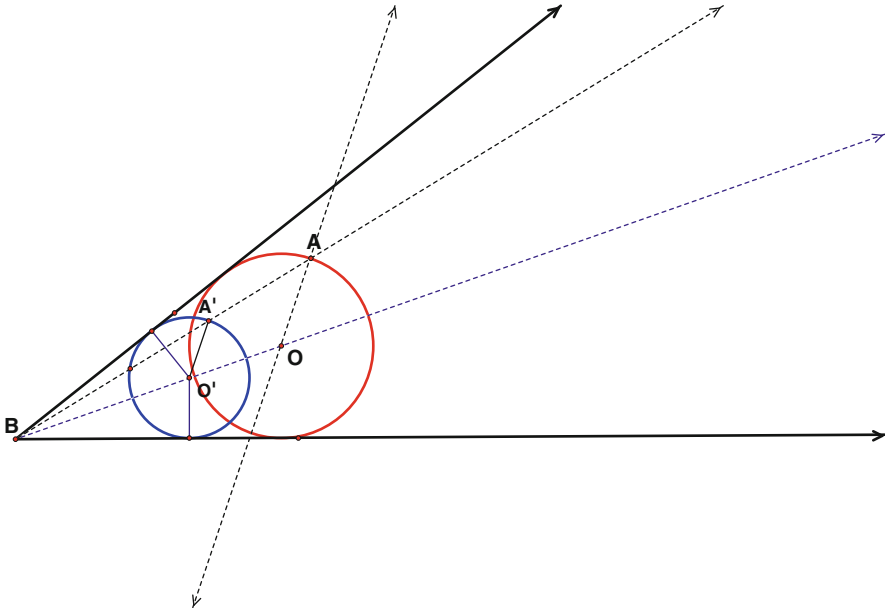
Fig. 4.33 Construction of a circle tangent to a given angle

Remark. If we use another point of intersection of the ray BA with the first circle, we would be able to draw the second circle passing through point A and tangent to the sides of the given angle.

19. Given segment $|MN| = \sqrt{5}$, construct a segment of length 2.

    Solution. Note that there are several possible solutions. This one uses similar triangles.

    Construct a right triangle with legs $AB = a$ and $BC = 2a$, where a is some arbitrary segment. Then the length of the hypotenuse equals $|AC| = a\sqrt{5}$. Place point D on the hypotenuse using a compass such that $|CD| = |MN|$. Drop a perpendicular DH from point D to side BC. Triangles DHC and ABC are similar so $|CH| = 2$.

20. (Budak). Points A and B lie on a circle. Draw a chord XY of the circle and parallel to a given line m such that the sum of chords AX and BY equals the length of a given segment n.

Solution. Connect points *A* and *B* (Fig. 4.34). Assume that *X* and *Y* are the points of interest. Construct chord *AA′* parallel to the given direction *m*. Segment $A'Y = AX$ as the sides of the inscribed isosceles trapezoid *AA′YX*. Therefore, we need to construct point *Y* such that $BY + YA' = n$.

m $\overline{AX}$= 2.48 cm
m $\overline{YA'}$= 2.48 cm
m $\overline{YB}$ = 5.80 cm
m $\overline{YA'}$= 2.48 cm
n = 8.28 cm
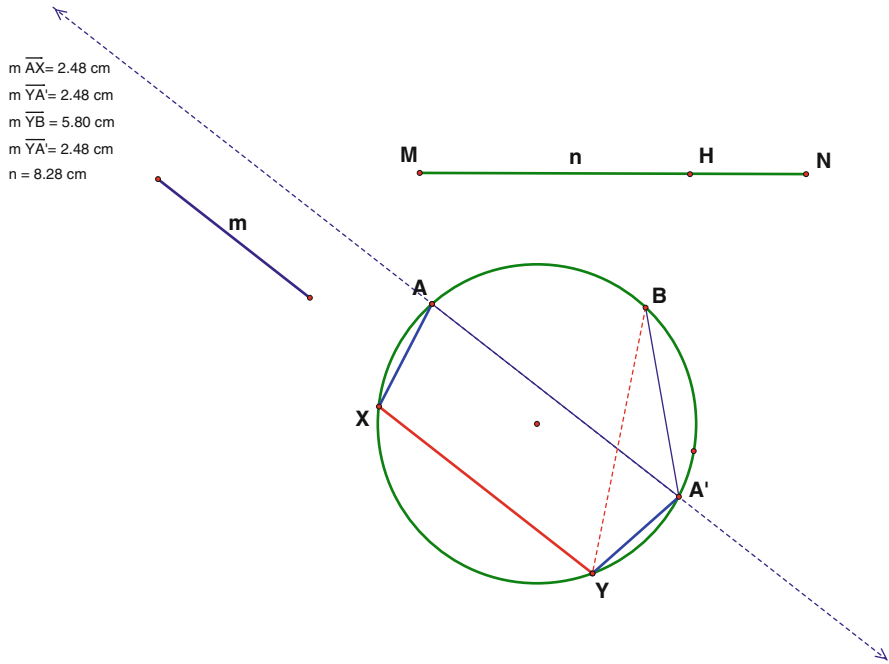
Fig. 4.34 Sketch for Homework Problem 20

Appendix A: Ratios and Proportion

A **ratio** represents a comparison between two numbers or quantities. For example, if a box has 5 red and 7 blue balls then the ratio between red and blue balls is 5–7 that also can be written as a fraction 5/7 or using a division sign as 5:7. Now, if we want to make another box with the same ratio of red and blue balls in it then we have to put in the new box the same multiple of 5 of the red and the same multiple of 7 of the blue balls. Thus, if we want to have 40 red balls (8·5) in the new box, then we need to add 56 blue balls (8·7) in order to keep the ratio 5:7

A ratio does not have any units of measure.

A **proportion** represents two equal ratios and can be written as $\frac{a}{b} = \frac{c}{d} = k$ where k is the coefficient of proportionality.

There are many problems involving proportions and their applications. I noticed that some students do not know how to use proportions correctly and try to memorize formulas—eventually applying them wrongly. When I was a student I used the following method that connects proportional objects. Maybe it can be useful for you as well.

For example, we need to solve the following problem.

Problem 82 Given a circle of radius R, evaluate the area of its sector with central angle $30°$
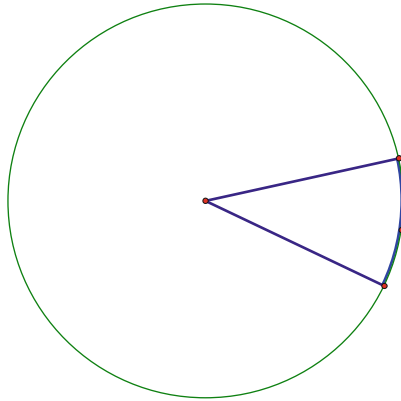
Fig. A.1 Sketch for Problem 82

Many of us do not remember the formula for a sector of a circle. However, most of us know that the area of a circle with radius R is $A = \pi R^2$. Can we derive the formula for a sector? Of course! Look at the picture above. A sector of degree θ is like the area of a piece of pie with central angel θ with respect to the area of the whole pie with central angle 2π. Denote the area of the sector by x and we know that the area of the pie (circle) is πR^2. Next, there is a very important step: instead of writing proportions, we will start from correspondence sentences such as

$$2\pi \text{ (angle): corresponds to } \pi R^2 \text{ (area)}$$
$$\theta \text{ (angle): corresponds to } x \text{ (area).}$$

It is important to relate angles to angles and areas to areas in the proportions. Mathematically, we can use arrows for simplification:

$$2\pi \rightarrow \pi R^2$$
$$\theta \rightarrow x$$

By cross multiplying the means and extremes, after canceling common factors, we will get the required formula for the area of the sector.

$$2\pi x = \theta \cdot \pi R^2$$
$$x = \frac{\theta \cdot \pi R^2}{2\pi} = \frac{\theta \cdot R^2}{2}$$

Note. We derived the sector area using proportions. If a central angle is given in degrees, the procedure of finding its area is the same. Just replace 2π by $360°$ and θ by the angle measure in degrees. Thus, for $30°$ sector, we have the following:

$$\left.\begin{array}{l} 360° \rightarrow 2\pi R \\ 30° \rightarrow x \end{array}\right\} \Rightarrow 360° \cdot x = 2\pi R \cdot 30° \Rightarrow x = \frac{2\pi R \cdot 30°}{360°} = \frac{\pi R}{6}$$

Of course, one would say that such areas can be evaluated mentally by dividing the area of the circle by 12 (since $360 = 12 \cdot 30$) which is fine, but the method of proportions is general and works for any angle measure, not necessarily a multiple of 360°.

Let us finalize the rule of proportion. If a is related to b as c is to d, then:

$$\left.\begin{array}{l} a \rightarrow b \\ c \rightarrow d \end{array}\right\} \Rightarrow a \cdot d = c \cdot b$$

If three or more ratios are equal, then an **extended proportion** exists,

$$\frac{a}{b} = \frac{c}{d} = \frac{f}{e} = \frac{m}{n} = k$$

Equal Ratios Property. If you know two ratios are equal (we'll call them k in the first equation below), then subtracting their numerators and subtracting their denominators give a ratio that's also equal to the first two.

If we have $\dfrac{a}{b} = \dfrac{c}{d} = k$, then $\dfrac{a-c}{b-d} = k$.

Proof. Since $a/b = k$, $a = k \cdot b$. If $c/d = k$, then $c = k \cdot d$.

Therefore, $a - c = k(b - d)$ and $\dfrac{a-c}{b-d} = \dfrac{k(b-d)}{b-d} = k$.

Multifactor Ratios

Sometimes we are given a condition such as $a : b : c : d = e : f : g : h$ or perhaps a proportion with a greater number of factors. How can you understand this relationship between quantities?

Problem 83 will help with this issue.

Problem 83. A segment is divided into five parts in the ratio 2:3:4:5:6. Find the length of each segment if the length of original is 15 inches.

Solution. Giving a similar problem, I often see mistakes in the students' solutions. They believed that given numbers of ratios 2:3:4:5:6 they could use the actual lengths of the segments. Of course, such ratios mean only that the length of the first

segment is proportional to 2, the length of the second segment is proportional to 3, the third to 4, and so on. Using algebra we can assume that the length of the first segment is $2x$, where x is some variable we don't know it yet. Then $3x$ is the length of the second segment, $4x$ of the third, $5x$ of the fourth, and $6x$ of the fifth segment.

How can we find x?

We will use the fact that the length of original segment is 15 inches. Putting together the lengths of all segments and solving the equation:

$$2x + 3x + 4x + 5x + 6x = 15$$

$$20x = 15, \quad x = 0.75$$

Because $2x = 1.5$, $3x = 2.25$, $4x = 3$, $5x = 3.75$, and $6x = 4.5$ we can conclude that a segment of 15 inches was divided into segments of 1.5, 2.25, 3, 3.75, and 4.5 inches.

Answer. 1.5, 2.25, 3, 3.75, and 4.25 inches.

Question. In the problem above, a segment is divided into five smaller segments in the ratio 2:3:4:5:6. How is the length of the first segment related to the length of the whole segment?

Answer. The smallest segment is only two parts of the big one. The big one is $2 + 3 + 4 + 5 + 6 = 20$ parts, then the proportion of the smallest to the biggest is $2:20 = 1:10$. It is one tenth the part of the whole segment.

Proof of the Vector Form of Menelaus Theorem

Connect points A_1, B_1, and C_1 and draw lines through B, C, and A parallel to line A_1C_1. Pick a point L on the line that goes through point A and draw another line through L (the line does not have to be parallel to AB).
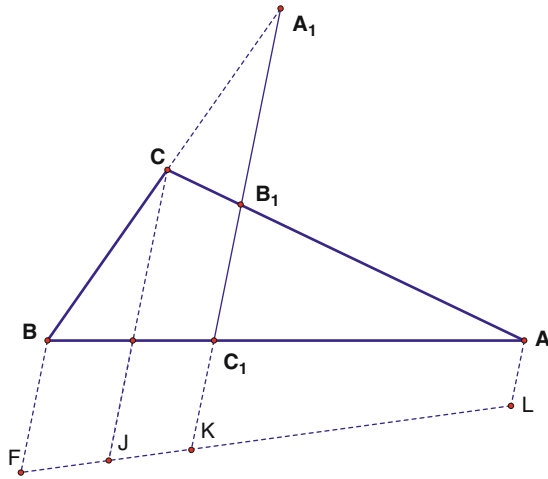
Fig. A.2 Proof of the vector form of the Menelaus Theorem using Thales's Theorem

The ratios of corresponding segments cut by parallel lines are obtained by Thales' Theorem. Multiplying the left and right sides of these ratios completes the proof.
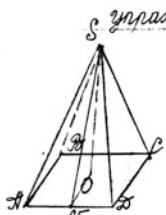
$$
\left.
\begin{aligned}
\frac{AC_1}{C_1B} &= \frac{LK}{KF} \\[4pt]
\frac{BA_1}{A_1C} &= \frac{FK}{KJ} \\[4pt]
\frac{CB_1}{B_1A} &= \frac{JK}{KL}
\end{aligned}
\right\}
\Rightarrow
\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = \frac{LK}{KF} \cdot \frac{FK}{KJ} \cdot \frac{JK}{KL}
$$

$$
= \frac{LK}{KL} \cdot \frac{FK}{KF} \cdot \frac{JK}{KJ} (-1) \cdot (-1) \cdot (-1) = -1
$$

Appendix B: My Ninth Grade Notebook Page

Fig. B.1 My ninth grade geometry notebook page

Appendix C: My Pictures

Fig. C.1 I am in my office

Fig. C.2 At an International Conference in honor of Suresh Sethi, France, 2005
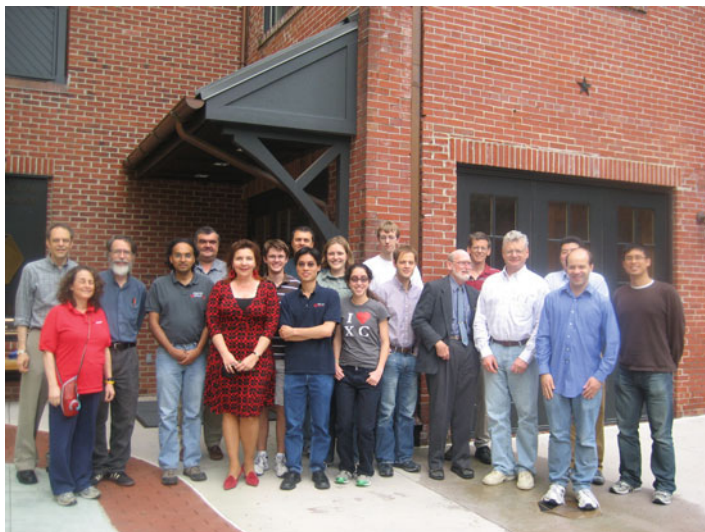
Fig. C.3 USAMO 2008 Graders, Washington, DC

Fig. C.4 USSR Math Olympiad Winners, circa 1979

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