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A NEW APPROACH TO ANALYZING REACTIONS AND DEFLECTIONS OF BEAMS: FORMULATION AND EXAMPLES

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ABSTRACT

This paper is aimed at developing a new approach to analyzing statically indeterminate reactions at supports, as well as the slopes and deflections, of beams. The approach uses a set of four general formulas, derived using singularity functions. These formulas are expressed in terms of shear forces, bending moments, distributed loads, slopes, and deflections of a beam having a constant flexural rigidity and carrying typical loads. These loads include *(a)* a bending moment and a shear force at the left, as well as at the right, end of the beam, *(b)* a concentrated force, as well as a concentrated moment, somewhere on the beam; and *(c)* a uniformly, as well as a linearly varying, distributed force over a portion of the beam. The approach allows one to treat reactions at supports (even supports not at the ends of a beam) as concentrated forces or moments, where corresponding boundary conditions at the points of supports are to be imposed. This feature allows one to readily determine reactions at supports as well as slopes and deflections of beams. A beam needs to be divided into segments for study if it contains discontinuities in slope at hinge connections or different flexural rigidities in different segments. Several examples are included to illustrate the new approach.

INTRODUCTION

There are different established methods for determining deflections of beams, which may be found in published papers and textbooks for the traditional undergraduate course in mechanics of materials. These methods may include the following [1-6]:

- *(a)* method of double integration (*with* or *without* the use of singularity functions),
- *(b)* method of superposition,
- *(c)* method using moment-area theorems,
- *(d)* method using Castigliano's theorem, and
- *(e)* conjugate beam method.

Naturally, there are advantages and disadvantages in using any of the above methods. This paper is aimed at expanding the mechanics literature by presenting a new approach and making available a new method to mechanics practitioners and educators for their choice and at their disposal when it comes to determining reactions and deflections of beams. It is also intended to contribute to the enrichment of one's learning experience.

The paper begins with the description of sign conventions and derives four general formulas for the slope and deflection of a beam segment having a constant flexural rigidity and carrying a variety of typical, applied loads. These formulas, derived using singularity functions, form the basis for a new approach to solving problems involving reactions and deflections of beams. This approach is consistent in philosophy with the approach presented in most mechanics of materials textbooks in treating axially loaded bars and torsionally loaded shafts. Application of these formulas is direct and requires no further integration or writing of continuity equations. This new approach can readily be extended to the analysis of beams having discontinuities in slope at hinge connections or different flexural rigidities in different segments. It can solve both statically determinate and statically indeterminate beam problems.

SIGN CONVENTIONS

A segment of beam *ab* having a constant flexural rigidity *EI* is shown in Fig. 1. Note that we adopt the positive directions of the shear forces, moments, and distributed loads as indicated.

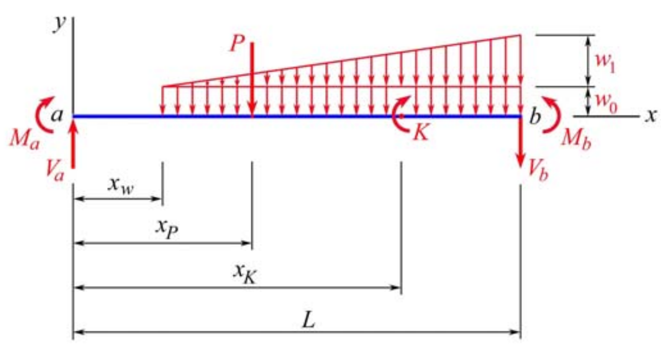

Fig. 1: Positive directions of shear forces, moments, and loads

As in most textbooks for mechanics of materials, notice in Fig. 1 the following conventions:

(a) a *positive shear force* is one that tends to rotate the beam segment clockwise (e.g., $\mathbf{V}_a$ at the left end a, and $\mathbf{V}_b$ at the right end b),

(b) a *positive moment* is one that tends to cause compression in the top fiber of the beam (e.g., $\mathbf{M}_a$ at the left end a, $\mathbf{M}_b$ at the right end b, and the applied moment **K** tending to cause compression in the top fiber of the beam just to the right of the position where the moment **K** acts),

(c) a *positive concentrated force* applied to the beam is one that is directed downward (e.g., the applied force **P**), and

(d) a *positive distributed load* is one that is directed downward (e.g., the uniformly distributed load with intensity w_0, and the linearly varying distributed load with highest intensity w_1).

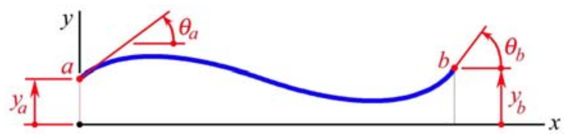

Fig. 2: Positive deflections and positive slopes of beam *ab*

The positive directions of deflections and slopes of the beam are defined as illustrated in Fig. 2. As in most textbooks for mechanics of materials, notice in Fig. 2 the following conventions:

(a) a *positive deflection* is an upward displacement (e.g., y_a at position *a*, and y_b at position *b*),

(b) a *positive slope* is a counterclockwise rotation (e.g., θ_a at position *a*, and θ_b at position *b*).

DERIVATION OF GENERAL FORMULAS

Using singularity functions [7, 8], we may write the loading function *q*, the shear force *V*, and the bending moment *M* for the beam *ab* in Fig. 1 as

$$q = V_a\langle x\rangle^{-1} + M_a\langle x\rangle^{-2} - P\langle x-x_P\rangle^{-1} + K\langle x-x_K\rangle^{-2} - w_0\langle x-x_w\rangle^{0} - \frac{w_1}{L-x_w}\langle x-x_w\rangle^{1} \tag{1}$$

$$V = V_a\langle x\rangle^{0} + M_a\langle x\rangle^{-1} - P\langle x-x_P\rangle^{0} + K\langle x-x_K\rangle^{-1} - w_0\langle x-x_w\rangle^{1} - \frac{w_1}{2(L-x_w)}\langle x-x_w\rangle^{2} \tag{2}$$

$$M = V_a\langle x\rangle^{1} + M_a\langle x\rangle^{0} - P\langle x-x_P\rangle^{1} + K\langle x-x_K\rangle^{0} - \frac{w_0}{2}\langle x-x_w\rangle^{2} - \frac{w_1}{6(L-x_w)}\langle x-x_w\rangle^{3} \tag{3}$$

Letting *EI* be the flexural rigidity, *y* be the deflection, y' be the slope, and y'' be the second derivative of *y* with respect to the abscissa *x* for the prismatic segment of beam *ab*, we write [4]

$$EIy'' = M$$

$$EIy'' = V_a\langle x\rangle^{1} + M_a\langle x\rangle^{0} - P\langle x-x_P\rangle^{1} + K\langle x-x_K\rangle^{0} - \frac{w_0}{2}\langle x-x_w\rangle^{2} - \frac{w_1}{6(L-x_w)}\langle x-x_w\rangle^{3} \tag{4}$$

$$EIy' = \frac{1}{2}V_a\langle x\rangle^{2} + M_a\langle x\rangle^{1} - \frac{1}{2}P\langle x-x_P\rangle^{2} + K\langle x-x_K\rangle^{1} - \frac{w_0}{6}\langle x-x_w\rangle^{3} - \frac{w_1}{24(L-x_w)}\langle x-x_w\rangle^{4} + C_1 \tag{5}$$

$$EIy = \frac{1}{6}V_a\langle x\rangle^{3} + \frac{1}{2}M_a\langle x\rangle^{2} - \frac{1}{6}P\langle x-x_P\rangle^{3} + \frac{1}{2}K\langle x-x_K\rangle^{2} - \frac{w_0}{24}\langle x-x_w\rangle^{4} - \frac{w_1}{120(L-x_w)}\langle x-x_w\rangle^{5} + C_1x + C_2 \tag{6}$$

The slope and deflection of the beam in Fig. 1 at its left end *a* (i.e., at $x = 0$) are θ_a and y_a, respectively, as illustrated in Fig. 2. Imposition of these two boundary conditions on Eqs. (5) and (6) allows us to obtain the values for the constants of integration C_1 and C_2 as follows:

$$C_1 = EI\theta_a \tag{7}$$

$$C_2 = EIy_a \tag{8}$$

Substituting Eqs. (7) and (8) into Eqs. (5) and (6), we obtain the **general formulas** for the slope y' and deflection *y*, at any position *x*, of the beam *ab* in Fig. 1 as follows:

$$y' = \theta_a + \frac{V_a}{2EI}x^2 + \frac{M_a}{EI}x - \frac{P}{2EI}\langle x-x_P\rangle^{2} + \frac{K}{EI}\langle x-x_K\rangle^{1} - \frac{w_0}{6EI}\langle x-x_w\rangle^{3} - \frac{w_1}{24EI(L-x_w)}\langle x-x_w\rangle^{4} \tag{9}$$

$$y = y_a + \theta_a x + \frac{V_a}{6EI}x^3 + \frac{M_a}{2EI}x^2 - \frac{P}{6EI}\langle x-x_P\rangle^{3} + \frac{K}{2EI}\langle x-x_K\rangle^{2} - \frac{w_0}{24EI}\langle x-x_w\rangle^{4} - \frac{w_1}{120EI(L-x_w)}\langle x-x_w\rangle^{5} \tag{10}$$

By letting $x = L$ in Eqs. (9) and (10), we obtain the **general formulas** for the slope θ_b and deflection y_b, at the right end *b*, of the beam *ab* in Fig. 1, as illustrated in Fig. 2, as follows:

$$\theta_b = \theta_a + \frac{V_aL^2}{2EI} + \frac{M_aL}{EI} - \frac{P}{2EI}(L-x_P)^2 + \frac{K}{EI}(L-x_K) - \frac{4w_0+w_1}{24EI}(L-x_w)^3 \tag{11}$$

$$y_b = y_a + \theta_a L + \frac{V_aL^3}{6EI} + \frac{M_aL^2}{2EI} - \frac{P}{6EI}(L-x_P)^3 + \frac{K}{2EI}(L-x_K)^2 - \frac{5w_0+w_1}{120EI}(L-x_w)^4 \tag{12}$$

A NEW APPROACH TO ANALYZING BEAMS

The set of four **general formulas** given by Eqs. (9) through (12) may be used as the basis upon which to formulate a new approach to analyzing statically indeterminate reactions at supports, as well as the slopes and deflections, of beams. The beams may carry a variety of loads, as illustrated in Fig. 1.

Note that *L* in the general formulas represents the total length of the beam segment, to which the general formulas are to be applied. These formulas have already taken into account the boundary conditions of the beam at its ends. Furthermore, this approach allows one to treat reactions at interior supports (those *not* at the ends of the beam) as applied concentrated forces or moments. All one has to do is to simply impose the additional corresponding boundary conditions at the interior supports for the beam segment. Thus, the new approach allows one to readily determine statically indeterminate reactions as well as slopes and deflections of beams.

A beam needs to be divided into separate segments for study only if *(a)* it contains segments of different flexural rigidities, and *(b)* it is a combined beam (e.g., *Gerber beam*) having discontinuities in slope at hinge connections between segments. The new approach proposed in this paper can best be understood with illustrations. Therefore, simple as well as more challenging problems are included in the following examples.

Example 1. A beam *AB* with a constant flexural rigidity *EI*, a roller support at *A*, a fixed support at *B*, and carrying a linearly distributed load over a portion of its length is given in Fig. 3. Determine the vertical reaction force $\mathbf{A}_y$ and the slope θ_A at *A*.

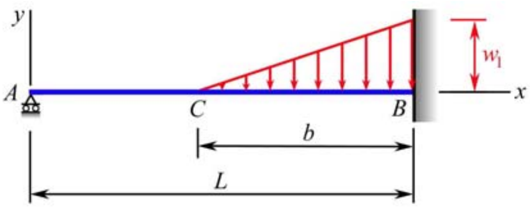

Fig. 3: Propped cantilever with a linearly distributed load

Solution. This beam is statically indeterminate to the *first* degree, and we have $x_w = L - b$. The boundary conditions reveal that the deflection and the moment at the roller support *A*, as well as the slope and deflection at the fixed support *B*, are all equal to zero. Applying the general formulas in Eqs. (11) and (12), successively, to the entire beam, we write

$$0 = \theta_A + \frac{A_y L^2}{2EI} + 0 - 0 + 0 - \frac{0 + w_1}{24EI}[L-(L-b)]^3$$

$$0 = 0 + \theta_A L + \frac{A_y L^3}{6EI} + 0 - 0 + 0 - \frac{0 + w_1}{120EI}[L-(L-b)]^4$$

The above two simultaneous equations yield

$$A_y = \frac{w_1 b^3(5L-b)}{40L^3} \qquad \theta_A = -\frac{w_1 b^3(5L-3b)}{240LEI}$$

Consistent with the defined sign conventions, we report that

$$\mathbf{A}_y = \frac{w_1 b^3(5L-b)}{40L^3} \uparrow \qquad \theta_A = -\frac{w_1 b^3(5L-3b)}{240LEI}$$

Example 2. A fixed-ended beam *AB* with a constant flexural rigidity *EI* and carrying a linearly distributed load over a portion of its length is given in Fig. 4. Determine *(a)* the vertical reaction force $\mathbf{A}_y$ and the reaction moment $\mathbf{M}_A$ at *A*, *(b)* the slope θ_C and deflection y_C at *C*.

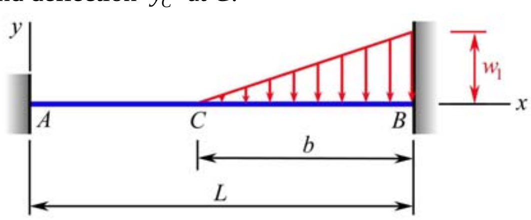

Fig. 4: Fixed-ended beam with a linearly distributed load

Solution. This beam is statically indeterminate to the *second* degree, and we have $x_w = L - b$. The boundary conditions reveal that the slope and deflection of the beam at *A* and *B* are all equal to zero. Applying the general formulas in Eqs. (11) and (12), successively, to the entire beam, we write

$$0 = 0 + \frac{A_y L^2}{2EI} + \frac{M_A L}{EI} - 0 + 0 - \frac{0 + w_1}{24EI}[L-(L-b)]^3$$

$$0 = 0 + 0 + \frac{A_y L^3}{6EI} + \frac{M_A L^2}{2EI} - 0 + 0 - \frac{0 + w_1}{120EI}[L-(L-b)]^4$$

The above two simultaneous equations yield

$$A_y = \frac{w_1 b^3(5L-2b)}{20L^3} \qquad M_A = -\frac{w_1 b^3(5L-3b)}{60L^2}$$

Consistent with the defined sign conventions, we report that

$$\mathbf{A}_y = \frac{w_1 b^3(5L-2b)}{20L^3} \uparrow \qquad \mathbf{M}_A = \frac{w_1 b^3(5L-3b)}{60L^2} \circlearrowleft$$

The position *C* of the beam is located at *x* = *L* – *b*. Applying the general formulas in Eqs. (9) and (10), successively, to the entire beam and, at the same time, utilizing the preceding solutions for A_y and M_A, we write

$$\theta_C = 0 + \frac{A_y(L-b)^2}{2EI} + \frac{M_A(L-b)}{EI} - 0 + 0 - 0 - \frac{w_1}{24EIb}(0)^4$$

$$\theta_C = \frac{w_1 b^3(L-b)(5L^2+6b^2-15bL)}{120L^3EI}$$

$$y_C = 0 + 0 + \frac{A_y(L-b)^3}{6EI} + \frac{M_A(L-b)^2}{2EI} - 0 + 0 - 0 - \frac{w_1}{120EIb}(0)^5$$

$$y_C = -\frac{w_1 b^4(2L-b)(L-b)^2}{60L^3EI}$$

Example 3. A fixed-ended beam *AB* with a constant flexural rigidity *EI* and carrying a uniformly distributed load over a portion of its length is given in Fig. 5. Determine *(a)* the vertical reaction force $\mathbf{A}_y$ and the reaction moment $\mathbf{M}_A$ at *A*, *(b)* the slope θ_C and deflection y_C at *C*.

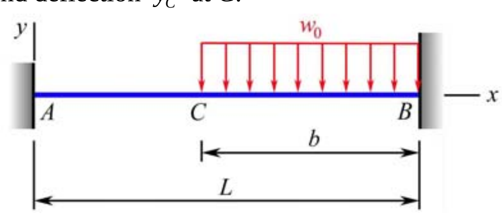

Fig. 5: Fixed-ended beam with a uniformly distributed load

Solution. This beam is statically indeterminate to the *second* degree, and we have $x_w = L - b$. The boundary conditions reveal that the slope and deflection of the beam at *A* and *B* are all equal to zero. Applying the general formulas in Eqs. (11) and (12) to the entire beam, we write

$$0 = 0 + \frac{A_y L^2}{2EI} + \frac{M_A L}{EI} - 0 + 0 - \frac{4w_0 + 0}{24EI}[L-(L-b)]^3$$

$$0 = 0 + 0 + \frac{A_y L^3}{6EI} + \frac{M_A L^2}{2EI} - 0 + 0 - \frac{5w_0 + 0}{120EI}[L-(L-b)]^4$$

The above two simultaneous equations yield

$$A_y = \frac{w_0 b^3(2L-b)}{2L^3} \qquad M_A = -\frac{w_0 b^3(4L-3b)}{12L^2}$$

Consistent with the defined sign conventions, we report that

$$\mathbf{A}_y = \frac{w_0 b^3(2L-b)}{2L^3} \uparrow \qquad \mathbf{M}_A = \frac{w_0 b^3(4L-3b)}{12L^2} \circlearrowleft$$

The position C of the beam is located at $x = L - b$. Applying the general formulas in Eqs. (9) and (10), successively, to the entire beam and, at the same time, utilizing the preceding solutions for A_y and M_A, we write

$$\theta_C = 0 + \frac{A_y(L-b)^2}{2EI} + \frac{M_A(L-b)}{EI} - 0 + 0 - \frac{w_0}{6EI}(0)^3 - 0$$

$$\theta_C = \frac{w_0 b^3(L-b)(2L^2+3b^2-6bL)}{12L^3EI}$$

$$y_C = 0 + 0 + \frac{A_y(L-b)^3}{6EI} + \frac{M_A(L-b)^2}{2EI} - 0 + 0 - \frac{w_0}{24EI}(0)^4 - 0$$

$$y_C = -\frac{w_0 b^3(L-b)^2(2L^2-b^2)}{12L^3EI}$$

Example 4. A fixed-ended beam AB with a constant flexural rigidity EI and carrying a concentrated moment $\mathbf{K}$ at C is given in Fig. 6. Determine *(a)* the vertical reaction force $\mathbf{A}_y$ and the reaction moment $\mathbf{M}_A$ at A, *(b)* the slope θ_C and deflection y_C at C.

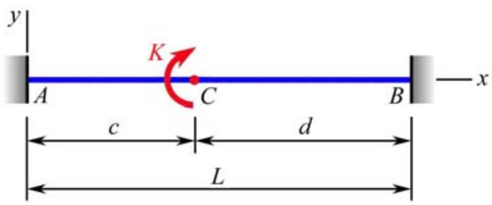

Fig. 6: Fixed-ended beam carrying a concentrated moment

Solution. This beam is statically indeterminate to the *second* degree, and we have $x_K = c$. The boundary conditions reveal that the slope and deflection of the beam at A and B are all equal to zero. Applying the general formulas in Eqs. (11) and (12) to the entire beam, we write

$$0 = 0 + \frac{A_y L^2}{2EI} + \frac{M_A L}{EI} - 0 + \frac{K}{EI}(L-c) - 0$$

$$0 = 0 + 0 + \frac{A_y L^3}{6EI} + \frac{M_A L^2}{2EI} - 0 + \frac{K}{2EI}(L-c)^2 - 0$$

Since $L = c + d$, the above two simultaneous equations yield

$$A_y = -\frac{6Kcd}{L^3} \qquad M_A = \frac{Kd(2c-d)}{L^2}$$

Consistent with the defined sign conventions, we report that

$$\mathbf{A}_y = \frac{6Kcd}{L^3} \downarrow \qquad \mathbf{M}_A = \frac{Kd(2c-d)}{L^2} \circlearrowright$$

The position C is located at $x = c$. Applying the general formulas in Eqs. (9) and (10), successively, to the entire beam and, at the same time, utilizing the preceding solutions for A_y and M_A, we write

$$\theta_C = 0 + \frac{A_y c^2}{2EI} + \frac{M_A c}{EI} - 0 + \frac{K}{EI}(0)^1 - 0 - 0$$

$$\theta_C = -\frac{Kcd[(c-d)^2+cd]}{L^3EI}$$

$$y_C = 0 + 0 + \frac{A_y c^3}{6EI} + \frac{M_A c^2}{2EI} - 0 + \frac{K}{2EI}(0)^2 - 0 - 0$$

$$y_C = \frac{Kc^2d^2(c-d)}{2L^3EI}$$

Example 5. A beam AB, which has a constant flexural rigidity EI, a fixed support at B, and a roller support at C, is loaded as shown in Fig. 7, where $M_0 = 3wL^2$. Determine *(a)* the slope θ_A and deflection y_A at A, *(b)* the vertical reaction force $\mathbf{C}_y$ and the slope θ_C at C.

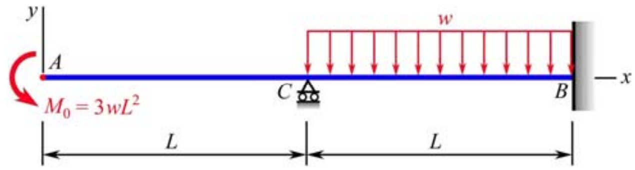

Fig. 7: Propped cantilever beam carrying loads

Solution. This beam is statically indeterminate to the *first* degree. There is no need to divide this beam into two segments for analysis in the solution by the proposed new approach. We can simply treat the vertical reaction force $\mathbf{C}_y$ at C as an unknown applied concentrated force directed upward and regard the beam AB as one that has a total length of $2L$, which is to be used as the value for the parameter L in the general formulas in Eqs. (9) through (12). The boundary conditions of this beam reveal that the slope and deflection at B are both equal to zero, the shear force at A is zero, the moment at A is $-3wL^2$, and the deflection at C is zero. Applying the general formulas in Eqs. (11), (12), and (10), in that *order*, to the entire beam, we write

$$0 = \theta_A + 0 + \frac{-3wL^2(2L)}{EI} - \frac{-C_y}{2EI}(2L-L)^2 + 0 - \frac{4w+0}{24EI}(2L-L)^3$$

$$0 = y_A + \theta_A(2L) + 0 + \frac{-3wL^2(2L)^2}{2EI} - \frac{-C_y}{6EI}(2L-L)^3 + 0 - \frac{5w+0}{120EI}(2L-L)^4$$

$$0 = y_A + \theta_A L + 0 + \frac{-3wL^2}{2EI}(L^2) - 0 + 0 - 0 - 0$$

The above three simultaneous equations yield

$$\theta_A = \frac{179wL^3}{48EI} \qquad y_A = -\frac{107wL^4}{48EI} \qquad C_y = \frac{39wL}{8}$$

Consistent with the defined sign conventions, we report that

$$\theta_A = \frac{179wL^3}{48EI} \qquad y_A = -\frac{107wL^4}{48EI} \qquad \mathbf{C}_y = \frac{39wL}{8} \uparrow$$

The position C is located at $x = L$. Applying the general formula in Eq. (9) to the entire beam and, at the same time, utilizing the preceding solutions for θ_A, we write

$$\theta_C = \theta_A + 0 + \frac{-3wL^2}{EI}(L) - 0 + 0 - 0 - 0$$

$$\theta_C = \frac{35wL^3}{48EI}$$

Example 6. A combined beam (*Gerber beam*), with a constant flexural rigidity EI, fixed supports at its ends A and D, a hinge connection at B, and carrying a concentrated force $\mathbf{P}$ at C, is given in Fig. 8. Determine *(a)* the vertical reaction force $\mathbf{A}_y$ and the reaction moment $\mathbf{M}_A$ at A, *(b)* the deflection y_B of the hinge at B, *(c)* the slopes θ_{BL} and θ_{BR} just to the left and just to the right of the hinge at B, respectively, and *(d)* the slope θ_C and the deflection y_C at C.

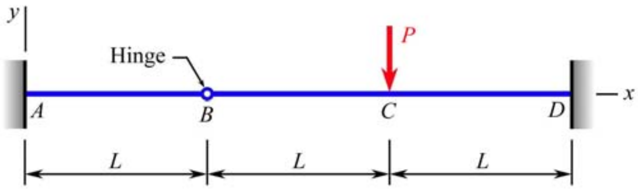

Fig. 8: Fixed-ended beam with a hinge connector

Solution. This beam is statically indeterminate to the *first* degree. Nevertheless, because of the discontinuity in slope at the hinge connection B, this beam needs to be divided into two segments AB and BD for analysis in the solution. The boundary conditions of this beam reveal that slope and deflection at A and D are all equal to zero.

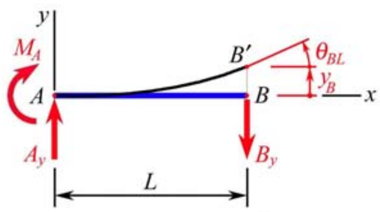

Fig. 9: Free-body diagram for segment AB and its deflections

Applying the general formulas in Eqs. (11) and (12), successively, to segment AB, as shown in Fig. 9, we write

$$\theta_{BL} = 0 + \frac{A_y L^2}{2EI} + \frac{M_A L}{EI} - 0 + 0 - 0 \qquad (a)$$

$$y_B = 0 + 0 + \frac{A_y L^3}{6EI} + \frac{M_A L^2}{2EI} - 0 + 0 - 0 \qquad (b)$$

For equilibrium of segment AB, we write

$$+\!\uparrow \Sigma F_y = 0: \qquad A_y - B_y = 0 \qquad (c)$$

$$+\circlearrowleft \Sigma M_B = 0: \qquad -M_A - LA_y = 0 \qquad (d)$$

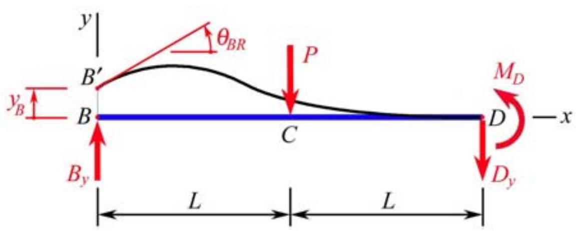

Fig. 10: Free-body diagram for segment BD and its deflections

Applying the general formulas in Eqs. (11) and (12), successively, to segment BD, as shown in Fig. 10, we write

$$0 = \theta_{BR} + \frac{B_y (2L)^2}{2EI} + 0 - \frac{P}{2EI}(2L - L)^2 + 0 - 0 \qquad (e)$$

$$0 = y_B + \theta_{BR}(2L) + \frac{B_y (2L)^3}{6EI} + 0 - \frac{P}{6EI}(2L - L)^3 + 0 - 0 \qquad (f)$$

For equilibrium of segment BD, we write

$$+\!\uparrow \Sigma F_y = 0: \qquad B_y - P - D_y = 0 \qquad (g)$$

$$+\circlearrowleft \Sigma M_B = 0: \qquad -LP - 2LD_y + M_D = 0 \qquad (h)$$

The above eight simultaneous equations yield

$$A_y = \frac{5P}{18} \qquad B_y = \frac{5P}{18} \qquad M_A = -\frac{5PL}{18}$$

$$\theta_{BL} = -\frac{5PL^2}{36EI} \qquad \theta_{BR} = -\frac{PL^2}{18EI}$$

$$D_y = -\frac{13P}{18} \qquad M_D = -\frac{4PL}{9} \qquad y_B = -\frac{5PL^3}{54EI}$$

Consistent with the defined sign conventions, we report that

$$\mathbf{A}_y = \frac{5P}{18} \uparrow \qquad \mathbf{M}_A = \frac{5PL}{18} \circlearrowleft \qquad y_B = -\frac{5PL^3}{54EI}$$

$$\theta_{BL} = -\frac{5PL^2}{36EI} \qquad \theta_{BR} = -\frac{PL^2}{18EI} = -\frac{2PL^2}{36EI}$$

The position C is located at $x = L$ in Fig. 10. Applying the general formulas in Eqs. (9) and (10), successively, to the segment BD in this figure and, at the same time, utilizing the preceding solutions for θ_{BR} and B_y, we write

$$\theta_C = \theta_{BR} + \frac{B_y}{2EI}(L^2) + 0 - \frac{P}{2EI}(0)^2 + 0 - 0 - 0$$

$$\theta_C = \frac{PL^2}{12EI}$$

$$y_C = y_B + \theta_{BR} L + \frac{B_y}{6EI}(L^2) + 0 - \frac{P}{6EI}(0)^3 + 0 - 0 - 0$$

$$y_C = -\frac{11PL^3}{108EI}$$

Based on the preceding solutions, the deflections of the combined beam AD may be illustrated as shown in Fig. 11.

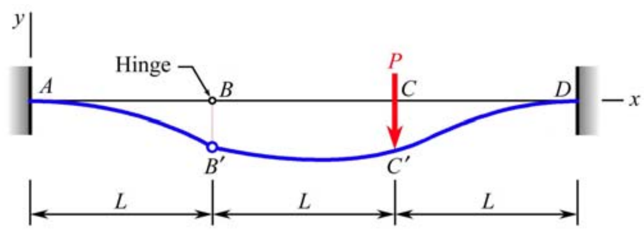

Fig. 11: Deflections of the beam AD

CONCLUDING REMARKS

This paper presents the formulation of a new approach to analyzing statically indeterminate reactions at supports, as well as the slopes and deflections, of beams. A set of four general formulas, derived using singularity functions, is used as the basic tools for providing the material equations, besides the equations of static equilibrium, for the solution of indeterminate reactions, slopes, and reactions at supports of beams. These formulas are expressed in terms of *(a)* a bending moment and a shear force at the left, as well as at the right, end of the beam, *(b)* a concentrated force, as well as a concentrated moment, somewhere on the beam; and (*c*) a uniformly, as well as a linearly varying, distributed force over a portion of the beam.

The proposed approach depends heavily on a collection of material formulas. In a way, it is a *method of formulary*. It allows one to treat unknown reactions at supports (even supports not at the ends of a beam) as concentrated forces or moments. A beam needs to be divided into separate segments for study only if *(a)* it contains segments of different flexural rigidities, and *(b)* it is a combined beam having discontinuities in slope at hinge connections between segments. Note that *L* in the general formulas represents the total length of the beam segment, to which the general formulas are to be applied.

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