Conjugacy

Orientation. The subject corresponds to material treated in Hirsch-Smale-Devaney as follows.

- Linear Conjugacy: §3.4
- Topological Conjugacy definition and Linear Systems: §4.2
- Hartman-Grobman Linearization Theorem: §8.2
- Flow Box Theorem: §10.2

The notion of conjugacy reappears in the treatment of Chaos in Chapter 15.

1 Linear conjugacy

Definition 1.1 *Two linear systems $X' = AX$ and $Y' = BY$ are linearly conjugate if and only if there is an invertible linear change of variables, $Y = HX$ that converts one to the other.*

Theorem 1.1 *Two systems are linearly conjugate if and only if $A = H^{-1}BH$.*

Remark 1.1 **i.** Linear conjugacy is the same as similarity of matrices.
ii. If two systems are linearly conjugate then the characteristic polynomials of A and B are equal. Indeed

$$\begin{aligned} \det(zI - B)) &= \det(zHH^{-1} - HAH^{-1}) = \det H\,(zI - A)H^{-1} \\ &= \det H \ \det(zI - A) \ \det H^{-1} = \det(zI - A)\,. \end{aligned}$$

Therefore the eigenvalues of the matrices A and B must be the same with the same multiplicities as roots.
iii. The equality of characteristic polynomial is necessary but not sufficient for conjugacy. The matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

have characteristic polynomial equal to z^2 yet are not similar.

Example 1.1 *The equations $x' = 2x$ and $x' = x$ are* **not** *linearly conjugate since the eigenvalues 1 and 2 are not equal.*

This equivalence relation between linear equations is **too strong** *since $x' = 2x$ and $x' = x$ have qualitative behavior that is so similar that you would like to classify them as equivalent.*

Exercise 1.1 *Show that a system $X' = AX$ on $\mathbb{C}^N$ is conjugate to a system $Y' = DY$ with diagonal matrix D if and only if A has a basis of eigenvectors.* **Discussion.** In this case solving the simple diagonal system exactly recover's the eigenvalue-eigenvector algorithm for $X' = AX$. This is a discovery method for that algorithm complementary to beginning with Euler's analysis of scalar constant coefficient linear homogeneous equations.

2 Differentiable conjugacy

For linear system a linear change of coordinates yields a new linear system. For nonlinear equations making a nonlinear change of coordinates $Y = h(X)$ with h and its inverse $X = h^{-1}(Y)$ continuously differentiable yields a new system for Y. Indeed, the equation $X' = F(X)$ after substitution of $X = h^{-1}(Y)$ reads

$$\frac{d}{dt}\, h^{-1}(Y) \;=\; F(h^{-1}(Y))\,. \tag{1}$$

Evaluating the left hand side using the chain rule yields

$$D_Y h^{-1}(Y)\;\frac{dY}{dt}\,.$$

The formula for the derivative of the inverse is

$$D_Y h^{-1}(Y) \;=\; \Big(D_X h\big|_{X=h^{-1}(Y)}\Big)^{-1}\,.$$

Multiplying (1) by $D_X h(h^{-1}(Y))$ yields the differential equation

$$Y' \;=\; G(Y)\,,$$

with

$$G(Y) \;:=\; D_X h(h^{-1}(Y))\; F(h^{-1}(Y))\,. \tag{2}$$

This is the equation in the new coordinate.

Definition 2.1 *The equation $X' = F(X)$ on the open set $\mathcal{O}$ is differentiably conjugate to $Y' = G(Y)$ on Ω when there is a continuously differentiable invertible map $h : \mathcal{O} \to \Omega$ with differentiable inverse so that the change of variable $Y = h(X)$ converts one system to the other. This holds if and only if* (2) *holds.*

Remark 2.1 *If one has a differentiable conjugacy $h : \mathcal{O} \to \Omega$ and ω is an open subset of Ω then $h : h^{-1}(\omega) \to \omega$ is a differentiable conjugacy of the smaller sets.*

Definition 2.2 *If $\underline{X}$ and $\underline{Y}$ are points the differential equations are said to be locally differentiably conjugate when there are exist neighborhood $\mathcal{O} \ni \underline{X}$ and $\Omega \ni \underline{Y}$ and a conjugacy as in the preceding definition.*

Exercise 2.1 *Show that local differentiable conjugacy is an equivalence relation. Need to verify reflexivity and transitivity. The first asserts that an equation is conjugate to itself. The second that if one has three equations with the first conjugate to the second and the second conjugate to the third, then the first is conjugate to the third.*

Equation (2) makes sense only for differentiable h because of the $D_X h$. There is an equivalent version that does not involve derivative of h. Denote by $\phi_t(X)$ and $\psi_t(Y)$ the flows of the respective differential equations.

Theorem 2.1 *The map $Y = h(X)$ is a differentiable conjugacy if and only if for all $X \in \mathcal{O}$ and t one has $\phi_t(X) \in \mathcal{O}$ if and only if $\psi_t(h(X)) \in \Omega$ and*

$$h(\phi_t(X)) \;=\; \psi_t(h(X))\,. \tag{3}$$

Proof. If the mapping $Y = h(X)$ converts the differential equation to $Y' = G(Y)$ and equivalently $X = h^{-1}(Y)$ converts the Y equation to that in X it follows that the solution curve $\phi_t(\underline{X})$ is mapped to the solution curve $\psi_t(h(\underline{X}))$. This is (3).

Conversely, differentiating (3) with respect to time using the chain rule yields

$$D_X h(\phi_t(X))\,\frac{d}{dt}\,\phi_t(X) \;=\; \frac{d\psi_t(h(X))}{dt}\,.$$

Using the two differential equations yields

$$D_X h(\phi_t(X))\;F(\phi_t(X)) \;=\; G(\psi_t(h(X))\,.$$

Setting $t = 0$ yields

$$D_X h(X)\ F(X) \ = \ G(h(X))\,.$$

This is equivalent to (2). $\square$

Remark 2.2 **i.** Formula (3) does not involve the derivatives of h. It allows us to generalize to conjugacies that are not differentiable.

ii. Instead of converting the X differential equation to the Y equation it says that h converts the flow ϕ_t to ψ_t.

Equation (3) *implies that if* h *is a differentiable conjugacy and you know the value of* $h(\underline{X})$ *then the value of* $h(X)$ *is determined along the orbit through* $\underline{X}$.

Indeed, suppose that $h(\underline{X}) = \underline{Y}$ is known. Denote by $X(t) = \phi_t(\underline{X})$ and $Y(t) = \psi_t(\underline{Y})$ the orbits through $\underline{X}$ and $\underline{Y}$. Equation 3 evaluated at $\underline{X}$ asserts that $h(X(t)) = Y(t)$ determining h on the orbit $X(t)$.

It suffices to know the value of $h(X)$ *on one point of each orbit in* $\mathcal{O}$ *to determine entirely the values of* h.

2.1 Differentiable conjugacy away from equilibria

The next result asserts that away from equilibria all differential equations are differentiably conjugate.

Theorem 2.2 (Flow Box Theorem) *If* $F(\underline{X}) \neq 0$ *then there are open sets* $\mathcal{O} \ni \underline{X}$ *and* $\Omega \ni 0$ *so that* $X' = F(X)$ *on* $\mathcal{O}$ *is differentially conjugate to* $Y' = (1, 0, \ldots, 0)$ *on* Ω.

Remark 2.3 **i.** Any two systems are differentiably conjugate at points that are not equilibrium points. Differentiable conjugacy is a good notion away from equilibria.

ii. The next section shows that differentiable conjugacy is too strong an equivalence relation at equilibria. The equivalence classes are too small.

Proof of the Flow Box Theorem. Change basis in $\mathbb{R}^d$ so that $F(\underline{X})$ is the first basis element. Then translate coordinates so that $\underline{X} = 0$. In this way reduce to the case

$$\underline{X} \ = \ 0, \qquad F(0) \ = \ \big(1, 0, 0, \ldots, 0\big)\,.$$

Near the origin, orbits of both equations have velocity close to $\big(1,0,0,\dots,0\big)$ so each orbit crosses $\{x_1=0\}$. Thus any conjugacy is uniquely determined by its values on $\{x_1=0\}$. Construct $h(X)$ so that

$$h(0,x_1,\dots,x_d) \;=\; (0,x_1,\dots,x_d)\,.$$

Denote by ϕ_t and ψ_t the flows of the X and Y differential equations respectively. To be a conjugacy requires that

$$h\big(\phi_t(0,x_2,\dots,x_d)\big) \;=\; \psi_t(h(0,x_2,\dots,x_d) \;=\; \big(t,x_2,\dots,x_d)\,.$$

To show that this uniquely determines h on a neighborhood of the origin it suffices to show that the map

$$\mathbb{R}^1\times\mathbb{R}^{d-1} \;\ni\; t,x_2,\dots,x_d \quad\mapsto\quad \phi_t(0,x_2,\dots,x_d)\;\in\mathbb{R}^d$$

is invertible on a neighborhood of the origin. It suffices to show that the Jacobian matrix at the origin is invertible.

The first column of the matrix is the derivative with respect to time of $\phi_t(0,0,\dots,0)$ at $t=0$. That is the tangent vector to the solution at 0 so is equal to $F(0)=(1,0,\dots,0)$.

The next $d-1$ columns of Jacobian are the x derivatives of $\phi(0,x_2,\dots,x_d)=(0,x_2,\dots,x_d)$. Thus, the Jacobian is equal to the identity so is invertible. Thus h is uniquely determined and the construction guarantees that h satisfies the conjugacy relation (3). $\square$

2.2 Differentiable conjugacy at equilibria

Theorem 2.3 *If $X'=F(X)$ and $Y'=G(Y)$ are differentiably conjugate by $Y=h(X)$ on neighborhoods of equilibria $\underline{X}$ and $\underline{Y}$, then the linearizations*

$$W'=AW,\quad A:=D_XF(\underline{X})\qquad\text{and}\qquad Z'=BZ,\quad B:=D_YG(\underline{Y})$$

are linearly conjugate by $H=D_Xh(\underline{X})W$.

Proof. Compute the Taylor expansion of the left hand side of (2) at $\underline{Y}$ to find,

$$B\,(Y-\underline{Y})\;+\;\text{higher order terms}\,.$$

Denote $H:=D_Xh(\underline{X})$. The Taylor expansion of the right hand side of (2) is

$$H\,A\,H^{-1}(Y-\underline{Y})\;+\;\text{higher order terms}\,.$$

The leading order terms must be the same so $B = HAH^{-1}$. This is the desired linear conjugacy. $\square$

Example 2.1 *Therefore* $x' = 2x$ *and* $y' = y$ *are* **not differentiably conjugate** *near the origin. As in Remark* (1.1)*, this shows that differentiable conjugacy is* **too strong** *a condition as soon as there are equilibria.*

3 Topological conjugacy in dimension 1

Example 3.1 *Continuing the preceding example, if* λ_j *are nonzero real numbers with the same sign then the linear equations* $x' = \lambda_1 x$ *and* $y' = \lambda_2 y$ *are topologically conjugate.*

Indeed if $0 < \lambda_2 < \lambda_1$*, the example in §4.2 of Hirsch-Smale-Devaney shows that the unique topological conjugacy* h *between* $x' = \lambda_1 x$ *and* $y' = \lambda_2 y$ *on* $[0, \infty[$ *satisfying* $h(\pm 1) = \pm 1$ *is given by*

$$h(x) \;=\; x^{\lambda_2/\lambda_1} \,.$$

This is not differentiable at $x = 0$*. It is Hölder continuous. Defining* h *at* ± 1 *gives its value at one point on each orbit.*

This section shows that in dimension 1, conjugacy by continuous but not differentiable maps has many desirable properties. That continues to be true in higher dimension.

Suppose that the flow of the ordinary differential equation

$$\dot{x} = f(x) \tag{4}$$

maps the interval I to itself for $t > 0$. We say that I in invariant. The interval may be open, closed, or half open. It can be infinite on one side or both.

Similarly suppose that J in an invariant interval for the differential equation

$$\dot{y} = g(y)\,. \tag{5}$$

Denote by $\phi(t, x)$ the flow of the x equation and $\psi(t, y)$ the flow of the y equation. Invariance says that for $t \geq 0$ and $x \in I$, $\phi(t, x) \in I$. Similarly for J.

Definition 3.1 *The differential equations on I and J are* **topologically conjugate** *when there is a continuous $h : I \to J$ that is one to one, onto, with continuous inverse so that for all $t \geq 0$ and $x \in I$,*

$$h(\phi(t,x)) \;=\; \psi(t, h(x))\,.$$

Exercise 3.1 *Show that this holds if and only if for all $t \geq 0$ and $y \in J$,*

$$h^{-1}(\psi(t,y)) \;=\; \phi(t, h^{-1}(y))\,.$$

Discussion. This shows that the definition is symmetric on interchange of the equations.

Theorem 3.1 *If $-\infty < a < b < \infty$, $-\infty < \widetilde{a} < \widetilde{b} < \infty$, $f(a) = f(b) = g(\widetilde{a}) = g(\widetilde{b}) = 0$, $f \neq 0$ on $]a,b[$, and $g \neq 0$ on $]\widetilde{a},\widetilde{b}[$, then equation* (4) *on $[a,b]$ is topologically conjugate to* (5) *on $[\widetilde{a},\widetilde{b}]$.*

Remark 3.1 In general one cannot do better than this with regards to differentiability. The proof constructs h that is differentiable on $]a,b[$ with differentiable inverse. Typically at least one of h and h^{-1} is not differentiable at a.

Proof of Theorem. We prove the case where both f and g are positive on the interior of the intervals. The three other sign possibilities are similar.

The idea is to let the dynamics define the conjugacy. From the Fundamental Theorem of the Phase Line, we know that the intervals I and J are invariant. As $t \to \infty$ orbits approach the right hand equilibria and and $t \to -\infty$ the left hand.

Pick a point $x_1 \in]a,b[$ and $y_1 \in]\widetilde{a},\widetilde{b}[$. We show that there is a unique conjugacy $h(x)$ with $h(x_1) = y_1$.

For any $x \in]a,b[$ there is a unique time $-\infty < t(x) < \infty$ so that

$$\phi(t(x), x_1) = x\,.$$

The function $t(x)$ is a strictly increasing function of x. The function $t(x)$ is determined by the one equation

$$\phi(t, x_1) \;= x \tag{6}$$

for unknown t. Since

$$\frac{\partial \phi(t,x)}{\partial t} \;=\; f\big(\phi(t,x)\big) \;>\; 0\,, \tag{7}$$

the implicit function theorem implies that $t(x)$ is a differentiable function of x. Suppose that $\underline{x}$ is an arbitrary point of $]a, b[$ and $\phi(\underline{t}, x_1) \;=\; \underline{x}$. The in equation (7) is the main hypothesis of the implicit function theorem that guarantees that near $\underline{t}, \underline{x}$ there is a unique solution $\widetilde{t}(x)$ to (6) that is continuously differentiable. This local solution provided by the Implicit Function Theorem must agree with the previously found solution $t(x)$ near $\underline{x}$ by uniqueness of solutions (6). Therefore $t(x)$ is continuously differentiable near $\underline{x}$.

If there were a conjugacy with $h(x_1) = y_1$ it would satisfy

$$h(x) \;=\; h(\phi(t(x), x_1)) \;=\; \psi(t(x), h(x_1)) \;=\; \psi(t(x), y_1)\,. \tag{8}$$

This shows that there is only one possibility and that it is given by (8).

Since $t(x) \to \infty$ as $x \to b$ it follows that h has a continuous extension to $x = b$ by setting $h(b) = \widetilde{b}$. Similarly defining $h(a) = \widetilde{a}$ yields a continuous strictly increasing map of $[a, b]$ onto $[\widetilde{a}, \widetilde{b}]$.

The inverse of a strictly increasing continuous function is also a continuous strictly increasing function proving the invertibility of h.

It remains to show that

$$h(\phi(t, x)) \;=\; \psi(t, h(x))\,. \tag{9}$$

If x is an endpoint this is immediate.

Equation (8) implies that for all $-\infty < t < \infty$

$$h(\phi(t, x_1)) \;=\; \psi(t, h(x_1)) \tag{10}$$

If x is not an endpoint, write $x = \phi(t(x), x_1)$ so $\phi(t, x) = \phi(t + t(x), x_1)$. Compute using (8)-(10),

$$h(\phi(t, x)) = h(\phi(t{+}t(x), x_1)) = \psi(t{+}t(x), y_1)) = \psi(t, \psi(t(x), y_1)) = \psi(t, h(x)).$$

completing the proof that h is a conjugacy. $\square$

Exercise 3.2 i. *With λ_j and equations as in the Example 3.1, find the unique conjugacy $g : [0, \infty[\to [0, \infty[$ that satisfies $g(1) = 2$.*

ii. *Is it g or g^{-1} that is not differentiable at the origin?*

One can glue conjugacies on adjacent intervals to yield conjugacies on the union. In this way the result just proved puts meat on the bones of the definition of equivalence of phase portraits in the handout on Dynamics in Dimension 1. What was "phase portraits look alike" can now be strengthened to topological conjugacy.

4 Linearization or Hartman-Grobman Theorem

4.1 Main result

Definition 4.1 *The equations $X' = F(X)$ on $\mathcal{O}$ and $Y' = G(Y)$ on Ω are topologically conjugate if and only if there is a continuous invertible $h : \mathcal{O} \to \Omega$ so that for all $X \in \mathcal{O}$ and t one has $\phi_t(X) \in \mathcal{O}$ if and only if $\psi_t(h(X)) \in \Omega$ and*

$$h(\phi_t(X)) \;=\; \psi_t(h(X)) . \tag{11}$$

Remark 4.1 **i.** As in the case of differentiable conjugacy, a topological conjugacy is determined by its values at one point on each orbit. To take advantage of this one often defines a conjugacy on a section transverse to the flow and lets the dynamics extend to a much larger set.

ii. As for differentiable conjugacy in Section 2, one defines local topological conjugacy. The local concept defines an equivalence relation.

Theorem 4.1 *If the system $X' = F(X)$ has dimension $k_1 + k_2$ and an equilibrium $\underline{X}$ at which the linearization has k_1 eigenvalues with strictly negative real part and k_2 eigenvalues with strictly positive real part then it is topologically conjugate to the system*

$$Y := (Y_I, Y_{II}) \in \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}, \qquad Y_I' \;=\; -Y_I, \qquad Y_{II}' \;=\; Y_{II} .$$

Example 4.1 *The example of a center with small outward or inward cubic perturbation*

$$X' \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X \;+\; \epsilon\, |X|^2\, X$$

is a sink for $\epsilon < 0$ and a source for $\epsilon > 0$ and in all cases linearizes to the same center. This shows that the hyperbolicity hypothesis is essential.

Example 4.2 *Topological conjugacy can strikingly deform images. For example a linear spiral sink in $\mathbb{R}^2$ is topologically conjugate to $X' = -X$, while the phase diagrams don't look very similar with the usual criteria of pattern recognition. It is the lack of differentiability of the conjugacies at the equilibria that allows for this distortion.*

4.2 Proof of topological conjugacy of sinks

We prove the Linearization Theorem only in the case that $X' = F(X)$ has a sink. By time reversal this implies the case of sources. It also yields a complete topological classification of *linear* hyperbolic dynamics.

Proof of linearization at a sink. Translate coordinates so that $\underline{X} = 0$. Denote by $A := D_X F(0)$ the coefficient matrix of the linearized equation. By hypothesis, A has only eigenvalues with strictly negative real part. Choose a scalar product Q and an $c, r > 0$ so that if $X' = F(X)$ with $Q(X(\underline{t}), X(\underline{t})) \leq r^2$ one has

$$\frac{dQ(X(t), X(t))}{dt}\bigg|_{t=\underline{t}} \;\leq\; -c\, Q(X(\underline{t}), X(\underline{t}))\,.$$

This inequality shows that if $-\infty < t_1 < t_2 < \infty$ and $X(t)$ is a solution with $X(t) \in B$ for $t_1 \leq t \leq t_2$ then

$$Q(X(t_2), X(t_2)) \;\leq\; e^{-c(t_2 - t_1)}\, Q(X(t_1), X(t_1)) \tag{12}$$

Define the ellipsoid S and solid ellipsoid B, by

$$S \;:=\; \{X \,:\, Q(X, X) = r^2\}\,, \qquad B \;:=\; \{X \,:\, Q(X, X) \leq r^2\}\,.$$

With $0 = t_1 < t_2 < \infty$, (12) shows that orbits starting on S say inside B and tend to the origin as $t \to \infty$. Taking $t_2 < 0 = t_1$ shows that orbits starting at any point of $B \backslash 0$ when followed backward in time have Q growing exponentially till the orbit reaches S. This shows that *the map*

$$[0\,,\,\infty[\times S \;\ni\; t, w \;\mapsto\; \phi_t(w) \;:= g(t, w) \;\in B \setminus 0$$

is a one to one and onto continuously differentiable map.

To show that the inverse is continuously differentiable it suffices, by the Inverse Function Theorem, to show that for every $\underline{t}, \underline{w}$ the derivative $D_{t,w} g(\underline{t}, \underline{w})$ is an invertible linear map from $\mathbb{R}^d$ to itself.

Exercise 4.1 *Give more detail of this application of the Inverse Function Theorem.* **Hint.** There is an analogous argument in the proof of Theorem 3.1.

For $\underline{w} \in S$, choose a basis $v_1, \ldots, v_{d-1}$ of the set of vectors tangent to S at $\underline{w}$. Denote by v_0 the vector $(1, 0, , \ldots, 0)$ corresponding to a unit change in t and no change in w. Need to show that

$$\Big(D_{t,w} g(\underline{t}, \underline{w})\Big) v_j\,, \quad j = 0, 1, \ldots, d-1$$

is a basis for $\mathbb{R}^d$. Introduce the solutions of the linearized equation at $X(t) = \phi_t(\underline{w})$,

$$\Gamma(t) \;:=\; D_X F(X(t))\,, \qquad Z_j' \;=\; \Gamma(t)\, Z_j\,, \quad Z_j(0) \;=\; v_j\,.$$

The definition of the linearization shows that

$$\Big(D_{t,w} g(\underline{t}, \underline{w})\Big) v_j \;=\; Z_j(\underline{t})\,.$$

Exercise 4.2 *Explain this in a little more detail.*

Since the $Z_j(0)$ are a basis it follows from the Fundamental Theorem of Linear Systems that the $Z_j(t)$ are a basis for all t. Taking $t = \underline{t}$ yields the desired invertibility.

Define a conjugation h of ϕ_t on B to ψ_t on B. The equilibria must be mapped to each other, $h(0) \;=\; 0$. Each orbit in $B \setminus 0$ touches S so it suffices to define the values of h on S. Define the conjugation h to be equal to the identity on S. The conjugacy relation

$$h \circ \phi_t \;=\; \psi_t \circ h \tag{13}$$

then determines h as a one to one and onto map of B to itself that is continuously differentiable along with its inverse from $B \setminus 0$ to itself.

By construction, equation (13) is satisfied. To complete the proof one needs to show that h and h^{-1} are continuous. Only continuity at the origin remains unproved.

To prove continuity of h at 0, suppose that $X_n \in B$ with $X_n \to 0$. Need to show that $h(X_n) \to 0$. Since $h(0) = 0$ it is sufficient to treat the case where $X_n \neq 0$ for all n. Write $X_n = \phi_{t(n)} W_n$ with $W_n \in S$. Since $X_n \to 0$ one must have $t(n) \to \infty$.

Exercise 4.3 *Explain why.*

Therefore

$$h(X_n) \;:=\; \psi_{t(n)}(W_n) \;=\; e^{-t(n)}\, W_n \;\to\; 0\,.$$

Thus h is continuous.

The continuity of the inverse is proved by a nearly identical argument. Composing (13) on the left and right with h^{-1} yields

$$\phi_t \circ h^{-1} \;=\; h^{-1} \circ \psi_t\,. \tag{14}$$

Suppose that $Y_k \in B$ with $Y_k \to 0$. Need to show that $h^{-1}(Y_k) \to 0$. It suffices to consider $Y_k \neq 0$ for all n. Choose $U_k \in S$ and $t(k) \geq 0$ so that $Y_k = \psi_{t(k)}(U_k) = e^{-t(k)} U_k$. Therefore $Y_k \to 0$ implies $t(k) \to \infty$.

Using (14) shows that as $k \to \infty$,

$$h^{-1}(Y_k) \;=\; h^{-1}\big(\psi_{t(k)}(U_k)\big) \;=\; \phi_{t(k)}(h^{-1}(U_k)) \;=\; \phi_{t(k)}(U_k)) \;\to\; 0\,.$$

The proves the continuity of h^{-1} and completes the proof of the linearization of sinks. $\square$

4.3 Topological conjugacy of linear systems

Though we do not prove the Linearization Theorem in its full generality, the case of linear equations can be reduced to the case of sinks and sources.

Theorem 4.2 *Suppose that $X' = AX$ and $Y' = BY$ are hyperbolic linear systems on finite dimensional complex vector spaces $\mathbb{V}$ and $\mathbb{W}$ respectively. They are topologically conjugate if and only if*

$$\dim \mathbb{V}_u \;=\; \dim \mathbb{W}_u \qquad \text{and} \qquad \dim \mathbb{V}_s \;=\; \dim \mathbb{W}_s$$

where the subscripts u and s denote the unstable and stable subspaces.

Proof Decompose

$$\mathbb{V} \;=\; \mathbb{V}_u \;\oplus\; \mathbb{V}_s\,, \qquad A \;=\; A\big|_{\mathbb{V}_u} \;\oplus\; A\big|_{\mathbb{V}_s}\,,$$

$$\mathbb{W} \;=\; \mathbb{W}_u \;\oplus\; \mathbb{W}_s\,, \qquad B \;=\; B\big|_{\mathbb{W}_u} \;\oplus\; B\big|_{\mathbb{W}_s}\,.$$

Choose topological conjugacies

$$h_u \;:\; \mathbb{V}_u \;\to\; \mathbb{W}_u \qquad \text{and} \qquad h_s \;:\; \mathbb{V}_s \;\to\; \mathbb{W}_s$$

conjugating the flows on the unstable and stable spaces. This is possible since these are sources and sinks respectively.

The map

$$\mathbb{V} \ni v_u + v_s \;\mapsto\; h(v_u + v_s) \;:=\; h_u(v_u) \;+\; h_s(v_s)$$

conjugates the X and Y equations. $\square$

Example 4.3 *HSD page 67 Case 1, considers the case where the X equation is a* **linear** *saddle in dimension $d = 2$. It depends on Example 3.1 and is an elementary example of such a gluing.*

Exercise 4.4 *Explain why an analogous argument does not reduce the* **nonlinear** *Linearization Theorem for a general hyperbolic equilibrium to a pair of conjugations on the stable and unstable manifolds.*

Exercise 4.5 *Show that the conjugation h just constructed has a natural extension to a conjugation defined on the entire space $\mathbb{R}^d$.* **Hint.**Extend the maps h_u and h_s to the entire stable and unstable subspaces. Do this by considering the past of orbits starting on the ellipsoid S.

Exercise 4.6 *Show by example that for nonlinear problems the local conjugacy need not extend to a global one and explain where the proof analogous to that of the preceding exercise breaks down.*