Projectile Motion

Unit: Kinematics (Motion)

NGSS Standards: N/A

MA Curriculum Frameworks (2006): 1.2

AP Physics 1 Learning Objectives: 3.A.1.1, 3.A.1.3

Skills:

- solve problems involving linear motion in two dimensions

Language Objectives:

- Understand and correctly use the term "projectile."
- Set up and solve word problems involving projectiles.

Labs, Activities & Demonstrations:

- Play "catch."
- Drop one ball and punch the other at the same time.
- "Shoot the monkey."

Notes:

<u>projectile</u>: an object that is propelled (thrown, shot, *etc.*) horizontally and also falls due to gravity.

Gravity affects projectiles the same way regardless of whether the projectile is moving horizontally. Gravity does not affect the horizontal motion of the projectile. This means the vertical and horizontal motion of the projectile can be considered separately, using a separate set of equations for each.

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Assuming we can neglect friction and air resistance (which is usually the case in first-year physics problems), we make two important assumptions:

1. All projectiles have a constant horizontal velocity, v_h, in the positive horizontal direction. The equation for the horizontal motion (without acceleration) is:

   $$d = vt$$

   However, we have displacement and velocity in both the vertical and horizontal directions. This means we need add a subscript "*h*" to the horizontal quantities and a subscript "*v*" to the vertical quantities so we can tell them apart. The horizontal equation becomes:

   $$d_h = v_h t$$

2. All projectiles have a constant downward acceleration of $g = \pm 10\,\frac{\text{m}}{\text{s}^2}$ (in the vertical direction), due to gravity. (You need to choose whether the positive vertical direction is up or down, depending on the situation.) The equation for the vertical motion is:

   $$d = v_o t + \tfrac{1}{2} a t^2$$

   Adding a subscript "v", (and using "*g*" instead of "*a*" because gravity is causing the acceleration), this becomes:

   $$d_v = v_{o,v} t + \tfrac{1}{2} g t^2$$

   Notice that we have ***two*** subscripts on the velocity, because it is ***both*** the initial velocity v_o ***and also*** the vertical velocity v_v.

3. The time that the projectile spends falling must be the same as the time that the projectile spends moving horizontally. This means time (t) is the same in both equations, which means time is the variable that links the vertical problem to the horizontal problem.

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The consequences of these assumptions are:

- The *time* that the object takes to fall is determined by its movement *only* in the vertical direction. (*I.e.,* all motion stops when the object hits the ground.)
- The *horizontal distance* that the object travels is determined by the time (calculated above) and its velocity in the horizontal direction.

Therefore, the general strategy for most projectile problems is:

1. Solve the vertical problem first, to get the time.
2. Use the time from the vertical problem to solve the horizontal problem.

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Sample problem:

Q: A ball is thrown horizontally at a velocity of $5\,\frac{\text{m}}{\text{s}}$ from a height of 1.5 m. How far does the ball travel (horizontally)?

A: We're looking for the horizontal distance, d_h. We know the vertical distance, $d_v = 1.5\,\text{m}$, and we know that $v_{o,v} = 0$ (there is no initial vertical velocity because the ball is thrown horizontally), and we know that $a = g = 10\,\frac{\text{m}}{\text{s}^2}$.

We need to separate the problem into the horizontal and vertical components.

Horizontal:

$$d_h = v_h t$$
$$d_h = 5\,t$$

At this point we can't get any farther, so we need to turn to the vertical problem.

Vertical:

$$d_v = v_{o,v} t + \tfrac{1}{2} g t^2$$
$$d_v = \tfrac{1}{2} g t^2$$
$$1.5 = (\tfrac{1}{2})(10)\,t^2$$
$$\frac{1.5}{5.0} = 0.30 = t^2$$
$$t = \sqrt{0.30} = 0.55\,s$$

Now that we know the time, we can substitute it back into the horizontal equation, giving:

$$d_h = (5)(0.55) = 2.74\,\text{m}$$

A graph of the vertical *vs.* horizontal motion of the ball looks like this:

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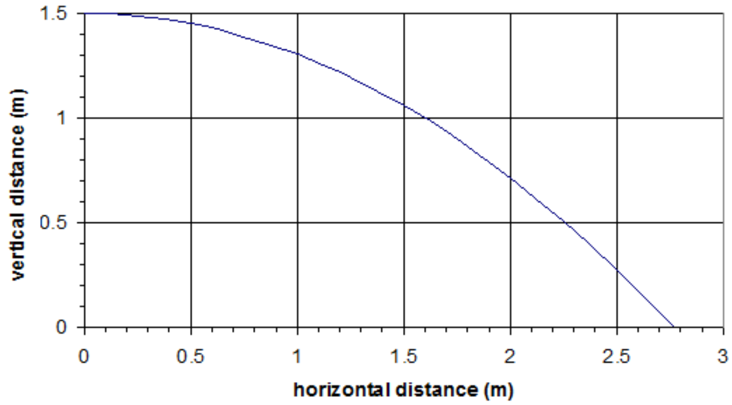

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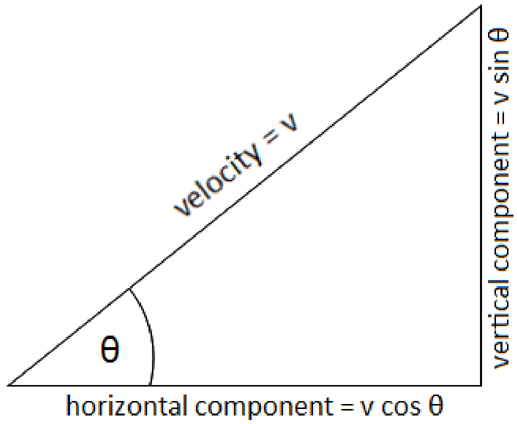
Projectiles Launched at an Angle

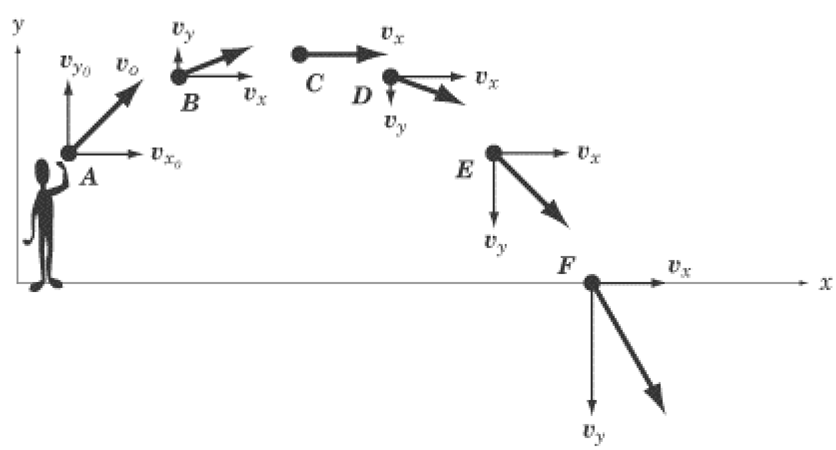
If the object is thrown/launched at an angle, you will need to use trigonometry to separate the velocity vector into its horizontal (x) and vertical (y) components:



Thus:

- horizontal velocity = $v_h = v \cos\theta$
- *initial* vertical velocity = $v_{o,v} = v \sin\theta$

Note that the vertical component of the velocity, v_y, is constantly changing because of acceleration due to gravity.



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Sample Problems:

Q: An Angry Bird is launched upward from a slingshot at an angle of 40° with a velocity of $20\,\frac{m}{s}$. The bird strikes the pigs' fortress at the same height that it was launched from. How far away is the fortress?

A: We are looking for the horizontal distance, d_h.

We know the magnitude and direction of the launch, so we can find the horizontal and vertical components of the velocity using trigonometry:

$$v_h = v\cos\theta = 20\cos 40° = (20)(0.766) = 15.3\,\tfrac{m}{s}$$
$$v_{o,v} = v\sin\theta = 20\sin 40° = (20)(0.643) = 12.9\,\tfrac{m}{s}$$

We call the vertical component $v_{o,v}$ because it is both the initial velocity (v_o) and the vertical velocity (v_v), so we need both subscripts.

Let's make upward the positive vertical direction.

We want the horizontal distance (d_h), which is in the horizontal equation, so we start with:

$$d_h = v_h t$$
$$d_h = 15.3\,t$$

At this point we can't get any farther because we don't know the time, so we need to get it by solving the vertical equation.

The Angry Bird lands at the same height as it was launched, which means the vertical displacement (d_v) is zero. We already calculated that the initial vertical velocity is $12.9\,\frac{m}{s}$. If upward is the positive direction, acceleration due to gravity needs to be negative (because it's downward), so $a = g = -10\,\frac{m}{s^2}$.

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The vertical equation is:

$$d_v = v_o t + \tfrac{1}{2} a t^2$$
$$0 = 12.9\,t + (\tfrac{1}{2})(-10)\,t^2$$
$$0 = 12.9\,t - 5\,t^2$$
$$0 = t\,(12.9 - 5\,t)$$
$$t = 0, \quad 12.9 - 5\,t = 0$$
$$12.9 = 5\,t$$
$$t = \frac{12.9}{5} = 2.58\text{ s}$$

Finally, we return to the horizontal equation to find d_h.

$$d_h = 15.3\,t$$
$$d_h = (15.3)(2.58) = 39.5\text{ m}$$

Q: A ball is thrown upward at an angle of 30° from a height of 1 m with a velocity of $18\,\tfrac{\text{m}}{\text{s}}$. How far does the ball travel?

A: As before, we are looking for the horizontal distance, d_h.

Again, we'll make upward the positive vertical direction.

Again we find the horizontal and vertical components of the velocity using trigonometry:

$$v_h = v\cos\theta = 18\cos 30° = (18)(0.866) = 15.6\,\tfrac{\text{m}}{\text{s}}$$
$$v_{o,v} = v\sin\theta = 18\sin 30° = (18)(0.500) = 9.0\,\tfrac{\text{m}}{\text{s}}$$

Starting with the horizontal equation:

$$d_h = v_h t$$
$$d_h = 15.6\,t$$

Again, we can't get any farther, so we need to get the time from the vertical problem.

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The ball moves 1 m downwards. Its initial position is $s_o = +1\,\text{m}$, and its final position is $s = 0$, so we can use the equation:

$$s - s_o = v_o t + \tfrac{1}{2} a t^2$$
$$0 - 1 = 9.0\,t + (\tfrac{1}{2})(-10)\,t^2$$
$$0 = 1 + 9.0\,t - 5\,t^2$$

This time we can't factor the equation, so we need to solve it using the quadratic formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$t = \frac{-9 \pm \sqrt{9^2 - (4)(-5)(1)}}{(2)(-5)}$$
$$t = \frac{9 \pm \sqrt{81 + 20}}{10} = \frac{9 \pm \sqrt{101}}{10}$$
$$t = \frac{9 \pm 10.049}{10} = \frac{19.049}{10} = 1.90\,\text{s}$$

Now we can go back to the horizontal equation and use the horizontal velocity $(15.6\,\frac{\text{m}}{\text{s}})$ and the time $(1.90\,\text{s})$ to find the distance:

$$d_h = v_h t = (15.6)(1.9) = 29.7\,\text{m}$$

Another way to solve the vertical problem is to realize that the ball goes up to its maximum height, then comes back down. The ball starts from a different height than it falls down to, so unfortunately we can't just find the time at the halfway point and double it.

At the maximum height, the vertical velocity is zero. For the ball going up, this gives:

$$v - v_o = at$$
$$0 - 9 = (-10)\,t_{up}$$
$$t_{up} = \frac{-9}{-10} = 0.9\,\text{s}$$

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At this point, the ball has reached a height of:

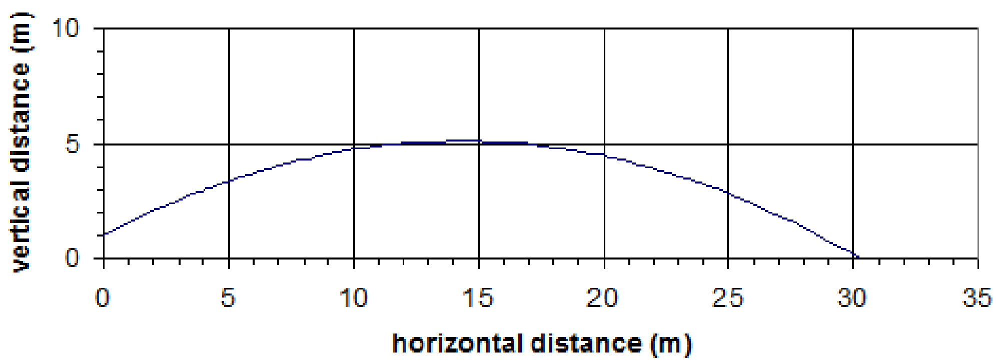
$$s = s_o + v_o t + \tfrac{1}{2}at^2$$
$$s = 1 + (9)(0.9) + (\tfrac{1}{2})(-10)(0.9)^2$$
$$s = 1 + 8.1 - 4.05 = 5.05\,\text{m}$$

Now the ball falls from its maximum height of 4.6 m to the ground. The time this takes is:

$$d = \tfrac{1}{2}at_{down}^2$$
$$5.05 = (\tfrac{1}{2})(10)\,t_{down}^2$$
$$t_{down}^2 = \frac{5.05}{5} = 1.01$$
$$t_{down} = \sqrt{1.01} = 1.00\,\text{s}$$

Thus the total elapsed time is $t_{up} + t_{down}$ = 0.90 + 1.00 = 1.90 s. (Q.E.D.)

The motion of this ball looks like this:



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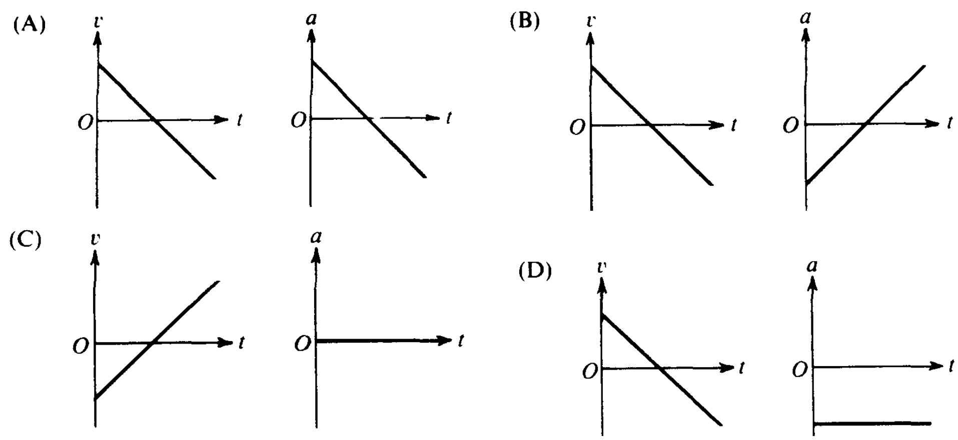
What AP Projectile Problems Look Like

AP motion and acceleration problems almost always involve graphs or projectiles. Here is an example that involves both:

Q:

A projectile is fired with initial velocity v_o at an angle θ_o with the horizontal and follows the trajectory shown above. Which of the following pairs of graphs best represents the vertical components of the velocity and acceleration, v and a, respectively, of the projectile as functions of time t?



A: *Because the object is a projectile:*

- *It can move both vertically and horizontally.*
- *It has a nonzero initial horizontal velocity. However, because the problem is asking about the vertical components, we can ignore the horizontal velocity.*
- *It has a constant acceleration of* $-g$ *(i.e., g in the downward direction) due to gravity.*

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For each pair of graphs, the first graph is velocity vs. time. The slope, $\frac{\Delta v}{\Delta t}$*, is acceleration. Because acceleration is constant, the graph has to have a constant. if we choose up to be the positive direction (which is the most common convention), correct answers would be (A), (B), and (D). If we choose down to be positive, only (C) would be correct.*

The second graph is acceleration vs. time. We know that acceleration is constant, which eliminates choices (A) and (B). We also know that acceleration is not zero, which eliminates choice (C). This leaves choice (D) as the only possible remaining answer. Choice (D) correctly shows a constant negative acceleration, because the slope of the first graph is negative, and the y-value of the second graph is also negative.

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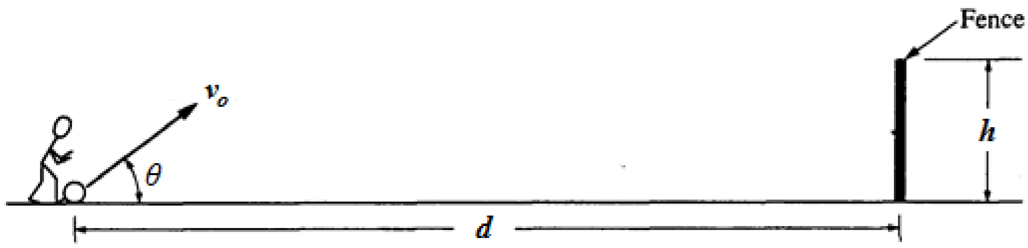
Q: A ball of mass *m*, initially at rest, is kicked directly toward a fence from a point that is a distance *d* away, as shown above. The velocity of the ball as it leaves the kicker's foot is v_o at an angle of θ above the horizontal. The ball just clears the top of the fence, which has a height of *h*. The ball hits nothing while in flight and air resistance is negligible.



a. Determine the time, *t*, that it takes for the ball to reach the plane of the fence, in terms of v_o, θ, *d*, and appropriate physical constants.

The horizontal component of the velocity is $v_o \cos\theta = \frac{d}{t}$.

Solving this expression for t gives $t = \frac{d}{v_o \cos\theta}$.

b. What is the vertical velocity of the ball when it passes over the top of the fence?

The velocity equation is $v = v_o + at$. *Substituting* $a = -g$, *and* $t = \frac{d}{v_o \cos\theta}$ *gives:*

$$v = v_o \sin\theta - \frac{gd}{v_o \cos\theta}$$

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Homework Problems

Horizontal (level) projectile problems:

1. A diver running $1.6\,\frac{\text{m}}{\text{s}}$ dives out horizontally from the edge of a vertical cliff and reaches the water below 3.0 s later.
   a. How high was the cliff?

      Answer: 44 m

   b. How far from the base did the diver hit the water?

      Answer: 4.8 m

2. A tiger leaps horizontally from a 7.5 m high rock with a speed of $4.5\,\frac{\text{m}}{\text{s}}$. How far from the base of the rock will he land?

   Answer: 5.6 m

Use this space for summary and/or additional notes.

3. A tiger leaps horizontally from a rock with height *h* at a speed of v_o. What is the distance, *d*, from the base of the rock where the tiger lands? (You may use your work from problem #2 above to guide your algebra.)

   Answer: $d = v_o\sqrt{\frac{2h}{g}}$

4. A ball is thrown horizontally from the roof of a building 56 m tall and lands 45 m from the base. What was the ball's initial speed?

   Answer: $13\,\frac{\text{m}}{\text{s}}$

5. The pilot of an airplane traveling $45\,\frac{\text{m}}{\text{s}}$ wants to drop supplies to flood victims isolated on a patch of land 160 m below. The supplies should be dropped when the plane is how far from the island?

   Answer: 257 m

Use this space for summary and/or additional notes.

Problems involving projectiles launched at an angle:

6. A ball is shot out of a slingshot with a velocity of $10.0\,\frac{\text{m}}{\text{s}}$ at an angle of 40.0° above the horizontal. How far away does it land?

   Answer: 10.05 m

7. A ball is shot out of a slingshot with a velocity of v_o at an angle of θ above the horizontal. How far away does it land? (You may use your work from problem #6 above to guide your algebra.)

   Answer: $v_o \cos\theta \left(\dfrac{v_o \sin\theta}{\frac{1}{2} g} \right) = \dfrac{v_o^2}{g}(2\sin\theta\cos\theta) = \dfrac{v_o^2}{g}\sin(2\theta)$

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8. The 12 Pounder Napoleon Model 1857 was the primary cannon used during the American Civil War. If the cannon had a muzzle velocity of $439\,\frac{\text{m}}{\text{s}}$ and was fired at a 5.00° angle, what was the effective range of the cannon (the distance it could fire)? (Neglect air resistance.)

Answer: 3 415 m (Note that this is more than 2 miles!)

Use this space for summary and/or additional notes.

9. A physics teacher is designing a ballistics event for a science competition. The ceiling is 3.00 m high, and the maximum velocity of the projectile will be $20.0\,\frac{m}{s}$.

   a. What is the maximum that the vertical component of the projectile's initial velocity could have?

   Answer: $7.67\,\frac{m}{s}$

   b. At what angle should the projectile be launched in order to achieve this maximum height?

   Answer: 22.5°

   c. What is the maximum horizontal distance that the projectile could travel?

   Answer: 28.9 m

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