Math 113 Homework 4 Solutions

Solutions by Guanyang Wang, with edits by Tom Church.

Exercises from the book.

Exercise 3.E.13 Suppose U is a subspace of V and $v_1+U,...,v_m+U$ is a basis of V/U and $u_1,...,u_n$ is a basis of U. Prove that $v_1,...,v_m$, $u_1,...,u_n$ is a basis of V.

Proof. First we prove that $v_1,...,v_m,u_1,...,u_n$ is a linearly independent list in V.

Suppose that $a_1,...,a_m,b_1,...,b_n \in \mathbb{F}$ are scalars such that

$$a_1v_1+...+a_mv_m+b_1u_1+...+b_nu_n=0$$

Thus $a_1v_1+...+a_mv_m=-(b_1u_1+...+b_nu_n)\in U$, which implies that

$$a_1(v_1+U)+...+a_m(v_m+U)=0+U$$

Because $v_1+U,...,v_m+U$ is a basis of V/U, this implies $a_1=...=a_m=0$. Therefore we have

$$b_1u_1+...+b_nu_n=0$$

Since $u_1,...,u_n$ is a basis of V, this implies $b_1=...=b_n=0$. Thus $a_1=...=a_m=b_1=...=b_n=0$, which implies that $v_1,...,v_m,u_1,...,u_n$ is a linearly independent list in V.

Then we prove that $v_1,...,v_m,u_1,...,u_n$ is a basis of V. Now suppose $v\in V$. Because the list $v_1+U,...,v_m+U$ spans V/U, there exist $c_1,...,c_m\in\mathbb{F}$ such that

$$v+U=c_1(v_1+U)+...+c_m(v_m+U)$$

Thus

$$v-c_1v_1-...-c_mv_m\in U$$

Because the list $u_1,...,u_n$ spans V/U, there exist $d_1,...,d_n\in\mathbb{F}$ such that

$$v-c_1v_1-...-c_mv_m=d_1u_1+...+d_nu_n.$$

Hence

$$v=c_1v_1+...+c_mv_m+d_1u_1+...+d_nu_n.$$

Then the list $v_1,...,v_m,u_1,...,u_n$ spans V and hence is a basis of V, as desired. □

Exercise 3.F.7 Suppose m is a positive integer. Show that the dual basis of the basis $1,x,...,x^m$ of $\mathcal{P}_m(\mathbb{R})$ is $\varphi_0,\varphi_1,...,\varphi_m$, where $\varphi_j(p)=\frac{p^{(j)}(0)}{j!}$. Here $p^{(j)}$ denotes the j^{th} derivative of p, with the understanding that the 0^{th} derivative of p is p.

Proof. From Proposition 3.98 we know that the dual basis is a basis of dual space. By definition of dual basis (3.96), we just need to check if

$$\varphi_j(x^k)=\begin{cases}1 \ (j=k)\\ 0 \ (j\neq k)\end{cases} \tag{0.1}$$

Note that $\varphi_j(x^k)=\frac{(x^k)^{(j)}(0)}{j!}$, hence if $j=k$, $\varphi_j(x^k)=1$, if $j\neq k$, $\varphi_j(x^k)=0$. Therefore we know that $\varphi_0,...,\varphi_m$ is the dual basis of $\mathcal{P}_m(\mathbb{R})$.

□

Exercise 3.F.8 Suppose m is a positive integer.
(a) Show that $1, x-5, ..., (x-5)^m$ is a basis of $\mathcal{P}_m(\mathbb{R})$.
(b) What is the dual basis of the basis in part(a)?

Proof. (a) Define $\varphi_0, \varphi_1, ..., \varphi_m \in (\mathcal{P}_m(\mathbb{R}))'$ by

$$\varphi_j(p) = \frac{p^{(j)}(5)}{j!}.$$

So suppose $a_0, ..., a_m \in \mathbb{F}$ and

$$a_0 + a_1(x-5) + ... + a_m(x-5)^m = 0.$$

Then for $j = 0, 1, ..., m$, we have

$$a_j = \varphi_j(a_0 + a_1(x-5) + ... + a_m(x-5)^m) = \varphi_j(0) = 0$$

.

Thus $a_0 = a_1 = ... = a_m = 0$. Hence $1, x-5, ..., (x-5)^m$ is a linearly independent list in $\mathcal{P}_m(\mathbb{R})$ of length $m+1$, which equals the dimension of $\mathcal{P}_m(\mathbb{R})$. Thus $1, x-5, ..., (x-5)^m$ is a basis of $\mathcal{P}_m(\mathbb{R})$ (by 2.39).

(b) Let $\varphi_0, \varphi_1, ..., \varphi_m \in (\mathcal{P}_m(\mathbb{R}))'$ be defined as in part (a). Then we have

$$\varphi_j((x-5)^k) = \begin{cases} 1 \ (j = k) \\ 0 \ (j \neq k) \end{cases} \tag{0.2}$$

From Proposition 3.98 we know that $\varphi_0, \varphi_1, ..., \varphi_m$ is the dual basis of the basis in part (a). □

Exercise 3.F.15 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T' = 0$ if and only if $T = 0$.

Proof. First suppose $T = 0$. For any $\varphi \in W'$, then $T'(\varphi) = \varphi \circ T = 0$, and thus $T' = 0$.

To prove the other direction, now suppose $T' = 0$. Thus

$$0 = T'(\varphi) = \varphi \circ T$$

for every $\varphi \in W'$.

If $T \neq 0$, we can find some $v \in V$ such that $Tv = w \neq 0$. We can extend Tv to a basis $Tv, w_2, ..., w_n$ of W. Now Proposition 3.5 implies that there exists a $\tilde{\varphi}$ such that $\tilde{\varphi}(Tv) = 1$ (and $\tilde{\varphi}(v_j)$ equals whatever we want for $j = 2, 3, ..., n$). Therefore $(T'(\tilde{\varphi}))(v) = \tilde{\varphi}(Tv) = 1$. Which contradicts the fact that $0 = T'(\varphi) = \varphi \circ T$ for every $\varphi \in W'$. So we must have $T = 0$, as desired. □

Exercise 5.A.12 Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbb{R}))$ by

$$(Tp)(x) = xp'(x)$$

for all $x \in \mathbb{R}$. Find all eigenvalues and eigenvectors of T.

Answer. A typical element p of $\mathcal{P}_4(\mathbb{R})$ is given by expression

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4,$$

where $a_0, ..., a_4 \in \mathbb{R}$.

With that expression, the eigenvalue-eigenvector equation $Tp = \lambda p$, which in this case is $xp'(x) = \lambda p(x)$, becomes

$$a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 = \lambda(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4)$$

Comparing coefficients in the equation above, we see that the eigenvalue-eigenvector equation is equivalent to the system of equations

$$\begin{aligned} 0 &= \lambda a_0 \\ a_1 &= \lambda a_1 \\ 2a_2 &= \lambda a_2 \\ 3a_3 &= \lambda a_3 \\ 4a_4 &= \lambda a_4. \end{aligned}$$

From the equations above, we can see that if $j \in \{0, 1, 2, 3, 4\}$ and $a_j \neq 0$, then we have $\lambda = j$ and $a_k = 0$ for any $k \neq j$. Thus the eigenvalue of T are $0, 1, 2, 3, 4$ and the corresponding eigenvectors are of the form c, cx, cx^2, cx^3, cx^4, where $c \in \mathbb{R}$ and $c \neq 0$. □

Exercise 5.A.15 Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

Answer. Suppose $v \in V$ and $\lambda \in \mathbb{F}$. Then we have

$$Tv = \lambda v \iff (S^{-1}TS)(S^{-1}v) = \lambda S^{-1}v$$

This is because if $Tv = \lambda v$, then $(S^{-1}TS)(S^{-1}v) = S^{-1}Tv = \lambda S^{-1}v$, on the other hand, if $(S^{-1}TS)(S^{-1}v) = \lambda S^{-1}v$, then $\lambda v = \lambda S(S^{-1}v) = S((S^{-1}TS)(S^{-1}v)) = Tv$.

Thus we see that T and $S^{-1}TS$ have the same eigenvalues, and furthermore, v is an eigenvector of T if and only if $S^{-1}v$ is an eigenvector of $S^{-1}TS$. □

Exercise 5.A.18 Show that the operator $T \in \mathcal{L}(\mathbb{C}^\infty)$ defined by

$$T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$$

has no eigenvalues.

Answer. The eigenvalue-eigenvector equation $Tz = \lambda z$ for this operator is

$$(0, z_1, z_2, \dots) = (\lambda z_1, \lambda z_2, \lambda z_3, ...)$$

which is equivalent to

$$0 = \lambda z_1, z_1 = \lambda z_2, z_2 = \lambda z_3, ...$$

The first equation implies $z_1 = 0$ or $\lambda = 0$. If $\lambda = 0$, then the rest of the equations implies $0 = z_1 = z_2 = ...$, which eliminates 0 as the possible eigenvalue. If $\lambda \neq 0$, then $z_1 = 0$, then the rest of the equations also implies $z_2 = z_3 = ... = 0 = z_1$, which eliminates all nonzero complex numbers λ as possible eigenvalues. Thus we conclude that T has no eigenvalues. □

Exercise 5.A.20 Find all eigenvalues and eigenvectors of the backward shift operator $T \in \mathcal{L}(\mathbb{F}^\infty)$ defined by

$$T(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots)$$

Answer. We will show that all $\lambda \in \mathbb{F}$ are eigenvalues of T, and the set of eigenvectors of T with eigenvalue λ is the set $V_\lambda = \{(z, \lambda z, \lambda^2 z, \dots) \mid z \in \mathbb{F}\}$.

First we show that if v is an eigenvector of T, then $v \in V_\lambda$ for some λ. That is, we show that $v = (z, \lambda z, \lambda^2 z, \dots)$ for some z and some λ. Suppose $v = (z_1, z_2, z_3, \dots)$ is an eigenvector for T with eigenvalue λ. Then the eigenvalue equation $T(v) = \lambda v$ takes the form

$$(\lambda z_1, \lambda z_2, \lambda z_3, \dots) = (z_2, z_3, z_4, \dots)$$

Since two vectors in $\mathbb{F}^\infty$ are equal if and only if their terms are all equal, this yields an infinite sequence of equations:

$$z_2 = \lambda z_1, \quad z_3 = \lambda z_2, \dots, \quad z_n = \lambda z_{n-1}, \dots$$

From this, we can repeatedly substitute $z_n = \lambda z_{n-1} = \lambda^2 z_{n-2} = \dots$, so in fact (by a simple induction)

$$z_n = \lambda^{n-1} z_1$$

So every eigenvector v with eigenvalue λ is of the form $v = (z_1, \lambda z_1, \lambda^2 z_1, \dots)$. Furthermore, for any $z \in \mathbb{F}$, if we set $z_1 = z$, $z_2 = \lambda z$, $\dots$, $z_n = \lambda^n z$, the vector

$$v = (z, \lambda z, \lambda^2 z, \dots)$$

satisfies the equations above and is an eigenvector of T with eigenvalue λ Therefore, the eigenspace V_λ of T with eigenvalue λ is the set of vectors

$$V_\lambda = \left\{(z, \lambda z, \lambda^2 z, \dots) \,\middle|\, z \in \mathbb{F}\right\}.$$

Finally, we show that every single $\lambda \in \mathbb{F}$ occurs as an eigenvalue of T. Given $\lambda \in \mathbb{F}$, consider the vector $v = (1, \lambda, \lambda^2, \dots)$. Applying T to v, we get

$$\begin{aligned} T(v) = (1, \lambda, \lambda^2, \dots) &= (\lambda, \lambda^2, \lambda^3, \dots) \\ &= \lambda(1, \lambda, \lambda^2, \dots) \end{aligned}$$

Thus $T(v) = \lambda v$ for this vector. We have thus shown that all $\lambda \in \mathbb{F}$ are eigenvalues for T, and the eigenspace for λ is $V_\lambda = \{(z, \lambda z, \lambda^2 z, \dots) \mid z \in \mathbb{F}\}$. □

Exercise 5.A.22 Suppose $T \in \mathcal{L}(V)$ and there exist nonzero vectors v and w in V such that

$$Tv = 3w \text{ and } Tw = 3v.$$

Prove that 3 or -3 is an eigenvalue of T.

Proof. The equations above imply that

$$T(v + w) = 3(v + w) \text{ and } T(v - w) = -3(v - w).$$

The vectors $v + w$ and $v - w$ cannot both be 0 (because otherwise we would have $v = w = 0$). Thus the equations above imply that 3 or -3 is an eigenvalue of T. □

Exercise 5.A.30 Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ and $4, -5$ and $\sqrt{7}$ are the eigenvalues of T. Prove that there exists $x \in \mathbb{R}^3$ such that $Tx - 9x = (4, -5, \sqrt{7})$

Proof. Since T has at most 3 distinct eigenvalues (by 5.13), the hypothesis imply that 9 is not an eigenvalue of T. Thus $T - 9I$ is surjective. In particular, there exists $x \in \mathbb{R}^3$ such that $(T - 9I)x = Tx - 9x = (4, -5, \sqrt{7})$. (The entries of this particular vector are a red herring: we could just as easily find a $y \in \mathbb{R}^3$ such that $Ty - 9y = (86, 75, 309)$ by the same argument.) □

Exercise 5.A.32 Suppose $\lambda_1, ..., \lambda_n$ is a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, ..., e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on $\mathbb{R}$.

Proof. Let $V = \text{span}(e^{\lambda_1 x}, ..., e^{\lambda_n x})$, and define $T \in \mathcal{L}(V)$ by $Tf = f'$. This linear map does map V into V because

$$T(e^{\lambda_j x}) = \lambda_j e^{\lambda_j x}.$$

This equation above also shows that for each $j = 1, ..., n$, the vector $e^{\lambda_j x}$ is an eigenvector of T with eigenvalue λ_j. Thus Proposition 5.10 implies that $e^{\lambda_1 x}, ..., e^{\lambda_n x}$ is linearly independent. □

Exercise 5.B.1 Suppose $T \in \mathcal{L}(V)$ and there exists a positive integer n such that $T^n = 0$.

(a) Prove that $I - T$ is invertible and that

$$(I - T)^{-1} = I + T + ... + T^{n-1}$$

(b) Explain how you would guess the formula above.

Proof. We have

$$\begin{aligned}
&(I - T)(I + T + ... + T^{n-1}) \\
&\quad = I + T + ... + T^{n-1} - T - T^2 - ... - T^{n-1} - T^n \\
&\quad = I - T^n = I
\end{aligned}$$

since $T^n = 0$.

Similarly, we have

$$\begin{aligned}
&(I + T + ... + T^{n-1})(I - T) \\
&\quad = I + T + ... + T^{n-1} - T - T^2 - ... - T^{n-1} - T^n \\
&\quad = I - T^n = I
\end{aligned}$$

Therefore $(I - T)$ is invertible and $(I - T)^{-1} = I + T + ... + T^{n-1}$.

(b) If $r \in \mathbb{C}$ and $|r| < 1$, then we might be familiar with the usual formula for the sum of a geometric series:

$$(1 - r)^{-1} = 1 + r + r^2 + ... + r^n + r^{n+1} + ...$$

If we guess that in the formula above, we can replace 1 with I and r with T, then we would have

$$(I - T)^{-1} = I + T + ... + T^{n-1}$$

where the sum becomes finite because $0 = T^n = T^{n+1} = \cdots$. $\square$

Exercise 5.B.2 Suppose $T \in \mathcal{L}(V)$ and $(T-2I)(T-3I)(T-4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Proof. Let $v \in V$ be a eigenvector of T corresponding to the eigenvalue λ. Then

$$0 = (T-2I)(T-3I)(T-4I)v = (\lambda-2)(\lambda-3)(\lambda-4)v.$$

Since $v \neq 0$, the equation above implies that

$$(\lambda-2)(\lambda-3)(\lambda-4) = 0$$

Thus $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$, as desired. $\square$

Question 1. Suppose U is a subspace of V such that $\dim V/U = 1$. Prove that there exists a linear functional $f \in V'$ such that

$$\operatorname{null} f = U$$

Proof. Since V/U is a 1-dimensional linear space, we can construct an arbitrary nonzero linear map $g \in (V/U)'$. Definition 3.88 says we have a quotient map $\pi\colon V \to V/U$ which sends $v \in V$ to $v + U \in V/U$. Now let $f = g \circ \pi$. We claim that $\operatorname{null} f = U$

On the one hand, for any $u \in U$, $\pi(u) = 0 + U = 0$ in V/U, so we have $f(u) = g(\pi(u)) = g(0) = 0$, therefore $\operatorname{null} f \supset U$.

On the other hand, since $g \neq 0$, we can find $v+U \in V/U$ such that $g(v+U) \neq 0$, so $f(v) = g(\pi(v)) = g(v+U) \neq 0$. Since $\dim V/U = 1$, $v+U$ is the basis of V/U. Therefore for any $w \notin U$, we can find a non-zero $\lambda \in \mathbb{F}$ and such that $\pi(w) = w + U = \lambda(v+U) = \lambda v + U$. So we have

$$\begin{aligned} f(w) =& g(\pi(w)) \\ =& g(w+U) \\ =& g(\lambda(v+U)) \\ =& \lambda g(v+U) \neq 0 \end{aligned}$$

because $\lambda \neq 0$ and $g(v+U) \neq 0$. So $\operatorname{null} f \subset U$.

Hence we have proved that $\operatorname{null} f = U$, as desired. $\square$

Question 2. Let $C^\infty(\mathbb{R})$ denote the vector space (over $\mathbb{R}$) of infinitely-differentiable real-valued functions $f : \mathbb{R} \to \mathbb{R}$.

a) Let U denote the subspace of $C^\infty(\mathbb{R})$ consisting of functions which vanish at 42 and at π:

$$U = \{f \in C^\infty(\mathbb{R}) \mid f(42) = 0, f(\pi) = 0\}$$

Prove that the quotient vector space $C^\infty(\mathbb{R})/U$ is finite dimensional. What is its dimension?

b) Let W denote the subspace of $C^\infty(\mathbb{R})$ consisting of functions which "vanish to second order at 0":

$$W = \{f \in C^\infty(\mathbb{R}) \mid f(0) = 0, f'(0) = 0, f''(0) = 0\}$$

Prove that the quotient vector space $C^\infty(\mathbb{R})/W$ is finite dimensional, and find a basis for $C^\infty(\mathbb{R})/W$.

Proof. a) Define the linear transformation $T\colon C^\infty(\mathbb{R}) \to \mathbb{R}^2$ by $T(f) = \big(f(42), f(\pi)\big)$.[1] The kernel of T is

$$\begin{aligned}\ker T &= \{f \in C^\infty(\mathbb{R}) \mid T(f) = 0\}\\ &= \{f \in C^\infty(\mathbb{R}) \mid f(2) = 0, f(7) = 0\}\\ &= U.\end{aligned}$$

The Quotient Isomorphism Theorem (Thm 3.91(d)) thus tells us that $\overline{T}\colon C^\infty(\mathbb{R})/U \to \text{Image } T$ is an isomorphism, so we need to understand Image T.

Choose two functions $f, g \in C^\infty(\mathbb{R})$ that satisfy $T(f) = (1, 0)$ and $T(g) = (0, 1)$, such as:[2]

$$\begin{aligned}f(x) &= \frac{x - \pi}{42 - \pi}\\ g(x) &= \frac{42 - x}{42 - \pi}\end{aligned}$$

Since

$$\begin{array}{ll} f(42) = 1 & f(\pi) = 0\\ g(42) = 0 & g(\pi) = 1\end{array}$$

we have $T(f) = (1, 0)$ and $T(g) = (0, 1)$. This shows that $(1, 0) \in \text{Image } T$ and $(0, 1) \in \text{Image } T$. Since these are the standard basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$, they span $\mathbb{R}^2$, and so Image $T = \mathbb{R}^2$.

Since Image$T = \mathbb{R}^2$, the Quotient Isomorphism Theorem (Thm 3.91(d)) states that $\overline{T}\colon C^\infty(\mathbb{R})/U \to \mathbb{R}^2$ is an isomorphism. Since $C^\infty(\mathbb{R})/U$ and $\mathbb{R}^2$ are isomorphic, they have the same dimension: therefore $C^\infty(\mathbb{R})/U$ has dimension 2.

b) Define the linear transformation $S\colon C^\infty(\mathbb{R}) \to \mathbb{R}^3$ by [3]

$$S(f) = \big(f(0), f'(0), f''(0)\big).$$

The kernel of S is

$$\begin{aligned}\ker S &= \{f \in C^\infty(\mathbb{R}) \mid S(f) = 0\}\\ &= \{f \in C^\infty(\mathbb{R}) \mid f(0) = 0, f'(0) = 0, f''(0) = 0\}\\ &= W.\end{aligned}$$

The Quotient Isomorphism Theorem (Thm 3.91(d)) thus tells us that $\overline{S}\colon C^\infty(\mathbb{R})/W \to \text{Image } S$ is an isomorphism, so we need to understand Image S. Consider the following functions in $C^\infty(\mathbb{R})$:

$$\begin{aligned}f_1 &= 1\\ f_2 &= x - 1\end{aligned}$$

[1] For example, if $f(x) = x$ then $T(f) = (42, \pi)$; if $g(x) = e^x$ then $T(g) = (e^{42}, e^\pi)$, if $h(x) = \sin x$ then $T(h) = (\sin 42, \sin \pi)$, etc.

[2] Many other choices are possible.

[3] For example, if $f(x) = x^2$ then $S(f) = (0, 0, 2)$; if $g(x) = e^x$ then $S(g) = (1, 1, 1)$; if $h(x) = \sin x$ then $S(h) = (0, 1, 0)$, etc.

$$f_3 = x^2 - 2x + 1$$

These three functions are infinitely differentiable, so they are in $C^\infty(\mathbb{R})$. Their only important properties are that

$$\begin{array}{lll} f_1(0) = 1 & f_1'(0) = 0 & f_1''(0) = 0 \\ f_2(0) = 0 & f_2'(0) = 1 & f_2''(0) = 0 \\ f_3(0) = 0 & f_3'(0) = 0 & f_3''(0) = 1 \end{array}$$

This implies that

$$S(f_1) = e_1, \quad S(f_2) = e_2, \quad S(f_3) = e_3.$$

Therefore e_1, e_2, and e_3 are all in Image$S$. Since e_1, e_2, e_3 is a basis for $\mathbb{R}^3$, this shows that Image$S = \mathbb{R}^3$.

Since Image$S = \mathbb{R}^3$, the Quotient Isomorphism Theorem (Thm 3.91(d)) states that $\overline{S}\colon C^\infty(\mathbb{R})/W \to \mathbb{R}^3$ is an isomorphism. Since $C^\infty(\mathbb{R})/W$ and $\mathbb{R}^3$ are isomorphic, they have the same dimension: therefore $C^\infty(\mathbb{R})/W$ has dimension 3.

Consider the elements $v_1 = f_1 + W$, $v_2 = f_2 + W$, and $v_3 = f_3 + W$ in the quotient space $C^\infty(\mathbb{R})/W$. We will show they are linearly independent. Assume that $av_1 + bv_2 + cv_3 = 0$ in $C^\infty(\mathbb{R})/W$. The above formula shows that

$$\overline{S}(v_1) = \overline{S}(f_1 + W) = e_1, \quad \overline{S}(v_2) = \overline{S}(f_2 + W) = e_2, \quad \overline{S}(v_3) = \overline{S}(f_3 + W) = e_3.$$

Since $\overline{S}$ is linear, $\overline{S}(av_1 + bv_2 + cv_3) = ae_1 + be_2 + ce_3$. But e_1, e_2, e_3 are linearly independent, so we conclude that $a = b = c = 0$. This shows that v_1, v_2, v_3 are linearly independent in the quotient space $C^\infty(\mathbb{R})/W$. Since this vector space has dimension 3, this implies that v_1, v_2, v_3 is a basis for $C^\infty(\mathbb{R})/W$. $\square$

Question 3. Let $C^\infty(\mathbb{R}, \mathbb{C})$ be the vector space (over $\mathbb{C}$) of complex-valued functions $f : \mathbb{R} \to \mathbb{C}$ that are infinitely differentiable. Let V be the space of functions $f \in C^\infty(\mathbb{R}, \mathbb{C})$ satisfying the equation $f'' = -f$:

$$V = \{f \in C^\infty(\mathbb{R}, \mathbb{C}) \mid f'' = -f\}$$

- Assume without proof that $\dim V \leq 2$. Prove that the functions $\sin x$ and $\cos x$ both lie in V, and moreover that $(\sin x, \cos x)$ form a basis for V.
- Let D be the operator on $\mathbb{C}^\infty(\mathbb{R}, \mathbb{C})$ defined by $D(f) = f'$. Prove that V is an invariant subspace for D.
- Now consider $D \in \mathcal{L}(V)$ as an operator on V. Find a basis for V consisting of eigenvectors for D. What are their eigenvalues?

Proof.

- Consider the functions $\sin x$ and $\cos x$. Then $\sin''(x) = (\cos'(x)) = -\sin(x)$ and $\cos''(x) = (-\sin(x))' = -\cos(x)$. Thus $\sin(x), \cos(x) \in V$.

  To show that $(\sin x, \cos x)$ form a basis for V, first we show that they are linearly independent. So suppose there are numbers $a, b \in \mathbb{C}$ s.t. $a\sin(x) + b\cos(x) = 0$. Then, plugging in $x = 0$, we get $b = 0$ since $\sin(0) = 0$ and $\cos(0) = 1$. Plugging in $x = \pi/2$, we get $a = 0$ since $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$. Thus $\sin(x)$ and $\cos(x)$ are linearly independent.

  Since $\sin(x)$ and $\cos(x)$ are linearly independent, the dimension of V must be at least 2. Since we were given that $\dim V$ is at most 2, we conclude that $\dim V = 2$. Thus $\sin(x)$ and $\cos(x)$ form a basis for V.
- Let D be the operator on $\mathbb{C}^\infty(\mathbb{R}, \mathbb{C})$ defined by $D(f) = f'$. To show that V is invariant under D, we must show that if $f \in V$ then $Df \in V$. So suppose that $f \in V$, and set $g = D(f)$. Then $f'' = -f$. Differentiating both sides

of this equation, we get that $f''' = -f'$, or in other words $g'' = -g$. Thus $g = D(f)$ lies in V. Therefore, V is invariant under D.

- Now consider $D \in \mathcal{L}(V)$ as an operator on V. Find a basis for V consisting of eigenvectors for D. What are their eigenvalues?

  The properties $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$ mean that

$$D\big(a\sin(x) + b\cos(x)\big) = -b\sin(x) + a\cos(x).$$

We have seen a similar linear transformation in class, namely

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2 \qquad T(x,y) = (-y,x).$$

However that operator has *no* eigenvalues because it is on a real vector space, and its minimal polynomial $p(x) = x^2 + 1$ has no real roots. In contrast, here we are working over the *complex numbers*, so we might imagine that the eigenvalues would be the complex roots of $p(x) = x^2 + 1$, namely i and $-i$.

The eigenvalue equation $D\big(a\sin(x) + b\cos(x)\big) = i\big(a\sin(x) + b\cos(x)\big)$ can be solved to find

$$f = \cos(x) + i\sin(x)$$

and similarly $D\big(a\sin(x)+b\cos(x)\big) = -i\big(a\sin(x)+b\cos(x)\big)$ can be solved to find

$$g = \cos(x) - i\sin(x).$$

Then we can check that

$$D(f) = -\sin(x) + i\cos(x) = if$$

and

$$D(g) = -\sin(x) - i\cos(x) = -ig.$$

Thus f and g are eigenvectors for D with eigenvalues i and $-i$. Since they have distinct eigenvalues, Theorem 5.6 in the book implies that they are linearly independent. Since $\dim V \leq 2$, any spanning list of length 2 forms a basis for V.

Remark by TC: you have probably learned what the eigenvectors of D as an operator on $\mathbb{C}^\infty(\mathbb{R},\mathbb{C})$ are in a previous class. For the eigenvalue a, the eigenvalue equation $D(f) = af$ becomes the differential equation $f' = af$, and you may already know that the solutions to this equation are (constant multiples of)

$$f(x) = e^{ax},$$

since the chain rule implies that

$$(e^{ax})' = a \cdot e^{ax}.$$

But the functions f and g you found above are eigenvectors with eigenvalues $a = i$ and $a = -i$, so they must be of the form Ce^{ix} and Ce^{-ix}! We can find the constants by plugging in 0, since $Ce^{i\cdot 0} = C$. By plugging in $f(0) = \cos(0) + i\sin 0 = 1 + i\cdot 0 = 1$ and $g(0) = \cos(0) - i\sin 0 = 1 - i\cdot 0 = 1$ we see that the constants are 1 for both f and g. Therefore you have proved the famous formula of Euler:

$$e^{ix} = \cos x + i\cdot\sin x \qquad e^{-ix} = \cos x - i\cdot\sin x.$$

In particular, if we evaluate the first eigenfunction at π we get

$$e^{i\pi} = \cos\pi + i \cdot \sin\pi = -1 + i \cdot 0,$$

or in other words

$$e^{i\pi} = -1. \qquad \square$$