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# PROOF OF LOCKE'S CONJECTURE, I†

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**Abstract**. Locke's conjecture $(L)$ states that the binary hypercube $Q_n$ with $k$ deleted vertices of each parity is Hamiltonian if $n \geq k+2$. In 2003, S. C. Locke and R. Stong published in *The American Mathematical Monthly* a proof of $(L)$ for the case $k = 1$. In 2007, in the paper *Path coverings with prescribed ends in faulty hypercubes*[1] the authors proved $(L)$ for every $k \leq 4$ and every $n \geq k+2$ and formulated the following conjecture $(CG)$: Let $n \geq k+3$ and $\mathcal{F}$ be a set of $k$ even (odd) and $k+1$ odd (even) vertices of $Q_n$. If $u, v$ are two even (odd) vertices of $\mathcal{Q}_n - \mathcal{F}$ then there exists a Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ which connects $u$ and $v$. $(CG)$ is known to be true for every $k \leq 3$ and every $n \geq k+3$.

In this paper we prove that if $n \geq 7$, $5 \leq k \leq n-2$ and $(L)$ is true for every $n_1 \leq n-1$ and every $k_1 \leq n_1 - 2$ and $(CG)$ is true for every $n_1 \leq n-1$ and every $k_1 \leq n_1 - 3$ then $(L)$ is also true for $n$ and $k$. To keep the paper shorter the proof that if $n \geq 7$, $4 \leq k \leq n-3$ and $(L)$ is true for every $n_1 \leq n-1$ and every $k_1 \leq n_1 - 2$ and $(CG)$ is true for every $n_1 \leq n-1$ and every $k_1 \leq n_1 - 3$ then $(CG)$ is also true for $n$ and $k$ will appear in a forthcoming paper. In that way the two papers together complete the proofs of $(L)$ and $(CG)$.

## 1 Introduction

The $n-$dimensional binary hypercube $\mathcal{Q}_n$ is the graph whose vertices are the binary sequences of length $n$ and whose edges are pairs of binary sequences that differ in exactly one position.

A given vertex is called *even* if it has an even number of 1's in its components; otherwise the vertex is called *odd*.

In 2001 S. Locke asked the following question in [8]: Let $k \geq 1$ and $n \geq k+2$ be integers. Let also $\mathcal{F}$ be a set of $k$ even and $k$ odd vertices of $\mathcal{Q}_n$. Is it true that the graph $\mathcal{Q}_n - \mathcal{F}$ has a Hamiltonian cycle?

Since S. Locke had a positive answer to the above question in the case when $k = 1$, most likely he anticipated a positive answer for his question for every integer $k \geq 1$. That is why in the literature the above problem is known as *Locke's conjecture* and we are going to denote it by $(L)$.

*The Monthly* published R. Stong's proof of $(L)$ for the case $k = 1$ and made the remark that Stong had also proved $(L)$ when $n \geq 2k + 3\log_2 k + 4$ (see [9]) .

In [5] we proved $(L)$ for every $k \leq 4$ and every $n \geq k+2$, formulated the following conjecture (henceforth denoted by $(CG)$) and proved it for $k \leq 2$ (the case $k = 0$ had appeared already in [7]).

**Conjecture 1.1.** Let $k \geq 0$ and $n \geq k+3$ be integers. Let also $\mathcal{F}$ be a set of $k$ even (odd) and $k+1$ odd (even) vertices of $Q_n$. If $u, v$ are two even (odd) vertices of $\mathcal{Q}_n - \mathcal{F}$ then there exists a Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ that connects $u$ and $v$.

The proof of $(CG)$ for the case $k = 3$ is contained in [2]. Therefore $(L)$ has already been verified for every $k \leq 4$ and every $n \geq k+2$ and $(CG)$ has been verified for every $k \leq 3$ and every $n \geq k+3$.

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[1] The paper *Path coverings with prescribed ends in faulty hypercubes* [5] was written and submitted for publication in 2007 and did not appear in print until 2015 because of its length, but the main results in that paper, summarized in tables, appeared in print in [1] and [2] in 2009, and in [3] and [6] in 2010.

In this paper we prove that if $n \geq 7$, $5 \leq k \leq n-2$ and $(L)$ is true for every $n_1 \leq n-1$ and every $k_1 \leq n_1 - 2$ and $(CG)$ is true for every $n_1 \leq n-1$ and every $k_1 \leq n_1 - 3$ then $(L)$ is also true for $n$ and $k$. To keep the paper shorter the proof that if $n \geq 7$, $4 \leq k \leq n-3$ and $(L)$ is true for every $n_1 \leq n-1$ and every $k_1 \leq n_1 - 2$ and $(CG)$ is true for every $n_1 \leq n-1$ and every $k_1 \leq n_1 - 3$ then $(CG)$ is also true for $n$ and $k$ will appear in a forthcoming paper [4]. In that way the two papers together complete the proofs of $(L)$ and $(CG)$ for all admissible values of $n$ and $k$.

In [5] the following conjecture $(T)$ was also formulated.

**Conjecture 1.2.** Let $k \geq 0$ and $n \geq k+3$ be integers. Let also $\mathcal{F}$ be a set of $k$ even and $k$ odd vertices of $Q_n$. If $u$ and $v$ are two vertices of $\mathcal{Q}_n - \mathcal{F}$ with different parity then there exists a Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ which connects $u$ and $v$.

For the proof of $(L)$ we need the following theorem which shows that $(T)$ is a simple corollary of $(CG)$.

**Theorem 1.3.** *Let $k \geq 0$ and $n \geq k+3$ be integers. Let also $\mathcal{F}$ be a set of $k$ even and $k$ odd vertices of $Q_n$ and suppose that $(CG)$ is true for $n$ and $k$. If $u$ and $v$ are two vertices of $\mathcal{Q}_n - \mathcal{F}$ with different parity then there exists a Hamiltonian path of $\mathcal{Q}_n - \mathcal{F}$ which connects $u$ and $v$.*

*Proof.* Let $v_1$ be a neighbor of $u$ which is not in $\mathcal{F}$. It follows from $(CG)$ that there exists a Hamiltonian path $\gamma_1$ for $Q_n - \mathcal{F} - \{u\}$ from $v_1$ to $v$. Therefore

$$u \longrightarrow v_1 \xrightarrow{\gamma_1} v$$

is a Hamiltonian path for $Q_n - \mathcal{F}$ from $u$ to $v$. $\square$

## 2 Preliminaries

To simplify the explanations and the proofs that follow we introduce some terminology.

Every vertex $u$ of $\mathcal{Q}_n$ is a binary sequence of length $n$. We refer to the $i-$th bit of $u$ as the $i-$th coordinate of $u$. All vertices of $\mathcal{Q}_n$ with identical $i-$th coordinates form $(n-1)$-dimensional hypercubes that we denote by $\mathcal{Q}_n^0$ and $\mathcal{Q}_n^1$, or $\mathcal{Q}_n^{bot}$ and $\mathcal{Q}_n^{top}$, and we call them *bottom and top plates*, respectively. Sometimes we say that *the $i-$th coordinate splits $\mathcal{Q}_n$ into two hypercubes*. If we split the hypercube using two coordinates then we get four $(n-2)$-dimensional hypercubes that we denote by $\mathcal{Q}_n^{00}$, $\mathcal{Q}_n^{01}$, $\mathcal{Q}_n^{10}$ and $\mathcal{Q}_n^{11}$. If $u$ and $v$ are two vertices of $\mathcal{Q}_n$ such that $u \in \mathcal{Q}_n^0$ and $v \in \mathcal{Q}_n^1$ then we say that the $i-$th coordinate separates $u$ and $v$. Thus, if $\mathcal{F}$ is any set of vertices of $\mathcal{Q}_n$ then each coordinate $i$ induces a partition $\{\mathcal{F}^0, \mathcal{F}^1\}$ of the set $\mathcal{F}$, where $\mathcal{F}^0$ and $\mathcal{F}^1$ are the set of vertices of $\mathcal{F}$ with the $i$-th coordinate equal to zero or one, respectively. In the special case when one of the sets $\mathcal{F}^0$ or $\mathcal{F}^1$ is empty we say that the $i$-th coordinate does not separate $\mathcal{F}$. We say that *the coordinate $i$ separates $\mathcal{F}$ in the way $(s,t)$ (or the separation type is $(s,t)$)* if the sets $\mathcal{F}^0$ and $\mathcal{F}^1$ are nonempty and have cardinalities $s$ and $t$. We do not make a difference between the types $(s,t)$ and $(t,s)$.

We say that *the $j-$th coordinate separates $\mathcal{F}$ in a different way than the $i-$th coordinate* if the partitions of $\mathcal{F}$ induced by $i$ and $j$ are different. More generally, we say that a set of $k$ coordinates separates $\mathcal{F}$ in $l$ different ways if the total number of different partitions of $\mathcal{F}$ induced by the $k$ coordinates is $l$. It is easy to see that if the coordinates $i$ and $j$ separate $\mathcal{F}$ in two different ways then there are vertices from $\mathcal{F}$ in at least three of the four $n-2-$dimensional hypercubes $\mathcal{Q}_n^{00}$, $\mathcal{Q}_n^{01}$, $\mathcal{Q}_n^{10}$ and $\mathcal{Q}_n^{11}$.

Given a vertex $a$ of $\mathcal{F}$ we say that a coordinate is *$\mathcal{F}$-special for $a$* if this coordinate separates $a$ from the rest of the vertices of $\mathcal{F}$.

We view $\mathcal{Q}_n$ as the graph $\mathcal{Q}_{n-1} \times K_2$, where $K_2$ is the edge $(0,1)$, and then $\mathcal{Q}_n^0 = \mathcal{Q}_{n-1} \times \{0\}$ and $\mathcal{Q}_n^1 = \mathcal{Q}_{n-1} \times \{1\}$. Also, we view $\mathcal{Q}_n$ as the graph $\mathcal{Q}_{n-2} \times \mathcal{Q}_2$ and then $\mathcal{Q}_n^{00} = \mathcal{Q}_{n-2} \times \{00\}$, $\mathcal{Q}_n^{01} = \mathcal{Q}_{n-2} \times \{01\}$, $\mathcal{Q}_n^{10} = \mathcal{Q}_{n-2} \times \{10\}$ and $\mathcal{Q}_n^{11} = \mathcal{Q}_{n-2} \times \{11\}$.

For the proof of $(L)$ we need the following separation lemmas.

**Lemma 2.1.** *Let $k \geq 3$, $n \geq k$, and $\mathcal{F}$ be a set of $k$ pairs of even and odd vertices of $Q_n$. Let also every coordinate which separates the even vertices in $\mathcal{F}$ separates also the odd vertices in $\mathcal{F}$ and vice versa. Then there exist two coordinates that separate the even and the odd vertices in $\mathcal{F}$ in different ways.*

*Proof.* Take one coordinate, say $A$, which separates the even and the odd vertices in $\mathcal{F}$. Since there are at least three odd vertices in $\mathcal{F}$, at least two of them are not separated by $A$. Take another coordinate, say $B$, which separates these two odd vertices. Then $B$ separates the odd vertices in $\mathcal{F}$ in a different way than $A$. If $B$ also separates the even vertices in $\mathcal{F}$ in a different way than $A$ then $A$ and $B$ are as required.

Suppose that $B$ separates the even vertices in $\mathcal{F}$ in the same way as $A$. Since there are at least three even vertices in $\mathcal{F}$, at least two of them are not separated by $A$ (and by $B$). Let $C$ be a coordinate that separates these two even vertices. Clearly, $C$ separates the even vertices in $\mathcal{F}$ in a different way than $A$ and $B$. Also, $C$ separates the odd vertices in $\mathcal{F}$. If $C$ separates the odd vertices in $\mathcal{F}$ in a different way than $A$ then $A$ and $C$ are as required, otherwise $C$ and $B$ are the required two coordinates. □

**Lemma 2.2.** *Let $k \geq 3$, $n \geq k$, and $\mathcal{F}$ be a set of $k$ even (odd) vertices of $Q_n$. If every coordinate which separates the vertices in $\mathcal{F}$ separates them in the way $(1, k-1)$ then either there exist $k$ coordinates that separate all vertices in $\mathcal{F}$ in $k$ different ways or there exist $2k-2$ coordinates that separate all vertices in $\mathcal{F}$ in $k-1$ different ways. These $2k-2$ coordinates can be arranged in $k-1$ pairs of coordinates such that the two coordinates of each pair separate $\mathcal{F}$ in the same way.*

*Proof.* It is clear that if $|\mathcal{F}| \geq 3$ then no two vertices can have a common $\mathcal{F}$-special coordinate. If for every vertex in $\mathcal{F}$ there is an $\mathcal{F}$-special coordinate then there are at least $k$ coordinates that separate the vertices of $\mathcal{F}$ in $k$ different ways.

Assume now that there is a vertex $a$ in $\mathcal{F}$ with no $\mathcal{F}$-special coordinates. If $b$ is any other vertex in $\mathcal{F}$ then it differs from $a$ in at least two coordinates and these two coordinates must by necessity be $\mathcal{F}$-special for $b$. Therefore we have $k-1$ pairs of coordinates with the properties stated in the lemma. □

**Lemma 2.3.** *Let $n \geq 3$ be a positive integer and $\mathcal{F}$ be a set of four odd (even) vertices of $Q_n$. Then there exist three coordinates that separate these vertices in three different ways or there are two pairs of coordinates such that these coordinates separate $\mathcal{F}$ in two different ways with the two coordinates of each pair separating $\mathcal{F}$ in the same way that is of type $(2, 2)$.*

*Proof.* Let $\mathcal{F} = \{a, b, c, d\}$. If at least one vertex in $\mathcal{F}$, say $a$, has a coordinate that is $\mathcal{F}$-special for $a$ then this coordinate together with any two coordinates that separate $\{b, c, d\}$ in two different ways (which exists by Lemma 2.2) form a group of three coordinates that separate $\mathcal{F}$ in three different ways. So, if $\mathcal{F}$ cannot be separated in three different ways then all the separations of $\mathcal{F}$ are of the type $(2, 2)$. Also, without loss of generality, we can assume that $d$ has no $\{b, c, d\}$-special coordinates and that one pair of coordinates separates $b$ from $\{c, d\}$ and another pair of coordinates separates $c$ from $\{b, d\}$ (see Lemma 2.2). It follows that the first pair of coordinates separate $\mathcal{F}$ as $\{a, b\}, \{c, d\}$ and the second pair of coordinates separates $\mathcal{F}$ as $\{a, c\}, \{b, d\}$. □

**Lemma 2.4.** *Let $n \geq 4$ be a positive integer and $\mathcal{F} = \{a, b, c, d, e\}$ be a set of five odd (even) vertices of $Q_n$. Then there exist four coordinates that separate these vertices in four different ways or there exist three pairs of coordinates that separate $\mathcal{F}$ in three different ways with the two coordinates of each pair separating $\mathcal{F}$ in the same way that is of type $(2, 3)$.*

*Proof.* There are two types of separations for a set of 5 vertices: $(1, 4)$, and $(2, 3)$. If all the separations of $\mathcal{F}$ are of type $(1, 4)$ then according to Lemma 2.2 there are at least four coordinates that separate $\mathcal{F}$ in four different ways. Now, suppose that there are separations of $\mathcal{F}$ of type $(2, 3)$. Without loss of generality we can assume that $\Pi = \{\{a, b\}, \{c, d, e\}\}$ is a separation of $\mathcal{F}$ produced by some coordinate $A$. If $c, d, e$ can be separated in three different ways by three coordinates then these three coordinates together with $A$ form a group of four coordinates that separate $\mathcal{F}$ in four different ways. If $c, d, e$ cannot be separated in three different ways then, according to Lemma 2.2 and without loss of generality, we can assume that there are two coordinates, say $C$ and $D$, that separate $c, d, e$ in the way $\{c\}, \{d, e\}$ and two coordinates, say $E$ and $F$, that separate $c, d, e$ in the way $\{d\}, \{c, e\}$. If $C$ ( $E$ ) separates $\mathcal{F}$ in a different way than $D$ ( $F$ ) then $A, C, D, E$ ( $A, C, E, F$ ) form a group of four coordinates that separate $\mathcal{F}$ in four different ways. Assume now that $C$ separates $\mathcal{F}$ in the same way as $D$ and that $E$ separates $\mathcal{F}$ in the same way as $F$. If neither $C$ nor $E$ separates $a$ from $b$ then any coordinate that separates $a$ from $b$ together with $A, C$ and $E$ form a group of four coordinates that separate $\mathcal{F}$ in four different ways.

Assume now that one of the coordinates $C$ or $E$ separates $a$ from $b$. Without loss of generality we can assume that that coordinate is $C$ and that $C$ separates $\mathcal{F}$ in the way $\{a, c\}, \{b, d, e\}$. If any of the coordinates that separate $b, d, e$ separates $\mathcal{F}$ in a different way than $A, C$, and $E$ then such coordinate together with $A, C$ and $E$ form a group of four coordinates that separate $\mathcal{F}$ in four different ways. If that is not the case then $b, d, e$ can be separated only in the ways produced by $A, C$ and $E$. In particular (by Lemma 2.2) there must be a coordinate $B$ which separates $\mathcal{F}$ in exactly the same way as $A$ does. □

**Corollary 2.5.** *Let $k \geq 5$, $n = k+2$, and $\mathcal{F}$ be a set of $k$ even and $k$ odd vertices of $Q_n$. If every coordinate which separates the even (odd) vertices in $\mathcal{F}$ separates them in the way $(1, k-1)$ then there exist two coordinates that separate the even and the odd vertices in different ways.*

*Proof.* Since every coordinate which separates the even vertices separates them in the way $(1, k-1)$, it follows from Lemma 2.2 that there are either $k$ coordinates that separate the even vertices in different ways or $2k-2$ coordinates that separate the even vertices in $k-1$ different ways. Since $k \geq 5$, it follows from Lemma 2.4 that either there exist at least four coordinates that separate the odd vertices in different ways or there exist three pairs of coordinates that separate the odd vertices in three different ways with the two coordinates from each pair separating the odd vertices in the same way. In either case, since $n = k+2$, there will be two coordinates that separate the even and the odd vertices in different ways. □

An important ingredient in the proofs of $(L)$ and $(CG)$ is the existence of a long enough path that avoids a set of faulty vertices as the one guaranteed by Lemma 2.7 below. For the proof of Lemma 2.7 we need the following result.

**Theorem 2.6** ([6]). *Let $n \geq 5$ and $f$ be integers with $0 \leq f \leq 3n-7$. Then for any set $\mathcal{F}$ of vertices of $\mathcal{Q}_n$ of cardinality $f$ there exists a cycle in $\mathcal{Q}_n - \mathcal{F}$ of length at least $2^n - 2f$.*

**Lemma 2.7.** *Let $n \geq 5$ be an integer and $\mathcal{F}$ be a set of $2n$ vertices of $\mathcal{Q}_n$. Then there exists a path $\gamma$ in $\mathcal{Q}_n - \mathcal{F}$ with length at least $2(n-3)+2$.*

*Proof.* We have $|\mathcal{F}| = 2n$.

If $n \geq 7$ then $3n - 7 \geq 2n$. Therefore, according to Theorem 2.6, there is a Hamiltonian cycle in $\mathcal{Q}_n - \mathcal{F}$ with length at least $2^n - 2(2n)$. Since $2^n - 2(2n) \geq 2(n-3)+3$, when $n \geq 5$, we conclude that if $n \geq 7$ there is a path $\gamma$ in $\mathcal{Q}_n - \mathcal{F}$ with length at least $2(n-3)+2$.

If $n = 6$ then it follows from Theorem 2.6 that we can find a cycle in $\mathcal{Q}_n$ with length at least $2^6 - 2 \cdot 11 = 42$, such that it contains at most one of the vertices from $\mathcal{F}$, for in this case $|\mathcal{F}| \leq 12$. Therefore, when $n = 6$ there exists a path $\gamma$ in $\mathcal{Q}_n - \mathcal{F}$ with length at least $2(n-3)+2 = 8 \leq 40$.

Finally, if $n = 5$, again using Theorem 2.6, we can find a cycle in $\mathcal{Q}_n$ with length at least $2^5 - 2 \cdot 8 = 16$ that contains at most two of the vertices from $\mathcal{F}$. Therefore, when $n = 5$, there exists a path $\gamma$ in $\mathcal{Q}_n - \mathcal{F}$ with length at least $6 = 2(n-3)+2$. □

As a corollary of Lemma 2.7 we obtain the following very useful lemma.

**Lemma 2.8.** *Let $k \geq 1$ and $n \geq 7$ be integers, with $n \geq k+2$, and $\mathcal{F}$ be a set of $k$ even and $k$ odd vertices of $Q_n$. Split $\mathcal{Q}_n$ using two coordinates and let $Q_{n-2}$ be one of the four hypercubes $\mathcal{Q}_n^{00}$, $\mathcal{Q}_n^{01}$, $\mathcal{Q}_n^{10}$ or $\mathcal{Q}_n^{11}$. Project all vertices from $\mathcal{F}$ onto $Q_{n-2}$ using the natural projections and denote the projection by $\mathcal{F}'$. Then there exists a path $\mu$ in $Q_{n-2} - \mathcal{F}'$ with length $2(n-5)+1$. Since the length of $\mu$ is an odd number, we can choose the beginning vertex of $\mu$ to be either even or odd depending on our needs.*

*Proof.* Since $|\mathcal{F}| \leq 2k \leq 2(n-2)$, we have $|\mathcal{F}'| \leq 2(n-2)$. Also $n-2 \geq 5$. Therefore, it follows from Lemma 2.7 that there exists a path $\gamma$ in $Q_{n-2} - \mathcal{F}'$ with length at least $2(n-5)+2$ and therefore there exists a path $\mu$ with length $2(n-5)+1$ which begins with an even or odd vertex, depending on our choice. □

## 3 Proof of Locke's conjecture

In this section we complete the prove of Locke's conjecture (L) under the assumption that (CG) is true for some appropriate values of $n$ and $k$. More specifically we prove the following theorem.

**Theorem 3.1.** *Let $n \geq 7$ and $5 \leq k \leq n-2$ be integers. Let also $\mathcal{F}$ be a set of $k$ even and $k$ odd vertices of $Q_n$ and suppose that $(L)$ is true for every $n_1 \leq n-1$ and every $k_1 \leq n_1 - 2$ and $(CG)$ is true for every $n_1 \leq n-1$ and every $k_1 \leq n_1 - 3$. Then $(L)$ is also true for $n$ and $k$.*

**Remark 3.2.** Notice that it follows immediately from the hypothesis of the above theorem and Theorem 1.3 that the conjecture $(T)$ is true for every $n_1 \leq n-1$ and every $k_1 \leq n_1 - 3$, as well.

Let $\mathcal{F} = \{u_1, ..., u_k, v_1, ..., v_k\}$, where all $u-$s are even and all $v-$s are odd vertices. Sometimes we call the elements of $\mathcal{F}$ *deleted* vertices.

The idea of the proof is to choose "appropriately" one or two coordinates that separate the deleted vertices "in a good way" and to split $\mathcal{Q}_n$ using them. Then by using $(L)$, $(CG)$ or $(T)$ for some $n_1 \leq n-1$ and $k_1 \leq k-1$ we construct the required Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$. In many cases it is impossible to find one coordinate such that immediately after the splitting we can use $(L)$, $(CG)$ or $(T)$ in the resulting hypercubes $\mathcal{Q}_n^{top}$ and $\mathcal{Q}_n^{bot}$ since usually there is a big difference (more than one) or *disbalance* between the number of the deleted even and odd vertices in these hypercubes. In such cases we choose "appropriately" two coordinates and using them we split $\mathcal{Q}_n$ into four hypercubes $\mathcal{Q}_n^{00}$, $\mathcal{Q}_n^{01}$, $\mathcal{Q}_n^{10}$ and $\mathcal{Q}_n^{11}$. Then, we start creating a path $\gamma_0$ that is the first part of the desired cycle of $\mathcal{Q}_n - \mathcal{F}$ by concatenating paths of the type

$$e_{01} \to o_{00} \to (e, o)_{10} \to e_{11} \to (o, e')_{01}$$

or other similar types of paths that we call *short cycles* (since the projection of each such path on $\mathcal{Q}_2$ is a cycle). We refer to all vertices used in those short cycles, except the starting one as *used* vertices. We stop the creation of $\gamma_0$ at a point when the set of deleted or used vertices in each of the four hypercubes is *balanced* or *semi-balanced* in the sense that the disbalance between even and odd vertices is at most one. Considering the originally deleted vertices and the used vertices as new deleted vertices we proceed by creating paths in each of the four hypercubes, applying $(L)$, $(CG)$ or $(T)$ as needed, to complete the desired cycle of $\mathcal{Q}_n - \mathcal{F}$.

The notation that we use in these short cycles is self-explanatory: $e_{01}$ represents an even (in $\mathcal{Q}_n$) vertex which is in the hypercube $\mathcal{Q}_n^{01}$; $o_{00}$ represents an odd (in $\mathcal{Q}_n$) vertex which is in the hypercube $\mathcal{Q}_n^{00}$ and is a neighbor of $e_{01}$ in $\mathcal{Q}_n$; $e_{01} \to o_{00}$ means that $(e_{01}, o_{00})$, which is an edge in $\mathcal{Q}_n$, is an edge in the constructed path; $(e, o)_{10}$ represents the edge $(e_{10}, o_{10})$ in the hypercube $\mathcal{Q}_n^{10}$ which is also an edge in the constructed path; $o_{00} \to (e, o)_{10}$ means that $o_{00}$ and $e_{10}$ are neighbors in $\mathcal{Q}_n$ and that $(o_{00}, e_{10})$ is an edge in the constructed path; and so on. We call the edges of the type $(e_{ij}, o_{i_1 j_1})$, where $ij \neq i_1 j_1$, *vertical*, and the edges of the type $(e_{ij}, o_{ij})$ *horizontal*.

Usually more than one short cycle is needed in order to (semi) balance the four plates. Since each one of those short cycles will be part of the required Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$, we do not want different short cycles to use the same vertices and to contain deleted vertices. In order to guarantee that, in the beginning of each construction we project all deleted vertices on one of the four hypercubes $\mathcal{Q}_n^{00}$, $\mathcal{Q}_n^{01}$, $\mathcal{Q}_n^{10}$ and $\mathcal{Q}_n^{11}$, where the construction of the Hamiltonian cycle begins, and using Lemma 2.8 we choose a path $\mu$ in that hypercube with length at least $2(n-5)+1$ which begins with an even or an odd vertex (in $\mathcal{Q}_n$), depending on our needs. Using the natural projections we identify all four hypercubes $\mathcal{Q}_n^{00}$, $\mathcal{Q}_n^{01}$, $\mathcal{Q}_n^{10}$ and $\mathcal{Q}_n^{11}$ and in that way we obtain four copies of $\mu$: $\mu_{00}$, $\mu_{01}$, $\mu_{10}$ and $\mu_{11}$ – one in each of the four hypercubes. Then, to construct the short cycles, we follow $\mu$, i.e. every vertex from each short cycle which is in $\mathcal{Q}_n^{ii}$ belongs to $\mu_{ii}$ and every horizontal edge which is in $\mathcal{Q}_n^{ii}$ belongs to $\mu_{ii}$. In each short cycle we use at least one and at most two horizontal edges, hence for each short cycle the first and the last vertex are different and for each short cycle we use at most two edges from $\mu$. Also, we always traverse $\mu$ in the same direction and therefore we never use the same edge from $\mu$ twice. Therefore, at the end of the construction, our short cycles do not contain deleted vertices and every undeleted vertex is contained in at most one short cycle. Clearly, the length of $\mu$ is enough to construct at least $n-5 \geq k-3$ such short cycles.

In the proofs below we refer to the path $\mu$ described above as a *model path* and shall not repeat each time how we choose $\mu$ when we use it. Also, whenever we construct short cycles in the proofs below we shall use the model path $\mu$ and the procedure described above without mentioning that specifically.

In order to explain how we choose the coordinates that we use to split $\mathcal{Q}_n$ we order all vertices from $\mathcal{F}$ in a column and let $M'$ be the $2k \times n$ matrix determined by the coordinates of those vertices (every row corresponds to a vertex). Then every coordinate corresponds to a column in $M'$ and every column in $M'$ corresponds to a coordinate, hence we shall not make a difference between columns and coordinates. Let $M_e$ be the submatrix of $M'$ determined by those columns in $M'$ that separate only the even vertices in $\mathcal{F}$ and $M_o$ be the submatrix of $M'$ determined by those columns in $M'$ that separate only the odd vertices in $\mathcal{F}$. For two disjoint submatrices $A$ and $B$ of $M'$ by $(A, B)$ we denote the submatrix of $M'$ determined by the columns that are in $A$ or in $B$. (The order of the columns in all matrices that we consider here is not important to us). Let also $M_2$ be the submatrix of $M'$ determined by those columns in $M'$ that separate simultaneously the even and the odd vertices in $\mathcal{F}$. Finally, set $M_1 = (M_e, M_o)$ and $M = (M_1, M_2)$.

Now we shall show that we can always choose one or two columns from $M$ which satisfy at least one of the cases $(A) - (J)$ considered below and therefore to complete the proof of $(L)$ it will be enough to show that in all those cases there is a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$.

In Cases $(A) - (C)$, the existence of one column in $M$ which separates the deleted vertices in a special way is sufficient for the construction of the required Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$.

(A) The case when there exists a column in $M_1$ which separates the vertices in $\mathcal{F}$ in the way $(1, 2k-1)$ is considered in Case $(A)$.

(B) The case when there exists a column in $M_1$ that separates the vertices in $\mathcal{F}$ in the way $(2, 2k-2)$ is considered in Case $(B)$.

(C) The case when there is a column in $M_2$ which separates the odd vertices in the way $(s, k-s)$, the even vertices in the way $(s, k-s)$, and all vertices in $\mathcal{F}$ in the way $(2s, 2k-2s)$, where $1 \leq s \leq k-1$, is considered in Case $(C)$.

**Remark 3.3.** In the remaining Cases $(D) - (J)$ two columns are required for the construction of the Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$. In those cases, without loss of generality, we assume that there are no columns in $M$ that could allow us to obtain any of the Cases $(A)-(C)$. In particular, we assume that every column which separates only the odd or only the even vertices in the way $(1, k-1)$ separates all vertices in $\mathcal{F}$ in the way $(k+1, k-1)$ and every column which separates only the odd or only the even vertices in the way $(2, k-2)$ separates all vertices in $\mathcal{F}$ in the way $(k+2, k-2)$.

(D) The case when there exist two columns in $M_2$ that separate the even and the odd vertices in different ways is considered in Case $(D)$.

**Remark 3.4.** In the remaining Cases $(E) - (J)$ we assume that columns as in $(D)$ do not exist and for Cases $(E) - (G)$ we assume that there exists a column $A$ in $M_2$ that separates the even vertices in the way $(r, k-r)$, where $2 \leq r \leq k-2$. Since $k > 2$, there is a column $B$ in $M$ which separates the odd vertices in a different way than $A$.

(E) The case when $B$ is in $M_2$ and separates the even vertices in the same way as $A$ is considered in Case $(E)$.

**Remark 3.5.** Now we suppose that there is no such column in $M_2$ as in (E). Hence every column in $M_2$ separates the odd vertices as $A$ does. Therefore $B$ is in $M_o$.

(F) The case when $A$ or $B$ separates the odd vertices in the way $(s, k-s)$, where $2 \leq s \leq k-2$, is considered in Case $(F)$.

(G) If neither $A$ nor any $B$ from $M_o$ separates the odd vertices in the way $(s, k-s)$, where $2 \leq s \leq k-2$, then every column that separates the odd vertices separates them in the way $(1, k-1)$[2]. Then, it follows from Corollary 2.5 that if $k = n-2$ then there are two columns in $M_2$ that separate the even and the odd vertices in different ways, which is case $(D)$. Therefore we can assume that $k \leq n-3$. This case is considered in Case $(G)$.

**Remark 3.6.** For the remaining cases $(H) - (J)$ we assume that every column in $M_2$ separates the even and the odd vertices in the way $(1, k-1)$. The case when there is a column in $M_2$ which separates all vertices in $\mathcal{F}$ in the way $(2, 2k-2)$ was considered in $(C)$. Therefore we can assume that every column in $M_2$ separates the vertices in $\mathcal{F}$ in the way $(k, k)$. If $M_1$ is empty, or equivalently, $M = M_2$, then according to Lemma 2.1, there exist two columns in $M$ that separate

[2] Recall that we are assuming that no two columns that separate both the even and the odd vertices in two different ways exist.

the even and the odd vertices in different ways, which is impossible according to Remark 3.4. Hence, we can assume that $M_1$ is non-empty and therefore, without loss of generality, we can assume that $M_o$ is non-empty.

(H) Suppose that there exists a column $A$ in $M_o$ such that $A$ separates the odd vertices in the way $(s, k-s)$, where $2 \le s \le k-2$.The case when there exists a column in $M_e$ which separates the even vertices in the way $(r, k-r)$, where $2 \le r \le k-2$, is considered in Case $(H)$.

**Remark 3.7.** Suppose now that every column which separates the odd vertices in $\mathcal{F}$ separates them in the way $(1, k-1)$ and since, according to our previous assumption $M_o$ is non-empty, we can fix a column $A$ from $M_o$. Notice that if $k = n-2$ then it follows from Corollary 2.5 that there are two columns in $M_2$ that separate the even and the odd vertices in different ways, which is impossible according to Remark 3.4. Therefore in the remaining Cases (I) and (J) we can assume that $k \le n-3$ and that there are no two columns in $M_2$ that separate the even and the odd vertices in different ways.

(I) The case when there is a column $B$ which separates only the even vertices is considered in Case (I).

(J) The case when there is a column $B$ which separates the even vertices and separates the odd vertices in a different way than $A$ is considered in Case (J).

Clearly, the above cases exhaust all possibilities that need to be considered in order to prove $(L)$.

Now in each of the Cases (A) – (J) we are going to construct a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$.

*Case* $(A)$ There is a column $A$ in $M_1$ which separates the vertices in $\mathcal{F}$ in the way $(1, 2k-1)$.

Use $A$ to split the hypercube. Without loss of generality we can assume that there are $k$ even and $k-1$ odd deleted vertices in the top plate and one odd vertex in the bottom plate. Use $(CG)$ for $n-1$ and $k-1$ to find a Hamiltonian cycle for the top plate that contains one of the deleted odd vertices. Then delete that odd vertex from the cycle and connect the resulting path to the bottom plate with two edges that we call bridges. Then use $(CG)$ for $n-1$ and 0 to find a Hamiltonian path for the bottom plate that connects the end vertices of the bridges and does not contain the deleted odd vertex. The result is the desired Hamiltonian cycle.

*Case* $(B)$ There exists a column $A$ in $M_1$ which separates the vertices in $\mathcal{F}$ in the way $(2, 2k-2)$.

Without loss of generality we can assume that $A$ belongs to $M_o$. Thus, if we split $\mathcal{Q}_n$ using $A$, there will be two odd vertices in one of the plates, say the top plate, and $2k-2$ deleted vertices in the bottom plate. Since there are at least five deleted even vertices in the bottom plate, there are two, say $e_1$ and $e_2$, that are at distance at least four. Use $(L)$ for $n-1$ and $k-2$ to find a Hamiltonian cycle $\gamma$ for the bottom plate that contains $e_1$ and $e_2$ and avoids all the other $k-2$ pairs of deleted even and odd vertices. Delete $e_1$ and $e_2$ from $\gamma$. In that way we obtain a 2−path covering for the bottom plate that does not contain any of the deleted vertices. Connect the end vertices of both paths with bridges to the top plate. Use [5, Lemma 4.3] to find a 2−path covering of the top plate that avoids the two deleted odd vertices, each path connects two end vertices of two of the bridges, and such that these two paths together with the bridges and the other two paths form the desired Hamiltonian cycle.

*Case* $(C)$ There exists a column $A$ in $M_2$ which separates the odd vertices in the way $(s, k-s)$, the even vertices in the way $(s, k-s)$, and all vertices in $\mathcal{F}$ in the way $(2s, 2k-2s)$, where $1 \le s \le k-1$.

Without loss of generality we can assume that $s \le k-s$. Since $k \ge 5$, $s$ and $k-s$ cannot be simultaneously equal to $k-1$. It follows from our hypothesis that if we split the hypercube using $A$, there will be $k-s \le k-1 \le (n-1)-2$ pairs of deleted even and odd vertices in one of the plates and $s \le k-2 \le (n-2)-2 = (n-1)-3$ pairs of deleted even and odd vertices in the other plate. Use $(L)$ for $n-1$ and $k-s$ to find a Hamiltonian cycle for the plate that contains $2k-2s$ deleted vertices which avoids all the deleted vertices. Cut that cycle at an edge whose end vertices are not neighbors of any of the deleted vertices on the other plate. Such edge exists since the length of the Hamiltonian cycle is $2^{n-1} - 2(k-s) > 4s$ and there are only $2s$ deleted vertices on the other plate. Connect the ends of the resulting path with bridges with the other plate. Use $(T)$ for $n-1$ and $s$ to find a Hamiltonian path for the plate that contains $s \le (n-1)-3$ deleted pairs of vertices which connects the end vertices of both bridges and avoids all deleted vertices. The result is the desired Hamiltonian cycle.

**Note 1.** For the remaining cases Remark 3.3 applies.

**Remark 3.8.** For easier explanation, for a hypercube $K$, we use the following terminology: if there are $s$ deleted even and $t$ deleted odd vertices in $K$ then $|s-t|$ is called *charge* of $K$; when $s-t>0$ we say that $K$ has a *positive charge*; when $s-t<0$ we say that $K$ has a *negative charge*; and when $s-t=0$ we say that $K$ is *neutral*.

*Case* $(D)$ There exist two columns $A$ and $B$ in $M_2$ that separate the odd and the even vertices in different ways.

We split the hypercube using $A$ and $B$ into the four plates $\mathcal{Q}_n^{00}$, $\mathcal{Q}_n^{01}$, $\mathcal{Q}_n^{10}$, and $\mathcal{Q}_n^{11}$. Then there will be deleted even (odd) vertices in at least three of the plates, hence the maximal number of deleted even or odd vertices in a given plate could be at most $k-2$. Also, there will be deleted even and odd vertices in at least two of the plates, hence there will be at least two pairs of even and odd vertices such that each one is contained in one of the four plates.

If there exist two plates at distance one which union is a neutral hypercube then that case was considered in $(C)$. Therefore, without loss of generality, we can make the following assumption:

**Assumption.** Every hypercube, which is the union of two of the four plates which are at distance one, is not neutral.

It follows from Assumption that there exists at least one plate $K_1$ with a positive charge and at least one plate $K_2$ with a negative charge. Let the charge of $K_1$ be $s>0$ and the charge of $K_2$ be $t>0$. We denote by $q_{ij}$ the maximal number of pairs of deleted even and odd vertices that can be formed in the plate $\mathcal{Q}_n^{ij}$.

We consider two subcases: (D)(1) The plates $K_1$ and $K_2$ are at distance two; and (D)(2) $K_1$ and $K_2$ are at distance one from each other.

(D)(1) $K_1$ and $K_2$ are at distance two.

Without loss of generality we can assume that $K_1=\mathcal{Q}_n^{00}$ and $K_2=\mathcal{Q}_n^{11}$. Then, up to symmetry and up to interchanging positive and negative charge, there are four different subcases: (D)(1)(a) $\mathcal{Q}_n^{01}$ and $\mathcal{Q}_n^{10}$ are neutral; (D)(1)(b) $\mathcal{Q}_n^{10}$ has a negative charge and $\mathcal{Q}_n^{01}$ is neutral; (D)(1)(c) $\mathcal{Q}_n^{01}$ and $\mathcal{Q}_n^{10}$ have negative charges; and (D)(1)(d) $\mathcal{Q}_n^{01}$ has a positive charge and $\mathcal{Q}_n^{10}$ has a negative charge.

(D)(1)(a) $\mathcal{Q}_n^{01}$ and $\mathcal{Q}_n^{10}$ are neutral.

Since there are even (odd) deleted vertices in at least three of the four plates, we have $s=t\le k-2$, $q_{01}+t\le k-1$, $q_{00}+s\le k-2$, $q_{10}+s\le k-1$ and $q_{11}+t\le k-2$.

Take a model path $\mu$ in $\mathcal{Q}_n^{01}$ which begins with an even vertex $e_{01}=u_{01}$ and following $\mu$ make $s-1$ short cycles of the type

$$e_{01}\to o_{00}\to (e,o)_{10}\to e_{11}\to (o,e')_{01}.$$

We denote the resulting path by $\gamma_0$, its end vertex by $a_{01}$ and let the odd neighbor of $a_{01}$ in $\mathcal{Q}_n^{00}$ be $v_{00}$. We extend the constructed path so far with the edge $(a_{01},v_{00})$.

The total number of the constructed short cycles is $s-1\le k-3$, hence the length of $\mu$ is enough for that construction.

Let $v'_{00}$ be any unused and undeleted odd vertex in $\mathcal{Q}_n^{00}$ different from $v_{00}$ whose even neighbor $u_{10}$ in $\mathcal{Q}_n^{10}$ is neither a deleted nor used vertex. There are $q_{00}+s$ deleted or used even and $q_{00}+s-1$ deleted or used odd vertices in $\mathcal{Q}_n^{00}$. Since $q_{00}+s-1\le k-3\le(n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_1$ for $\mathcal{Q}_n^{00}$ minus all deleted and used vertices which connects $v_{00}$ to $v'_{00}$.

To continue we need to construct paths $\gamma_2$ and $\gamma_4$ in $\mathcal{Q}_n^{10}$ and $\mathcal{Q}_n^{01}$, respectively. For that end we consider two subcases: (D)(1)(a)(i) there exists a deleted even vertex in $\mathcal{Q}_n^{11}$, hence $q_{11}>0$; and (D)(1)(a)(ii) there are no deleted even vertices in $\mathcal{Q}_n^{11}$, hence $q_{11}=0$.

(D)(1)(a)(i) $q_{11}>0$.

In this case we have $s=t\le k-3$ and $q_{01}+q_{11}+t\le k-1$, hence $q_{01}+t\le k-2$. Notice also that $q_{10}+q_{11}+t\le k-1$, hence $q_{10}+t\le k-2$.

There are $q_{10}+s-1=q_{10}+t-1$ even and odd used or deleted vertices in $\mathcal{Q}_n^{10}$ ($u_{10}$ is not counted). Let $v_{10}$ be any unused and undeleted odd vertex in $\mathcal{Q}_n^{10}$ whose even neighbor $u_{11}$ in $\mathcal{Q}_n^{11}$ is neither a deleted nor used vertex. Since $q_{10}+t-1\le k-3\le(n-2)-3$, it follows from $(T)$ that there exists a Hamiltonian path $\gamma_2$ for $\mathcal{Q}_n^{10}$ minus all deleted and used vertices which connects $u_{10}$ to $v_{10}$.

There are $q_{01} + s - 1 = q_{01} + t - 1$ deleted or used even and odd vertices in $\mathcal{Q}_n^{01}$ ($u_{01}$ is not counted). Let $v_{01}$ be any unused and undeleted odd vertex in $\mathcal{Q}_n^{01}$ which even neighbor $u'_{11}$ in $\mathcal{Q}_n^{11}$ is different from $u_{11}$ and is neither a deleted nor used vertex. Since $q_{01}+t-1 \leq k-3 \leq (n-2)-3$, it follows from $(T)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{01}$ minus all deleted and used vertices which connects $v_{01}$ to $u_{01}$.

(D)(1)(a)(ii) $q_{11} = 0$.

There are $q_{10} + s - 1 = q_{10} + t - 1$ even and odd used or deleted vertices in $\mathcal{Q}_n^{10}$ ($u_{10}$ is not counted). Since $q_{10} + s - 1 \leq k - 2 \leq (n-2) - 2$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma'$ for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices. This cycle contains $u_{10}$. Let $v_{10}$ be an odd neighbor of $u_{10}$ in $\gamma'$. Since $q_{11} = 0$, the even neighbor $u_{11}$ of $v_{10}$ in $\mathcal{Q}_n^{11}$ is neither a deleted nor used vertex. We denote by $\gamma_1$ the Hamiltonian path for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices that is defined by $\gamma'$ and connects $u_{10}$ to $v_{10}$.

There are $q_{01} + s - 1$ deleted or used even and odd vertices in $\mathcal{Q}_n^{01}$ ($u_{01}$ is not counted). Since $q_{01} + s - 1 \leq k - 2 \leq (n-2) - 2$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma''$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices. This cycle contains $u_{01}$. Let $v_{01}$ be an odd neighbor of $u_{01}$ in $\gamma''$ such that its even neighbor $u'_{11}$ of $v_{01}$ in $\mathcal{Q}_n^{11}$ is different from $u_{11}$. Since $q_{11} = 0$, $u'_{11}$ is neither a deleted nor used vertex. We denote by $\gamma_4$ the Hamiltonian path for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices that is defined by $\gamma''$ and connects $v_{01}$ to $u_{01}$.

Now we continue the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ for both subcases (D)(1)(a)(i) and (D)(1)(a)(ii).

There are $q_{11} + t - 1$ deleted or used even and $q_{11} + t$ deleted or used odd vertices in $\mathcal{Q}_n^{11}$. Since $q_{11} + t - 1 \leq k - 3 \leq (n-2) - 3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{11}$ minus all deleted and used vertices which connects $u_{11}$ to $u'_{11}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path $\gamma_0$ with the path

$$v_{00} \xrightarrow{\gamma_1} v'_{00} \to u_{10} \xrightarrow{\gamma_2} v_{10} \to u_{11} \xrightarrow{\gamma_3} u'_{11} \to v_{01} \xrightarrow{\gamma_4} u_{01}.$$

(D)(1)(b) $\mathcal{Q}_n^{10}$ has a negative charge and $\mathcal{Q}_n^{01}$ is neutral.

Let the charge of $\mathcal{Q}_n^{10}$ be $p > 0$, hence $p + t = s \leq k - 2$ and $s + q_{01} \leq k - 1$. Notice also that $q_{00} + s \leq k - 2$, $q_{10} + p + t \leq k - 1$ and $q_{11} + p + t \leq k - 1$ since there are even (odd) deleted vertices in at least three of the four plates.

Take a model path $\mu$ in $\mathcal{Q}_n^{11}$ which begins with an even vertex $e_{11} = u_{11}$ and following $\mu$ make $p - 1$ short cycles of the type

$$e_{11} \to (o, e)_{01} \to o_{00} \to e_{10} \to (o, e')_{11}.$$

We denote the end vertex of the resulting path by $a_{11}$.

Then begin with $e_{11} = a_{11}$ and following $\mu$ make $t - 1$ short cycles of the type

$$e_{11} \to (o, e)_{01} \to o_{00} \to (e, o)_{10} \to e'_{11}.$$

We denote the end vertex of the resulting path by $a'_{11}$.

Finally, begin with $e_{11} = a'_{11}$ and following $\mu$ make the following path with length four

$$e_{11} \to (o, e)_{01} \to o_{00} \to e_{10}.$$

We denote the resulting path by $\gamma_0$ and its end vertex by $u_{10}$.

The total number of the constructed short cycles is $p - 1 + t - 1 + 1 = p + t - 1 \leq k - 3$, hence the length of $\mu$ is enough for that construction.

There are $q_{01} + p - 1 + t - 1 + 1 = q_{01} + s - 1 \leq k - 2$ even and odd deleted or used vertices in $\mathcal{Q}_n^{01}$. Since $k - 2 \leq (n-2) - 2$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma'$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices. Let $(u_{01}, v_{01})$ be any edge in $\gamma'$ such that the even neighbor $u'_{11}$ of $v_{01}$ in $\mathcal{Q}_n^{11}$ and the odd neighbor $v_{00}$ of $u_{01}$ in $\mathcal{Q}_n^{00}$ are neither deleted nor used vertices. We denote by $\gamma_3$ the Hamiltonian path for $\mathcal{Q}_n^{01}$ minus all deleted and used vertices that is defined by $\gamma'$ and connects $u_{01}$ to $v_{01}$.

Let $v'_{00}$ be any unused and undeleted odd vertex in $\mathcal{Q}_n^{00}$ different from $v_{00}$ which even neighbor $u'_{10}$ in $\mathcal{Q}_n^{10}$ is neither deleted nor used vertex. There are $q_{00} + s$ used or deleted even and $q_{00} + p - 1 + t - 1 + 1 = q_{00} + s - 1$ used or deleted odd vertices in $\mathcal{Q}_n^{00}$. Since $q_{00} + s - 1 \leq k - 3 \leq (n-2) - 3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_2$ for $\mathcal{Q}_n^{00}$ minus all deleted and used vertices which connects $v'_{00}$ to $v_{00}$.

There are $q_{10} + p - 1 + t - 1 = q_{10} + p + t - 2$ used or deleted even and $q_{10} + p - 1 + t - 1 + 1 = q_{10} + p + t - 1$ deleted or used odd vertices in $\mathcal{Q}_n^{10}$ ($u_{10}$ and $u'_{10}$ are not counted). Since $q_{10}+p+t-2 \leq (k-1)-2 \leq (n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_1$ for $\mathcal{Q}_n^{10}$ minus all deleted and used vertices which connects $u_{10}$ to $u'_{10}$.

There are $q_{11} + p - 1 + t - 1 = q_{11} + p + t - 2$ deleted or used even and $q_{11} + p - 1 + t = q_{11}+p+t-1$ deleted or used odd vertices in $\mathcal{Q}_n^{11}$. Since $q_{11}+p+t-2 \leq (k-1)-2 \leq (n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{11}$ minus all deleted and used vertices which connects $u'_{11}$ to $u_{11}$.

Finally, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path $\gamma_0$ with the path

$$u_{10} \xrightarrow{\gamma_1} u'_{10} \to v'_{00} \xrightarrow{\gamma_2} v_{00} \to u_{01} \xrightarrow{\gamma_3} v_{01} \to u'_{11} \xrightarrow{\gamma_4} u_{11}.$$

(D)(1)(c) $\mathcal{Q}_n^{01}$ and $\mathcal{Q}_n^{10}$ have negative charges.

Let the charge of $\mathcal{Q}_n^{01}$ be $p$ and the charge of $\mathcal{Q}_n^{10}$ be $r$. Then $p + t + r = s \leq k - 2$ and $s + q_{11} \leq k - 1$. Notice also that $q_{00} + s \leq k - 2$, $q_{10} + s \leq k - 1$ and $q_{01} + s \leq k - 1$ since there are even (odd) deleted vertices in at least three of the four plates.

Take a model path $\mu$ in $\mathcal{Q}_n^{01}$ which begins with an even vertex $e_{01} = u_{01}$ and following $\mu$ make $p - 1$ short cycles of the type

$$e_{01} \to o_{00} \to (e, o)_{10} \to (e, o)_{11} \to e'_{01}.$$

We denote the end vertex of the resulting path by $a_{01}$.

Then begin with $e_{01} = a_{01}$ and following $\mu$ make $r - 1$ short cycles of the type

$$e_{01} \to o_{00} \to e_{10} \to (o, e)_{11} \to (o, e')_{01}.$$

We denote the end vertex of the resulting path by $a'_{01}$.

Now begin with $e_{01} = a'_{01}$ and following $\mu$ make $t$ short cycles of the type

$$e_{01} \to o_{00} \to (e, o)_{10} \to e_{11} \to (o, e')_{01}.$$

We denote the end vertex of the resulting path by $a''_{01}$.

Finally, begin with $e_{01} = a''_{01}$ and following $\mu$ make the following path with length three

$$e_{01} \to o_{00} \to e_{10} \to o_{11}.$$

We denote the end vertex of the resulting path by $v_{11}$.

The total number of the constructed short cycles is $p - 1 + r - 1 + t = s - 2 \leq k - 4$, hence the length of $\mu$ is enough for that construction.

There are $q_{10} + p - 1 + t + r - 1 + 1 = q_{10} + s - 1 \leq k - 2$ even and odd deleted or used vertices in $\mathcal{Q}_n^{10}$. Since $k - 2 \leq (n - 2) - 2$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma'$ for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices. Let $(u_{10}, v_{10})$ be any edge in $\gamma'$ such that the even neighbor $u_{11}$ of $v_{10}$ in $\mathcal{Q}_n^{11}$ and the odd neighbor $v_{00}$ of $u_{10}$ in $\mathcal{Q}_n^{00}$ are neither deleted nor used vertices. We denote by $\gamma_2$ the Hamiltonian path for $\mathcal{Q}_n^{10}$ minus all deleted and used vertices that is defined by $\gamma'$ and connects $v_{10}$ to $u_{10}$.

There are $q_{11} + p - 1 + t + r - 1 = q_{11} + s - 2$ deleted or used even and odd vertices in $\mathcal{Q}_n^{11}$. Since $q_{11} + s - 2 \leq (k-1) - 2 \leq (n-2) - 3$, it follows from $(T)$ that there exists a Hamiltonian path $\gamma_1$ for $\mathcal{Q}_n^{11}$ minus all deleted and used vertices which connects $v_{11}$ to $u_{11}$.

Let $v'_{00}$ be any unused and undeleted odd vertex in $\mathcal{Q}_n^{00}$ different from $v_{00}$ which even neighbor $u'_{01}$ in $\mathcal{Q}_n^{01}$ is neither deleted nor used vertex. There are $q_{00} + s$ used or deleted even and $q_{00} + p - 1 + t + r - 1 + 1 = q_{00} + s - 1$ used or deleted odd vertices in $\mathcal{Q}_n^{00}$. Since $q_{00} + s - 1 \leq k - 3 \leq (n - 2) - 3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{00}$ minus all deleted and used vertices which connects $v_{00}$ to $v'_{00}$.

There are $q_{01} + p - 1 + t + r - 1 = q_{01} + s - 2$ used or deleted even and $q_{01} + p + t + r - 1 = q_{01} + s - 1$ deleted or used odd vertices in $\mathcal{Q}_n^{01}$ ($u_{01}$ and $u'_{01}$ are not counted). Since $q_{01} + s - 2 \leq (k - 1) - 2 \leq (n - 2) - 3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{01}$ minus all deleted and used vertices which connects $u'_{01}$ to $u_{01}$.

Finally, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$v_{11} \xrightarrow{\gamma_1} u_{11} \to v_{10} \xrightarrow{\gamma_2} u_{10} \to v_{00} \xrightarrow{\gamma_3} v'_{00} \to u'_{01} \xrightarrow{\gamma_4} u_{01}.$$

(D)(1)(d) $\mathcal{Q}_n^{01}$ has a positive charge and $\mathcal{Q}_n^{10}$ has a negative charge.

Let the charge of $\mathcal{Q}_n^{01}$ be $p > 0$ and the charge of $\mathcal{Q}_n^{10}$ be $r > 0$. Since there are odd vertices in at least three of the four plates, either in $\mathcal{Q}_n^{00}$ or in $\mathcal{Q}_n^{01}$ there is a deleted odd vertex. Using the symmetry of this case, without loss of generality, we can assume that there is an odd deleted vertex in $\mathcal{Q}_n^{00}$. Then $q_{01} + p + q_{00} + s \leq k - 1$, hence $q_{01} + p + s \leq k - 2$ and therefore $p + s = t + r \leq k - 2$. Notice also that $q_{00} + p + s \leq k - 1$, $q_{10} + r + t \leq k - 1$ and $q_{11} + r + t \leq k - 1$, since there are even (odd) deleted vertices in at least three of the four plates.

Take a model path $\mu$ in $\mathcal{Q}_n^{11}$ which begins with an even vertex $e_{11} = u_{11}$ and following $\mu$ make a total of $t + r - 2 = p + s - 2$ short cycles of the types

$$e_{11} \to (o, e)_{01} \to o_{00} \to e_{10} \to (o, e')_{11};$$

$$e_{11} \to o_{01} \to (e, o)_{00} \to e_{10} \to (o, e')_{11};$$

$$e_{11} \to o_{01} \to (e, o)_{00} \to (e, o)_{10} \to e'_{11};$$

$$e_{11} \to (o, e)_{01} \to o_{00} \to (e, o)_{10} \to e'_{11};$$

such that at the end all plates have charge one. We denote the end vertex of the resulting path by $a_{11}$.

Finally, begin with $e_{11} = a_{11}$ and following $\mu$ make the following path with length four

$$e_{11} \to (o, e)_{01} \to o_{00} \to e_{10}.$$

We denote the end vertex of the resulting path by $u_{10}$.

The total number of the constructed short cycles is $p - 1 + s - 1 + 1 = p + s - 1 \leq k - 3$, hence the length of $\mu$ is enough for that construction.

There are $q_{00} + p - 1 + s = q_{00} + p + s - 1$ deleted or used even and $q_{00} + p - 1 + s - 1 + 1 = q_{00} + p + s - 1$ deleted or used odd vertices in $\mathcal{Q}_n^{00}$. Since $q_{00} + p + s - 1 \leq k - 2 \leq (n - 2) - 2$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma'$ for $\mathcal{Q}_n^{00}$ minus all deleted or used vertices. Let $(u_{00}, v_{00})$ be any edge in $\gamma'$ such that the even neighbor $u'_{10}$ of $v_{00}$ in $\mathcal{Q}_n^{10}$ and the odd neighbor $v_{01}$ of $u_{00}$ in $\mathcal{Q}_n^{01}$ are neither deleted nor used vertices. We denote by $\gamma_2$ the Hamiltonian path for $\mathcal{Q}_n^{00}$ minus all deleted and used vertices that is defined by $\gamma'$ and connects $v_{00}$ to $u_{00}$.

There are $q_{10} + r - 1 + t - 1 = q_{10} + r + t - 2$ used or deleted even and $q_{10} + r + t - 1$ deleted or used odd vertices in $\mathcal{Q}_n^{10}$ ($u_{10}$ and $u'_{10}$ are not counted). Since $q_{10} + r + t - 2 \leq (k - 1) - 2 \leq (n - 2) - 3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_1$ for $\mathcal{Q}_n^{10}$ minus all deleted and used vertices which connects $u_{10}$ to $u'_{10}$.

Let $v'_{01}$ be any unused and undeleted odd vertex in $\mathcal{Q}_n^{01}$ different from $v_{01}$ such that its even neighbor $u'_{11}$ in $\mathcal{Q}_n^{11}$ is neither a deleted nor used vertex. There are $q_{01} + p + s - 1 + 1 = q_{01} + p + s$ deleted or used even and $q_{01} + p - 1 + s - 1 + 1 = q_{01} + p + s - 1$ deleted or used odd vertices in $\mathcal{Q}_n^{01}$. Since $q_{01} + p + s - 1 \leq (k - 2) - 1 \leq (n - 2) - 3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{01}$ minus all deleted and used vertices which connects $v_{01}$ to $v'_{01}$.

There are $q_{11} + t - 1 + r - 1 = q_{11} + t + r - 2$ deleted or used even and $q_{11} + t + r - 1$ deleted or used odd vertices in $\mathcal{Q}_n^{11}$. Since $q_{11} + t + r - 2 \leq (k - 1) - 2 \leq (n - 2) - 3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices which connects $u'_{11}$ to $u_{11}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$u_{10} \xrightarrow{\gamma_1} u'_{10} \to v_{00} \xrightarrow{\gamma_2} u_{00} \to v_{01} \xrightarrow{\gamma_3} v'_{01} \to u'_{11} \xrightarrow{\gamma_4} u_{11}.$$

(D)(2) $K_1$ and $K_2$ are at distance one.

Without loss of generality we can assume that $K_1 = \mathcal{Q}_n^{00}$ and $K_2 = \mathcal{Q}_n^{01}$. Then, up to symmetry and up to interchanging positive and negative charge, there are six different cases: (a)

$\mathcal{Q}_n^{10}$ and $\mathcal{Q}_n^{11}$ are neutral; (b) $\mathcal{Q}_n^{10}$ is neutral and $\mathcal{Q}_n^{11}$ has a negative charge; (c) $\mathcal{Q}_n^{10}$ has a negative charge and $\mathcal{Q}_n^{11}$ is neutral; (d) $\mathcal{Q}_n^{10}$ and $\mathcal{Q}_n^{11}$ have negative charges; (e) $\mathcal{Q}_n^{10}$ has a negative charge and $\mathcal{Q}_n^{11}$ has a positive charge; and (f) $\mathcal{Q}_n^{10}$ has a positive charge and $\mathcal{Q}_n^{11}$ has a negative charge.

(D)(2)(a) $\mathcal{Q}_n^{10}$ and $\mathcal{Q}_n^{11}$ are neutral.

According to Assumption we do not need to consider this case.

(D)(2)(b) $\mathcal{Q}_n^{10}$ is neutral and $\mathcal{Q}_n^{11}$ has a negative charge.

This case is equivalent to case (D)(1)(b).

(D)(2)(c) $\mathcal{Q}_n^{10}$ has a negative charge and $\mathcal{Q}_n^{11}$ is neutral.

Let the charge of $\mathcal{Q}_n^{10}$ be $p > 0$, hence $p + t = s \leq k - 2$ and $s + q_{11} \leq k - 1$. Notice also that $q_{00} + s \leq k - 2$, $q_{10} + p + t \leq k - 1$ and $q_{01} + p + t \leq k - 1$ since there are even (odd) deleted vertices in at least three of the four plates.

Take a model path $\mu$ in $\mathcal{Q}_n^{01}$ which begins with an even vertex $e_{01} = u_{01}$ and following $\mu$ make $t - 1$ short cycles of the type

$$e_{01} \to o_{00} \to (e, o)_{10} \to (e, o)_{11} \to e'_{01}.$$

We denote the end vertex of the resulting path by $a_{01}$.

Then begin with $e_{01} = a_{01}$ and following $\mu$ make $p - 1$ short cycles of the type

$$e_{01} \to o_{00} \to e_{10} \to (o, e)_{11} \to (o, e')_{01}.$$

We denote the end vertex of the resulting path by $a'_{01}$.

Finally, begin with $e_{01} = a'_{01}$ and following $\mu$ make the following path with length three

$$e_{01} \to o_{00} \to e_{10} \to o_{11}.$$

We denote the end vertex of the resulting path by $v_{11}$.

The total number of the constructed short cycles is $t - 1 + p - 1 = t + p - 2 \leq k - 4$, hence the length of $\mu$ is enough for that construction.

There are $q_{10} + p - 1 + t - 1 + 1 = q_{10} + p + t - 1 \leq k - 2$ even and odd deleted or used vertices in $\mathcal{Q}_n^{10}$. Since $k - 2 \leq (n - 2) - 2$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma'$ for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices. Let $(u_{10}, v_{10})$ be any edge in $\gamma'$ such that the even neighbor $u_{11}$ of $v_{10}$ in $\mathcal{Q}_n^{11}$ and the odd neighbor $v_{00}$ of $u_{10}$ in $\mathcal{Q}_n^{00}$ are neither deleted nor used vertices. We denote by $\gamma_2$ the Hamiltonian path for $\mathcal{Q}_n^{10}$ minus all deleted and used vertices that is defined by $\gamma'$ and connects $v_{10}$ to $u_{10}$.

There are $q_{11} + p - 1 + t - 1 = q_{11} + p + t - 2$ deleted or used even and odd vertices in $\mathcal{Q}_n^{11}$. Since $q_{11} + p + t - 2 \leq (k - 1) - 2 \leq (n - 2) - 3$, it follows from $(T)$ that there exists a Hamiltonian path $\gamma_1$ for $\mathcal{Q}_n^{11}$ minus all deleted and used vertices which connects $v_{11}$ to $u_{11}$.

Let $v'_{00}$ be any unused and undeleted odd vertex in $\mathcal{Q}_n^{00}$ different from $v_{00}$ which even neighbor $u'_{01}$ in $\mathcal{Q}_n^{01}$ is neither deleted nor used vertex. There are $q_{00} + s$ used or deleted even and $q_{00} + p - 1 + t - 1 + 1 = q_{00} + s - 1$ used or deleted odd vertices in $\mathcal{Q}_n^{00}$. Since $q_{00} + s - 1 \leq k - 3 \leq (n-2) - 3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{00}$ minus all deleted and used vertices which connects $v_{00}$ to $v'_{00}$.

There are $q_{01} + p - 1 + t - 1 = q_{01} + p + t - 2$ used or deleted even and $q_{01} + p - 1 + t - 1 + 1 = q_{01} + p + t - 1$ deleted or used odd vertices in $\mathcal{Q}_n^{01}$ ($u_{01}$ and $u'_{01}$ are not counted). Since $q_{01} + p + t - 2 \leq (k - 1) - 2 \leq (n - 2) - 3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{01}$ minus all deleted and used vertices which connects $u'_{01}$ to $u_{01}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$v_{11} \xrightarrow{\gamma_1} u_{11} \to v_{10} \xrightarrow{\gamma_2} u_{10} \to v_{00} \xrightarrow{\gamma_3} v'_{00} \to u'_{01} \xrightarrow{\gamma_4} u_{01}.$$

(D)(2)(d) $\mathcal{Q}_n^{10}$ and $\mathcal{Q}_n^{11}$ have negative charges.

This case is equivalent to case (D)(1)(c).

(D)(2)(e) $\mathcal{Q}_n^{10}$ has a negative charge and $\mathcal{Q}_n^{11}$ has a positive charge.

Let the charge of $\mathcal{Q}_n^{10}$ be $p > 0$ and the charge of $\mathcal{Q}_n^{11}$ be $r > 0$. Since there are odd vertices in at least three of the four plates, either in $\mathcal{Q}_n^{00}$ or in $\mathcal{Q}_n^{11}$ there is a deleted odd vertex. Using the symmetry of this case, without loss of generality, we can assume that there is an odd deleted vertex in $\mathcal{Q}_n^{00}$. Then $q_{00} + s + q_{11} + r \leq k - 1$, hence $q_{11} + s + r \leq k - 2$ and therefore

$r+s=t+p\le k-2$. Notice also that $q_{00}+s+r\le k-1$, $q_{10}+t+p\le k-1$ and $q_{01}+t+p\le k-1$, since there are even (odd) deleted vertices in at least three of the four plates.

Take a model path $\mu$ in $\mathcal{Q}_n^{01}$ which begins with an even vertex $e_{01}=u_{01}$ and following $\mu$ make a total of $t+p-2=r+s-2$ short cycles of the types

$$e_{01}\to(o,e)_{00}\to(o,e)_{10}\to o_{11}\to e'_{01};$$

$$e_{01}\to o_{00}\to e_{10}\to(o,e)_{11}\to(o,e')_{01};$$

$$e_{01}\to o_{00}\to(e,o)_{10}\to(e,o)_{11}\to e'_{01};$$

$$(e,o)_{01}\to(e,o)_{00}\to e_{10}\to o_{11}\to e'_{01};$$

such that at the end all plates to have charge one. We denote the end vertex of the resulting path by $a_{01}$ and let its odd neighbor in $\mathcal{Q}_n^{00}$ be $v_{00}$. We extend the constructed path so far with the edge $(a_{01},v_{00})$.

The total number of the constructed short cycles is $r-1+s-1=r+s-2\le k-4$, hence the length of $\mu$ is enough for that construction.

Let $v'_{00}$ be any unused and undeleted odd vertex in $\mathcal{Q}_n^{00}$ different from $v_{00}$ whose even neighbor $u_{10}$ in $\mathcal{Q}_n^{10}$ is neither a deleted nor used vertex. There are $q_{00}+s+r-1$ deleted or used even and $q_{00}+s+r-2$ deleted or used odd vertices in $\mathcal{Q}_n^{00}$. Since $q_{00}+s+r-2\le k-3\le(n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_1$ for $\mathcal{Q}_n^{00}$ minus all deleted and used vertices which connects $v_{00}$ to $v'_{00}$.

Let $u'_{10}$ be any unused and undeleted odd vertex in $\mathcal{Q}_n^{10}$ different from $u_{10}$ whose odd neighbor $v_{11}$ in $\mathcal{Q}_n^{11}$ is neither a deleted nor used vertex. There are $q_{10}+p+t-2$ deleted or used even and $q_{10}+p+t-1$ deleted or used odd vertices in $\mathcal{Q}_n^{10}$. Since $q_{10}+p+t-2\le k-3\le(n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_2$ for $\mathcal{Q}_n^{10}$ minus all deleted and used vertices which connects $u_{10}$ to $u'_{10}$.

Let $v'_{11}$ be any unused and undeleted odd vertex in $\mathcal{Q}_n^{11}$ different from $v_{11}$ whose even neighbor $u'_{01}$ in $\mathcal{Q}_n^{01}$ is neither a deleted nor used vertex. There are $q_{11}+s+r-1$ deleted or used even and $q_{11}+s+r-2$ deleted or used odd vertices in $\mathcal{Q}_n^{11}$. Since $q_{11}+s+r-2\le k-3\le(n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{11}$ minus all deleted and used vertices which connects $v_{11}$ to $v'_{11}$.

There are $q_{01}+p+t-2$ deleted or used even and $q_{01}+p+t-1$ deleted or used odd vertices in $\mathcal{Q}_n^{01}$. Since $q_{01}+p+t-2\le k-3\le(n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{10}$ minus all deleted and used vertices which connects $u'_{01}$ to $u_{01}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n-\mathcal{F}$ we extend the previously constructed path with the path

$$v_{00}\xrightarrow{\gamma_1}v'_{00}\to u_{10}\xrightarrow{\gamma_2}u'_{10}\to v_{11}\xrightarrow{\gamma_3}v'_{11}\to u'_{01}\xrightarrow{\gamma_4}u_{01}.$$

(D)(2)(f) $\mathcal{Q}_n^{10}$ has a positive charge and $\mathcal{Q}_n^{11}$ has a negative charge.

This case is equivalent to case (D)(1)(d).

**Note 2.** For the remaining cases Remark 3.4 applies.

*Case* $(E)$ There exists a column $A$ in $M_2$ which separates the deleted even vertices in the way $(r,k-r)$, where $2\le r\le k-2$, and another column $B$ in $M_2$ which separates the deleted odd vertices in different way than $A$ and the deleted even vertices in the same way as $A$.

We split the hypercube using $A$ and $B$. Without loss of generality, we can assume that the deleted vertices are distributed as follows:

$$\{v_1,\dots,v_p\}\subset\mathcal{Q}_n^{00},\ \{u_1,\dots,u_r,v_{p+1},\dots,v_s\}\subset\mathcal{Q}_n^{10},\ \{v_{s+1},\dots,v_t\}\subset\mathcal{Q}_n^{11},\text{ and}$$
$$\{v_{t+1},\dots,v_k,u_{r+1},\dots,u_k\}\subset\mathcal{Q}_n^{01},$$

where $2\le r\le k-2$, $0\le p\le k-2$, $1\le s\le k-1$, and $2\le t\le k$, since there are odd vertices in at least three of the hypercubes.

If $r\le s-p<t$ then necessarily $k-r>k-t$. If $r>s-p$ then there are two possibilities: $k-r\le k-t$ or $k-r>k-t$. Since the cases $r\le s-p$, $k-r>k-t$ and $r>s-p$, $k-r\le k-t$ are symmetric, there are only two subcases to consider: (E)(1) $2\le r\le s-p\le k-2$; and (E)(2) $s-p<r\le k-2$ and $0\le k-t<k-r\le k-2$.

(E)(1) $2\le r\le s-p\le k-2$.

Take a model path $\mu$ in $\mathcal{Q}_n^{01}$ which begins with an odd vertex $o_{01} = v_{01}$.

We know that there are odd vertices in at least three of the hypercubes and since $s - p \geq 2$, we conclude that there are odd vertices in $\mathcal{Q}_n^{10}$. Hence there are three possibilities: (E)(1)(a) $t - s \geq 1$ and $k - t \geq 1$; (E)(1)(b) $t - s \geq 1$ and $p \geq 1$; and (E)(1)(c) $k - t \geq 1$ and $p \geq 1$. Since case (E)(1)(c) is symmetric to case (E)(1)(a), we consider only cases (E)(1)(a) and (E)(1)(b).

To construct a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ in each one of these cases we proceed as follows.

(E)(1)(a) $t - s \geq 1$ and $k - t \geq 1$.

In this case $3 \leq s + 1 \leq t \leq k - 1$.

Begin with $o_{01} = v_{01}$ and following $\mu$ make $s - p - r$ short cycles of the type

$$o_{01} \to (e, o)_{00} \to e_{10} \to (o, e)_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a'_{01}$.

Now begin with $o_{01} = a'_{01}$ and following $\mu$ make $p$ cycles of the type

$$o_{01} \to e_{00} \to (o, e)_{10} \to (o, e)_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a''_{01}$.

Finally, begin with $o_{01} = a''_{01}$ and following $\mu$ make $t - s - 1$ cycles of the type

$$o_{01} \to (e, o)_{00} \to (e, o)_{10} \to e_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a'''_{01}$.

Let the neighbor of $a'''_{01}$ in $\mathcal{Q}_n^{00}$ be $u_{00}$. We extend the constructed path with the edge $(a'''_{01}, u_{00})$.

The total number of the constructed short cycles is $s-p-r+p+t-s-1 = t-r-1 \leq k-3$, hence the length of $\mu$ is enough for that construction.

There are $s-p-r+p+r+t-s-1 = t-1$ used or deleted even and $s-p+p+t-s-1 = t-1$ odd used or deleted vertices in $\mathcal{Q}_n^{10}$. Since $t - 1 \leq (k-1) - 1 \leq (n-2) - 2$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma'$ for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices. Let $(u_{10}, v_{10})$ be any edge in $\gamma'$ such that the even neighbor $u_{11}$ of $v_{10}$ in $\mathcal{Q}_n^{11}$ and the odd neighbor $v_{00}$ of $u_{10}$ in $\mathcal{Q}_n^{00}$ are neither deleted nor used vertices. Such edge exists since there are only $p$ deleted odd vertices in $\mathcal{Q}_n^{00}$ that could be neighbors of $u_{10}$ and should be avoided. We denote by $\gamma_2$ the Hamiltonian path for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices that is defined by $\gamma'$ and connects $u_{10}$ to $v_{10}$.

There are $s - p - r + p + t - s - 1 = t - r - 1$ even and odd deleted or used vertices in $\mathcal{Q}_n^{00}$ ($u_{00}$ is not counted). Since $t - r - 1 \leq (k-1) - 2 - 1 \leq (n-2) - 4$, it follows from $(T)$ that there exists a Hamiltonian path $\gamma_1$ for $\mathcal{Q}_n^{00}$ minus all deleted or used vertices which connects $u_{00}$ to $v_{00}$.

There are $s-p-r+p+t-s-1 = t-r-1$ used even and $s-p-r+p+t-s = t-r$ odd deleted or used vertices in $\mathcal{Q}_n^{11}$. Let $u'_{11}$ be any unused even vertex in $\mathcal{Q}_n^{11}$ such that its odd neighbor $v'_{01}$ in $\mathcal{Q}_n^{01}$ is neither a deleted nor used vertex. Since $t - r - 1 \leq (k-1) - 2 - 1 \leq (n-2) - 4$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices which connects $u_{11}$ to $u'_{11}$.

There are $k - r$ deleted even and $s - p - r + p + t - s - 1 + k - t = k - r - 1$ deleted or used odd vertices in $\mathcal{Q}_n^{01}$. Since $k - r - 1 \leq k - 2 - 1 \leq (n-2) - 3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices which connects $v'_{01}$ to $v_{01}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$u_{00} \xrightarrow{\gamma_1} v_{00} \to u_{10} \xrightarrow{\gamma_2} v_{10} \to u_{11} \xrightarrow{\gamma_3} u'_{11} \to v'_{01} \xrightarrow{\gamma_4} v_{01}.$$

(E)(1)(b) $t - s \geq 1$ and $p \geq 1$.

In this case $t = k$.

Begin with $o_{01} = v_{01}$ and following $\mu$ make $s - p - r$ short cycles of the type

$$o_{01} \to (e, o)_{00} \to e_{10} \to (o, e)_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a'_{01}$.

Now begin with $o_{01} = a'_{01}$ and following $\mu$ make $p-1$ cycles of the type

$$o_{01} \to e_{00} \to (o,e)_{10} \to (o,e)_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a''_{01}$.

Finally, begin with $o_{01} = a''_{01}$ and following $\mu$ make $t-s-1$ cycles of the type

$$o_{01} \to (e,o)_{00} \to (e,o)_{10} \to e_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a'''_{01}$. Following $\mu$ we extend the constructed path with the path

$$a'''_{01} \to (e,o)_{00} \to e_{10}.$$

We denote the end vertex of the resulting path by $u_{10}$.

The total number of the constructed short cycles is $s-p-r+p-1+t-s-1 = t-r-2 \leq k-4$, hence the length of $\mu$ is enough for that construction.

There are $s-p-r+p-1+r+t-s-1 = t-2$ used or deleted even and $s-p+p-1+t-s-1 = t-2$ odd used or deleted vertices in $\mathcal{Q}_n^{10}$. Since $t-2 \leq k-2 \leq (n-2)-2$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma'$ for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices. This cycle contains the vertex $u_{10}$. Let $v_{10}$ be a vertex which is a neighbor of $u_{10}$ in $\gamma'$. Clearly, the even neighbor $u_{11}$ of $v_{10}$ in $\mathcal{Q}_n^{11}$ is neither a deleted nor used vertex. We denote by $\gamma_1$ the Hamiltonian path for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices that is defined by $\gamma'$ and connects $u_{10}$ to $v_{10}$.

There are $s-p-r+p-1+t-s-1 = t-r-2$ used even and $s-p-r+p-1+t-s = t-r-1$ odd deleted or used vertices in $\mathcal{Q}_n^{11}$. Let $u'_{11}$ be any unused even vertex in $\mathcal{Q}_n^{11}$ such that its odd neighbor $v'_{01}$ in $\mathcal{Q}_n^{01}$ is neither a deleted nor used vertex. Clearly, the even neighbor $u_{00}$ of $v'_{01}$ in $\mathcal{Q}_n^{00}$ is also neither a deleted nor used vertex. Since $t-r-2 \leq k-2-2 \leq (n-2)-4$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_2$ for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices which connects $u_{11}$ to $u'_{11}$.

There are $s-p-r+p-1+t-s-1 = t-r-2$ even and $s-p-r+t-s-1+p = t-r-1$ odd deleted or used vertices in $\mathcal{Q}_n^{00}$ ($u_{00}$ is not counted). Let $u'_{00}$ be any unused even vertex in $\mathcal{Q}_n^{00}$ such that its odd neighbor $v''_{01}$ in $\mathcal{Q}_n^{01}$ is neither a deleted nor used vertex. Since $t-r-2 \leq k-2-2 \leq (n-2)-4$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{00}$ minus all deleted or used vertices that connects $u_{00}$ to $u'_{00}$.

There are $k-r$ deleted even and $s-p-r+p-1+t-s-1+1+k-t = k-r-1$ deleted or used odd vertices in $\mathcal{Q}_n^{01}$. Since $k-r-1 \leq k-2-1 \leq (n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices which connects $v''_{01}$ to $v_{01}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$u_{10} \xrightarrow{\gamma_1} v_{10} \to u_{11} \xrightarrow{\gamma_2} u'_{11} \to v'_{01} \to u_{00} \xrightarrow{\gamma_3} u'_{00} \to v''_{01} \xrightarrow{\gamma_4} v_{01}.$$

(E)(2) $s-p < r \leq k-2$ and $0 \leq k-t < k-r \leq k-2$.

Since there are odd vertices in at least three of the hypercubes and because of the symmetry, without loss of generality, we can assume that $p \geq 1$. Again thanks to the symmetrical situation we can also assume that $p \geq t-s$ and $r-(s-p) \leq k-r-(k-t)$. Since $p+(t-s) = (r-(s-p)) + (k-r-(k-t))$, either $t-s \leq r-(s-p) \leq k-r-(k-t) \leq p$ or $r-(s-p) \leq t-s \leq p \leq k-r-(k-t)$. Since both cases are symmetrical, we consider only the case

$$t-s \leq r-(s-p) \leq k-r-(k-t) \leq p.$$

There are two possibilities: (E)(2)(a) $p \geq t-s \geq 1$; and (E)(2)(b) $p \geq 2$ and $t-s = 0$.

(E)(2)(a) $p \geq t-s \geq 1$.

Either $t < k$ or $s-p > 0$. Since both cases are symmetrical we consider below only the case $s-p > 0$, hence $k-(s-p)-2 \leq k-3$.

Begin with $o_{01} = v_{01}$ and following $\mu$ make $t-r-1$ short cycles of the type

$$o_{01} \to e_{00} \to (o,e)_{10} \to (o,e)_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a'_{01}$.

Now begin with $o_{01} = a'_{01}$ and following $\mu$ make $t-s-1$ cycles of the type

$$(o,e)_{01} \to (o,e)_{00} \to o_{10} \to e_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a''_{01}$.

Finally, begin with $o_{01} = a''_{01}$ and following $\mu$ make $p-t+r$ cycles of the type

$$o_{01} \to e_{00} \to o_{10} \to (e,o)_{11} \to (e,o')_{01}.$$

We denote the end vertex of the resulting path by $a'''_{01}$. Let the neighbor of $a'''_{01}$ in $\mathcal{Q}_n^{00}$ be $u_{00}$. We extend the constructed path with the edge $(a'''_{01}, u_{00})$.

The total number of the constructed short cycles is $t-r-1+t-s-1+p-t+r = t-(s-p)-2$. Since there are odd vertices in at least three hypercubes, either $t<k$ or $s>p$, hence $t-(s-p)-2 \le k-3$ and therefore the length of $\mu$ is enough for that construction.

There are $t-r-1+r = t-1$ even and $t-r-1+t-s-1+p-t+r+s-p = t-2$ used or deleted odd vertices in $\mathcal{Q}_n^{10}$. Also, there are $k-r+t-s-1+p-t+r = k-(s-p)-1$ deleted or used even and $k-t+t-r-1+t-s-1+p-t+r = k-(s-p)-2$ deleted or used odd vertices in $\mathcal{Q}_n^{01}$.

We fix one deleted vertex $u$ in $\mathcal{Q}_n^{10}$. Since $t-2 \le k-2 \le (n-2)-2$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma'$ for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices except $u$. This cycle contains $u$. Let $v_{10}$ and $v'_{10}$ be the neighbors of $u$ in $\gamma'$. Clearly, the even neighbor $u_{11}$ of $v'_{10}$ in $\mathcal{Q}_n^{11}$ and $u'_{00}$ of $v_{10}$ in $\mathcal{Q}_n^{00}$ are neither deleted nor used vertices. We denote by $\gamma_2$ the Hamiltonian path for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices that is defined by $\gamma'$ and connects $v_{10}$ to $v'_{10}$. Let $u'_{11}$ be any unused even vertex in $\mathcal{Q}_n^{11}$ such that its odd neighbor $v'_{01}$ in $\mathcal{Q}_n^{01}$ is neither deleted nor used vertex. Since $k-(s-p)-2 \le k-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices which connects $v'_{01}$ to $v_{01}$.

There are $t-r-1+t-s-1+p-t+r = t-(s-p)-2$ even and $t-s-1+p = t-(s-p)-1$ odd deleted or used vertices in $\mathcal{Q}_n^{00}$ ($u_{00}$ is not counted). Since $t-(s-p)-2 \le k-3 \le (n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_1$ for $\mathcal{Q}_n^{00}$ minus all deleted or used vertices which connects $u_{00}$ to $u'_{00}$.

There are $t-r-1+t-s-1+p-t+r = t-(s-p)-2$ used even and $t-r-1+p-t+r+t-s = t-(s-p)-1$ odd deleted or used vertices in $\mathcal{Q}_n^{11}$. Since $t-(s-p)-2 \le k-3 \le (n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices which connects $u_{11}$ to $u'_{11}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$u_{00} \xrightarrow{\gamma_1} u'_{00} \to v_{10} \xrightarrow{\gamma_2} v'_{10} \to u_{11} \xrightarrow{\gamma_3} u'_{11} \to v'_{01} \xrightarrow{\gamma_4} v_{01}.$$

(E)(2)(b) $p \ge 2$ and $t-s=0$.

Then $t<k$ and $s-p>0$, hence $t-1 \le k-2$ and $k-(s-p)-2 \le k-3$.

Begin with $o_{01} = v_{01}$ and following $\mu$ make $t-r-1$ short cycles of the type

$$o_{01} \to e_{00} \to (o,e)_{10} \to (o,e)_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a'_{01}$.

Finally, begin with $o_{01} = a'_{01}$ and following $\mu$ make $r-(s-p)-1$ cycles of the type

$$o_{01} \to e_{00} \to o_{10} \to (e,o)_{11} \to (e,o')_{01}.$$

We denote the end vertex of the resulting path by $a''_{01}$. Let the neighbor of $a''_{01}$ in $\mathcal{Q}_n^{00}$ be $e_{00}$, the neighbor of $e_{00}$ in $\mathcal{Q}_n^{10}$ be $o_{10}$, and the neighbor of $o_{10}$ in $\mathcal{Q}_n^{11}$ be $u_{11}$. We extend the constructed path with the path

$$v''_{01} \to e_{00} \to o_{10} \to u_{11}.$$

The total number of the constructed short cycles is $t-r-1+r-(s-p)-1 = t-s+p-2 = p-2 \le (k-2)-2 = k-4$, hence the length of $\mu$ is enough for that construction.

There are $t-r-1+r = t-1$ used or deleted even and $t-r-1+r-s+p-1+s-p+1 = t-1$ used or deleted odd vertices in $\mathcal{Q}_n^{10}$. Since $t-1 \le k-2 \le (n-2)-2$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma'$ for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices. Let $u_{10}$ and

$v_{10}$ be two neighbors in $\gamma'$. Clearly, the even neighbor $v_{11}$ of $u_{10}$ in $\mathcal{Q}_n^{11}$ and $u_{00}$ of $v_{10}$ in $\mathcal{Q}_n^{00}$ are neither deleted nor used vertices. We denote by $\gamma_2$ the Hamiltonian path for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices that is defined by $\gamma'$ and connects $u_{10}$ to $v_{10}$.

There are $t-r-1+r-s+p-1=t-s+p-2=p-2$ used even and odd deleted or used vertices in $\mathcal{Q}_n^{11}$. Since $p-2 \le k-4 \le (n-2)-4$, it follows from $(T)$ that there exists a Hamiltonian path $\gamma_1$ for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices which connects $u_{11}$ to $v_{11}$.

There are $t-r-1+r-s+p-1=t-s+p-2=p-2$ even and $t-s-1+p=t-s+p-1=p-1$ odd deleted or used vertices in $\mathcal{Q}_n^{00}$ ($u_{00}$ is not counted). Let $u'_{00}$ be any unused even vertex in $\mathcal{Q}_n^{00}$ such that its odd neighbor $v'_{01}$ in $\mathcal{Q}_n^{01}$ is neither deleted nor used vertex. Since $p-2 \le k-2-2 \le (n-2)-4$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{00}$ minus all deleted or used vertices which connects $u_{00}$ to $u'_{00}$.

Also, there are $k-r+r-s+p-1=k-(s-p)-1$ deleted or used even and $k-t+t-r-1+r-s+p-1=k-(s-p)-2$ deleted or used odd vertices in $\mathcal{Q}_n^{01}$. Since $k-(s-p)-2 \le k-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices which connects $v'_{01}$ to $v_{01}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n-\mathcal{F}$ we extend the previously constructed path with the path

$$u_{11} \xrightarrow{\gamma_1} v_{11} \to u_{10} \xrightarrow{\gamma_2} v_{10} \to u_{00} \xrightarrow{\gamma_3} u'_{00} \to v'_{01} \xrightarrow{\gamma_4} v_{01}.$$

**Note 3.** For the remaining cases Remark 3.5 applies.

*Case* $(F)$ There exists a column $A$ in $M_2$ which separates the even vertices in the way $(r,k-r)$, where $2 \le r \le k-2$, and a column $B$ in $M_o$ which separates the odd vertices in a different way than $A$. Also, either $A$ or $B$ separates the odd vertices in the way $(s,k-s)$, where $2 \le s \le k-2$.

If $B$ separates the deleted odd vertices in the way $(1,k-1)$ and all deleted vertices in the way $(1,2k-1)$ then that would be case $(B)$. If $B$ separates the deleted odd vertices in the way $(2,k-2)$ and all deleted vertices in the way $(2,2k-2)$ then that would be case $(C)$. Therefore we assume that $B$ separates the deleted odd vertices in the way $(s,k-s)$ and all deleted vertices in the way $(s,2k-s)$, where $3 \le s \le k-1$.

We split the hypercube using $A$ and $B$. Without loss of generality, we can assume that the deleted vertices are distributed as follows:

$$\{v_1,\dots,v_p\} \subset \mathcal{Q}_n^{00},\ \{v_{t+1},\dots,v_k,u_{r+1},\dots,u_k\} \subset \mathcal{Q}_n^{01},\ \{v_{p+1},\dots,v_s\} \subset \mathcal{Q}_n^{10}, \text{ and }\\ \{u_1,\dots,u_r,v_{s+1},\dots,v_t\} \subset \mathcal{Q}_n^{11},$$

where $2 \le r \le k-2$, $0 \le p \le k-2$, $3 \le s \le k-1$, and $2 \le t \le k$, since there are odd vertices in at least three of the hypercubes. Also, either $k-r > k-t$ or $r > t-s$ since in at least one of the hypercubes $\mathcal{Q}_n^{00}$ or $\mathcal{Q}_n^{10}$ there is an odd vertex. Without loss of generality, we assume that $k-r > k-t$. Finally, it follows from our assumptions that $A$ and $B$ separate the odd vertices in such a way that either $k-s \ge 2$ or $2 \le t-p \le k-2$.

We consider two cases.

(F)(1) $r \le t-s$.

There are two subcases to consider.

(F)(1)(a) $t \le k-1$, hence there is at least one odd vertex in $\mathcal{Q}_n^{01}$.

(F)(1)(a)(i) There is an odd vertex in $\mathcal{Q}_n^{00}$, hence $p \ge 1$.

Take a model path $\mu$ in $\mathcal{Q}_n^{01}$ which begins with an odd vertex $o_{01}=v_{01}$ and following $\mu$ make $p-1$ short cycles of the type

$$o_{01} \to e_{00} \to (o,e)_{10} \to (o,e)_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a'_{01}$.

Now begin with $o_{01}=a'_{01}$ and following $\mu$ make $s-p$ cycles of the type

$$o_{01} \to (e,o)_{00} \to e_{10} \to (o,e)_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a''_{01}$.

Finally, begin with $o_{01}=a''_{01}$ and following $\mu$ make $t-s-r$ cycles of the type

$$o_{01} \to (e,o)_{00} \to (e,o)_{10} \to e_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a'''_{01}$. Let the neighbor of $a'''_{01}$ in $\mathcal{Q}_n^{00}$ be $u_{00}$. We extend the constructed path with the edge $(a'''_{01}, u_{00})$.

The total number of the constructed short cycles is $p-1+s-p+t-s-r = t-r-1 \leq k-3$, hence the length of $\mu$ is enough for that construction.

There are $p-1+s-p+t-s-r+r = t-1$ deleted or used even and $p-1+s-p+t-s = t-1$ used or deleted odd vertices in $\mathcal{Q}_n^{11}$. Since $t-1 \leq k-1-1 \leq (n-2)-2$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma'$ for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices. Let $u_{11}$ and $v_{11}$ be two neighbors in $\gamma'$ such that the odd neighbor $v'_{01}$ of $u_{11}$ in $\mathcal{Q}_n^{01}$ is different from $v_{01}$ and is neither deleted nor used vertex. Clearly, the even neighbor $u_{10}$ of $v_{11}$ in $\mathcal{Q}_n^{10}$ is also neither deleted nor used vertex. We denote by $\gamma_3$ the Hamiltonian path for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices that is defined by $\gamma'$ and connects $v_{11}$ to $u_{11}$.

Let $u'_{00}$ be neither deleted nor used vertex in $\mathcal{Q}_n^{00}$ different from $u_{00}$, whose neighbor $v_{10}$ in $\mathcal{Q}_n^{10}$ is neither deleted nor used vertex. There are $p-1+s-p+t-s-r = t-r-1$ used even and $p+s-p+t-s-r = t-r$ deleted or used odd vertices in $\mathcal{Q}_n^{00}$. Since $t-r-1 \leq k-1-3 \leq (n-2)-4$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_1$ for $\mathcal{Q}_n^{00}$ minus all deleted or used vertices which connects $u_{00}$ to $u'_{00}$.

There are $p-1+s-p+t-s-r = t-r-1$ used even and $p-1+t-s-r+s-p = t-r-1$ deleted or used odd vertices in $\mathcal{Q}_n^{10}$. Since $t-r-1 \leq k-1-3 \leq (n-2)-4$, it follows from $(T)$ that there exists a Hamiltonian path $\gamma_2$ for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices which connects $v_{10}$ to $u_{10}$.

There are $k-r$ deleted even and $p-1+s-p+t-s-r+k-t = k-r-1$ odd deleted or used vertices in $\mathcal{Q}_n^{01}$ ($v_{01}$ is not counted). Since $k-r-1 \leq k-1-3 \leq (n-2)-4$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices which connects $v'_{01}$ to $v_{01}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - F$ we extend the previously constructed path with the path

$$u_{00} \xrightarrow{\gamma_1} u'_{00} \to v_{10} \xrightarrow{\gamma_2} u_{10} \to v_{11} \xrightarrow{\gamma_3} u_{11} \to v'_{01} \xrightarrow{\gamma_4} v_{01}.$$

(F)(1)(a)(ii) There is an odd vertex in $\mathcal{Q}_n^{10}$, hence $s-p \geq 1$.

This case is similar to the previous case in (i). To obtain a solution of that case just switch the roles of $\mathcal{Q}_n^{00}$ and $\mathcal{Q}_n^{10}$ in the above solution. Then in the beginning of the construction make $p$ cycles of the type

$$o_{01} \to e_{00} \to (o,e)_{10} \to (o,e)_{11} \to o'_{01},$$

instead of $p-1$ and then make $s-p-1$ cycles of the type

$$o_{01} \to (e,o)_{00} \to e_{10} \to (o,e)_{11} \to o'_{01},$$

instead of $s-p$ cycles. At the end finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ by extending the previously constructed path with the path

$$u_{00} \xrightarrow{\gamma_1} v_{00} \to u_{10} \xrightarrow{\gamma_2} u'_{10} \to v_{11} \xrightarrow{\gamma_3} u_{11} \to v'_{01} \xrightarrow{\gamma_4} v_{01}.$$

(F)(1)(b) $t = k$, hence there are no odd vertices in $\mathcal{Q}_n^{01}$.

Since there are no odd vertices in $\mathcal{Q}_n^{01}$, there must be odd vertices in the other three hypercubes. Also, since $2 \leq r \leq k-2$, the difference between deleted even and odd vertices in $\mathcal{Q}_n^{01}$ is $k-r \geq 2$.

Take a model path $\mu$ in $\mathcal{Q}_n^{00}$ which begins with an even vertex $e_{00} = u_{00}$ and following $\mu$ make $t-s-r$ short cycles of the type

$$e_{00} \to o_{01} \to e_{11} \to (o,e)_{10} \to (o,e')_{00}.$$

We denote the end vertex of the resulting path by $b'_{00}$.

Now begin with $e_{00} = b'_{00}$ and following $\mu$ make $s-p-1$ cycles of the type

$$e_{00} \to o_{01} \to (e,o)_{11} \to e_{10} \to (o,e')_{00}.$$

We denote the end vertex of the resulting path by $b''_{00}$.

Finally, begin with $e_{00} = b''_{00}$ and following $\mu$ make $p-1$ cycles of the type

$$e_{00} \to o_{01} \to (e,o)_{11} \to (e,o)_{10} \to e'_{00}.$$

We denote the end vertex of the resulting path by $b'''_{00}$. Let the neighbor of $b'''_{00}$ in $\mathcal{Q}_n^{01}$ be $v''_{01}$ and the neighbor of $v''_{01}$ in $\mathcal{Q}_n^{11}$ be $u_{11}$. We extend the constructed path with the edges $(b'''_{00}, v''_{01})$ and $(v''_{01}, u_{11})$.

The total number of the constructed short cycles is $p-1+s-p-1+t-s-r = t-r-2 \leq k-4$, hence the length of $\mu$ is enough for that construction.

There are $p-1+s-p-1+t-s-r+r = t-2 = k-2$ deleted or used even and $p-1+s-p-1+t-s = t-2 = k-2$ used or deleted odd vertices in $\mathcal{Q}_n^{11}$. Since $k-2 \leq (n-2)-2$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma'$ for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices. Clearly, $\gamma'$ contains $u_{11}$. Let $v_{11}$ be a neighbor of $u_{11}$ in $\gamma'$. Then the even neighbor $u_{10}$ of $v_{11}$ in $\mathcal{Q}_n^{10}$ is neither deleted nor used vertex. We denote by $\gamma_1$ the Hamiltonian path for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices that is defined by $\gamma'$ and connects $u_{11}$ to $v_{11}$.

Let $e''_{00}$ and $o''_{00}$ be two neighbors in $\gamma'$ such that the odd neighbor $v_{01}$ of $e''_{00}$ in $\mathcal{Q}_n^{01}$ is neither deleted nor used vertex. Clearly, the even neighbor $u'_{10}$ of $o''_{00}$ in $\mathcal{Q}_n^{10}$ is also neither deleted nor used vertex. There are $p-1+s-p-1+t-s-r = t-r-2$ used even and $p-1+t-s-r+s-p = t-r-1$ deleted or used odd vertices in $\mathcal{Q}_n^{10}$. Since $t-r-2 \leq k-2-2 \leq (n-2)-4$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_2$ for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices which connects $u_{10}$ to $u'_{10}$.

Let $v'_{01}$ be neither deleted nor used vertex in $\mathcal{Q}_n^{01}$ different from $v_{01}$, whose neighbor $u'_{00}$ in $\mathcal{Q}_n^{00}$ is neither deleted nor used vertex. There are $k-r$ deleted even and $p-1+s-p-1+t-s-r+1+k-t = k-r-1$ odd deleted or used vertices in $\mathcal{Q}_n^{01}$ ($v_{01}$ is not counted). Since $k-r-1 \leq k-2-1 \leq (n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices which connects $v_{01}$ to $v'_{01}$.

There are $p-1+s-p-1+t-s-r+1 = t-r-1$ used even and $p+s-p-1+t-s-r+1 = t-r$ deleted or used odd vertices in $\mathcal{Q}_n^{00}$. Since $t-r-1 \leq k-2-1 \leq (n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{00}$ minus all deleted or used vertices which connects $u'_{00}$ to $u_{00}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$u_{11} \xrightarrow{\gamma_1} v_{11} \to u_{10} \xrightarrow{\gamma_2} u'_{10} \to o''_{00} \to e''_{00} \to v_{01} \xrightarrow{\gamma_3} v'_{01} \to u'_{00} \xrightarrow{\gamma_4} u_{00}.$$

(F)(2) $r > t-s$.

We have $k-r > k-t$ and $r > t-s$. Since $(t-r)+r-(t-s) = s \geq 3$, we have either $(k-r)-(k-t) = t-r \geq 2$ or $r-(t-s) \geq 2$. Also, since $s \geq 3$, we have either $p \geq 2$ or $s-p \geq 2$. There are two possibilities: either $(k-r)-(k-t) = t-r \geq 2$ and $p \geq 2$ (or equivalently, $r-(t-s) \geq 2$ and $s-p \geq 2$) or we do not have any of the previous cases and we have $(k-r)-(k-t) = t-r \geq 2$ and $p \leq 1$ (or $r-(t-s) \geq 2$ and $s-p \leq 1$), instead.

Since the cases in each group are symmetric of each other, we consider only the first cases from each group.

(F)(2)(a) $(k-r)-(k-t) = t-r \geq 2$ and $p \geq 2$.

For easier explanation of how we balance the plates we assume that $(k-r)-(k-t) = t-r \geq r-(t-s)$. The other case is similar: the balancing of the plates is slightly different but the rest of the construction is the same.

There are three possibilities:

(F)(2)(a)(i) $t-r \geq \max(p, s-p)$, (F)(2)(a)(ii) $s-p \leq t-r \leq p$, and (F)(2)(a)(iii) $p \leq t-r \leq s-p$.

We consider all cases below.

In case (F)(2)(a)(i) we have $t-r \geq \max(p, s-p) \geq \min(p, s-p) \geq r-t+s$.

Take a model path $\mu$ in $\mathcal{Q}_n^{01}$ which begins with an odd vertex $o_{01} = v_{01}$ and following $\mu$ make $r-t+s-1$ short cycles of the type

$$o_{01} \to (e,o)_{00} \to e_{10} \to o_{11} \to (e,o')_{01}.$$

We denote the end vertex of the resulting path by $a'_{01}$.

Now begin with $o_{01} = a'_{01}$ and following $\mu$ make $t-p-r$ cycles of the type

$$o_{01} \to (e,o)_{00} \to e_{10} \to (o,e)_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a''_{01}$.

Then, begin with $o_{01} = a''_{01}$ and following $\mu$ make $p-2$ cycles of the type

$$o_{01} \to e_{00} \to (o,e)_{10} \to (o,e)_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a'''_{01}$.

Finally, begin with $o_{01} = a'''_{01}$ and following $\mu$, extend the resulting path with the following path with length four

$$o_{01} \to e_{00} \to (o,e)_{10} \to o_{11}.$$

We denote the end vertex of the resulting path by $v_{11}$.

In case (F)(2)(a)(ii) we have

$$s-p \le r-t+s \le t-r \le p.$$

In that case, begin with $o_{01} = v_{01}$ and following $\mu$ make $s-p-1$ short cycles of the type

$$o_{01} \to (e,o)_{00} \to e_{10} \to o_{11} \to (e,o')_{01}.$$

We denote the end vertex of the resulting path by $a'_{01}$.

Now begin with $o_{01} = a'_{01}$ and following $\mu$ make $p+r-t$ cycles of the type

$$o_{01} \to e_{00} \to (o,e)_{10} \to o_{11} \to (e,o')_{01}.$$

We denote the end vertex of the resulting path by $a''_{01}$.

Then, begin with $o_{01} = a''_{01}$ and following $\mu$ make $t-r-2$ cycles of the type

$$o_{01} \to e_{00} \to (o,e)_{10} \to (o,e)_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a'''_{01}$. Let the neighbor of $a'''_{01}$ in $\mathcal{Q}_n^{00}$ be $u_{00}$.

Finally, begin with $o_{01} = a'''_{01}$ and following $\mu$, extend the resulting path with the following path with length four

$$o_{01} \to e_{00} \to (o,e)_{10} \to o_{11}.$$

We denote the end vertex of the resulting path by $v_{11}$.

In case (F)(2)(a)(iii) we have

$$p \le r-t+s \le t-r \le s-p.$$

In that case, begin with $o_{01} = v_{01}$ and following $\mu$ make $p-2$ short cycles of the type

$$o_{01} \to e_{00} \to (o,e)_{10} \to o_{11} \to (e,o')_{01}.$$

We denote the end vertex of the resulting path by $a'_{01}$.

Now begin with $o_{01} = a'_{01}$ and following $\mu$ make $r-t+s-p+1$ cycles of the type

$$o_{01} \to (e,o)_{00} \to e_{10} \to o_{11} \to (e,o')_{01}.$$

We denote the end vertex of the resulting path by $a''_{01}$.

Then, begin with $o_{01} = a''_{01}$ and following $\mu$ make $t-r-2$ cycles of the type

$$o_{01} \to (e,o)_{00} \to e_{10} \to (o,e)_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a'''_{01}$. Let the neighbor of $a'''_{01}$ in $\mathcal{Q}_n^{00}$ be $u_{00}$.

Finally, begin with $o_{01} = a'''_{01}$ and following $\mu$, extend the resulting path with the following path with length four

$$o_{01} \to e_{00} \to (o,e)_{10} \to o_{11}.$$

We denote the end vertex of the resulting path by $v_{11}$.

The total number of the constructed short cycles is $r-t+s-1+t-p-r+p-2=s-3$ in (i), $s-p-1+p+r-t+t-r-2=s-3$ in (ii), and $p-2+r-t+s-p+1+t-r-2=s-3$ in (iii). Since $s-3\leq k-4\leq n-6$, the length of $\mu$ is enough for that construction.

Let $v'_{11}$ be neither deleted nor used odd vertex in $\mathcal{Q}_n^{00}$ different from $v_{11}$. Clearly, its neighbor $u_{10}$ in $\mathcal{Q}_n^{10}$ is not an used vertex. There are $r-t+s+t-p-r+p-2=s-2$ deleted or used even and $r-t+s-1+t-p-r+p-2=s-3$ used or deleted odd vertices in $\mathcal{Q}_n^{11}$. Since $s-3\leq k-1-3\leq(n-2)-4$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma'$ for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices which connects $v_{11}$ to $v'_{11}$. Let $v''_{11}$ and $u_{11}$ be two neighbors in $\gamma'$ such that $v''_{11}$ is closer to $v''_{11}$ and the odd neighbor $v'_{01}$ of $u_{11}$ in $\mathcal{Q}_n^{01}$ is different from $v_{01}$ and is neither deleted nor used vertex. Clearly, the even neighbor $u'_{10}$ of $v''_{11}$ in $\mathcal{Q}_n^{10}$ is also neither deleted nor used vertex. We denote by $\gamma_1$ the path in $\mathcal{Q}_n^{11}$ defined by $\gamma'$ which connects $v_{11}$ to $v''_{11}$ and by $\gamma_2$ the path defined by $\gamma'$ which connects $v'_{11}$ to $u_{11}$.

There are $r-t+s-1+t-p-r+p-2+1=s-2$ used even and $s-p+p-2+1=s-1$ deleted or used odd vertices in $\mathcal{Q}_n^{10}$. Since $s-2\leq k-1-2\leq(n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices which connects $u'_{10}$ to $u_{10}$.

Let $u_{00}$ be the even neighbor of $v'_{01}$ in $\mathcal{Q}_n^{00}$. Clearly, $u_{00}$ is not a deleted vertex. Let $u'_{00}$ be another unused even vertex in $\mathcal{Q}_n^{10}$ different from $u_{00}$ and such that its odd neighbor $v''_{01}$ in $\mathcal{Q}_n^{10}$ is neither deleted nor used vertex. There are $r-t+s-1+t-p-r+p-2+1=s-2$ used even and $r-t+s-1+t-p-r+p=s-1$ deleted or used odd vertices in $\mathcal{Q}_n^{00}$. Since $s-2\leq k-1-2\leq(n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{00}$ minus all deleted or used vertices which connects $u_{00}$ to $u'_{00}$.

There are $k-r+r-t+s-1=k-t+s-1\leq s-1$ deleted even and $r-t+s-1+t-p-r+p-2+1=s-2$ odd deleted or used vertices in $\mathcal{Q}_n^{01}$ ($v_{01}$ is not counted). Since $s-2\leq k-1-2\leq(n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_5$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices which connects $v''_{01}$ to $v_{01}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n-\mathcal{F}$ we extend the previously constructed path with the path

$$v_{11}\xrightarrow{\gamma_1}v''_{11}\to u'_{10}\xrightarrow{\gamma_3}u_{10}\to v'_{11}\xrightarrow{\gamma_2}u_{11}\to$$

$$v'_{01}\to u_{00}\xrightarrow{\gamma_4}u'_{00}\to v''_{01}\xrightarrow{\gamma_5}v_{01}.$$

(F)(2)(b) $(k-r)-(k-t)=t-r\geq 2$ and $p\leq 1$.

Since $s\geq 3$ and $p\leq 1$, we have $s-p\geq 2$. Therefore if $r-t+s\geq 2$ then that would be case (a). Thus, we have $r-t+s=1\geq p$ and since $r\geq 2$, we conclude that $t-s\geq 1$, hence there exists at least one odd deleted vertex in $\mathcal{Q}_n^{11}$. Also, it follows that $t-r>r-t+s$. Finally, since $s\geq 3$, we have $s-p\geq 2$, hence $p\leq s-p$. Therefore we have $p\leq r-t+s\leq t-r\leq s-p$.

There are two cases: $t-s\geq 2$ or $t-s=1$. If $t-s\geq 2$ then $s\leq k-2$. Let $t-s=1$. Since there exists at most one deleted odd vertex in $\mathcal{Q}_n^{00}$, there must be at least one deleted odd vertex in $\mathcal{Q}_n^{01}$ for at least one of both coordinates $A$ or $B$ separates the deleted vertices in two groups with at least two deleted odd vertises in each group. Hence, again $s\leq k-2$. Therefore in either case we have $s\leq k-2$.

We consider two cases below: (F)(2)(b)(i) $t=k$ and therefore there are no deleted odd vertices in $\mathcal{Q}_n^{01}$, hence $p=1$; and (F)(2)(b)(ii) $t\leq k-1$ and therefore there is at least one deleted odd vertex in $\mathcal{Q}_n^{01}$, hence $p\leq 1$.

(F)(2)(b)(i) $t=k$ and therefore there are no deleted odd vertices in $\mathcal{Q}_n^{01}$, hence $p=1$.

Take a model path $\mu$ in $\mathcal{Q}_n^{01}$ which begins with an odd vertex $o_{01}=v_{01}$ and following $\mu$ make $t-r-2=k-r-2$ short cycles of the type

$$o_{01}\to(e,o)_{00}\to e_{10}\to(o,e)_{11}\to o'_{01}.$$

We denote the end vertex of the resulting path by $a'_{01}$.

Then begin with $o_{01}=a'_{01}$ and following $\mu$ make $r-t+s=r+s-k$ cycles of the type

$$o_{01}\to(e,o)_{00}\to e_{10}\to o_{11}\to(e,o')_{01}.$$

We denote the end vertex of the resulting path by $a''_{01}$.

Finally, begin with $o_{01} = a''_{01}$ and following $\mu$, extend the resulting path with the following path with length four

$$o_{01} \to (e,o)_{00} \to e_{10} \to o_{11}.$$

We denote the end vertex of the resulting path by $v_{11}$.

The total number of the constructed short cycles is $k - r - 2 + r + s - k + 1 = s - 1 \leq (k-2) - 1 \leq k - 3$, hence the length of $\mu$ is enough for that construction.

There are $k - r - 2 + r = k - 2$ deleted or used even and $k - r - 2 + r + s - k + k - s = k - 2$ used or deleted odd vertices in $\mathcal{Q}_n^{11}$ ($v_{11}$ is not counted). Since $k - 2 \leq (n-2) - 2$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma'$ for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices. This cycle contains $v_{11}$. Let $u_{11}$ be a neighbor of $v_{11}$ in $\gamma'$. We denote by $\gamma_1$ the Hamiltonian path for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices defined by $\gamma'$ which connects $v_{11}$ to $u_{11}$.

Clearly, the odd neighbor $v'_{01}$ in $\mathcal{Q}_n^{01}$ of $u_{11}$ is not an used vertex and not a deleted vertex since there are no deleted odd vertices in $\mathcal{Q}_n^{01}$. Let $u_{00}$ be the even neighbor of $v'_{01}$ in $\mathcal{Q}_n^{00}$. Then $u_{00}$ is neither deleted nor used even vertex in $\mathcal{Q}_n^{00}$ (there are no deleted even vertices in $\mathcal{Q}_n^{00}$). There are $k - r - 2 + r + s - k + 1 = s - 1 \leq k - 3 \leq (n-2) - 3$ used or deleted odd vertices in $\mathcal{Q}_n^{00}$. Therefore there exists an odd neighbor $v_{00}$ of $u_{00}$ in $\mathcal{Q}_n^{00}$ which is neither deleted nor used vertex.

Let $u_{10}$ be the even neighbor of $v_{00}$ in $\mathcal{Q}_n^{10}$. Clearly, $u_{10}$ is neither used nor deleted vertex (there are no deleted even vertices in $\mathcal{Q}_n^{10}$). Let also $v_{10}$ be an undeleted odd vertex in $\mathcal{Q}_n^{10}$ whose even neighbor $u'_{00}$ in $\mathcal{Q}_n^{00}$ is not an used vertex. There are $k - r - 2 + r + s - k + 1 = s - 1$ used even and $s - 1$ deleted odd vertices in $\mathcal{Q}_n^{10}$. Since $s - 1 \leq k - 3 \leq (n-2) - 3$, it follows from $(T)$ that there exists a Hamiltonian path $\gamma_2$ for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices which connects $u_{10}$ to $v_{10}$.

Let $u''_{00}$ be an unused even vertex in $\mathcal{Q}_n^{00}$ different from $u'_{10}$, whose odd neighbor $v''_{01}$ in $\mathcal{Q}_n^{01}$ is neither deleted nor used vertex. There are $k - r - 2 + r + s - k + 1 = s - 1$ used even and $k - r - 2 + r + s - k + 1 + 1 = s$ deleted or used odd vertices in $\mathcal{Q}_n^{00}$. Since $s - 1 \leq (k-2) - 1 \leq (n-2) - 3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{00}$ minus all deleted or used vertices which connects $u'_{00}$ to $u''_{00}$.

There are $k - r + r + s - k = s$ deleted or used even and $k - r - 2 + r + s - k + 1 = s - 1$ used or deleted odd vertices in $\mathcal{Q}_n^{01}$ ($v_{01}$ is not counted). Since $s - 1 \leq (k-2) - 1 \leq (n-2) - 3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices which connects $v''_{01}$ to $v_{01}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$v_{11} \xrightarrow{\gamma_1} u_{11} \to v'_{01} \to (u_{00}, v_{00}) \to u_{10} \xrightarrow{\gamma_2} v_{10} \to$$

$$u'_{00} \xrightarrow{\gamma_3} u''_{00} \to v''_{01} \xrightarrow{\gamma_4} v_{01}.$$

(F)(2)(b)(ii) $t \leq k - 1$ and therefore there is at least one deleted odd vertex in $\mathcal{Q}_n^{01}$, hence $p \leq 1$.

Take a model path $\mu$ in $\mathcal{Q}_n^{01}$ which begins with an odd vertex $o_{01} = v_{01}$ and following $\mu$ make $r - t + s - 1$ short cycles of the type

$$o_{01} \to (e,o)_{00} \to e_{10} \to o_{11} \to (e,o')_{01}.$$

We denote the end vertex of the resulting path by $a'_{01}$.

Then begin with $o_{01} = a'_{01}$ and following $\mu$ make $p$ short cycles of the type

$$o_{01} \to e_{00} \to (o,e)_{10} \to (o,e)_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a''_{01}$ (if $p = 0$ we do not make such cycles).

Next, begin with $o_{01} = a''_{01}$ and following $\mu$ make $t - r - p - 1$ cycles of the type

$$o_{01} \to (e,o)_{00} \to e_{10} \to (o,e)_{11} \to o'_{01}.$$

We denote the end vertex of the resulting path by $a'''_{01}$.

Finally, begin with $o_{01} = a'''_{01}$ and following $\mu$, extend the resulting path with the following path with length four

$$o_{01} \to (e,o)_{00} \to e_{10} \to o_{11}.$$

We denote the end vertex of the resulting path by $v_{11}$.

The total number of the constructed short cycles is $r-t+s-1+p+t-r-p-1+1=s-1\leq(k-2)-1\leq k-3$, hence the length of $\mu$ is enough for that construction.

Let $v'_{11}$ be neither deleted nor used odd vertex in $\mathcal{Q}_n^{11}$ different from $v_{11}$, whose even neighbor $u_{10}$ in $\mathcal{Q}_n^{10}$ is not an used vertex. There are $p+t-r-p-1+r=t-1$ used even and $r-t+s-1+p+t-r-p-1+t-s=t-2$ used or deleted odd vertices in $\mathcal{Q}_n^{11}$. Since $t-2\leq(k-1)-1=k-3\leq(n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_1$ for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices.

Let $u'_{10}$ be an unused even vertex in $\mathcal{Q}_n^{10}$ different from $u_{10}$, whose odd neighbor $v_{00}$ in $\mathcal{Q}_n^{00}$ is neither deleted nor used vertex. There are $r-t+s-1+p+t-r-p-1+1=s-1$ used even and $p+s-p=s$ used or deleted odd vertices in $\mathcal{Q}_n^{10}$. Since $s-1\leq k-3\leq(n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_2$ for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices which connects $u_{10}$ to $u'_{10}$.

Let $u_{00}$ be an unused even vertex in $\mathcal{Q}_n^{00}$ whose odd neighbor $v'_{01}$ in $\mathcal{Q}_n^{01}$ is neither deleted nor used vertex. There are $r-t+s-1+p+t-r-p-1+1=s-1$ used even and $r-t+s-1+t-r-p-1+1+p=s-1$ deleted or used odd vertices in $\mathcal{Q}_n^{10}$. Since $s-1\leq(k-2)-1\leq(n-2)-3$, it follows from $(T)$ that there exists a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{00}$ minus all deleted or used vertices which connects $v_{00}$ to $u_{00}$.

There are $r-t+s-1+k-r=k-t+s-1$ deleted or used even and $r-t+s-1+p+t-r-p-1+k-t=k-t+s-2$ used odd vertices in $\mathcal{Q}_n^{01}$ ($v_{01}$ is not counted). Since $t-s>0$, we have $k-(t-s)-2\leq k-3\leq(n-2)-3$. Then it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices which connects $v'_{01}$ to $v_{01}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n-\mathcal{F}$ we extend the previously constructed path with the path

$$v_{11}\xrightarrow{\gamma_1}v'_{11}\to u_{10}\xrightarrow{\gamma_2}u'_{10}\to v_{00}\xrightarrow{\gamma_3}u_{00}\to v'_{01}\xrightarrow{\gamma_4}v_{01}.$$

*Case* $(G)$ $k\leq n-3$ and there exists a column $A$ in $M_2$ which separates the even vertices in the way $(r,k-r)$, where $2\leq r\leq k-2$, and the odd vertices in the way $(1,k-1)$. Also, there exists a column $B$ in $M_o$ which separates the odd vertices in the way $(1,k-1)$ but in a different way than $A$.

Since $B$ separates the deleted odd vertices in the way $(1,k-1)$, according to Remark 3.3, $B$ separates all deleted vertices in the way $(k+1,k-1)$.

We split $\mathcal{Q}_n$ using $A$ and $B$. Without loss of generality, we can assume that the deleted vertices are distributed as follows:

$$\{v_1\}\subset\mathcal{Q}_n^{00},\ \{u_{r+1},\dots,u_k\}\subset\mathcal{Q}_n^{01},\ \{v_2,\dots,v_{k-1}\}\subset\mathcal{Q}_n^{10},\text{ and }\{u_1,\dots,u_r,v_k\}\subset\mathcal{Q}_n^{11}.$$

Take a model path $\mu$ in $\mathcal{Q}_n^{01}$ which begins with an odd vertex $o_{01}=v_{01}$ and following $\mu$ make $k-r-1$ short cycles of the type

$$o_{01}\to(e,o)_{00}\to e_{10}\to(o,e)_{11}\to o'_{01}.$$

We denote the end vertex of the resulting path by $a'_{01}$.

Then begin with $o_{01}=a'_{01}$ and following $\mu$ make $r-1$ short cycles of the type

$$o_{01}\to(e,o)_{00}\to e_{10}\to o_{11}\to(e,o')_{01}.$$

We denote the end vertex of the resulting path by $a''_{01}$.

Let the neighbor of $a''_{01}$ in $\mathcal{Q}_n^{00}$ be $u_{00}$. We extend the constructed path with the edge $(a''_{01},u_{00})$.

The total number of the constructed short cycles is $k-r-1+r-1=k-2\leq(n-3)-2$, hence the length of $\mu$ is enough for that construction.

There are $r+k-r-1=k-1$ deleted or used even and $k-r-1+r-1+1=k-1$ used or deleted odd vertices in $\mathcal{Q}_n^{11}$. Since $k-1\leq(n-3)-1=(n-2)-2$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma'$ for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices. Let $u_{11}$ and $v_{11}$ be two neighbors in $\gamma'$ such that the odd neighbor $v'_{01}$ of $u_{11}$ in $\mathcal{Q}_n^{01}$ is different from $v_{01}$ and is neither deleted nor used vertex. Clearly, the even neighbor $u_{10}$ of $v_{11}$ in $\mathcal{Q}_n^{10}$ is also neither deleted nor used vertex. We denote by $\gamma_3$ the Hamiltonian path for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices that is defined by $\gamma'$ and connects $v_{11}$ to $u_{11}$.

Let $u'_{00}$ be neither deleted nor used vertex in $\mathcal{Q}_n^{00}$ different from $u_{00}$, whose neighbor $v_{10}$ in $\mathcal{Q}_n^{10}$ is neither deleted nor used vertex. There are $k-r-1+r-1=k-2$ used even and $k-r-1+r-1+1=k-1$ used or deleted odd vertices in $\mathcal{Q}_n^{00}$. Since $k-2 \leq (n-3)-2 = (n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_1$ for $\mathcal{Q}_n^{00}$ minus all deleted or used vertices which connects $u_{00}$ to $u'_{00}$.

There are $k-r-1+r-1=k-2$ used even and $k-2$ deleted odd vertices in $\mathcal{Q}_n^{10}$. Since $k-2 \leq (n-2)-3$, it follows from $(T)$ that there exists a Hamiltonian path $\gamma_2$ for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices which connects $v_{10}$ to $u_{10}$.

There are $k-r+r-1=k-1$ deleted or used even and $k-r-1+r-1=k-2$ used odd vertices in $\mathcal{Q}_n^{01}$ ($v_{01}$ is not counted). Since $k-2 \leq (n-2)-3$, it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices which connects $v'_{01}$ to $v_{01}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$u_{00} \xrightarrow{\gamma_1} u'_{00} \to v_{10} \xrightarrow{\gamma_2} u_{10} \to v_{11} \xrightarrow{\gamma_3} u_{11} \to v'_{01} \xrightarrow{\gamma_4} v_{01}.$$

**Note 4.** For the remaining cases Remark 3.6 applies.

*Case* $(H)$ There exists a column $A$ in $M_o$ which separates the deleted odd vertices in the way $(s, k-s)$, where $2 \leq s \leq k-2$, and a column $B$ in $M_e$ which separates the deleted even vertices in the way $(r, k-r)$, where $2 \leq r \leq k-2$.

We split $\mathcal{Q}_n$ using $A$ and $B$. Without loss of generality we can assume that $r \geq s$ and that the deleted vertices are distributed as follows:

$$\{v_1, v_2, \ldots, v_s\} \subset \mathcal{Q}_n^{00},\ \{u_1, u_2, \ldots, u_r\} \subset \mathcal{Q}_n^{11}, \text{ and } \{v_{s+1}, \ldots, v_k, u_{r+1}, \ldots, u_k\} \subset \mathcal{Q}_n^{01}.$$

Take a model path $\mu$ in $\mathcal{Q}_n^{10}$ which begins with an even vertex $e_{10} = u_{10}$ and following $\mu$ make $r-s$ short cycles of the type

$$e_{10} \to o_{11} \to e_{01} \to (o,e)_{00} \to (o,e')_{10}$$

(if $r-s=0$ we do not make such cycles). We denote the end vertex of the resulting path by $a'_{10}$.

There are $k-s$ pairs of even and odd deleted or used vertices in $\mathcal{Q}_n^{01}$. Since $2 \leq s \leq k-2$, we have $2 \leq k-s \leq k-2 \leq (n-2)-2$. Then, it follows from $(L)$ that there exists a Hamiltonian cycle $\mu'$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices. Let $\mu''$ be the projection of $\mu'$ on $\mathcal{Q}_n^{10}$. Clearly, $a'_{10}$ belongs to $\mu''$.

Now begin with $e_{10} = a'_{10}$ and following $\mu''$ (and $\mu'$), continue with $s-1$ cycles of the type

$$e_{10} \to o_{11} \to (e,o)_{01} \to e_{00} \to (o,e')_{10}.$$

We denote the end vertex of the resulting path by $a''_{10}$.

Let the neighbor of $a''_{10}$ in $\mathcal{Q}_n^{11}$ be $v_{11}$. We extend the constructed path with the edge $(a''_{10}, v_{11})$.

Since we have been following $\mu'$, $2(s-1)$ consecutive vertices from $\mu'$ have been used in these short cycles for all edges of the type $(e,o)_{01}$. The length of $\mu'$ is $2^{n-2} - 2(k-s)$ and since $2^k > 4k$ for $k \geq 5$, we have

$$2^{n-2} - 2(k-s) > 2^k - 2k > 2k > 2(s-1).$$

Therefore what remains unused from $\gamma_1$ forms a path $\gamma_2$. Notice that the end vertices of $\gamma_2$ have different parity. We denote these end vertices by $u_{01}$ and $v_{01}$.

Denote the neighbor of $u_{01}$ in $\mathcal{Q}_n^{11}$ by $v'_{11}$. Clearly, $v'_{11}$ has not been used so far and is not a deleted vertex. There are $r$ deleted even vertices and $r-s+s-1=r-1$ used odd vertices in $\mathcal{Q}_n^{11}$. Since $r-1 \leq k-3$, we can use $(CG)$ to find a Hamiltonian path $\gamma_1$ for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices which connects $v_{11}$ to $v'_{11}$.

Denote the neighbor of $v_{01}$ in $\mathcal{Q}_n^{00}$ by $u_{00}$ and let $u'_{00}$ be any other undeleted and unused even vertex in $\mathcal{Q}_n^{00}$. Then its neighbor $v_{10}$ in $\mathcal{Q}_n^{10}$ has not been used so far. There are $r-s+s=r$ deleted or used odd vertices and $r-s+s-1=r-1$ used even vertices in $\mathcal{Q}_n^{00}$. Since $r-1 \leq k-3$, we can use $(CG)$ to find a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{00}$ minus all deleted and used vertices which connects $u_{00}$ to $u'_{00}$.

There are $r-s+s-1=r-1$ pairs of used even and odd vertices in $\mathcal{Q}_n^{10}$. Since $r-1 \leq k-3$, we can use $(T)$ to find a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{10}$ minus all used vertices which connects $v_{10}$ to $u_{10}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$v_{11} \xrightarrow{\gamma_1} v'_{11} \to u_{01} \xrightarrow{\gamma_2} v_{01} \to u_{00} \xrightarrow{\gamma_3} u'_{00} \to v_{10} \xrightarrow{\gamma_4} u_{10}.$$

**Note 5.** For the remaining cases Remark 3.7 applies. Therefore $k \leq n-3$ and that every coordinate which separates only the deleted odd vertices separates them in the way $(1, k-1)$, and therefore it separates all vertices in the way $(k+1, k-1)$.

Case (I) $k \leq n-3$, there is a column $A$ in $M_o$ and there is a column $B$ in $M_e$.

We split $\mathcal{Q}_n$ using $A$ and $B$. Without loss of generality, we can assume that the deleted vertices are distributed as follows:

$$\{v_1, v_2, \dots, v_s\} \subset \mathcal{Q}_n^{00},\ \{u_1, u_2, \dots, u_{k-1}\} \subset \mathcal{Q}_n^{11}, \text{ and } \{v_{s+1}, \dots, v_k, u_k\} \subset \mathcal{Q}_n^{01}.$$

Take a model path $\mu$ in $\mathcal{Q}_n^{10}$ which begins with an even vertex $e_{10} = u_{10}$ and following $\mu$ make $k-1-s$ short cycles of the type

$$e_{10} \to o_{11} \to e_{01} \to (o,e)_{00} \to (o,e')_{10}$$

(if $k-1-s=0$ we do not make such cycles). We denote the end vertex of the resulting path by $a'_{10}$.

Then begin with $e_{10} = a'_{10}$ and following $\mu$ make $s-1$ cycles of the type

$$e_{10} \to o_{11} \to (e,o)_{01} \to e_{00} \to (o,e')_{10}.$$

We denote the end vertex of the resulting path by $a''_{10}$.

Let the neighbor of $a''_{10}$ in $\mathcal{Q}_n^{11}$ be $v_{11}$. We extend the constructed path with the edge $(a''_{10}, v_{11})$.

The total number of the constructed short cycles is $k-1-s+s-1 = k-2 \leq (n-3)-2$, hence the length of $\mu$ is enough for that construction.

There are $k-s-1+s-1+1 = k-1$ deleted or used even and $k-s+s-1 = k-1$ used or deleted odd vertices in $\mathcal{Q}_n^{01}$. Since $k-1 \leq (n-3)-1 = (n-2)-2$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma'$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices. Let $u_{01}$ and $v_{01}$ be two neighbors in $\gamma'$ such that the odd neighbor $v'_{11}$ of $u_{01}$ in $\mathcal{Q}_n^{11}$ is different from $v_{11}$ and is neither deleted nor used vertex and the even neighbor $u_{00}$ of $v_{01}$ in $\mathcal{Q}_n^{00}$ is also neither deleted nor used vertex. We denote by $\gamma_2$ the Hamiltonian path for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices that is defined by $\gamma'$ and connects $u_{01}$ to $v_{01}$.

There are $k-1$ deleted even vertices and $k-s-1+s-1 = k-2$ used odd vertices in $\mathcal{Q}_n^{11}$. Since $k-2 \leq (n-3)-2 = (n-2)-3$, we can use $(CG)$ to find a Hamiltonian path $\gamma_1$ for $\mathcal{Q}_n^{11}$ minus all deleted and used vertices which connects $v_{11}$ to $v'_{11}$.

Let $u'_{00}$ be any undeleted and unused odd vertex in $\mathcal{Q}_n^{00}$ different from $u_{00}$. Then its neighbor $v_{10}$ in $\mathcal{Q}_n^{10}$ has not been used so far. There are $k-s-1+s-1 = k-2$ used even vertices and $k-s-1+s = k-1$ deleted or used odd vertices in $\mathcal{Q}_n^{00}$. Since $k-2 \leq (n-3)-2 = (n-2)-3$, we can use $(CG)$ to find a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{00}$ minus all deleted and used vertices which connects $u_{00}$ to $u'_{00}$.

There are $k-s-1+s-1 = k-2$ pairs of used even and odd vertices in $\mathcal{Q}_n^{10}$. Since $k-2 \leq (n-2)-3$, we can use $(T)$ to find a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{10}$ minus all used vertices which connects $v_{10}$ to $u_{10}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$v_{11} \xrightarrow{\gamma_1} v'_{11} \to u_{01} \xrightarrow{\gamma_2} v_{01} \to u_{00} \xrightarrow{\gamma_3} u'_{00} \to v_{10} \xrightarrow{\gamma_4} u_{10}.$$

Case (J) $k \leq n-3$, there is a column $A$ in $M_o$ and there is a column $B$ in $M_2$ which separates the odd vertices in a different way than $A$.

It follows from our hypotheses that $A$ separates the deleted odd vertices in the way $(1, k-1)$ and all vertices in the way $(k+1, k-1)$. Also, $B$ separates the odd and the even vertices in the way $(1, k-1)$ and all vertices in the way $(k, k)$. Therefore, without loss of generality, we can assume that the deleted vertices are distributed as follows:

$$\{v_1\} \subset \mathcal{Q}_n^{00},\ \{v_3,\ldots,v_k\} \subset \mathcal{Q}_n^{10},\ \{u_1,v_2\} \subset \mathcal{Q}_n^{11}, \text{ and } \{u_2,\ldots,u_k\} \subset \mathcal{Q}_n^{01}.$$

Take a model path $\mu$ in $\mathcal{Q}_n^{10}$ which begins with an even vertex $e_{10} = u_{10}$ and following $\mu$ make $k-2$ short cycles of the type

$$e_{10} \to (o,e)_{11} \to o_{01} \to (e,o)_{00} \to e'_{10}.$$

We denote the end vertex of the resulting path by $u'_{10}$. Let the neighbor of $u'_{10}$ in $\mathcal{Q}_n^{11}$ be $v_{11}$. We extend the constructed path with the edge $(u'_{10}, v_{11})$.

The total number of the constructed short cycles is $k-2 \le (n-3)-2$, hence the length of $\mu$ is enough for that construction.

There are $k-s+s-1 = k-1$ pairs of even and odd deleted or used vertices in $\mathcal{Q}_n^{01}$. Since $k-1 \le (n-3)-1$, it follows from $(L)$ that there exists a Hamiltonian cycle $\gamma'$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices.

There are $k-2+1 = k-1 \le (n-2)-2$ pairs of deleted or used even and odd vertices in $\mathcal{Q}_n^{11}$. Therefore, according to $(L)$, there exists a Hamiltonian cycle for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices. This cycle contains $v_{11}$. Let $u_{11}$ be a neighbor of $v_{11}$ in that cycle and let $\gamma_1$ be the Hamiltonian path for $\mathcal{Q}_n^{11}$ minus all deleted or used vertices determined by this cycle which connects $v_{11}$ to $u_{11}$.

The neighbor $v_{01}$ of $u_{11}$ is clearly neither deleted nor used odd vertex in $\mathcal{Q}_n^{01}$. Let $v'_{01} \ne v_{01}$ be any unused odd vertex in $\mathcal{Q}_n^{01}$. There are $k-1$ deleted even vertices and $k-2 \le (n-2)-3$ used odd vertices in $\mathcal{Q}_n^{01}$. Then it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_2$ for $\mathcal{Q}_n^{01}$ minus all deleted or used vertices which connects $v_{01}$ to $v'_{01}$.

The neighbor $u_{00}$ of $v_{01}$ is clearly neither deleted nor used even vertex in $\mathcal{Q}_n^{00}$. Let $u'_{00} \ne u_{00}$ be any unused even vertex in $\mathcal{Q}_n^{00}$ whose neighbor $v_{10}$ is not a deleted even vertex in $\mathcal{Q}_n^{10}$. There are $k-2 \le (n-2)-3$ used even vertices and $k-2+1 = k-1$ deleted or used even vertices in $\mathcal{Q}_n^{00}$. Then it follows from $(CG)$ that there exists a Hamiltonian path $\gamma_3$ for $\mathcal{Q}_n^{00}$ minus all deleted or used vertices which connects $u_{00}$ to $u'_{00}$.

There are $k-2$ deleted odd and $k-2 \le (n-2)-3$ used even vertices in $\mathcal{Q}_n^{10}$. Then it follows from $(T)$ that there exists a Hamiltonian path $\gamma_4$ for $\mathcal{Q}_n^{10}$ minus all deleted or used vertices which connects $v_{10}$ to $u_{10}$.

Then, to finish the construction of a Hamiltonian cycle for $\mathcal{Q}_n - \mathcal{F}$ we extend the previously constructed path with the path

$$v_{11} \xrightarrow{\gamma_1} u_{11} \to v_{01} \xrightarrow{\gamma_2} v'_{01} \to u_{00} \xrightarrow{\gamma_3} u'_{00} \to v_{10} \xrightarrow{\gamma_4} u_{10}.$$

The proof of Theorem 3.1 is completed.

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