PROPERTIES AND STRENGTH OF MATERIALS

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References

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SIMPLE STRESS AND STRAIN

1. Load (P) (N)

In any engineering structure or mechanism the individual components will be subjected to external forces arising from the service conditions or environment in which the component works.

$$\Sigma P_X = 0 \quad , \Sigma P_Y = 0 \quad , \Sigma M_o = 0$$

If a cylindrical bar is subjected to a direct pull or push along its axis as shown in Figure (1), then it is said to be subjected to ***tension*** or ***compression.***

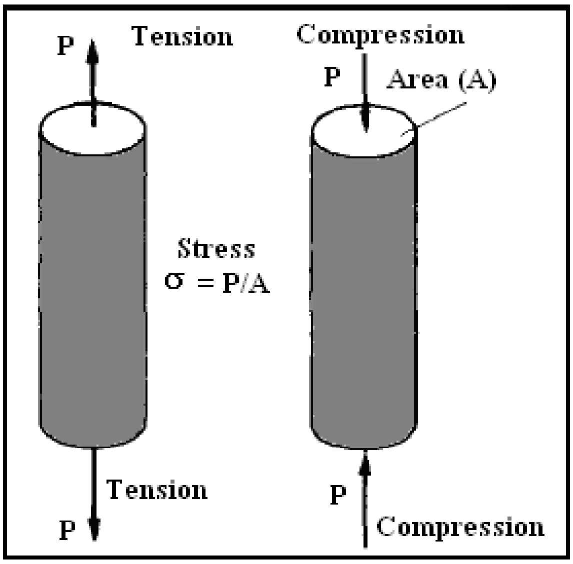

Figure (1) Types of direct stress (Tension or Compression)

In the SI system of units load is measured in newtons, loads appear in SI multiples, i.e. kilonewtons (kN) or meganewtons (MN). There are a number of different ways in which load can be applied to a member. Typical loading types are:

(a) Static or dead loads, i.e. non-fluctuating loads, generally caused by gravity effects.

(b) Liue loads, as produced by, for example, lorries crossing a bridge.

(c) Impact or shock loads caused by sudden blows.

(d) Fatigue, fluctuating or alternating loads.

2. Direct or normal stress (σ), (N/m^2)

A bar is subjected to a uniform tension or compression, i.e. a direct force, which is uniformly or equally applied across the cross section, then the internal forces set up are also distributed uniformly and the bar is said to be subjected to a uniform **direct or normal stress,** the stress being defined as

$$Stress = \frac{Load}{area} = \frac{P}{A}$$

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Stress (σ) may thus be (i) compressive stress or (ii) tensile stress depending on the nature of the load and will be measured in units of (N/m^2).

3. Direct strain *(e)*

Figure (2) show a bar is subjected to a direct load, and hence a stress, the bar will change in length. If the bar has an original length L and changes in length by an amount ***dL,*** the ***strain*** produced is defined as follows:

$$\text{strain } (\varepsilon) = \frac{\text{change in length}}{\text{original length}} = \frac{\delta L}{L}$$

Strain is thus a measure of the deformation of the material and is non-dimensional,

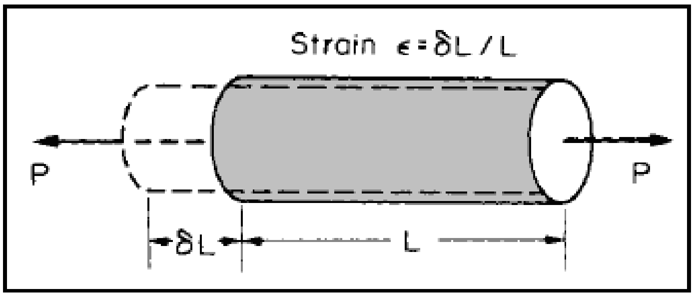

Figure (2)

Alternatively, strain can be expressed as a percentage strain

$$\text{strain } (\varepsilon) = \frac{\delta L}{L} \times 100\%$$

4. Sign convention for direct stress and strain

Tensile stresses and strains are considered **POSITIVE** in sense producing an ***increase*** in length. Compressive stresses and strains are considered **NEGATIVE** in sense producing a ***decrease*** in length.

5. Elastic materials - Hooke's law (E), (N/m^2)

A material is said to be ***elastic*** if it returns to its original, unloaded dimensions when load is removed. A particular form of elasticity which applies to a large range of engineering materials, at least over part of their load range, produces deformations which are proportional to the loads producing them. stress is proportional to strain. ***Hooke's law,*** in its simplest form*, therefore states that

$$\text{stress } (\sigma) \propto \text{strain } (\varepsilon)$$

$$\frac{\text{stress}}{\text{strain}} = \text{constant*}$$

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Other classifications of materials with which the reader should be acquainted are as follows:

A material which has a uniform structure throughout without any flaws or discontinuities is termed a ***homogeneous*** material. ***Non-homogeneous*** or ***inhomogeneous*** materials such as concrete and poor-quality cast iron will thus have a structure which varies from point to point depending on its constituents and the presence of casting flaws or impurities.

If a material exhibits uniform properties throughout in all directions it is said to be ***isotropic;*** conversely one which does not exhibit this uniform behaviour is said to be ***nonisotropic*** or ***anisotropic*.**

An ***orthotropic*** material is one which has different properties in different planes. A typical example of such a material is wood, although some composites which contain systematically orientated "***inhomogeneities***" may also be considered to fall into this category.

6. Modulus of elasticity - Young's modulus (E), (N/m^2)

Within the elastic limits of materials, i.e. within the limits in which Hooke's law applies, it has been shown that:

$$\frac{\text{stress}}{\text{strain}} = \text{constant}$$

This constant is given the symbol E and termed the ***modulus of elasticity*** or ***Young's modulus***, Thus

$$E = \frac{Stress}{Strain} = \frac{s}{e} \qquad .....(1)$$

$$E = \frac{P.L}{A.dL} \qquad .....(2)$$

Young's modulus E is generally assumed to be the same in tension or compression and for most engineering materials has a high numerical value. Typically, E = 200 x 10^9 N/m^2 for steel.

$$e = \frac{s}{E} \qquad ......(3)$$

In most common engineering applications strains do not often exceed 0.003 or 0.3 % so that the assumption used later in the text that deformations are small in relation to original dimensions is generally well founded.

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7. <u>Tensile test</u>

The standard tensile test in which a circular bar of uniform cross-section is subjected to a gradually increasing tensile load until failure occurs. Measurements of the change in length of a selected ***gauge length*** of the bar are recorded throughout the loading operation by means of extensometers and a graph of load against extension or stress against strain is produced as shown in Fig. (3); this shows a typical result for a test on a mild (low carbon) steel bar; other materials will exhibit different graphs but of a similar general form see Figures (5) to(7).

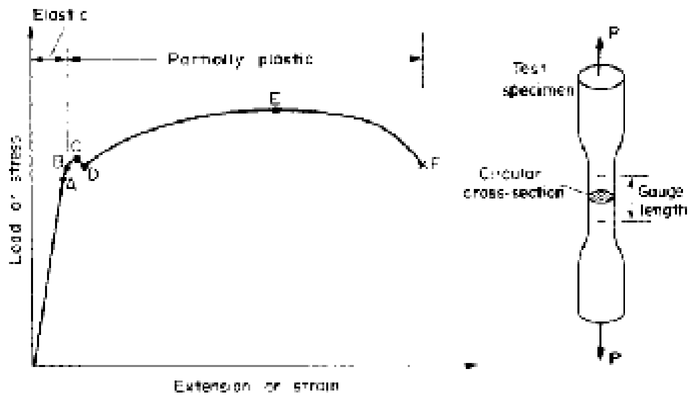

Figure (3) Typical tensile test curve for mild steel.

For the first part of the test it will be observed that Hooke's law is obeyed, the material behaves elastically and stress is proportional to strain, giving the straight-line graph indicated. Some point A is eventually reached, however, when the linear nature of the graph ceases and this point is termed the ***limit*** of ***proportionality.***

C, termed the ***upper yield point***

D, the ***lower yield point***

That stress which, when removed, produces a permanent strain or "set" of 0.1 % of the original gauge length-see Fig. (4a).

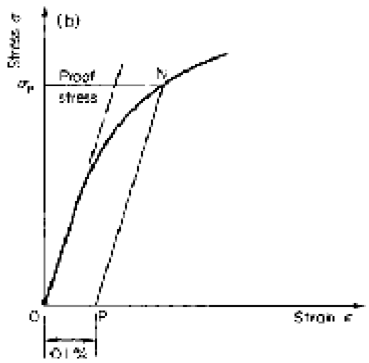

Figure (4a) Determination of 0.1 % proof stress.

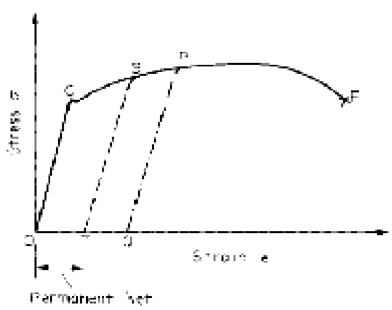

Figure (4b) Permanent deformation or "set" after straining beyond the yield point.

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Typical stress-strain curves resulting from tensile tests on other engineering materials are shown in Figs. (5) to (7).

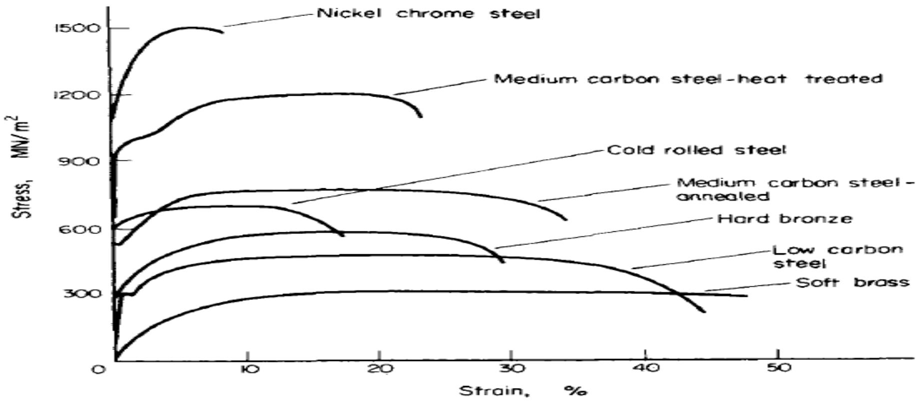


Figure (5)Tensile test curves for various metals.

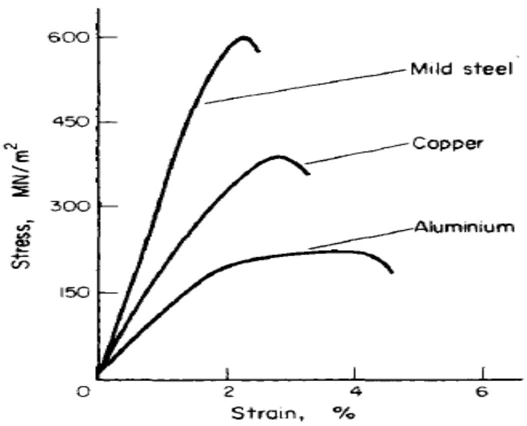
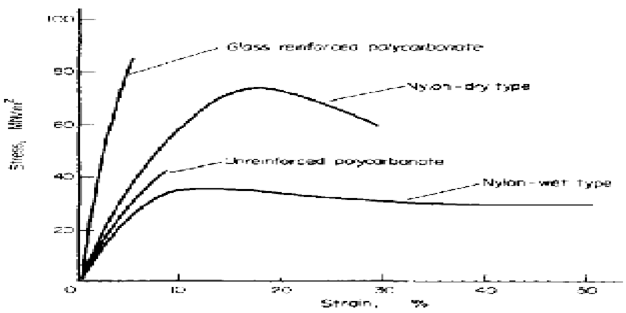


Figure (6) Typical stress - strain curves for hard drawn wire material-note large reduction in strain values from those of Figure (5)



Figure(7) Typical tension test results for various types of nylon and polycarbonate.

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8. Ductile materials

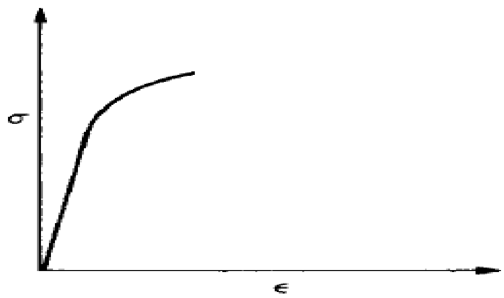
It has been observed above that the partially plastic range of the graph of Figure (3) covers a much wider part of the strain axis than does the elastic range. Thus the extension of the material over this range is considerably in excess of that associated with elastic loading. The capacity of a material to allow these large extensions, i.e. the ability to be drawn out plastically, is termed its ***ductility.*** Materials with high ductility are termed ***ductile*** materials, members with low ductility are termed ***brittle*** materials. **A** quantitative value of the ductility is obtained by measurements of the ***percentage elongation*** or ***percentage reduction in area,*** both
being defined below.

$$\text{Percentage elongation} = \frac{\text{increase in gauge length to fracture}}{\text{original gauge length}} \times 100$$

$$\text{Percentage reduction in area} = \frac{\text{reduction in cross-sectional area of necked portion}}{\text{original area}} \times 100$$

9. Brittle materials

A brittle material is one which exhibits relatively small extensions to fracture so that the partially plastic region of the tensile test graph is much reduced (Fig. **8).** Whilst Fig. (3) referred to a low carbon steel, Fig. **(8)** could well refer to a much higher strength steel with a higher carbon content. There is little or no necking at fracture for brittle materials.



Figure(8) Typical tensile test curve for a brittle material

10. Poisson's ratio (n)

Consider the rectangular bar of Figure (9) subjected to a tensile load. Under the action of this load the bar will increase in length by an amount ***dL*** giving a longitudinal strain in the bar .

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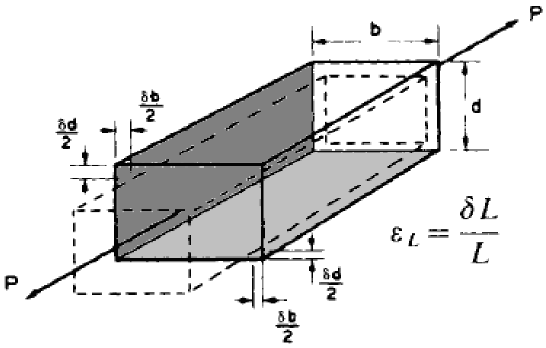

Figure (9)

The bar will also exhibit, however, a ***reduction*** in dimensions laterally, i.e. its breadth and depth will both reduce. The associated lateral strains will both be equal, will be of opposite sense to the longitudinal strain, and will be given by

$$\varepsilon_{lat} = -\frac{\delta b}{b} = -\frac{\delta d}{d}$$

Provided the load on the material is retained within the elastic range the ratio of the lateral and longitudinal strains will always be constant. This ratio is termed ***Poisson's ratio.***

$$\text{Poisson's ratio } (\nu) = \frac{\text{lateral strain}}{\text{longitudinal strain}} = \frac{(-\delta d/d)}{\delta L/L} \qquad \text{.... (4)}$$

The negative sign of the lateral strain is normally ignored to leave Poisson's ratio simply as a ratio of strain magnitudes. It must be remembered, however, that the longitudinal strain induces a lateral strain of opposite sign. For most engineering materials the value of *v* lies between 0.25 and 0.33.

Since

$$\text{longitudinal strain} = \frac{\text{longitudinal stress}}{\text{Young's modulus}} = \frac{\sigma}{E} \qquad \text{..... (4a)}$$

$$\text{lateral strain} = \nu\frac{\sigma}{E} \qquad \text{..... (4b)}$$

11. Application of Poisson's ratio to a two-dimensional stress system

A two-dimensional stress system is one in which all the stresses lie within one plane such as the X-Y plane as shown in figure (10).

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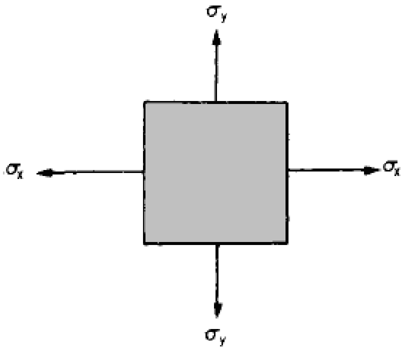

Figure (10) Simple two-dimensional system of direct stresses.

The following strains will be produced
(a) in the X direction resulting from $e_x = s_x /E$
(b) in the Y direction resulting from $e_y = s_y/E$.
(c) in the X direction resulting from $e_y = - v(s_y /E)$,
(d) in the Y direction resulting from $e_x = - v(s_x /E)$.
strains (c) and (d) being the so-called ***Poisson's ratio strain,*** opposite in sign to the applied strains, i.e. compressive.
The total strain in the X direction will therefore be given by:

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad ..... (5)$$

and the total strain in the Y direction will be:

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} = \frac{1}{E}(\sigma_y - \nu\sigma_x) \qquad ..... (6)$$

If any stress is, in fact, compressive its value must be substituted in the above equations together with a negative sign following the normal sign convention.

12. Shear stress(t) , (N/m^2)

Consider a block or portion of material as shown in Figure (11) subjected to a set of **equal** and opposite forces **Q.** (Such a system could be realised in a bicycle brake block when contacted with the wheel.) then a ***shear stress*** *t* is set up, defined as follows:

$$\text{shear stress}\,(\tau) = \frac{\text{shear load}}{\text{area resisting shear}} = \frac{Q}{A}$$

This shear stress will always be ***tangential*** to the area on which it acts; direct stresses, however, are always ***normal*** to the area on which they act.

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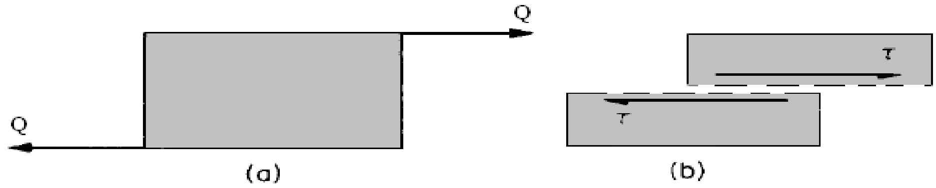

Figure(11) Shear force and resulting shear stress system showing typical form of failure by relative sliding of planes.

13. Shear strain (g)

If one again considers the block of Figure (11a)to be a bicycle brake block it is clear that the rectangular shape of the block will not be retained as the brake is applied and the shear forces introduced. The block will in fact change shape or "strain" into the form shown in Figure (12) The angle of deformation y is then termed the ***shear strain.***
Shear strain is measured in radians and hence is non-dimensional, i.e. it has no units.

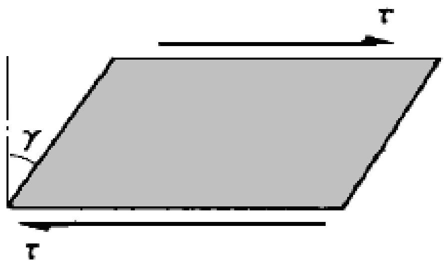

Figure (12) Deformation (shear strain) produced by shear stresses.

14. Modulus of rigidity (G), (N/m^2)

For materials within the elastic range the shear strain is proportional to the shear stress producing it, The constant G is termed the ***modulus of rigidity*** or ***shear modulus*** and is directly comparable to the modulus of elasticity used in the direct stress application.

$$\frac{\text{shear stress}}{\text{shear strain}} = \frac{\tau}{\gamma} = \text{constant} = G \qquad \text{..... (7)}$$

15. Double shear

Consider the simple riveted lap joint shown in Figure (13a) When load is applied to the plates the rivet is subjected to shear forces tending to shear it on one plane as indicated. In the butt joint with two cover plates of Figure (13b), however, each rivet is subjected to possible shearing on two faces, i.e. ***double shear.*** In such cases twice the area of metal is resisting the applied forces so that the shear stress set up is given by

$$\text{shear stress } \tau \text{ (in double shear)} = \frac{P}{2A} \qquad \text{..... (8)}$$

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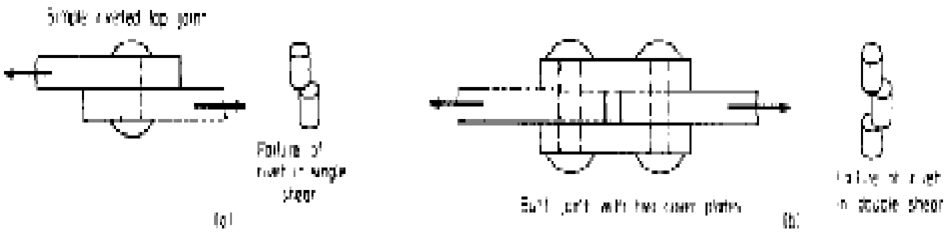

Figure (13) (a) Single shear. (b) Double shear.

16. Allowable working stress-factor of safety

The most suitable strength or stiffness criterion for any structural element or component is normally some maximum stress or deformation which must not be exceeded. In the case of stresses the value is generally known as the ***maximum allowable working stress.*** Because of uncertainties of loading conditions, design procedures, production methods, etc., designers generally introduce a ***factor of safety*** into their designs, defined as follows

:

$$\text{factor of safety} = \frac{\text{maximum stress}}{\text{allowable working stress}} \quad \text{..... (9)}$$

$$\text{factor of safety} = \frac{\text{yield stress (or proof stress)}}{\text{allowable working stress}}$$

$$\text{load factor} = \frac{\text{load at failure}}{\text{allowable working load}} \quad \text{...... (10)}$$

18. Temperature stresses

When the temperature of a component is increased or decreased the material respectively expands or contracts. If this expansion or contraction is not resisted in any way then the processes take place free of stress. If, however, the changes in dimensions are restricted then stresses termed ***temperature stresses*** will be set up within the material.

Consider a bar of material with a linear coefficient of expansion a. Let the original length of the bar be ***L*** and let the temperature ***increase*** be ***t.*** If the bar is free to expand the change in length would be given by

$$\Delta L = L\alpha t \quad \text{..... (11)}$$

and the new length

$$L' = L + L\alpha t = L(1 + \alpha t)$$

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$$\varepsilon = \frac{\Delta L}{L} = \frac{L\alpha t}{L(1+\alpha t)}$$

$$\varepsilon = \frac{L\alpha t}{L} = \alpha t$$

But $\frac{\sigma}{\varepsilon} = E$

$\therefore$ stress $\sigma = E\varepsilon = E\alpha t$

Examples

Example 1

Determine the stress in each section of the bar shown in Figure (14) when subjected to an axial tensile load of 20 kN. The central section is 30 mm square cross-section; the other portions are of circular section, their diameters being indicated. What will be the total extension of the bar? For the bar material E = 210GN/m^2.

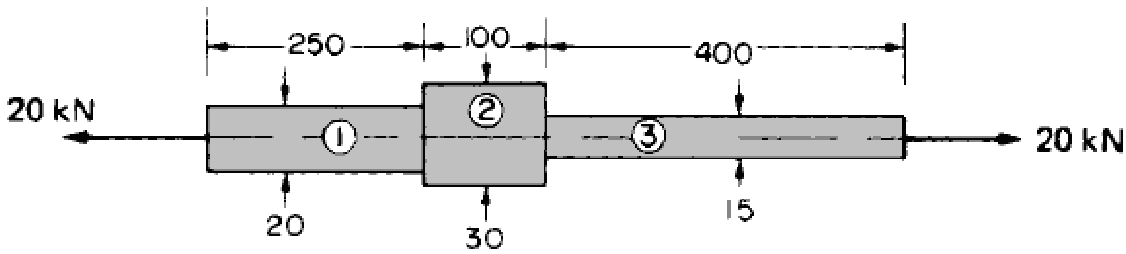

Figure (14) all dimensions mm

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Solution

$$\text{Stress} = \frac{\text{force}}{\text{area}} = \frac{P}{A}$$

$$\text{Stress in section (1)} = \frac{20 \times 10^3}{\frac{\pi(20 \times 10^{-3})^2}{4}} = \frac{80 \times 10^3}{\pi \times 400 \times 10^{-6}} = 63.66\ \text{MN/m}^2$$

$$\text{Stress in section (2)} = \frac{20 \times 10^3}{30 \times 30 \times 10^{-6}} = 22.2\ \text{MN/m}^2$$

$$\text{Stress in section (3)} = \frac{20 \times 10^3}{\frac{\pi(15 \times 10^{-3})^2}{4}} = \frac{80 \times 10^3}{\pi \times 225 \times 10^{-6}} = 113.2\ \text{MN/m}^2$$

Now the extension of a bar can always be written in terms of the stress in the bar since

$$E = \frac{\text{stress}}{\text{strain}} = \frac{\sigma}{\delta/L}$$

i.e.

$$\delta = \frac{\sigma L}{E}$$

$\therefore$

$$\text{extension of section (1)} = 63.66 \times 10^6 \times \frac{250 \times 10^{-3}}{210 \times 10^9} = 75.8 \times 10^{-6}\ \text{m}$$

$$\text{extension of section (2)} = 22.2 \times 10^6 \times \frac{100 \times 10^{-3}}{210 \times 10^9} = 10.6 \times 10^{-6}\ \text{m}$$

$$\text{extension of section (3)} = 113.2 \times 10^6 \times \frac{400 \times 10^{-3}}{210 \times 10^9} = 215.6 \times 10^{-6}\ \text{m}$$

$\therefore$

$$\begin{aligned}\text{total extension} &= (75.8 + 10.6 + 215.6)10^{-6} \\ &= 302 \times 10^{-6}\ \text{m} \\ &= \mathbf{0.302\ mm}\end{aligned}$$

Example 2

(a) A 25 mm diameter bar is subjected to an axial tensile load of 100 kN. Under the action of this load a **200mm** gauge length is found to extend 0.19 **x** 10^{-3}mm. Determine the modulus of elasticity for the bar material.

(b) If, in order to reduce weight whilst keeping the external diameter constant, the bar is bored axially to produce a cylinder of uniform thickness, what is the maximum diameter of bore possible given that the maximum allowable stress is 240MN/m^2? The load can be assumed to remain constant at 100 kN.

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(c) What will be the change in the outside diameter of the bar under the limiting stress quoted in (b)? (E = ***210GN/m2*** and ***v*** = 0.3).

Solution

(a) From eqn. (1.2),

$$\text{Young's modulus } E = \frac{PL}{A\,\delta L}$$

$$= \frac{100 \times 10^3 \times 200 \times 10^{-3}}{\frac{\pi(25 \times 10^{-3})^2}{4} \times 0.19 \times 10^{-3}}$$

$$= \mathbf{214\ GN/m^2}$$

(b) Let the required bore diameter be d mm; the cross-sectional area of the bar will then be reduced to

$$A = \left[\frac{\pi \times 25^2}{4} - \frac{\pi d^2}{4}\right]10^{-6} = \frac{\pi}{4}(25^2 - d^2)10^{-6}\ \text{m}^2$$

$$\therefore \qquad \text{stress in bar} = \frac{P}{A} = \frac{4 \times 100 \times 10^3}{\pi(25^2 - d^2)10^{-6}}$$

But this stress is restricted to a maximum allowable value of 240 MN/m^2.

$$\therefore \qquad 240 \times 10^6 = \frac{4 \times 100 \times 10^3}{\pi(25^2 - d^2)10^{-6}}$$

$$\therefore \qquad 25^2 - d^2 = \frac{4 \times 100 \times 10^3}{240 \times 10^6 \times \pi \times 10^{-6}} = 530.5$$

$$\therefore \qquad d^2 = 94.48 \quad \text{and} \quad d = 9.72\ \text{mm}$$

The maximum bore possible is thus **9.72 mm**.

(c) The change in the outside diameter of the bar will be obtained from the lateral strain,

i.e. $$\text{lateral strain} = \frac{\delta d}{d}$$

But $$\text{Poisson's ratio } \nu = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

and $$\text{longitudinal strain} = \frac{\sigma}{E} = \frac{240 \times 10^6}{210 \times 10^9}$$

$$\therefore \qquad \frac{\delta d}{d} = -\nu\frac{\sigma}{E} = -\frac{0.3 \times 240 \times 10^6}{210 \times 10^9}$$

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$$\therefore \quad \text{change in outside diameter} = -\frac{0.3 \times 240 \times 10^6}{210 \times 10^9} \times 25 \times 10^{-3}$$

$$= \mathbf{-8.57 \times 10^{-6}\ m} \text{ (a reduction)}$$

<u>Example 3</u>

The coupling shown in Figure (15) is constructed from steel of rectangular cross-section and is designed to transmit a tensile force of 50 kN. If the bolt is of **15** mm diameter calculate:

(a) the shear stress in the bolt;

(b) the direct stress in the plate;

(c) the direct stress in the forked end of the coupling.

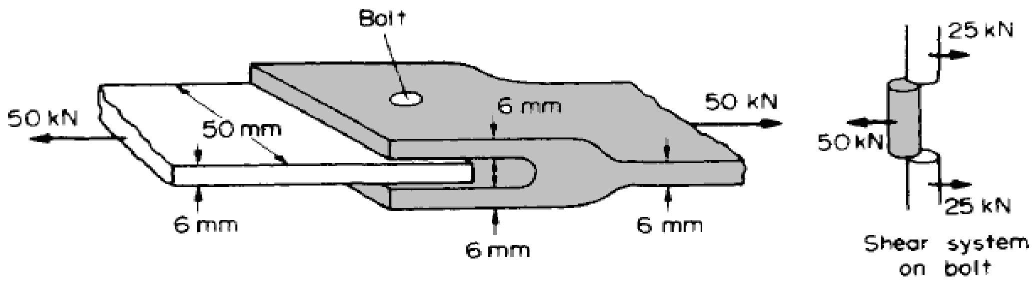

Figure (15)

Solution

(a) The bolt is subjected to double shear, tending to shear it as shown in Fig. 1.14b. There is thus twice the area of the bolt resisting the shear and from eqn. (1.8)

$$\text{shear stress in bolt} = \frac{P}{2A} = \frac{50 \times 10^3 \times 4}{2 \times \pi(15 \times 10^{-3})^2}$$

$$= \frac{100 \times 10^3}{\pi(15 \times 10^{-3})^2} = \mathbf{141.5\ MN/m^2}$$

(b) The plate will be subjected to a direct tensile stress given by

$$\sigma = \frac{P}{A} = \frac{50 \times 10^3}{50 \times 6 \times 10^{-6}} = \mathbf{166.7\ MN/m^2}$$

(c) The force in the coupling is shared by the forked end pieces, each being subjected to a direct stress

$$\sigma = \frac{P}{A} = \frac{25 \times 10^3}{50 \times 6 \times 10^{-6}} = \mathbf{83.3\ MN/m^2}$$

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Example 4

Derive an expression for the total extension of the tapered bar of circular cross-section shown in Figure (16) when it is subjected to an axial tensile load W.

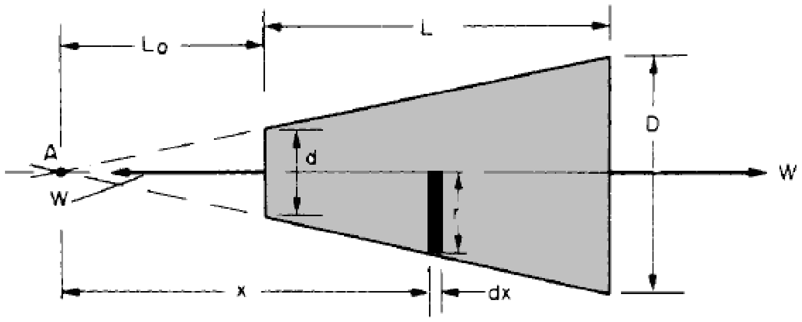

Figure (16)

Solution

From the proportions of Fig. 1.21,

$$\frac{d/2}{L_0} = \frac{(D-d)/2}{L}$$

$\therefore$ $$L_0 = \frac{d}{(D-d)} L$$

Consider an element of thickness dx and radius r, distance x from the point of taper A.

$$\text{Stress on the element} = \frac{W}{\pi r^2}$$

But $$\frac{r}{x} = \frac{d}{2L_0}$$

$\therefore$ $$r = d\left(\frac{D-d}{2dL}\right)x = \frac{x(D-d)}{2L}$$

$\therefore$ $$\text{stress on the element} = \frac{4WL^2}{\pi(D-d)^2x^2}$$

$\therefore$ $$\text{strain on the element} = \frac{\sigma}{E}$$

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and extension of the element $= \frac{\sigma dx}{E}$

$$= \frac{4WL^2}{\pi(D-d)^2 x^2 E} dx$$

$\therefore$ total extension of bar $= \int_{L_0}^{L_0+L} \frac{4WL^2}{\pi(D-d)^2 E} \frac{dx}{x^2}$

$$= \frac{4WL^2}{\pi(D-d)^2 E} \left[-\frac{1}{x} \right]_{L_0}^{L_0+L}$$

$$= \frac{4WL^2}{\pi(D-d)^2 E} \left[-\frac{1}{(L_0+L)} - \left(-\frac{1}{L_0} \right) \right]$$

But $$L_0 = \frac{d}{(D-d)} L$$

$$\therefore \quad L_0 + L = \frac{d}{(D-d)} L + L = \frac{(d+D-d)}{D-d} L = \frac{DL}{(D-d)}$$

$\therefore$ total extension

$$= \frac{4WL^2}{\pi(D-d)^2 E} \left[-\frac{(D-d)}{DL} + \frac{(D-d)}{dL} \right] = \frac{4WL}{\pi(D-d)E} \left[\frac{(-d+D)}{Dd} \right]$$

$$= \frac{4WL}{\pi DdE}$$

Example 5

The following figures were obtained in a standard tensile test on a specimen of low carbon steel:

diameter of specimen, 11.28 mm;

gauge length, 56mm;

minimum diameter after fracture, **6.45** mm.

Using the above information and the table of results below, produce:

(1) a load/extension graph over the complete test range;

(2) a load/extension graph to an enlarged scale over the elastic range of the specimen.

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Load (kN)	2.47	4.97	7.4	9.86	12.33	14.8	17.27	19.74	22.2	24.7
Extension ($m \times 10^{-6}$)	5.6	11.9	18.2	24.5	31.5	38.5	45.5	52.5	59.5	66.5
Load (kN)	27.13	29.6	32.1	33.3	31.2	32	31.5	32	32.2	34.5
Extension ($m \times 10^{-6}$)	73.5	81.2	89.6	112	224	448	672	840	1120	1680
Load (kN)	35.8	37	38.7	39.5	40	39.6	35.7	28		
Extension ($m \times 10^{-6}$)	1960	2520	3640	5600	7840	11200	13440	14560		

Using the two graphs and other information supplied, determine the values of

(a) Young's modulus of elasticity;

(b) the ultimate tensile stress;

(c) the stress at the upper and lower yield points;

(d) the percentage reduction of **area;**

(e) the percentage elongation;

(f) the nominal and actual stress at fracture.

Solution

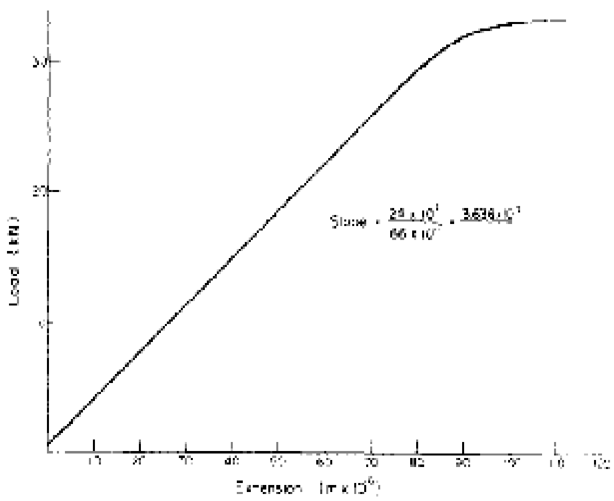

Figure (17) Load-extension graph for elastic range.

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(a)

$$\text{Young's modulus } E = \frac{\sigma}{\varepsilon} = \frac{\text{load}}{\text{area}} \times \frac{\text{gauge length}}{\text{extension}}$$

$$= \frac{\text{load}}{\text{extension}} \times \frac{\text{gauge length}}{\text{area}}$$

$$E = \text{slope of graph} \times \frac{L}{A} = 3.636 \times 10^8 \times \frac{56 \times 10^{-3}}{100 \times 10^{-6}}$$

$$= 203.6 \times 10^9 \text{ N/m}^2$$

$$\therefore \quad \mathbf{E = 203.6 \text{ GN/m}^2}$$

(b)

$$\text{Ultimate tensile stress} = \frac{\text{maximum load}}{\text{cross-section area}} = \frac{40.2 \times 10^3}{100 \times 10^{-6}} = \mathbf{402 \text{ MN/m}^2}$$

(c)

$$\text{Upper yield stress} = \frac{33.3 \times 10^3}{100 \times 10^{-6}} = \mathbf{333 \text{ MN/m}^2}$$

$$\text{Lower yield stress} = \frac{31.2 \times 10^3}{100 \times 10^{-6}} = \mathbf{312 \text{ MN/m}^2}$$

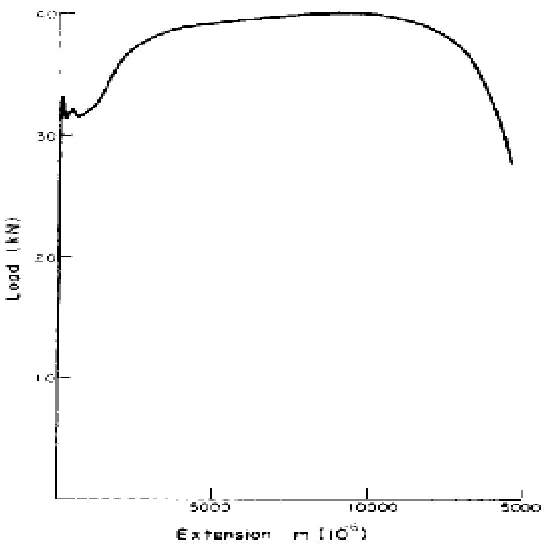

Figure (18) Load-extension graph for complete load

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(d) Percentage reduction of area $= \dfrac{\left(\frac{\pi}{4}D^2 - \frac{\pi}{4}d^2\right)}{\frac{\pi}{4}D^2} \times 100$

$$= \frac{(D^2 - d^2)}{D^2} \times 100$$

$$= \frac{(11.28^2 - 6.45^2)}{11.28^2} = \mathbf{67.3\%}$$

(e) Percentage elongation $= \dfrac{(70.56 - 56)}{56} \times 100$

$$= \mathbf{26\%}$$

(f) Nominal stress at fracture $= \dfrac{28 \times 10^3}{100 \times 10^{-6}} = \mathbf{280\ MN/m^2}$

Actual stress at fracture $= \dfrac{28 \times 10^3}{\frac{\pi}{4}(6.45)^2 \times 10^{-6}} = \mathbf{856.9\ MN/m^2}$

Problems

1. (A). A 25mm square cross-section bar of length 300mm carries an axial compressive load of 50kN. Determine the stress **set** up in the bar and its change of length when the load is applied. For the bar material E = 200 GN/m^2. [80 MN/m^2; 0.12mm]

2. (A). A steel tube, **25** mm outside diameter and 12mm inside diameter, cames an axial tensile load of 40 kN. What will be the stress in the bar? What further increase in load is possible if the stress in the bar is limited **to** 225 MN/m^2? [l06 MN/m^2; 45 kN]

3. (A). Define the terms shear stress and shear strain, illustrating your answer by means of a simple sketch. Two circular bars, one of brass and the other of steel, are to be loaded by a shear load of 30 kN. Determine the necessary diameter of the bars (a) in single shear, (b) in double shear, if the shear stress in the two materials must not exceed 50 MN/m^2 and 100 MN/m^2 respectively. [27.6, 19.5, 19.5, 13.8mm]

4. (A). Two forkend pieces are to be joined together by a single steel pin of 25mm diameter and they are required to transmit 50 kN. Determine the minimum cross-sectional area of material required in one branch of either fork if the stress in the fork material is not to exceed 180 MN/m^2. What will be the maximum shear stress in the pin?

[1.39 **x** $10^{-4}$$m^2$; 50.9MN/$m^2$.]

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5. (A). A simple turnbuckle arrangement is constructed from a 40 mm outside diameter tube threaded internally at each end to take two rods of 25 mm outside diameter with threaded ends. What will be the nominal stresses set up in the tube and the rods, ignoring thread depth, when the turnbuckle cames an axial load of 30 kN? Assuming a sufficient strength of thread, what maximum load can be transmitted by the turnbuckle if the maximum stress is limited to 180 MN/m^2?

[39.2, 61.1 MN/m2, 88.4 kN]

6. (A). A bar ABCD consists of three sections: AB is 25 mm square and 50 mm long, BC is of 20 mm diameter and40 mm long and CD is of 12 mm diameter and 50 mm long. Determine the stress set up in each section of the bar when it is subjected to an axial tensile load of 20 kN. What will be the total extension of the bar under this load? For the bar material, E = 210GN/m^2. [32,63.7, 176.8 MN/m^2, 0.062mm]

7.(A). A steel bar ABCD consists of three sections: AB is of 20mm diameter and 200 mm long, BC is 25 mm square and 400 mm long, and CD is of 12 mm diameter and 200mm long. The bar is subjected to an axial compressive load which induces a stress of 30 MN/m^2 on the largest cross-section. Determine the total decrease in the length of the bar when the load is applied. For steel E = 210GN/m^2.

[0.272 mm.]

8. Figure (19) shows a special spanner used to tighten screwed components. A torque is applied at the tommy -bar and is transmitted to the pins which engage into holes located into the end of a screwed component.

(a) Using the data given in Figure (19) calculate:

(i) the diameter D of the shank if the shear stress is not to exceed 50N/mm^2,

(ii) the stress due to bending in the tommy-bar,

(iii) the shear stress in the pins.

[9.14mm; 254.6 MN/m^2; 39.8 MN/m^2.]

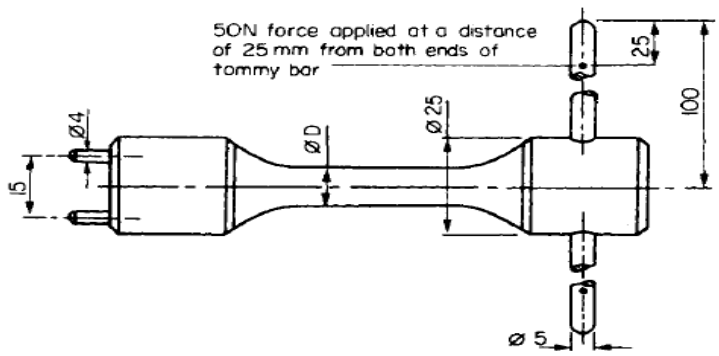

Figure (19)

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Q9-A punch for making holes in steel plates is shown in Figure (1a). Assume that a punch having diameter d =20 mm is used to punch a hole in an 8-mm plate, as shown in the cross-sectional view Figure (1b). If a force P= 110 kN is required to create the hole, what is the average shear stress in the plate and the average compressive stress in the punch?

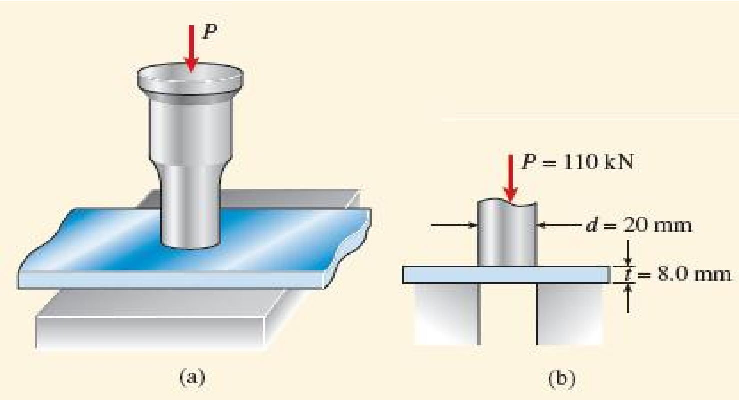

Figure (1)

Q10-A bearing pad of the kind used to support machines and bridge girders consists of a linearly elastic material (usually an elastomer, such as rubber) capped by a steel plate Figure (2a). Assume that the thickness of the elastomer is h, the dimensions of the plate are (a . b), and the pad is subjected to a horizontal shear force V. Obtain formulas for the average shear stress (τ) in the elastomer and the horizontal displacement d of the plate Figure (2b).

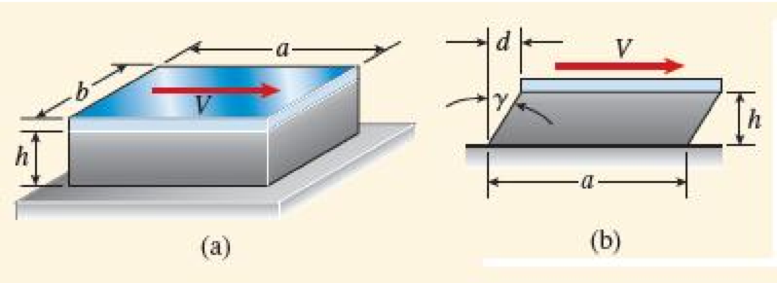

Figure (2)

Q11-A force P of 70 N is applied by a rider to the front hand brake of a bicycle (*P* is the resultant of an evenly distributed pressure). As the hand brake pivots at A, a tension T develops in the 460-mm long brake cable (Ae =1.075 mm^2) which elongates by δ= 0.214 mm. Find normal stress (σ) and strain (ε) in the brake cable.

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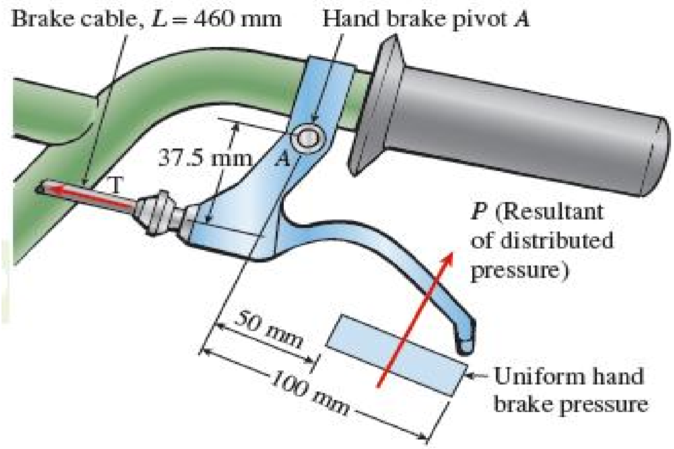

Figure (3)

Q12-A circular aluminum tube of length L = 400 mm is loaded in compression by forces P Figure (4). The outside and inside diameters are 60 mm and 50 mm, respectively. A strain gage is placed on the outside of the bar to measure normal strains in the longitudinal direction.

(a) If the measured strain is $e = 550 \times 10^{-6}$, what is the shortening δ of the bar?

(b) If the compressive stress in the bar is intended to be 40 MPa, what should be the load P?

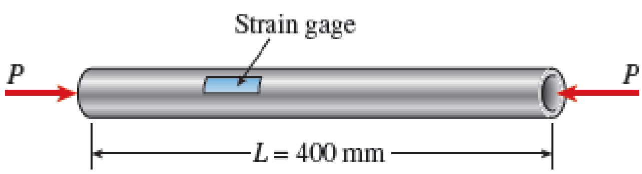

Figure (4)

Q13-(a) A test piece is cut from a brass bar and subjected to a tensile test. With a load of 6.4 kN the test piece, of diameter 11.28 mm, extends by 0.04 mm over a gauge length of 50 mm. Determine:

(i) the stress, (ii) the strain, (iii) the modulus of elasticity.

(b) A spacer is turned from the same bar. The spacer has a diameter of 28 mm and a length of 250mm. both measurements being made at 20°C. The temperature of the spacer is then increased to 100°C, the natural expansion being entirely prevented. Taking the coefficient of linear expansion to be18 x 10^{-6}/°C determine:

(i) the stress in the spacer, (ii) the compressive load on the spacer.

Ans. [64MN/m^2, 0.0008, 80GN/m^2, 115.2 MN/m^2, 71 kN.]

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COMPOUND BARS

1. Compound bars subjected to external load

In certain applications it is necessary to use a combination of elements or bars made from different materials. In overhead electric cables, for example, it is often convenient to carry the current in a set of copper wires surrounding steel wires, the latter being designed to support the weight of the cable over large spans. Such combinations of materials are generally termed ***compound*** burs. This chapter is concerned with compound bars which are symmetrically proportioned such that no bending results, when an external load is applied to such a compound bar it is shared between the individual component materials in proportions depending on their respective lengths, areas and Young's moduli. A compound bar consisting of n members, each having a different length and cross-sectional area and each being of a different material as shown in figure (1)

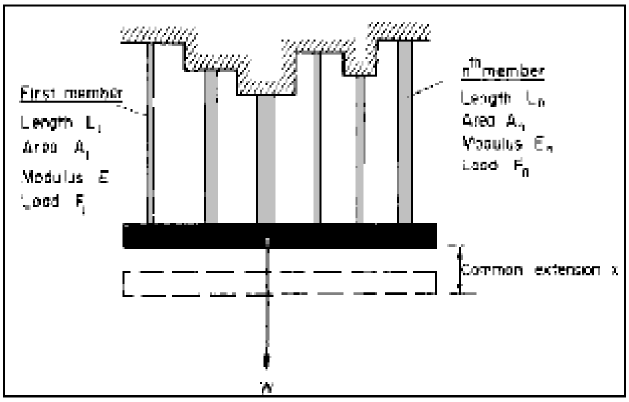

Figure (1) Compound bar formed of different materials

For the nth member,

$$\frac{stress}{strain} = E_n = \frac{F_n.L_n}{A_n.x_n}$$

$$F_n = \frac{E_n.A_n.x}{L_n} \qquad ....(1)$$

where F, is the force in the nth member ,A its cross-sectional area and L_n are its length. The total load carried will be the sum of all such loads for all the members

$$W = \sum \frac{E_n.A_n.x}{L_n} = x.\sum \frac{E_n.A_n}{L_n} \qquad ...(2)$$

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Now from equation (1) the force in member 1 is given by

$$F_1 = \frac{E_1.A_1.x}{L_1}$$

But, from equation (2),

$$x = \frac{W}{\sum \frac{E_n.A_n}{L_n}}$$

$$F_1 = \frac{\frac{E_1.A_1}{L_1}}{\sum \frac{E.A}{L}} W \quad .....(3)$$

i.e. each member carries a portion of the total load W proportional to its EAIL value. If the wires are all of equal length the above equation reduces to

$$F_1 = \frac{E_1.A_1}{\sum E.A} W \quad ....(4)$$

The stress in member 1 is then given by

$$s_1 = \frac{F_1}{A_1} \qquad ....(5)$$

2. <u>Compound bars - “equivalent” or “combined” modulus</u>

In order to determine the common extension of a compound bar it is convenient to consider it as a single bar of an imaginary material with an ***equivalent*** or ***combined*** modulus ***E*,.** Here it is necessary to assume that both the extension and the original lengths of the individual members of the compound bar are the same; the strains in all members will then be equal.

Now total load on compound bar **= F1 + F2 + F3 + . . . + F,** where **F1, F2,** etc., are the loads in members 1, 2, etc.

But

force = stress x area

$$s\ (A_1 + A_2 + ... + A_n) = s_1\ A_1 + s_2\ A_2 + ... + s_n\ A_n$$

Where: s is the stress in the equivalent single bar. Dividing through by the common strain ***e*,**

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$$\frac{s}{e}(A_1 + A_2 + ...+ A_n) = \frac{s_1}{e}A_1 + \frac{s_2}{e}A_2 + ...+ \frac{s_n}{e}A_n$$

$$E_c(A_1 + A_2 + ...+ A_n) = E_1 A_1 + E_2 A_2 + ...+ E_n A_n$$

where E_c**,** is the ***equivalent*** or ***combined E*** of the single bar.

$$combined \quad E = \frac{E_1.A_1 + E_2.A_2 + ... + E_n.A_n}{A_1 + A_2 + ... + A_n}$$

$$E_c = \frac{\sum E.A}{\sum A} \qquad ....(6)$$

With an external load W applied,

$$Stress\ in\ the\ equivalent\ bar = \frac{W}{\sum A}$$

$$Strain\ in\ the\ equivalent\ bar = \frac{W}{E_c.\sum A} = \frac{x}{L}$$

$$common\ extension \quad x = \frac{W.L}{E_c.\sum A} \qquad ....(7)$$

=extension of single bar

3. <u>Compound bars subjected to temperature change</u>

When a material is subjected to a change in temperature its length will change by an amount

$$a.L.\Delta T$$

where a is the coefficient of linear expansion for the material, L is the original length and ΔT the temperature change. (An increase in temperature produces an increase in length and a decrease in temperature a decrease in length except in very special cases of materials with zero or negative coefficients of expansion which need not be considered here.) If, however, the free expansion of the material is prevented by some external force, then a stress is set up in the material. This stress is equal in magnitude to that which would be produced in the bar by initially allowing the free change of length and then applying sufficient force to return the bar to its original length.

Now:

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$$Change\ in\quad Length = a.L.\Delta T$$

$$Strain = \frac{a.L.\Delta T}{L} = a.\Delta T$$

Therefore, the stress created in the material by the application of sufficient force to remove this strain

$$= strain\ x\ E = E.a.\Delta T$$

Consider now a compound bar constructed from two different materials rigidly joined together as shown in Figure (2) and Figure (3a). For simplicity of description consider that the materials in this case are steel and brass.

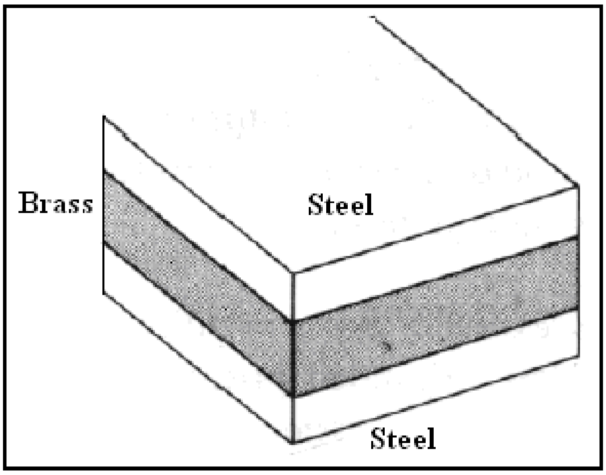

Figure (2)

In general, the coefficients of expansion of the two materials forming the compound bar will be different so that as the temperature rises each material will attempt to expand by different amounts. Figure (3b) shows the positions to which the individual materials will extend if they are completely free to expand (i.e. not joined rigidly together as a compound bar). The extension of any length L is given by $a.L.\Delta T$

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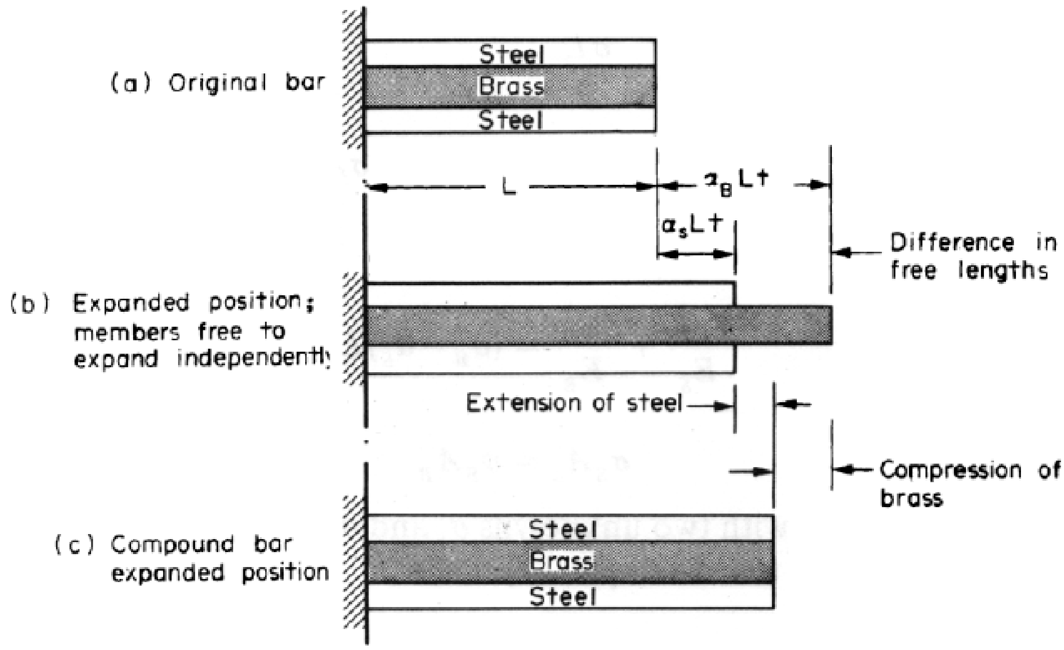

Figure (3) Thermal expansion of compound bar.

Thus the difference of "free" expansion lengths or so-called free lengths

$$= a_B.L.\Delta T - a_s.L.\Delta T = (a_B - a_s).L.\Delta T$$

since in this case the coefficient of expansion of the brass (a_B) is greater than that for the steel (a_s). The initial lengths L of the two materials are assumed equal. If the two materials are now rigidly joined as a compound bar and subjected to the same temperature rise, each material will attempt to expand to its free length position but each will be affected by the movement of the other, The higher coefficient of expansion material (brass) will therefore seek to pull the steel up to its free length position and conversely the lower position. In practice a compromise is reached, the compound bar extending to the position shown in Figure (3c), resulting in an effective compression of the brass from its free length position and an effective extension of the steel from its free length position. From the diagram it will be seen that the following rule holds.

Rule 1

(Extension of steel + compression of brass = difference in “free” lengths).

Referring to the bars in their free expanded positions the rule may be written as

(Extension of “short” member + compression of“1ong” member = difference in free lengths).

Applying Newton’s law of equal action and reaction the following second rule also applies.

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Rule 2

The tensile force applied to the short member by the long member is equal in magnitude to the compressive force applied to the long member by the short member.
Thus, in this case,

tensile force in steel = compressive force in brass

Now, from the definition of Young's modulus

$$E = \frac{stress}{strain} = \frac{s}{\Delta L / L}$$

where D**L** is the change in length.

$$\Delta L = \frac{s.L}{E}$$

Also,

force = stress **x** area = $s.\boldsymbol{A}$

where: A is the cross-sectional area, Therefore Rule 1 becomes

$$\frac{s_s.L}{E_s} + \frac{s_B.L}{E_B} = (a_B - a_s).L.\Delta T \quad ....(8)$$

and Rule 2 becomes

$$s_s.A_s = s_B.A_B \quad ....(9)$$

4. Compound bar (tube and rod)

Consider now the case of a hollow tube with washers or endplates at each end and a central threaded rod as shown in Figure (4) At first sight there would seem to be no connection with the work of the previous section, yet, in fact, the method of solution to determine the stresses set up in the tube and rod when one nut is tightened.
The compound bar which is formed after assembly of the tube and rod, i.e. with the nuts tightened, is shown in Figure (**4c),** the rod being in a state of tension and the tube in compression. Once again Rule 2 applies, i.e.

compressive force in tube = tensile force in rod

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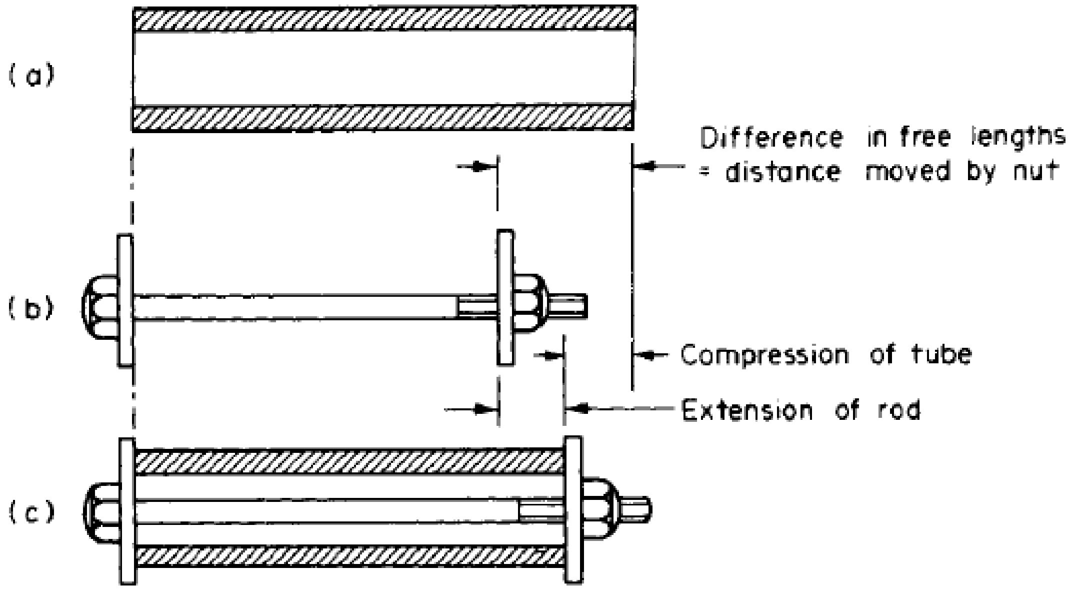

Figure (4)

Figure **(4a)** and b show, *diagrammatically*, the effective positions of the tube and rod before the nut is tightened and the two components are combined. As the nut is turned there is a simultaneous compression of the tube and tension of the rod leading to the final state shown in Figure (4c) **.** As before, however, the diagram shows that Rule **1** applies:

compression of tube +extension of rod = difference in free lengths
= axial advance of nut

i.e. the axial movement of the nut (= number of turns n x threads per metre) is taken up by combined compression of the tube and extension of the rod.
Thus, with suffix (t) for tube and (R) for rod,

$$\frac{\sigma_t L}{E_t} + \frac{\sigma_R L}{E_R} = n \times \text{threads/metre} \qquad \text{...... (10)}$$

$$\sigma_R A_R = \sigma_t A_t \qquad \text{....... (11)}$$

If the tube and rod are now subjected to a change of temperature they may be treated as a normal compound bar and Rules **1** and **2** again apply
Figure (5),

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$$\frac{\sigma'_t L}{E_t} + \frac{\sigma'_R L}{E_R} = (\alpha_t - \alpha_R) L t \qquad \text{...... (12)}$$

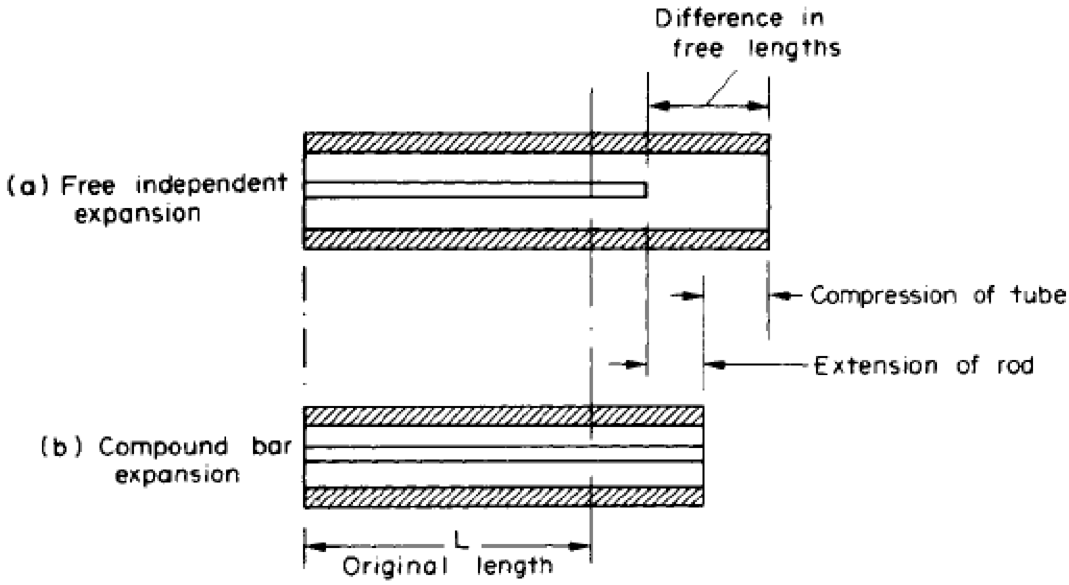

Figure (5)

Where (s'_R) ; and (s'_R) ; are the stresses in the tube and rod due to temperature change only and (a_t), is assumed greater than (a_R). If the latter is not the case the two terms inside the final bracket should be interchanged.

Also

$$\sigma'_R A_R = \sigma'_t A_t$$

Examples

Example 1

(a) A compound bar consists of four brass wires of 2.5 mm diameter and one steel wire of 1.5 mm diameter. Determine the stresses in each of the wires when the bar supports a load of 500 N. Assume all of the wires are of equal lengths.

(b) Calculate the "equivalent" or "combined modulus for the compound bar and determine its total extension if it is initially 0.75 m long. Hence check the values of the stresses obtained in part (a).

For brass E = **100** GN/m^2 and for steel E = 200 GN/m^2.

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Solution

(a) the force in the steel wire is given by

$$F_s = \frac{E_s A_s}{\Sigma EA} W$$

$$= \left[\frac{200 \times 10^9 \times \frac{\pi}{4} \times 1.5^2 \times 10^{-6}}{200 \times 10^9 \times \frac{\pi}{4} \times 1.5^2 \times 10^{-6} + 4(100 \times 10^9 \times \frac{\pi}{4} \times 2.5^2 \times 10^{-6})}\right] 500$$

$$= \left[\frac{2 \times 1.5^2}{(2 \times 1.5^2) + (4 \times 2.5^2)}\right] 500 = 76.27 \text{ N}$$

$\therefore$ total force in brass wires $= 500 - 76.27 = 423.73$ N

$$\therefore \quad \text{stress in steel} = \frac{\text{load}}{\text{area}} = \frac{76.27}{\frac{\pi}{4} \times 1.5^2 \times 10^{-6}} = \mathbf{43.2\ MN/m^2}$$

and

$$\text{stress in brass} = \frac{\text{load}}{\text{area}} = \frac{423.73}{4 \times \frac{\pi}{4} \times 2.5^2 \times 10^{-6}} = \mathbf{21.6\ MN/m^2}$$

(b) From eqn. (2.6)

$$\text{combined } E = \frac{\Sigma EA}{\Sigma A} = \frac{200 \times 10^9 \times \frac{\pi}{4} \times 1.5^2 \times 10^{-6} + 4(100 \times 10^9 \times \frac{\pi}{4} \times 2.5^2 \times 10^{-6})}{\frac{\pi}{4}(1.5^2 + 4 \times 2.5^2)^{10-6}}$$

$$= \frac{(200 \times 1.5^2 + 400 \times 2.5^2)}{(1.5^2 + 4 \times 2.5^2)} 10^9 = \mathbf{108.26\ GN/m^2}$$

Now $E = \dfrac{\text{stress}}{\text{strain}}$

and the stress in the equivalent bar

$$= \frac{500}{\Sigma A} = \frac{500}{\frac{\pi}{4}(1.5^2 + 4 \times 2.5^2) 10^{-6}} = 23.36 \text{ MN/m}^2$$

$$\therefore \quad \text{strain in the equivalent bar} = \frac{\text{stress}}{E} = \frac{23.36 \times 10^6}{108.26 \times 10^9} = 0.216 \times 10^{-3}$$

$\therefore$ common extension = strain × original length

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$$= 0.216 \times 10^{-3} \times 0.75 = 0.162 \times 10^{-3}$$
$$= \mathbf{0.162\ mm}$$

This is also the extension of any single bar, giving a strain in any bar

$$= \frac{0.162 \times 10^{-3}}{0.75} = 0.216 \times 10^{-3} \text{ as above}$$

$\therefore$ stress in steel $= \text{strain} \times E_s = 0.216 \times 10^{-3} \times 200 \times 10^9$
$$= \mathbf{43.2\ MN/m^2}$$

and stress in brass $= \text{strain} \times E_B = 0.216 \times 10^{-3} \times 100 \times 10^9$
$$= \mathbf{21.6\ MN/m^2}$$

These are the same values as obtained in part (a).

Example 2

(a) A compound bar is constructed from three bars 50 mm wide by 12 mm thick fastened together to form a bar 50 mm wide by 36 mm thick. The middle bar is of aluminium alloy for which E = 70 GN/m^2 and the outside bars are of brass with E = 100 GN/m^2. If the bars are initially fastened at 18°C and the temperature of the whole assembly is then raised to 50^{o}C, determine the stresses set up in the brass and the aluminium.

$$a_B = 18 \text{ x per } ^oC \text{ and } a_A = 22 \text{ x per } ^oC$$

(b) What will be the changes in these stresses if an external compressive load of 15 kN is applied to the compound bar at the higher temperature?

Solution

With any problem of this type it is convenient to let the stress in one of the component members or materials, e.g. the brass, be **x.**

Then, since

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$$\text{force in brass} = \text{force in aluminium}$$

and $$\text{force} = \text{stress} \times \text{area}$$

$$x \times 2 \times 50 \times 12 \times 10^{-6} = \sigma_A \times 50 \times 12 \times 10^{-6}$$

i.e. $$\text{stress in aluminium } \sigma_A = 2x$$

Now, from eqn. (2.8),

$$\text{extension of brass} + \text{compression of aluminium} = \text{difference in free lengths}$$

$$= (\alpha_A - \alpha_B)(T_2 - T_1)L$$

$$\frac{xL}{100 \times 10^9} + \frac{2xL}{70 \times 10^9} = (22 - 18)10^{-6}(50 - 18)L$$

$$\frac{(7x + 20x)}{700 \times 10^9} = 4 \times 10^{-6} \times 32$$

$$27x = 4 \times 10^{-6} \times 32 \times 700 \times 10^9$$

$$\mathbf{x = 3.32\ MN/m^2}$$

The stress in the brass is thus **3.32 MN/m² (tensile)** and the stress in the aluminium is $2 \times 3.32 =$ **6.64 MN/m² (compressive).**

(b) With an external load of 15 kN applied each member will take a proportion of the total load given

$$\text{Force in aluminium} = \frac{E_A A_A}{\Sigma EA} W$$

$$= \left[\frac{70 \times 10^9 \times 50 \times 12 \times 10^{-6}}{(70 \times 50 \times 12 + 2 \times 100 \times 50 \times 12)10^9 \times 10^{-6}}\right] 15 \times 10^3$$

$$= \left[\frac{70}{(70 + 200)}\right] 15 \times 10^3$$

$$= 3.89\ \text{kN}$$

$$\text{force in brass} = 15 - 3.89 = 11.11\ \text{kN}$$

$$\text{stress in brass} = \frac{\text{load}}{\text{area}} = \frac{11.11 \times 10^3}{2 \times 50 \times 12 \times 10^{-6}}$$

$$= \mathbf{9.26\ MN/m^2\ (compressive)}$$

These stresses represent the changes in the stresses owing to the applied load. The total or resultant stresses owing to combined applied loading plus temperature effects are, therefore,

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$$\text{stress in aluminium} = -6.64 - 6.5 = -13.14 \text{ MN/m}^2$$
$$= \mathbf{13.14 \text{ MN/m}^2 \text{ (compressive)}}$$
$$\text{stress in brass} = +3.32 - 9.26 = -5.94 \text{ MN/m}^2$$
$$= \mathbf{5.94 \text{ MN/m}^2 \text{ (compressive)}}$$

Example 3

A **25** mm diameter steel rod passes concentrically through a bronze tube 400 mm long, 50 mm external diameter and **40** mm internal diameter. The ends of the steel rod are threaded and provided with nuts and washers which are adjusted initially so that there is no end play at **20°C.**

(a) Assuming that there is no change in the thickness of the washers, find the stress produced in the steel and bronze when one of the nuts is tightened by giving it one tenth of a turn, the pitch of the thread being **2.5** mm.

(b) If the temperature of the steel and bronze is then raised to **50°C** find the changes that will occur in the stresses in both materials.

The coefficient of linear expansion per ^{o}C is 11 x 10^{-6} for steel and for bronze and 18 x10^{-6}. E for steel = **200** GN/m^2. E for bronze = **100** GN/m^2.

Solution

(a) Let **x** be the stress in the tube resulting from the tightening of the nut and s_R the stress in the rod

Then, from eqn. (11),

$$\text{force (stress} \times \text{area) in tube} = \text{force (stress} \times \text{area) in rod}$$

$$x \times \tfrac{\pi}{4}(50^2 - 40^2)10^{-6} = \sigma_R \times \tfrac{\pi}{4} \times 25^2 \times 10^{-6}$$

$$\sigma_R = \frac{(50^2 - 40^2)}{25^2} x = 1.44x$$

And since compression of tube + extension of rod = axial advance of nut, from eqn. (10),

$$\frac{x \times 400 \times 10^{-3}}{100 \times 10^9} + \frac{1.44x \times 400 \times 10^{-3}}{200 \times 10^9} = \frac{1}{10} \times 2.5 \times 10^{-3}$$

$$400 \frac{(2x + 1.44x)}{200 \times 10^9} 10^{-3} = 2.5 \times 10^{-4}$$

$$\therefore \quad 6.88x = 2.5 \times 10^8$$

$$x = 36.3 \text{ MN/m}^2$$

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The stress in the tube is thus **36.3 MN/m²** **(compressive)** and the stress in the rod is $1.44 \times 36.3 =$ **52.3 MN/m² (tensile)**.

(b) Let p be the stress in the tube resulting from temperature change. The relationship between the stresses in the tube and the rod will remain as in part (a) so that the stress in the rod is then $1.44p$. In this case, if free expansion were allowed in the independent members, the bronze tube would expand more than the steel rod and from eqn. (8)

$$\text{compression of tube} + \text{extension of rod} = \text{difference in free length}$$

$$\therefore \qquad \frac{pL}{100 \times 10^9} + \frac{1.44pL}{200 \times 10^9} = (\alpha_B - \alpha_S)(T_2 - T_1)L$$

$$\frac{(2p + 1.44p)}{200 \times 10^9} = (18 - 11)10^{-6}(50 - 20)$$

$$3.44p = 7 \times 10^{-6} \times 30 \times 200 \times 10^9$$

$$p = 12.21 \text{ MN/m}^2$$

and

$$1.44p = 17.6 \text{ MN/m}^2$$

The changes in the stresses resulting from the temperature effects are thus 12.2 MN/m² (compressive) in the tube and 17.6 MN/m² (tensile) in the rod.

The final, resultant, stresses are thus:

$$\text{stress in tube} = -36.3 - 12.2 = \mathbf{48.5 \text{ MN/m}^2 \text{ (compressive)}}$$

$$\text{stress in rod} = 52.3 + 17.6 = \mathbf{69.9 \text{ MN/m}^2 \text{ (tensile)}}$$

Problems

1 (A). A power transmission cable consists of ten copper wires each of 1.6 mm diameter surrounding three steel wires each of 3 mm diameter. Determine the combined E for the compound cable and hence determine the extension of a 30 m length of the cable when it is being laid with a tension of 2 kN.
For steel, $E = 200$ GN/m²; for copper, $E = 100$ GN/m². [151.3 GN/m²; 9.6 mm.]

2 (A). If the maximum stress allowed in the copper of the cable of problem 1 is 60 MN/m², determine the maximum tension which the cable can support. [3.75 kN.]

3 (A). What will be the stress induced in a steel bar when it is heated from 15°C to 60°C, all expansion being prevented?
For mild steel, $E = 210$ GN/m² and $\alpha = 11 \times 10^{-6}$ per °C. [104 MN/m²°.]

4 (A). A 75 mm diameter compound bar is constructed by shrinking a circular brass bush onto the outside of a 50 mm diameter solid steel rod. If the compound bar is then subjected to an axial compressive load of 160 kN determine the load carried by the steel rod and the brass bush and the compressive stress set up in each material.
For steel, $E = 210$ GN/m²; for brass, $E = 100$ GN/m². [I. Struct. E.] [100.3, 59.7 kN; 51.1, 24.3 MN/m².]

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SHEARING FORCE AND BENDING MOMENT DIAGRAMS

DEFINITION OF A BEAM

A bar subject to forces or couples that lie in a plane containing the longitudinal axis of the bar is called a *beam*. The forces are understood to act perpendicular to the longitudinal axis.

1. Types of Beams

CANTILEVER BEAMS

If a beam is supported at only one end and in such a manner that the axis of the beam cannot rotate at that point, it is called a *cantilever beam*. This type of beam is illustrated in Fig. 1. The left end of the bar is free to deflect but the right end is rigidly clamped. The right end is usually said to be "restrained." The reaction of the supporting wall at the right upon the beam consists of a vertical force together with a couple acting in the plane of the applied loads shown.

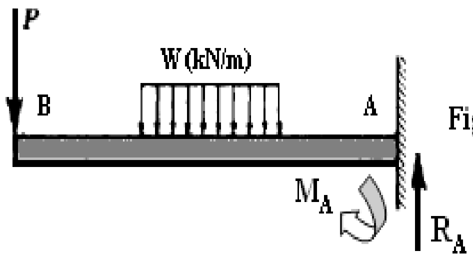

Figure (1)

SIMPLE BEAMS

A beam that is freely supported at both ends is called a *simple beam*. The term "freely supported" implies that the end supports are capable of exerting only forces upon the bar and are not capable of exerting any moments. Thus there is no restraint offered to the angular rotation of the ends of the bar at the supports as the bar deflects under the loads. Two simple beams are sketched in Fig. 2.

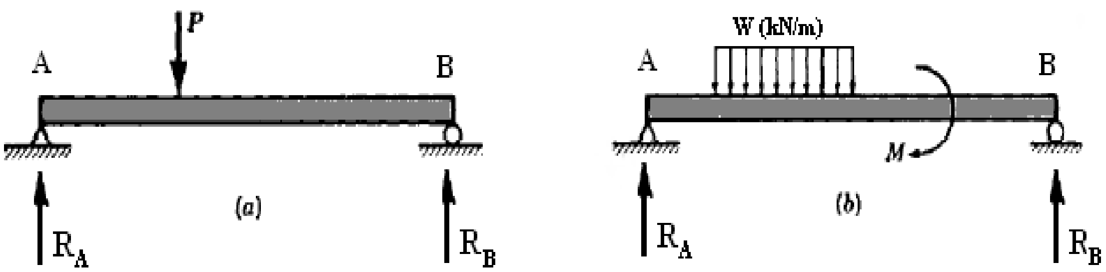

Figure (2)

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OVERHANGING BEAMS

A beam freely supported at two points and having one or both ends extending beyond these supports is termed an *overhanging beam*. Two examples are given in Fig. 3.

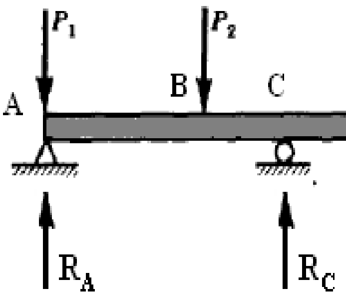
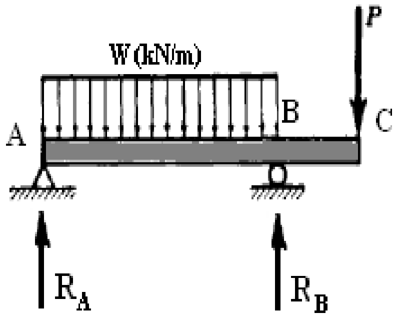


Figure (3)



2. Shearing force and bending moment

At every section in a beam carrying transverse loads there will be resultant forces on either side of the section which, for equilibrium, must be equal and opposite, and whose combined action tends to shear the section in one of the two ways shown in Figure (4 **a** and **b**). The shearing force (S.F.) at the section is defined therefore as the algebraic sum of the forces taken on one side of the section.

2.1. Shearing force (S.F.) sign convention

Forces upwards to the left of a section or downwards to the right of the section are positive. Thus Figure (4a) shows a positive S.F. system at X-X and Figure (4b) shows a negative S.F. system.

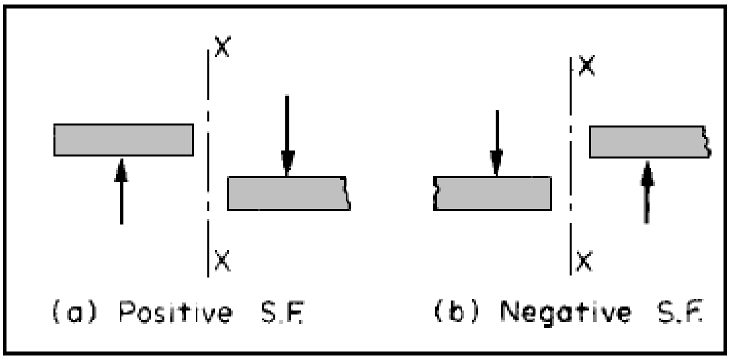


Figure (4) S.F. sign convention

2.2. Bending moment (B.M.) sign convention

Clockwise moments to the left and counterclockwise to the right are positive. Thus Figure (5a) shows a positive bending moment system resulting in sagging of the beam at X-X and Figure (5b) illustrates a negative B.M. system with its associated hogging beam.

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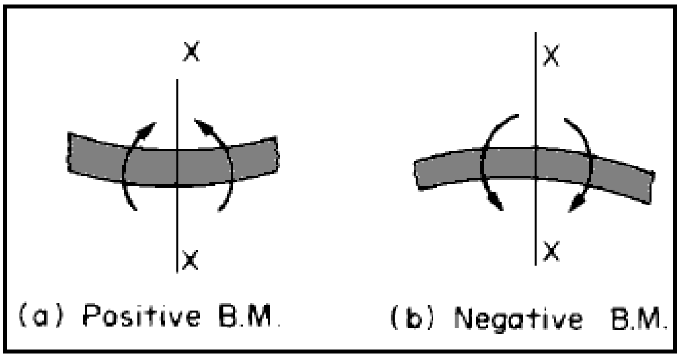

Figure (5) B.M. sign convention.

It should be noted that whilst the above sign conventions for S.F. and B.M. are somewhat arbitrary and could be completely reversed, the systems chosen here are the only ones which yield the mathematically correct signs for slopes and deflections of beams in subsequent work and therefore are highly recommended.

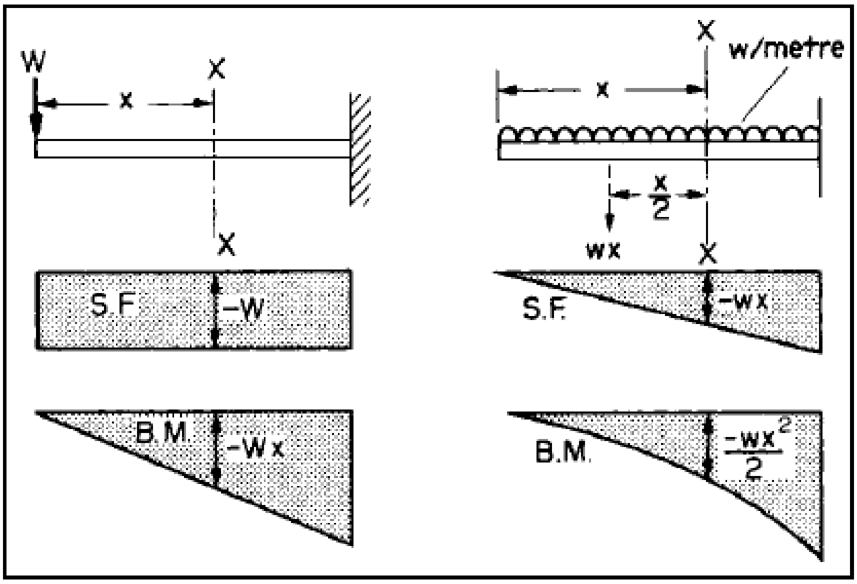

Figure (6) S.F.-B.M. diagrams for standard cases.

Thus in the case of a cantilever carrying a concentrated load (**W**) at the end Figure (6), the S.F. at any section X-X, distance x from the free end, is S.F. = - W. This will be true whatever the value of x, and so the S.F. diagram becomes a rectangle. The **B.M.** at the same section X-X is- **W.x** and this will increase linearly with **x.** The **B.M.** diagram is therefore a triangle. If the cantilever now carries a uniformly distributed load, the S.F. at X-X is the net load to one side of X-X, i.e. -wx. In this case, therefore, the S.F. diagram becomes triangular, increasing to a maximum value of - ***WL*** at the support. The **B.M.** at X-X is obtained by treating the load to the left of X-X as a concentrated load of the same value acting at the centre of gravity,

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$$\text{B.M. at } X\text{–}X = -wx\frac{x}{2} = -\frac{-wx^2}{2}$$

Plotted against x this produces the parabolic **B.M.** diagram shown.

3. S.F. and B.M. diagrams for beams carrying concentrated loads only

In order to illustrate the procedure to be adopted for the determination of S.F. and **B.M.** values for more complicated load conditions, consider the simply supported beam shown in Figure (4) carrying concentrated loads only. (The term ***simply*** supported means that the beam can be assumed to rest on knife-edges or roller supports and is free to bend at the supports without any restraint.)

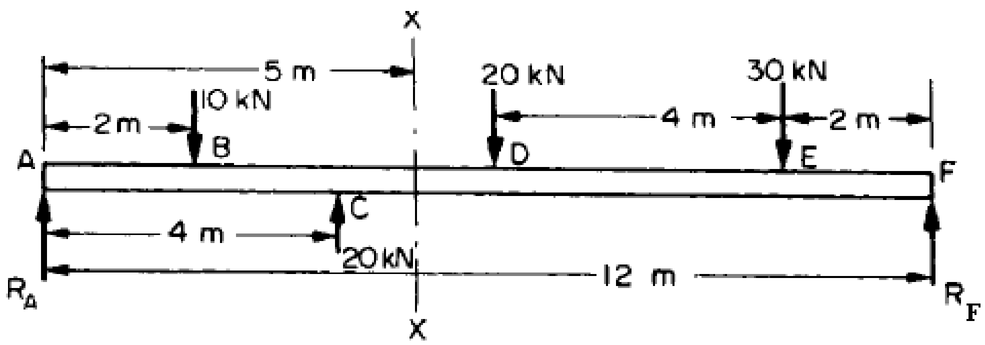

Figure (7)

The values of the reactions at the ends of the beam may be calculated by applying normal equilibrium conditions, i.e. by taking moments about F.
Thus

$$R_A \times 12 = (10 \times 10) + (20 \times 6) + (30 \times 2) - (20 \times 8) = 120$$

R_A = 10 kN
For vertical equilibrium
total force up = total load down
R_A+R_F = 10+20+30-20 = **40**
R_F= 30 kN

At this stage it is advisable to check the value of RF by taking moments about A. Summing up the forces on either side of X-X we have the result shown in Figure (8) Using the sign convention listed above, the shear force at X-X is therefore +20kN,i.e the resultant force at X-X tending to shear the beam is 20 kN.

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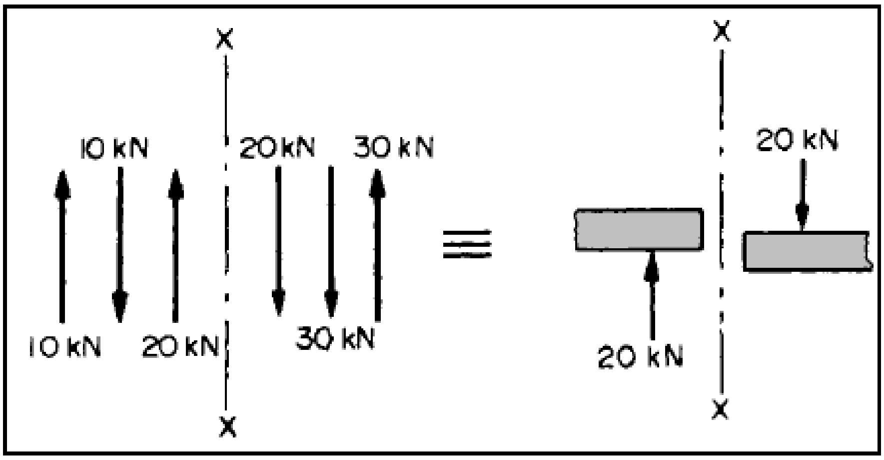

Figure (8) Total S.F. at X-X.

Similarly, Figure (9) shows the summation of the moments of the forces at ***X-X*,** the resultant **B.M.** being **40** kNm.

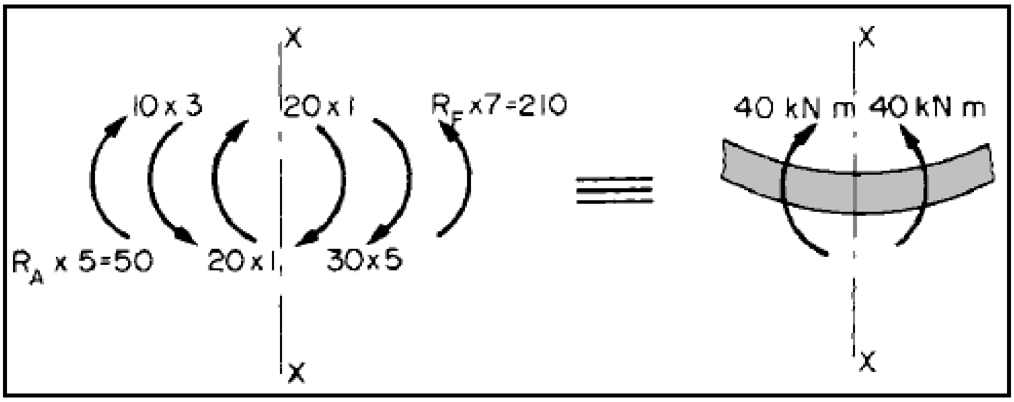

Figure (9)

In practice only one side of the section is normally considered and the summations involved can often be completed by mental arithmetic. The complete S.F. and **B.M.** diagrams for the beam are shown in Figure (9).

B.M. at A = o
B.M. at B = + (10 x 2) = +20 kN.m
B.M. at C= +(l0 x 4)-(10 x 2) = +20kN.m
B.M. at *D* = +(l0 x 6)+ (20 x 2)- (10 x 4) = +60 kN.m
B.M. at E = + (30 x 2) = +60 kN.m
B.M. at F = 0

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All the above values have been calculated from the moments of the forces to the left of each section considered except for E where forces to the right of the section are taken

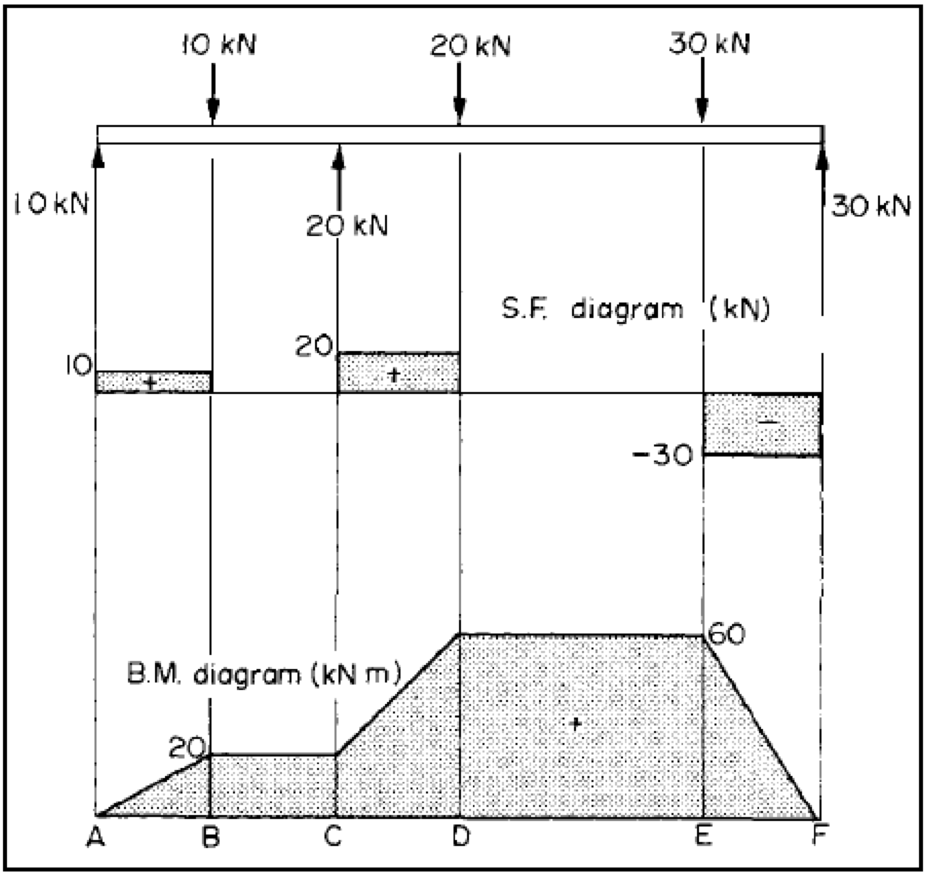

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Figure (10)

It may be observed at this stage that the S.F. diagram can be obtained very quickly when working from the left-hand side, since after plotting the S.F. value at the support all subsequent steps are in the direction of and equal in magnitude to the applied loads, e.g. 10 kN up at A, down 10 kN at B, up 20 kN at ***C,*** etc., with horizontal lines joining the steps to show that the S.F. remains constant between points of application of concentrated loads.

The S.F. and B.M. values at the left-hand support are determined by considering a section an infinitely small distance to the right of the support. The only load to the left (and hence the S.F.) is then the reaction of 10 kN upwards, Le. positive, and the bending moment

= reaction x zero distance = zero.

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The following characteristics of the two diagrams are now evident and will be explained
later in this chapter:
(a) between B and ***C*** the S.F. is zero and the B.M. remains constant;
(b) between A and ***B*** the S.F. is positive and the slope of the B.M. diagram is positive; vice
(c) the difference in B.M. between A and ***B*** = 20 kN m = area of S.F. diagram between A and B.

4. <u>S.F. and B.M. diagrams for uniformly distributed loads</u>
Consider now the simply supported beam shown in Figure (11) carrying a u.d.1. **w** = 25 kN/m across the complete span.

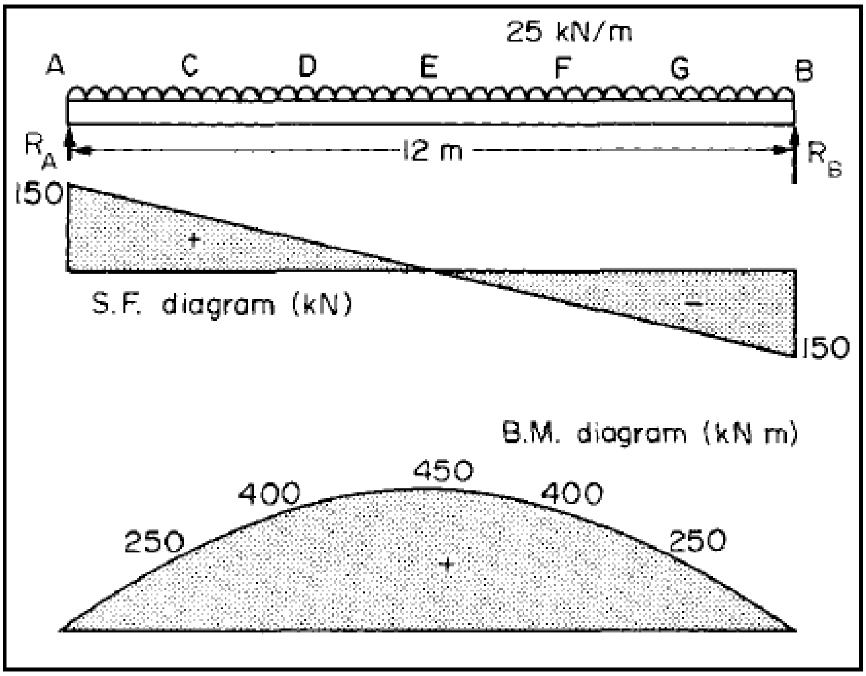

Figure (11)

Here again it is necessary to evaluate the reactions, but in this case the problem is simplified by the symmetry of the beam. Each reaction will therefore take half the applied load,
i.e.

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$$R_A = R_B = \frac{25 \times 12}{2} = 150\,\mathrm{kN}$$

The S.F. at A, using the usual sign convention, is therefore + 150kN.
is, therefore, Consider now the beam divided into six equal parts 2 m long. The S.F. at any other point ***C***

150 - load downwards between A and C
= 150 - (25 **x** 2) = + 100 kN

The whole diagram may be constructed in this way, or much more quickly by noticing that the S.F. at A is + 150 kN and that between A and ***B*** the S.F. decreases uniformly, producing the required sloping straight line, shown in Fig. 3.7. Alternatively, the S.F. at A is + 150 kN and between A and B this decreases gradually by the amount of the applied load (By 25 x 12 = 300kN) to - 150kN at B. When evaluating B.M.'s it is assumed that a u.d.1. can be replaced by a concentrated load of equal value acting at the middle of its spread. When taking moments about ***C,*** therefore, the portion of the u.d.1. between A and ***C*** has an effect equivalent to that of a concentrated load of
25 x 2 = 50 kN acting the centre of ***AC,*** i.e. 1 m from ***C.***

$$\text{B.M. at } C = (R_A \times 2) - (50 \times 1) = 300 - 50 = 250\,\mathrm{kN\,m}$$

Similarly, for moments at ***D*** the u.d.1. on ***AD*** can be replaced by a concentrated load of

$$25 \times 4 = 100\,\mathrm{kN} \text{ at the centre of } AD\text{, i.e. at } C.$$

$$\text{B.M. at } D = (R_A \times 4) - (100 \times 2) = 600 - 200 = 400\,\mathrm{kN\,m}$$

Similarly,

$$\text{B.M. at } E = (R_A \times 6) - (25 \times 6)3 = 900 - 450 = 450\,\mathrm{kN\,m}$$

The B.M. diagram will be symmetrical about the beam centre line; therefore the values of B.M. at F and G will be the same as those at ***D*** and ***C*** respectively. The final diagram is therefore as shown in Figure (11) and is parabolic.

Point (a) of the summary is clearly illustrated here, since the B.M. is a maximum when the S.F. is zero. Again, the reason for this will be shown later.

5. S.F. and B.M. diagrams for combined concentrated and uniformly distributed loads

Consider the beam shown in Figure (12) loaded with a combination of concentrated loads and u.d.1.s.
Taking moments about E

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$$(R_A \times 8) + (40 \times 2) = (10 \times 2 \times 7) + (20 \times 6) + (20 \times 3) + (10 \times 1) + (20 \times 3 \times 1.5)$$

$$8R_A + 80 = 420$$

$$R_A = 42.5\,\text{kN}\,(= \text{S.F. at } A)$$

Now
$$R_A + R_E = (10 \times 2) + 20 + 20 + 10 + (20 \times 3) + 40 = 170$$

$$R_E = 127.5\,\text{kN}$$

Working from the left-hand support it is now possible to construct the S.F. diagram, as indicated previously, by following the direction arrows of the loads. In the case of the u.d.l.'s the S.F. diagram will decrease gradually by the amount of the total load until the end of the u.d.1. or the next concentrated load is reached. Where there is no u.d.1. the S.F. diagram remains horizontal between load points. In order to plot the B.M. diagram the following values must be determined:

B.M. at A $= 0$

B.M. at $B = (42.5 \times 2) - (10 \times 2 \times 1) = 85 - 20 = 65\,\text{kN m}$

B.M. at $C = (42.5 \times 5) - (10 \times 2 \times 4) - (20 \times 3) = 212.5 - 80 - 60 = 72.5\,\text{kN m}$

B.M. at $D = (42.5 \times 7) - (10 \times 2 \times 6) - (20 \times 5) - (20 \times 2)$
$- (20 \times 2 \times 1) = 297.5 - 120 - 100 - 40 - 40 = 297.5 - 300 = -2.5\,\text{kN m}$

B.M. at $E = (-40 \times 2)$ working from r.h.s. $= -80\,\text{kN m}$

B.M. at F $= 0$

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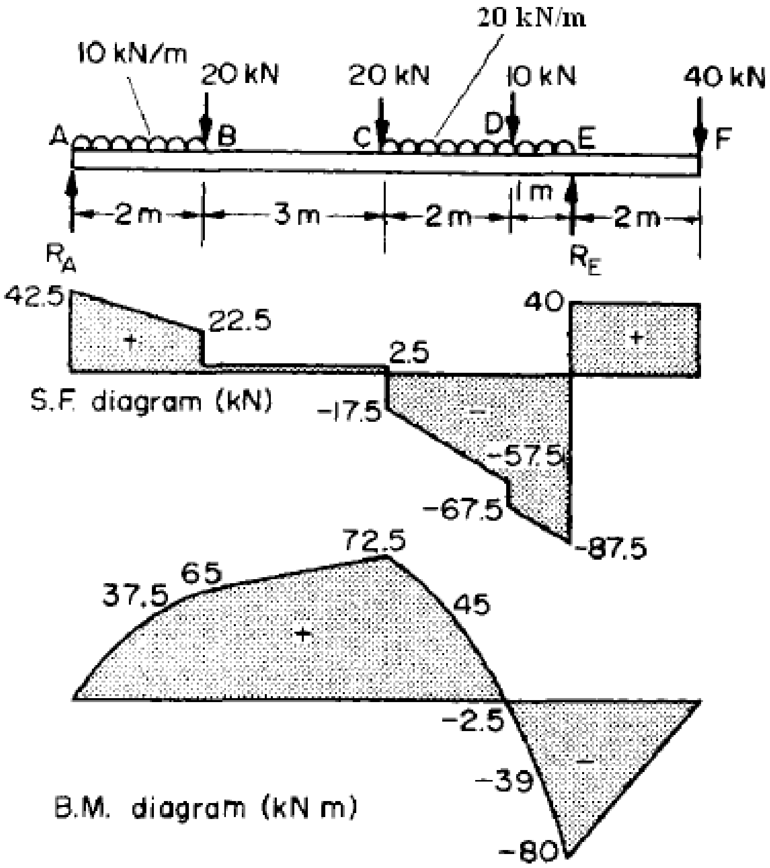

Figure (12)

For complete accuracy one or two intermediate values should be obtained along each u.d.l. portion of the beam,

e.g. B.M. midway between A and $B = (42.5 \times 1) - (10 \times 1 \times \frac{1}{2})$

$= 42.5 - 5 = 37.5\,\text{kN m}$

Similarly, B.M. midway between C and $D = 45\,\text{kN m}$

B.M. midway between D and $E = -39\,\text{kN m}$

The B.M. and S.F. diagrams are then as shown in Figure (12)

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5. Points of contraflexure

A point of contraflexure is a point where the curvature of the beam changes sign. It is sometimes referred to as a point of inflexion and will be shown later to occur at the point, or points, on the beam where the B.M. is zero.

For the beam of Figure **(9**) therefore, it is evident from the **B.M.** diagram that this point lies somewhere between C and D (B.M. at C is positive, B.M. at D is negative). If the required point is a distance **x** from C then at that point

$$\text{B.M.} = (42.5)(5+x) - (10 \times 2)(4+x) - 20(3+x) - 20x - \frac{20x^2}{2}$$

$$= 212.5 + 42.5x - 80 - 20x - 60 - 20x - 20x - 10x^2$$

$$= 72.5 - 17.5x - 10x^2$$

Thus the B.M. is zero where

$$0 = 72.5 - 17.5x - 10x^2$$

i.e. where $x = 1.96$ or -3.7

Since the last answer can be ignored (being outside the beam), the point of contraflexure must be situated at 1.96 m to the right of ***C.***

6. Relationship between shear force Q, bending moment *M* and intensity of loading W (kN/m)

Consider the beam ***AB*** shown in Figure (10) carrying a uniform loading intensity (uniformly distributed load) of W (kN/m). By symmetry, each reaction takes half the total load, i.e., WL/2.

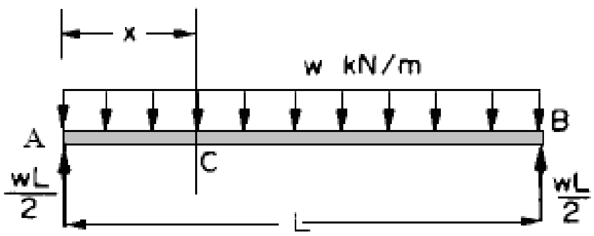

Figure (10)

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The B.M. at any point C, distance x from A, is given by

$$M = \frac{wL}{2}x - (wx)\frac{x}{2}$$

i.e.
$$M = \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2$$

Differentiating,
$$\frac{dM}{dx} = \tfrac{1}{2}wL - wx$$

Now
$$\text{S.F. at } C = \tfrac{1}{2}wL - wx = Q \qquad \text{.....(1)}$$

$\therefore$
$$\frac{dM}{dx} = Q \qquad \text{.....(2)}$$

Differentiating equation (1),

$$\frac{dQ}{dx} = -W \qquad \text{.....(3)}$$

These relationships are the basis of the rules stated in the summary, the proofs of which are as follows:

(a) The maximum or minimum B.M. occurs where $\frac{dM}{dx} = 0$

But $\frac{dM}{dx} = Q$

Thus where S.F. is zero B.M. is a maximum or minimum.

(b) The slope of the B.M. diagram $= \frac{dM}{dx} = Q$

Thus where Q = 0 the slope of the B.M. diagram is zero, and the B.M. is therefore constant.

(c) Also, since Q represents the slope of the B.M. diagram, it follows that where the S.F. is positive the slope of the B.M. diagram is positive, and where the S.F. is negative the slope of the B.M. diagram is also negative.

(d) The area of the S.F. diagram between any two points, from basic calculus, is

$$\int Q\,dx$$

But, $\frac{dM}{dx} = Q$ or $M = \int Q\,dx$

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i.e. the B.M. change between any two points is the area of the S.F. diagram between these points.
This often provides a very quick method of obtaining the B.M. diagram once the S.F. diagram has been drawn.

(e) With the chosen sign convention, when the B.M. is positive the beam is sagging and when it is negative the beam is hogging. Thus when the curvature of the beam changes from sagging to hogging, as at x-x in Figure (11), or vice versa, the B.M. changes sign, i.e. becomes instantaneously zero. This is termed a point of inflexion or contra flexure. Thus a point of contra flexure occurs where the B.M. is zero.

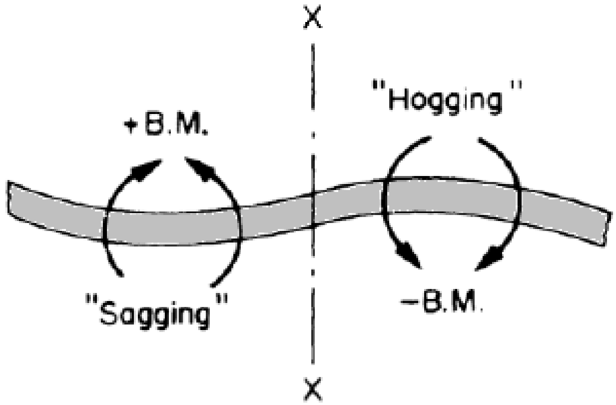

Figure (11) Beam with point contraflexure at X-X

7. S.F. and B.M. diagrams for an applied couple or moment

In general there are two ways in which the couple or moment can be applied: (a) with horizontal loads and (b) with vertical loads, and the method of solution is different for each.

Type (a): couple or moment applied with horizontal loads

Consider the beam AB shown in Figure (12) to which a moment (F.d) is applied by means of horizontal loads at a point C, distance a from A.

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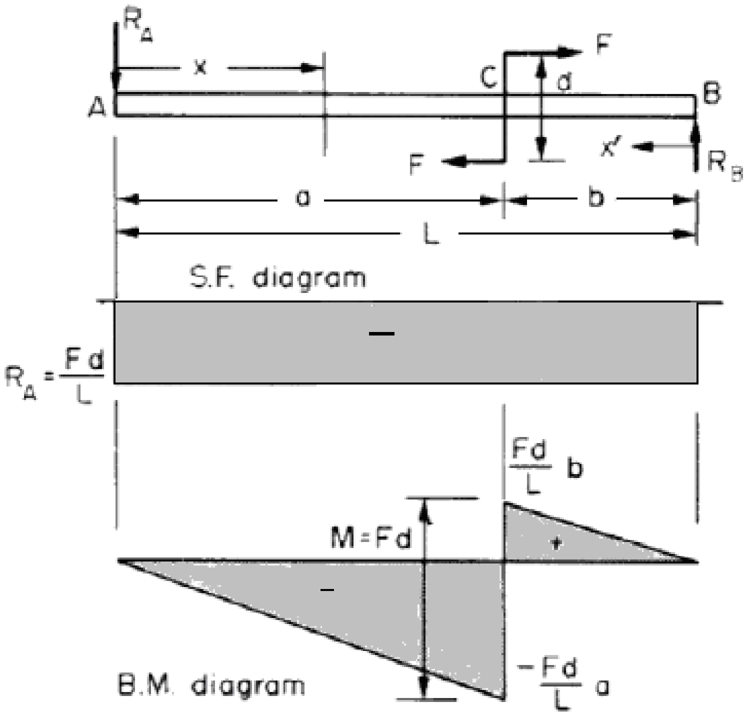

Figure (12)

Since this will tend to lift the beam at A, R_A acts downwards.

Moments about B: $R_A.L = F.d$, $R_A = \frac{F.d}{L}$

and for vertical equilibrium $R_B = R_A = \frac{F.d}{L}$

The S.F. diagram can now be drawn as the horizontal loads have no effect on the vertical shear.

The B.M. at any section between A and C is

$$M = -R_A.x = \frac{-F.d}{L}.x$$

Thus the value of the B.M. increases linearly from zero at A to $\frac{-F.d}{L}.a$ at C

Similarly, the B.M. at any section between C and B is

$$M = -R_A.x + F.d = R_B.x' = \frac{-F.d}{L}.x$$

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i.e. the value of the B.M. again increases linearly from zero at B to - b at C. The B.M. diagram is therefore as shown in Figure (12).

Type (b): <u>moment applied with vertical loads</u>

Consider the beam AB shown in Figure (13); taking moments about B:

$$R_A L = F(d+b)$$

$$\therefore \qquad R_A = \frac{F(d+b)}{L}$$

$$\text{Similarly,} \qquad R_B = \frac{F(a-d)}{L}$$

The S.F. diagram can therefore be drawn as in Figure **(**13**)** and it will be observed that in this case (F) does affect the diagram. For the B.M. diagram an equivalent system is used. The offset load ***F*** is replaced by a moment and a force acting at C, as shown in Figure (13). Thus

B.M. between A and $C = R_A x = \dfrac{F(d+b)}{L} x$

i.e. increasing linearly from zero to $\dfrac{F(d+b)}{L} a$ at C.

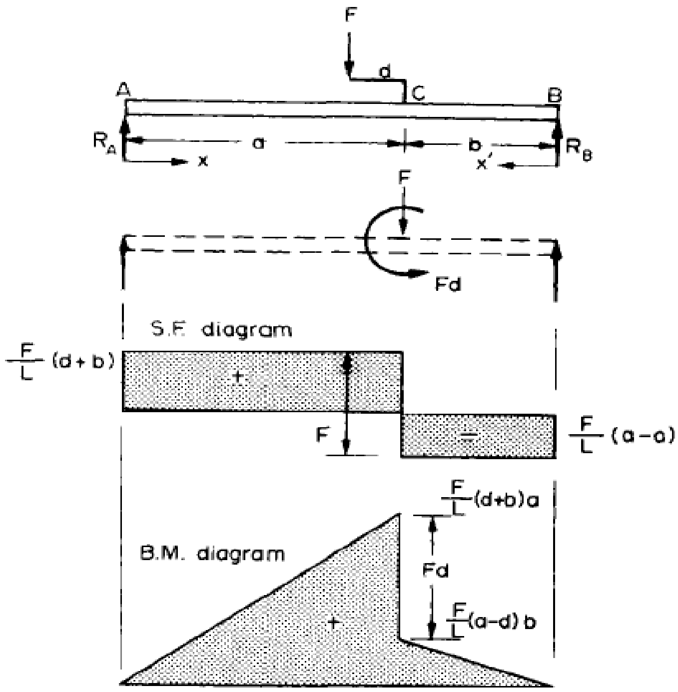

Figure (13)

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B.M. between C and $B = R_B x' = \frac{F(a-d)}{L} x'$

i.e. increasing linearly from zero to $\frac{F(a-d)}{L} b$ at C.

Examples

Example 1

Draw the S.F. and **B.M.** diagrams for the beam loaded as shown in Figure (14), and determine(a) the position and magnitude of the maximum **B.M.,** and (b) the position of any point of contraflexure.

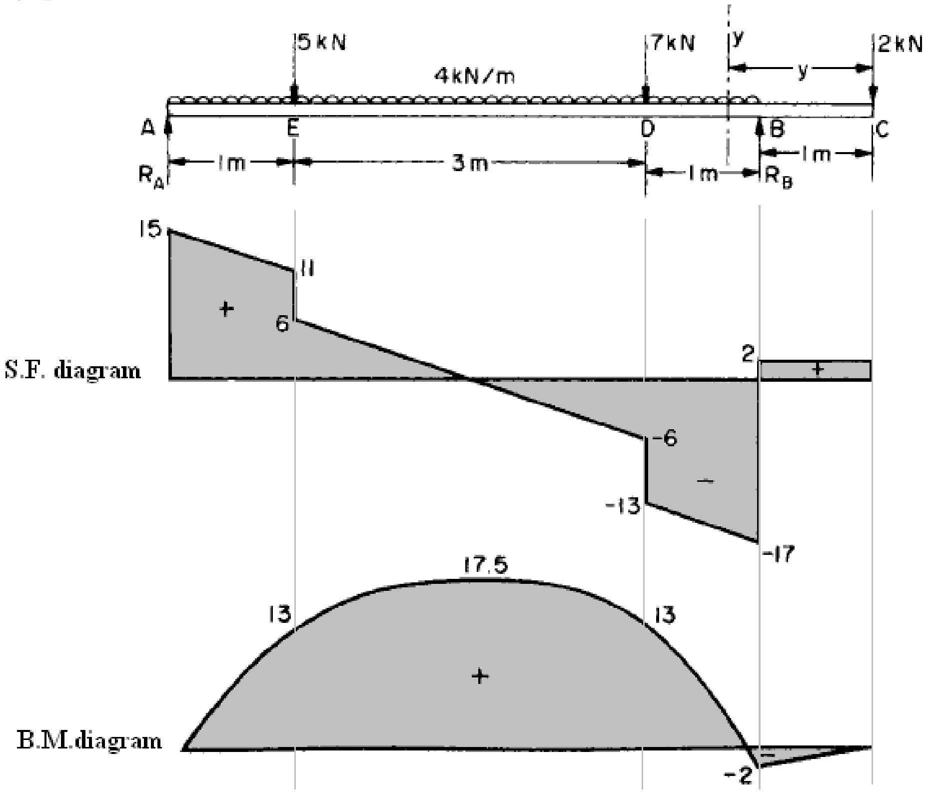

Figure (14)

Solution

Taking the moments about A,

$$5R_B = (5 \times 1) + (7 \times 4) + (2 \times 6) + (4 \times 5) \times 2.5$$

$$\therefore \quad R_B = \frac{5 + 28 + 12 + 50}{5} = \mathbf{19\,kN}$$

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$$R_A + R_B = 5 + 7 + 2 + (4 \times 5) = 34$$
$$R_A = 34 - 19 = \mathbf{15\,kN}$$

The S.F. diagram may now be constructed as shown in Figure (14) .

Calculation of bending moments

B.M. at A and $C = 0$

B.M. at B $= -2 \times 1 = -2$ kN.m

B.M. at D $= -(2 \times 2) + (19 \times 1) - (4 \times 1 \times \frac{1}{2}) = +13$ kN.m

B.M. at E $= +(15 \times 1) - (4 \times 1 \times \frac{1}{2}) = +13$ kN.m

The maximum B.M. will be given by the point (or points) at which dM/dx (Le. the shear force) is zero. By inspection of the S.F. diagram this occurs midway between D and E, i.e. at1.5 m from E.

$$\text{B.M. at this point} = (2.5 \times 15) - (5 \times 1.5) - \left(4 \times 2.5 \times \frac{2.5}{2}\right)$$
$$= +\mathbf{17.5}\ \text{kN.m}$$

The B.M. diagram is therefore as shown in Figure (14) Alternatively, the B.M. at any point between D and E at a distance of x from A will be given by

$$M_{xx} = 15x - 5(x-1) - \frac{4x^2}{2} = 10x + 5 - 2x^2$$

The maximum B.M. position is then given where $\frac{dM}{dx} = 0$.

$$\frac{dM}{dx} = 10 - 4x = 0 \qquad \therefore \qquad x = 2.5\,\text{m}$$

1.5 m from *E*, as found previously.

(b) Since the B.M. diagram only crosses the zero axis once there is only one point of contraflexure, i.e. between B and D. Then, B.M. at distance y from C will be given by

$$M_{yy} = -2y + 19(y-1) - 4(y-1)\tfrac{1}{2}(y-1)$$
$$= -2y + 19y - 19 - 2y^2 + 4y - 2 = 0$$

The point of contraflexure occurs where B.M. = 0, i.e. where $M_{yy} = 0$,

$$0 = -2y^2 + 21y - 21$$
$$2y^2 - 21y + 21 = 0$$
$$y = \frac{21 \pm \sqrt{(21^2 - 4 \times 2 \times 21)}}{4} = 1.12\,\text{m}$$

i.e. point of contraflexure occurs 0.12 m to the left of B.

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Example 2

A beam ABC is 9 m long and supported at B and C, **6** m apart as shown in Figure **(15).** The beam carries a triangular distribution of load over the portion ***BC*** together with an applied counterclockwise couple of moment 80 kN m at B and a uniform distributed load (u.d.1.) of 10 kN/m over AB, as shown. Draw the S.F. and B.M. diagrams for the beam.

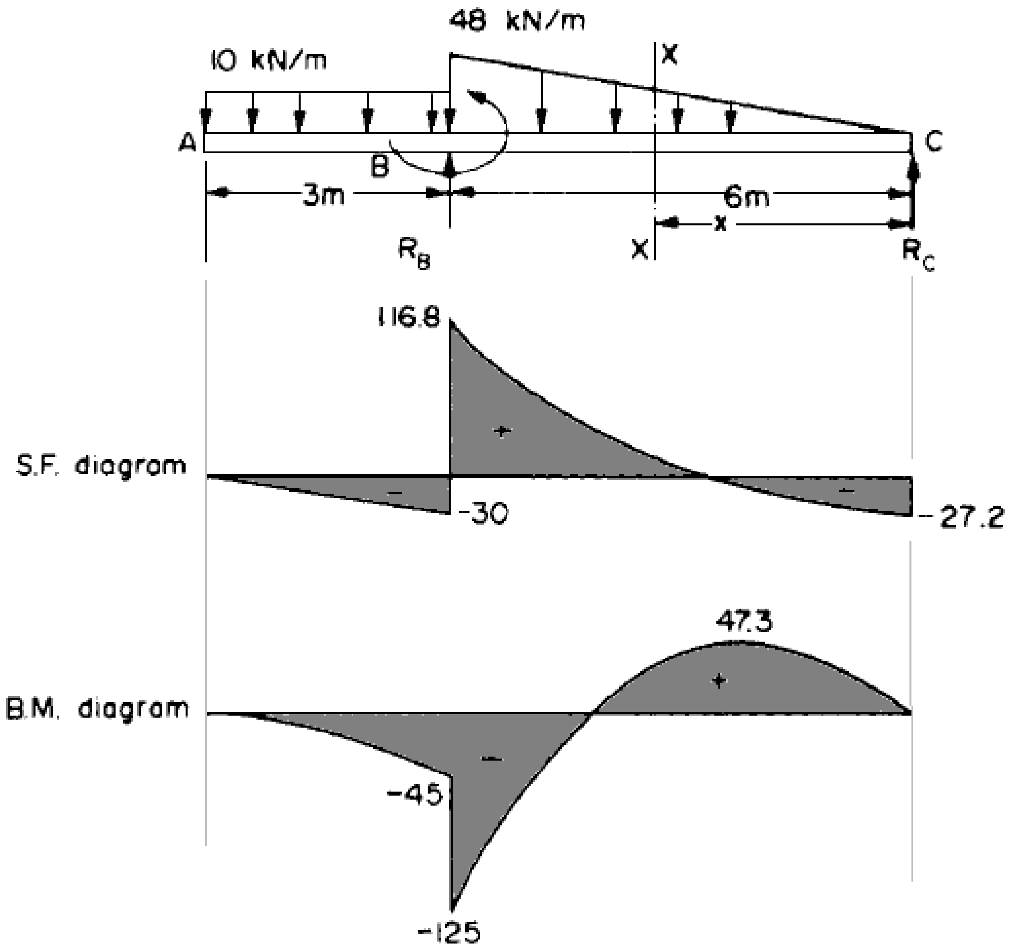

Figure **(15)**

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Solution

Taking moments about B,

$$(R_C \times 6) + (10 \times 3 \times 1.5) + 80 = (\tfrac{1}{2} \times 6 \times 48) \times \tfrac{1}{3} \times 6$$

$$6R_C + 45 + 80 = 288$$

$$R_C = 27.2\,\text{kN}$$

and $\quad R_C + R_B = (10 \times 3) + (\tfrac{1}{2} \times 6 \times 48)$

$$= 30 + 144 = 174$$

$\therefore \quad R_B = 146.8\,\text{kN}$

At any distance x from C between C and B the shear force is given by

$$\text{S.F.}_{XX} = -\tfrac{1}{2}wx + R_C$$

and by proportion $\quad \dfrac{w}{x} = \dfrac{48}{6} = 8$

i.e. $\quad w = 8x\,\text{kN/m}$

$\therefore \quad \text{S.F.}_{XX} = -(R_C - \tfrac{1}{2} \times 8x \times x)$

$$= -R_C + 4x^2$$

$$= -27.2 + 4x^2$$

The S.F. diagram is then as shown in Figure (15)

Also $\quad \text{B.M.}_{XX} = -(\tfrac{1}{2}wx)\dfrac{x}{3} + R_C x$

$$= 27.2x - \frac{4x^3}{3}$$

For a maximum value, $\dfrac{d\,(\text{B.M.})}{dx} = \text{S.F.} = 0$

i.e.,where $\quad 4x^2 = 27.2$

or $\quad x = 2.61\text{ m from } C$

$$\text{B.M.}_{max} = 27.2(2.61) - \frac{4}{3}(2.61)^3$$

$$= 47.3\,\text{kN m}$$

B.M. at A and $C = 0$

B.M. immediately to left of $B = -(10 \times 3 \times 1.5) = -45\,\text{kN m}$

At the point of application of the applied moment there will be a sudden change in B.M. of 80 kN. m. (There will be no such discontinuity in the S.F. diagram; the effect of the moment will merely be reflected in the values calculated for the reactions.) The B.M. diagram is therefore as shown in Figure (15).

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Problems

1. **A** beam AB, 1.2 m long, is simply-supported at its ends A and B and carries two concentrated loads, one of 10 kN at C, the other 15 kN at D. Point C is 0.4 m from A, point D is 1 m from A. Draw the S.F. and B.M. diagrams for the beam inserting principal values.

[9.17, - 0.83, -15.83 kN; 3.67, 3.17 kN.m]

2 . The beam of question (1) carries an additional load of **5** kN upwards at point E, 0.6 m from A. Draw the S.F. and B.M. diagrams for the modified loading. What is the maximum B.M.?

[6.67, -3.33, 1.67, -13.33 kN; 2.67, 2, 2.67 kN.m.]

3. **A** cantilever beam AB, 2.5 m long is rigidly built in at A and carries vertical concentrated loads of 8 kN at B and 12 kN at C, 1 m from A. Draw **S.F.** and **B.M.** diagrams for the beam inserting principal values.

[-8, -20 kN; -11.2, -31.2kN.m]

4. A beam AB, 5 m long, is simply-supported at the end B and at a point C, 1 m from A. It carries vertical loads of 5 kN at A and 20kN at D, the centre of the span BC. Draw S.F. and B.M. diagrams for the beam inserting principal values. [- 5 , 11.25, -8.75kN; - 5 , 17.5 kN.m]

5. A beam AB, 3 m long, is simply-supported at A and E. It carries a 16 kN concentrated load at C, 1.2 m from A, and a u.d.1. of 5 kN/m over the remainder of the beam. Draw the S.F. and B.M. diagrams and determine the value of the maximum B.M.

[12.3, -3.7, -12.7kN; 14.8 kN.m.]

6. A simply supported beam has a span of 4m and carries a uniformly distributed load of 60 kN/m together with a central concentrated load of 40 kN. Draw the S.F. and B.M. diagrams for the beam and hence determine the maximum B.M. acting on the beam.

[S.F. 140, k20, -140 kN; B.M. 0, 160,0 kN.m]

7. A 2 m long cantilever is built-in at the right-hand end and carries a load of 40 kN at the free end. In order to restrict the deflection of the cantilever within reasonable limits an upward load of 10 kN is applied at mid-span. Construct the S.F. and B.M.

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diagrams for the cantilever and hence determine the values of the reaction force and moment at the support. [30 kN, 70 kN. m.]

8. A beam 4.2 m long overhangs each of two simple supports by 0.6 m. The beam carries a uniformly distributed load of 30 kN/m between supports together with concentrated loads of 20 kN and 30 kN at the two ends. Sketch the S.F. and B.M. diagrams for the beam and hence determine the position of any points of contraflexure.
[S.F. -20, +43, -47, +30 kN; B.M. - 12, 18.75, - 18kN.m; 0.313 and 2.553 from left hand support.]

9. A beam ABCDE, with A on the left, is 7 m long and is simply supported at Band E. The lengths of the various portions are AB = 1.5 m, BC = 1.5 m, CD = 1 m and DE = 3 m. There is a uniformly distributed load of 15 kN/m between B and a point 2 m to the right of B and concentrated loads of 20 kN act at A and D with one of 50 kN at C.

(a) Draw the S.F. diagrams and hence determine the position from A at which the S.F. is zero.

(b) Determine the value of the B.M. at this point.

(c) Sketch the B.M. diagram approximately to scale, quoting the principal values.
[3.32 m; 69.8 kN,m; 0, -30, 69.1, 68.1, 0 kN.m]

10. A beam ABCDE is simply supported at A and D. It carries the following loading: a distributed load of 30 kN/m between A and B a concentrated load of 20 kN at B; a concentrated load of 20 kN at C; a concentrated load of 10 kN at E; a distributed load of 60 kN/m between D and E. Span AB = 1.5 m, BC = CD = DE = 1 m. Calculate the value of the reactions at A and D and hence draw the S.F. and B.M. diagrams. What are the magnitude and position of the maximum B.M. on the beam?
[41.1, 113.9kN; 28.15kN.m; 1.37 m from A.]

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TORSION

Simple torsion theory

When a uniform circular shaft is subjected to a torque it can be shown that every section of the shaft is subjected to a state of pure shear Figure (1), the moment of resistance developed by the shear stresses being everywhere equal to the magnitude, and opposite in sense, to the applied torque. For the purposes of deriving a simple theory, to make the following basic assumptions:

(1) The material is homogeneous, i.e. of uniform elastic properties throughout.

(2) The material is elastic, following Hooke's law with shear stress proportional to shear strain.

(3) The stress does not exceed the elastic limit or limit of proportionality.

(4) Circular Sections remain circular.

(5) Cross-sections remain plane. (This is certainly not the case with the torsion of non circular Sections.)

(6) Cross-sections rotate as if rigid, i.e. every diameter rotates through the same angle.

Practical tests carried out on circular shafts have shown that the theory developed below on the basis of these assumptions shows excellent correlation with experimental results.

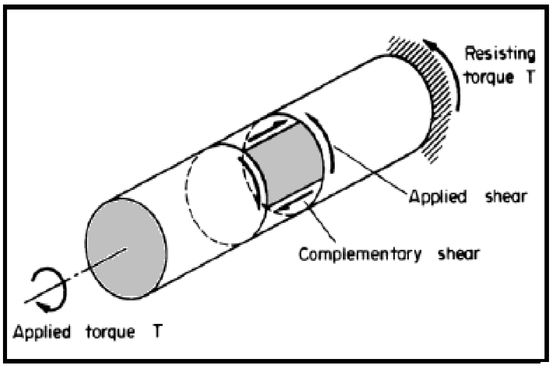

Figure (1) Shear system set up on an element in the surface of a shaft subjected to torsion.

(a) Angle of twist

Consider now the solid circular shaft of radius (R) subjected to a torque (T) at one end, the other end being fixed Figure (2). Under the action of this torque a radial line at the free end of the shaft twists through an angle (θ), point A moves to B, and AB subtends an angle (γ) at the fixed end. This is then the angle of distortion of the shaft, i.e. the shear strain.

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angle in radians = arc /radius

arc $AB = R\theta = L\gamma$

$\therefore \qquad \gamma = R\theta/L \qquad$ ... (1)

From the definition of rigidity modulus

$$G = \frac{\text{shear stress } \tau}{\text{shear strain } \gamma}$$

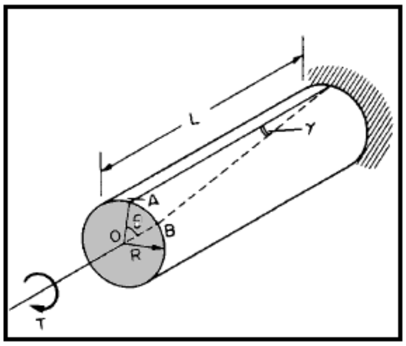

Figure (2)

$$\therefore \quad \gamma = \frac{\tau}{G} \qquad \text{....(2)}$$

where τ is the shear stress set up at radius R.

Therefore equating eqns. (1) and (2),

$$\frac{R\theta}{L} = \frac{\tau}{G}$$

$$\frac{\tau}{R} = \frac{G\theta}{L}\left(= \frac{\tau'}{r}\right) \qquad \text{....(3)}$$

where τ' is the shear stress at any other radius r.

(b) Stresses

Let the cross-section of the shaft be considered as divided into elements of radius r and thickness (dr) as shown in Figure (3) each subjected to a shear stress (τ').

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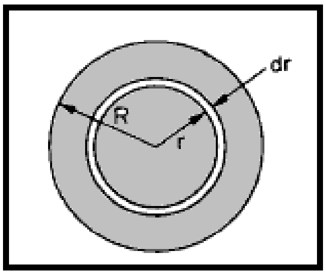

Figure (3) Shaft cross-section

The force set up on each element,

= stress x area= τ' x 2.*p*.*r* dr (approximately)

This force will produce a moment about the centre axis of the shaft, providing a contribution to the torque

= (τ' x 2.*p*.*r* dr).r= τ' x 2.*p*.r^2 dr

The total torque on the section **(T)** will then be the sum of all such contributions across the section,

$$T = \int_0^R 2\pi\tau' r^2 \, dr$$

Now the shear stress (t**'**) will vary with the radius rand must therefore be replaced in terms of ***r*** before the integral is evaluated. From eqnuation (3)

$$\tau' = \frac{G\theta}{L} r$$

$$T = \int_0^R 2\pi \frac{G\theta}{L} r^3 \, dr$$

$$= \frac{G\theta}{L} \int_0^R 2\pi r^3 \, dr$$

The integral $\int_0^R 2.p.r^3.dr$ is called the polar second moment of area (J), and may be evaluated as a

standard form for solid and hollow shafts .

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$$\therefore \quad T = \frac{G\theta}{L} J$$

$$\text{or} \quad \frac{T}{J} = \frac{G\theta}{L} \quad \text{....(4)}$$

Combining eqns. ***(3)*** and ***(4)*** produces the so-called simple theory of torsion:

$$\frac{T}{J} = \frac{\tau}{R} = \frac{G\theta}{L} \quad \text{.....(5)}$$

Polar second moment of area

As stated above the polar second moment of area ***J*** is defined as

$$J = \int_0^R 2\pi r^3 \, dr$$

For a *solid shaft,*

$$J = 2\pi \left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{2\pi R^4}{4} \quad \text{or} \quad \frac{\pi D^4}{32} \quad \text{.... (6)}$$

For a hollow ***shaft*** of internal radius ***r,***

$$J = 2\pi \int_r^R r^3 \, dr = 2\pi \left[\frac{r^4}{4}\right]_r^R$$

$$= \frac{\pi}{2}(R^4 - r^4) \quad \text{or} \quad \frac{\pi}{32}(D^4 - d^4) \quad \text{.... (7)}$$

For thin-walled hollow shafts the values of (D) and (d) may be nearly equal, and in such cases there can be considerable errors in using the above equation involving the difference of two large quantities of similar value. It is therefore convenient to obtain an alternative form of expression for the polar moment of area.

Now

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$$J = \int_0^R 2\pi r^3\,dr = \Sigma(2\pi r\,dr)r^2$$

$$= \Sigma\,Ar^2$$

Where; A = 2πr dr is the area of each small element of Figure (3).

If a thin hollow cylinder is therefore considered as just one of these small elements with its wall thickness t = dr, then

$$J = Ar^2 = (2\pi r\,t)r^2$$

$$= 2\pi r^3 t \text{ (approximately)} \qquad .... (8)$$

Shear stress and shear strain in shafts

The shear stresses which are developed in a shaft subjected to pure torsion are indicated in Figure (1),their values being given by the simple torsion theory as

$$\tau = \frac{G\theta}{L}R$$

Now from the definition of the shear or rigidity modulus (**G),**

$$\tau = G\gamma$$

It therefore follows that the two equations may be combined to relate the shear stress and strain in the shaft to the angle of twist per unit length, thus

$$\tau = \frac{G\theta}{L}R = G\gamma \qquad .... (9)$$

or, in terms of some internal radius ***r,***

$$\tau' = \frac{G\theta}{L}r = G\gamma \qquad .... (10)$$

These equations indicate that the shear stress and shear strain vary linearly with radius and have their maximum value at the outside radius Figure (4) .

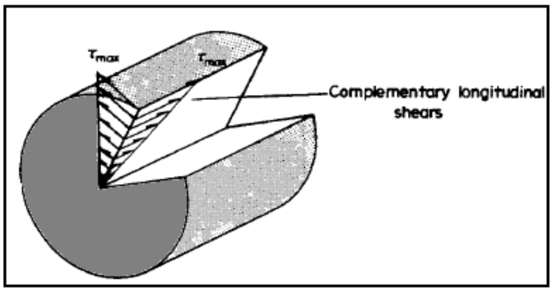

Figure (4) Complementary longitudinal shear stress in a shaft subjected to torsion.

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Section modulus

It is sometimes convenient to re-write part of the torsion theory formula to obtain the maximum shear stress in shafts as follows:

$$\frac{T}{J} = \frac{\tau}{R}$$

$$\tau = \frac{TR}{J}$$

With (R) the outside radius of the shaft the above equation yields the greatest value possible for **T,** Figure (4).

$$\tau_{max} = \frac{TR}{J}$$

$$\tau_{max} = \frac{T}{Z} \qquad \text{.... (11)}$$

Where; z = J/R is termed the polar section modulus. It will be seen from the preceding section that:

for solid shafts, $$Z = \frac{\pi D^3}{16} \qquad \text{..... (12)}$$

and for hollow shafts, $$Z = \frac{\pi(D^4 - d^4)}{16D} \qquad \text{.... (13)}$$

Torsional rigidity

The angle of twist per unit length of shafts is given by the torsion theory as

$$\frac{\theta}{L} = \frac{T}{GJ}$$

The quantity (GJ) is termed the *torsional rigidity* of the shaft and is thus given by

$$GJ = \frac{T}{\theta / L} \qquad \text{.... (14)}$$

i.e. the torsional rigidity is the torque divided by the angle of twist (in radians) per unit length.

Torsion of hollow shafts

It has been shown above that the maximum shear stress in a solid shaft is developed in the outer surface, values at other radii decreasing linearly to zero at the centre. In applications where weight reduction is of prime importance as in the aerospace industry, for instance, it is often found advisable to use hollow shafts.

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Composite shafts - series connection

If two or more shafts of different material, diameter or basic form are connected together in such a way that each carries the same torque, then the shafts are said to be connected in series and the composite shaft so produced is therefore termed ***series-connected*** Figure (5) .

$$T = \frac{G_1 J_1 \theta_1}{L_1} = \frac{G_2 J_2 \theta_2}{L_2} \qquad \text{.... (15)}$$

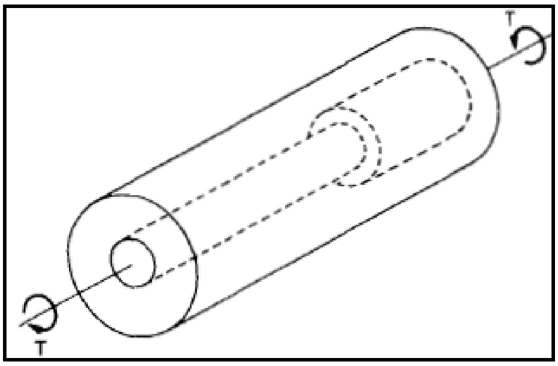

Figure (5)"Series-connected" shaft – common torque.

$$\frac{J_1}{L_1} = \frac{J_2}{L_2}$$

$$\frac{L_1}{L_2} = \frac{J_1}{J_2} \qquad \text{.... (16)}$$

Composite shafts - parallel connection

If two or more materials are rigidly fixed together such that the applied torque is shared between them then the composite shaft so formed is said to be connected in parallel Figure (6).

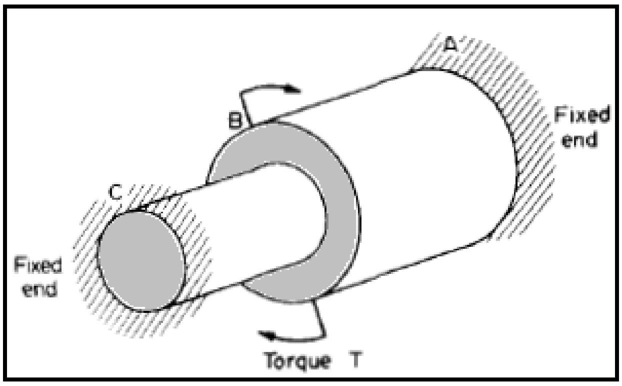

Figure (6)"Parallel-connected" shaft – shared torque.

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For parallel connection,

total torque $T = T_1 + T_2$ ....(17)

In this case the angles of twist of each portion are equal and

$$\frac{T_1 L_1}{G_1 J_1} = \frac{T_2 L_2}{G_2 J_2} \quad .....(18)$$

i.e. for equal lengths (as is normally the case for parallel shafts)

$$\frac{T_1}{T_2} = \frac{G_1 J_1}{G_2 J_2} \quad ....(19)$$

The maximum stresses in each part can then be found from

$$\tau_1 = \frac{T_1 R_1}{J_1} \quad \text{and} \quad \tau_2 = \frac{T_2 R_2}{J_2}$$

Strain energy in torsion

The strain energy stored in a solid circular bar or shaft subjected to a torque (T) is given by the alternative expressions.

$$U = \tfrac{1}{2} T\theta = \frac{T^2 L}{2GJ} = \frac{GJ\theta^2}{2L} = \frac{\tau^2}{4G} \times \text{volume} \quad .....(20)$$

Power transmitted by shafts

If a shaft carries a torque T Newton metres and rotates at ***o*** rad/s it will do work at the rate of;

T.ω Nm/s (or joule/s).

Now the rate at which a system works is defined as its power, the basic unit of power being the

Watt (1 Watt = 1 N.m/s).

Thus, the power transmitted by the shaft:

= T.ω Watts.

Since the Watt is a very small unit of power in engineering terms use is normally made of SI. multiples, i.e. kilowatts (kW) or megawatts (MW).

Combined bending and torsion - equivalent bending moment

For shafts subjected to the simultaneous application of a bending moment (**M*)*** and torque (T) the principal stresses set up in the shaft can be shown to be equal to those produced by an ***equivalent bending moment,*** of a certain value (**M_e)** acting alone.

From the simple bending theory the maximum direct stresses set up at the outside surface of the shaft owing to the bending moment (M) are given by

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$$\sigma = \frac{My_{max}}{I} = \frac{MD}{2I}$$

Similarly, from the torsion theory, the maximum shear stress in the surface of the shaft is given by

$$\tau = \frac{TR}{J} = \frac{TD}{2J}$$

But for a circular shaft J = 2I,

$$\tau = \frac{TD}{4I}$$

The principal stresses for this system can now be obtained by applying the formula derived in

$$\sigma_1 \text{ or } \sigma_2 = \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \tfrac{1}{2}\sqrt{[(\sigma_x - \sigma_y)^2 + 4\tau^2]}$$

and, with $s_y = 0$, the maximum principal stress (s_1) is given by

$$\sigma_1 = \frac{1}{2}\left(\frac{MD}{2I}\right) + \frac{1}{2}\sqrt{\left[\left(\frac{MD}{2I}\right)^2 + 4\left(\frac{TD}{4I}\right)^2\right]}$$

$$= \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}]$$

Now if (M_e) is the bending moment which, acting alone, will produce the same maximum stress, then

$$\sigma_1 = \frac{M_e y_{max}}{I} = \frac{M_e D}{2I}$$

$$\frac{M_e D}{2I} = \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}]$$

i.e. the equivalent bending moment is given by

$$M_e = \tfrac{1}{2}[M + \sqrt{(M^2 + T^2)}] \qquad \text{.....(21)}$$

and it will produce the same maximum direct stress as the combined bending and torsion effects.

Combined bending and torsion - equivalent torque

Again considering shafts subjected to the simultaneous application of a bending moment (M) and a torque (T) the ***maximum shear stress*** set up in the shaft may be determined by the application of an ***equivalent torque*** of value (T_e) acting alone. From the preceding section the principal stresses in the shaft are given by

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$$\sigma_1 = \frac{1}{2}\left(\frac{D}{2I}\right)[M + \sqrt{(M^2 + T^2)}] = \frac{1}{2}\left(\frac{D}{J}\right)[M + \sqrt{(M^2 + T^2)}]$$

$$\sigma_2 = \frac{1}{2}\left(\frac{D}{2I}\right)[M - \sqrt{(M^2 + T^2)}] = \frac{1}{2}\left(\frac{D}{J}\right)[M - \sqrt{(M^2 + T^2)}]$$

Now the maximum shear stress is given by equation (12)

$$\tau_{max} = \tfrac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\left(\frac{D}{J}\right)\sqrt{(M^2 + T^2)}$$

But, from the torsion theory, the equivalent torque T_e , will set up a maximum shear stress of

$$\tau_{max} = \frac{T_e D}{2J}$$

Thus if these maximum shear stresses are to be equal,

$$T_e = \sqrt{(M^2 + T^2)} \qquad .....(22)$$

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Examples

Example 1

(a) A solid shaft, 100 mm diameter, transmits 75 kW at 150 rev/min. Determine the value of the maximum shear stress set up in the shaft and the angle of twist per metre of the shaft length if G = 80 GN/m^2.

(b) If the shaft were now bored in order to reduce weight to produce a tube of 100 mm outside diameter and 60mm inside diameter, what torque could be carried if the same maximum shear stress is not to be exceeded? What is the percentage increase in power/weight ratio effected by this modification?

Solution

(a) Power $= T\omega \quad \therefore$ torque $T = \dfrac{\text{power}}{\omega}$

$$\therefore \quad T = \frac{75 \times 10^3}{150 \times 2\pi/60} = 4.77 \text{ kN m}$$

From the torsion theory ·

$$\frac{T}{J} = \frac{\tau}{R} \quad \text{and} \quad J = \frac{\pi}{32} \times 100^4 \times 10^{-12} = 9.82 \times 10^{-6} \text{ m}^4$$

$$\therefore \tau_{max} = \frac{TR_{max}}{J} = \frac{4.77 \times 10^3 \times 50 \times 10^{-3}}{9.82 \times 10^{-6}} = \mathbf{24.3\ MN/m^2}$$

Also from the torsion theory

$$\theta = \frac{TL}{GJ} = \frac{4.77 \times 10^3 \times 1}{80 \times 10^9 \times 9.82 \times 10^{-6}} = 6.07 \times 10^{-3} \text{ rad/m}$$

$$= 6.07 \times 10^{-3} \times \frac{360}{2\pi} = \mathbf{0.348\ degrees/m}$$

(b) When the shaft is bored, the polar moment of area J is modified thus:

$$J = \frac{\pi}{32}(D^4 - d^4) = \frac{\pi}{32}(100^4 - 60^4)10^{-12} = 8.545 \times 10^{-6} \text{ m}^4$$

The torque carried by the modified shaft is then given by

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$$T = \frac{\tau J}{R} = \frac{24.3 \times 10^6 \times 8.545 \times 10^{-6}}{50 \times 10^{-3}} = \mathbf{4.15 \times 10^3\ Nm}$$

Now, weight/metre of original shaft

$$= \frac{\pi}{4}(100)^2 \times 10^{-6} \times 1 \times \rho g = 7.854 \times 10^{-3}\ \rho g$$

where ρ is the density of the shaft material.

Also, weight/metre of modified shaft $= \frac{\pi}{4}(100^2 - 60^2)10^{-6} \times 1 \times \rho g$

$$= 5.027 \times 10^{-3}\ \rho g$$

Power/weight ratio for original shaft $= \frac{T\omega}{\text{weight/metre}}$

$$= \frac{4.77 \times 10^3\ \omega}{7.854 \times 10^{-3}\ \rho g} = 6.073 \times 10^5 \frac{\omega}{\rho g}$$

Power/weight ratio for modified shaft

$$= \frac{4.15 \times 10^3\ \omega}{5.027 \times 10^{-3}\ \rho g} = 8.255 \times 10^5 \frac{\omega}{\rho g}$$

Therefore percentage increase in power/weight ratio

$$= \frac{(8.255 - 6.073)}{6.073} \times 100 = \mathbf{36\%}$$

Example 2

Determine the dimensions of a hollow shaft with a diameter ratio of 3:4 which is to transmit 60 kW at 200 rev/min. The maximum shear stress in the shaft is limited to 70 MN/m^2 and the angle of twist to 3.8^o in a length of 4 m. For the shaft material G = 80 GN/m^2.

Solution

Maximum shear stress condition

Since power $= T\omega$ and $\omega = 200 \times \frac{2\pi}{60} = 20.94$ rad/s

$$T = \frac{60 \times 10^3}{20.94} = 2.86 \times 10^3\ \text{Nm}$$

From the torsion theory $J = \frac{TR}{\tau}$

$$\therefore \quad \frac{\pi}{32}(D^4 - d^4) = \frac{2.86 \times 10^3 \times D}{70 \times 10^6 \times 2}$$

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But $d/D = 0.75$

$$\therefore \frac{\pi}{32} D^4(1 - 0.75^4) = 20.43 \times 10^{-6} D$$

$$D^3 = \frac{20.43 \times 10^{-6}}{0.0671} = 304.4 \times 10^{-6}$$

$$D = 0.0673 \text{ m} = 67.3 \text{ mm}$$

$$d = 50.5 \text{ mm}$$

<u>*Angle of twist condition*</u>

Again from the torsion theory $J = \frac{TL}{G\theta}$

$$\frac{\pi}{32}(D^4 - d^4) = \frac{2.86 \times 10^3 \times 4 \times 360}{80 \times 10^9 \times 3.8 \times 2\pi}$$

$$\frac{\pi}{32} D^4(1 - 0.75^4) = \cdot 2.156 \times 10^{-6}$$

$$\frac{\pi}{32}(D^4 - d^4) = \frac{2.86 \times 10^3 \times 4 \times 360}{80 \times 10^9 \times 3.8 \times 2\pi}$$

$$\frac{\pi}{32} D^4(1 - 0.75^4) = \cdot 2.156 \times 10^{-6}$$

$$D^4 = \frac{2.156 \times 10^{-6}}{0.0671} = 32.12 \times 10^{-6}$$

$$D = 0.0753 \text{ m} = 75.3 \text{ mm}$$

$$d = 56.5 \text{ mm}$$

Thus the dimensions required for the shaft to satisfy both conditions are outer diameter 75.3mm; inner diameter 56.5 mm.

<u>Example 3</u>

(a) A steel transmission shaft is 510 mm long and 50 mm external diameter. For part of its length it is bored to a diameter of 25 mm and for the rest to 38 mm diameter. Find the maximum power that may be transmitted at a speed of 210 rev/min if the shear stress is not to exceed 70 MN/m^2.

(b) If the angle of twist in the length of 25 mm bore is equal to that in the length of 38 mm bore, find the length bored to the latter diameter.

<u>Solution</u>

(a) This is, in effect, a question on shafts in series since each part is subjected to the same torque. From the torsion theory ;

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$$T = \frac{\tau J}{R}$$

and as the maximum stress and the radius at which it occurs (the outside radius) are the same for both shafts the torque allowable for a known value of shear stress is dependent only on the value of (J). This will be least where the internal diameter is greatest since

$$J = \frac{\pi}{32}(D^4 - d^4)$$

$$\text{least value of } J = \frac{\pi}{32}(50^4 - 38^4)10^{-12} = 0.41 \times 10^{-6}\,\text{m}^4$$

Therefore maximum allowable torque if the shear stress is not to exceed 70 MN/m^2 (a 25 mm radius) is given by

$$T = \frac{70 \times 10^6 \times 0.41 \times 10^{-6}}{25 \times 10^{-3}} = 1.15 \times 10^3\,\text{Nm}$$

$$\text{Maximum power} = T\omega = 1.15 \times 10^3 \times 210 \times \frac{2\pi}{60}$$

$$= 25.2 \times 10^3 = \mathbf{25.2\,kW}$$

(b)

Let suffix 1 refer to the 38 mm diameter bore portion and suffix 2 to the other part. Now for shafts in series, equation (16) applies,

$$\frac{J_1}{L_1} = \frac{J_2}{L_2}$$

$$\frac{L_2}{L_1} = \frac{J_2}{J_1} = \frac{\frac{\pi}{32}(50^4 - 25^4)10^{-12}}{\frac{\pi}{32}(50^4 - 38^4)10^{-12}} = 1.43$$

$$L_2 = 1.43\,L_1$$

$$L_1 + L_2 = 510\,\text{mm}$$

$$L_1(1 + 1.43) = 510$$

$$L_1 = \frac{510}{2.43} = \mathbf{210\,mm}$$

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PROBLEMS

1 - A solid steel shaft (A) of 50 mm diameter rotates at 250 rev/min. Find the greatest power that can be transmitted for a limiting shearing stress of 60 MN/m^2 in the steel.
It is proposed to replace (A) by a hollow shaft (B), of the Same external diameter but with a limiting shearing stress of 75 MN/m^2. Determine the internal diameter of (B) to transmit the same power at the same speed. **Ans.[38.6kW, 33.4 mm]**

2 - Calculate the dimensions of a hollow steel shaft which is required to transmit 750 kW at a speed of 400 rev/min if the maximum torque exceeds the mean by 20 % and the greatest intensity of shear stress is limited to75 MN/m^2. The internal diameter of the shaft is to be 80 % of the external diameter. (The mean torque is that derived from the horsepower equation.)
Ans. [135.2mm, 108.2 mm.]

3 - A steel shaft 3 m long is transmitting 1 MW at 240 rev/min. The working conditions to be satisfied by the
shaft are:
(a) that the shaft must not twist more than 0.02 radian on a length of 10 diameters;
(b) that the working stress must not exceed 60 MN/m^2.
If the modulus of rigidity of steel is 80 GN/m^2 what is
(i) the diameter of the shaft required
(ii) the actual working stress;
(iii) the angle of twist of the 3 m length? **Ans. [l50 mm; 60MN/m^2; 0.030 rad.]**

4 - A hollow shaft has to transmit 6MW at 150 rev/min. The maximum allowable stress is not to exceed 60 MN/m^2 and the angle of twist 0.3^o per metre length of shafting. If the outside diameter of the shaft is 300 mm find the minimum thickness of the hollow shaft to satisfy the above conditions. G = 80 GN/m^2. **Ans. [61.5mm.]**

5 - A flanged coupling having six bolts placed at a pitch circle diameter of 180 mm connects two lengths of solid steel shafting of the same diameter. The shaft is required to transmit 80 kW at 240 rev/min. Assuming the allowable intensities of shearing stresses in the shaft and bolts are 75 MN/m^2 and 55 MN/m^2 respectively, and the maximum torque is 1.4 times the mean torque, calculate:
(a) the diameter of the shaft;
(b) the diameter of the bolts. **Ans. [67.2mm, 13.8 mm.]**

6 - A hollow low carbon steel shaft is subjected to a torque of 0.25 MN. m. If the ratio of internal to external diameter is 1 to 3 and the shear stress due to torque has to be limited to 70 MN/m^2 determine the required diameters and the angle of twist in degrees per metre length of shaft.
G = 80GN/m^2. **Ans. [264mm, 88 mm; 0.38^o]**

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CRYSTALLINE STRUCTURE OF METALS

The arrangement of atoms in a material determines the behavior and properties of that material. Most of the materials used in the construction of a nuclear reactor facility are metals. In this chapter, we will discuss the various types of bonding that occurs in material selected for use in a reactor facility.

1- Atomic Bonding

There are three common states, these three states are solid, liquid, and gas. The atomic or molecular interactions that occur within a substance determine its state. In this chapter, we will deal primarily with solids because solids are of the most concern in engineering applications of materials. Liquids and gases will be mentioned for comparative purposes only. Solid matter is held together by forces originating between neighboring atoms or molecules. These forces arise because of differences in the electron clouds of atoms. In other words, the valence electrons, or those in the outer shell, of atoms determine their attraction for their neighbors. When physical attraction between molecules or atoms of a material is great, the material is held tightly together. Molecules in solids are bound tightly together. When the attractions are weaker, the substance may be in a liquid form and free to flow. Gases exhibit virtually no attractive forces between atoms or molecules, and their particles are free to move independently of each other. The types of bonds in a material are determined by the manner in which forces hold matter together. Figure (1) illustrates several types of bonds and their characteristics are listed below.

a. Ionic bond - In this type of bond, one or more electrons are wholly transferred from an atom of one element to the atom of the other, and the elements are held together by the force of attraction due to the opposite polarity of the charge.

b. Covalent bond - A bond formed by shared electrons. Electrons are shared when an atom needs electrons to complete its outer shell and can share those electrons with its neighbor. The electrons are then part of both atoms and both shells are filled.

c. Metallic bond - In this type of bond, the atoms do not share or exchange electrons to bond together. Instead, many electrons (roughly one for each atom) are more or less free to move throughout the metal, so that each electron can interact with many of the fixed atoms.

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d. Molecular bond - When the electrons of neutral atoms spend more time in one region of their orbit, a temporary weak charge will exist. The molecule will weakly attract other molecules. This is sometimes called the van der Waals or molecular bonds.

e. Hydrogen bond - This bond is similar to the molecular bond and occurs due to the ease with which hydrogen atoms are willing to give up an electron to atoms of oxygen, fluorine, or nitrogen.

Some examples of materials and their bonds are identified in Table (1).

TABLE 1

Examples of Materials and Their Bonds

Material	Bond
Sodium chloride	Ionic
Diamond	Covalent
Sodium	Metallic
Solid H_2	Molecular
Ice	Hydrogen

The type of bond not only determines how well a material is held together, but also determines what microscopic properties the material possesses. Properties such as the ability to conduct heat or electrical current are determined by the freedom of movement of electrons. This is dependent on the type of bonding present. Knowledge of the microscopic structure of a material allows us to predict how that material will behave under certain conditions. Conversely, a material may be synthetically fabricated with a given microscopic structure to yield properties desirable for certain engineering applications.

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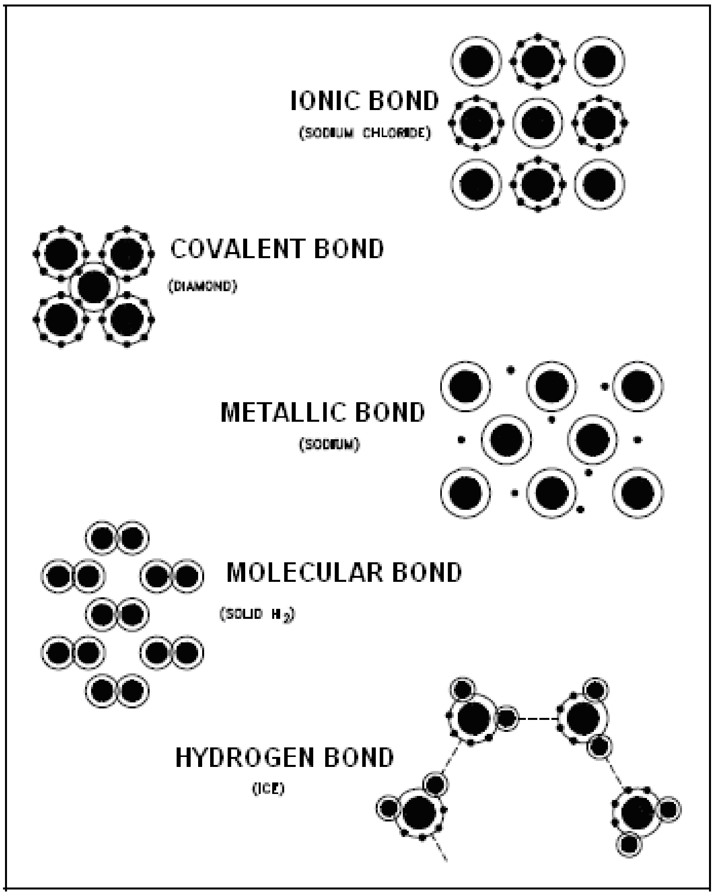

Figure (1) Bounding Types

2- Microstructures

Solids have greater inter atomic attractions than liquids and gases. However, there are wide variations in the properties of solid materials used for engineering purposes. The properties of materials depend on their inter atomic bonds. These same bonds also dictate the space between the configuration of atoms in solids. All solids may be classified as either amorphous or crystalline.

3- Amorphous

Amorphous materials have no regular arrangement of their molecules. Materials like glass and paraffin are considered amorphous. Amorphous materials have the properties of solids. They have definite shape and volume and diffuse slowly. These materials also lack sharply defined melting points. In many respects, they resemble liquids that flow very slowly at room temperature.

4- Crystalline

In a crystalline structure, the atoms are arranged in a three-dimensional array called a lattice. The lattice has a regular repeating configuration in all directions. A group of

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particles from one part of a crystal has exactly the same geometric relationship as a group from any other part of the same crystal.

COMMON LATTICE TYPES

All metals used in a reactor have crystalline structures. Crystalline microstructures are arranged in three-dimensional arrays called lattices.

1- Crystal structure

In metals, and in many other solids, the atoms are arranged in regular arrays called crystals. A *crystal structure* consists of atoms arranged in a pattern that repeats periodically in a three-dimensional geometric lattice. The forces of chemical bonding causes this repetition. It is this repeated pattern which control properties like strength, ductility, density (described in Module 2, Properties of Metals), conductivity (property of conducting or transmitting heat, electricity, etc.), and shape.

In general, the three most common basic crystal patterns associated with metals are: (I) the body-centered cubic, (II) the face-centered cubic, and (III) the hexagonal close-packed. Figure (2) shows these three patterns.

(I) Body-centered cubic structure

In a *body-centered cubic* (BCC) arrangement of atoms, the unit cell consists of eight atoms at the corners of a cube and one atom at the body center of the cube.

(II) Face-centered cubic structure

In a *face-centered cubic* (FCC) arrangement of atoms, the unit cell consists of eight atoms at the corners of a cube and one atom at the center of each of the faces of the cube.

(III) Hexagonal close-packed structure

In a *hexagonal close-packed* (HCP) arrangement of atoms, the unit cell consists of three layers of atoms. The top and bottom layers contain six atoms at the corners of a hexagon and one atom at the center of each hexagon. The middle layer contains three atoms nestled between the atoms of the top and bottom layers, hence, the name close-packed.

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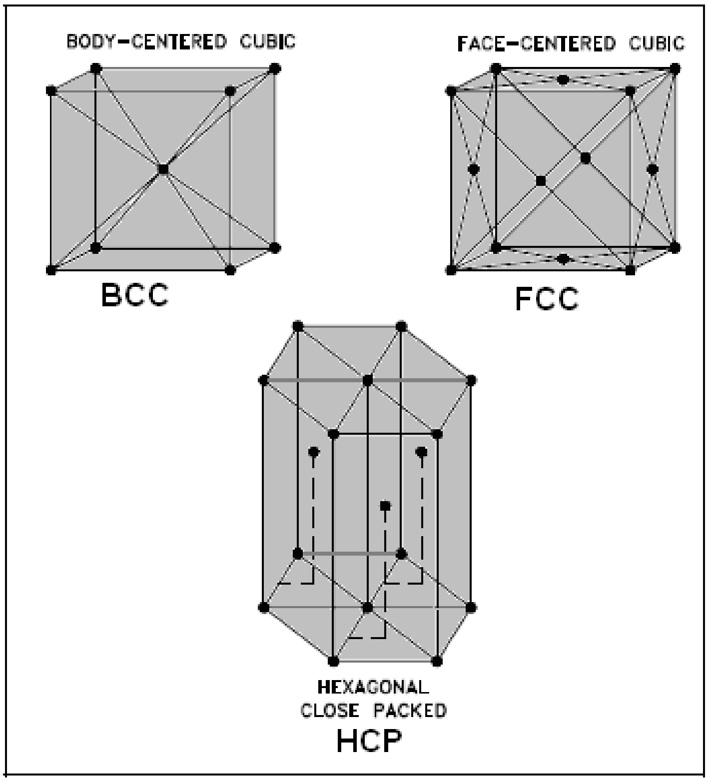

Figure (2) Common Lattice Types

Most diagrams of the structural cells for the BCC and FCC forms of iron are drawn as though they are of the same size, as shown in Figure (2) , but they are not. I n th e BCC arrangement, the structural cell, which uses only nine atoms, is much smaller.

Metals such as α-iron (Fe) (ferrite), chromium (Cr), vanadium (V), molybdenum (Mo), and tungsten (W) possess BCC structures. These BCC metals have two properties in common, high strength and low ductility (which permits permanent deformation). FCC metals such as γ-iron(Fe) (austenite), aluminum (Al), copper (Cu), lead (Pb), silver (Ag), gold (Au), nickel (Ni), platinum (Pt), and thorium (Th) are, in general, of lower strength and higher ductility than BCC metals. HCP structures are found in beryllium (Be), magnesium (Mg), zinc (Zn), cadmium (Cd), cobalt (Co), thallium (Tl), and zirconium (Zr). The important information in this chapter is summarized below.

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GRAIN STRUCTURE AND BOUNDARY

Metals contain grains and crystal structures. The individual needs a microscope to see the grains and crystal structures. Grains and grain boundaries help determine the properties of a material.

1- Grain Structure and Boundary

If you were to take a small section of a common metal and examine it under a microscope, you would see a structure similar to that shown in Figure 3(a). Each of the light areas is called a *grain*, or crystal, which is the region of space occupied by a continuous crystal lattice. The dark lines surrounding the grains are grain boundaries. The *grain structure* refers to the arrangement of the grains in a metal, with a grain having a particular crystal structure. The *grain boundary* refers to the outside area of a grain that separates it from the other grains. The grain boundary is a region of misfit between the grains and is usually one to three atom diameters wide. The grain boundaries separate variously-oriented crystal regions(polycrystalline) in which the crystal structures are identical. Figure 3(b) represents four grains of different orientation and the grain boundaries that arise at the interfaces between the grains. A very important feature of a metal is the average size of the grain. The size of the grain determines the properties of the metal. For example, smaller grain size increases tensile strength and tends to increase ductility. A larger grain size is preferred for improved high-temperature creep properties. *Creep* is the permanent deformation that increases with time under constant load or stress. Creep becomes progressively easier with increasing temperature. Stress and strain are covered in Module 2, Properties of Metals, and creep is covered in Module 5, Plant Materials.

Another important property of the grains is their orientation. Figure 4(a) represents a random Figure (3) Grains and Boundaries (a) Microscopic (b) Atomic arrangement of the grains such that no one direction within the grains is aligned with the external boundaries of the metal sample. This random orientation can be obtained by cross rolling the material. If such a sample were rolled sufficiently in one direction, it might develop a grain-oriented structure in the rolling direction as shown in Figure 4(b). This is called preferred orientation. In many cases, preferred orientation is very desirable, but in other instances, it can be most harmful. For example, preferred orientation in uranium fuel elements can result in catastrophic changes in dimensions during use in a nuclear reactor.

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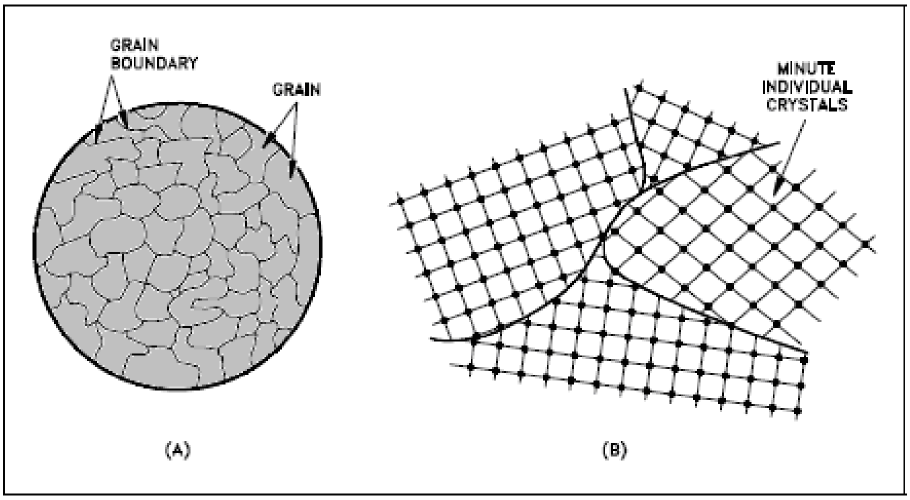

Figure (3) Grains and Boundaries
(a) Microscopic (b) Atomic

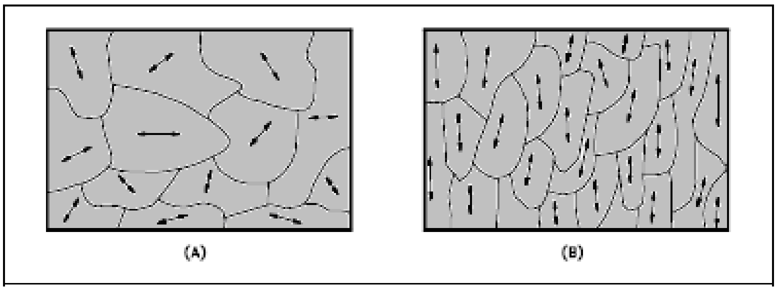

Figure (4) Grain Orientation
(a) Random (b) Preferred

POLYMORPHISM

Metals are capable of existing in more than one form at a time. This chapter will discuss this property of metals.

1- Polymorphism Phases

Polymorphism is the property Figure 5 Cooling Curve for Unalloyed Uranium or ability of a metal to exist in two or more crystalline forms
depending upon temperature and composition. Most metals and metal alloys exhibit this property. Uranium is a good example of a metal that exhibits polymorphism. Uranium metal can exist in three different crystalline structures. Each structure exists at a specific phase, as illustrated in Figure 5.

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1. The alpha phase, from room temperature to 663°C
2. The beta phase, from 663°C to 764°C
3. The gamma phase, from 764°C to its melting point of 1133°C

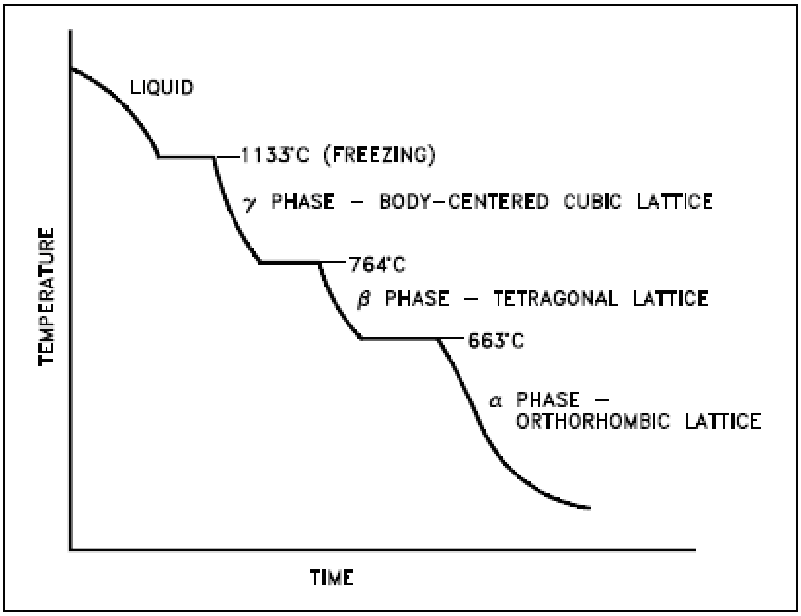

Figure (5) Cooling Curve for Unalloyed Uranium

2- Alpha (a) Phase

The alpha (α) phase is stable at room temperature and has a crystal system characterized by three unequal axes at right angles.

In the alpha phase, the properties of the lattice are different in the X, Y, and Z axes. This is because of the regular recurring state of the atoms is different. Because of this condition, when heated the phase expands in the X and Z directions and shrinks in the Y direction. Figure 6 shows what happens to the dimensions (Å = angstrom, one hundred-millionth of a centimeter) of a unit cell of alpha uranium upon being heated. As shown, heating and cooling of alpha phase uranium can lead to drastic dimensional changes and gross distortions of the metal. Thus, pure uranium is not used as a fuel, but only in alloys or compounds. Figure 6 Change in Alpha Uranium Upon Heating From 0 to 300°C The beta (β) phase of uranium occurs at elevated temperatures. This phase has a tetragonal (having four angles and four sides) lattice structure and is quite complex.

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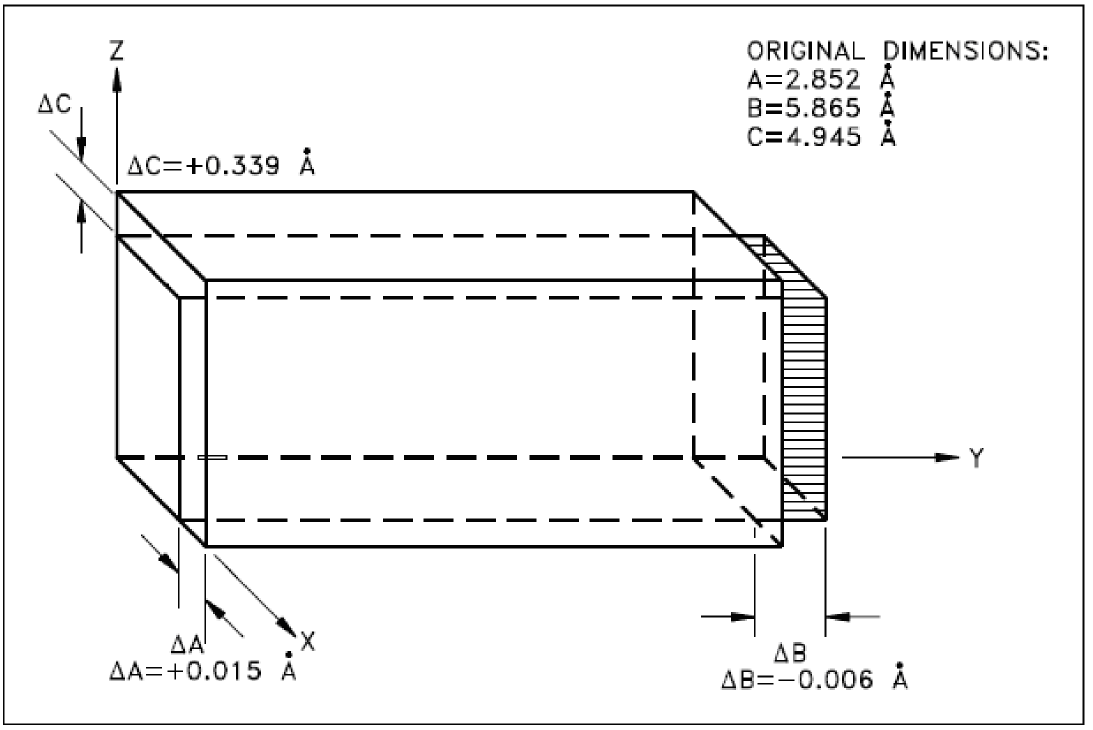

Figure (6) Change in Alpha Uranium Heating From 0 to 300C

3- Beta (b) Phase

The beta (β) phase of uranium occurs at elevated temperatures. This phase has a tetragonal (having four angles and four sides) lattice structure and is quite complex.

4- Gamma (g) Phase

The gamma (γ) phase of uranium is formed at temperatures above those required for beta phase stability. In the gamma phase, the lattice structure is BCC and expands equally in all directions when heated.

ALLOYS

Most of the materials used in structural engineering or component fabrication are metals. Alloying is a common practice because metallic bonds allow joining of different types of metals.

1- Alloy

An *alloy* is a mixture of two or more materials, at least one of which is a metal. Alloys can have a microstructure consisting of solid solutions, where secondary atoms are introduced as substitutionals or interstitials (discussed further in the next chapter and Module 5, Plant Materials) in a crystal lattice. An alloy might also be a crystal with a metallic compound at each lattice point. In addition, alloys may be composed of

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secondary crystals imbedded in a primary polycrystalline matrix. This type of alloy is called a composite (although the term "composite" does not necessarily imply that the component materials are metals). Module2, Properties of Metals, discusses how different elements change the physical properties of a metal.

2- Common Characteristics of Alloys

Alloys are usually stronger than pure metals, although they generally offer reduced electrical and thermal conductivity. Strength is the most important criterion by which many structural materials are judged. Therefore, alloys are used for engineering construction. Steel, probably the most common structural metal, is a good example of an alloy. It is an alloy of iron and carbon, with other elements to give it certain desirable properties. As mentioned in the previous chapter, it is sometimes possible for a material to be composed of several solid phases. The strengths of these materials are enhanced by allowing a solid structure to become a form composed of two interspersed phases. When the material in question is an alloy, it is possible to quench (discussed in more detail in Module 2, Properties of Metals) the metal from a molten state to form the interspersed phases. The type and rate of quenching determines the final solid structure and, therefore, its properties.

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BENDING

Simple bending theory

If a piece of rubber, most conveniently of rectangular cross-section, is bent between one's fingers it is readily apparent that one surface of the rubber is stretched, i.e. put into tension, and the opposite surface is compressed. In order for this to be achieved it is necessary to make certain simplifying assumptions. The assumptions are as follows:

(1) The beam is initially straight and unstressed.
(2) The material of the beam is perfectly homogeneous and isotropic, i.e. of the same density and elastic properties throughout.
(3) The elastic limit is nowhere exceeded.
(4) Young's modulus for the material is the same in tension and compression.
(5) Plane cross-sections remain plane before and after bending.
(6) Every cross-section of the beam is symmetrical about the plane of bending, i.e. about an
(7) There is no resultant force perpendicular to any cross-section.

If we now consider a beam initially unstressed and subjected to a constant (B.M.) along its length, i.e. pure bending, as would be obtained by applying equal couples at each end, it will bend to a radius (R) as shown in Figure (1b). As a result of this bending the top fibres of the beam will be subjected to tension and the bottom to compression. It is reasonable to suppose, therefore, that somewhere between the two there are points at which the stress is zero. The locus of all such points is termed the neutral axis (N.A). The radius of curvature R is then measured to this axis. For symmetrical sections the N.A. is the axis of symmetry, but whatever the section the N.A. will always pass through the centre of area or centroid.

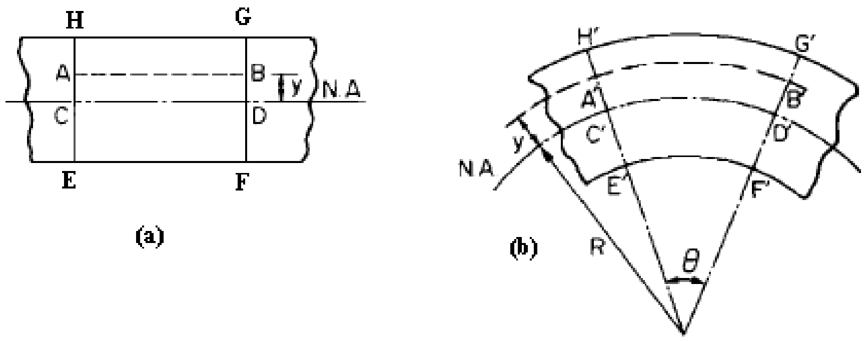

Figure (1) Beam subjected to pure bending
(a) before, and (b) after, the moment

Consider now two cross-sections of a beam, HE and GF, originally parallel Figure (1a) When the beam is bent Figure (1b). it is assumed that these sections remain

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plane; i.e. H' E' and G'F', the final positions of the sections, are still straight lines. They will then subtend some angle (q). Consider now some fibre AB in the material, distance y from the N.A. When the beam is bent this will stretch to A'B'.

$$\text{Strain in fibre } AB = \frac{\text{extension}}{\text{original length}} = \frac{A'B' - AB}{AB}$$

But $AB = CD$, and, since the N.A. is unstressed, $CD = C'D'$.

$$\therefore \quad \text{strain} = \frac{A'B' - C'D'}{C'D'} = \frac{(R+y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

$$\frac{\text{stress}}{\text{strain}} = \text{Young's modulus } E$$

$$\therefore \quad \text{strain} = \frac{\sigma}{E}$$

Equating the two equations for strain,

$$\frac{\sigma}{E} = \frac{y}{R} \quad \text{or} \quad \frac{\sigma}{y} = \frac{E}{R}$$

……..(1)

Consider now a cross-section of the beam Figure (2) From equation (1) the stress on a fibre at distance (*y)* from the N.A. is

$$\sigma = \frac{E}{R} y$$

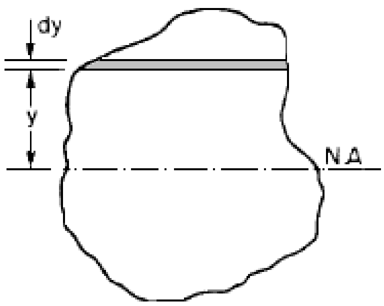

Figure (2) Beam cross-section.

If the strip is of area ***dA*** the force on the strip is

$$F = \sigma \delta A = \frac{E}{R} y \delta A$$

This has a moment about the **N.A.** of

$$Fy = \frac{E}{R} y^2 \delta A$$

The total moment for the whole cross-section is therefore

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$$M = \sum \frac{E}{R} y^2 \delta A$$
$$= \frac{E}{R} \Sigma y^2 \delta A$$

since (E) and (R) are assumed constant.

The term $\sum y^2.dA$ is called the ***second moment*** of ***area*** of the cross-section and given the symbol (I).

$$\therefore \quad M = \frac{E}{R} I \quad \text{and} \quad \frac{M}{I} = \frac{E}{R} \qquad \text{.....(2)}$$

Combining eqns. **(1)** and **(2)** we have the bending theory equation

$$\frac{M}{I} = \frac{s}{y} = \frac{E}{R} \qquad \text{......(3)}$$

Neutral axis

In bending, one surface of the beam is subjected to tension and the opposite surface to compression there must be a region within the beam cross-section at which the stress changes sign, i.e. where the stress is zero, and this is termed the neutral axis (N.A). Further, equation (3) may be re-written in the form

$$\sigma = \frac{M}{I} y \qquad \text{.....(4)}$$

which shows that at any section the stress is directly proportional to y, the distance from the N.A., i.e. (s) varies linearly with (y), the maximum stress values occurring in the outside surface of the beam where (y) is a maximum.

The force on the small element of area is (s.*dA)* acting perpendicular to the cross-section, i.e. parallel to the beam axis. The total force parallel to the beam axis is therefore $\int s\,.dA$. The tensile force on one side of the N.A. must exactly balance the compressive force on the other side

$$\int \sigma dA = 0$$

Substituting from equation (1)

$$\int \frac{E}{R} y dA = 0 \quad \text{and hence} \quad \frac{E}{R} \int y dA = 0$$

Typical stress distributions in bending are shown in Figure (4). In order to obtain the maximum resistance to bending it is advisable therefore to use sections which have large areas as far away from the N.A. as possible. For this reason beams with **I-** or T-sections find considerable favour in present engineering applications, such as girders, where bending plays a large part. Such beams have large moments of area about one axis and, provided that it is ensured that bending takes place about this axis, they will have a high resistance to bending stresses.

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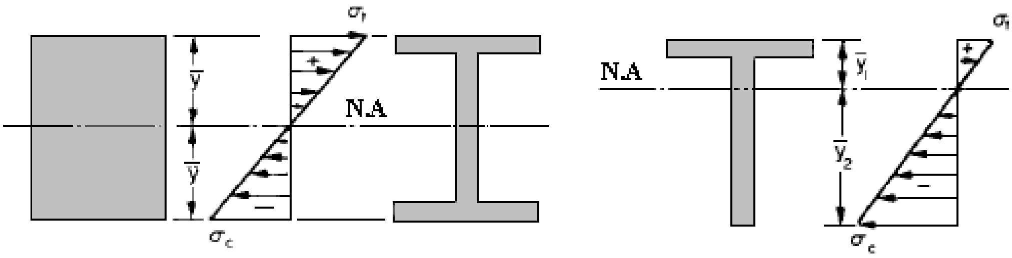

Figure (4) Typical bending stress distributions.

Section modulus

From equation (4) the maximum stress obtained in any cross-section is given by

$$\sigma_{max} = \frac{M}{I} y_{max} \qquad \text{.....(5)}$$

For any given allowable stress the maximum moment which can be accepted by a particular shape of cross-section is therefore

$$M = \frac{I}{y_{max}} \sigma_{max}$$

For ready comparison of the strength of various beam cross-sections this is sometimes written in the form

$$M = Z\sigma_{max} \qquad \text{......(6)}$$

where $Z(= I/y_{max.})$ is termed the section modulus. The higher the value of Z for a particular cross-section the higher the B.M. which it can withstand for a given maximum stress. This is particularly important in the case of unsymmetrical sections such as T-sections where the values of (y_{max}) wi1l also be different on each side of the N.A. Figure (4) and here two values of section modulus are often quoted,

$$Z_1 = I/y_1 \quad \text{and} \quad Z_2 = I/y_2 \qquad \text{......(7)}$$

each being then used with the appropriate value of allowable stress.

Second moment of area

Consider the rectangular beam cross-section shown in Figure (5) and an element of area (dA), thickness (dy), breadth (B) and distance (y) from the N.A. which by symmetry passes through the centre of the section. The second moment of area (I) has been defined earlier as

$$I = \int y^2 dA$$

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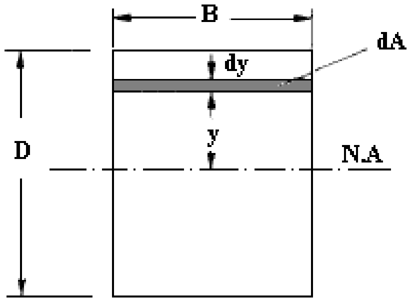

Figuer (5)

Thus for the rectangular section the second moment of area about the N.A., i.e. an axis through the centre, is given by

$$I_{N.A.} = \int_{-D/2}^{D/2} y^2 B dy = B \int_{-D/2}^{D/2} y^2 dy$$

$$= B\left[\frac{y^3}{3}\right]_{-D/2}^{D/2} = \frac{BD^3}{12} \qquad \text{..... (8)}$$

Similarly, the second moment of area of the rectangular section about an axis through the lower edge of the section would be found using the same procedure but with integral limits of 0 to D.

$$I = B\left[\frac{y^3}{3}\right]_0^D = \frac{BD^3}{3} \qquad \text{...... (9)}$$

These standard forms prove very convenient in the determination of ($I_{N.A.}$) values for built-up sections which can be conveniently divided into rectangles. For symmetrical sections as, for instance, the I-section shown in Figure (6)

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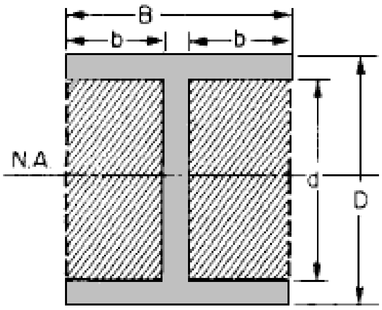


Figuer (6)

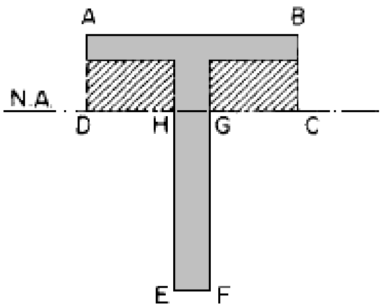
$$I_{N.A.} = I \text{ of dotted rectangle} - I \text{ of shaded portions}$$

$$= \frac{BD^3}{12} - 2\left(\frac{bd^3}{12}\right) \qquad \text{.... (10)}$$

It will be found that any symmetrical section can be divided into convenient rectangles with the N.A. running through each of their centroids and the above procedure can then be employed to effect a rapid solution. For unsymmetrical sections such as the T-section of Figure (7) it is more convenient to divide the section into rectangles with their edges in the N.A., when the second type of standard form may be applied.

$$I_{N.A.} = \underset{\text{(about } DC)}{I_{ABCD}} - \underset{\text{(about } DC)}{I_{\text{shaded areas}}} + \underset{\text{(about } HG)}{I_{EFGH}}$$

(each of these quantities may be written in the form $BD^3/3$).



Figuer (7)

As an alternative procedure it is possible to determine the second moment of area of each rectangle about an axis through its own centroid ($I_G = 8D^3/12$) and to “shift” this value to the equivalent value about the N.A. by means of the parallel axis theorem.

$$I_{N.A.} = I_G + Ah^2 \qquad \text{.... (11)}$$

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Where;(A) is the area of the rectangle and (h) the distance of its centroid (G) from the N.A. Whilst this is perhaps not so quick or convenient for sections built-up from rectangles.

Bending of composite or flitched beams

(a) A composite beam is one which is constructed from a combination of materials. If such a beam is formed by rigidly bolting together two timber joists and reinforcing steel plate, a then it is termed a flitched beam.

Since the bending theory only holds good when a constant value of Young's modulus applies across a section it cannot be used directly to solve composite-beam problems where two different materials, and therefore different values of (E), are present. The method of solution in such a case is to replace one of the materials by an equivalent section of the other.

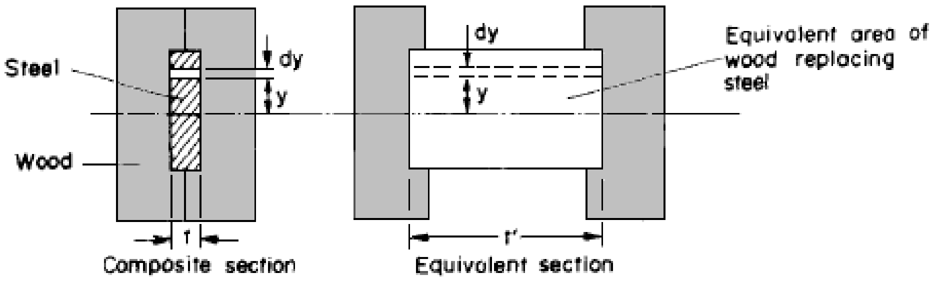

Figure (8) Bending of composite or flitched beams: original beam cross-section and equivalent of uniform material (wood) properties.

Consider, therefore, the beam shown in Figure (8) in which a steel plate is held centrally in an appropriate recess between two blocks of wood. Here it is convenient to replace the steel by an equivalent area of wood, retaining the same bending strength, i.e. the moment at any section must be the same in the equivalent section as in the original so that the force at any given (dy) in the equivalent beam must be equal to that at the strip it replaces.

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$$\therefore \quad \sigma t dy = \sigma' t' dy$$

$$\sigma t = \sigma' t' \quad \text{.....(12)}$$

$$\varepsilon E t = \varepsilon' E' t'$$

since $\frac{\sigma}{\varepsilon} = E$

Again, for true similarity the strains must be equal,

$$\therefore \quad \varepsilon = \varepsilon'$$

$$\therefore \quad Et = E't' \quad \text{or} \quad \frac{t'}{t} = \frac{E}{E'} \quad \text{......(13)}$$

i.e. $t' = \frac{E}{E'} t$ .....(14)

Thus to replace the steel strip by an equivalent wooden strip the thickness must be multiplied by the modular ratio E/E'. The equivalent section is then one of the same material throughout and the simple bending theory applies. The stress in the wooden part of the original beam is found directly and that in the steel found from the value at the same point in the equivalent material as follows:

from eqn. (12) $\frac{\sigma}{\sigma'} = \frac{t'}{t}$

and from eqn. (13) $\frac{\sigma}{\sigma'} = \frac{E}{E'}$ or $\sigma = \frac{E}{E'} \sigma'$ ......(15)

stress in steel = modular ratio x stress in equivalent wood

Strain energy in bending

For beams subjected to bending the total strain energy of the system is given by

$$U = \int_0^L \frac{M^2 ds}{2EI}$$

For uniform beams, or parts of beams, subjected to a constant (B.M).

$$U = \frac{M^2 L}{2EI}$$

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Examples

Example (1)

An I-section girder, 200 mm wide by 300 mm deep, with flange and web of thickness 20 mm is used as a simply supported beam over a span of 7 m. The girder carries a distributed load of 5 kN/m and a concentrated load of 20 kN at mid-span. Determine: (a) the second moment of area of the cross-section of the girder, (b) the maximum stress set-up.

Solution

(a) The second moment of area of the cross-section may be found in two ways.

Method 1 -Use of standard forms

For sections with symmetry about the N.A., use can be made of the standard *I* value for a rectangle about an axis through its centroid, i.e. $bd^3/12$. The section can thus be divided into convenient rectangles for each of which the N.A. passes through the centroid, e.g. in this case, enclosing the girder by a rectangle Figure (16).

$$I_{girder} = I_{rectangle} - I_{shaded\ portions}$$

$$= \left[\frac{200 \times 300^3}{12}\right]10^{-12} - 2\left[\frac{90 \times 260^3}{12}\right]10^{-12}$$

$$= (4.5 - 2.64)10^{-4} = \mathbf{1.86 \times 10^{-4}\ m^4}$$

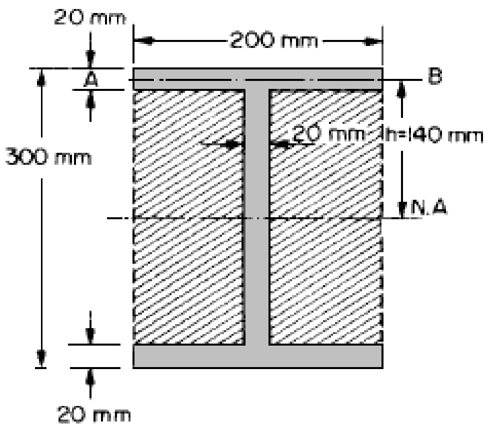

Figure (16)

Method 2 - Parallel axis theorem

Consider the section divided into three parts - the web and the two flanges.

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$$I_{N.A.} \text{ for the web} = \frac{bd^3}{12} = \left[\frac{20 \times 260^3}{12}\right] 10^{-12}$$

$$I \text{ of flange about } AB = \frac{bd^3}{12} = \left[\frac{200 \times 20^3}{12}\right] 10^{-12}$$

Therefore using the parallel axis theorem

$$I_{N.A.} \text{ for flange} = I_{AB} + Ah^2$$

where h is the distance between the N.A. and AB,

$$I_{N.A.} \text{ for flange} = \left[\frac{200 \times 20^3}{12}\right] 10^{-12} + [(200 \times 20)140^2] 10^{-12}$$

Therefore total $I_{N.A.}$ of girder

$$= 10^{-12}\left\{\left[\frac{20 \times 260^3}{12}\right] + 2\left[\frac{200 \times 20^3}{12}\right] + 200 \times 20 \times 140^2\right\}$$

$$= 10^{-6}(29.3 + 0.267 + 156.8)$$

$$= \mathbf{1.86 \times 10^{-4}\ m^4}$$

Both methods thus yield the same value and are equally applicable in most cases. Method **1,** however, normally yields the quicker solution.

(b) The maximum stress may be found from the simple bending theory .

$$\sigma_{max} = \frac{M_{max}\, y_{max}}{I}$$

Now the maximum B.M. for a beam carrying a u.d.1. is at the centre and given by ($wL^2/8$). Similarly, the value for the central concentrated load is ($WL/4$) also at the centre. Thus, in this case,

$$M_{max} = \frac{WL}{4} + \frac{WL^2}{8} = \left[\frac{20 \times 10^3 \times 7}{4}\right] + \left[\frac{5 \times 10^3 \times 7^2}{8}\right] \text{N m}$$

$$= (35.0 + 30.63)10^3 = 65.63 \text{ kN m}$$

$$\sigma_{max} = \frac{65.63 \times 10^3 \times 150 \times 10^{-3}}{1.9 \times 10^{-4}} = \mathbf{51.8\ MN/m^2}$$

The maximum stress in the girder is 52 MN/m^2, this value being compressive on the upper surface and tensile on the lower surface.

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Example (2)

A uniform T-section beam is 100 mm wide and 150 mm deep with a flange thickness of 25 mm and a web thickness of 12 mm. If the limiting bending stresses for the material of the beam are 80 MN/m^2 in compression and 160 MN/m^2 in tension, find the maximum u.d.1. that the beam can carry over a simply supported span of 5 m.

Solution

The second moment of area value I used in the simple bending theory is that about the N.A. Thus, in order to determine the I value of the T-section shown in Figure (17), it is necessary first to position the N.A.

Since this always passes through the centroid of the section we can take moments of area about the base to determine the position of the centroid and hence the N.A.

Thus

$$(100 \times 25 \times 137.5)10^{-9} + (125 \times 12 \times 62.5)10^{-9} = 10^{-6}[(100 \times 25) + (125 \times 12)\bar{y}]$$

$$(343750 + 93750)10^{-9} = 10^{-6}(2500 + 1500)\bar{y}$$

$$\bar{y} = \frac{437.5 \times 10^{-6}}{4000 \times 10^{-6}} = 109.4 \times 10^{-3} = 109.4 \text{ mm.}$$

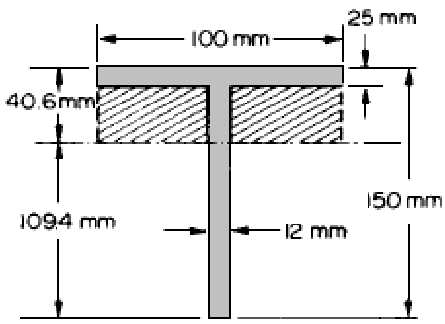

Figure (17)

Thus the N.A. is positioned, as shown, a distance of 109.4 mm above the base. The second moment of area I can now be found as suggested in Example **(1)** by dividing the section into convenient rectangles with their edges in the neutral axis.

$$I = \tfrac{1}{3}[(100 \times 40.6^3) - (88 \times 15.6^3) + (12 \times 109.4^3)]10^{-12}$$

$$= \tfrac{1}{3}(6.69 - 0.33 + 15.71)10^{-6} = 7.36 \times 10^{-6} \text{ m}^4$$

Now the maximum compressive stress will occur on the upper surface where y = **40.6** mm, and, using the limiting compressive stress value quoted,

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$$M = \frac{\sigma I}{y} = \frac{80 \times 10^6 \times 7.36 \times 10^{-6}}{40.6 \times 10^{-3}} = 14.5\ \mathbf{kN\,m}$$

This suggests a maximum allowable B.M. of 14.5 kN m. It is now necessary, however, to check the tensile stress criterion which must apply on the lower surface,

$$M = \frac{\sigma I}{y} = \frac{160 \times 10^6 \times 7.36 \times 10^{-6}}{109.4 \times 10^{-3}} = 10.76\ \mathbf{kN\,m}$$

The greatest moment that can therefore be applied to retain stresses within both conditions quoted is therefore M = 10.76 kNm. But for a simply supported beam with u.d.l.,

$$M_{max} = \frac{wL^2}{8}$$

$$w = \frac{8M}{L^2} = \frac{8 \times 10.76 \times 10^3}{5^2}$$

$$= \mathbf{3.4\ kN/m}$$

The u.d.1. must be limited to 3.4 kN m.

Example (3)

A flitched beam consists of two 50 mm x 200 mm wooden beams and a 12 mm x 80 mm steel plate. The plate is placed centrally between the wooden beams and recessed into each so that, when rigidly joined, the three units form a 100 mm x 200 mm section as shown in Figure(18). Determine the moment of resistance of the flitched beam when the maximum bending stress in the timber is 12 MN/m^2. What will then be the maximum bending stress in the steel?

For steel E = 200 GN/m^2; for wood E = 10 GN/m^2.

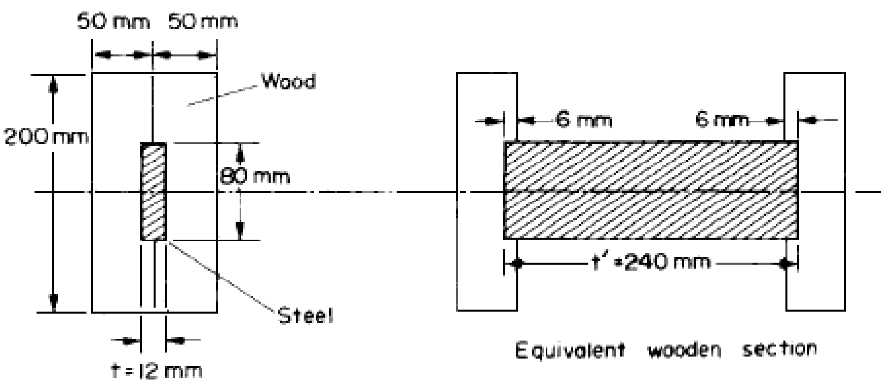

Figure (18)

Solution

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The flitched beam may be considered replaced by the equivalent wooden section shown in Figure (18). The thickness t' of the wood equivalent to the steel which it replaces is given by

$$t' = \frac{E}{E'} t = \frac{200 \times 10^9}{10 \times 10^9} \times 12 = 240 \text{ mm}$$

Then, for the equivalent section

$$I_{\text{N.A.}} = 2\left[\frac{50 \times 200^3}{12}\right] - 2\left[\frac{6 \times 80^3}{12}\right] + \left[\frac{240 \times 80^3}{12}\right] 10^{-12}$$

$$= (66.67 - 0.51 + 10.2)10^{-6} = 76.36 \times 10^{-6} \text{ m}^4$$

Now the maximum stress in the timber is **12** MN/m^2, and this will occur at y = **100** mm; thus, from the bending theory,

$$M = \frac{\sigma I}{y} = \frac{12 \times 10^6 \times 76.36 \times 10^{-6}}{100 \times 10^{-3}} = \mathbf{9.2 \text{ kN m}}$$

The moment of resistance of the beam, i.e. the bending moment which the beam can withstand within the given limit, is 9.2 kN m.
The maximum stress in the steel with this moment applied is then determined by finding first the maximum stress in the equivalent wood at the same position, i.e. at y = 40 mm. Therefore maximum stress in equivalent wood

$$\sigma'_{\max} = \frac{My}{I} = \frac{9.2 \times 10^3 \times 40 \times 10^{-3}}{76.36 \times 10^{-6}} = 4.82 \times 10^6 \text{ N/m}^2$$

The maximum stress in the steel is given by

$$\sigma_{\max} = \frac{E}{E'} \sigma'_{\max} = \frac{200 \times 10^9}{10 \times 10^9} \times 4.82 \times 10^6$$

$$= 96 \times 10^6 = \mathbf{96 \text{ MN/m}^2}$$

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Problems

1 - Determine the second moments of area about the axes *XX* for the sections shown in Figure (19). Ans. [15.69, 7.88, 41.15, 24; all x $10^{-6}m^4$]

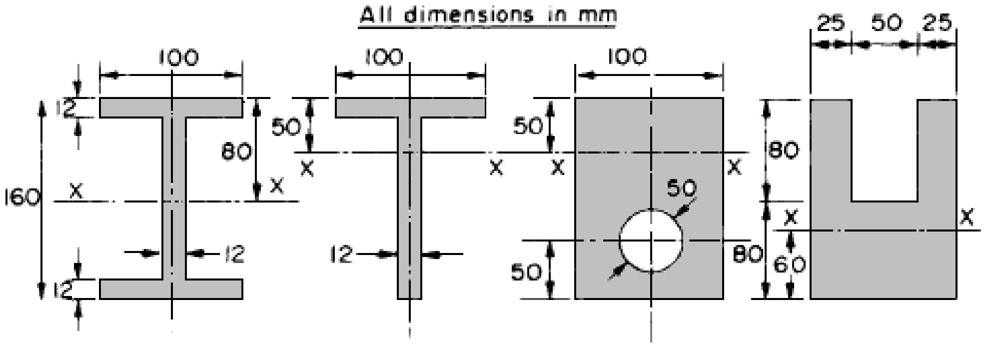

Figure (19)

2- A rectangular section beam has a depth equal to twice its width. It is the same material and mass per unit length as an I-section beam 300 mm deep with flanges 25 mm thick and 150 mm wide and a web 12 mm thick. Compare the flexural strengths of the two beams. Ans. [8.59: 1]

3- A conveyor beam has the cross-section shown in Figure (20) and it is subjected to a bending moment in the plane YK Determine the maximum permissible bending moment which can be applied to the beam (a) for bottom flange in tension, and (b) for bottom flange in compression, if the safe stresses for the material in tension and compression are 30 MN/m^2 and 150 MN/m^2 respectively. Ans.[32.3, 84.8 kN m.]

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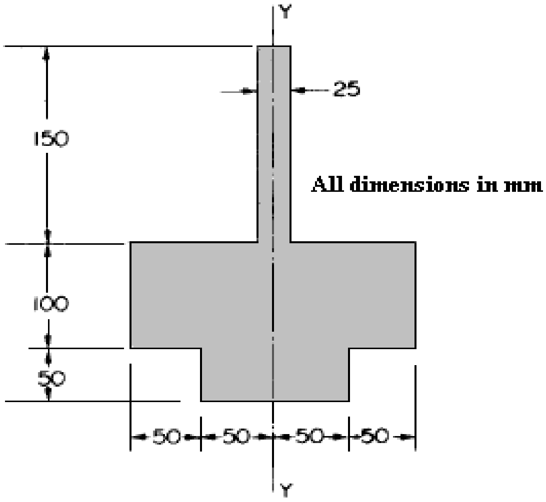

Figure (20)

4- A horizontal steel girder has a span of 3 m and is built-in at the left-hand end and freely supported at the other end. It carries a uniformly distributed load of 30 kN/m over the whole span, together with a single concentrated load of 20 kN at a point 2 m from the left-hand end. The supporting conditions are such that the reaction at the left-hand end is 65 kN.

(a) Determine the bending moment at the left-hand end and draw the B.M. diagram.

(b) Give the value of the maximum bending moment.

(c) If the girder is 200 mm deep and has a second moment of area of $40 \times 10^{-6} m^4$ determine the maximum stress resulting from bending.

Ans. [40 kN m; 100 MN/m^2]

5- Figure (21) represents the cross-section of an extruded alloy member which acts as a simply supported beam with the 75 mm wide flange at the bottom. Determine the moment of resistance of the section if the maximum permissible stresses in tension and compression are respectively 60 MN/m^2 and 45 MN/m^2.

Ans.[62 kN m]

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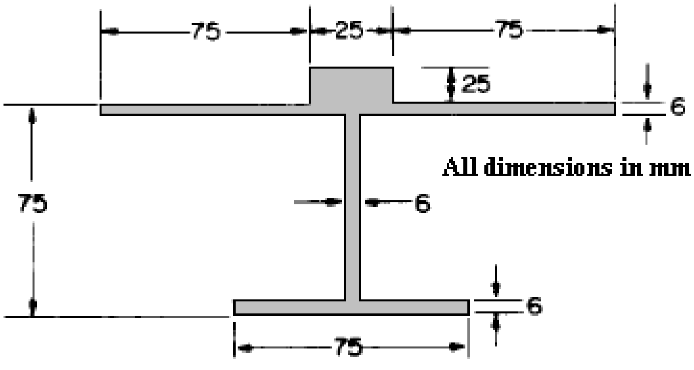

Figure (21)

6- A brass strip, 50 mm x 12 mm in section, is riveted to a steel strip, 65 mm x 10 mm in section, to form a compound beam of total depth 22 mm, the brass strip being on top and the beam section being symmetrical about the vertical axis. The beam is simply supported on a span of 1.3 m and carries a load of 2 kN at mid-span.
(a) Determine the maximum stresses in each of the materials owing to bending.
(b) Make a diagram showing the distribution of bending stress over the depth of the beam. Take E for steel = 200 GN/m^2 and E for brass = 100 GN/m^2.

Ans.[σ_b = 130 MN/m^2; σ_s = 162.9 MN/m^2]

7- A composite beam is of the construction shown in Figure (22). Calculate the allowable u.d.1. that the beam can carry over a simply supported span of 7 m if the stresses are limited to 120 MN/m^2 in the steel and 7 MN/m^2 in the timber.
Modular ratio = 20. Ans.[1.13 kN/m.]

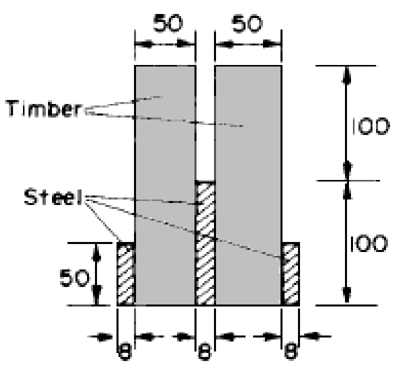

Figure (22)

8- Two bars, one of steel, the other of aluminium alloy, are each of 75 mm width and are rigidly joined together to form a rectangular bar 75 mm wide and of depth (t_s + t_A),

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where t_s = thickness of steel bar and t_A = thickness of alloy bar. Determine the ratio oft, to t,, in order that the neutral axis of the compound bar is coincident with the junction of the two bars. (E_s = 210 GN/m^2; E_A = 70 GN/m^2) If such a beam is 50 mm deep determine the maximum bending moment the beam can withstand if the maximum stresses in the steel and alloy are limited to 135 MN/m^2 and 37 MN/m^2 respectively.

Ans.[0.577; 1.47 kNm.]

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SLOPE AND DEFLECTION OF BEAMS

Introduction

In practically all engineering applications limitations are placed upon the performance and behavior of components and normally they are expected to operate within certain set limits of for example, stress or deflection. The stress limits are normally set so that the component does not yield or fail under the most severe load conditions which it is likely to meet in service.

Relationship between loading, S.F., B.M., slope and deflection

Consider a beam (**AB)** which is initially horizontal when unloaded. If this deflects to a new position (**$A^/B^/$)** under load, the slope at any point ***C*** is

$$i = \frac{dy}{dx}$$

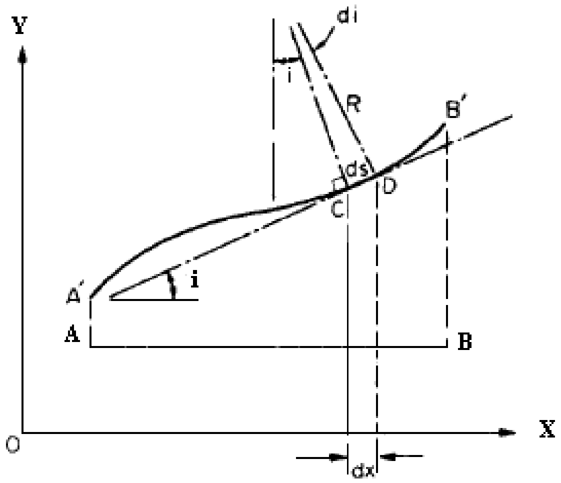

Figure (1) Unloaded beam AB deflected to $A^/B^/$ under load

This is usually very small in practice, and for small curvatures

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$$ds = dx = Rdi$$

$$\therefore \quad \frac{di}{dx} = \frac{1}{R}$$

But $\quad i = \frac{dy}{dx}$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{1}{R} \qquad \text{..... (1)}$$

Now from the simple bending theory

$$\frac{M}{I} = \frac{E}{R}$$

$$\therefore \quad \frac{1}{R} = \frac{M}{EI}$$

Therefore substituting in eqn. (1)

$$M = EI\frac{d^2y}{dx^2} \qquad \text{..... (2)}$$

This is the basic differential equation for the deflection of beams.

If the beam is now assumed to carry a distributed loading which varies in intensity over the length of the beam, then a small element of the beam of length (**dx)** will be subjected to the loading condition shown in Figure (2). The parts of the beam on either side of the element (**EFGH)** carry the externally applied forces, while reactions to these forces are shown on the element itself. Thus for vertical equilibrium of **(EFGH),**

$$Q - wdx = Q - dQ$$

$$dQ = wdx$$

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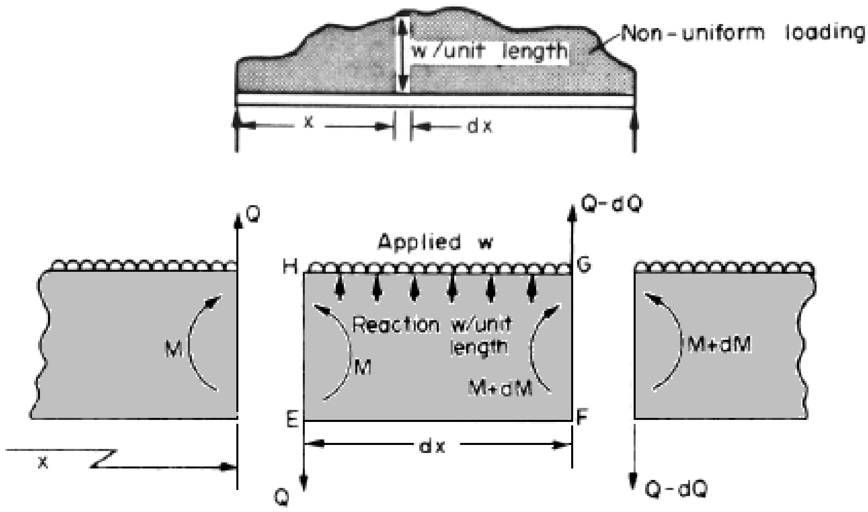

Figure (2) Small element of beam subjected to non-uniform loading (effectively uniform over small length dx).

and integrating,

$$Q = \int w dx \qquad \text{..... (3)}$$

Also, for equilibrium, moments about any point must be zero.
Therefore taking moments about F,

$$(M + dM) + wdx\frac{dx}{2} = M + Qdx$$

Therefore neglecting the square of small quantities,

$$dM = Qdx$$

$$M = \int Q dx$$

deflection = y

$$\text{slope} = \frac{dy}{dx}$$

$$\text{bending moment} = EI\frac{d^2y}{dx^2}$$

$$\text{shear force} = EI\frac{d^3y}{dx^3}$$

$$\text{load distribution} = EI\frac{d^4y}{dx^4}$$

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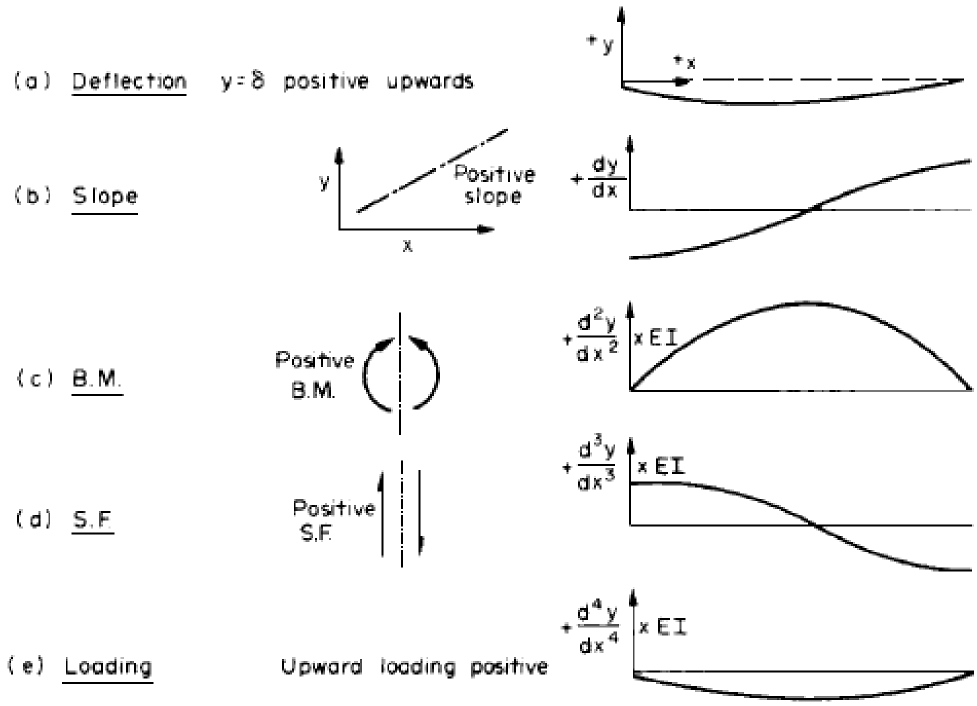

Figure (3) Sign conventions for load, S.F., B.M., slope and deflection.

In order that the above results should agree algebraically, i.e. that positive slopes shall have the normal mathematical interpretation of the positive sign and that B.M. and S.F. conventions are consistent with those introduced earlier, it is imperative that the sign convention illustrated in Figure (3) be adopted.

Direct integration method

If the value of the **B.M.** at any point on a beam is known in terms of ***x*,** the distance along the beam, and provided that the equation applies along the complete beam, then integration of eqn. (5.4a) will yield slopes and deflections at any point,
i.e.

$$M = EI\frac{d^2y}{dx^2} \quad \text{and} \quad \frac{dy}{dx} = \int \frac{M}{EI}dx + A$$

$$y = \int\int\left(\frac{M}{EI}dx\right)dx + Ax + B$$

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where **A** and **B** are constants of integration evaluated from known conditions of slope and deflection for particular values of **x**.

(a) Cantilever with concentrated load at the end

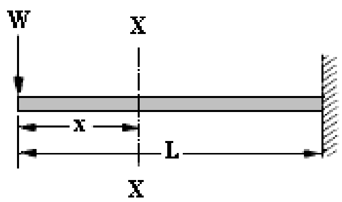

Figure (4)

$$M_{xx} = EI\frac{d^2y}{dx^2} = -Wx$$

$$\therefore \quad EI\frac{dy}{dx} = -\frac{Wx^2}{2} + A$$

assuming EI is constant.

$$EIy = -\frac{Wx^3}{6} + Ax + B$$

Now when $x = L, \quad \frac{dy}{dx} = 0 \quad \therefore \quad A = \frac{WL^2}{2}$

and when $x = L, y = 0 \quad \therefore \quad B = \frac{WL^3}{6} - \frac{WL^2}{2}L = -\frac{WL^3}{3}$

$$y = \frac{1}{EI}\left[-\frac{Wx^3}{6} + \frac{WL^2x}{2} - \frac{WL^3}{3}\right] \quad \text{..... (5)}$$

This gives the deflection at all values of x and produces a maximum value at the tip of the cantilever when x = 0,

$$Maximum\ deflection = y_{\max.} = -\frac{W.L^3}{3E.I} \quad \text{...... (6)}$$

The negative sign indicates that deflection is in the negative y direction, i.e. downwards.

$$\frac{dy}{dx} = \frac{1}{EI}\left[-\frac{Wx^2}{2} + \frac{WL^2}{2}\right] \quad \text{..... (7)}$$

and produces a maximum value again when x = 0.

$$\textbf{Maximum slope} = \left(\frac{dy}{dx}\right)_{\max} = \frac{WL^2}{2EI} \quad \textbf{(positive)} \quad \text{..... (8)}$$

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(b) Cantilever with uniformly distributed load

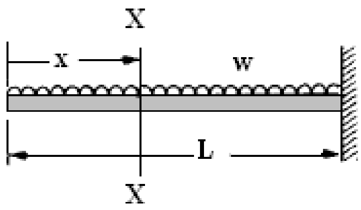

Figure (5)

$$M_{xx} = EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2}$$

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + A$$

$$EIy = -\frac{wx^4}{24} + Ax + B$$

Again, when $x = L, \quad \frac{dy}{dx} = 0 \quad \text{and} \quad A = \frac{wL^3}{6}$

$x = L, \quad y = 0 \quad \text{and} \quad B = \frac{wL^4}{24} - \frac{wL^4}{6} = -\frac{wL^4}{8}$

$$\therefore \qquad y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8}\right] \qquad \text{..... (9)}$$

At $x = 0$, $\quad y_{max} = -\frac{wL^4}{8EI}$ **and** $\left(\frac{dy}{dx}\right)_{max} = \frac{wL^3}{6EI}$ ..... (10)

(c) Simply-supported beam with uniformly distributed load

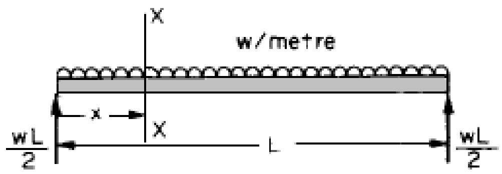

Figure (6)

$$M_{xx} = EI\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2}$$

$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + A$$

$$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + Ax + B$$

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At $x = 0, \quad y = 0 \quad \therefore \quad B = 0$

At $x = L, \quad y = 0 \quad \therefore \quad 0 = \frac{wL^4}{12} - \frac{wL^4}{24} + AL$

$$\therefore \quad A = -\frac{wL^3}{24}$$

$$\therefore \quad y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right] \qquad \text{...... (11)}$$

In this case the maximum deflection will occur at the centre of the beam where x = L/2.

$$y_{max} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$= -\frac{5wL^4}{384EI} \qquad \text{...... (12)}$$

$$\left(\frac{dy}{dx}\right)_{max} = \pm\frac{wL^3}{24EI} \text{ at the ends of the beam.} \qquad \text{..... (13)}$$

(d) Simply supported beam with central concentrated load

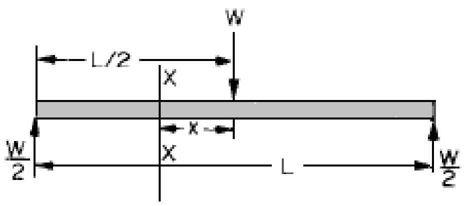

Figure (7)

In order to obtain a single expression for B.M. which will apply across the complete beam in this case it is convenient to take the origin for x at the centre, then:

$$M_{xx} = EI\frac{d^2y}{dx^2} = \frac{W}{2}\left(\frac{L}{2} - x\right) = \frac{WL}{4} - \frac{Wx}{2}$$

$$EI\frac{dy}{dx} = \frac{WL}{4}x - \frac{Wx^2}{4} + A$$

$$EIy = \frac{WLx^2}{8} - \frac{Wx^3}{12} + Ax + B$$

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At $x = 0,\ \dfrac{dy}{dx} = 0 \quad \therefore \quad A = 0$

$$x = \frac{L}{2},\quad y = 0 \quad \therefore \quad 0 = \frac{WL^3}{32} - \frac{WL^3}{96} + B$$

$$\therefore \quad B = -\frac{WL^3}{48}$$

$$y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48}\right] \qquad \text{...... (14)}$$

$$y_{max} = -\frac{WL^3}{48EI} \quad \text{at the centre} \qquad \text{..... (15)}$$

$$\left(\frac{dy}{dx}\right)_{max} = \pm\frac{WL^2}{16EI} \quad \text{at the ends} \qquad \text{...... (16)}$$

(e) Cantilever subjected to non-uniform distributed load

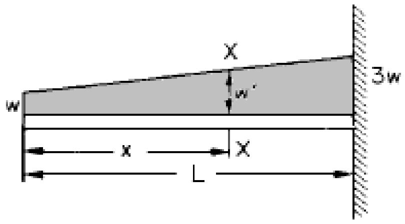

Figure (8)

The loading at section X X is

$$w' = EI\frac{d^4y}{dx^4} = -\left[w + (3w - w)\frac{x}{L}\right] = -w\left(1 + \frac{2x}{L}\right)$$

$$EI\frac{d^3y}{dx^3} = -w\left(x + \frac{x^2}{L}\right) + A \qquad (1)$$

$$EI\frac{d^2y}{dx^2} = -w\left(\frac{x^2}{2} + \frac{x^3}{3L}\right) + Ax + B \qquad (2)$$

$$EI\frac{dy}{dx} = -w\left(\frac{x^3}{6} + \frac{x^4}{12L}\right) + \frac{Ax^2}{2} + Bx + C \qquad (3)$$

$$EIy = -w\left(\frac{x^4}{24} + \frac{x^5}{60L}\right) + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D \qquad (4)$$

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Thus, before the slope or deflection can be evaluated, four constants have to be determined; therefore four conditions are required. They are:

At x = 0, S.F. is zero

from (1); A = O

At x = 0, B.M. is zero

from (2); B = O

At x = L, slope dy/dx = 0 (slope normally assumed zero at a built-in support)

from (3)

$$0 = -w\left(\frac{L^3}{6} + \frac{L^3}{12}\right) + C$$

$$C = \frac{wL^3}{4}$$

At $x = L$, $y = 0$

from (4) $$0 = -w\left(\frac{L^4}{24} + \frac{L^4}{60}\right) + \frac{wL^4}{4} + D$$

$$D = -\frac{23wL^4}{120}$$

$$EIy = -\frac{wx^4}{24} - \frac{wx^5}{60L} + \frac{wL^3x}{4} - \frac{23wL^4}{120}$$

Then, for example, the deflection at the tip of the cantilever, where *x* = 0, is

$$y = -\frac{23wL^4}{120EI}$$

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SLOPE AND DEFLECTION OF BEAMS

Macaulay's method

The simple integration method used in the previous examples can only be used when a single expression for B.M. applies along the complete length of the beam. In general this is not the case, and the method has to be adapted to cover all loading conditions. Consider, therefore, a small portion of a beam in which, at a particular section A, the shearing force is Q and the B.M. is M, as shown in Figure (9). At another section B, distance a along the beam, a concentrated load W is applied which will change the B.M. for points beyond B.

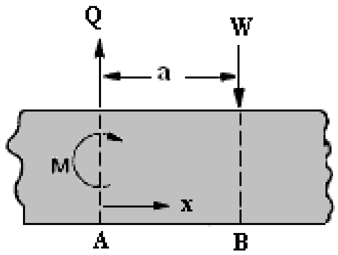

Figure (9)

Between A and B,

$$M = EI\frac{d^2y}{dx^2} = M + Qx \qquad (1)$$

$$EI\frac{dy}{dx} = Mx + Q\frac{x^2}{2} + C_1 \qquad (2)$$

$$EIy = M\frac{x^2}{2} + Q\frac{x^3}{6} + C_1x + C_2 \qquad (3)$$

Beyond, B

$$M = EI\frac{d^2y}{dx^2} = M + Qx - W(x-a) \qquad (4)$$

$$EI\frac{dy}{dx} = Mx + Q\frac{x^2}{2} - W\frac{x^2}{2} + Wax + C_3 \qquad (5)$$

$$EIy = M\frac{x^2}{2} + Q\frac{x^3}{6} - W\frac{x^3}{6} + Wa\frac{x^2}{2} + C_3x + C_4 \qquad (6)$$

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Now for the same slope at B, equating (2) and (5),

$$Mx + Q\frac{x^2}{2} + C_1 = Mx + Q\frac{x^2}{2} - W\frac{x^2}{2} + Wax + C_3$$

But at B, x = a

$$C_1 = -\frac{Wa^2}{2} + Wa^2 + C_3$$

$$C_3 = C_1 - \frac{Wa^2}{2}$$

Substituting in (5)

$$EI\frac{dy}{dx} = Mx + Q\frac{x^2}{2} - W\frac{x^2}{2} + Wax + C_1 - \frac{Wa^2}{2}$$

$$EI\frac{dy}{dx} = Mx + Q\frac{x^2}{2} - \frac{W}{2}(x-a)^2 + C_1 \qquad (7)$$

Also, for the same deflection at (B) equating (3) and (6), with x = a

$$\frac{Ma^2}{2} + \frac{Qa^3}{6} + C_1a + C_2 = \frac{Ma^2}{2} + \frac{Qa^3}{6} - \frac{Wa^3}{6} + \frac{Wa^3}{2} + C_3a + C_4$$

$$C_1a + C_2 = -\frac{Wa^3}{6} + \frac{Wa^3}{2} + C_3a + C_4$$

$$= -\frac{Wa^3}{6} + \frac{Wa^3}{2} + \left(C_1 - \frac{Wa^2}{2}\right)a + C_4$$

$$C_4 = C_2 + \frac{Wa^3}{6}$$

Substituting in (6),

$$EIy = M\frac{x^2}{2} + Q\frac{x^3}{6} - W\frac{x^3}{6} + Wa\frac{x^2}{2}\left(C_1 - \frac{Wa^2}{2}\right)x + W\frac{a^3}{6} + C_2$$

$$= M\frac{x^2}{2} + Q\frac{x^3}{6} - W\frac{(x-a)^3}{6} + C_1x + C_2 \qquad (8)$$

Thus, inspecting (4), (7) and (8), we can see that the general method of obtaining slopes and deflections (i.e. integrating the equation for M) will still apply provided that the term W(x -a) is integrated with respect to (x -a) and not x. Thus, when integrated, the term becomes

$$W\frac{(x-a)^2}{2} \quad \text{and} \quad W\frac{(x-a)^3}{6}$$

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successively.

In addition, since the term W(x - a) applies only after the discontinuity, i.e. when x > a, it should be considered only when x > a or when (x - a) is positive. For these reasons such terms are conventionally put into square or curly brackets and called Macaulay terms.

Thus Macaulay terms must be (a) integrated with respect to themselves and (b) neglected when negative.

For the whole beam, therefore,

$$EI\frac{d^2y}{dx^2} = M + Qx - W[(x-a)]$$

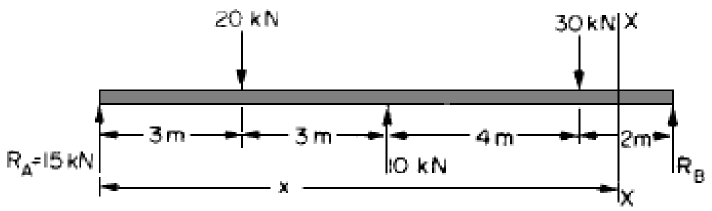

Figure (10)

As an illustration of the procedure consider the beam loaded as shown in Figure (10) for which the central deflection is required. Using the Macaulay method the equation for the B.M. at any general section XX is then given by

$$\text{B.M.}_{XX} = 15x - 20[(x-3)] + 10[(x-6)] - 30[(x-10)]$$

Care is then necessary to ensure that the terms inside the square brackets (Macaulay terms) are treated in the special way noted on the previous page.

Here it must be emphasised that all loads in the right-hand side of the equation are in units of kN (i.e. newtons X 10^3).I n subsequent working, therefore, it is convenient to carry through this factor as a denominator on the left-hand side in order that the expressions are dimensionally correct.

Integrating,

$$\frac{EI}{10^3}\frac{dy}{dx} = 15\frac{x^2}{2} - 20\left[\frac{(x-3)^2}{2}\right] + 10\left[\frac{(x-6)^2}{2}\right] - 30\left[\frac{(x-10)^2}{2}\right] + A$$

$$\frac{EI}{10^3}y = 15\frac{x^3}{6} - 20\left[\frac{(x-3)^3}{6}\right] + 10\left[\frac{(x-6)^3}{6}\right] - 30\left[\frac{(x-10)^3}{6}\right] + Ax + B$$

where A and B are two constants of integration.

Now when x =0, y = *0* .’. B = 0

and when x = 12, y = *0*

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$$0 = \frac{15 \times 12^3}{6} - 20\left[\frac{9^3}{6}\right] + 10\left[\frac{6^3}{6}\right] - 30\left[\frac{2^3}{6}\right] + 12A$$

$$= 4320 - 2430 + 360 - 40 + 12A$$

$$12A = -4680 + 2470 = -2210$$

$$A = -184.2$$

The deflection at any point is given by

$$\frac{EI}{10^3} y = 15\frac{x^3}{6} - 20\left[\frac{(x-3)^3}{6}\right] + 10\left[\frac{(x-6)^3}{6}\right] - 30\left[\frac{(x-10)^3}{6}\right] - 184.2x$$

The deflection at mid-span is thus found by substituting x = 6 in the above equation, bearing in mind that the dimensions of the equation are kN.m^3.

N.B.-Two of the Macaulay terms then vanish since one becomes zero and the other negative and therefore neglected.

$$\text{central deflection} = \frac{10^3}{EI}\left[\frac{15 \times 6^3}{6} - \frac{20 \times 3^3}{6} - 184.2 \times 6\right]$$

$$= -\frac{655.2 \times 10^3}{EI}$$

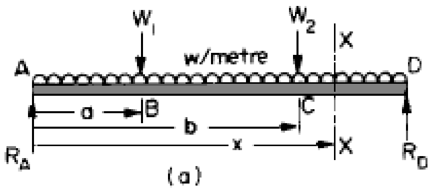
With typical values of E = 208 GN/m^2 and I = 82 x m^4

central deflection = 38.4 x l0-3 m = 38.4 mm

Macaulay's method for uniformly distributed load (u.d.l)

If a beam carries a uniformly distributed load over the complete span as shown in Figure (11a).

$$\text{B.M.}_{xx} = EI\frac{d^2y}{dx^2} = R_A x - \frac{wx^2}{2} - W_1[(x-a)] - W_2[(x-b)]$$



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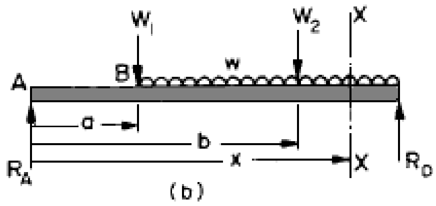

Figure (11)

The u.d.l. term applies across the complete span and does not require the special treatment associated with the Macaulay terms. If, however, the u.d.1. starts at B as shown in Figure (11b) the B.M. equation is modified and the u.d.1. term becomes a Macaulay term and is written inside square brackets.

$$\text{B.M.}_{XX} = EI\frac{d^2y}{dx^2} = R_A x - W_1[(x-a)] - w\left[\frac{(x-a)^2}{2}\right] - W_2[(x-b)]$$

Integrating,

$$EI\frac{dy}{dx} = R_A\frac{x^2}{2} - W_1\left[\frac{(x-a)^2}{2}\right] - w\left[\frac{(x-a)^3}{6}\right] - W_2\left[\frac{(x-b)^2}{2}\right] + A$$

$$EIy = R_A\frac{x^3}{6} - W_1\left[\frac{(x-a)^3}{6}\right] - w\left[\frac{(x-a)^4}{24}\right] - W_2\left[\frac{(x-b)^3}{6}\right] + Ax + B$$

Note that Macaulay terms are integrated with respect to, for example, ***(x** -a)* and they must be ignored when negative. Substitution of end conditions will then yield the values of the constants ***A*** and ***B*** in the normal way and hence the required values of slope or deflection. It must be appreciated, however, that once a term has been entered in the B.M. expression it will apply across the complete beam. The modifications to the procedure required for cases when u.d.1.s. are applied over part of the beam only are introduced in the following theory.

Macaulay's method for beams with u.d.1. applied over part of the beam

Consider the beam loading case shown in Figure (12).

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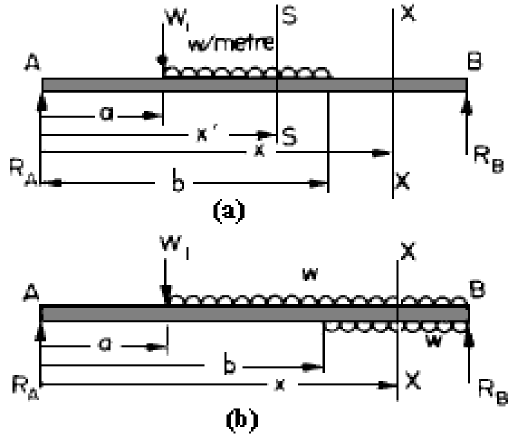

Figure (12)

The B.M. at the section SS is given by the previously introduced procedure as

$$\text{B.M.}_{SS} = R_A x' - W_1[(x'-a)] - W\left[\frac{(x'-a)^2}{2}\right]$$

Having introduced the last (u.d.1.) term, however, it will apply for all values of x' greater than a, i.e. across the rest of the span to the end of the beam. (Remember, Macaulay terms are only neglected when they are negative, e.g. with $x' < a$.) The above equation is NOT therefore the correct equation for the load condition shown. The Macaulay method requires that this continuation of the u.d.1. be shown on the loading diagram and the required loading condition can therefore only be achieved by introducing an equal and opposite u.d.1. over the last part of the beam to cancel the unwanted continuation of the initial distributed load. This procedure is shown in Figure (12b).
The correct B.M. equation for any general section XX is then given by

$$\text{B.M.}_{XX} = EI\frac{d^2y}{dx^2} = R_A x - W_1[(x-a)] - w\left[\frac{(x-a)^2}{2}\right] + w\left[\frac{(x-b)^2}{2}\right]$$

This type of approach can be adopted for any beam loading cases in which u.d.1.s are stopped or added to.

Example (1)
Determine the deflection at a point 1 m from the left-hand end of the beam loaded as shown in Figure (13a) using Macaulay’s method. EI = 0.65 MN m^2.

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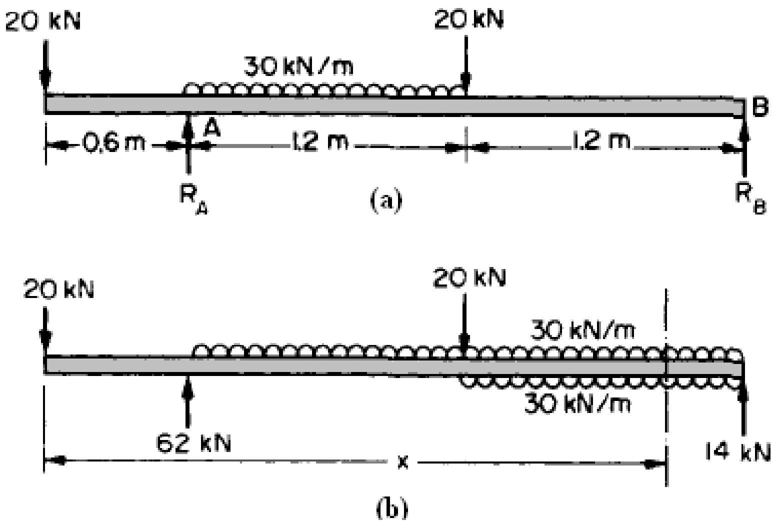

Figure (13)

Solution

Taking moments about **B**

$$(3 \times 20) + (30 \times 1.2 \times 1.8) + (1.2 \times 20) = 2.4R_A$$

$$R_A = 62\,\text{kN} \quad \text{and} \quad R_B = 20 + (30 \times 1.2) + 20 - 62 = 14\,\text{kN}$$

Using the modified Macaulay approach for distributed loads over part of a beam introduced in Figure (13b),

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$$M_{xx} = \frac{EI}{10^3}\frac{d^2y}{dx^2} = -20x + 62[(x-0.6)] - 30\left[\frac{(x-0.6)^2}{2}\right] + 30\left[\frac{(x-1.8)^2}{2}\right] - 20[(x-1.8)]$$

$$\frac{EI}{10^3}\frac{dy}{dx} = \frac{-20x^2}{2} + 62\left[\frac{(x-0.6)^2}{2}\right] - 30\left[\frac{(x-0.6)^3}{6}\right] + 30\left[\frac{(x-1.8)^3}{6}\right] - 20\left[\frac{(x-1.8)^2}{2}\right] + A$$

$$\frac{EI}{10^3}y = \frac{-20x^3}{6} + 62\left[\frac{(x-0.6)^3}{6}\right] - 30\left[\frac{(x-0.6)^4}{24}\right] + 30\left[\frac{(x-1.8)^4}{24}\right] - 20\left[\frac{(x-1.8)^3}{6}\right] + Ax + B$$

Now when x = 0.6, y = 0

$$0 = -\frac{20 \times 0.6^3}{6} + 0.6A + B$$

$$0.72 = 0.6A + B \qquad (1)$$

and when $x = 3$, $y = 0$,

$$0 = -\frac{20 \times 3^3}{6} + \frac{62 \times 2.4^3}{6} - \frac{30 \times 2.4^4}{24} + \frac{30 \times 1.2^4}{24} - \frac{20 \times 1.2^3}{6} + 3A + B$$

$$= -90 + 142.848 - 41.472 + 2.592 - 5.76 + 3A + B$$

$$-8.208 = 3A + B \qquad (2)$$

$$(2) - (1)$$

$$-8.928 = 2.4A \qquad \therefore \quad A = -3.72$$

Substituting in (1),

$$B = 0.72 - 0.6(-3.72) \qquad B = 2.952$$

Substituting into the Macaulay deflection equation,

$$\frac{EI}{10^3}y = -\frac{20x^3}{6} + 62\left[\frac{(x-0.6)^3}{6}\right] - 30\left[\frac{(x-0.6)^4}{24}\right] + 30\left[\frac{(x-1.8)^4}{24}\right] - 20\left[\frac{(x-1.8)^3}{6}\right] - 3.72x + 2.952$$

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At $\quad x = 1$

$$y = \frac{10^3}{EI}\left[-\frac{20}{6} + \frac{62}{6} \times 0.4^3 - \frac{30 \times 0.4^4}{24} - 3.72 \times 1 + 2.952\right]$$

$$= \frac{10^3}{EI}[-3.33 + 0.661 - 0.032 - 3.72 + 2.952]$$

$$= -\frac{10^3 \times 3.472}{0.65 \times 10^6} = -5.34 \times 10^{-3}\,\text{m} = \mathbf{-5.34\,mm}$$

The beam therefore is deflected downwards at the given position.

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SLOPE AND DEFLECTION OF BEAMS

3- Finite difference method

A numerical method for the calculation of beam deflections which is particularly useful for non-prismatic beams or for cases of irregular loading is the so-called finite difference method. The basic principle of the method is to replace the standard differential equation (1) by its finite difference approximation, obtain equations for deflections in terms of moments at various points along the beam and solve these simultaneously to yield the required deflection values.

Consider, therefore, Figure (1) which shows part of a deflected beam with the x-axis divided into a series of equally spaced intervals. By convention, the ordinates are numbered with respect to the Central ordinate *B*.

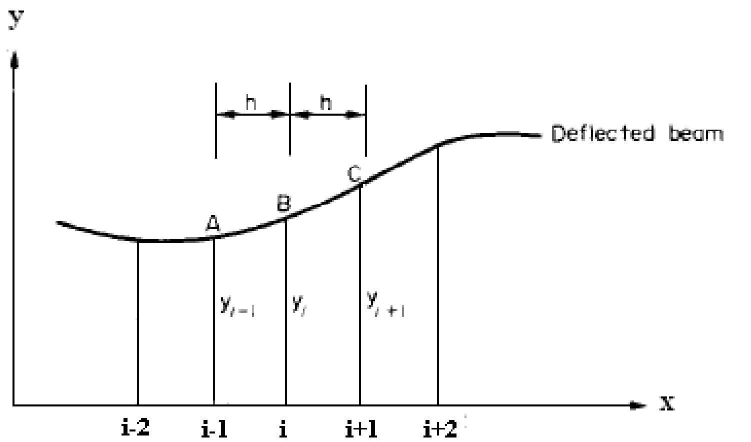

Figure (1) Deflected beam curve divided into a convenient number of equally spaced intervals

If the equation for the deflected curve of the beam is taken to be $y = f(x)$ then the first derivative dy/dx at B is the slope of the curve at B. Approximately (provided h is small) this can be taken to be the slope of the chord joining A and C so that:

$$\left(\frac{dy}{dx}\right)_i \doteq \frac{(y_{i+1} - y_{i-1})}{2h} = \frac{1}{2h}(y_{i+1} - y_{i-1}) \qquad \text{.....(1)}$$

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The rate of change of the first derivative, i.e. the rate of change of the slope $= \frac{d^2y}{dx^2}$ given in the same way approximately as the slope to the right of (i) minus the slope to the left of (i) divided by the interval between them.

$$\left(\frac{d^2y}{dx^2}\right)_i = \frac{\frac{(y_{i+1}-y_i)}{h} - \frac{(y_i - y_{i-1})}{h}}{h} = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \qquad \text{.....(2)}$$

Equations (1) and (2) are the finite difference approximations of the standard beam deflection differential equations and, because they are written in terms of ordinates on either side of the central point (i), they are known as central differences. Alternative expressions which can be formed to contain only ordinates at, or to the right of (i), or ordinates at, or to the left of (i) are known as forward and backward differences, respectively but these will not be considered here.

Now from equation , $M = EI\frac{d^2y}{dx^2}$

.'. At position (i), combining equations between above equation and equation (2).

$$\left(\frac{M}{EI_i}\right) = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) \qquad \text{.....(3)}$$

A solution for any of the deflection (y) values can then be obtained by applying the finite difference equation at a series of points along the beam and solving the resulting simultaneous equations. The higher the number of points selected the greater the accuracy of solution but the more the number of equations which are required to be solved. The method thus lends itself to computer-assisted evaluation. In addition to the solution of statically determinate beam problems of the type treated in example (1) the method is also applicable to the analysis of statically indeterminate beams.

The principal advantages of the finite difference method are thus:

(a) that it can be applied to statically determinate or indeterminate beams,

(b) that it can be used for non-prismatic beams,

(c) that it is amenable to computer solutions.

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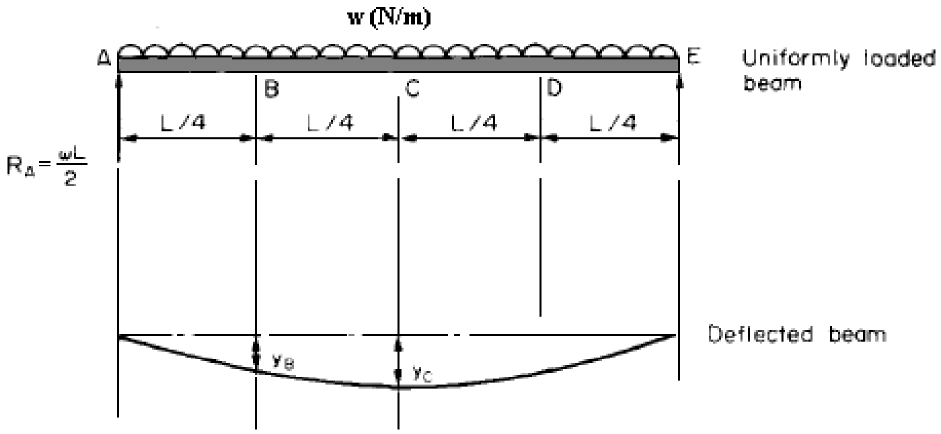
Example (1)

Using the finite difference method, determine the central deflection of a simply-supported beam carrying a uniformly distributed load over its complete span. The beam can be assumed to have constant flexural rigidity El throughout.

Solution



As a simple demonstration of the finite difference approach, assume that the beam is divided into only four equal segments (thus reducing the accuracy of the solution from that which could be achieved with a greater number of segments).

The n,

$$\text{B.M. at B} = \frac{\omega L}{2} \times \frac{L}{4} - \frac{\omega L}{4}\,\frac{L}{8} = \frac{3\omega L^2}{32} = M_B$$

but, from equation (3)

$$\frac{M_B}{EI} = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$

$$\frac{I}{EI}\left(\frac{3\omega L^2}{32}\right) = \frac{1}{(L/4)^2}(y_c - 2y_B + y_A)$$

and, since y, = 0,

$$\frac{3\omega L^2}{512\,EI} = y_c - 2y_B \qquad \text{.....(1)}$$

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$$\text{B.M. at } C = \frac{\omega L}{2}.\frac{L}{2} - \frac{\omega L}{2}.\frac{L}{4} = \frac{\omega L^2}{8} = M_c.$$

and, from equation (3)

$$\frac{1}{EI}\left(\frac{\omega L^2}{8}\right) = \frac{1}{(L/4)^2}(y_B - 2y_c + y_D)$$

Now, from symmetry, $y_D = y_B$

$$\frac{\omega L^4}{128EI} = 2y_B - 2y_c \qquad \text{......(2)}$$

Adding eqns. (1) and (2);

$$-y_c = \frac{\omega L^4}{128EI} + \frac{3\omega L^4}{512EI}$$

$$y_c = \frac{-7\omega L^4}{512EI} = -0.0137\frac{\omega L^4}{EI}$$

the negative sign indicating a downwards deflection as expected. This value compares with the "exact method (by direct method)" value of:

$$y_c = \frac{5\omega L^4}{384EI} = -0.01302\frac{\omega L^4}{EI}$$

a difference of about 5 %. As stated earlier, this comparison could be improved by selecting more segments but, nevertheless, it is remarkably accurate for the very small number of segments chosen.

Problems

1-A beam of length 10 m is symmetrically placed on two supports 7m apart. The loading is 15 kN/m between the supports and 20 kN at each end. What is the central deflection of the beam? E = 210 GN/m^2; I = 200 x $10^{-6}$$m^4$.

Ans. [6.8 mm.]

2-Derive the expression for the maximum deflection of a simply supported beam of negligible weight carrying a point load at its mid-span position. The distance between

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the supports is L, the second moment of area of the cross-section is (I) and the modulus of elasticity of the beam material is (E).

The maximum deflection of such a simply supported beam of length 3 m is 4.3 mm when carrying a load of 200 kN at its mid-span position. What would be the deflection at the free end of a cantilever of the same material, length and cross-section if it carries a load of l00kN at a point 1.3m from the free end? Ans.[13.4 mm]

3-A horizontal beam, simply supported at its ends, carries a load which varies uniformly from 15 kN/m at one end to 60 kN/m at the other. Estimate the central deflection if the span is 7 m, the section 450 mm deep and the maximum bending stress 100 MN/m^2, E = 210 GN/m^2. Ans. [21.9 mm.]

4-A beam AB, 8 m long, is freely supported at its ends and carries loads of 30 kN and 50 kN at points 1 m and 5 m respectively from A. Find the position and magnitude of the maximum deflection. E = 210 GN/m^2; I = 200 x $10^{-6}m^4$.

Ans.[14.4 mm.]

5-A beam 7 m long is simply supported at its ends and loaded as follows: 120 kN at 1 m from one end A, 20 kN at 4 m from A and 60 kN at *5* m from A. Calculate the position and magnitude of the maximum deflection. The second moment of area of the beam section is 400 x$10^{-6}m^4$ and (E) for the beam material is 210GN/m^2.

Ans. [9.8 mm at 3.474m.]

6- A beam ABCD, 6 m long, is simply-supported at the right-hand end *D* and at a point B 1 m from the left hand end A. It carries a vertical load of 10 kN at A, a second concentrated load of 20 kN at C, 3 m from *D*, and a uniformly distributed load of 10 kN/m between C and *D.* Determine the position and magnitude of the maximum deflection if E = 208 GN/m^2 and *1* = 35 x 10^{-6} m^4 from A,

Ans. [3.553 m from A, 11.95 mm.]

7- A 3 m long cantilever ABC is built-in at A, partially supported at B, 2 m from A, with a force of 10 kN and carries a vertical load of 20 kN at C. A uniformly distributed load of *5* kN/m is also applied between A and B.

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Determine(a) the values of the vertical reaction and built-in moment at A and (b) the deflection of the free end C of the cantilever. Develop an expression for the slope of the beam at any position and hence plot a slope diagram. E = 208 GN/m^2 and I = 24 x 10^{-6} m^4. Ans. [20 kN, 50 kN.m, -15mm]

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FATIGUE, CREEP AND FRACTURE

Fatigue

Fracture of components due to fatigue is the most common cause of service failure, particularly in shafts, axles, aircraft wings, etc., where cyclic stressing is taking place. With static loading of a ductile material, plastic flow precedes final fracture, the specimen necks and the fractured surface reveals a fibrous structure, but with fatigue, the crack is initiated from points of high stress concentration on the surface of the component such as sharp changes in cross-section, slag inclusions, tool marks, etc., and then spreads or propagates under the influence of the load cycles until it reaches a critical size when fast fracture of the remaining cross-section takes place.

The Stress to number of cycles (S/N) curve

Fatigue tests are usually carried out under conditions of rotating - bending and with a zero mean stress as obtained by means of a Wohler machine. From Figure (1), it can be seen that the top surface of the specimen, held "cantilever fashion" in the machine, is in tension, whilst the bottom surface is in compression. As the specimen rotates, the top surface moves to the bottom and hence each segment of the surface moves continuously from tension to compression producing a stress-cycle curve as shown in Figure (2). In order to understand certain terms in common usage, let us consider a stress-cycle curve where there is a positive tensile mean stress as may be obtained using other types of fatigue machines such as a Haigh "push-pull" machine.

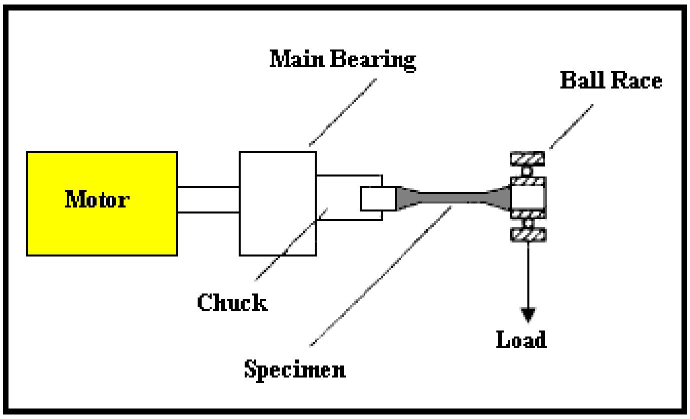

Figure (1) Single point load arrangement in a Wohler machine for zero mean stress fatigue testing

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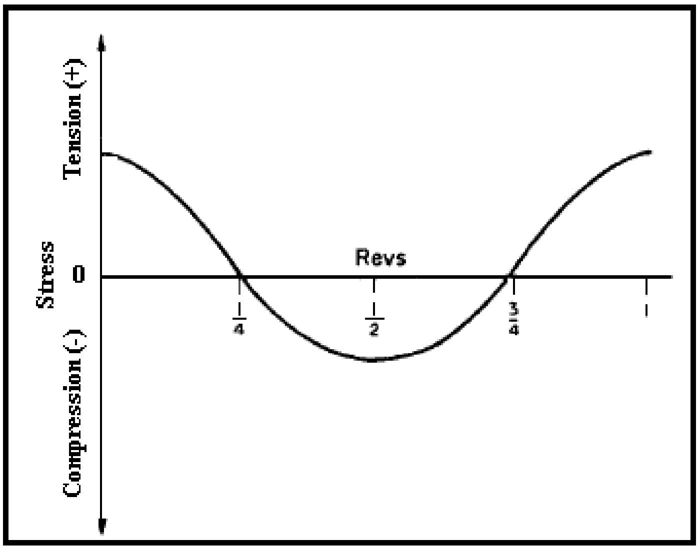

Figure (2) Simple sinusoidal (zero mean) stress fatigue curve, "reversed-symmetrical"

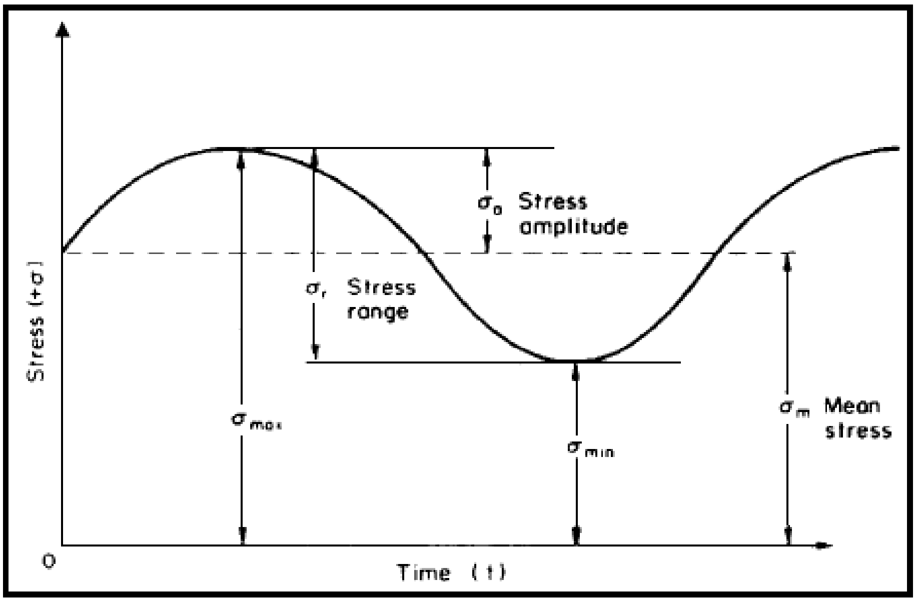

Figure (3) Fluctuating tension stress cycle producing positive mean stress

The stress-cycle curve is shown in Figure (3), and from this diagram it can be seen that:

$$\textbf{Stress range, } \sigma_r = 2\sigma_a. \qquad \text{......(1)}$$

$$\textbf{Mean stress, } \sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2} \qquad \text{........(2)}$$

$$\textbf{Alternating stress amplitude, } \sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2} \qquad \text{.......(3)}$$

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If the mean stress is not zero, we sometimes make use of the "stress ratio" ***R,*** where

$$R_s = \frac{\sigma_{\min}}{\sigma_{\max}} \qquad \text{....... (4)}$$

The most general method of presenting the results of a fatigue test is to plot a graph of the stress amplitude as ordinate against the corresponding number of cycles to failure.

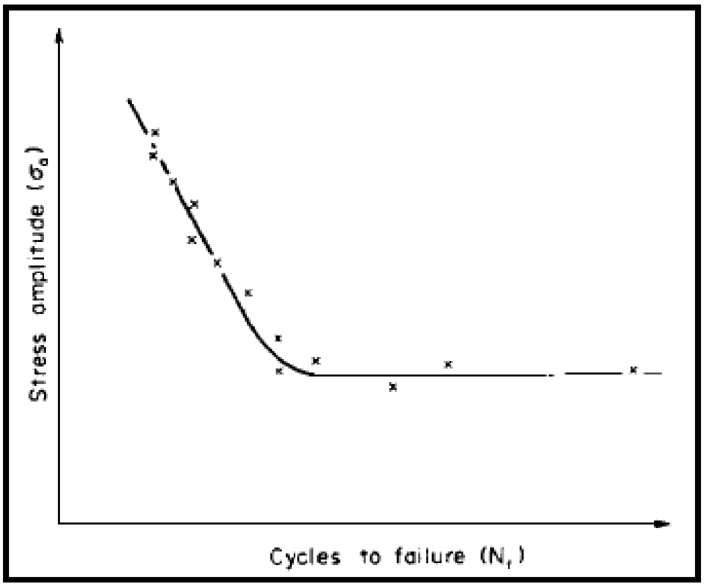

Figure (4) Typical S/N curve fatigue life curve

In using the **S/N** curve for design purposes it may be advantageous to express the relationship between (a) and (***N_f)***, the number of cycles to failure.

$$\sigma_r^a N_f = K \qquad \text{...... (5)}$$

Where: (a) is a constant which varies from 8 to 15 and (K) is a second constant depending on the material. From the S/N curve the "fatigue limit" or "endurance limit" may be ascertained. The "**fatigue limit**" is the stress condition below which a material may endure an infinite number of cycles prior to failure. Ferrous metal specimens often produce S/N curves which exhibit fatigue limits as indicated in Figure (5a) The '**fatigue strength**" or "**endurance limit**", is the stress condition under which a specimen would have a fatigue life of N cycles as shown in Figure (5B) Non-ferrous metal specimens show this type of curve and hence components made from aluminium, copper and nickel, etc., must always be designed for a finite life.

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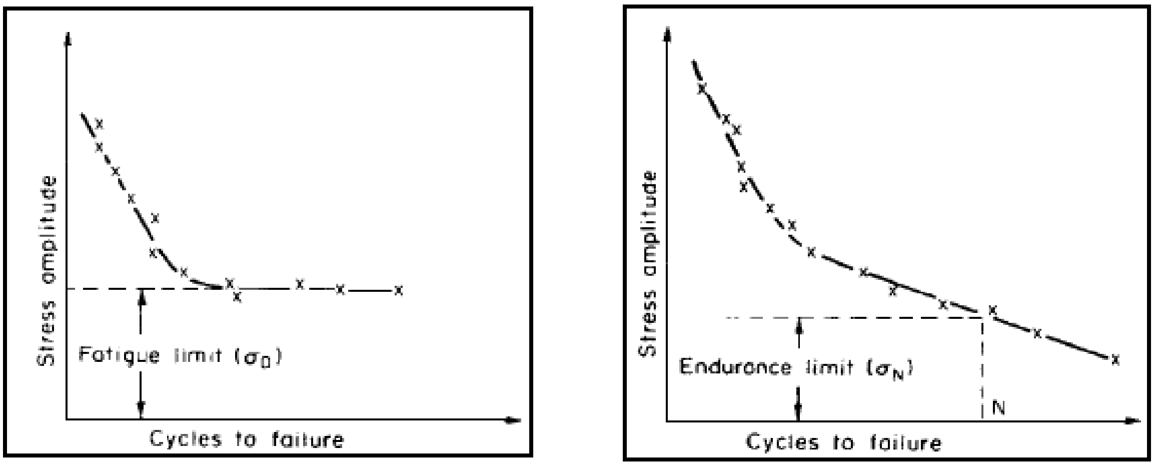

Figure (5) S/N curve showing (a) fatigue limit (b) endurance limit

$$\sigma'_N = \frac{\sigma_N}{K_f}[C_a \cdot C_b \cdot C_c] \qquad \text{......(6)}$$

Where (s_N) oh is the "modified fatigue strength" or "modified fatigue limit", (s_N)is the intrinsic value, (K_f)is the fatigue strength reduction factor and C_a, C_b and C_c are factors allowing for size, surface finish, type of loading, etc.

The types of fatigue loading in common usage include direct stress, where the material is repeatedly loaded in its axial direction; plane bending, where the material is bent about its neutral plane; rotating bending, where the specimen is being rotated and at the same time subjected to a bending moment; torsion, where the specimen is subjected to conditions which produce reversed or fluctuating torsional stresses and, finally, combined stress conditions, where two or more of the previous types of loading are operating simultaneously.

P/S/N curves

The fatigue life of a component as determined at a particular stress level is a very variable quantity so that seemingly identical specimens may give widely differing results. This scatter arises from many sources including variations in material composition and heterogeneity, variations in surface finish, variations in axiality of loading, etc.

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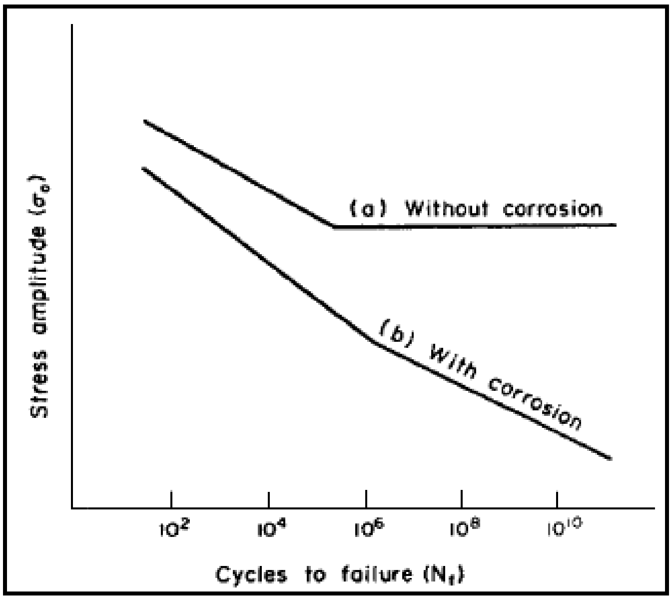

Figure (6) The effect of corrosion on fatigue life. S/N Curve for (a) material showing fatigue limit; (b) same material under corrosion conditions

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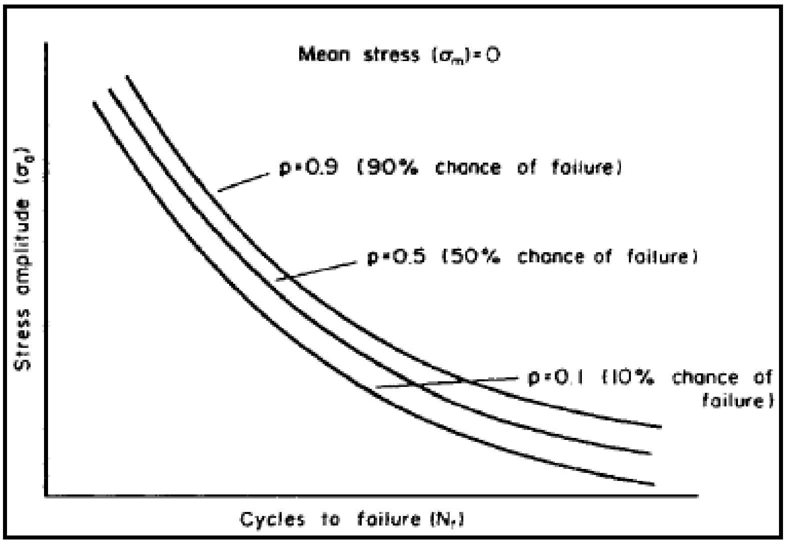

Figure (7) P/S/N curves indicating percentage chance of failure for given stress level after known number of cycles(zero mean stress)

To overcome this problem, a number of test pieces should be tested at several different stresses and then an estimate of the life at a particular stress level for a given probability can be made.

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Effect of mean stress

If the fatigue test is carried out under conditions such that the mean stress is tensile Figure (3), then, in order that the specimen will fail in the same number of cycles as a similar specimen tested under zero mean stress conditions, the stress amplitude in the former case will have to be reduced. The fact that an increasing tensile mean stress lowers the fatigue or endurance limit is important, and all **S/N** curves should contain information regarding the test conditions Figure (8).

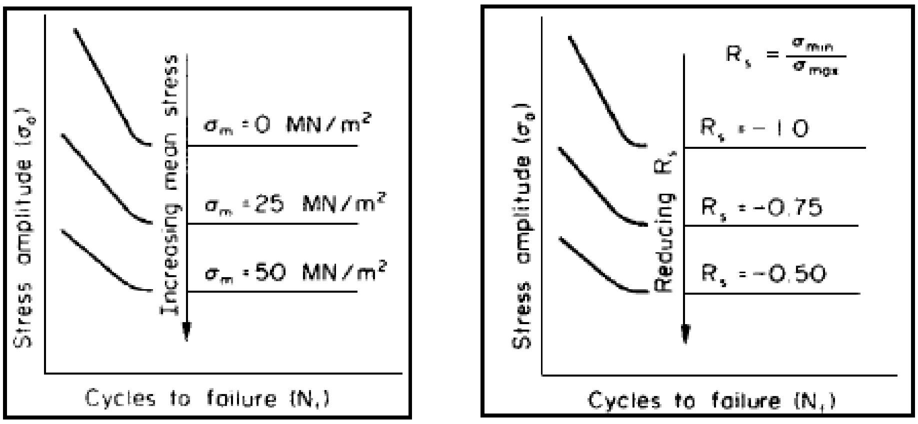

Figure (8) Effect of mean stress on the S/N curve expressed in alternative ways

A number of investigations have been made of the quantitative effect of tensile mean stress resulting in the following equations:

Goodman $$\sigma_a = \sigma_N\left[1-\left(\frac{\sigma_m}{\sigma_{TS}}\right)\right] \quad \text{...... (7)}$$

Geber $$\sigma_a = \sigma_N\left[1-\left(\frac{\sigma_m}{\sigma_{TS}}\right)^2\right] \quad \text{...... (8)}$$

Soderberg $$\sigma_a = \sigma_N\left[1-\left(\frac{\sigma_m}{\sigma_y}\right)\right] \quad \text{....... (9)}$$

Where;

s_N = the fatigue strength for *N* cycles under zero mean stress conditions.

s_a = the fatigue strength for *N* cycles under condition of mean stress **a,** .

s_{TS} = tensile strength of the material.

s_y = yield strength of the material.

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The above equations may be shown in graphical form Figure (9) and in actual practice it has been found that most test results fall within the envelope formed by the parabolic curve of Geber and the straight line of Goodman. However, because the use of Soderberg gives an additional margin of safety, this is the equation often preferred .

Even when using the Soderberg equation it is usual to apply a factor of safety (***F)*** to both the alternating and the steady component of stress, in which case equation (9) becomes:

$$\sigma_a = \frac{\sigma_N}{F}\left(1 - \frac{\sigma_m \times F}{\sigma_y}\right) \qquad \text{...... (10)}$$

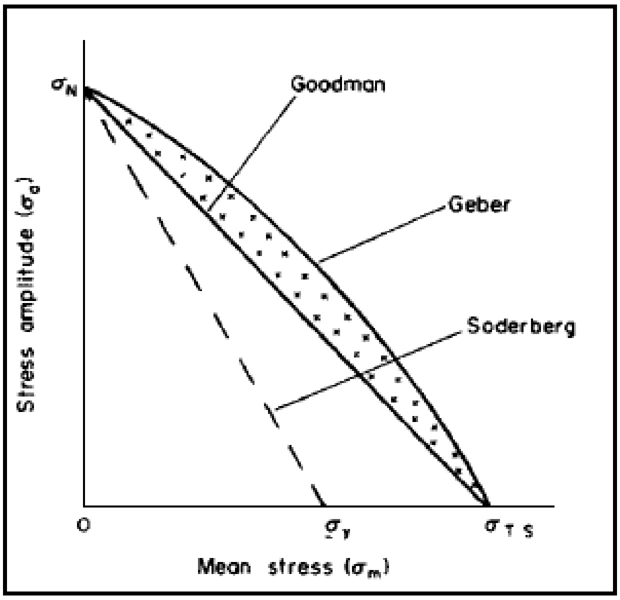

Figure (9) Amplitude/mean stress relationships as per Goodman. Geber and Soderberg

Effect of stress concentration

The influence of stress concentration can be illustrated by consideration of an elliptical crack in a plate subjected to a tensile stress. Provided that the plate is very large, the "**theoretical stress concentration**" factor (K_t) is given by:

$$K_t = 1 + \frac{2A}{B} \qquad \text{...... (11)}$$

Where; **"A" and *"B'*** are the crack dimensions as shown in Figure (10). If the crack is perpendicular to the direction of stress, then **A** is large compared with **B** and hence (K_t)

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will be large. If the crack is parallel to the direction of stress, then **A** is very small compared with ***B*** and hence K_t = 1. If the dimensions of **A** and ***B*** are equal such that the crack becomes a round hole, then K_t = **3** and a maximum stress of ($3\sigma_{nom.}$), acts at the sides of the hole.

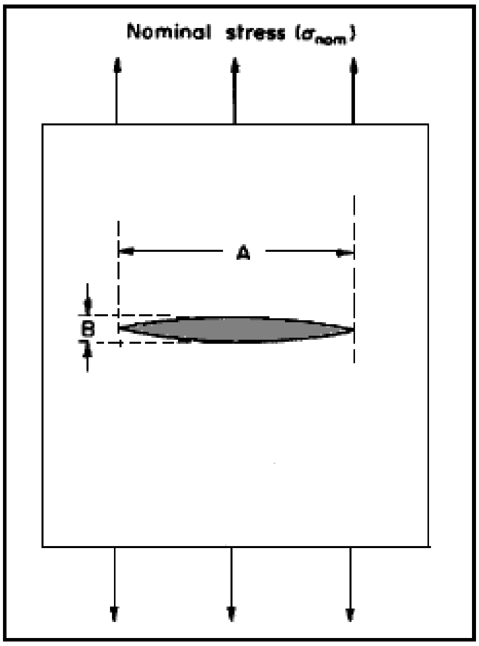

Figure (10) Elliptical crack in semi-infinite plate

The effect of sudden changes of section, notches or defects upon the fatigue performance of a component may be indicated by the "***fatigue notch***" or "***fatigue strength reduction***" factor (K_f) which is the ratio of the stress amplitude at the fatigue limit of an un-notched specimen.

(K_f) is always less than the static theoretical stress concentration factor referred to above because under the compressive part of a tensile-compressive fatigue cycle, a fatigue crack is unlikely to grow.

The extent to which the stress concentration effect under fatigue conditions approaches that for static conditions is given by the "***notch sensitivity factor***" (q), and the relationship between them may be simply expressed by:

$$q = \frac{K_f - 1}{K_t - 1} \qquad \text{...... (12)}$$

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thus q is always less than 1.

Notch sensitivity is a very complex factor depending not only upon the material but also upon the grain size, a finer grain size resulting in a higher value of ***q*** than a coarse grain size.

Examples

Example (1)

The fatigue behavior of a specimen under alternating stress conditions with zero mean stress is given by the expression:

$$\sigma_r^a \cdot N_f = K$$

Where; σ_r is the range of cyclic stress, $\mathbf{N}_f$ is the number of cycles to failure and (**K**) and (**a**) are material constants.

It is known that $\mathbf{N}_f = 10^6$ when, $\sigma_r = 300$ MN/m^2 and $\mathbf{N}_f = 10^8$ when $\sigma_r = 200$ MN/m^2.

Calculate the constants (**K**) and (**a**) and hence the life of the specimen when subjected to a stress range of 100 MN/m^2.

Solution

Taking logarithms of the given expression we have:

$$a \log \sigma_r + \log N_f = \log K \qquad (1)$$

Substituting the two given sets of condition for N_f and σ_r:

$$2.4771a + 6.0000 = \log K \qquad (2)$$

$$2.3010a + 8.0000 = \log K \qquad (3)$$

$\therefore$ (3) − (2)

$$-0.1761a + 2.0000 = 0$$

$$a = \frac{2.0000}{0.1761} = \mathbf{11.357}$$

Substituting in eqn. (2)

$$11.357 \times 2.4771 + 6.000 = \log K = 34.1324$$

$$\therefore \mathbf{K} = \mathbf{1.356 \times 10^{33}}$$

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Hence, for stress range of 100 MN/m^2, from equation (1);

$$11.357 \times 2.0000 + \log N_f = 34.1324$$
$$22.714 + \log N_f = 34.1324$$
$$\log N_f = 11.4184$$
$$N_f = \mathbf{262.0 \times 10^9 \text{ cycles}}$$

Example (2)

A steel bolt 0.003 m^2 in cross-section is subjected to a static mean load of 178 kN. What value of completely reversed direct fatigue load will produce failure in l0^7 cycles? Use the Soderberg relationship and assume that the yield strength of the steel is 344 MN/m^2 and the stress required to produce failure at l0^7 cycles under zero mean stress conditions is 276 MN/m^2.

Solution

From equation of Soderberg

$$\sigma_a = \sigma_N \left[1 - \left(\frac{\sigma_m}{\sigma_y}\right)\right]$$

$$\text{Now, mean stress } \sigma_m \text{ on bolt } = \frac{178 \times 10^{-3}}{3 \times 10^{-3}}$$
$$= 59.33 \text{ MN/m}^2$$
$$\therefore \quad \sigma_a = 276\left(1 - \frac{59.33}{344}\right)$$
$$= 276(1 - 0.172)$$
$$= 276 \times 0.828$$
$$= 228.5 \text{ MN/m}^2$$
$$\therefore \quad \text{Load} = 228.5 \times 0.003 \text{ MN}$$
$$= 0.6855 \text{ MN}$$
$$= \mathbf{685.5 \text{ kN}}$$

Example (3)

A stepped steel rod, the smaller section of which is 50 mm in diameter, is subjected to a fluctuating direct axial load which varies from +178 kN to -178 kN. If the theoretical stress concentration due to the reduction in section is 2.2, the notch sensitivity factor is

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0.97, the yield strength of the material is 578 MN/m^2 and the fatigue limit under rotating bending is 347 MN/m^2, calculated the factor of safety if the fatigue limit in tension-compression is 0.85 of that in rotating bending.

Solution

From equation (12)

$$q = \frac{K_f - 1}{K_t - 1}$$

$$\therefore\ K_f = q(K_t - 1) + 1$$

$$= 0.97(2.2 - 1) + 1$$

$$= 2.16$$

But $$\sigma_{\max} = \frac{178 \times 4}{\pi \times (0.05)^2}$$

$$= 90642 \text{ kN/m}^2$$

$$= \mathbf{90.64\ MN/m^2}$$

$$\therefore\quad \sigma_{\min} = -90.64 \text{ MN/m}^2 \text{ and } \sigma_{\text{mean}} = 0$$

:. Under direct stress conditions

$$\sigma_N = 0.85 \times 347$$

$$= \mathbf{294.95\ MN/m^2}$$

From equation (13)

$$\sigma_a = \frac{\sigma_N}{K_f F}\left(1 - \frac{\sigma_m \times F}{\sigma_y}\right)$$

:. With common units of MN/m^2:

$$90.64 = \frac{294.95}{2.16 \times F}\left(1 - \frac{0 \times F}{578}\right)$$

$$\therefore\quad F = \frac{294.95}{2.16 \times 90.64}$$

$$\mathbf{F = 1.5}$$

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Creep

Creep is the time-dependent deformation which accompanies the application of stress to a material. At room temperatures, apart from the low-melting-point metals such as lead, most metallic materials show only very small creep rates which can be ignored. With increase in temperature, however, the creep rate also increases and above approximately 0.4 T_m where T_m, is the melting point on the Kelvin scale, creep becomes very significant.

The creep test

The creep test is usually carried out at a constant temperature and under constant load conditions rather than at constant stress conditions. This is acceptable because it is more representative of service conditions. A typical creep testing machine is shown in Figure (1) Each end of the specimen is screwed into the specimen holder which is made of a creep resisting alloy and thermocouples and accurate extensometers are fixed to the specimen in order to measure temperature and strain. The electric furnace is then lowered into place and when all is ready and the specimen is at the desired temperature, the load is applied by adding weights to the lower arm and readings are taken at periodic intervals of extension against time. It is important that accurate control of temperature is possible and to facilitate this the equipment is often housed in a temperature-controlled room. The results from the creep test are plotted in graphical form to produce a typical curve as shown in Figure (2) After the initial extension OA which is produced as soon as the test load is applied, and which is not part of the creep process proper (but which nevertheless should not be ignored), the curve can be divided into three stages. In the first or *primary* stage AB, the movement of dislocations is very rapid, any barriers to movement caused by work-hardening being overcome by the recovery processes, albeit at a decreasing rate. Thus the initial creep strain rate is high but it rapidly decreases to a constant value. In the secondary stage BC**,** the work-hardening process of “dislocation pile-up” and “entanglement” are balanced by the recovery processes of “dislocation climb” and “cross-slip”, to give a straight-line relationship and the slope of the graph in this steady-state portion of the curve is equal to the secondary creep rate. Since, generally, the primary and tertiary stages occur quickly, it is the secondary creep rate which is of prime importance to the design engineer. The third or tertiary stage CD coincides with the formation of internal voids within the specimen and this leads to “necking”, causing the stress to increase and rapid failure to result.

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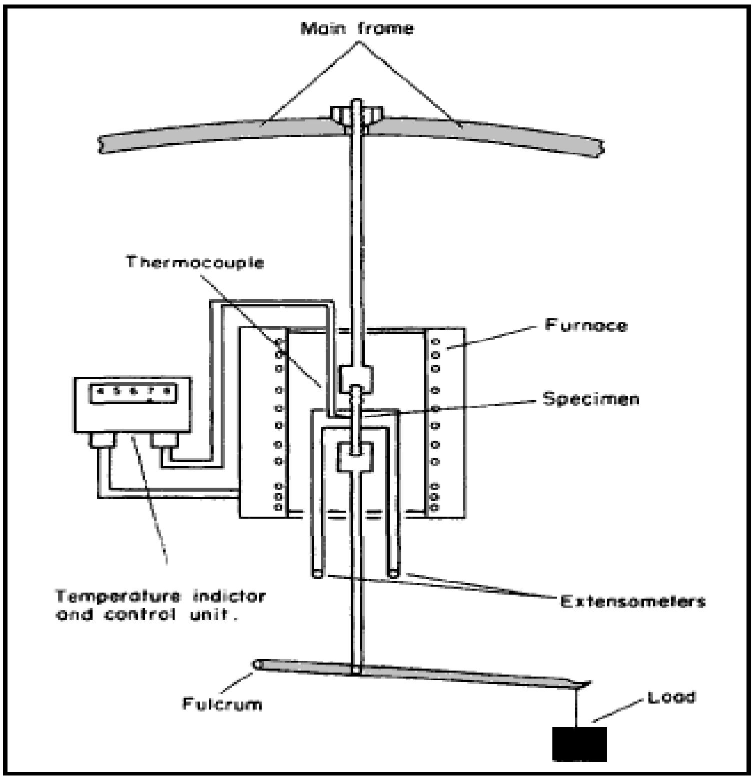

Figure (1) Schematic diagram of a typical creep testing machine

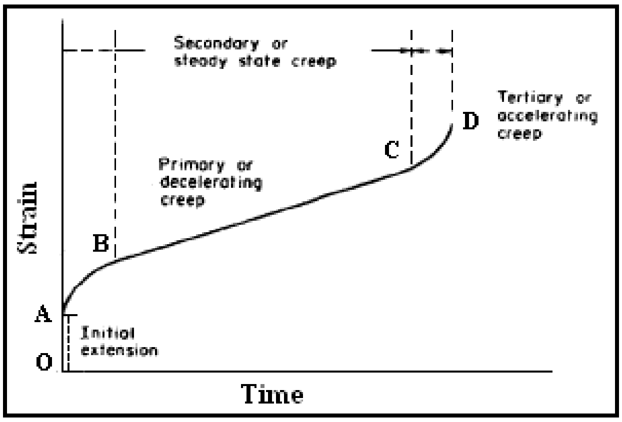

Figure (2) Typical creep curve

The shape of the creep curve for any material will depend upon the temperature of the test and the stress at any time since these are the main factors controlling the work-hardening and recovery processes. With increase in temperature, the creep rate increases because the softening processes such as "dislocation climb" can take place more easily,

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being diffusion controlled and hence a thermally activated process. It is expected, therefore, that the creep rate is closely related to the ***Arrhenius equation***,
viz.:

$$\varepsilon_s^0 = Ae^{-H/RT} \qquad \text{....... (1)}$$

where (e_s^0): is the secondary creep rate, H is the activation energy for creep for the material under test, R is the universal gas constant, T is the absolute temperature and A is a constant. It should be noted that both A and H are not true constants, their values depending upon stress, temperature range and metallurgical variables.

The secondary creep rate also increases with increasing stress, the relationship being most commonly expressed by the power law equation:

$$\varepsilon_s^0 = \beta\sigma^n \qquad \text{....... (2)}$$

Where; (*b*) and (n) are constants, the value of n usually varying between 3 and 8.

Equations (1) and (2) may be combined to give:

$$\varepsilon_s^0 = K\sigma^n e^{-H/RT} \qquad \text{...... (3)}$$

Figure (3) illustrates the effect of increasing stress or temperature upon the creep curve and it can be seen that increasing either of these two variables results in a similar change of creep behavior, that is, an increase in the secondary or minimum creep rate, a shortening of the secondary creep stage, and the earlier onset of tertiary creep and fracture.

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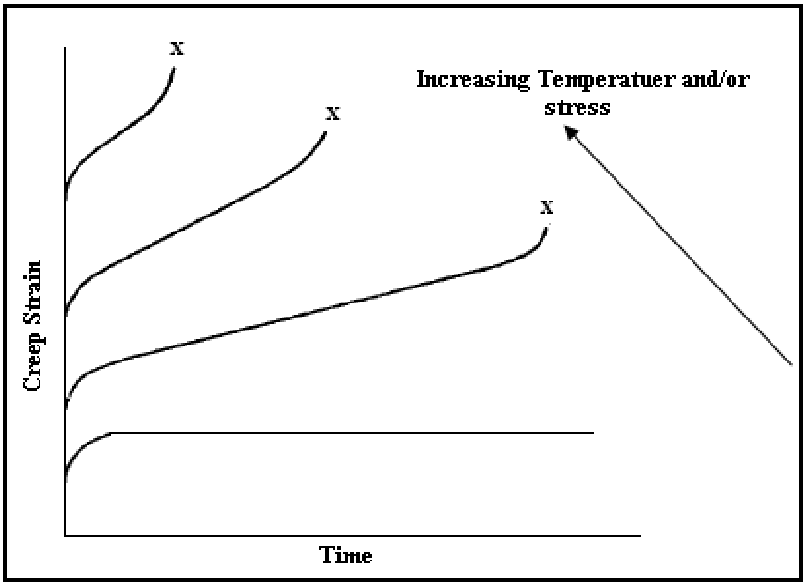

Figure (3) Creep curves showing effect of increasing temperature or stress

Fracture mechanics

The study of how materials fracture is known as **fracture mechanics** and the resistance of a material to fracture is colloquially known as its **"toughness".** No structure is entirely free of defects and even on a microscopic scale these defects act as stress-raisers which initiate the growth of cracks. The theory of fracture mechanics therefore assumes the pre-existence of cracks and develops criteria for the catastrophic growth of these cracks. In a stressed body, a crack can propagate in a combination of the three opening modes shown in Figure (1) **Mode I** represents opening in a purely tensile field while **Modes II** and **III** are in-plane and anti-plane shear modes respectively.

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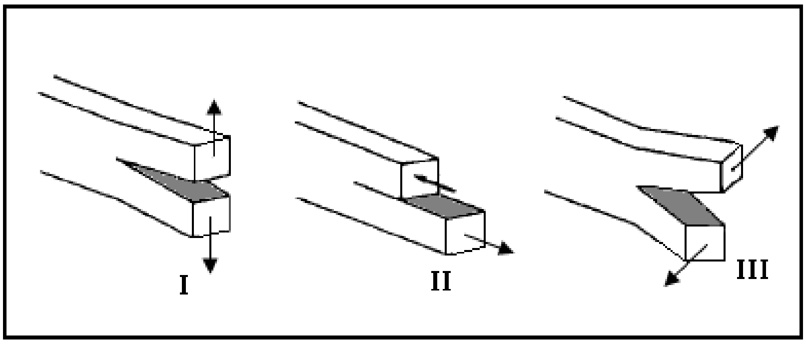

Figure (1) The three opening modes, associated with crack growth: mode I-tensile; mode 11-in-plan

Energy variation in cracked bodies

It is assumed that a crack will only grow if there is a decrease in the free energy of the system which comprises the cracked body and the loading mechanism. The first usable criterion for fracture was developed from this assumption by Griffith(").For a clearer understanding of Griffith's theory it is necessary to examine the changes in stored elastic energy as a crack grows. Consider, therefore, the simple case of a strip containing an edge crack of length (***a)*** under uniaxial tension as shown in Figure (2) If load (W) is applied gradually, the load points will move a distance (**x)** and the strain energy (***U)* ,** stored in the body will be given by

$$U = \tfrac{1}{2} W x$$

for purely elastic deformation. The load and displacement are related by the ***"compliance" C,***

$$x = CW$$

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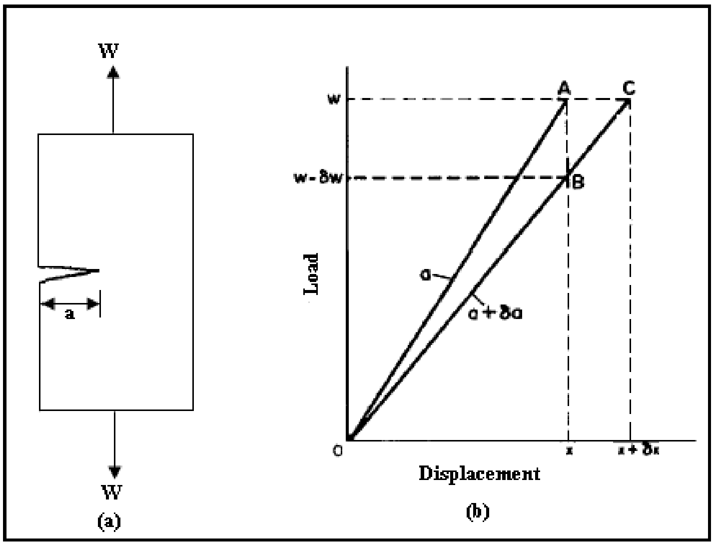

Figure (2) (a) Cracked body under tensile load W;
(b) force-displacement curves for a body with crack lengths *a* and a + da.

The compliance is itself a function of the crack length but the exact relationship varies with the geometry of the cracked body. However, if the crack length increases, the body will become less stiff and the compliance will increase.

There are two limiting conditions to be considered depending on whether the cracked body is maintained at (a) constant displacement or (b) constant loading. Generally a crack will grow with both changing loads and displacement but these two conditions represent the extreme constraints.

(a) Constant displacement

Consider the case shown in Figure (2)(b). If the body is taken to be perfectly elastic then the load-displacement relationship will be linear. With an initial crack length (*a)* loading will take place along the line OA. If the crack extends a small distance (δa) while the points of application of the load remain fixed, there will be a small increase in the compliance resulting in a decrease in the load of (δW). The load and displacement are then given by the point B. The change in stored energy will then be given by

$$\delta U_x = \tfrac{1}{2}(W - \delta W)x - \tfrac{1}{2}Wx$$

$$\delta U_x = -\tfrac{1}{2}\delta W x \qquad \text{...... (2)}$$

(b) Constant loading

In this case, if the crack again extends a small distance (δa) the loading points must move through an additional displacement (δx) in order to keep the load

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constant. The load and displacement are then represented by the point C. There would appear to be an increase in stored energy given by

$$\delta U = \tfrac{1}{2}W(x + \delta x) - \tfrac{1}{2}Wx$$
$$= \tfrac{1}{2}W\delta x$$

However, the load has supplied an amount of energy

$$= W\,\delta x$$

This has to be obtained from external sources so that there is a total reduction in the potential energy of the system of

$$\delta U_W = \tfrac{1}{2}W\,\delta x - W\,\delta x$$
$$\delta U_W = -\tfrac{1}{2}W\,\delta x \qquad \text{...... (3)}$$

For infinitesimally small increases in crack length the compliance C remains essentially constant so that

$$\delta x = C\,\delta W$$

Substituting in equation (3)

$$\delta U_W = -\tfrac{1}{2}WC\,\delta W = -\tfrac{1}{2}x\,\delta W$$

Comparison with equation (2) shows that, for small increases in crack length,

$$\delta U_w = \delta U_x$$

It is therefore evident that for small increases in crack length there is a similar decrease in potential energy no matter what the loading conditions. If there is a decrease in potential energy when a crack grows then there must be an energy requirement for the production of a crack - otherwise all cracked bodies would fracture instantaneously.

Linear elastic fracture mechanics (L.E.F.M.)

(a) Griffrth's criterion for fracture

Griffith's thermodynamics approach was the first to produce a usable theory of fracture mechanics. His theoretical model shown in Figure (3) was of an infinite sheet under a remotely applied uniaxial stress (σ) and containing a central crack of length (2a). Griffith, by a more mathematically rigorous treatment, was able to show that if that decrease in energy is greater than the energy required to produce new crack faces then there will be a net decrease in energy and the crack will propagate. For an increase in crack length of (δa).

$$\delta U = 2\gamma b\,\delta a$$

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g; is the surface energy of the crack faces;
b; is the thickness of the sheet.
At the onset of crack growth, (*d**a)*** is small and we have

$$\frac{dU}{da} = 2b\gamma$$

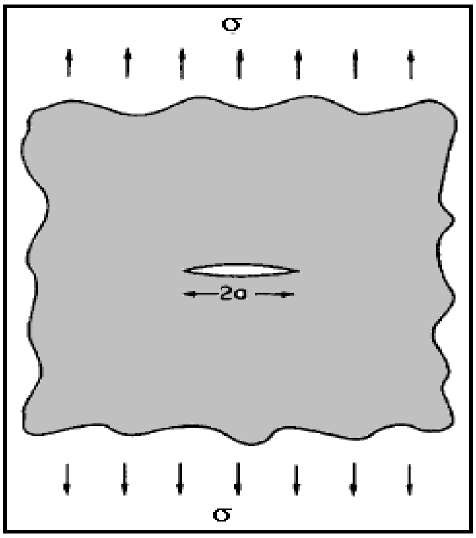

Figure (3) Mathematical model for Griffith's analysis.

The expression on the left-hand side of the above equation is termed the ***"critical strain energy release"*** (with respect to crack length) and is usually denoted as ***Gc,***

$$G_c = \frac{\partial U}{\partial a} = 2b\gamma \quad .....(4)$$

This is the ***Griflth criterion for fracture.***
Griffith's analysis gives ***G,*** in terms of the fracture stress *of*

$$G_c = \frac{\sigma_f^2 \pi a}{E} \quad \text{in plane stress} \quad ..... (5)$$

$$G_c = \frac{\sigma_f^2 \pi a}{E}(1 - \nu^2) \quad \text{in plane strain.} \quad ...... (6)$$

From equations (4) and (6) we can predict that, *for plane strain*, the fracture stress

$$\sigma_f^2 = \frac{2bE\gamma}{\pi a(1 - \nu^2)} \quad ...... (7)$$

or, for plane stress:

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$$\sigma_f^2 = \frac{2bE\gamma}{\pi a}$$

(b) Stress intensity factor

The elastic crack was developed by Irwin, who used a similar mathematical model to that employed by Griffith except in this case the remotely applied stress is biaxial - see Figure (4). Irwin's theory obtained expressions for the stress components near the crack tip. The most elegant expression of the stress field is obtained by relating the Cartesian components of stress to polar coordinates based at the crack tip as shown in Figure (5).

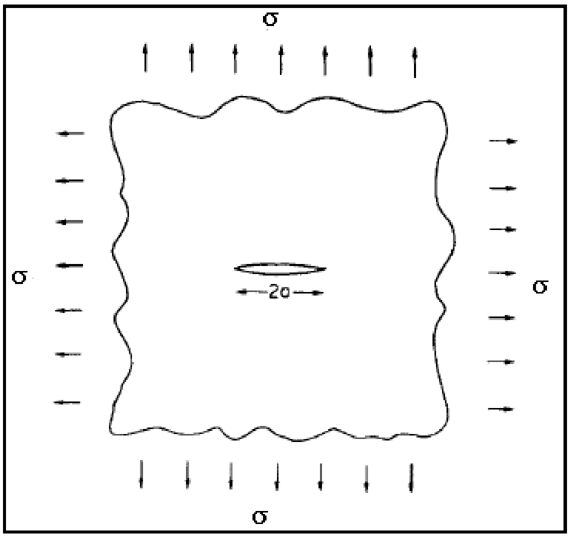

Figure (4) Mathematical model for Irwin's analysis.

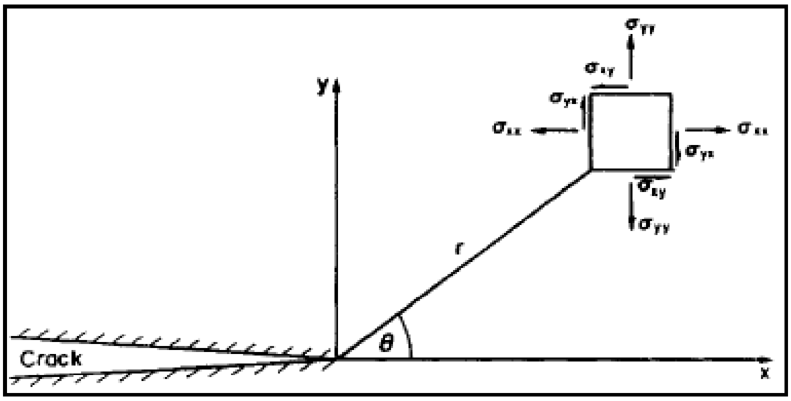

Figure (5) Coordinate system for stress components in Irwin's analysis.

Then we have:

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$$\left.\begin{aligned}\sigma_{yy} &= \frac{K}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]\\ \sigma_{xx} &= \frac{K}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left[1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]\\ \sigma_{xy} &= \frac{K}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}\end{aligned}\right\} \qquad \text{..... (8)}$$

With, for plane stress,

$$\sigma_{zz}=0$$

or, for plane strain,

$$\sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})$$
$$\sigma_{zx} = \sigma_{zy} = 0 \quad \text{for both cases.}$$

If more than one crack opening mode is to be considered then K sometimes carries the suffix I, II or III corresponding to the three modes shown in Figure (1). However since this text is restricted to consideration of mode I crack propagation only, the formulae have been simplified by adopting the symbol K without its suffix. K, in development of similar formulae.
For Irwin's model, K is given by

$$\boldsymbol{K = \sigma\sqrt{\pi a}} \qquad \text{......(9a)}$$

For an edge crack in a semi-infinite sheet

$$\boldsymbol{K = 1.12\sigma\sqrt{\pi a}} \qquad \text{...... (9b)}$$

To accommodate different crack geometries a flaw shape parameter Q is sometimes introduced thus

$$K = \sigma\sqrt{\frac{\pi a}{Q}} \qquad \text{...... (9c)}$$

or, for an edge crack

$$K = 1.12\sigma\sqrt{\frac{\pi a}{Q}} \qquad \text{........ (9d)}$$

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Values of Q for various aspect (depth to width) ratios of crack can be obtained from standard texts, but typically, they range from 1.0 for an aspect ratio of zero to 2.0 for an aspect ratio of 0.4.

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