

# PURE MATHEMATICS Unit 1 FOR CAPE<sup>®</sup> EXAMINATIONS

DIPCHAND BAHALL

CAPE<sup>®</sup> is a registered trade mark of the Caribbean Examinations Council (CXC). **Pure Mathematics for CAPE<sup>®</sup> Examinations Unit 1** is an independent publication and has not been authorised, sponsored, or otherwise approved by CXC.



Macmillan Education  
4 Crinan Street, London N1 9XW  
A division of Macmillan Publishers Limited  
Companies and representatives throughout the world

[www.macmillan-caribbean.com](http://www.macmillan-caribbean.com)

ISBN 978-0-2304-6575-6 AER

Text © Dipchand Bahall 2013  
Design and illustration © Macmillan Publishers Limited 2013

First published in 2013

All rights reserved; no part of this publication may be reproduced, stored in a retrieval system, transmitted in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without the prior written permission of the publishers.

These materials may contain links for third party websites. We have no control over, and are not responsible for, the contents of such third party websites. Please use care when accessing them.

Designed by Tech Type and Oxford Designers and Illustrators  
Typeset and illustrated by MPS Limited  
Cover design by Clare Webber  
Cover photo: Alamy/Science Photo Library

# Contents

---

INTRODUCTION	xii
MATHEMATICAL MODELLING	xiii
<b>MODULE 1 BASIC ALGEBRA AND FUNCTIONS</b>	
CHAPTER 1 REASONING AND LOGIC	2
Notation	4
Simple statement	4
Negation	4
Truth tables	4
Compound statements	5
Connectives	6
Conjunction	6
Disjunction ('or')	7
Conditional statements	11
Interpretation of $p \rightarrow q$	12
The contrapositive	12
Converse	13
Inverse	13
Equivalent propositions	14
Biconditional statements	15
Tautology and contradiction	17
Algebra of propositions	18
CHAPTER 2 THE REAL NUMBER SYSTEM	24
Subsets of rational numbers	25
Real numbers	26
Operations	26
Binary operations	26
Closure	26
Commutativity	27
Associativity	28
Distributivity	29
Identity	30
Inverse	31
Constructing simple proofs in mathematics	33
Proof by exhaustion	33
Direct proof	33
Proof by contradiction	35
Proof by counter example	36

CHAPTER 3	PRINCIPLE OF MATHEMATICAL INDUCTION	44
	Sequences and series	45
	Finding the general term of a series	45
	Sigma notation	47
	Expansion of a series	47
	Standard results	48
	Summation results	49
	Mathematical induction	53
	Divisibility tests and mathematical induction	57
CHAPTER 4	POLYNOMIALS	62
	Review of polynomials	63
	Degree or order of polynomials	63
	Algebra of polynomials	63
	Evaluating polynomials	64
	Rational expressions	64
	Comparing polynomials	65
	Remainder theorem	69
	The factor theorem	74
	Factorising polynomials and solving equations	77
	Factorising $x^n - y^n$	82
CHAPTER 5	INDICES, SURDS AND LOGARITHMS	88
	Indices	89
	Laws of indices	89
	Surds	91
	Rules of surds	92
	Simplifying surds	93
	Conjugate surds	94
	Rationalising the denominator	94
	Exponential functions	98
	Graphs of exponential functions	98
	The number $e$	100
	Exponential equations	102
	Logarithmic functions	104
	Converting exponential expressions to logarithmic expressions	104
	Changing logarithms to exponents using the definition of logarithm	105
	Properties of logarithms	107
	Solving logarithmic equations	108
	Equations involving exponents	110
	Change of base formula (change to base $b$ from base $a$ )	113
	Logarithms and exponents in simultaneous equations	115

	Application problems	117
	Compound interest	118
	Continuous compound interest	120
CHAPTER 6	FUNCTIONS	128
	Relations and functions	129
	Describing a function	130
	The vertical line test	130
	One-to-one function (injective function)	132
	Onto function (surjective function)	134
	Bijective functions	137
	Inverse functions	139
	Graphs of inverse functions	141
	Odd and even functions	144
	Odd functions	144
	Even functions	144
	Periodic functions	144
	The modulus function	145
	Graph of the modulus function	145
	Composite functions	146
	Relationship between inverse functions	149
	Increasing and decreasing functions	152
	Increasing functions	152
	Decreasing functions	152
	Transformations of graphs	153
	Vertical translation	153
	Horizontal translation	154
	Horizontal stretch	155
	Vertical stretch	157
	Reflection in the $x$ -axis	158
	Reflection in the $y$ -axis	158
	Graphs of simple rational functions	160
	Piecewise defined functions	162
CHAPTER 7	CUBIC POLYNOMIALS	171
	Review: Roots of a quadratic and the coefficient of the quadratic	172
	Cubic equations	173
	Notation	175
	Finding $\alpha^3 + \beta^3 + \gamma^3$ , using a formula	175
	Finding a cubic equation, given the roots of the equation	176

CHAPTER 8	INEQUALITIES AND THE MODULUS FUNCTION	185
	Theorems of inequalities	186
	Quadratic inequalities	186
	Sign table	188
	Rational functions and inequalities	191
	General results about the absolute value function	196
	Square root of $x^2$	201
	The triangle inequality	201
	Applications problems for inequalities	203
MODULE 1 TESTS		208

## MODULE 2 TRIGONOMETRY AND PLANE GEOMETRY

CHAPTER 9	TRIGONOMETRY	212
	Inverse trigonometric functions and graphs	213
	Inverse sine function	213
	Inverse cosine function	213
	Inverse tangent function	214
	Solving simple trigonometric equations	214
	Graphical solution of $\sin x = k$	214
	Graphical solution of $\cos x = k$	216
	Graphical solution of $\tan x = k$	217
	Trigonometrical identities	218
	Reciprocal identities	218
	Pythagorean identities	219
	Proving identities	220
	Solving trigonometric equations	224
	Further trigonometrical identities	229
	Expansion of $\sin(A \pm B)$	229
	Expansion of $\cos(A \pm B)$	230
	Expansion of $\tan(A + B)$	234
	Double-angle formulae	236
	Half-angle formulae	238
	Proving identities using the addition theorems and the double-angle formulae	238
	The form $a \cos \theta + b \sin \theta$	241
	Solving equations of the form $a \cos \theta + b \sin \theta = c$	244
	Equations involving double-angle or half-angle formulae	249
	Products as sums and differences	253
	Converting sums and differences to products	254
	Solving equations using the sums and differences as products	258

CHAPTER 10	COORDINATE GEOMETRY	266
	Review of coordinate geometry	267
	The equation of a circle	267
	Equation of a circle with centre $(a, b)$ and radius $r$	268
	General equation of the circle	269
	Intersection of a line and a circle	275
	Intersection of two circles	276
	Intersection of two curves	277
	Parametric representation of a curve	278
	Cartesian equation of a curve given its parametric form	279
	Parametric equations in trigonometric form	280
	Parametric equations of a circle	282
	Conic sections	285
	Ellipses	286
	Equation of an ellipse	286
	Equation of an ellipse with centre $(h, k)$	289
	Focus–directrix property of an ellipse	291
	Parametric equations of ellipses	291
	Equations of tangents and normals to an ellipse	293
	Parabolas	294
	Equation of a parabola	295
	Parametric equations of parabolas	296
	Equations of tangents and normals to a parabola	296
CHAPTER 11	VECTORS IN THREE DIMENSIONS ( $\mathbb{R}^3$ )	303
	Vectors in 3D	304
	Plotting a point in three dimensions	304
	Algebra of vectors	304
	Addition of vectors	304
	Subtraction of vectors	305
	Multiplication by a scalar	305
	Equality of vectors	305
	Magnitude of a vector	306
	Displacement vectors	306
	Unit vectors	307
	Special unit vectors	308
	Scalar product or dot product	309
	Properties of the scalar product	310
	Angle between two vectors	310
	Perpendicular and parallel vectors	312
	Perpendicular vectors	312
	Parallel vectors	313

Equation of a line	316
Finding the equation of a line given a point on a line and the direction of the line	316
Finding the equation of a line given two points on the line	317
Vector equation of a line	319
Parametric equation of a line	319
Cartesian equation of a line	320
Finding the angle between two lines, given the equations of the lines	322
Skew lines	323
Equation of a plane	326
Equation of a plane, given the distance from the origin to the plane and a unit vector perpendicular to the plane	327
Equation of a plane, given a point on the plane and a normal to the plane	328
Cartesian equation of a plane	330
MODULE 2 TESTS	338
<b>MODULE 3    CALCULUS I</b>	
CHAPTER 12    LIMITS AND CONTINUITY	342
Limits	343
The existence of a limit	345
Limit laws	345
Evaluating limits	347
Direct substitution	347
Factorising method	349
Conjugate method	350
Tending to infinity	351
Limits at infinity	352
Special limits	354
Continuity	358
Types of discontinuity	359
Infinite discontinuity	359
Point discontinuity	359
Jump discontinuity	359
Removable and non-removable discontinuity	360
CHAPTER 13    DIFFERENTIATION 1	366
Differentiation	367
The difference quotient	368
Existence of a derivative	368



	Notation for derivatives	368
	Interpretations of derivatives	369
	Finding derivatives using first principles	369
	Differentiation of $ag(x)$ where $a$ is a constant	372
	Differentiation of sums and differences of functions	373
	First principle and sums and differences of functions of $x$	375
	Rate of change	377
	Chain rule	379
	Product rule	382
	Quotient rule	383
	Differentiation of trigonometric functions	385
	Higher derivatives	393
CHAPTER 14	APPLICATIONS OF DIFFERENTIATION	399
	Tangents and normals	400
	Equations of tangents and normals	401
	Increasing and decreasing functions	403
	Stationary points/second derivatives	407
	Maximum and minimum values	407
	Stationary points	407
	Classification of turning points	409
	First derivative test	409
	Second derivative test	413
	Inflexion points	416
	Practical maximum and minimum problems	419
	Parametric differentiation	424
	Rate of change	425
	Curve sketching	430
	Polynomials, rational functions, trigonometric functions	430
	Graph of a polynomial	434
	Graphs of functions of the form $f(x) = x^n$ where $n$ is an even integer	434
	Graphs of functions of the form $f(x) = x^n$ where $n$ is an odd integer greater than 1	435
	Graphs of polynomials	435
	Zeros of a polynomial	438
	Graphing functions	440
	Graphing functions with a table of values	440
	Solving simultaneous equations graphically	444
	Solving inequalities graphically	447
	Review of trigonometry	449
	Sine, cosine and tangent of $45^\circ$ , $30^\circ$ and $60^\circ$	449

	Graph of cosec $x$	450
	Graph of sec $x$	451
	Graph of cot $x$	451
	Properties and graphs of trigonometric functions	452
	Transformations of trigonometric functions	456
	$y = a \sin(bx) + c$ and $y = a \cos(bx) + c$	459
	$y = a \tan(bx) + c$	460
	Graphs of rational functions	461
	Vertical asymptotes	462
	Horizontal asymptotes	462
	Sketching graphs of rational functions	463
	Shape of a curve for large values of the independent variable	466
CHAPTER 15	INTEGRATION	473
	Anti-derivatives (integrations)	474
	The constant of integration	474
	Integrals of the form $ax^n$	474
	Integration theorems	475
	Integration of polynomial functions	477
	Integration of a function involving a linear factor	482
	Integration of trigonometric functions	483
	Integration of more trigonometric functions	487
	Integrating $\sin^2 x$ and $\cos^2 x$	488
	Integration of products of sines and cosines	491
	The definite integral	492
	Integration by substitution	498
	Substituting with limits	501
	The equation of a curve	504
CHAPTER 16	APPLICATIONS OF INTEGRATION	509
	Approximating the area under a curve, using rectangles	510
	Estimating the area under a curve using $n$ rectangles of equal width	512
	Using integration to find the area under a curve	514
	Area between two curves	516
	Area below the $x$ -axis	517
	Area between the curve and the $y$ -axis	519
	Volume of solids of revolution	524
	Rotation about the $x$ -axis	524
	Rotation about the $y$ -axis	527
	Volume generated by the region bounded by two curves	530

CHAPTER 17	DIFFERENTIAL EQUATIONS	543
	Families of curves	544
	Classifying differential equations	544
	Linear versus non-linear differential equations	544
	Practical applications of differential equations	544
	First order differential equations	545
	Solutions of variable-separable differential equations	547
	Modelling problems	549
	Second order differential equations	552
MODULE 3 TESTS		558
UNIT 1—MULTIPLE CHOICE TESTS		561
INDEX		575

Answers are available online at  
[www.macmillan-caribbean.com/resources](http://www.macmillan-caribbean.com/resources)

# Introduction

---

These two volumes provide students with an understanding of pure mathematics at the CAPE® level taken from both a theoretical and an application aspect and encourage the learning of mathematics. They provide the medium through which a student can find problems applied to different disciplines. The concepts are developed step by step; they start from the basics (for those who did not do additional mathematics) and move to the more advanced content areas, thereby satisfying the needs of the syllabus. Examination questions all seem to have answers that are considered ‘nice’ whole numbers or small fractions that are easy to work with; not all real-world problems have such answers and these books have avoided that to some extent. Expect any kind of numbers for your answers; there are no strange or weird numbers.

The objectives are outlined at the beginning of each chapter, followed by the keywords and terms that a student should be familiar with for a better understanding of the subject. Every student should have a section of their work book for the language of the subject. I have met many students who do not understand terms such as ‘root’ and ‘factor’. A dictionary developed in class from topic to topic may assist the students in understanding the terms involved. Each objective is fulfilled throughout the chapters with examples clearly explained. Mathematical modelling is a concept that is developed throughout, with each chapter containing the relevant modelling questions.

The exercises at the end of each section are graded in difficulty and have adequate problems so that a student can move on once they feel comfortable with the concepts. Additionally, review exercises give the student a feel for solving problems that are varied in content. There are three multiple choice papers at the end of each Unit, and at the end of each module there are tests based on that module. For additional practice, the student can go to the relevant past papers and solve the problems given. After going through the questions in each chapter, a student should be able to do past paper questions from different examining boards for further practice.

A checklist at the end of each chapter enables the student to note easily what is understood and to what extent. A student can identify areas that need work with proper use of this checklist. Furthermore, each chapter is summarised as far as possible as a diagram. Students can use this to revise the content that was covered in the chapter.

The text provides all the material that is needed for the CAPE® syllabus so that teachers will not have to search for additional material. Both new and experienced teachers will benefit from the text since it goes through the syllabus chapter by chapter and objective to objective. All objectives in the syllabus are dealt with in detail and both students and teachers can work through the text, comfortably knowing that the content of the syllabus will be covered.