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CONTINUOUS TRACE MEASUREMENT TECHNIQUE EXPERIMENTAL TEST PHASE II

Final Report

CONTRACT NO. P.O. #70-60218MT

Prepared For
UNIVERSITY OF SOUTH FLORIDA
OFFICE OF SPONSORED RESEARCH
TAMPA, FLORIDA 33620

MAY 1970

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COMPUTER SCIENCES CORPORATION

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by

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ABSTRACT

Performance and accuracy of the Continuous Trace Geometric Model are tested by means of a satellite and station positioning experiment which uses actual observations. The data are right ascension and declination from three passes of PAGEOS observed with BC-4 cameras. The cameras were located at Beltsville, Maryland; Moses Lake, Washington; and Revilla Gigedo Is., Mexico.

The data are preprocessed to eliminate time synchronization and to recover camera orientations and plate continuous traces. Coordinates and relative covariance matrices of the satellite are computed assuming Beltsville and Moses Lake stations to be perfectly located. The position of Revilla Gigedo is evaluated by adjusting its coordinates using a least square procedure applied to the Continuous Trace Technique.

Modifications to and extentions of the analytical development of the Continuous Trace Technique as discussed in the Phase I Report are presented.

The results of the experiment show that the accuracy and practicality of the Continuous Trace Technique is comparable with the accuracy and practicality of other geometrical techniques employing the same data with synchronization. Satellite and station positions are obtained with a precision of a few parts per million which is in agreement with the precision of the measured input data.

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SECTION 1

INTRODUCTION AND SUMMARY

INTRODUCTION

Geodetic tracking cameras are used to observe sunlight reflected from passive satellites to obtain accurate data useful for adjusting the tracking station location. Usual methods of adjustment make use of the geometric aspects of spatial triangulation theory and require highly accurate timing to recover the simultaneous observations. Simultaneity is accomplished through the synchronization of the shutters of the cameras to a common time reference system. As an alternative, the Continuous Trace Technique employs a continuous observation of satellite passes with Earth fixed cameras. Thus, continuous traces are generated on the camera plates. This technique avoids any requirement for synchronization, and the complexity and cost of tracking cameras are thereby reduced.

The objective of the present study is to test the performance and accuracy of the geometric model of the Continuous Trace Technique using actual data.

Two questions are of primary interest for testing purposes. First, is the model capable of producing satellite and station positions from the geometric configurations found in practice? Second, what is the magnitude of the actual error? Error propagation is highly dependent on the geometry of the station/satellite system and on the number of observations. Therefore, the use of real data is preferable to the utilization of simulated observations because (a) no hypothesis is necessary to reproduce the actual satellite trajectory, (b) the data are affected by the actual errors, and (c) a comparison with other methods is not affected by peculiarities in the testing hypotheses.

To the best of our knowledge, no continuous trace observations have been reduced to date. However, a large number of optical observations of passive, sun-reflecting satellites have been made with time synchronized cameras. Among the data available, BC-4 camera observations of PAGEOS, made and reduced by USC&GS, are particularly suitable for testing purposes. The data are expressed as right ascension and declination of satellite position and an average of 200 points per plate are available. Therefore, a continuous trace representation of the satellite plate image is derivable without introducing bias.

Descriptions of the data employed and of the analysis performed are reported in Section 4. Section 5 describes the implementation of the testing method with computer programs. The final results are presented and discussed in Section 6. The mathematical formulation of the equations employed in the programs and the derivation of the covariance matrix analysis can be found in Section 7. The appendices contain mathematical details of the analytical derivations.

SUMMARY

The methodology employed by Computer Sciences Corporation to perform the test consisted of three steps:

1. Data associated with three events observed from three stations were transformed to Cartesian camera plate coordinates and timing correspondence was eliminated. Thus, we reproduced the original plate measurements of the satellite track without time reference. The orientation of the camera at the epoch was reduced to an Earth fixed system.

2. Plate measurements and camera orientations from two stations were employed to compute the coordinates of some satellite positions on the observed orbital arcs and to estimate the covariance matrices of the coordinate errors.

3. The position of the third station was adjusted by means of a least square procedure to obtain the best fit between plate measurements and computed satellite positions.

Particular care was devoted to the derivation of error covariance matrix estimates to assure the validity of the testing methodology. This derivation was the result of an original study of error propagation theory applied to the Continuous Trace Technique.

The results of the computations show a good agreement between theoretical expectations and practical outputs. Satellite positions were

obtained with an accuracy comparable to the estimated precision of plate measurements. No degradation was observed. The actual measured data gave the satellite coordinates with a precision of a few parts per million.

SECTION 2

CONCLUSIONS

Testing of the Continuous Trace Method with real data and comparison of the results with those derived from the same data with established methods has demonstrated the effectiveness of the method for both orbit determination and station location.

The principal limitation occurs with the use of parallel orbital arcs for station adjustment. This is expected from the theory since it corresponds to an insufficient amount of information. Mathematically, parallel arcs create systems of equations which are ill-conditioned and thus give meaningless answers.

The test shows that the Continuous Trace Technique is capable of solving the station adjustment problem when the proper amount of information is available. This conclusion has been reached from single event solutions assuming that only one coordinate is to be corrected. The resulting precision of the correction has an average value of about 20 meters which is in agreement with the precision of the initial data.

Computation of satellite positions shows that the Continuous Trace Technique will be useful for orbit determination. The coordinates of observed orbit points have been recovered with an accuracy of about 60 meters. This error reflects the direct propagation of plate measurements errors since no a priori constraint was available for the orbits.

The tests reported herein demonstrated that the Continuous Trace Technique is a valuable tool for orbit determination and geodetic station adjustment. Use of the technique should significantly reduce the cost of satellite photogrammetry.

SECTION 3

RECOMMENDATIONS

With the conclusion of this Phase II effort, we have shown that the Continuous Trace Technique (CONTRA) is both feasible and practical for geodetic satellite data processing. The theoretical basis for CONTRA has been analyzed (Phase I) and the geometrical aspects tested experimentally as described in the remainder of this report (Phase II).

The experimental test employed only the geometrical information in the satellite data. This limitation was imposed in order to avoid interference between the problems associated with the determination of orbit parameters and those associated with station positioning. If orbital constraints can be included, the station location accuracy could be improved substantially. Therefore, modification of the technique to include orbital constraints is the logical next step in evaluating the overall accuracy and practicality of CONTRA. A five step approach is recommended:

1. Define an orbit model suitable for use with CONTRA data which can be related to an Earth-fixed coordinate system without requiring a timing reference (see reference 5, pages 16-20).

2. Derive an error model which converts uncertainties in the measured data into uncertainties in the orbital parameters.

3. Develop a statistical method to discriminate those orbit model parameters which can be meaningfully determined.

4. Implement a testing program which provides a definitive understanding of the overall accuracy of CONTRA for station location and orbit determination.

5. Provide NASA with recommendations concerning future utilization of existing camera networks. Determine the realizable benefits associated with maintaining and operating inexpensive camera stations throughout the world.

SECTION 4

DESCRIPTION OF TESTING METHODOLOGY

In this section we review the fundamental concepts of the Continuous Trace Technique, and we summarize the theory of error propagation as it is applied to the Continuous Trace case. Further, we describe the original input data used to test the Technique.

4.1 REVIEW OF BASIC CONCEPTS

During Phase I of the Continuous Trace Technique analysis, Computer Sciences Corporation investigated the theoretical possibility of deriving geodetic information from continuous trace observations of passive satellites. The theory is based on the following hypotheses:

1. At least three Earth fixed cameras observe overlapping arcs of a satellite pass.

2. The orientation of each camera in an Earth fixed system has been established by independent means to astronomical accuracy.

3. The coordinates of at least two stations are known to geodetic accuracy.

To find the position of the satellite, we employ the observations made from the two known stations. A homological correspondence is established between points on the two associated traces.* Once the homology has been established, the satellite position is obtained by means of an elementary triangulation procedure.

In practice, we represent the trace by tabularizing the trace point coordinates measured from the plate. We proceed as follows (Fig 1):

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* The homological correspondence between points on the two traces does not exist if the stations and the observed orbital arc lie in the same plane. In this case, each point on one trace corresponds to a point on the other trace and the Continuous Trace Technique cannot be applied. This difficulty can be avoided by careful observational planning.

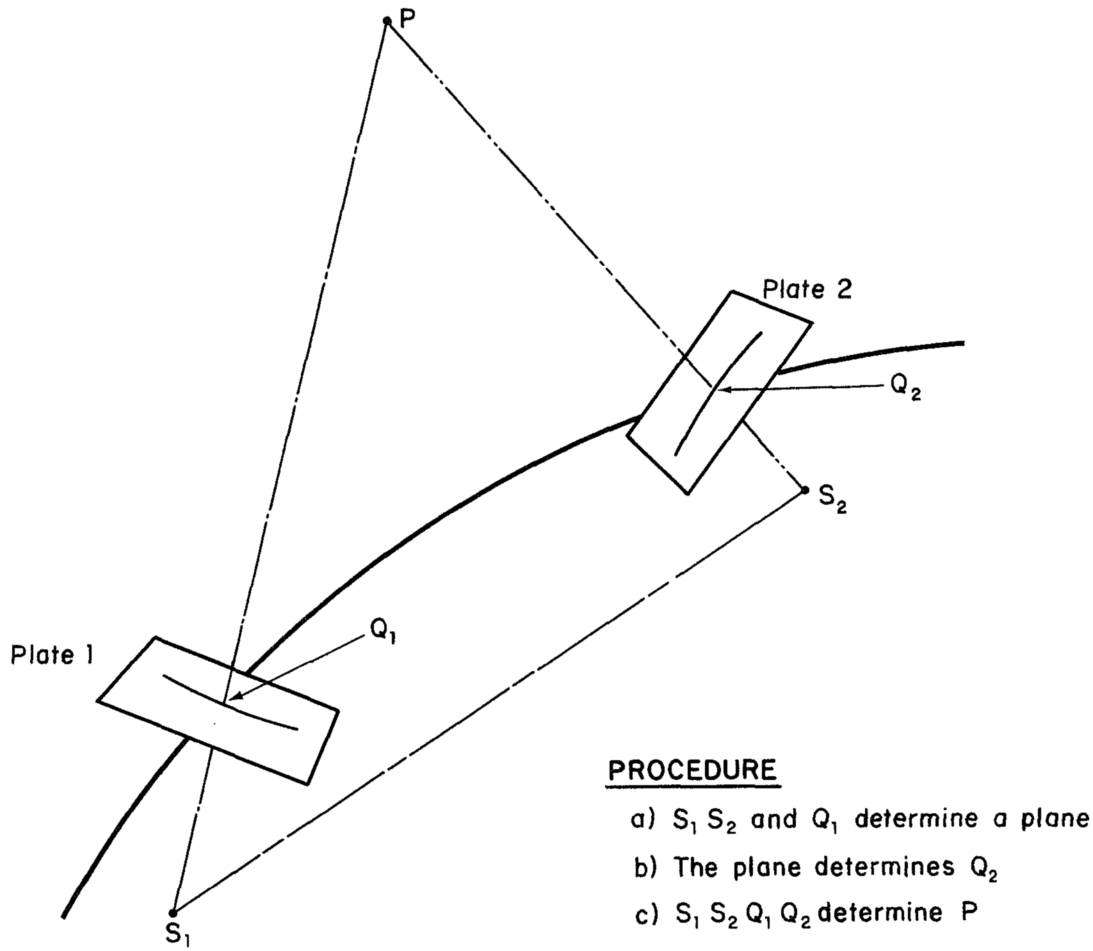

Figure 1 Orbit Points Determination

1. Point Q_1 is taken from the table corresponding to trace-1. Its coordinates are transformed to the geocentric reference system.

2. The two stations S_1 and S_2 determine a plane through Q_1.

3. The plane intersects the trace-2 at a point Q_2. This point is the homolog of Q_1.

4. The two rays S_1Q_1 and S_2Q_2 then determine the satellite position P.

In theory, the third station coordinates are completely determined if three positions of the satellite are known. In fact, the continuous trace obtained from a station S_3 defines a cone that has its vertex in the third station unknown position. This cone is analytically represented by an equation that relates the coordinates of the station, say X_S, Y_S, Z_S, and the coordinates of the observed satellite, say X, Y, Z.

$$F\ (X_S, Y_S, Z_S,\ X, Y, Z) = 0$$

Since we know the coordinates of three satellite positions, we have a system of three equations which can be solved for the unknown X_S, Y_S, Z_S.

In practice, we can obtain more than three satellite positions from the same observed arc. Therefore, we can employ the redundancy of the system of equations to adjust the third station using a least square procedure. However, if the orbital arc is very short we face the problem of an ill-conditioned system because the arc is almost a straight line and the cone becomes indistinguishable from a plane. In this case, only that component of the adjustment normal to the plane can be reliably evaluated. At least three non-parallel arcs are necessary for a complete adjustment of the unknown stations.

4.2 MEASUREMENT ERROR PROPAGATION

There are various errors associated with the measurement of the coordinates from the photographic plate such as those due to shimmer effect, comparator irregularities, and human error. However, we consider only the random residual of these errors. In practice, this residual has an rms amplitude of a few parts per million.

To determine a satellite position, we need two points - one from each of two plates. There are, of course, four linear components of the error, one for each trace coordinate. However, some of these components do not affect the accuracy of the satellite position determination. The number of error components which must be considered depends on the geometric characteristics of the tracking system as explained in the next paragraph.

First, let us consider the conventional case of simultaneous observations. In this case, we know a priori which pair of images corresponds to the same satellite position. The satellite is at the intersection of two straight lines. The system is over-determined because we have four equations with three unknown coordinates. The redundancy can be employed to reduce the four linear components of error to three. It follows that measurement errors affect all three coordinates of the satellite position and the resulting covariance matrix has rank three. On the other hand, the geometry of the Continuous Trace Technique is very different from the conventional case. The satellite coordinates are evaluated via the procedure delineated in Section 4.1 by determining the homologous points. To determine them, we establish an arbitrary plane through the two stations. Since the placement of the plane is arbitrary, measurement errors normal to the plane are of no consequence. Therefore, only two coordinates of the satellite position are affected by the measurement errors - namely, those which position the satellite within the plane. However, when we use the Continuous Trace Technique we consider the trace as a whole and we disregard the particular identity of any specific point. In other words, if we interchange the positions of two nearby points on the trace, the point coordinates will vary but we cannot recognize any variation in the trace as a whole. Therefore, the error component which interchanges two nearby points does not affect the trace, and we should take into account only that component which is normal to the trace. It follows that only two coordinates of satellite position are affected by measurement errors and that only two of the four measurement error components have some effects. The corresponding covariance matrix will have rank two. The rigorous computation of this covariance matrix is offered in Section 7.3.

4.3 DATA EMPLOYED IN THE TESTING

An experimental test of the performance of the Continuous Trace Technique requires input data coming from real observations of actual satellite orbits. However, to our knowledge, no optical observations in a continuous mode have been reduced to date. Therefore, we decided to employ the results of plate reductions made by USC&GS from observations of PAGEOS passes with BC-4 ballistic cameras located in a worldwide network of stations.

Advantages of employing these data are:

1. Each plate contains an average of 200-400 satellite images spaced every 0.8 sec in time. Therefore, it is feasible to consider these images as belonging to a continuous track of the satellite.

2. Plate measurements have been reduced to right ascension and declination of station-satellite directions with a highly accurate procedure that takes into account astronomic refraction, lens and scale distortions, and plate inclination.

3. The same data have been employed by independent investigators in station positioning problems whose result can be employed for comparison purposes. (Ref 1 & 4)

4. Among the observed events, three events are observed from three stations. This is the minimum of observations necessary to perform the station adjustment.

A disadvantage of using these data is that the plate reduction procedure introduces a statistical correlation among the data of the same plate. However, we remark that: 1) the final solution is still unbiased even if the correlation is neglected, 2) the data we use are plate coordinates which are less correlated than astronomical coordinates, and 3) we employ well separated points of the trace. Therefore, we assume that the correlation among the measured data points is negligible.

The satellite data employed are the observations of the events 4182, 4236, and 4267 from stations located at Beltsville, Maryland; Moses Lake, Washington; and Revilla Gigedo Is., Mexico.

We assume that we know the positions of Beltsville, Maryland and Moses Lake, Washington exactly. These two stations are used as baseline to evaluate the satellite positions. The third station, Revilla Gigedo Is., Mexico is considered unknown in the testing. Therefore, we do not constrain the adjusted values of its coordinates to any preassumed value. However, we use very good estimates of coordinate starting values in order to avoid the necessity of an iterative procedure in the least square adjustment. Using a good estimate does not harm the test validity because we are mainly interested in the degree to which the covariance matrix of the adjustment reflects the global effect of error propagation.

The station coordinates which were used were obtained from the C-7 Datum (Ref 4) and are given in Table 1.

TABLE 1

	X (m)	Y (m)	Z (m)
Beltsville, Md.	1130773	-4830833	3994706
Moses Lake, Wash.	-2127831	-3785842	4656029
Revilla Gigedo Is., Mex.	-2160983	-5642717	2035347

SECTION 5

DESCRIPTION OF THE COMPUTER PROGRAMS

The program package employed to test the Continuous Trace Technique reflects the various aspects of the testing methodology described in Section 4.

The operations necessary to perform the test are:

1. Extract the satellite event data from the National Space Science Data Center magnetic tape.

2. Recover plate coordinates of the satellite images and camera orientations.

3. Rotate the plates axes to be parallel to the trace, and evaluate the interpolation polynomials used in computing the derivatives.

4. Find the homological correspondence between the two traces from the known stations for the same event, determine satellite position and evaluate the covariance matrix.

5. Perform the least square adjustment of the third station.

A block diagram that shows the data flow among the programs is given in Fig. 2.

The programs present no particular difficulty from a data processing point of view. Therefore, we present only a brief summary of the operations in the following sections. In Section 7, we describe the mathematical formulas involved.

5.1 DATA EXTRACTION

This program extracts the observational data from the NSSDC Tape and loads a disk file after checking for sequence, format, and completeness. The data stored on the disk are: 1) event and station identifiers, 2) event date, and 3) time, right ascension and declination of each trace point.

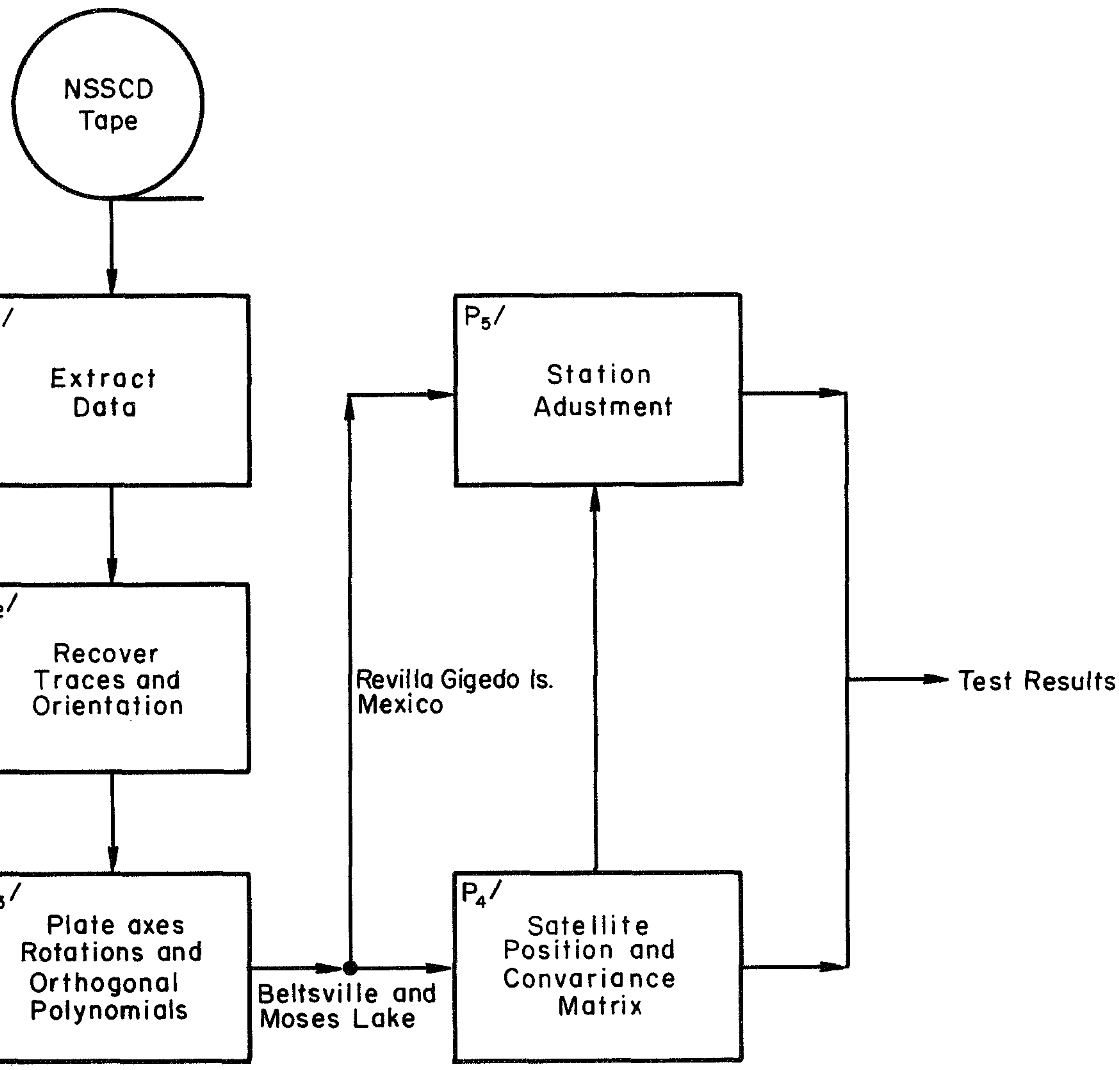

Figure 2 Program Data Flow

5.2 CONTINUOUS TRACE AND ORIENTATION

This program performs the following operations:

1. Corrects the data for phase angle, parallactic refraction and planetary aberration.

2. Transforms right ascension and declination system to Greenwich hour angle and declination system.

3. Evaluates the direction cosines of the rays which project the satellite positions onto the plate.

4. Evaluates the matrix orientation of the established camera system. This system is arbitrarily chosen so that the optical axis is pointing toward the approximate center of the observed arc and is normal to the plate.

5. Calculates the intersections of the projecting rays with the camera plate and computes the image coordinates.

The output data are stored in a disk file. These data for each plate are: 1) Camera Orientation Matrix, and 2) (x, y) coordinates of each trace point.

The time information is employed by this program to convert from the inertial to an Earth fixed system of reference. Time is not employed in subsequent computations. We remark that time inaccuracies are equivalent to an error along the trace and do not effect the trace accuracy as discussed in Section 4.2.

5.3 PLATE AXES ROTATION AND ORTHOGONAL POLYNOMIAL FITTING

The axes of each camera reference system are rotated around the optical axis, z, so that the x axis is parallel to a linear average of the trace. In this way, the straight line component of the continuous trace is reduced to a constant and the possible curvature is easily discriminated.

The coordinates of each trace point are correspondingly transformed. These points are the sampled representation of the continuous trace.

A polynomial is fitted through the points using the method of orthogonal polynomials (Ref 2). This method yields the automatic choice of the best polynomial degree. The fitting is employed to evaluate the direction cosines of tangents to the trace. These direction cosines are necessary to evaluate the covariance matrix.

5.4 SATELLITE POSITION AND COVARIANCE MATRIX

For each event, ten satellite positions are evaluated. Trace points are randomly taken from the first trace (Beltsville, Maryland) in such a way that the ten points are approximately equispaced.

The homologous point on the second trace (Moses Lake, Washington) is evaluated with respect to each selected point on the first trace. Since the second trace is represented with a large number of sampled points, we employ a simple linear interpolation to solve the homology equation. We avoid using the polynomial fitting for interpolation purposes in order to maintain the statistical independence among the evaluated satellite positions.

The covariance matrix corresponding to each satellite position is evaluated in a rigorous way, taking into account the points discussed in Section 4.2.

5.5 STATION ADJUSTMENT

The coordinates of the third station (Revilla Gigedo Is., Mexico) are adjusted in this program by means of a rigorous least square procedure.

The procedure simultaneously corrects satellite, trace, and third station coordinates by minimizing the weighted squared values of the satellite and trace coordinate corrections. The third station coordinates are considered as unconstrained parameters because: 1) there is no covariance matrix available for the third station, and 2) it would be difficult to isolate the effects of the a priori constraints on the final results of the adjustment.

The equations employed in the adjustment assure that the differences between measured and computed values of the trace coordinates are zero after the adjustment. We do not employ any iteration to solve the equations since the starting values of the third station coordinates are very accurate and, therefore, the evaluated corrections are known to be within the linearity limits. In fact, no correction resulted to be larger than 40 m which corresponds to 10 pp million of the station coordinates.

The principal difficulty in the adjustment stems from the ill-conditioned matrix of the station coordinates in their normal system. The traces are almost parallel to each other. An approximate method for checking the matrix condition is employed. For a severely ill-conditioned matrix, only the station coordinates that appear to be meaningful are adjusted. Although we do not claim to have adjusted the station coordinates to their best values, the results are valid for testing purposes.

SECTION 6

DISCUSSION OF THE RESULTS

Processing real camera data provided significant results for the following three areas:

1. Values of the coordinates for satellite positions and the corresponding covariance matrices.

2. Corrections to the third station coordinates and the corresponding covariance matrix.

3. Estimates of the standard deviations and the degree of the polynomials obtained from the trace fitting procedure. These estimates are employed to evaluate the observational data.

6.1 TRACE CHARACTERISTICS

Estimates of the standard deviation, σ, and the degree of the fitting polynomial, Pd, associated with each event and station are given in Table 2.

TABLE 2

Event	4182		4235		4267	
	$\sigma\ 10^6$	Pd	$\sigma\ 10^6$	Pd	$\sigma\ 10^6$	Pd
Station						
6002	9.32	2	5.81	3	10.46	2
6003	7.12	2	12.86	2	6.44	2
6038	29.3	2	9.40	2	7.85	3

The standard deviation estimates are normalized to a focal length value of 1. The average value of the estimates is

$$\bar{\sigma} = 8.23\ 10^{-6}$$

this is equivalent to a plate measurement error of

$$\sigma_P = 3.7\ \mu m$$

for a 450 mm focal length camera.

This value is in close agreement with values found in the BC-4 camera literature.

The best fitting curve is second degree, a parabola which represents the image of a short arc of a Keplerian orbit. The two third degree cases correspond to the longest arcs observed and probably represent earth rotation effects. However, the coefficients of both second and third degree have very small values since the average curvature of the trace is less than 0.1%.

6.2 SATELLITE POSITIONS

The printouts of the program that computes the satellite position coordinates and correlation matrices are shown in Figures 3, 4, and 5 for the three events analyzed.

The data for each computed point is contained on four lines of printout with the event number and point and line indices being listed at the end of each line. In each group of four lines, the data is presented as follows:

Line 1: Identifiers

Line 2: Geocentric position coordinates X, Y, Z (m)

Line 3: Covariance matrix diagonal terms CV_{11}, CV_{22}, CV_{33} (m^2)

Line 4: Covariance matrix off-diagonal terms CV_{12}, CV_{13}, CV_{23} (m^2)

The most important data are the correlation matrix values. Some of the matrices in the figures are highly correlated having factors as great as 0.9 between the Y and Z coordinates. This is due to the error propagation characteristics of the Continuous Trace Technique discussed in Section 4.2.

The squared estimates of the standard deviations of the coordinates are represented by the diagonal elements of the matrix. Average rms errors in position coordinates are:

Event 4182	$\bar{\sigma}$ = 51 m
Event 4236	$\bar{\sigma}$ = 88 m
Event 4267	$\bar{\sigma}$ = 44 m

These values agree with the results of the error estimates obtained by fitting polynomials to plate measurements. They correspond to an average

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					Event No.	Point Index	Line Index
2	3	1 62		6	4182	1	1
-.29099622810D	07	-.61867654748D	07	.73103176893D	074182	1	2
.11982854353D	04	.27179139010D	04	.38151255318D	044182	1	3
.11606372150D	04	-.16971087490D	04	-.31436666987D	044182	1	4
2	3	1 75		20	4182	2	1
-.29314795194D	07	-.62373183425D	07	.72691981933D	074182	2	2
.12297887540D	04	.28299147667D	04	.36971409944D	044182	2	3
.12155479517D	04	-.17055259441D	04	-.31584701392D	044182	2	4
2	3	1 88		32	4182	3	1
-.29563384042D	07	-.62955015011D	07	.72216453950D	074182	3	2
.12665457706D	04	.29594829675D	04	.35612792049D	044182	3	3
.12795282221D	04	-.17132213919D	04	-.31706787147D	044182	3	4
2	3	1 101		46	4182	4	1
-.29776030995D	07	-.63455938618D	07	.71801089003D	074182	4	2
.12980973968D	04	.30709852331D	04	.34429164264D	044182	4	3
.13347408271D	04	-.17173179942D	04	-.31761164427D	044182	4	4
2	3	1 114		53	4182	5	1
-.30004240714D	07	-.63992295439D	07	.71348493408D	074182	5	2
.13319738169D	04	.31903158582D	04	.33148626501D	044182	5	3
.13941900533D	04	-.17193062579D	04	-.31767381142D	044182	5	4
2	3	1 127		67	4182	6	1
-.30214756252D	07	-.64487303402D	07	.70925940383D	074182	6	2
.13632876503D	04	.33001980699D	04	.31964014729D	044182	6	3
.14492285929D	04	-.17190563086D	04	-.31728507281D	044182	6	4
2	3	1 140		81	4182	7	1
-.30471459915D	07	-.65091941332D	07	.70400410760D	074182	7	2
.14013745875D	04	.34336614988D	04	.30507726083D	044182	7	3
.15164294596D	04	-.17156727692D	04	-.31617724069D	044182	7	4
2	3	1 153		94	4182	8	1
-.30679941020D	07	-.65583586060D	07	.69972448093D	074182	8	2
.14324805512D	04	.35417164184D	04	.29339848711D	044182	8	3
.15711802883D	04	-.17111864002D	04	-.31489332955D	044182	8	4
2	3	1 166		105	4182	9	1
-.30949132620D	07	-.66219550120D	07	.69406660315D	074182	9	2
.14723440012D	04	.36800545624D	04	.27821501933D	044182	9	3
.16416712257D	04	-.17016999327D	04	-.31253210351D	044182	9	4
2	3	1 179		118	4182	10	1
-.31153091089D	07	-.66701391075D	07	.68970218706D	074182	10	2
.15023448417D	04	.37837125474D	04	.26673254786D	044182	10	3
.16948488130D	04	-.16924792051D	04	-.31025356598D	044182	10	4

Figure 3 Event 4182 Satellite Positions

					Event No.	Point Index	Line Index
2	3	1	654	35	4236	1	1
.52889638598D	06	-.92875306525D	07	.46450490811D	074236	1	2
.69789134352D	03	.20011229551D	05	.39314532313D	034236	1	3
.17454264388D	04	-.36533432027D	03	-.26910292383D	044236	1	4
2	3	1	674	56	4236	2	1
.53566738190D	06	-.93466817421D	07	.45441011580D	074236	2	2
.71128424534D	03	.21042873300D	05	.28797593388D	034236	2	3
.17769430949D	04	-.32694489222D	03	-.23288088066D	044236	2	4
2	3	1	694	76	4236	3	1
.54248154265D	06	-.94072515934D	07	.44387961314D	074236	3	2
.72516400041D	03	.22130203537D	05	.19477008207D	034236	3	3
.18073065455D	04	-.28616867868D	03	-.19142564611D	044236	3	4
2	3	1	714	93	4236	4	1
.54866532850D	06	-.94618200344D	07	.43412847972D	074236	4	2
.73774972502D	03	.23139213010D	05	.12504908332D	034236	4	3
.18325228305D	04	-.24783674386D	03	-.14969720953D	044236	4	4
2	3	1	734	117	4236	5	1
.55560054425D	06	-.95215601260D	07	.42311748488D	074236	5	2
.75154807665D	03	.24277080288D	05	.67281888634D	024236	5	3
.18571871154D	04	-.20405890861D	03	-.98780414411D	034236	5	4
2	3	1	754	131	4236	6	1
.56053489654D	06	-.95649603021D	07	.41494248464D	074236	6	2
.76171469263D	03	.25122248906D	05	.39850834003D	024236	6	3
.18737894295D	04	-.17136413107D	03	-.58425465253D	034236	6	4
2	3	1	774	150	4236	7	1
.56696345098D	06	-.96201187652D	07	.40428762102D	074236	7	2
.77462225015D	03	.26221697697D	05	.25213001688D	024236	7	3
.18920648433D	04	-.12867460712D	03	-.26350419643D	024236	7	4
2	3	1	794	172	4236	8	1
.57346650426D	06	-.96757980537D	07	.39317816485D	074236	8	2
.78777084154D	03	.27356787545D	05	.36911871854D	024236	8	3
.19078674832D	04	-.84173602151D	02	.59288879825D	034236	8	4
2	3	1	814	191	4236	9	1
.57888867154D	06	-.97216042908D	07	.38371176194D	074236	9	2
.79864992769D	03	.28315513099D	05	.69786186697D	024236	9	3
.19185167352D	04	-.46399493865D	02	.11496911942D	044236	9	4
2	3	1	834	212	4236	10	1
.58474742602D	06	-.97706049166D	07	.37338702861D	074236	10	2
.81030349986D	03	.29356972164D	05	.13083004612D	034236	10	3
.19272622251D	04	-.55324339162D	01	.17868769973D	044236	10	4

Figure 4 Event 4236 Satellite Positions

					Event No.	Point Index	Line Index
2	3	1	1444	41	4267	1	1
-.80660136353D	06	-.78556724458D	07	.64922347464D	074267	1	2
.61467802371D	03	.34871676517D	04	.10975561470D	044267	1	3
-.34953799410D	03	-.36770560045D	02	-.18769713210D	044267	1	4
2	3	1	1459	57	4267	2	1
-.80798364166D	06	-.79257579407D	07	.64229959712D	074267	2	2
.62246120916D	03	.36384102310D	04	.10394159539D	044267	2	3
-.33864369978D	03	-.50209620479D	02	-.18638039564D	044267	2	4
2	3	1	1474	68	4267	3	1
-.80899252271D	06	-.79779585049D	07	.63707628458D	074267	3	2
.62829233029D	03	.37530814455D	04	.99614863242D	034267	3	3
-.33009596207D	03	-.59866356612D	02	-.15516933521D	044267	3	4
2	3	1	1489	86	4267	4	1
-.81019617467D	06	-.80432517369D	07	.63042427399D	074267	4	2
.63558819298D	03	.39013990692D	04	.94165849646D	034267	4	3
-.31882523374D	03	-.71544192011D	02	-.18330835205D	044267	4	4
2	3	1	1504	100	4267	5	1
-.81135419963D	06	-.81077402744D	07	.62372725483D	074267	5	2
.64279775543D	03	.40560037866D	04	.88768006367D	034267	5	3
-.30708551684D	03	-.82605055132D	02	-.18109337452D	044267	5	4
2	3	1	1519	114	4267	6	1
-.81223520995D	06	-.81581147497D	07	.61840810804D	074267	6	2
.64842810926D	03	.41695873950D	04	.84548726509D	034267	6	3
-.29750894069D	03	-.90897232442D	02	-.17909249413D	044267	6	4
2	3	1	1534	127	4267	7	1
-.81318060641D	06	-.82149355012D	07	.61235606283D	074267	7	2
.65481037775D	03	.43067280776D	04	.79837213207D	034267	7	3
-.28639621592D	03	-.99824638001D	02	-.17659615299D	044267	7	4
2	3	1	1549	141	4267	8	1
-.81405755325D	06	-.82643953607D	07	.60697661138D	074267	8	2
.66032654401D	03	.44281457206D	04	.75716801310D	034267	8	3
-.27626951353D	03	-.10729285549D	03	-.17412093492D	044267	8	4
2	3	1	1564	153	4267	9	1
-.81492312910D	06	-.83165510120D	07	.60122366663D	074267	9	2
.66613746150D	03	.45584801655D	04	.71396518143D	034267	9	3
-.26528669663D	03	-.11479115376D	03	-.17124326985D	044267	9	4
2	3	1	1579	169	4267	10	1
-.81583713048D	06	-.83747836448D	07	.59470693110D	074267	10	2
.67261724779D	03	.47060607467D	04	.66617064620D	034267	10	3
-.25267195027D	03	-.12269190672D	03	-.16770251155D	044267	10	4

Figure 5 Event 4267 Satellite Positions

of 8 parts per million of the satellite slant range. (Slant range between PAGEOS and the two stations analyzed was approximately 6500 km.)

6.3 STATION ADJUSTMENT

The principal difficulty encountered in adjusting the coordinates of the third station was caused by the ill-conditioned system of equations. This ill-conditioning was caused by: 1) each trace being almost a straight line, thus limiting the observer to adjusting only one coordinate of the station with each event, and 2) the three observed arcs of the satellite being almost parallel.

The results of adjusting the station coordinates with a single trace are presented in Figure 6. The first three lines for each event correspond to the normal equation* $N_S \, \Delta S = E_S$ where:

Line 1 contains the known terms E_{S1}, E_{S2}, E_{S3}

Line 2 contains the diagonal entries of the normal matrix N_S
N_{S11}, N_{S22}, N_{S33}

Line 3 contains the off-diagonal terms N_{S12}, N_{S13}, N_{S23}

The second group of three lines for each event correspond to the inverse solution of the equation above; that is,

$$\Delta S = (N_S)^{-1} E_S = CV_S \cdot E_S$$

where Line 1 contains ΔS_1, ΔS_2, ΔS_3, in meters

Line 2 contains CV_{S11}, CV_{S22}, CV_{S33}, in squared meters

Line 3 contains CV_{S12}, CV_{S13}, CV_{S23}, in squared meters

All data are expressed with reference to the camera system corresponding to the event.

To avoid serious ill-conditioning effects, we solved only for the station coordinate along the normal to the trace. The undetermined values were left

* Refer to Section 7.6 for a complete development of the normal equation.

EVENT 4182 NO OF POINTS 10

0.15685087913D-04	-0.17114868074D-01	0.22538303723D-03	E_{S1}, E_{S2}, E_{S3}
0.12405254510D-08	0.41894067422D-03	0.72629682179D-07	N_{S11}, N_{S22}, N_{S33}
-0.30627081124D-06	0.40435798638D-08	-0.55160924985D-05	N_{S12}, N_{S13}, N_{S23}

DEGREES OF FREEDOM 9 SIG 0.1101D 01

0.00000000000D 00	-0.40852724805D 02	0.00000000000D 00	ΔS_1, ΔS_2, ΔS_3
0.00000000000D 00	0.23869728139D 04	0.00000000000D 00	CV_{S11}, CV_{S22}, CV_{S33}
0.00000000000D 00	0.00000000000D 00	0.00000000000D 00	CV_{S12}, CV_{S13}, CV_{S23}

EVENT 4236 NO OF POINTS 10

-0.61880113899D-03	-0.24712784506D-01	-0.10543406759D-03
0.19547541187D-06	0.97426069434D-03	0.17199300528D-07
0.11254631736D-05	0.55317858474D-08	0.40750437277D-05

DEGREES OF FREEDOM 8 SIG 0.2709D 01

-0.30397947709D 04	-0.21854117240D 02	0.00000000000D 00
0.51499861356D 07	0.10332918764D 04	0.00000000000D 00
-0.59492492862D 04	0.00000000000D 00	0.00000000000D 00

EVENT 4267 NO OF POINTS 10

-0.30712963126D-03	0.56185960991D-01	0.11712172487D-02
0.31044155580D-06	0.42621018099D-02	0.18507386841D-05
-0.22172583384D-04	-0.46850321863D-06	0.88807385247D-04

DEGREES OF FREEDOM 9 SIG 0.1254D 01

0.00000000000D 00	0.13182688612D 02	0.00000000000D 00
0.00000000000D 00	0.23462602364D 03	0.00000000000D 00
0.00000000000D 00	0.00000000000D 00	0.00000000000D 00

Figure 6 Single Event Station Adjustment Results

equal to zero. Although the normal matrix for event 4236 appeared to be better conditioned, a solution for the component along the trace obtained a meaningless result.

Since we employed very reliable station coordinates as starting values in the adjustment, we should expect very small values for the resulting corrections. In fact, the corrections in the direction of the normal to the trace range from -40.8 m to 13.1 m with estimated errors ranging from 52.9 m (r.m.s.) to 17.5 m (r.m.s.) respectively. These values are in agreement with the observation precisions (Section 6.1). However, the value of the along trace correction for event 4236 is very large which shows the ill-conditioning effects. In fact, we obtained anomalous results: -3039.8 m for the correction, and 3750 m (r.m.s.) for the estimated error.

The printouts present also the degrees of freedom and estimates of the standard deviation (SIG) of measurements having unit weight. The values of SIG are nearly 1 because we employed the inverse of the covariance matrices as weighting matrices. The higher value of SIG for event 4236 is a further indication that we cannot solve for two coordinates. Solving for only one coordinate rather than two would reduce value of SIG for event 4236 from 2.7 to 2.3.

A completely different pattern is shown by the output of the global adjustment. In this case, the degrading effect of the ill-conditioned normal matrix on the computations could not be avoided, because we did not have a-priori information about the significance of individual coordinates. The results are presented in Figure 7 with the same reading key as for Figure 6. The principal characteristics of this adjustment are the following:

1. The vector of known terms has an amplitude comparable to that defined for the case of a single event adjustment.

2. The normal matrix is highly correlated.

3. The resulting correction to the coordinates and the corresponding estimated errors are enormous.

4. The standard deviation of the unit weight is much less than 1.

These characteristics confirm that we cannot rely upon the global adjustment results for testing purposes.

```
GLOBAL  SYSTEM
  -0.63940762221D-01   0.50976465868D-01   0.13241829054D-01
   0.50453933401D-02   0.40972590587D-03   0.20263165756D-03
   0.85222919931D-03  -0.94292006178D-03  -0.23668413280D-03

DEGREES OF FREEDOM 27   SIG  0.2397D 0
   0.63542253947D 03   0.16865339766D 04   0.49921632835D 04
   0.52127229279D 04   0.25723462068D 05   0.25884685142D 06
   0.97454770550D 04   0.35639943420D 05   0.75395628318D 05
```

Figure 7 Global Adjustment Results

SECTION 7

MATHEMATICAL DERIVATIONS

This section develops the analytical expressions of the equations employed in implementing the programs.

The equations are, in general, a straightforward derivation of spherical astronomy, projective geometry and Continuous Trace theory as derived in Phase I report. In this Phase II report, particular emphasis is given to analysis and evaluation of satellite position covariance matrices (Section 7. 5) and of station coordinates' adjustment (Section 7. 6 and 7. 7), work which was not treated in detail in Phase I.

7. 1 TRACE AND CAMERA ORIENTATION RECOVERY

The first step in the analysis of the Continuous Trace Method consists in recovering the traces on the BC-4 camera plates corresponding to each satellite observation and determining the camera orientation in an Earth fixed reference system.

The data available are right ascention, α, declination, δ, and Greenwich mean time, t(GMT), of the satellite observations provided from the plate reduction procedure by USC&GS [Ref. 3].

Correction to the data should be made for parallactic refraction, aberration, travel time and phase. We employed the values obtained from the results of [Ref. 4]. No correction for nutation was considered [Ref. 5].

The Earth fixed reference system is the Geocentric mean equatorial system (X, Y, Z)

where X is in the Greenwich meridian and in the mean equatorial plane

Y is in the mean equatorial plane and eastward

Z is the mean polar axis

In this system the direction from the station to the satellite is expressed by means of spherical longitude ψ and latitude φ. Neglecting the effects of precession, nutation and acceleration of the Earth center gives [Ref. 6]:

1) the spherical longitude

$$\psi = \alpha - GSTO - 1.0027379t \text{ (sec)}$$

where GSTO is the Greenwich mean time at 0^h UT and 1.0027379 converts the UT to a mean sidereal interval,

and 2) the spherical latitude

$$\varphi = \delta$$

Therefore, the unit vector of the direction from station to satellite has components V_1, V_2, V_3:

$$V_1 = \cos\varphi \cos\psi$$

$$V_2 = \cos\varphi \sin\psi$$

$$V_3 = \sin\varphi$$

in the average terrestrial system.

The actual orientations of the cameras are not contained in the data available to us. Therefore, it is assumed that each camera is generally aimed toward a point near the middle of the observed satellite orbital arc. This point is somehow arbitrarily chosen since its position is immaterial for generating traces.

We indicate with α_o and δ_o the right ascension and declination of the camera axis and with ψ_o and φ_o the corresponding spherical longitude and latitude. The orientation of the camera system is therefore assumed to be

z the optical axis

y normal to the optical axis in the mean equatorial plane eastward

x right-handed complement axis

From this definition the unit vectors of the camera axes are:

$$x_1 = \sin\varphi_o \cos\psi_o$$
$$x_2 = \sin\varphi_o \sin\psi_o$$
$$x_3 = -\cos\varphi_o$$
$$y_1 = -\sin\psi_o$$
$$y_2 = \cos\psi_o$$
$$y_3 = 0$$
$$z_1 = \cos\varphi_o \cos\psi_o$$
$$z_2 = \cos\varphi_o \sin\psi_o$$
$$z_3 = \sin\varphi_o$$

in the spherical longitude and latitude Earth fixed system.

The system of unit vectors generates the orthogonal matrix that transforms the coordinates of a general vector from the camera system to the Earth spherical system.

This transformation can be expressed by the matrix relationship

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

where v_1, v_2 and v_3 are the components in the camera system. In matrix form we have

$$V = T_{CG} v$$

To recover the coordinates of the trace points on the plate, we assume that the camera system has its origin in the optical center of projection and that the plate plane is normal to the z axis and has equations

$$z = f$$

where f is the camera focal length. A trace point is the intercept of the plate plane with a ray from the camera optical center to the satellite. The coordinates (x, y, z) of the trace points are easily obtained in the camera system by means of:

1) evaluating the components of the unit vector corresponding to a ray direction in the camera system.

$$v = T'_{CG} V$$

2) evaluating the trace coordinates

$$x = f \frac{v_1}{v_3}$$

$$y = f \frac{v_2}{v_3}$$

$$z = f$$

A certain number of randomly chosen points have been generated for each trace. The coordinates of these points and the matrices of the camera systems are the starting data for the numerical analysis of the Continuous Trace Method.

7.2 GENERATION OF TRACE RELATED CAMERA AXES

The camera axes defined in Section 7.1 give a coordinate system which is referenced to the photographic plate by means of the axes x and y. These axes have no relation to the trace of the satellite orbit arc because we chose them arbitrarily.

However, the trace is better represented by means of two axes that are related to the global pattern of the trace. The trace is, in general, almost rectilinear and thus we choose one of the axes, ξ, to represent the linearity of the shape and the other, η, to account for the deviations from linearity (Fig. 8).

We obtain two advantages by using the ξ, η plate system: 1) the numerical computations are more easily programmed, and 2) the deviations from linearity are easily identified and recognized in the final least square adjustment where they have a fundamental effect on the ill-condition of the normal matrix of the station correction system.

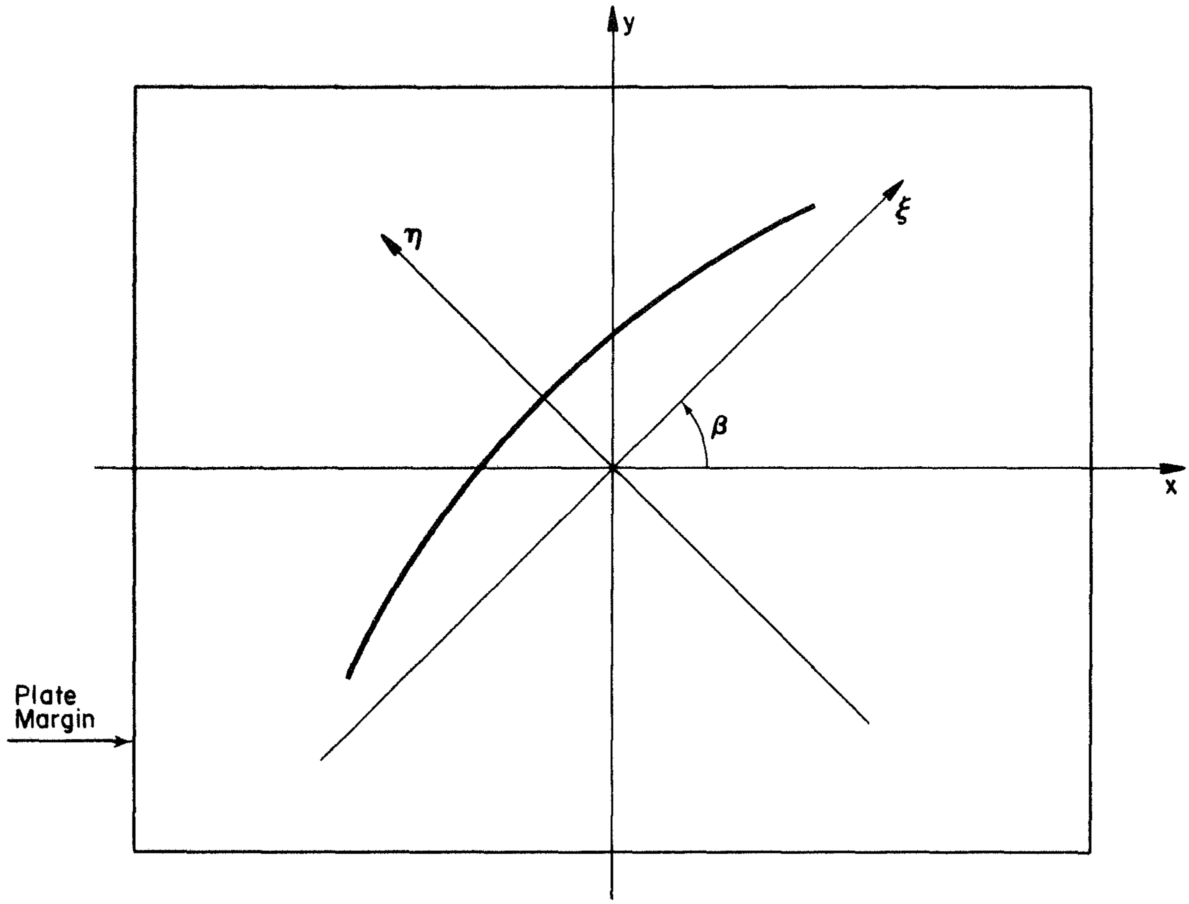

Figure 8 Trace Related Axes

To generate the ξ, η axes, we consider the trace points as mass points. Thereafter, we choose ξ, η to be parallel respectively to the minimum and maximum principal axes of inertia of the mass points. In this way, the coordinates η of the trace contains no component of the linear part. In fact, the coordinate η would have the randomly distributed values of the measurement errors if the true trace were exactly a straight line. The analytical details of the transformation are in Appendix A.

The result of the transformation is a rotation by an angle β around the z axis of the camera and thus it is

$$\nu = \begin{pmatrix} \xi \\ \eta \\ z \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = T_B v$$

The new matrix T_{RG} that relates the new camera axes to the system is obtained from T_{CG} (previously computed, Section 7.1) and T_B by means of the relations

$$v = T'_B \nu$$

$$V = T_{CG} \nu$$

$$V = T_{CG}\, T'_B \nu = T_{RG} \nu$$

and, thus

$$T_{RG} = T_{CG}\, T'_B$$

The covariance matrix of the measurements will accordingly transform as

$$C_{(\xi,\eta)} = T_B\, C_{(x,y)}\, T'_B$$

7.3 REPRESENTATION OF THE CONTINUOUS TRACE

The traces of the observed satellite orbital arcs consist of a set of points which sample the continuous trace on the photographic plate. However, we need the local tangent of the trace to compute the effects that a variation of the trace itself has on the adjustment of station coordinates. The tangent is evaluated by means of derivatives and this is accomplished by representing the trace with a continuous line.

In general a two dimension continuous line is represented implicitly with an equation relating the coordinates of the points on the line:

$$F(x, y) = 0$$

In the cases at hand, the trace is almost a straight line in the (ξ, η) coordinates (Section 7.2) and, the values of ξ are never greater than the focal length, f, (actually we have $-0.15\ f < \xi < +0.15\ f$). Therefore, it is possible to neglect every contribution from powers of η higher than one and from products of η by powers of ξ and to represent the trace with η as an explicit function of ξ. Expanding this function in a power series of ξ results in a polynomial representation:

$$\eta = a_o + a_1 \xi + a_2 \xi^2 + \ldots$$

The coefficients of the polynomial are computed by means of a least square fitting through the points of the trace in the (ξ, η) coordinates by means of the method of orthogonal polynomials [Ref. 2]. This method retains only those terms of the series that are significant and minimizes the error resulting from the combination of series truncation and of measurement random noise. Some details are in Appendix B.

The polynomial fittings will furnish us with an estimate of the standard deviation of the measurement. However, the polynomial representation is not employed to evaluate satellite positions and station adjustment in order to maintain the statistical independence of the trace points.

7.4 SATELLITE POSITION EVALUATION

This section deals with the problem of finding the satellite positions corresponding to the traces obtained at two known stations. Satellite positions are necessary to establish the corrections of the third station coordinates, and they are also used in independent investigations such as orbit determination and geopotential analysis. Therefore, the problem is considered as a separate aspect of the Continuous Trace Technique.

We start from the hypothesis that we know:

1) the position of two stations S_1 and S_2 which we assume coincident with the optical centers of the corresponding cameras,

2) the matrices T_1 and T_2 that relate the camera axes of each station to the Geocentric System, and

3) the focal lengths f_1 and f_2 of each camera, and the relationships $\{\eta, \xi\}_i ; i = 1, 2,$ among the plate plane coordinates of the points of each trace.

Positions of the satellite on the observed orbital arc are obtained in two steps. The first step is the identification of homologous points Q_1 and Q_2 on the two traces. Homologous points are those points on each trace that correspond to the same satellite position. The second step is the evaluation of the coordinates of a satellite position corresponding to two homologous points.

Homologous points are derived from a one-to-one correspondence between the two traces. We consider any plane in the family of planes through S_1 and S_2, and we assume that the orbital arc intersects this plane at B [Figure 9]. The plane will in general intersect each photographic trace at a point, say Q_1 and Q_2 respectively for S_1 and S_2. Since the points Q_1 and Q_2 are on the same plane through S_1, S_2 and B, it follows that Q_1 and Q_2 are the images of B. Therefore, Q_1 and Q_2 are homologous points.

To identify homologous points in practice, we choose any point Q_1 on the first trace and we consider the plane through S_1, S_2 and Q_1. The intersection between the plane S_1, S_2, Q_1 and the second trace is the homologous Q_2 of Q_1.

The point Q_2 is obtained by means of the homology equation, that in vectorial notation has the form

$$(\overline{S_1Q_1} \times \overline{S_1S_2}) \cdot \overline{S_2Q_2} = 0$$

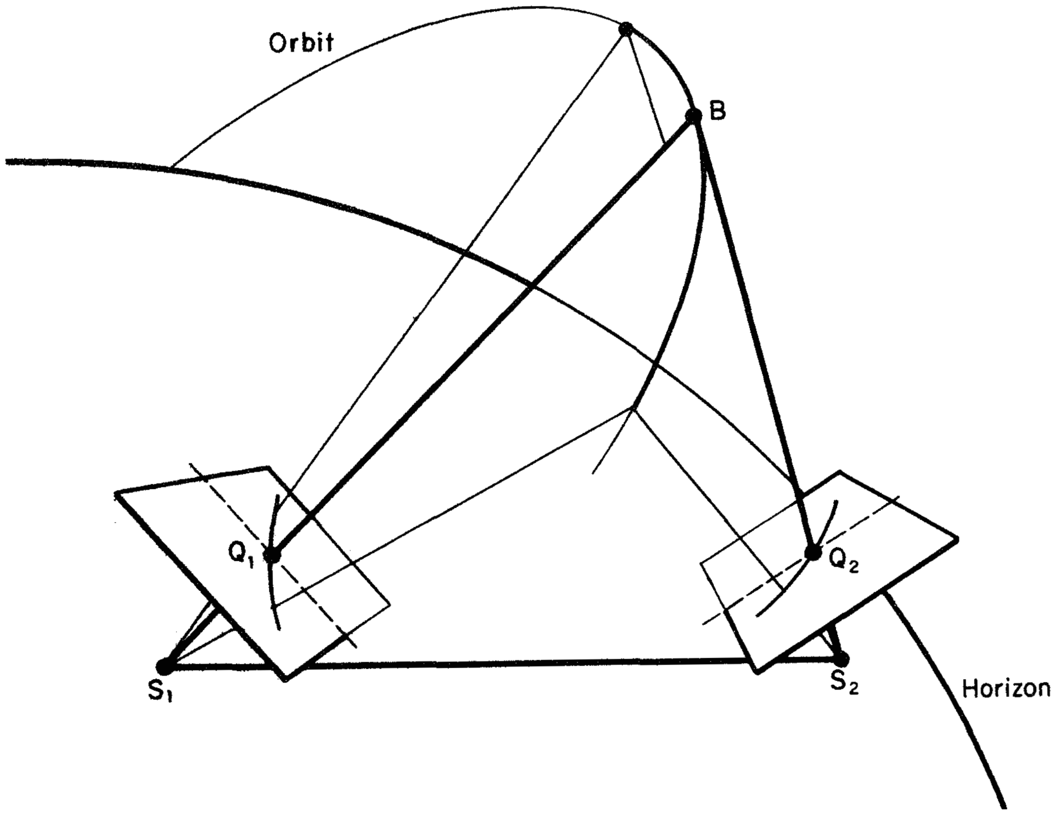

Figure 9 Trace Homology Determination

The position B of the satellite is obtained directly by means of triangulation in the plane through S_1, S_2, Q_1 and Q_2. It is the intersection of the two straight lines through S_1 and Q_1 and through S_2 and Q_2 [Figure 10]. In vectorial notation we have

$$\overline{B} = \overline{S}_1 + C_B \, \overline{S_1 Q_1}$$

where the coefficient C_B is:

$$C_B = \frac{D}{|\overline{S_1 Q_1}|} \frac{\sin \gamma_2}{\sin \gamma_3}$$

Details for the solution of the homology equation are reported in Appendix C.

The Continuous Trace Technique fails if the homology equation vanishes or is unsolvable. The equation vanishes if all points of the two traces lie on the same plane through S_1 and S_2. This is the case when the observed orbital arc is entirely contained in a plane through S_1 and S_2. The equation is unsolvable if no homology exists between the two traces. This occurs if the two stations observe nonoverlapping portions of the same orbit.

7.5 MEASUREMENT ERRORS AND SATELLITE POSITION COVARIANCE MATRIX

The covariance matrix of satellite position is the statistical representation of the random errors in the coordinates of the satellite. It is defined as the statistical average of the product of the random error vector and its transponse. In symbols

$$CV = \langle \partial B \cdot \partial B' \rangle$$

These random errors in the satellite coordinates are due to the errors in measuring the coordinates of the traces on the plates. In order to derive the covariance matrix of satellite position, we assume that the covariance matrix of each plate measurement is known. In order to simplify the computations, we further assume: 1) that the measurement errors have a

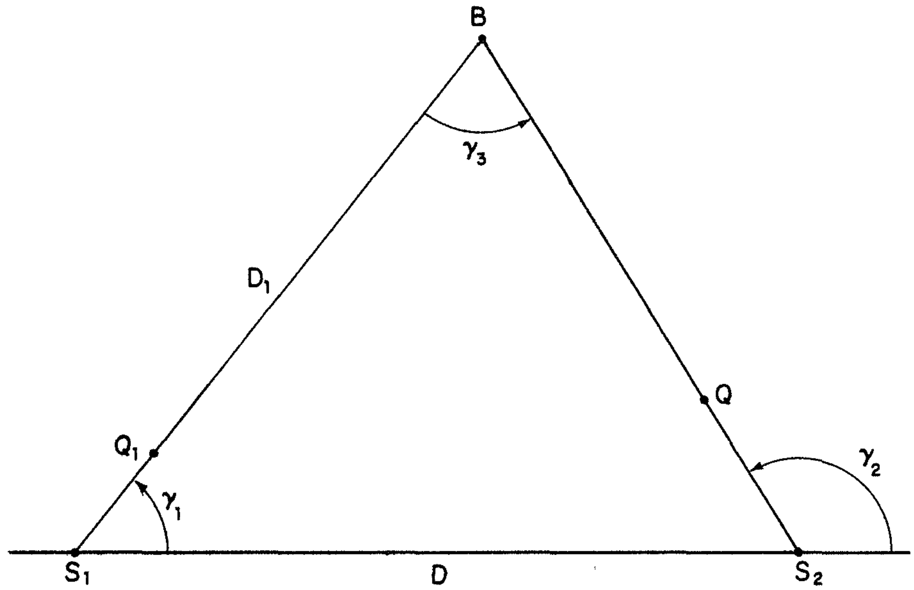

Figure 10 Plane Triangulation

zero average and are statistically independent, and 2) that the standard deviation, σ^2, is a constant, independent of the direction, for each plate. Therefore, the covariance matrix for every point of a plate is

$$CT = \sigma^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in the camera reference system.

To evaluate CV we should compute the error ∂B as a function of the measurement errors. This is accomplished by differentiating the expression for B obtained in Section 7.4

$$\overline{B} = \overline{S}_1 + C_B \, \overline{S_1 Q_1}$$

To obtain B we follow four steps:

1) Arbitrarily establish a plane through S_1 and S_2.

2) Find the homologous points Q_1 and Q_2 on the two traces.

3) Evaluate the lengths of the two vectors S_1Q_1 and S_2Q_2 and the angles γ_1 and γ_2 between these vectors and the vector S_1S_2.

4) Compute the coordinates of B.

The variables affected by measurement errors are the lengths and angles evaluated in step 3. These variables affect the position of B in the plane through S_1 and S_2. Therefore, the error ∂B is entirely in this plane and has no component along the normal to the plane. Actually, the two angles γ_1 and γ_2 are the only independent variables of the triangulation, and ∂B can be completely defined in terms of the errors $\partial \gamma_1$ and $\partial \gamma_2$.

The analytical details and the relationship between angle errors and measurement errors are described in Appendix D. The resulting covariance matrix of the satellite position can be represented by

$$CV = a^2 \left(b_1^2 M_1 C_1 M_1' + b_2^2 M_2 C_2 M_2'\right)$$

where a, b_1, b_2 are coefficients

M_1 and M_2 are two matrices

The definition of these quantities is presented in Appendix D.

The matrices C_1 and C_2 represent the covariance matrices of measurements errors in the two plates relative to the continuous traces. The derivation of these matrices deserves a particular discussion.

The coordinates ξ, η of a point on a photographic plate are affected by measurements error $\partial\xi$, $\partial\eta$ that we assumed to be statistically independent with a constant standard deviation. This is also true if we consider a particular point of the trace. However, we do not use the points of a continuous trace as if they have an individual identity; we use the trace as a whole. In other words, if we let every point of the trace slide along the trace itself, we change the coordinates of each point of the trace but the trace as a whole is completely unaffected. Therefore, we can disregard those measurement errors that do not affect the trace considered as a continuous curve.

To state this observation analytically, let us consider point p on the trace and assume that measurement errors have only first order effects. Construct two axes t and n which are tangent and normal to the trace at point p (see Figure 11). An error in measuring the coordinates of p has components ∂n normal to and ∂t along the trace. The component ∂t does not affect the trace and should be deleted. Thus in the local axes t, n, the covariance matrix for each point of the trace is

$$< \begin{pmatrix} 0 \\ \partial\eta \\ 0 \end{pmatrix} (0 \; \partial\eta \; 0) > = \sigma^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

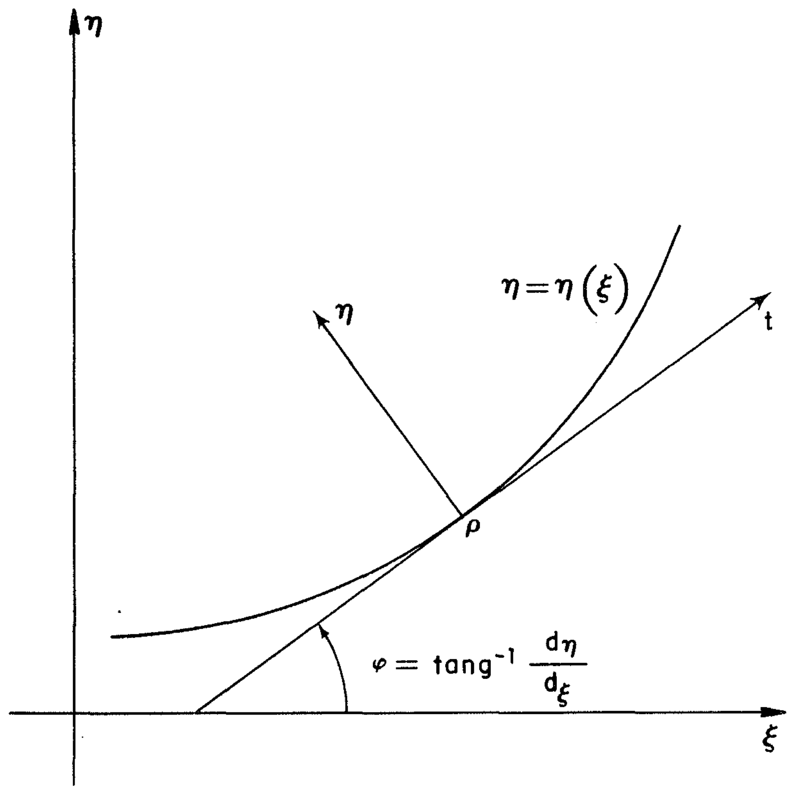

Figure 11 Propagation of Plate Error to the Trace

Transformation to plate axes ξ, η

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \frac{1}{\sqrt{1 + \eta'^2}} \begin{pmatrix} 1 & - \eta' \\ \eta' & 1 \end{pmatrix} \begin{pmatrix} t \\ n \end{pmatrix}$$

gives the covariance matrix

$$C = \frac{\sigma^2}{1 + \eta'^2} \begin{pmatrix} \eta'^2 & - \eta' \\ \eta' & 1 \end{pmatrix} \begin{pmatrix} t \\ n \end{pmatrix}$$

where
$$\eta' = \frac{d}{d\xi} \eta(\xi)$$

is the derivative of the function $\eta = \eta(\xi)$ which represents the trace.

In summary, the analysis of the errors in the satellite position shows that:

1) Satellite position error has only two components different from zero, and

2) One of the components of each plate measurement error has no effect.

7.6 LEAST SQUARE ADJUSTMENT OF STATION COORDINATES

Station coordinates are adjusted by means of a least square fitting between the measured trace relative to the station and the trace computed from the satellite positions. We assume that the orientation of the camera is known along with the coordinates of the trace points in the camera system. Furthermore, we assume that the station coordinates are accurately known a priori.

Let us consider N positions of the satellite on the observed orbital arc, say $P_1, \ldots, P_N$. In order to simplify the analytical expressions, we will express station and satellite positions in the camera system. In this system, a satellite point P will have coordinates

$$P = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

the station will have coordinates

$$S = \begin{pmatrix} x_s \\ y_s \\ x_s \end{pmatrix}$$

and a point on the plate will have coordinates

$$p = \begin{pmatrix} \xi \\ \eta \\ f \end{pmatrix}$$

The true values of these coordinates should satisfy the constraints that: 1) the points S, p, and P lie on a straight line, and 2) the point p lies on the trace. Therefore, we have the straight line equations

$$\begin{cases} G\ (S,\ P,\ p) = f\ (y - y_s) - \eta\ (z - z_s) = 0 \\ \xi = f \dfrac{(x - x_s)}{(y - y_s)} \end{cases}$$

and the trace equation

$$\eta = \eta(\xi)$$

The true values of the coordinates are given by the approximate values plus a correction

$$P = P_o + \Delta P$$

$$S = S_o + \Delta S$$

$$p = p_o + \Delta p$$

where index o refers to the approximate values.

By means of least square method, we compute ΔP, ΔS, and Δp such that the variance

$$V^2 = \sum_{i=1}^{N} \Delta P_i' w_i \Delta P_i + \sum_{i=1}^{N} \Delta p_i' v_i \Delta p_i + \Delta S' w_S \Delta S$$

is a minimum, with the linearized constraints

$$G(S, P_i, p_i) = G_o + G_P' \Delta P_i + G_S' \Delta S + G_p' \Delta p_i = 0 \qquad i = 1, \ldots, N$$

We employ Lagrange multipliers, λ_i, to solve the problem of minimization. Thus, we have the system of linear equations

$$\begin{pmatrix} w_i \Delta P_i & & + G_P \lambda_i & = 0 \\ & v_i \Delta p_i & + G_p \lambda_i & = 0 \end{pmatrix} \quad i = 1, \ldots, N$$

$$w_S \Delta S + \sum_{i=1}^{N} G_S \lambda_i = 0$$

$$(G_P' \Delta P_i + G_p' \Delta p_i + G_S' \Delta S + G_o = 0) \quad i = 1, \ldots, N$$

In these equations G_P, G_p, G_S indicate the vectors of partial derivatives of the function G(S, P, p) with respect to the coordinates of P, p, and S respectively. The matrices w_i, v_i, w_S are the weighting matrices of the corrections ΔP, Δp and ΔS respectively.

Since we are interested in determining station positions, we solve the system for the unknown ΔS. The weighting matrices are the inverses of the covariance matrices of the errors of the corresponding quantities. Therefore,

$$\Delta P_i = - CV_i G_P \lambda_i$$

$$\Delta p_i = - C_i G_p \lambda_i$$

and

$$\lambda_i = \frac{1}{C_{\lambda i}} [G_o + G_S' \Delta S]$$

The coefficient $C_{\lambda i}$ is given by

$$C_{\lambda i} = G_P' CV_i G_p + G_p' C_i G_p$$

Substitution of the expression of λ_i in the equation for ΔS gives

$$\left[w_S + \sum_{i=1}^{N} \frac{1}{C_{\lambda i}} G_S G_S' \right] \Delta S = - \sum_{i=1}^{N} \frac{1}{C_{\lambda i}} G_o G_S$$

This is the normal equation of ΔS

$$N_S \Delta S = -E_S$$

where
$$N_S = w_S + \sum_{i=1}^{N} \frac{1}{C_{\lambda i}} G_S G_S'$$

$$E_S = \sum_{i=1}^{N} \frac{1}{C_{\lambda i}} G_o G_S$$

If the normal equation can be solved, we have the corrections to the station coordinates. The covariance matrix of the station coordinates is given by

$$CV_S = \sigma_o^2 N_S^{-1}$$

where σ_o^2 is the standard deviation of the unit weight observation

$$\sigma_o^2 = \frac{V^2}{N - 3}$$

and

$$V^2 = \sum_{i=1}^{N} \frac{G_o^2}{C_{\lambda i}} - \Delta S' N_S \Delta S$$

The number of degrees of freedom of the adjustment is clearly N-3.

The results are extended to the observations of n passes if we add, term by term, the equations relative to each pass. However, we should express the adjustment ΔS to the same system of coordinates by means of the camera orientation matrices T_j, $j = 1, \ldots n$. Therefore, we obtain the compound system of equations

$$N_{ST} \Delta S = - E_{ST}$$

$$N_{ST} = \sum_{j=1}^{n} T_j N_{Sj} T'_j$$

$$E_{ST} = \sum_{j=1}^{n} T_j E_{Sj}$$

Details of the computations of the derivatives are reported in Appendix E.

The normal equation relative to a single pass is singular if the trace is a straight line. In this case, only one component of the correction ΔS can be evaluated, namely the component along the η axes.

In practice, the trace is almost a straight line and the normal equation can be ill-conditioned. To solve the normal equation, we employ a test that detects the ill-conditioned cases. Thereafter, we solve the equation only for the meaningful correction components.

7.7 CASE OF A STRAIGHT LINE TRACE

We can easily discuss the case where the trace is a straight line because the plate axes are parallel to the trace [Section 7.2].

The trace equation is

$$\eta = \eta_o$$

where η_o is a constant, and the trace derivative $\eta' = 0$.

Let us assume that no reliable information about ΔS is known. Therefore, $w_S = 0$ and the normal matrix is singular. In fact [Appendix E]

$$N_S = \sum_{i=1}^{N} \frac{1}{C_{\lambda i}} G_S G_S'$$

and

$$G_S G_S' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & f^2 & -f\eta_o \\ 0 & -f\eta_o & \eta_o^2 \end{pmatrix}$$

when $\eta' = 0$.

By means of a rotation around the ξ axes we can obtain $\eta_o = 0$. The system thus becomes

$$f^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sum_{i=1}^{N} \frac{1}{C_{\lambda i}} \Delta S = f \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \sum_{i=1}^{N} \frac{G_o}{C_{\lambda i}}$$

It is apparent that we have sufficient information to adjust only one coordinate of ΔS, namely the coordinate along the η axes. The correction is

$$(\Delta S)_\eta = \frac{1}{f} \frac{\sum_{i=1}^{N} \frac{G_o}{C_{\lambda i}}}{\sum_{i=1}^{N} \frac{1}{C_{\lambda i}}}$$

The standard deviation for the unique component is

$$\sigma^2_{\Delta S\eta} = \frac{\sigma^2_{o\eta}}{f^2 \sum\limits_{i=1}^{N} \frac{1}{C_{\lambda i}}}$$

where the standard deviation of the unit weight observation is

$$\sigma^2_{o\eta} = \frac{1}{N-1}\left(\sum_{i=1}^{N} \frac{G_o^2}{C_{\lambda i}} - \left(\sum_{i=1}^{N} \frac{G_o}{C_{\lambda i}}\right)^2 \left(\sum_{i=1}^{N} \frac{1}{C_{\lambda i}}\right)^{-1}\right)$$

since we have in this case N-1 degrees of freedom.

All the coordinates of ΔS can be adjusted only if we have at least three observed passes which are <u>not</u> parallel to a common plane.

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APPENDICES

APPENDIX A

TRANSFORMATION TO PRINCIPAL AXES OF INERTIA

Measurement of a photographic plate containing a continuous trace results in a set of n points P_i, each of which can be expressed in terms of the plate coordinates (x, y)

$$P_i = (x_i, y_i) \qquad i = 1, \ldots n$$

By considering each of the observed points to be a unit of mass, we can define a set of principle axes of inertia for the entire set. To apply this data to the continuous trace, we need then to refer the individual points to a system of axes (ξ, η) which are parallel to the principal axes of inertia and which produce a minimum moment of inertia about ξ. This is obtained by a rotation of the x, y system by an angle β:

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Since the principal axes of inertia have their origin in the center of mass of the system, we compute the coordinates of the center by means of

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = \frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

Therafter, the principal axes of inertia are obtained by the equation:

$$\sum_{i=1}^{n} (\xi_i - \xi_c)(\eta_i - \eta_c) = 0$$

and the moment of inertia around ξ is a minimum if

$$\sum_{i=1}^{n} (\xi_i - \xi_c)^2 > \sum_{i=1}^{n} (\eta_i - \eta_c)^2$$

Substituting the values of ξ and η as a function of x and y reduces the equation defining the principal axes to

$$b^2 J_{xy} + b (J_{xx} - J_{yy}) - J_{xy} = 0$$

where $$b = \frac{\sin \beta}{\cos \beta}$$

$$J_{uv} = \sum_{i=1}^{n} (u_i - u_c)(v_i - v_c) \qquad u = x, y \quad v = x, y$$

Solving the equation gives

$$\sin \beta = \operatorname{sign}(J_{xy}) \sqrt{\frac{\sqrt{D} - (J_{xx} - J_{yy})}{2\sqrt{D}}}$$

$$\cos \beta = \sqrt{\frac{\sqrt{D} + (J_{xx} - J_{yy})}{2\sqrt{D}}}$$

where $$D = (J_{xx} - J_{yy})^2 + 4 J_{xy}^2$$

The inequality, which assures a minimum moment of inertia about ξ, is always satisfied since $\sin \beta$ has the sign of J_{xy}.

APPENDIX B

BEST POLYNOMIAL FITTING BY MEANS OF ORTHOGONAL POLYNOMIALS

We assume that y is a function of x and can be represented with a polynomial series of x.

$$y = \sum_{k=0}^{\infty} d_k x^k \qquad |x| \leq X_M$$

If we retain only the first n terms of the series

$$y_{(n)T} = \sum_{k=0}^{n} d_k x^k$$

we have a truncation error which is given by

$$(y - y_{(n)T}) = \sum_{k=n+1}^{\infty} d_k x^k$$

We know N measured values of y, say $\left\{y_i^o\right\}$ i = 1, . . . , N corresponding to N values of x, say $\left\{x_i\right\}$ i = 1, . . . , N, and each measurement of y_i has a standard deviation σ.

We fit a polynomial to the data in order to: 1) evaluate the best estimates of the truncated series and 2) determine what value of n minimizes the sum of squared truncation error and standard deviation of the truncated series. In other words, we want to find those terms of the series which are significant with respect to the measured data.

The first step is to represent the truncated polynomial series by means of a set of orthogonal polynomials. A set of orthogonal polynomials over the N points $\left\{x_i\right\}$ is computed by means of the recursive relationships:

$$p_o(x) = 1$$

$$p_k(x) = (x - a_k)\, p_{k-1}(x) - b_k p_{k-2}(x) \qquad k = 1, 2, 3, \ldots$$

$$a_k = \frac{\left[x\, p_{k-1}^2(x)\right]}{\left[p_{k-1}^2(x)\right]} \qquad b_k = \frac{\left[p_{k-1}^2(x)\right]}{\left[p_{k-2}^2(x)\right]}$$

$$a_o = b_o = b_1 = 0$$

where $p_k(x)$ is a k-degree polynomial

and [ ] means summation over the N points $\{x_i\}$

These polynomials are orthogonal, that is

$$\left[p_h\, p_k\right] = 0 \qquad h \neq k$$

The truncated series

$$y_{(n)T} = \sum_{k=0}^{n} d_k x^k$$

can be represented by means of the set of orthogonal polynomials with

$$y_{(n)} = \sum_{k=0}^{n} c_k p_k(x)$$

where the coefficients c_k are linearly related to the coefficients d_k through the system of equations

$$c_k = \frac{1}{\left[p_k^2(x)\right]} \sum_{h=k}^{n} d_h \left[x^h\, p_k(x)\right]$$

The average of the squared truncation error is computed by means of

$$r_{(n)}^2 = \frac{1}{N}\left[\left(y - y_{(n)}\right)^2\right] = \frac{1}{N}\sum_{k=n+1}^{\infty} c_k^2 \left[p_k^2(x)\right]$$

The best estimates $\overline{c}_k$ of the coefficients c_k k=1, . . . , n are immediately evaluated with the measured values $\{y_i^o\}$ i=1, . . . N from the relationships

$$\overline{c}_k = \frac{\left[y^o p_k(x)\right]}{\left[p_k^2(x)\right]}$$

with a standard deviation given by

$$\sigma_k^2 = \frac{\sigma^2}{\left[p_k^2(x)\right]}$$

where σ is the standard deviation of the measured values y_i^o .

Employing the estimates $\bar{c}_k$ of c_x gives estimates $\bar{y}_{(n)}$ of $y_{(n)}$

$$\bar{y}_{(n)} = \sum_{k=0}^{n} \bar{c}_k \, p_k(x)$$

with an average standard deviation given by

$$\sigma_{\bar{y}_{(n)}}^2 = \frac{n}{N}\sigma^2$$

The best value of n is therefore the one that minimizes the sum of standard deviation and truncation:

$$r_{(n)}^2 + \sigma_{\bar{y}_{(n)}}^2 = \min$$

Since

$$r_{(n)}^2 = r_{(n+1)}^2 + \frac{1}{N}\bar{c}_{n\ 1}^2 \left[p_{n+1}^2(x)\right]$$

$$r_{(n)}^2 = r_{(n-1)}^2 - \frac{1}{N}\bar{c}_n^2 \left[p_n^2(x)\right]$$

the minimum is obtained when

$$\bar{c}_{n+1}^2 \left[p_{n\ 1}^2(x)\right] < \sigma^2 < \bar{c}_n^2 \left[p_n^2(x)\right]$$

For practical purposes, an estimate of σ^2 is obtained from

$$\sigma^2 = \frac{1}{N-n}\left[\left(y^o - \bar{y}_{(n)}\right)^2\right]$$

$$= \frac{1}{N-n}\left(\left[(y^o)^2\right] - \sum_{k=0}^{n} \bar{c}_k^2\left[p_k(x)\right]\right)$$

APPENDIX C

SOLUTION OF THE HOMOLOGY EQUATION

The homology equation identifies the point Q_2 on the second trace which corresponds to a given point Q_1 on the first trace. From the notation of Section 7.4, we have

$$\overline{S_2Q_2} \cdot (\overline{S_1S_2} \times \overline{S_1Q_1}) = 0$$

In this appendix we show the explicit derivation of the homology equation and a method for its solution.

Consider a point Q_1 on the first trace. The vector $\overline{S_1Q_1}$ has the coordinates

$$q_1 = \begin{pmatrix} \xi_1 \\ \eta_1 \\ f_1 \end{pmatrix}$$

in the first camera system and has coordinates

$$S_1Q_1 \equiv T_1 q_1$$

in the geocentric system.

Let us indicate with E the geocentric system coordinates of the vector $\overline{E}$ that results from the cross product of $\overline{S_1S_2}$ and $\overline{S_1Q_1}$

$$\overline{E} = (\overline{S_1S_2} \times \overline{S_1Q_1})$$

We change the coordinates of $\overline{E}$ to the second camera system

$$e = T_2' E = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

In this same system, the second trace is represented by the relations

$$\left[\begin{array}{l} q_2 = \begin{pmatrix} \xi_2 \\ \eta_2 \\ f_2 \end{pmatrix} \\ \eta_2 = \eta_2\,(\xi_2) \end{array}\right.$$

Therefore, the homology equation becomes

$$e_1\xi_2 + e_2\eta_2(\xi_2) + e_3 f_2 = 0$$

where ξ_2 is the unknown quantity.

This equation can be solved by means of the Newton-Raphson method if the trace is represented with an analytical function of ξ_2, i.e., if $\eta_2 = \eta_2(\xi_2)$ is analytical.

We will use, instead, a method of searching and linear interpolation since in general the trace is represented with a tabular function of ξ_2, i.e., we have a table $\left\{\xi_2, \eta_2\right\}$.

We use c to indicate the value of the function of ξ_2

$$c = e_1\xi_2 + e_2\eta_2 + e_3 f_2$$

We search the table of q_2 for two consecutive entries q_{2i} and q_{2s} such that

$$c_i = e_1\xi_{2i} + e_2\eta_{2i} + e_3 f_2 \leq 0$$

and

$$c_s = e_1\xi_{2s} + e_2\eta_{2s} + e_3 f_2 \geq 0$$

Once we have found q_{2i} and q_{2s}, we assume q_2 has the interpolated value

$$q_2 = q_{2i} - \frac{c_i}{c_s - c_i}\,(q_{2s} - q_{2i})$$

If the value of c does not change sign we recognize that there is no homologous point on trace-2 relative to the point Q_1 on trace-1. If the value of c is a constant equal to zero, we recognize that the homology equation vanishes and, thus, the orbital arc lies on a plane through S_1 and S_2.

APPENDIX D

DERIVATION OF THE COVARIANCE MATRIX OF SATELLITE POSITION

This appendix presents a derivation of the covariance matrix of satellite position based on data obtained with continuous trace camera observations. Figure 12 presents the geometry and some of the notation used.

Also

$p_1 \to$ the vector $\overline{S_1}Q_1$

$p_2 \to$ the vector $\overline{S_2}Q_2$

$D \to$ the distance S_1S_2

$b \to$ the unit vector from S_1 toward S_2

$h \to$ the unit vector normal to b, in the plane (S_1, S_2, B) and toward the same side of B

$n \to$ the vector $(b x h)$

We define the angles γ_1 γ_2 γ_3 by means of the vector products

$$e_1 = (p_1 \times b) = - |p_1| \sin \gamma_1 n$$

$$e_2 = (p_2 \times b) = - |p_2| \sin \gamma_2 n$$

$$e_3 = (p_2 \times p_1) = - |p_1| |p_2| \sin \gamma_3 n$$

where $|p|$ stands for magnitude of vector p.

The unit vectors along p_1 and p_2 are given by

$$\frac{p_1}{|p_1|} = \cos \gamma_1 b + \sin \gamma_1 h$$

$$\frac{p_2}{|p_2|} = \cos \gamma_2 b + \sin \gamma_2 h$$

From Figure 12 and Section 7.4 we have the position of the satellite

$$B = S_1 + \Gamma_B \frac{p_1}{|p_1|}$$

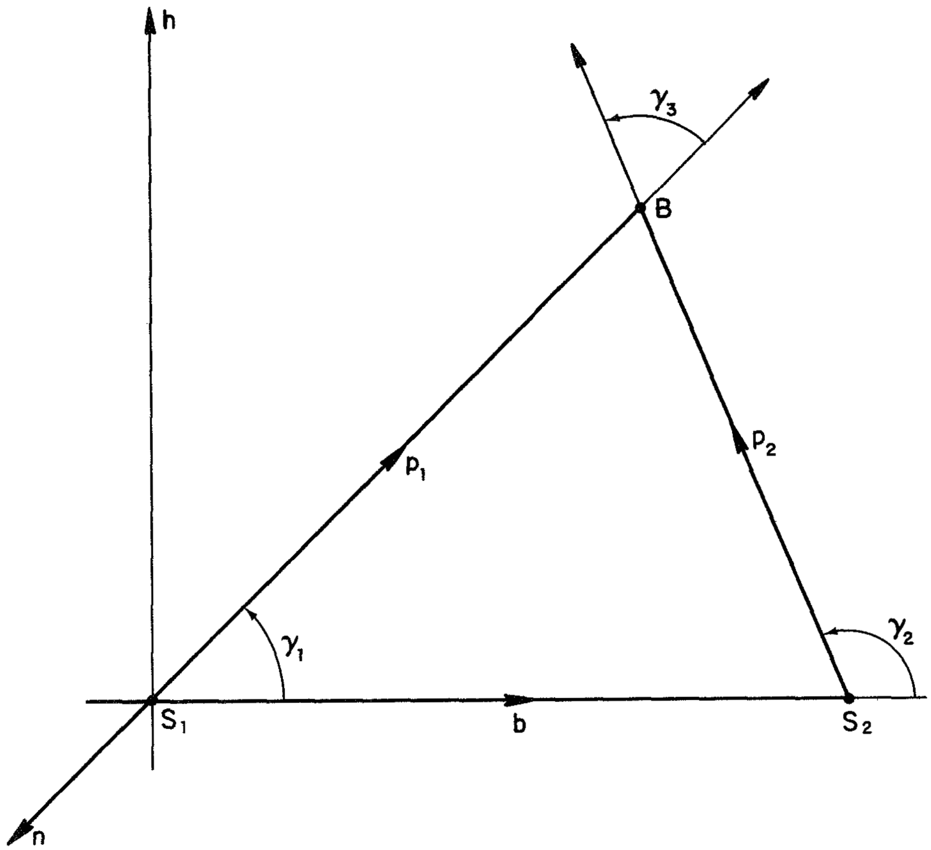

Figure 12 Geometry of Error Propagation to Satellite Position

where $$\Gamma_B = D \frac{\sin \gamma_2}{\sin \gamma_3}$$

The differential of B is therefore given by

$$\partial B = \Gamma_B \left(\frac{p_1}{|p_1|} \frac{\partial \Gamma_B}{\Gamma_B} + \partial \left(\frac{p_1}{|p_1|} \right) \right)$$

The second term within the parentheses can be computed from

$$\partial \left(\frac{p_1}{|p_1|} \right) = (-\sin \gamma_1 b + \cos \gamma_1 h) \, \partial \gamma_1$$

since b and h are not affected by measurement errors.

The first term within the parentheses can be computed from

$$\frac{\partial \Gamma_B}{\Gamma_B} = \frac{\cos \gamma_2}{\sin \gamma_2} \partial \gamma_2 - \frac{\cos \gamma_3}{\sin \gamma_3} \partial \gamma_3$$

Since

$$\gamma_3 = \gamma_2 - \gamma_1$$

the differential can be reduced to

$$\frac{\partial \Gamma_B}{\Gamma_B} = \frac{\cos \gamma_3}{\sin \gamma_3} \partial \gamma_1 - \frac{\sin \gamma_1}{\sin \gamma_2 \sin \gamma_3} \partial \gamma_2$$

Substituting this expression in the equation of ∂B and taking into account the equation of $\partial(p_1/|p_1|)$, reduces the equation of ∂B to

$$\partial B = D \frac{\sin \gamma_1 \sin \gamma_2}{\sin^2 \gamma_3} \left[\frac{p_2}{|p_2|} \frac{\partial \gamma_1}{\sin \gamma_1} - \frac{p_1}{|p_1|} \frac{\partial \gamma_2}{\sin \gamma_2} \right]$$

We note that the difference between the two indices is due to the assumed direction for b that is from S_1 to S_2. Interchanging the indices will also invert the direction of b and therefore the sign.

To evaluate $\partial \gamma / \sin \gamma$ we note that

$$\frac{\partial \gamma}{\sin \gamma} = -\frac{1}{2} \frac{\cos \gamma}{\sin^2 \gamma} \partial \left(\ln \cos^2 \gamma\right)$$

and

$$\cos^2 \gamma = \frac{(p \cdot b)^2}{(p \cdot p)}$$

Therefore

$$\partial(\ln \cos^2 \gamma) = \partial \ln \left(\frac{(p \cdot b)^2}{(p \cdot p)} \right)$$

$$= 2 \, \frac{\partial(p \cdot b)}{(p \cdot b)} - \frac{\partial(p \cdot p)}{(p \cdot p)}$$

$$= 2 \left(\frac{b'}{(p \cdot b)} - \frac{p'}{(p \cdot p)} \right) \partial p$$

since the scalar product of two vectors is the product of the transpose of the first times the second.

Finally, we obtain

$$\frac{\partial \gamma}{\sin \gamma} = \frac{\cos \gamma}{\sin^2 \gamma} \left(\frac{p'}{(p \cdot p)} - \frac{b'}{(p \cdot b)} \right) \partial p$$

Substitution in the last expression of ∂B gives

$$\partial B = D \, \frac{\sin \gamma_1}{\sin \gamma_3} \, \frac{\sin \gamma_2}{\sin \gamma_3} \, \frac{p_2}{|p_2|} \, \frac{\cos \gamma_1}{\sin^2 \gamma_1} \left(\frac{p_1{}'}{(p_1 \cdot p_1)} - \frac{b'}{(p_1 \cdot b)} \right) \partial p_1$$

$$- D \, \frac{\sin \gamma_2}{\sin \gamma_3} \, \frac{\sin \gamma_1}{\sin \gamma_3} \, \frac{p_1}{|p_1|} \, \frac{\cos \gamma_2}{\sin^2 \gamma_2} \left(\frac{p_2{}'}{(p_2 \cdot p_2)} - \frac{b'}{(p_2 \cdot b)} \right) \partial p_2$$

We note that

$$S_1 B = D \, \frac{\sin \gamma_2}{\sin \gamma_3} \, \frac{p_1}{|p_1|} = D \, \frac{(e_1 \cdot e_2)}{(e_1 \cdot e_3)} \, p_1$$

$$S_2 B = D \, \frac{\sin \gamma_1}{\sin \gamma_3} \, \frac{p_2}{|p_2|} = D \, \frac{(e_1 \cdot e_2)}{(e_2 \cdot e_3)} \, p_2$$

$$\frac{\sin \gamma_2}{\sin \gamma_3} \cos \gamma_1 = \frac{(e_1 \cdot e_2)}{(e_1 \cdot e_3)} \, (p_1 \cdot b)$$

$$\frac{\sin \gamma_1}{\sin \gamma_3} \cos \gamma_2 = \frac{(e_1 \cdot e_2)}{(e_2 \cdot e_3)} (p_2 \cdot b)$$

$$\sin^2 \gamma_1 = \frac{(e_1 \cdot e_1)}{(p_1 \cdot p_1)}$$

$$\sin^2 \gamma_2 = \frac{(e_2 \cdot e_2)}{(p_2 \cdot p_2)}$$

Therefore, the relationship between variation of satellite coordinates and variation of trace points is

$$\partial B = \frac{(e_1 \cdot e_2)}{(e_1 \cdot e_3)\,(e_1 \cdot e_1)} S_2 B \left((p_1 \cdot b)\, p_1' - (p_1 \cdot p_1)\, b' \right) \partial p_1$$

$$- \frac{(e_1 \cdot e_2)}{(e_2 \cdot e_3)\,(e_2 \cdot e_2)} S_1 B \left((p_2 \cdot b)\, p_2' - (p_2 \cdot p_2)\, b' \right) \partial p_2$$

This relationship can be expressed by

$$\partial B = a\,(b_1 M_1 \partial p_1 - b_2 M_2 \partial p_2)$$

where $a = (e_1 \cdot e_2)$

$$b_i = \frac{1}{(e_i \cdot e_3)\,(e_i \cdot e_i)}$$

and

$$M_i = (p_i \cdot b)\, S_j B p_i' - (p_i \cdot p_i)\, S_j B b'$$

$i = 1, 2 \qquad j = 1, 2 \qquad i \neq j$

The covariance matrix of the satellite coordinates' random errors is

$$CV = < \partial B\, \partial B' >$$

$$= a^2 \left(b_1^2 M_1 C_1 M_1' + b_2^2 M_2 C_2 M_2' \right)$$

where $C_i = < \partial p_i\, \partial p_i' > \qquad i = 1, 2$

are the covariance matrices of the random errors of the trace points'.

To evaluate the product $M_i C_i M_i'$ in practice, we observe that covariance matrices of the C_i's are known in the camera coordinates [Section 7.5]

$$C_i = \frac{\sigma_i^2}{1 + \eta_i'^2} \begin{pmatrix} \eta_i'^2 & -\eta_i' & 0 \\ -\eta_i' & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad i = 1, 2$$

We can express them as the product of a vector d times its transposed vector d'

$$C_i = d_i d_i' \qquad i = 1, 2$$

where d_i is the vector representing ∂p_i in the camera system

$$d_i = \frac{\sigma_i}{\sqrt{1 + \eta_i'^2}} \begin{pmatrix} -\eta_i' \\ 1 \\ 0 \end{pmatrix} \quad i = 1, 2$$

Therefore, we operate directly on the vector d before performing the products M C M'. We compute the vector

$$D_i = M_i T_i \begin{pmatrix} -\eta_i' \\ 1 \\ 0 \end{pmatrix} \qquad i = 1, 2$$

where T_i transforms the coordinates from camera system to geocentric system. Doing so, we obtain

$$M_i C_i M_i' = \frac{\sigma_i^2}{1 + \eta_i^2} D_i D_i' \qquad i = 1, 2$$

Finally, the covariance matrix CV is given by

$$CV = a^2 (k_1^2 D_1 D_1' + k_2^2 D_2 D_2')$$

where

$$k_i = \frac{\sigma_i}{\left(e_i \cdot e_3\right) (e_i \cdot e_i) \left(1 + \eta_i'^2\right)^{1/2}} \qquad i = 1, 2$$

This method reduces the direct product of square matrices to axes transformations and product of vectors. Thus, we can obtain a considerable saving of computer time.

The values of σ_i and η_i' are evaluated in practice by means of fitting a polynomial through the trace with the procedure described in Appendix B.

APPENDIX E

COMPUTATION OF MATRICES IN THE LEAST SQUARE SOLUTION

The matrices involved in the normal equation of the least square procedure are evaluated from the partial derivatives of the constraints equations. These equations are given by [Section 7.6]

$$G(S, P, p) = f\,(y - y_s) - \eta\,(z - z_s)$$

$$\eta = \eta\,(\xi)$$

$$\xi = f\,\frac{x - x_s}{z - z_s}$$

To evaluate the partial derivatives we use the fact that η is a function of ξ, and ξ is a function of $(x - x_s)$ and $(z - z_s)$.

We indicate the derivative of the trace equation by

$$\eta' = \frac{d\eta}{d\xi}$$

Therefore, we have

$$G_P = \begin{pmatrix} -\eta'\,(z - z_s)\,\dfrac{\partial\xi}{\partial x} \\ f - \eta'\,(z - z_s)\,\dfrac{\partial\xi}{\partial y} \\ -\eta'\,(z - z_s)\,\dfrac{\partial\xi}{\partial z} - \eta \end{pmatrix}$$

$$G_p = \begin{pmatrix} -\eta'\,(z - z_s) \\ -\,(z - z_s) \\ 0 \end{pmatrix}$$

$$G_S = -\,G_P$$

To complete the evaluation

$$\frac{\partial \xi}{\partial x} = \frac{f}{z - z_s} \; ; \; \frac{\partial \xi}{\partial y} = 0 \; ; \; \frac{\partial \xi}{\partial z} = - \frac{\xi}{z - z_s}$$

Substitution in the expression of G_P gives

$$G_P = \begin{pmatrix} -f\eta' \\ f \\ -\eta + \xi\eta' \end{pmatrix}$$

$$G_p = - (z - z_s) \begin{pmatrix} \eta' \\ 1 \\ 0 \end{pmatrix}$$

$$G_S = \begin{pmatrix} f\eta' \\ - f \\ \eta - \xi\eta' \end{pmatrix}$$

The value of the coefficients C_λ [Section 7.6] can be directly computed from

$$C_\lambda = \overline{G}'_P \, CV \, \overline{G}_P + G'_p \, C \, G_p$$

CV is given in the geocentric system; therefore, we have to convert G_P from camera system to geocentric system

$$\overline{G}_P = T \, G_p$$

Further, from Section 7.5

$$C = \frac{\sigma^2}{1 + \eta'^2} \begin{pmatrix} \eta'^2 & -\eta' & 0 \\ -\eta' & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where σ^2 is the standard deviation of the plate measurements. Thus,

$$G'_p \, C \, G_p = \sigma^2 \, (z - z_s)^2 \, \frac{(1 - \eta'^2)^2}{(1 + \eta'^2)}$$

Once the values of C_λ are known, the normal matrix is

$$N_S = \sum_{i=1}^{N} \frac{1}{C_{\lambda i}} G_S G_S' + W_S$$

where

$$G_S G_S' = \begin{pmatrix} f^2\eta'^2 & -f^2\eta' & f\eta' (\eta - \xi\eta') \\ -f^2\eta' & f^2 & -f (\eta - \eta'\xi) \\ f\eta' (\eta - \eta'\xi) & -f (\eta - \eta'\xi) & (\eta - \xi\eta')^2 \end{pmatrix}$$

and the known vector is

$$E_S = \sum_{i=1}^{N} \frac{G_o}{C_{\lambda i}} G_S$$