Chapter 9

Quadratic equations

Vocabulary

- Plus-or-minus ($\pm$) notation
- Hypotenuse
- Pythagorean theorem
- Completing the square
- Quadratic formula
- Discriminant
- Zero product property
- Parabola

9.1 Solving quadratic equations I. A first strategy

In the last chapter, we introduced the notation and manipulation of symbols representing irrational square roots and complex numbers. Both of these types of numbers arise from the simplest quadratic equations, and we will meet them over and over again for the remainder of this chapter.

Now we are ready to return to the main theme: solving quadratic equations in one variable. So far, we have seen one strategy to solve the simplest quadratic equations. Let us recall that strategy, which we repeat again here for emphasis:

First strategy to solve quadratic equations of the form $x^2 = k$

An equation having the form $x^2 = k$ has two solutions, written symbolically as $\sqrt{k}$ and $-\sqrt{k}$.

This first strategy only applies to quadratic equations in a very special form. In particular, the x^2 term is by itself on one side of the equation, and the other side has no variable terms. The main point in this section is to see that the first strategy can in fact be modified to solve other quadratic equations in one variable, as long as there is any perfect square involving the variable on one side of the equation and no variable terms on the other.

Recall that two algebraic statements are *equivalent* when they have the same solutions. Most of the statements we have seen up to now have been *simple* statements, in the sense that they involve just one equation or inequality. A *compound statement* is formed by several simple statements, joined by the words AND, OR or NOT. For example, the system of linear equations $\begin{cases} x + y = 1 \\ x - y = 5 \end{cases}$ is a compound statement of the form $x + y = 1$ AND $x - y = 5$; a solution to the system (a compound statement) must be a solution BOTH $x + y = 1$ AND $x - y = 5$. As another example, a solution to the statement "Either $x = 5$ or $x = 2$" must be EITHER a solution to the simple statement $x = 5$ OR it must be a solution to $x = 2$.

With this is mind, we will introduce a new notation. The symbol $\pm$ (which is unfortunately[1] read "plus or minus") will be used to indicate an either/or statement:

Plus-or-minus ($\pm$) notation

$\pm k$ means the same as "either k or $-k$."

So, for example, ± 3 means "either 3 or -3." Similarly, $\pm\sqrt{5}$ means the same as "either $\sqrt{5}$ or $-\sqrt{5}$.

The "plus-or-minus" notation is often seen in the context of equations.

[1] It should really be read as "positive or negative."

Compound plus-or-minus statements

The statement $x = \pm k$ means the same as the compound statement "either $x = k$ or $x = -k$."

With this new notation, we can rephrase our basic strategy to solve simple quadratic equations in one variable:

First strategy to solve quadratic equations of the form $x^2 = k$ (revisited)

The equation $x^2 = k$ is equivalent to the statement $x = \pm\sqrt{k}$.

In particular, it has two solutions, $\sqrt{k}$ and $-\sqrt{k}$. (When $k = 0$, the two solutions are the same.)

The reader may refer to Example 8.2.1, where this strategy was illustrated in the simplest cases.

The rephrasing of our basic strategy for solving quadratic equation is useful because it can be applied to any equation having the form $\square^2 = k$: one side of the equation (represented by the box) is a perfect square, while the other side is a constant. The next example shows this principle in action, in a sightly more complicated setting.

Example 9.1.1. *Solve:* $16x^2 = 25$.

Answer. *We will show two ways that our "first strategy" can be used to solve this equation, which is should be noticed involves more than just x^2 on the left side of the equation.*

Method 1: "Solve for x^2"

Although the x^2 is not "by itself," we can use the multiplication principle to write an equivalent equation in the special form to apply the basic strategy.

$$\begin{array}{rcl} 16x^2 & = & 25 \\ \frac{16x^2}{16} & = & \frac{25}{16} \\ x^2 & = & \frac{25}{16} \end{array}$$

The resulting equivalent equation is in the special form to apply our basic strategy. The equation $x^2 = 25/16$ is equivalent to the compound statement $x = \sqrt{\frac{25}{16}}$ or $x = -\sqrt{\frac{25}{16}}$.

The solutions are (after simplifying the radical expressions) $5/4$ *and* $-5/4$.

Method 2: Perfect square form

Notice that even though original equation $16x^2 = 25$ *is not in the simplest form (with* x^2 *by itself on one side of the equation), the left hand side involving the variable is, in fact, a perfect square!*

$$\begin{array}{rcll} 16x^2 & = & 25 & \\ (4x)^2 & = & 25 & \textit{Emphasizing the perfect square} \end{array}$$

We can apply our basic strategy. The quadratic equation $(4x)^2 = 25$ *is equivalent to the compound statement*

$$4x = \sqrt{25} \qquad OR \qquad 4x = -\sqrt{25}.$$

After simplifying the radicals, we see that we have a compound statement involving two linear *equations:* $4x = 5$ *or* $4x = -5$. *Each of them is easily solved.*

The solutions are $5/4$ *and* $-5/4$.

The next example gives another illustration of Method 2, where we apply our basic strategy to a perfect square more complicated than just x^2.

Example 9.1.2. *Solve:* $(x+4)^2 = 7$.

Answer. *The left side of the equation is a perfect square, so we may apply the same basic strategy we have seen in the simpler case. Namely, the equation* $(x+4)^2 = 7$ *is equivalent to the statement*

$$x + 4 = \pm\sqrt{7},$$

or, what is the same, to the compound statement

$$x + 4 = \sqrt{7} \qquad OR \qquad x + 4 = -\sqrt{7}.$$

Each of the two equations are linear, and can be solved using our standard techniques for solving linear equations:

$$\begin{array}{cccccc} x & + & 4 & = & & \sqrt{7} \\ & & -4 & \vdots & -4 & \\ \hline x & & & = & -4 & +\sqrt{7} \end{array}$$

The solution to $x + 4 = \sqrt{7}$ *is* $-4 + \sqrt{7}$. *Similarly, solving the other equation in the compound statement:*

$$\begin{array}{cccccc} x & + & 4 & = & & -\sqrt{7} \\ & & -4 & \vdots & -4 & \\ \hline x & & & = & -4 & -\sqrt{7} \end{array}$$

The solution to $x + 4 = -\sqrt{7}$ *is* $-4 - \sqrt{7}$.

Combining to obtain the solutions to the compound equation, the solutions to $(x+4)^2 = 7$ *are* $-4 + \sqrt{7}$ *and* $-4 - \sqrt{7}$.

It is common, although a little lazy, to combine the two solving steps in the previous example as follows:

$$\begin{array}{cccccc} x & + & 4 & = & & \pm\sqrt{7} \\ & & -4 & \vdots & -4 & \\ \hline x & & & = & -4 & \pm\sqrt{7}, \end{array}$$

and then to say that the solutions are $-4 \pm \sqrt{7}$. Keep in mind these are two different solutions!

9.1.1 The Pythagorean theorem

Even though we have only been dealing with quadratic equations of the simplest form, these simple equations occur naturally in the context of using the Pythagorean theorem. Given the importance of this theorem in a wide variety of contexts (it is used to obtain a formula for measuring distances, for example), we illustrate the theorem and several examples of its application here.

The Pythagorean[2] theorem is a theorem—a mathematical statement which can be proved, or deduced, from definitions and previously-proved statements—about right triangles. Recall that a right triangle is a triangle which includes one right angle. The side opposite to the right angle is called the *hypotenuse*. The other two sides are referred to as the *legs* of the right triangle. See Figure 9.1.

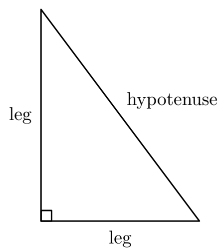


Figure 9.1: A right triangle. The right angle is indicated with the small square symbol.

[2]Although the theorem is named after Pythagorus and his followers, the relationship it describes had been noticed by the earliest civilizations in Egypt, Mesopotamia, India and China. The oldest proof of the statement is from Euclid, around 300 BCE.

The Pythagorean theorem

Given a right triangle whose hypotenuse has length h and whose legs have lengths a and b, the following relationship holds:

$$h^2 = a^2 + b^2.$$

In fact, if the lengths of the three sides of a triangle satisfy this equation, then the triangle must be a right triangle.

Notice that in the equation relating the side lengths in the theorem, the square of the length of the special side, the hypotenuse, is by itself on one side. The squares of the lengths of the other two sides that are not the hypotenuse are added (in either order) on the other side of the equation.

As a consequence of the Pythagorean theorem, the hypotenuse must be the longest side of a right triangle. (In fact, sometimes the hypotenuse is *defined* to be the longest side of a right triangle. In that case, the Pythagorean theorem would emphasize that the hypotenuse must be opposite the right angle.)

For our purposes, the main consequence of the Pythagorean theorem is that if the lengths of two sides of a right triangle are known, then the length of the third side can be found as well. We illustrate this fact with several examples.

Example 9.1.3. *In a right triangle, the length of the hypotenuse is* 13 *inches, while the length of one of the legs is* 5 *inches. Find the length of the remaining side.*

Answer. *In this example, we use the values* $h = 13$ *and* $a = 5$ *in the Pythagorean theorem and attempt to solve for* b*.*

$$\begin{aligned} (13)^2 &= (5)^2 + b^2 \\ 169 &= 25 + b^2 \end{aligned}$$

The resulting equation (in the variable b*) can be written in the special form we have been considering by subtracting* 25 *from both sides:*

$$\begin{array}{ccccc} 169 & = & 25 & + & b^2 \\ -25 & \vdots & -25 & & \\ \hline 144 & = & & & b^2. \end{array}$$

The solutions to this equation (in which b^2 *is by itself on one side of the equation) are* $\sqrt{144}$ *and* $-\sqrt{144}$*, or* 12 *and* -12*. However, since* b *is to represent a length, we ignore the negative solution as being meaningless in the given problem.*

The length of the remaining side is 12 *inches.*

Example 9.1.4. *In a right triangle, one leg has length* 4 *cm and the other has length* 6 *cm. Find the length of the hypotenuse.*

Answer. *We substitute the values* $a = 4$ *and* $b = 6$ *into the equation* $h^2 = a^2+b^2$ *in order to solve for the value of* h.

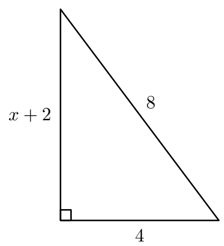
$$\begin{aligned} h^2 &= (4)^2 + (6)^2 \\ h^2 &= 16 + 36 \\ h^2 &= 52. \end{aligned}$$

According to our basic strategy, the solutions to this equation are $\sqrt{52}$ *and* $-\sqrt{52}$*; we ignore the negative solution since our answer is to represent a length, which is always a positive quantity. We will simplify our result:*

$$\sqrt{52} = \sqrt{4 \cdot 13} = \sqrt{4} \cdot \sqrt{13} = 2\sqrt{13}.$$

The length of the hypotenuse is $2\sqrt{13}$ *cm. (This is approximately* 7.21 *cm.)*

Example 9.1.5. *In the figure below, find the value of* x *so that the triangle is a right triangle.*



Answer. *Since we require the triangle to be a right triangle, the Pythagorean theorem must hold:*

$$\begin{aligned} (8)^2 &= (4)^2 + (x+2)^2 \\ 64 &= 16 + (x+2)^2. \end{aligned}$$

We will rewrite the equation so that the perfect square involving x, $(x+2)^2$, *is by itself on one side of the equation, and then proceed as in Example 9.1.2.*

$$\begin{array}{ccccc} 64 & = & 16 & + & (x+2)^2 \\ -16 & \vdots & -16 & & \\ \hline 48 & = & & & (x+2)^2. \end{array}$$

According to our basic strategy, this equation is equivalent to the compound statement $x + 2 = \pm\sqrt{48}$, *or* $x = -2 \pm \sqrt{48}$. *Simplifying the radical expression* $\sqrt{48} = \sqrt{16 \cdot 3}$, *this is equivalent to* $x = -2 \pm 4\sqrt{3}$, *which has two solutions,* $-2 + 4\sqrt{3}$ *and* $-2 - 4\sqrt{3}$. *However, the second solution would make one side of the triangle have a length of* $-4\sqrt{3}$, *which makes no geometric sense.*

The value of x *which makes the triangle a right triangle is* $-2+4\sqrt{3}$. *(Check on your calculator that this is approximately* 4.93.*)*

9.1.2 Exercises

Solve the following quadratic equations.

1. $x^2 = 150$
2. $4x^2 = 3$
3. $x^2 = -49$
4. $x^2 + 18 = 0$
5. $12x^2 = 75$
6. $(x + 3)^2 = 12$
7. $(2x - 1)^2 = 144$
8. $(x - 5)^2 = -18$

Find the length of the third of the right triangle whose other two side lengths are given. (Here h represents the length of the hypotenuse and a and b represent the lengths of the other two sides, as shown in Figure 9.2.)

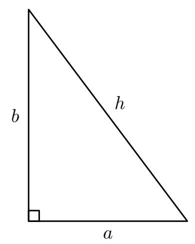


Figure 9.2: Diagram for Exercises 8–10. (Not drawn to scale)

9. $a = 4$, $b = 6$.
10. $h = 10$, $b = 5$.
11. $a = 12$, $b = 4$.

9.2 Solving quadratic equations II. Completing the square

We saw in the previous section that even if a quadratic equation was not as simple as $x^2 = k$, we can still apply the same basic strategy as long as we have a perfect square on one side of the equation and a constant on the other. But what about equations like

$$x^2 + 6x - 4 = 0?$$

Because of the $6x$ term, there is little hope that we can obtain a perfect square on one side and a constant on the other by the same kind of simple manipulations as in the previous section.

We will now develop a procedure where *any* quadratic equation of the form $ax^2 + bx + c = 0$ can be written in the special form $\boxed{}\,^2 = k$, where one side is a perfect square and the other side is a constant.

Consider the following identities (which you can confirm my multiplication):

$$\begin{aligned}
(x+1)^2 &= x^2 + 2x + 1\\
(x+2)^2 &= x^2 + 4x + 4\\
(x+3)^2 &= x^2 + 6x + 9\\
(x+4)^2 &= x^2 + 8x + 16\\
(x+5)^2 &= x^2 + 10x + 25\\
(x+6)^2 &= x^2 + 12x + 36\\
&\vdots
\end{aligned}$$

Do you see any patterns on the right hand side? There are several: the constant terms are all perfect squares, and the coefficient of x is always an even number. To be more specific,

- The constant term in the expansion of $(x + n)^2$ is n^2;
- The coefficient of x in the expansion of $(x + n)^2$ is $2n$ (twice the value of n).

Can you use these two facts to write down the next five expansions in the list, without actually multiplying polynomials?

Now, using the patterns we noticed above, let's try to answer the following question: How can I fill in the blank with a number in the expression

$$x^2 + 56x + \underline{\qquad}$$

is a perfect square of the form $(x + n)^2$? We see the coefficient of x is 56, which according to the pattern above, should be twice the value of n. In other words, n should be $56/2 = 28$. But that tells us what to fill in as the constant, according to the other pattern: the constant terms should be $(28)^2 = 784$. In other words, the polynomial

$$x^2 + 56x + 784$$

is a perfect square: it is $(x + 28)^2$. No other constant would have worked other than 784; our choice was determined by the coefficient of x. This process of determining a constant to add to a quadratic expression to obtain a perfect square is known as *completing the square*.

The next example shows how this technique, combined with the strategy we have already developed, can be used to solve a quadratic equation which is not initially written in the special form $(x + n)^2 = k$.

Example 9.2.1. *Solve:* $x^2 - 6x + 4 = 0$.

Answer. *If the goal is to write the equation in the form* $(x + n)^2 = k$*, the first step will be to arrange to write the constant term by itself on one side of the equation. This is not hard to do:*

$$\begin{array}{ccccccc} x^2 & - & 6x+ & 4 & = & 0 \\ & & & -4 & \vdots & -4 \\ \hline x^2 & - & 6x & & = & -4 \end{array}$$

We are now in a position to complete the square on the left side. To illustrate what we are about to do, let's write

$$x^2 \quad - \quad 6x \quad + \quad \underline{\quad} \quad = \quad -4 \quad + \quad \underline{\quad}$$

to indicate that we will "fill in the blanks" to complete the square on the left side, and add the same quantity to the right side to guarantee that the new equation will be equivalent. The coefficient of x *on the left side is* -6*. As before, we find half of* -6*:*

$$\frac{-6}{2} = -3.$$

We square the result:

$$(-3)^2 = 9.$$

In other words, we will add 9 *to both sides (to "fill in the blank"):*

$$x^2 \quad - \quad 6x \quad + \quad 9 \quad = \quad -4 \quad + \quad 9.$$

The left-hand side is now a perfect square of the form $(x+n)^2$*, where* $n = -3$*. The right-hand side can be combined to obtain:*

$$(x - 3)^2 = 5.$$

We are now in a position to apply our basic strategy. The equation $(x-3)^2 = 5$ *is equivalent to the compound statement*

$$x - 3 = \pm\sqrt{5}.$$

Using the shorthand notation discussed above, we can solve this compound statement:

$$\begin{array}{ccccccc} x & - & 3 & = & & \pm\sqrt{5} \\ & & +3 & \vdots & +3 & \\ \hline x & & & = & 3 & \pm\sqrt{5} \end{array}$$

The two solutions are $3+\sqrt{5}$ *and* $3-\sqrt{5}$.

Exercise 9.2.2. *Before continuing, apply the completing the square method to solve the following quadratic equations. Be sure to simplify square roots when possible.*

(a) $x^2 - 8x + 12 = 0$

(b) $x^2 + 4x + 5 = 0$

(c) $x^2 - 2x - 6 = 0$

With a little thinking, we can see that here was nothing really special about the equation that we started with in the previous example. It was written at the outset in general form $ax^2+bx+c=0$. The technique of completing the square was applied to write it in the general form $(x+n)^2 = k$. From there we applied our basic strategy, which had started off as a special strategy only for equations involving x^2 by itself on one side of the equation. **In fact, the technique of completely the square can be used to solve any quadratic equation in one variable.**

The next two examples show that the calculations involved in completing the square can become tedious.

Example 9.2.3. *Solve:* $x^2 + 5x + 7 = 0$.

Answer. *As before, we begin by arranging the equation so that the variable terms are on one side of the equation and the constant term is on the other:*

$$\begin{array}{cccccc} x^2 & + & 5x+ & 7 & = & 0 \\ & & & -7 & \vdots & -7 \\ \hline x^2 & + & 5x & & = & -7 \end{array}$$

Write

$$x^2 \quad + \quad 5x+ \quad \underline{} \quad = \quad -7 \quad +\underline{}.$$

The coefficient of x *this time is an odd number, which means that half of it will be a fraction. This doesn't stop us, though: we find half of* 5*:*

$$\frac{1}{2}\cdot 5 = \frac{5}{2}.$$

We square the result:

$$\left(\frac{5}{2}\right)^2 = \frac{25}{4}.$$

We add this quantity to both sides to both sides (to "fill in the blank"):

$$x^2 \quad + \quad 5x \quad + \quad \tfrac{25}{4} \quad = \quad -7 \quad + \tfrac{25}{4}.$$

The left-hand side is by design a perfect square of the form $(x+n)^2$*, where now* $n = 5/2$ *(notice this was the result of the first calculation, finding half of the coefficient of* x*). The right-hand side can be combined by finding a common denominator:*

$$-7 + \frac{25}{4} = -\frac{28}{4} + \frac{25}{4} = -\frac{3}{4}.$$

Hence our original equation is equivalent to

$$\left(x + \frac{5}{2}\right)^2 = -\frac{3}{4}.$$

Applying our basic strategy, this equation is equivalent to the compound statement

$$x + \frac{5}{2} = \pm\sqrt{\frac{-3}{4}},$$

which, after simplifying the right side, becomes

$$x + \frac{5}{2} = \pm\frac{i\sqrt{3}}{2}.$$

Solving the compound statement:

$$\begin{array}{cccccc} x & + & \frac{5}{2} & = & & \pm\frac{i\sqrt{3}}{2} \\ & & -\frac{5}{2} & \vdots & -\frac{5}{2} & \\ \hline x & & & = & -\frac{5}{2} & \pm\frac{i\sqrt{3}}{2}. \end{array}$$

The right hand side, while a little complicated, at least has the virtue of having a common denominator. It is customary to combine this sum into a single fraction:

$$x = \frac{-5 \pm i\sqrt{3}}{2}.$$

The two solutions (which are complex numbers!) are $\dfrac{-5 + i\sqrt{3}}{2}$ *and* $\dfrac{-5 - i\sqrt{3}}{2}$*.*

Example 9.2.4. *Solve* $2x^2 - 5x = 3$*.*

Answer. *The first thing to notice is that the coefficient of* x^2 *is not* 1*. Although we have not made a point of it until now, the fact that we are aiming for an equation involving an expression of the form* $(x+n)^2$ *makes it essential that the coefficient of the* x^2 *term is* 1*. In order to address this problem, we will simply divide both sides of the equation by the coefficient* 2*:*

$$\begin{aligned} \frac{2x^2 - 5x}{2} &= \frac{3}{2} \\ x^2 - \frac{5}{2}x &= \frac{3}{2}. \end{aligned}$$

(Despite the fractions, notice that we already have the constant on one side of the equation with the variable terms on the other.)

We are aiming to complete the square on the left side:

$$x^2 \quad - \quad \tfrac{5}{2}x \quad + \quad \underline{\quad} \quad = \quad \tfrac{3}{2} \quad + \underline{\quad}.$$

The coefficient of x is now $-\dfrac{5}{2}$. We will still aim to find half of this coefficient:

$$\frac{1}{2}\left(-\frac{5}{2}\right) = -\frac{5}{4}.$$

When we square the result we obtain

$$\left(-\frac{5}{4}\right)^2 = \frac{25}{16}.$$

This is the number we add to both sides to obtain a perfect square:

$$x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{3}{2} + \frac{25}{16}.$$

The left side is a perfect square by design: it is $\left(x - \dfrac{5}{4}\right)^2$. The right side can be combined:

$$\frac{3}{2} + \frac{25}{16} = \frac{24}{16} + \frac{25}{16} = \frac{49}{16}.$$

Hence our equation now has the form

$$\left(x - \frac{5}{4}\right)^2 = \frac{49}{16}.$$

According to our basic strategy, this is equivalent to the statement

$$x - \frac{5}{4} = \pm\sqrt{\frac{49}{16}}.$$

Notice that the square root on the right side can be simplified:

$$x - \frac{5}{4} = \pm\frac{7}{4}.$$

Solving the compound statement

$$\begin{array}{ccccc} x & - & \tfrac{5}{4} & = & & \pm\tfrac{7}{4} \\ & & +\tfrac{5}{4} & \vdots & +\tfrac{5}{4} & \\ \hline x & & & = & \tfrac{5}{4} & \pm\tfrac{7}{4}. \end{array}$$

In this case, though, we can do better; since no radicals appear in the expression, we can look at the two solutions separately to write them in simpler form:

$$\frac{5}{4} + \frac{7}{4} = \frac{12}{4} = 3,$$

and

$$\frac{5}{4} - \frac{7}{4} = -\frac{2}{4} = -\frac{1}{2}.$$

The two solutions are 3 *and* $-1/2$.

We end this section by summarizing the method of completing the square.

Solving quadratic equations by completing the square

To apply the technique of completing the square to solve a quadratic equation of the form

$$ax^2 + bx + c = 0 :$$

1. Divide both side of the equation by the coefficient of x^2 to obtain an equivalent equation whose leading coefficient is 1;
2. Use the addition principle to obtain an equivalent equation having all variable terms on one side and the constant term on the other;
3. Complete the square based on the new coefficient of the x-term: Add the square of half the coefficient of x to both sides;
4. Apply the basic strategy of solving equations of the form $x^2 = k$ to obtain two solutions.

9.2.1 Exercises

Solve the following quadratic equations by completing the square.

1. $x^2 - 4x - 5 = 0$
2. $x^2 + 2x = 24$
3. $x^2 - 6x = 6$
4. $x^2 + 10x + 21 = 0$
5. $x^2 + 5x + 6 = 0$
6. $x^2 - 6x + 3 = 0$
7. $x^2 + 3x = 0$
8. $2x^2 + x = 6$
9. $x^2 + 4x + 6 = 0$
10. $3x^2 - 4x = 1$

9.3 Solving quadratic equations III. The quadratic formula

Completing the square is a powerful tool for solving quadratic equations in one variable. In fact, if our only goal was to solve quadratic equations, we could stop here: completing the square will always work. However, because quadratic equations occur relatively frequently, both in algebra and its applications, several other techniques have been developed to solve them.

One of the most well-known techniques arises directly from completing the square. The *quadratic formula* is a formula which expresses the solutions to a quadratic equation completely in terms of the equations' coefficients. To derive this formula, let's apply the technique of completing the square to the quadratic equation

$$ax^2 + bx + c = 0, \qquad (a \neq 0). \tag{9.1}$$

We begin by ensuring a leading coefficient of 1;

$$\frac{ax^2 + bx + c}{a} = \frac{0}{a}$$

$$x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0.$$

Arranging the constant on the right side and the variable terms on the left:

$$\begin{array}{ccccccc} x^2 & + & \left(\frac{b}{a}\right)x & + & \frac{c}{a} & = & 0 \\ & & & & -\left(\frac{c}{a}\right) & \vdots & -\frac{c}{a} \\ \hline x^2 & + & \left(\frac{b}{a}\right)x & + & \_\_ & = & -\frac{c}{a} + \_\_ \end{array}$$

To complete the square, we find half the coefficient of x and square the result:

$$\frac{1}{2} \cdot \left(\frac{b}{a}\right) = \frac{b}{2a},$$

$$\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}.$$

(We used properties of exponents in the squaring operation.)

Completing the square:

$$\begin{array}{ccccc} x^2 + \left(\frac{b}{a}\right)x & + & \frac{b^2}{4a^2} & = & -\frac{c}{a} + \frac{b^2}{4a^2} \\ & \left(x + \frac{b}{2a}\right)^2 & & = & \frac{b^2 - 4ac}{4a^2}. \end{array}$$

(On the right side of the last equation, we rewrote the first fraction using common denominator $4a^2$ by multiplying numerator and denominator by $4a$. We also reordered the terms in the numerator.)

According to our basic strategy, the resulting equation is equivalent to the compound statement

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}.$$

This can be simplified using properties of square roots:

$$\begin{aligned} x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \\ x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

Solving the resulting (compound) linear equation:

$$\begin{array}{ccccccc} x & + & \frac{b}{2a} & = & & & \pm\frac{\sqrt{b^2-4ac}}{2a} \\ & & -\frac{b}{2a} & \vdots & & -\frac{b}{2a} & \\ \hline x & & & = & & -\frac{b}{2a} & \pm\frac{\sqrt{b^2-4ac}}{2a} \\ x & & & = & & \frac{-b\pm\sqrt{b^2-4ac}}{2a}. & \end{array}$$

Let's summarize the result of this somewhat tedious calculation.

The quadratic formula

The quadratic equation

$$ax^2 + bx + c = 0$$

(where the coefficient $a \neq 0$) is equivalent to the compound statement

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In particular, the two solutions are $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Notice: The quadratic equation was derived from a quadratic equation having a very particular form, the standard form Equation 9.1 above. In particular, to use the quadratic formula, *one side of the quadratic equation must be equal to 0*!

The quadratic equation is not particularly pleasant-looking. That is not surprising, given the fact that we began from a general quadratic equation with arbitrary coefficients. However, an enormous amount of material can be learned just from studying this formula:

- The solutions to a quadratic equation can be expressed completely as algebraic expressions involving its coefficients a, b and c.
- The nature of the solutions of a quadratic equation depend on its *discriminant* $b^2 - 4ac$:
  - If the discriminant is positive, the two solutions will be *real numbers*.
  - If the discriminant is negative, the two solutions will be *complex numbers*.
  - If the discriminant is zero, the two solutions will coincide, so that there is only one distinct solution.
  - If the discriminant is a (non-negative) perfect square, the solutions will be rational; otherwise, they will be irrational.

We illustrate the quadratic formula by solving the same three examples we solved in the last section by completing the square. Keep in mind that the quadratic formula involves *nine* separate operations. A look back at Chapter 1 should remind you that some of these operations can be performed at the same step.

Example 9.3.1. *Solve:* $x^2 - 6x + 4 = 0$.

Answer. *We see that the equation is in standard form, since one side is equal to zero, with* $a = 1$, $b = -6$ *and* $c = 4$.

Substituting into the quadratic formula and evaluating:

$$\begin{aligned} x &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(4)}}{2(1)} \\ &= \frac{6 \pm \sqrt{36 - 16}}{2} \\ &= \frac{6 \pm \sqrt{20}}{2} \\ &= \frac{6 \pm \sqrt{4 \cdot 5}}{2} && \textit{Simplifying the square root} \\ &= \frac{6 \pm 2\sqrt{5}}{2} \\ &= \frac{6}{2} \pm \frac{2\sqrt{5}}{2} \\ &= 3 \pm \sqrt{5}. \end{aligned}$$

The solutions are $3 + \sqrt{5}$ *and* $3 - \sqrt{5}$.

Example 9.3.2. *Solve:* $x^2 + 5x + 7 = 0$.

Answer. *The equation is in standard form, since one side is equal to zero, with* $a = 1$, $b = 5$ *and* $c = 7$.

Substituting into the quadratic formula and evaluating:

$$\begin{aligned}
x &= \frac{-(5) \pm \sqrt{(5)^2 - 4(1)(7)}}{2(1)} \\
&= \frac{-5 \pm \sqrt{25 - 28}}{2} \\
&= \frac{-5 \pm \sqrt{-3}}{2} \\
&= \frac{-5 \pm \sqrt{-1 \cdot 3}}{2} && \textit{Simplifying the square root} \\
&= \frac{-5 \pm i\sqrt{3}}{2}
\end{aligned}$$

(Notice that compared to the previous example, the numerator and denominator have no factors in common, so there is no need to take an extra step to divide.

The solutions are $\dfrac{-5 + i\sqrt{3}}{2}$ *and* $\dfrac{-5 - i\sqrt{3}}{2}$.

Example 9.3.3. *Solve:* $2x^2 - 5x = 3$.

Answer. *In this example, the quadratic equation is* not *in standard form, since neither side is equal to zero. This is not hard to fix, though, using the addition principle:*

$$\begin{array}{ccccccc}
2x^2 & - & 5x & & & = & 3 \\
 & & & & -3 & \vdots & -3 \\
\hline
2x^2 & - & 5x & - & 3 & = & 0.
\end{array}$$

We can now apply the quadratic formula with $a = 2$, $b = -5$ *and* $c = -3$.

Substituting and evaluating:

$$\begin{aligned}
x &= \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(-3)}}{2(2)} \\
&= \frac{5 \pm \sqrt{25 + 24}}{4} \\
&= \frac{5 \pm \sqrt{49}}{4} \\
&= \frac{5 \pm 7}{4}
\end{aligned}$$

In the case when the discriminant is a perfect square, as in this example, it *is useful to write the compound statement more explicitly:*

$$x = \frac{5+7}{4} \quad OR \quad x = \frac{5-7}{4}$$

$$x = \frac{12}{4} \quad OR \quad x = \frac{-2}{4}$$

$$x = 3 \quad OR \quad x = -\frac{1}{2}.$$

The solutions are 3 *and* $-1/2$*.*

9.3.1 Exercises

Solve each of the following quadratic equations by using the quadratic formula. (These are the same equations as in Exercises 9.2.1. Compare your answers obtained by using the quadratic formula with those you obtained by completing the square.)

1. $x^2 - 4x - 5 = 0$
2. $x^2 + 2x = 24$
3. $x^2 - 6x = 6$
4. $x^2 + 10x + 21 = 0$
5. $x^2 + 5x + 6 = 0$
6. $x^2 - 6x + 3 = 0$
7. $x^2 + 3x = 0$
8. $2x^2 + x = 6$
9. $x^2 + 4x + 6 = 0$
10. $3x^2 - 4x = 1$

9.4 Solving quadratic equations IV. Factoring

In this section, we describe a completely different approach to solving quadratic equations by using the techniques of factoring developed in Chapter 7. The factoring technique has the advantage of being less tedious (in many cases) than using either the quadratic formula or completing the square. In addition, unlike the methods we have seen so far, the new method can be applied to solve polynomial equation of higher degree (a few examples are given in the challenge exercises at the end of the section). It has the disadvantage, however, that it cannot be applied to solve *every* quadratic equation.

Solving quadratic equations by factoring is based on the following property of numbers, sometimes called the *zero product property*: *If a and b are two numbers having the property that $a \cdot b = 0$, then either a or b is* 0. Said in another way, the only way that the product of two numbers can be zero is if one of the two numbers is zero.

This property of numbers leads immediately to the following strategy for solving polynomial equations, including quadratic equations. We will state this strategy using function notation (see Chapter 1).

Solving polynomial equations by factoring

Suppose $P(x)$ is a polynomial which can be factored as a product of two polynomials, $P(x) = F(x) \cdot G(x)$. Then then polynomial equation

$$P(x) = 0$$

is equivalent to the compound polynomial statement

$$F(x) = 0 \quad \text{OR} \quad G(x) = 0.$$

In other words, the solutions to $P(x) = 0$ are exactly the solutions to $F(x) = 0$ and the solutions to $G(x) = 0$.

The idea behind this strategy is that instead of solving a quadratic polynomial equation (having degree 2), we will try to factor the polynomial and then solve two *linear* equations (of degree 1).

Notice: As in the situation when we solved a quadratic equation using the quadratic formula, it is essential that the polynomial be set equal to zero! After all, the strategy is based on the zero product property.

The following examples illustrate the technique of factoring to solve a quadratic equation.

Example 9.4.1. *Solve:* $x^2 - 3x - 10 = 0$.

Answer. *The quadratic equation does have zero on one side of the equation. So we will decide whether it is possible to factor the polynomial* $x^2 - 3x - 10$. *In fact, factoring the monic quadratic trinomial yields* $(x + 2)(x - 5)$.

Hence, using our factoring strategy, the equation $x^2-3x-10 = 0$ *is equivalent to the compound statement*

$$x + 2 = 0 \quad OR \quad x - 5 = 0.$$

Each of the two linear equations is quite simple to solve:

$$\begin{array}{cccc} x & + & 2 & = & 0 \\ & & -2 & \vdots & -2 \\ \hline x & & & = & -2 \end{array}$$

and

$$\begin{array}{cccc} x & - & 5 & = & 0 \\ & & +5 & \vdots & +5 \\ \hline x & & & = & 5. \end{array}$$

The solutions of $x^2 - 3x - 10 = 0$ *are* -2 *and* 5.

Example 9.4.2. *Solve:* $16x^2 = 25$.

Answer. *We solved this equation (two different ways!) in Example 9.1.1. However, we will now see that it can also be solved by factoring.*

First, the equation is not in the required form, having zero on one side of the equation. We apply the addition principle:

$$\begin{array}{ccccc} 16x^2 & & & = & 25 \\ & & -25 & \vdots & -25 \\ \hline 16x^2 & - & 25 & = & 0. \end{array}$$

Notice that the left side of the equivalent equation $16x-25 = 0$ *can be factored as a difference of squares:*

$$(4x+5)(4x-5) = 0.$$

So, using the zero product property, the equation is equivalent to the compound statement

$$4x + 5 = 0 \quad OR \quad 4x - 5 = 0.$$

The first equation has solution $-5/4$, *while the second has solution* $5/4$.

The solutions are $5/4$ *and* $-5/4$.

Example 9.4.3. *Solve:* $4x^2 + 12x = 0$

Answer. *The quadratic equation does have zero on one side, so we can ask whether it is possible to factor the polynomial* $4x^2 + 12x$. *In fact, the terms of this polynomial have a common factor of* $4x$, *giving a factorization of* $4x(x+3)$.

So the solutions of $4x^2 + 12x = 0$ *are the same as those of*

$$4x = 0 \quad OR \quad x + 3 = 0.$$

The first equation has solution 0; *the second equation has solution* -3.

The solutions are 0 *and* -3.

Example 9.4.4. *Solve:* $2x^2 + 5x = 3$.

Answer. *In this case, the quadratic equation is not in standard form, since neither side of the equation is zero. However, we can rewrite the equation by subtracting* 3 *from both sides:*

$$2x^2 + 5x - 3 = 0.$$

We attempt to factor the polynomial $2x^2 + 5x - 3$*. This is a non-monic quadratic trinomial; we will need to rely on the ac-method. We look for factors of* $(2)(-3) = -6$ *whose sum is* 5*. Such a pair is* 6 *and* -1*. Splitting the middle term we obtain*

$$2x^2 + 6x - 1x - 3.$$

Factoring by groups yields

$$\begin{gathered} 2x(x+3) - 1(x+3) \\ (x+3)(2x-1). \end{gathered}$$

All this, combined with our strategy to solve by factoring, means that the solutions of $2x^2 + 5x - 3 = 0$ *are the same as those of the compound statement*

$$x + 3 = 0 \quad OR \quad 2x - 1 = 0.$$

The solutions are -3 *and* $1/2$*.*

The reader might compare this example with Example 9.3.3, where we solved the same equation by using the quadratic formula, to see which method is more comfortable.

Example 9.4.5. *Solve:* $x^2 - 6x + 4 = 0$*.*

Answer. *The quadratic equation is in standard form, with one side being zero. We try to factor the polynomial* x^2-6x+4*. Its terms have no factor in common, but it is a monic quadratic trinomial. However, there are no factors of* 4 *whose sum is* -6*. This polynomial cannot be factored.*

In this case, our quadratic equation cannot be solved by factoring. We are forced to rely on either completing the square or the quadratic formula. Fortunately, we already did that in Example 9.2.1.

The solutions (which did not come from factoring!) are $3 + \sqrt{5}$ *and* $3 - \sqrt{5}$*.*

As the reader might guess from the last example, when solving quadratic equations with integer coefficients, the factoring technique is only helpful when the solutions are rational numbers.

9.4.1 Exercises

Solve the following quadratic equations by factoring and using the zero product property.

1. $x^2 - 4x + 3 = 0$

2. $x^2 - x - 12 = 0$

3. $x^2 + 7x = 0$

4. $x^2 + 9x = -20$

5. $2x^2 - 3x - 2 = 0$

6. $4x^2 + 12x + 8 = 0$

7. $4x^2 + 4x = 15$

8. $9x^2 - 25 = 0$

The following exercises explore the zero product property further.

9. (*) The zero product property may be applied to polynomials in any degree. Use it to solve the following cubic (degree 3) polynomial equations.

   (a) $x^3 - 6x^2 + 8x = 0$

   (b) $(x + 3)(x - 1)(2x + 7) = 0$

10. (*) The zero product property is crucial to solving polynomial equations by factoring. To see what happens without it, consider the "clock number system." In this number system, the only numbers allowed are the integers from 0 (represented by 12 on the clock) to 11. We can add and multiply these numbers normally, except if the result is greater than or equal to 12, we use instead the remainder when the result is divided by 12. For example, in the clock number system, $9 + 8 = 5$, since 17 divided by 12 has remainder 5. Likewise, $5 \cdot 8 = 4$, since $40 \div 12 = 3\ R\ 4$. Note that in the clock number system, the zero product property does not hold. For example, $3 \cdot 4 = 0$, but neither 3 nor 4 is zero.

    Solve the following quadratic equations by guessing and checking all possible solutions. Remember, only numbers from 0 to 11 are allowed!

    (a) $x^2 - 6x = 0$

    (b) $x^2 + 4x + 3 = 0$

    (c) $x^2 + x + 1 = 0$.

9.5 Summary and applications

In the previous sections, we have seen a number of different strategies to solve quadratic equations in one variable. Broadly speaking, these strategies fall into two categories. On the one hand, there are the strategies based on the basic principle that equations of the form $x^2 = k$ have two solutions, $\pm\sqrt{k}$. Under this broad principle are included the strategies of completing the square and the quadratic formula, both of which apply to any quadratic equation in one variable. On the other hand, there is the strategy of factoring, based on the zero product property of numbers. While this strategy cannot be applied to every quadratic equation in one variable, it is generally more efficient in the cases where it can be applied.

Only practice can guide the student to an intuition as to which strategy best applies in solving a given equation. With that said, the following can be used as a rough guide toward solving quadratic equations in one variable.

Guide to solving quadratic equations in one variable

To solve a quadratic equation in one variable x, first write the equation in the standard form

$$ax^2 + bx + c = 0,$$

that is, with one side of the equation being zero. Consider the following questions:

1. *Can the equation be easily written in the form $x^2 = k$?* This is always the case when $b = 0$.
   - If so, apply the basic strategy to write the solutions $\sqrt{k}$ and $-\sqrt{k}$.
2. *Can the quadratic expression be easily factored?* Look especially for common factors and monic trinomials with integer coefficients.
   - If so, solve the equation by factoring.
3. *If neither of the two preceding cases hold, either apply the completing the square technique, or use the quadratic formula.*
   - Completing the square is most practical when $a = 1$ and b is an even integer; otherwise the quadratic formula is more efficient.

Always make sure that radical expressions are simplified!

At the end of this section, there are more exercises with which to practice deciding which technique to apply to solve quadratic equations.

We conclude our discussion of quadratic equations in one variable with some examples of how these equations arise in the context of word problems. The reader should review the four-step strategy for handling word problems in Section 5.3.5.

Example 9.5.1. *One number is three more than twice another number. Their product is seven more than their sum. Find the two numbers.*

Answer. ***Step 1: Create a dictionary.*** *The problem involves two unknown* *quantities. As usual, one of them will be denoted x, in this case the quantity referred to as "another number" in the first sentence. In that case, the first sentence describes the other quantity as three more than twice x, or $2x+3$. The dictionary can be written as*

One number	$2x+3$
Another number	x

Step 2: Write an equation. *We have used the first sentence to create the dictionary. The second sentence describes a relationship between them as an equation ("is"):*

$$(2x+3)(x) = [(2x+3)+(x)]+7.$$

Step 3: Solve the equation. *We will first simplify both sides of the equation separately, distributing and combining like terms where possible:*

$$\begin{aligned}(2x+3)(x) &= (2x+3)+(x)+7\\ (2x)(x)+(3)(x) &= 2x+3+x+7\\ 2x^2+3x &= 3x+10.\end{aligned}$$

It is clear now we are working with a quadratic equation. Let's rewrite it in standard form, so that one side is zero.

$$\begin{array}{ccccccccc} 2x^2 & + & 3x & & & = & 3x & + & 10\\ & & -3x & & -10 & \vdots & -3x & & -10\\ \hline 2x^2 & & & - & 10 & = & 0. & & \end{array}$$

Since the x-term does not appear, we can rewrite this equation again in the special form $x^2=k$:

$$\begin{array}{ccccc} 2x^2 & - & 10 & = & 0\\ & & +10 & \vdots & +10\\ \hline 2x^2 & & & = & 10\\ \frac{2x^2}{2} & & & = & \frac{10}{2}\\ x^2 & & & = & 5.\end{array}$$

(The reader may have noticed that the work we did to rewrite the equation in the standard form $2x^2-10=0$ was really unnecessary in this problem. We could have proceeded to isolate the x^2 term as soon as we noticed that there was no linear term, in other words no term involving x^1.)

Now that the equation is written in the special form $x^2=5$, we can apply the basic strategy directly.

The solutions are $\sqrt{5}$ and $-\sqrt{5}$.

Step 4: Answer the question. *Notice right away: The two solutions to the equation in Step 3 are not the two numbers we are looking for! The solutions represented the value of x, which, according to our dictionary, represent* *"another number." In other words, each solution to the quadratic equation will correspond to a* pair *of numbers as an answer.*

To find the other number in each pair, we rely on the dictionary: the "one number" is given by $2x+3$. *So for the solution* $\sqrt{5}$, *the corresponding "one number" would be* $2(\sqrt{5})+3$, *or* $3+2\sqrt{5}$. *For the solution* $-\sqrt{5}$, *the corresponding "one number" would be* $2(-\sqrt{5})+3$, *or* $3-2\sqrt{5}$.

The problem has two possible answers. Either one number is $3+2\sqrt{5}$ *and another number is* $\sqrt{5}$, *or one number is* $3-2\sqrt{5}$ *and another number is* $-\sqrt{5}$.

Example 9.5.2. *The length of a rectangle is four feet less than twice the width. The area of the rectangle is* 70 *square feet. Find the dimensions of the rectangle.*

Answer. ***Step 1: Create a dictionary.*** *The problem involves two unknown quantities, the length and the width. The first sentence relates the two. We will denote the width by* x, *so that according to the first sentence, the length will be* $2x-4$. *The dictionary can be written as*

$$\begin{array}{cc} \textit{Length} & 2x-4 \\ \textit{Width} & x \end{array}$$

Step 2: Write an equation. *We recall from geometry that the area of a rectangle is given by the formula*

$$Area = (Length)(Width).$$

Using the dictionary and the second sentence, we substitute the information into this formula to obtain the equation

$$70 = (2x-4)(x).$$

Step 3: Solve the equation. *Let's simplify the right-hand side first by distributing:*

$$\begin{aligned} 70 &= (2x-4)(x) \\ 70 &= (2x)(x)+(-4)(x) \\ 70 &= 2x^2-4x. \end{aligned}$$

We see that this is a quadratic equation. We write it in standard form by subtracting 70 *from both sides:*

$$\begin{array}{ccccccc} 70 & = & 2x^2 & - & 4x & & \\ -70 & \vdots & & & & & -70 \\ \hline 0 & = & 2x^2 & - & 4x & - & 70. \end{array}$$

The right-hand side can be factored:

$$\begin{aligned} 0 &= 2x^2-4x-70 \\ 0 &= 2(x^2-2x-35) \\ 0 &= 2(x+5)(x-7). \end{aligned}$$

Applying the zero product property, this equation is equivalent to the compound statement

$$x + 5 = 0 \quad OR \quad x - 7 = 0.$$

(Notice the factor of 2 *does not contribute any solutions to the equation.) This statement has solutions* -5 *and* 7*.*

The solutions are -5 *and* 7*.*

Step 4: Answer the question. *Recall from our dictionary that* x *represents the width of the rectangle. For that reason, we will ignore the negative solution to the equation as being physically meaningless, and only consider the solution* 7*. From the dictionary, we know that the length is expressed by* $2x - 4$*. Substituting the solution* 7*, we find that the length is* $2(7) - 4 = 14 - 4 = 10$*.*

The width is 7 *and the length is* 10*.*

Example 9.5.3. *An object is thrown straight upwards from the ground with an initial velocity of* 60 *ft/sec. It's height* h *above the ground is related to the number* t *of seconds that pass by the equation* $h = -16t^2 + 60t$*. When will the object be* 12 *feet above the ground?*

Answer. ***Step 1: Create a dictionary.*** *In this case, the only unknown quantity is the time* t*. (The height is also unknown, but the problem specifically asks for the time ("when").) Writing the dictionary would simply mean writing*

$$\textit{Time} \quad t$$

Step 2: Write an equation. *The problem gives a relationship between the height* h *and the time* t*. Using the fact that the problem asks specifically about the object when the height is* 12 *feet, we substitute to obtain the quadratic equation*

$$12 = -16t^2 + 60t.$$

Step 3: Solve the equation. *We begin by writing the equation in standard form with one side being zero. For the sake of having the leading coefficient positive, we will add* $16t^2 - 60t$ *to both sides:*

$$\begin{array}{ccccccccc}
 & & & & 12 & = & -16t^2 & + & 60t \\
+16t^2 & & -60t & & & \vdots & +16t^2 & & -60t \\
\hline
16t^2 & - & 60t & + & 12 & = & 0. & &
\end{array}$$

The terms on the left side have a common factor of 4*:*

$$4(4t^2 - 15t + 3) = 0.$$

By the zero product property (or by dividing both sides of the equation by 4*), this is equivalent to the equation*

$$4t^2 - 15t + 3 = 0.$$

Since the quadratic trinomial cannot be factored any further, we will apply the quadratic formula with $a = 4$*,* $b = -15$ *and* $c = 3$*:*

$$\begin{aligned} t &= \frac{-(-15) \pm \sqrt{(-15)^2 - 4(4)(3)}}{2(4)} \\ &= \frac{15 \pm \sqrt{225 - 48}}{8} \\ &= \frac{15 \pm \sqrt{177}}{8}. \end{aligned}$$

The solutions are $\dfrac{15 + \sqrt{177}}{8}$ *and* $\dfrac{15 - \sqrt{177}}{8}$

Step 4: Answer the question. *Since* $\sqrt{177} \approx 13.304$*, both solutions are positive numbers and so are physically meaningful in the problem. Using the approximate value (again rounding to three decimal places), substituting, and evaluating, we find that both after* 0.212 *seconds and again after* 3.538 *seconds, the object is* 12 *feet above ground. (This corresponds to the object "going up" and then "coming down.")*

9.5.1 Exercises

Solve the following quadratic equations by any method.

1. $x^2 - 4x + 4 = 0$
2. $3x^2 + 5x - 2 = 0$
3. $6x^2 - 6x = 6$
4. $2x^2 + 6x + 2 = 0$
5. $4b^2 + 8b = 0$
6. $(x + 6)(x - 4) = -9$
7. $x^2 = -16$
8. $2x^2 - 3x = 2$
9. $6x^2 + 15x - 9 = 0$
10. $(x - 2)^2 = 3$
11. $3x^2 + 24 = 0$
12. $4x^2 - 3x = 1$

In each of the following, solve a quadratic equation to answer the question.

13. A rectangular apartment is designed so that the length is 10 feet more than three times the width. If the area enclosed is 800 square feet, find the dimensions of the apartment.

14. In an isosceles triangle, two of the three sides of the triangle have equal length. If the hypotenuse of an isosceles right triangle measures 10 units more than the length of either side, find the lengths of all three sides.

15. The product of the first two of three consecutive integers is 16 more than 10 times the third. Find the three integers.

16. The length of a rectangle measures 6 inches longer than twice the width. Find the perimeter of the rectangle if the length of the diagonal is 25 inches.

17. An object is tossed straight upwards from the ground with an initial velocity of 25 ft/sec. It's height h above the ground is related to the number t of seconds that pass by the formula $h = -16t^2 + 25t$. How long will it take for the object to fall back to the ground?

18. Taissha is standing on top of a building 100 feet tall. She drops her cell phone, whose height from the ground after t seconds is given by the formula $h = -16t^2 + 100$. How long does it take for her phone to hit the ground?

19. (*) If you want to divide a line segment into two parts (not necessarily equal), what is the perfect way to do it? For centuries, many artists and mathematicians thought the answer was the related to "golden ratio:" the ratio of the length of the larger part to the smaller part should be the same as the ratio of the length of the whole segment to the length of the larger part. If a segment of length 10 is divided according to the golden ratio, find the length of the two parts.

9.6 Introduction to quadratic equations in two variables

In this section, we will consider the simplest type of quadratic equations in two variables. As might be expected after Chapter 5, these equations will generally have infinitely many solutions, which we will graph in the xy-plane. The main feature of these equations that we want to point out in this section is that their graphs have a distinct shape *which is not a line.*

The simplest case of such an equation already illustrates the essential features of these graphs. Consider the equation

$$y = x^2.$$

This equation is quadratic (degree 2) in the variable x, but linear (degree 1) in the variable y. Nevertheless, the equation is a quadratic equation since the highest degree of *any* term is 2.

We will start off exactly as we did in the case of linear equations in two variables. We will begin finding solutions by choosing values for x (since the equation is written with y by itself on one side of the equation) and finding the corresponding value of y. By considering the resulting ordered pairs, we will plot the solutions and see if we can determine a pattern to "connect the dots."

Choosing 0 for x, we substitute:

$$\begin{aligned} y &= (0)^2 \\ y &= 0. \end{aligned}$$

Choosing 1 for x, we obtain

$$\begin{aligned} y &= (1)^2 \\ y &= 1. \end{aligned}$$

Choosing 2 for x,

$$\begin{aligned} y &= (2)^2 \\ y &= 4. \end{aligned}$$

So far, we can summarize our results in the following chart:

x	y	Solution
$\boxed{0}$	0	$(0,0)$
$\boxed{1}$	1	$(1,1)$
$\boxed{2}$	4	$(2,4)$

(Recall we write a box around the values we chose, as opposed to those we obtained by solving the equation for the chosen values.)

In Figure 9.3, we plot these three solutions.

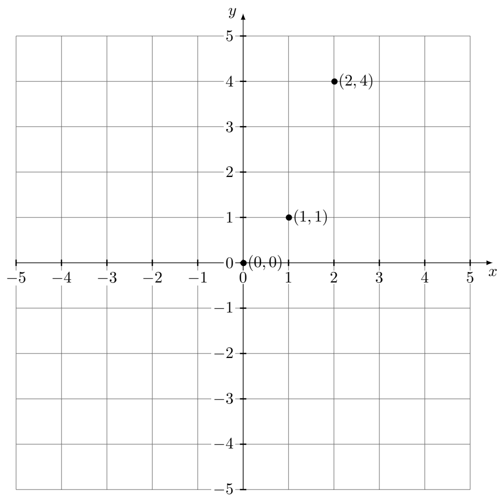


Figure 9.3: Three solutions of $y = x^2$.

We can notice one thing right away about Figure 9.3: no line will pass through all three of these points on the graph! Since it is not clear what the relationship is between these three solutions, we will continue to find more solutions until a pattern begins to emerge.

So far, we have chosen positive values for x. Let's see what happens when we chose some negative values. Choosing -1 for x, for example, we obtain

$$\begin{aligned} y &= (-1)^2 \\ y &= 1. \end{aligned}$$

Choosing -2 for x,

$$\begin{aligned} y &= (-2)^2 \\ y &= 4. \end{aligned}$$

Our table of solutions now appears as

x	y	Solution
$\boxed{0}$	0	$(0, 0)$
$\boxed{1}$	1	$(1, 1)$
$\boxed{2}$	4	$(2, 4)$
$\boxed{-1}$	1	$(-1, 1)$
$\boxed{-2}$	1	$(-2, 4)$

The graph of these five solutions is given in Figure 9.4.

At this point, some patterns start to take shape. First, there is some symmetry to the solutions: the y-axis appears to be a "mirror," with pairs of solutions appearing at equal distances from this axis. Furthermore, the origin $(0, 0)$, which is a solution to the equation, appears to be a "turning point" for the graph of the

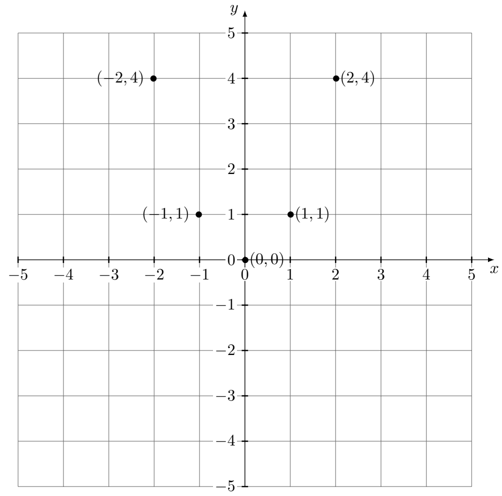


Figure 9.4: Five solutions of $y = x^2$.

solutions: the y-values seem to be decreasing as the x-values increase (through negative values) to 0, then begin to increase as the x-values increase past zero (through positive values).

Based on this information, we might "connect the dots" as in Figure 9.5.

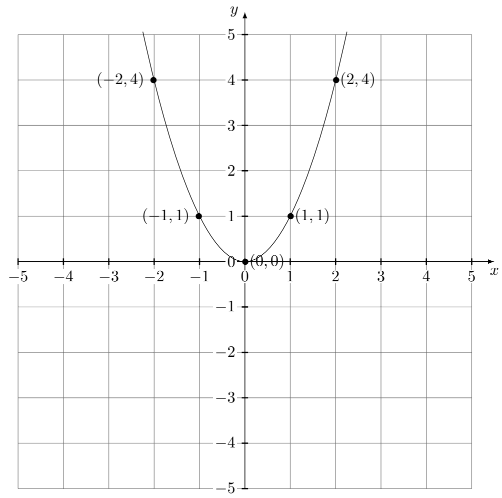


Figure 9.5: All solutions of $y = x^2$.

The shape of this graph is what is known as a *parabola*. For our purposes, the important features of a parabola is that it has one turning point (called the *vertex*) and a line (or *axis*) of symmetry. It is worth pointing out that the parabola's "U" shape does not make it a parabola; there are graphs that also have a "U" shape but are not parabolas.

Parabolas

The graph of solutions to a quadratic equation of the form

$$y = ax^2 + bx + c \quad (a \neq 0)$$

is a parabola.

In a more detailed study of these equations, we could ask whether it is possible, given such an equation, we could find the essential features of its graph, namely its vertex and axis of symmetry. For our limited purposes, though, our strategy will be to simply find enough solutions until the vertex and axis of symmetry are apparent, then connect the solutions by drawing a parabola. In general, we will choose at least five solutions, although if a pattern does not emerge, we may even choose to find more.

Example 9.6.1. *Graph the solutions: $y = -x^2 + 2x$.*

Answer. *We begin by finding solutions by choosing values of x close to 0, keeping the previous example in mind.*

Choosing -2 for x:

$$\begin{aligned} y &= -(-2)^2 + 2(-2) \\ y &= -4 + (-4) \\ y &= -8. \end{aligned}$$

Choosing -1 for x:

$$\begin{aligned} y &= -(-1)^2 + 2(-1) \\ y &= -1 + (-2) \\ y &= -3. \end{aligned}$$

Choosing 0 for x:

$$\begin{aligned} y &= -(0)^2 + 2(0) \\ y &= 0. \end{aligned}$$

Choosing 1 for x:

$$\begin{aligned} y &= -(1)^2 + 2(1) \\ y &= -1 + (2) \\ y &= 1. \end{aligned}$$

Choosing 2 for x:

$$\begin{aligned} y &= -(2)^2 + 2(2) \\ y &= -4 + (4) \\ y &= 0. \end{aligned}$$

So for our first five choices of values of x, we have obtained the following table:

x	y	*Solution*
$\boxed{-2}$	-8	$(-2,-8)$
$\boxed{-1}$	-3	$(-1,-3)$
$\boxed{0}$	0	$(0,0)$
$\boxed{1}$	1	$(1,1)$
$\boxed{2}$	0	$(2,0)$

We plot these five solutions in Figure 9.6.

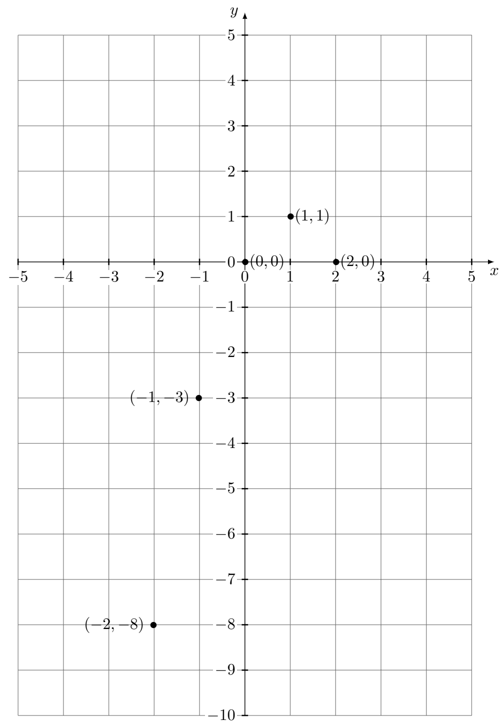


Figure 9.6: Five solutions of $y = -x^2 + 2x$.

Judging from the solutions plotted so far, we might guess that the vertex is the point corresponding to $(1, 1)$. We might confirm that by checking that $(3, -3)$ and $(4, -8)$ are also solutions. We also see that unlike the previous example, a parabola joining these solutions will face downwards.

We conclude by drawing the parabola as in Figure 9.7.

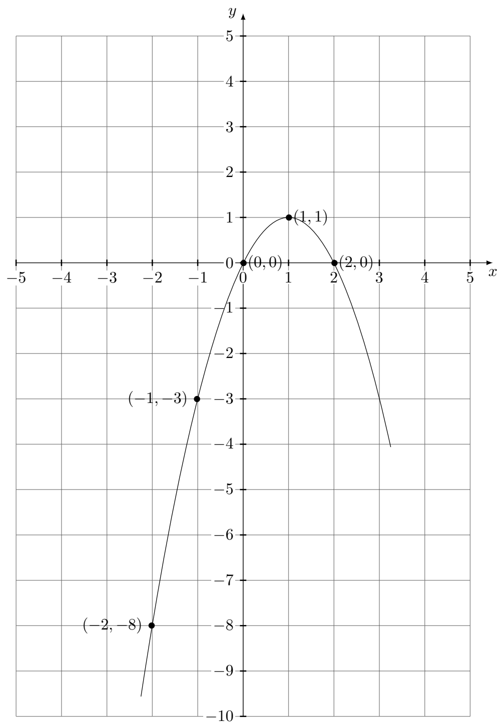


Figure 9.7: All solutions of $y = -x^2 + 2x$.

9.6.1 Exercises

Graph the solutions to the following equations.

1. $y = x^2 + 2$
2. $y = x^2 - 1$
3. $y = 2x^2$
4. $y = -x^2$
5. $y = x^2 - 4x$
6. $y = 1 - x^2$

9.7 Chapter summary

- A typical quadratic equation in one variable will have two solutions.
- Quadratic equations of the form $x^2 = k$ have two solutions, represented symbolically as $\sqrt{k}$ and $-\sqrt{k}$. (The two solutions are the same when $k = 0$.)

- Any quadratic equation in one variable can be solved by either the technique of completing the square or by using the quadratic formula.
- Some quadratic equations (but not all!) can be solved by the alternate technique of factoring and using the zero product property of numbers.
- The graph of an equation of the form $y = ax^2 + bx + c$ in the xy-plane (with $a \neq 0$) will be a parabola.