Quadratic Equations

Key Topics

- **Quadratic Equations:** A quadratic equation is an equation with a degree of two, meaning that at least one term in the equation is raised to the second power. The standard form is $ax^2 + bx + c = 0 \; where \; a \neq 0$.

Solving Quadratic Equations

- **Factoring Method:** This method uses the zero product property which states that if a product is zero, then at least one of its factors has to be zero.
  Example: Solve the equation $x(x+4) = 12$.
  Step 1: Put the equation in standard form.
  $$x(x+4) = 12 \to \; x^2 + 4x = 12 \to \; x^2 + 4x - 12 = 0$$
  Step 2: Factor the equation.
  $$x^2 + 4x - 12 = 0 \to \; (x+6)(x-2) = 0$$
  Step 3: If a product equals zero, one of its factors is equal to zero.
  $$x + 6 = 0 \;\; or \;\; x - 2 = 0$$
  $$x = -6 \; or \; x = 2$$
  $So \; the \; solution \; set \; is \; \{-6, 2\}.$

- **Square Root Method:** If an expression squared is equal to a constant, then that expression is equal to the positive or negative square root of the constant.
  Example: Solve the equation $3(x-1)^2 - 75 = 0$.
  Step 1: Isolate the squared expression.
  $$3(x-1)^2 - 75 = 0 \to \; 3(x-1)^2 = 75 \to \; (x-1)^2 = 25$$
  Step 2: Apply the square root property.
  $$(x-1)^2 = 25 \to \; \sqrt{(x-1)^2} = \sqrt{25} \to x - 1 = \pm\sqrt{25} \to x - 1 = \pm 5$$
  Step 3: Solve for x.
  $$x - 1 = \pm 5 \to x = \pm 5 + 1$$
  $If \; it \; is -5, then \; x = -5 + 1 = -4.$ *If it is* $5, then \; x = 5 + 1 = 6.$

  $The \; solution \; set \; is \; \{-4, 6\}.$

- **Completing the Square:** Not all quadratic equations can use the factoring or square root methods. Completing the square aims to transform a standard quadratic equation $ax^2 + bx + c = 0$ into a perfect square $(x + A)^2 = B$.
  Example: Solve the equation $2x^2 - 4x + 3 = 0$.
  Step 1: If the leading coefficient is not 1, divide by the leading coefficient.
  $$\frac{1}{2}(2x^2 - 4x + 3) = \frac{1}{2}(0) \to x^2 - 2x + \frac{3}{2} = 0$$

Step 2: Add the opposite of the constant term to both sides.

$$x^2 - 2x + \frac{3}{2} - \frac{3}{2} = 0 - \frac{3}{2} \rightarrow x^2 - 2x = -\frac{3}{2}$$

Step 3: Add $\left(\frac{b}{2}\right)^2$ to both sides where b is the coefficient of x.

$$x^2 - 2x + \left(\frac{-2}{2}\right)^2 = -\frac{3}{2} + \left(\frac{-2}{2}\right)^2 \rightarrow x^2 - 2x + 1 = -\frac{1}{2}$$

Step 4: Write the left side of the equation as a perfect square.

$$(x-1)^2 = -\frac{1}{2}$$

Step 5: Apply the square root method to solve.

$$x - 1 = \pm\sqrt{-\frac{1}{2}} \rightarrow x = 1 \pm \sqrt{-\frac{1}{2}} \rightarrow x = 1 \pm i\frac{1}{\sqrt{2}} \rightarrow x = 1 \pm \frac{i}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \rightarrow x = 1 \pm \frac{i\sqrt{2}}{2}$$

The solution set is $\left\{1 - \frac{i\sqrt{2}}{2}, 1 + \frac{i\sqrt{2}}{2}\right\}$

Quadratic Formula

- **Quadratic Formula:** For any equation of the form $ax^2 + bx + c = 0$ (with $a \neq 0$), the solution can be found using the following formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula is called the *quadratic formula* and can be used to solve any quadratic equation.

Example: Solve the equation $3x^2 - 2x + 9 = 0$.

Step 1: Identify a, b, and c.

$$a = 3, \qquad b = -2, \qquad c = 9$$

Step 2: Substitute the values of a, b, and c into the quadratic equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(9)}}{2(3)}$$

Step 3: Simplify.

$$x = \frac{2 \pm \sqrt{4 - 108}}{6} = \frac{2 \pm \sqrt{-104}}{6} = \frac{2 \pm i\sqrt{4 \cdot 26}}{2 \cdot 3} = \frac{2 \pm 2i\sqrt{26}}{2 \cdot 3} = \frac{\cancel{2}(1 \pm i\sqrt{26})}{\cancel{2} \cdot 3}$$
$$= \frac{1 \pm i\sqrt{26}}{3}$$

The solution set is $\left\{\frac{1 - i\sqrt{26}}{3}, \frac{1 + i\sqrt{26}}{3}\right\}$

- **Discriminant:** The discriminant is $b^2 - 4ac$. This is the term found inside the radical of the quadratic equation and gives information about the solutions of any equation of the form $ax^2 + bx + c = 0$ where a, b, and c are real numbers.

$b^2 - 4ac$	Solutions (Roots)
Positive	Two distinct real roots
0	One real root
Negative	Two complex roots

Example: Check the discriminant of the example above to see if the solution set we found matches the information in the table above.

Step 1: Identify a, b, and c.

$$a = 3, \qquad b = -2, \qquad c = 9$$

Step 2: Substitute the values of $a, b,$ and c into the discriminant.

$$b^2 - 4ac \to (-2)^2 - 4(3)(9)$$

Step 3: Simplify.

$$(-2)^2 - 4(3)(9) = 4 - 12(9) = 4 - 108 = -104$$

Since $b^2 - 4ac$ simplifies to a negative for $3x^2 - 2x + 9 = 0$ the table says it should have two complex roots, and we found that the solutions for $3x^2 - 2x + 9 = 0$ are

$$\left\{\frac{1 - i\sqrt{26}}{3}, \frac{1 + i\sqrt{26}}{3}\right\}$$

which are in fact complex.

Other Types of Equations

Radical Equations

- **<u>Radical Equation</u>:** A radical equation is an equation where the variable is inside a radical (i.e. $\sqrt{3x - 4} = 7$; $\sqrt[5]{x^5 + x^2 - 7} = 9x$).
- **<u>Procedure for Solving Radical Equations</u>:** The following example shows how to solve radical equations.

  Example: Solve the equation $\sqrt{2x - 6} = x - 3$

  Step 1: Square both sides of the equation.

  $$\left(\sqrt{2x - 6}\right)^2 = (x - 3)^2$$

  Step 2: Simplify.

  $$2x - 6 = x^2 - 6x + 9$$

  Step 3: Write the equation in standard form.

  $$x^2 - 6x + 9 - 2x + 6 = 0$$
  $$x^2 - 8x + 15 = 0$$

  Step 4: Factor.

  $$(x - 3)(x - 5) = 0$$

Step 5: Use the zero product property.

$$x = 3 \text{ } or \text{ } x = 5$$

Step 6: Check the solutions.

$$\sqrt{2(3) - 6} = 3 - 3 \to \sqrt{6 - 6} = 0 \to \sqrt{0} = 0 \to 0 = 0 \checkmark$$
$$\sqrt{2(5) - 6} = 5 - 3 \to \sqrt{10 - 6} = 2 \to \sqrt{4} = 2 \to 2 = 2 \checkmark$$

u-Substitution

- Equations that are higher order or that have fractional powers often can be transformed into a quadratic equation through substitution. When this is the case, we say these equations are quadratic in form.
- **Procedure for Solving Equation through u-substitution:** The following example shows how to solve equations using u-substitution.

Example: Solve $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} - 10 = 0$

Step 1: Identify the substitution.

$$u = x^{\frac{1}{3}}$$

Step 2: Transform the equation into a quadratic equation.

$$(x^{\frac{1}{3}})^2 - 3x^{\frac{1}{3}} - 10 = 0$$
$$u^2 - 3u - 10 = 0$$

Step 3: Solve the quadratic equation.

$$(u - 5)(u + 2) = 0$$
$$u = 5 \text{ } or \text{ } u = -2$$

Step 4: Apply the substitution to rewrite the solution in terms of the original variable.

$$x^{\frac{1}{3}} = 5 \text{ } or \text{ } x^{\frac{1}{3}} = -2$$

Step 5: Solve the resulting equation.

$$\left(x^{\frac{1}{3}}\right)^3 = 5^3 \to x = 125$$
$$(x^{\frac{1}{3}})^3 = (-2)^3 \to x = -8$$

Step 6: Check the solutions in the original equation.

$$(125)^{\frac{2}{3}} - 3(125)^{\frac{1}{3}} - 10 = 0 \to \sqrt[3]{(125)^2} - 3\sqrt[3]{125} - 10 = 0 \to 25 - 3(5) - 10 = 0$$
$$\to 25 - 15 - 10 = 0 \to 0 = 0 \checkmark$$
$$(-8)^{\frac{2}{3}} - 3(-8)^{\frac{1}{3}} - 10 = 0 \to \sqrt[3]{(-8)^2} - 3\sqrt[3]{-8} - 10 = 0 \to 4 - 3(-2) - 10 = 0 \to 0 = 0 \checkmark$$

Factorable Equations

- Some equations (both polynomial and with rational exponents) that are factorable can be solved using the zero product property.
- **Example with Rational Exponents:** Solve $x^{\frac{7}{3}} - 3x^{\frac{4}{3}} - 4x^{\frac{1}{3}} = 0$

$$x^{\frac{1}{3}}(x^2 - 3x - 4) = 0 \to x^{\frac{1}{3}}(x - 4)(x + 1) = 0$$
$$x^{\frac{1}{3}} = 0 \to x = 0$$
$$x - 4 = 0 \to x = 4$$

$$x + 1 = 0 \rightarrow x = -1$$

- **Example of Polynomial:** Solve $x^3 - 5x^2 - 9x + 45 = 0$

$$(x^3 - 5x^2) + (-9x + 45) = 0 \rightarrow\ x^2(x-5) + (-9)(x-5) = 0 \rightarrow (x-5)(x^2 - 9) = 0$$

$$\rightarrow (x-5)(x+3)(x-3) = 0$$

$$x = 5\ or\ x = -3\ or\ x = 3$$