Quadratic Functions and Complex Numbers

Unit 3

Unit Overview

This unit focuses on quadratic equations and functions. You will study solutions of quadratic equations: methods of finding them, determining the nature of solutions, interpreting the meaning of solutions, and writing equations given solutions. You will graph quadratic equations and inequalities. In addition, you will extend your knowledge of the number systems to the complex numbers.

Unit 3 Academic Vocabulary

Add these words and others you encounter in this unit to your vocabulary notebook.

- completing the square
- complex conjugate
- complex number
- discriminant
- imaginary number
- root (zero)

Essential Questions

- Why are some solutions of quadratic equations meaningful in real-life applications while others are not?
- How do graphic, symbolic, and numeric methods of solving quadratic equations compare to one another?

EMBEDDED ASSESSMENTS

This unit has two embedded assessments, following Activities 3.4 and 3.6. By completing these embedded assessments, you will demonstrate your understanding of solutions to quadratic equations, the graphs of quadratic equations and inequalities, and the system of complex numbers.

Embedded Assessment 1

Applications of Quadratic Functions—p. 173

Embedded Assessment 2

Graphs of Quadratic Functions—p. 191

UNIT 3

Getting Ready

Write your answers on notebook paper or grid paper. Show your work.

Factor the expressions in Items 1–4 completely.

1. $6x^3y + 12x^2y^2$
2. $x^2 + 3x - 40$
3. $x^2 - 49$
4. $x^2 - 6x + 9$
5. Graph $f(x) = \frac{3}{4}x - \frac{3}{2}$.
6. Graph a line that has an x-intercept of 5 and a y-intercept of -2.
7. Graph $y = |x|$, $y = |x + 3|$, and $y = |x| + 3$ on the same grid.
8. Solve $x^2 - 3x - 5 = 0$.

Applications of Quadratic Functions
Fences

ACTIVITY
3.1

SUGGESTED LEARNING STRATEGIES: Marking the Text, Activating Prior Knowledge, Group Presentation, Look for a Pattern, Guess and Check

My Notes

Fence Me In is a business that specializes in building fenced enclosures. One client has purchased 100 ft of fencing to enclose the largest possible rectangular area in her yard.

1. If the width of the rectangular enclosure is 20 ft, what must be the length? Find the area of this rectangular enclosure.

2. Choose several values for the width of a rectangle with a perimeter of 100 ft. Determine the corresponding length and area of each rectangle. Share your values with members of your class. Then record each set of values in the table below.

Width (ft)	Length (ft)	Area (ft^2)

3. What is the relationship between the length and width of a rectangle with perimeter of 100 ft?

4. Based on your observations, predict if it is possible for a rectangle with perimeter of 100 ft to have each area. Explain your reasoning.

a. 400 ft^2

b. 500 ft^2

c. 700 ft^2

SUGGESTED LEARNING STRATEGIES: Create Representations, Marking the Text, Summarize/Paraphrase/Retell, Group Presentation

My Notes

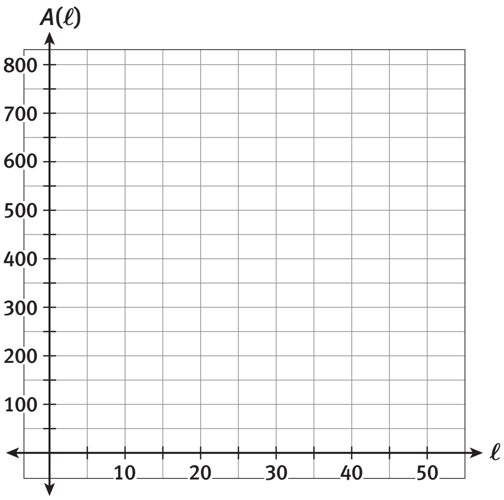
TECHNOLOGY TIP

You can also use a graphing calculator to graph your function $A(l)$. Set your window to correspond to the values on the axes on the graph.

5. Let l represent the length of a rectangle with a perimeter of 100 ft. Write an expression for the width of the rectangle in terms of l.

6. Express the area $A(l)$ for a rectangle with a perimeter of 100 ft as a function of its length, l.

7. Graph the quadratic function $A(l)$ on the coordinate grid.



8. Use the graph of the function to revise or confirm your predictions from Item 4. If a rectangle is possible, write the dimensions and explain how to use the graph to determine the dimensions.

a. 400 ft^2

b. 500 ft^2

c. 700 ft^2

SUGGESTED LEARNING STRATEGIES: Marking the Text, Summarize/Paraphrase/Retell, Quickwrite, Self/Peer Revision

My Notes

9. You may have been unable to use the graph to determine exact values for the dimensions in the previous item. Use the function you found in Item 6 to *algebraically* determine the length l of a rectangle with a perimeter of 100 ft and the areas given below, if possible. Show your work.

a. 400 ft^2

b. 500 ft^2

c. 700 ft^2

MATH TERMS

Quadratic equations can be solved by factoring, completing the square, or using the Quadratic Formula. Use the **Quadratic Formula** $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve quadratic equations in the form $ax^2 + bx + c = 0$, where $a \neq 0$.

10. What is the maximum value of the function in Item 6? Explain how you arrived at this conclusion and what this maximum represents.

CONNECT TO AP

The process of finding the maximum (or minimum) value of a function is called *optimization*, a topic addressed in calculus.

11. For the maximum value that you found in Item 10, what is the corresponding length l of the rectangle? Explain how you arrived at this conclusion.

12. What is the specific name of the rectangle with a perimeter of 100 ft and maximum area?

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern

My Notes

13. Another customer bought 180 ft of fencing to build a rectangular enclosure. Support your answers with explanations.

a. Write a function $R(l)$ that represents the area of the rectangle.

b. What are the dimensions of a rectangle with an area of 1925 ft^2?

c. What is the maximum area of the rectangular enclosure?

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or on grid paper. Show your work.

A rectangle has a perimeter of 120 ft.

1. Write a function $B(l)$ that represents the area of the rectangle with length l.

2. Graph the function $B(l)$, using a graphing calculator. Then copy it on your paper, labeling axes and using an appropriate scale.

3. Use $B(l)$ to find the dimensions of the rectangle with a perimeter of 120 feet that has each area. Explain you answer.

a. 500 ft^2 **b.** 700 ft^2

4. An area of 1000 ft^2 is not possible. Give two explanations for why this is true.

5. MATHEMATICAL REFLECTION How is the maximum value of a function shown on the graph of the function? How would a minimum value be shown?

Factoring and Solving Quadratics
Factor Fever

ACTIVITY
3.2

SUGGESTED LEARNING STRATEGIES: Marking the Text, Summarize/Paraphrase/Retell, Activating Prior Knowledge, Guess and Check

My Notes

Factoring quadratic expressions is an important skill in algebra. One of the most common uses of factoring is to solve quadratic equations. You can use the graphic organizer below to recall factoring trinomials of the form $x^2 + bx + c = 0$.

EXAMPLE 1

Factor $x^2 + 12x + 32$.

Step 1: *Place* x^2 *in the upper left box and the constant term 32 in the lower right.*

x^2	
	32

Step 2: *List factor pairs of 32, the constant term. Choose the pair that has a sum equal to 12, the coefficient* b *of the* x*-term.*

Factors		Sum
32	1	$32 + 1 = 33$
16	2	$16 + 2 = 18$
8	**4**	**$8 + 4 = 12$**

Step 3: *Write each factor as coefficients of* x *and place them in the two empty boxes. Write common factors from each row to the left and common factors for each column above.*

	x	8
x	x^2	$8x$
4	$4x$	32

Step 4: *Write the sum of the common factors as binomials. Then write the factors as a product.*

$(x + 4)(x + 8)$

Solution: $x^2 + 12x + 32 = (x + 4)(x + 8)$

MATH TIP

To check that your factoring is correct, multiply the two binomials by distributing.

$(x + 4)(x + 8)$

$= x^2 + 4x + 8x + 32$

$= x^2 + 12x + 32$

TRY THESE A

a. Factor $x^2 - 7x + 12$, using the graphic organizer. Then check by multiplying.

Factor, and then check by multiplying. Write your answers on notebook paper. Show your work.

b. $x^2 + 9x + 14$

c. $x^2 - 7x - 30$

d. $x^2 - 12x + 36$

e. $x^2 - 144$

f. $5x^2 + 40x + 75$

g. $-12x^2 + 108$

MATH TIP

Recall that a difference of squares $a^2 - b^2$ is equal to $(a - b)(a + b)$. A perfect square trinomial $a^2 + 2ab + b^2$ is equal to $(a + b)^2$.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Activating Prior Knowledge, Summarize/Paraphrase/Retell, Work Backward

My Notes

Before factoring quadratic expressions $ax^2 + bx + c$, where the leading coefficient $a \neq 1$, consider how multiplying binomial factors results in that form of a quadratic expression.

1. Use a graphic organizer to multiply $(2x + 3)(4x + 5)$.

a. Complete the display at the right by filling in the two empty boxes.

	$2x$	3
$4x$	$8x^2$	
5		15

b. $(2x + 3)(4x + 5)$

$= 8x^2 +$ ______ $+$ ______ $+ 15$

$= 8x^2 +$ __________ $+ 15$

Using the distributive property, you can see the relationship between the numbers in the binomial factors and the terms of the trinomial.

x^2 term, $8x^2$, is product of x-terms

constant term, 15, is product of constants

$(2x + 3)(4x + 5)$

x^2 term, $22x^2$, is sum of product of x-terms and constants

To factor a quadratic expression $ax^2 + bx + c$, work backward from the coefficients of the terms.

EXAMPLE 2

Factor $6x^2 + 13x - 5$. Use a table to organize your work.

Step 1: *Identify the factors of 6, which is* a, *the coefficient of the* x^2*-term.*

Step 2: *Identify the factors of* -5*, which is* c, *the coefficient of the constant term.*

Step 3: *Find the numbers whose products add together to equal 13, which is* b, *the coefficient of the* x*-term.*

Step 4: *Then write the binomial factors.*

Factors of 6	Factors of -5	Sum = 13?
1 and 6	-1 and 5	$1(5) + 6(-1) = -1$
1 and 6	5 and -1	$1(-1) + 6(5) = 29$
2 and 3	-1 and 5	$2(5) + 3(-1) = 7$
2 and 3	5 and -1	$2(-1) + 3(5) = 13$ ✓

Solution: $6x^2 + 13x - 5 = (2x + 5)(3x - 1)$

MATH TIP

Check your answer by multiplying the two binomials

$(2x + 5)(3x - 1)$

$= 6x^2 - 2x + 15x - 5$

$= 6x^2 + 13x - 5$

SUGGESTED LEARNING STRATEGIES: Work Backward, Group Presentation, Marking the Text, Summarize/Paraphrase/Retell, Identify a Subtask, Quickwrite

My Notes

TRY THESE B

Factor, and then check by multiplying. Write your answers on notebook paper. Show your work.

a. $10x^2 + 11x + 3$ **b.** $4x^2 + 17x - 15$

c. $2x^2 - 13x + 21$ **d.** $6x^2 - 19x - 36$

To solve a **quadratic equation** $ax^2 + bx + c = 0$ by factoring, the equation must be in factored form to use the Zero Product Property.

MATH TIP

The *Zero Product Property* states that if $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

EXAMPLE 3

Solve $x^2 + 5x - 14 = 0$ by factoring.

	Original equation	$x^2 + 5x - 14 = 0$
Step 1:	*Factor the left side.*	$(x + 7)(x - 2) = 0$
Step 2:	*Apply the Zero Product Property.*	$x + 7 = 0$ *or* $x - 2 = 0$
Step 3:	*Solve each equation for* x.	
Solution:	$x = -7$ or $x = 2$	

TRY THESE C

a. Solve $3x^2 - 17x + 10 = 0$ and check by substitution.

	Original equation
	Factor the left side.
	Apply the Zero Product Property.
	Solve each equation for x.

Solve each equation by factoring. Write your answers on notebook paper. Show your work.

b. $12x^2 - 7x - 10 = 0$ **c.** $x^2 + 8x - 9 = 0$ **d.** $4x^2 + 12x + 9 = 0$

e. $18x^2 - 98 = 0$ **f.** $x^2 + 6x = -8$ **g.** $5x^2 + 2x = 3$

MATH TIP

You can check your solutions by substituting the values into the original equation.

2. The equation $2x^2 + 9x - 3 = 0$ cannot be solved by factoring. Explain why this is true.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Summarize/Paraphrase/Retell, Work Backward

My Notes

MATH TIP

Recall that the *standard form* of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$.

If you know the solutions to a quadratic equation, then you can write the equation.

EXAMPLE 4

Write a quadratic equation in standard form with the solutions $x = 4$ and $x = -5$.

Step 1:	*Write linear equations that correspond to the solutions.*	$x - 4 = 0$ or $x + 5 = 0$
Step 2:	*Write the linear expressions as factors.*	$(x - 4)$ and $(x + 5)$
Step 3:	*Multiply the factors to write the equation in factored form.*	$(x - 4)(x + 5) = 0$
Step 4:	*Multiply the binomials and write the equation in standard form.*	$x^2 + x - 20 = 0$

TRY THESE D

a. Write a quadratic equation in standard form with the solutions $x = -1$ and $x = -7$.

	Write linear equations that correspond to the solutions.
	Write the linear expressions as factors.
	Multiply the factors to write the equation in factored form.
	Multiply the binomials and write the equation in standard form.

MATH TIP

To avoid fractions as coefficients, Multiply the coefficients by the LCD.

b. Write a quadratic equation in standard form whose solutions are $x = \frac{2}{5}$ and $x = -\frac{1}{2}$. How is your result different from those in Example 4?

Write a quadratic equation in standard form with integer coefficients for each pair of solutions. Write your answers in the My Notes space. Show your work.

c. $x = \frac{2}{3}, x = 2$

d. $x = -\frac{3}{2}, x = \frac{5}{2}$

SUGGESTED LEARNING STRATEGIES: Marking the Text, Summarize/Paraphrase/Retell, Identify a Subtask, Guess and Check

My Notes

Factoring is also useful for solving quadratic inequalities.

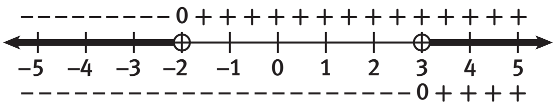
EXAMPLE 5

Solve $x^2 - x - 6 > 0$.

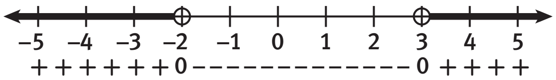
Step 1: *Factor the quadratic expression on the left side.* $(x + 2)(x - 3) > 0$

Step 2: *Determine where each factor equals zero.* $(x + 2) = 0$ at $x = -2$; $(x - 3) = 0$ at $x = 3$

Step 3: *Use a number line to visualize the intervals for which each factor is positive and negative. (Test a value in each interval to determine the signs.)* $(x + 2)$, $(x - 3)$



Step 4: *Identify the sign of the product of the two factors on each interval.* $(x + 2)(x - 3)$

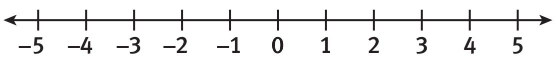


Step 5: *Choose the appropriate interval.* *Since $x^2 - x - 6$ is positive (>0), the intervals that show $(x + 2)(x - 3)$ as positive represent the solutions.*

Solution: $x < -2$ or $x > 3$

TRY THESE E

a. Use the number line provided to solve $2x^2 + x - 10 \leq 0$.



Solve each quadratic inequality.

b. $x^2 + 3x - 4 < 0$

c. $3x^2 + x - 10 \geq 0$

MATH TIP

For a product of two numbers to be positive, both factors must be the same sign. If the product is negative, then the factors must have opposite signs.

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper. Show your work.

1. Factor each quadratic expression.
   a. $2x^2 + 15x + 28$
   b. $3x^2 + 25x - 18$
   c. $6x^2 - 7x - 5$
   d. $12x^2 - 43x + 10$
2. Solve each quadratic equation by factoring.
   a. $2x^2 - 11x + 5 = 0$
   b. $x^2 + 8x = -15$
   c. $3x^2 + x - 4 = 0$
   d. $6x^2 - 13x - 5 = 0$
3. Write a quadratic equation in standard form with integer coefficients for which the given numbers are solutions.
   a. $x = 2$ and $x = -5$
   b. $x = -\frac{2}{3}$ and $x = -5$
   c. $x = \frac{3}{5}$ and $x = 3$
   d. $x = -\frac{1}{2}$ and $x = \frac{3}{4}$
4. Solve each inequality.
   a. $x^2 + 3x - 10 \geq 0$
   b. $2x^2 + 3x - 9 < 0$
5. MATHEMATICAL REFLECTION Explain how the method for factoring trinomials of the form $ax^2 + bx + c$ when $a \neq 1$ can be used to factor trinomials when $a = 1$. Give an example in your explanation.

Introduction to Complex Numbers
Complex Numbers

ACTIVITY 3.3

SUGGESTED LEARNING STRATEGIES: Summarize/Paraphrase/Retell, Activating Prior Knowledge, Create Representations

My Notes

The equation $x^2 + 1 = 0$ has special historical and mathematical significance. At the beginning of the sixteenth century, mathematicians believed that the equation had no solutions.

1. Why would mathematicians of the early sixteenth century think that $x^2 + 1 = 0$ had no solutions?

A breakthrough occurred in 1545 when the talented Italian mathematician Girolamo Cardano (1501–1576) published his book, *Ars Magna (The Great Art)*. In the process of solving one cubic (third degree) equation, he encountered—and was required to make use of—the square roots of negative numbers. While skeptical of their existence, he demonstrated the situation with this famous problem: Find two numbers with the sum 10 and the product 40.

2. To better understand this problem, first find two numbers with the sum 10 and the product 21.

3. Letting x represent one number, write an expression for the other number in terms of x. Use the expressions to write an equation that models the problem in Item 2.

4. Solve your equation in Item 3 in two different ways. Explain each method.

SUGGESTED LEARNING STRATEGIES: Create Representations, Activating Prior Knowledge, Interactive Word Wall, Marking the Text

My Notes

CONNECT TO HISTORY

When considering his solutions, Cardano dismissed "mental tortures" and ignored the fact that $\sqrt{x} \cdot \sqrt{x} = x$ only when $x \geq 0$.

5. Write an equation that represents the problem that Cardano posed.

6. Cardano claimed that the solutions to the problem are $x = 5 + \sqrt{-15}$ and $x = 5 - \sqrt{-15}$. Verify his solutions by using the Quadratic Formula with the equation in Item 5.

ACADEMIC VOCABULARY

An **imaginary number** is any number of the form *bi*, where *b* is a real number and $i = \sqrt{-1}$.

Cardano avoided any more problems in *Ars Magna* involving the square root of a negative number. However, he did demonstrate an understanding about the properties of such numbers. Solving the equation $x^2 + 1 = 0$ yields the solutions $x = \sqrt{-1}$ and $x = -\sqrt{-1}$. The number $\sqrt{-1}$ is represented by the symbol i, the **imaginary unit**. You can say $i = \sqrt{-1}$. The imaginary unit i is considered the solution to the equation $i^2 + 1 = 0$, or $i^2 = -1$.

To simplify an imaginary number $\sqrt{-s}$, where s is a positive number, you can write $\sqrt{-s} = i\sqrt{s}$.

CONNECT TO HISTORY

René Descartes (1596–1650) was the first to call these numbers *imaginary*. Although his reference was meant to be derogatory, the term *imaginary number* persists. Leonhard Euler (1707–1783) introduced the use of ***i*** for the imaginary unit.

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Interactive Word Wall, Marking the Text

My Notes

EXAMPLE 1

Write the numbers $\sqrt{-17}$ and $\sqrt{-9}$ in terms of i.

		$\sqrt{-17}$	$\sqrt{-9}$
Step 1:	*Definition of* $\sqrt{-s}$	$= i \cdot \sqrt{17}$	$= i \cdot \sqrt{9}$
Step 2:	*Take the square root of 9.*	$= i\sqrt{17}$	$= i \cdot 3$
			$= 3i$

TRY THESE A

Write each number in terms of i.

a. $\sqrt{-25}$ **b.** $\sqrt{-7}$

c. $\sqrt{-12}$ **d.** $\sqrt{-150}$

WRITING MATH

Write $i\sqrt{17}$ instead of $\sqrt{17}i$, which may be confused with $\sqrt{17i}$.

7. Write the solutions to Cardano's problem, $x = 5 + \sqrt{-15}$ and $x = 5 - \sqrt{-15}$, using the imaginary unit i.

The **set of complex numbers** consists of the real numbers and the imaginary numbers. A **complex number** is a number in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. Each complex number has two parts: the **real part** a and the **imaginary part** b. For example, in $2 + 3i$, the real part is 2 and the imaginary part is 3.

ACADEMIC VOCABULARY

complex number

TRY THESE B

Identify the real part and the imaginary part of each complex number.

a. $5 + 8i$ **b.** 8

c. $i\sqrt{10}$ **d.** $\dfrac{5 + 3i}{2}$

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Group Presentation

My Notes

8. Using the definition of complex numbers, show that the set of real numbers is a subset of the set of complex numbers.

Perform addition of complex numbers as you would for addition of binomials of the form $a + bx$. To add such binomials, you collect like terms.

EXAMPLE 2

		Addition of Binomials	Addition of Complex Numbers
		$(5 + 4x) + (-2 + 3x)$	$(5 + 4i) + (-2 + 3i)$
Step 1	*Collect like terms.*	$= (5 - 2) + (4x + 3x)$	$= (5 - 2) + (4i + 3i)$
Step 2	*Simplify.*	$= 3 + 7x$	$= 3 + 7i$

TRY THESE C

Add the complex numbers.

a. $(6 + 5i) + (4 - 7i)$

b. $(-5 + 3i) + (-3 - i)$

c. $(2 + 3i) + (-2 + 3i)$

9. Use Example 2 above and your knowledge of operations of real numbers to write general formulas for the sum and difference of two complex numbers.

$(a + bi) + (c + di) =$

$(a + bi) - (c + di) =$

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Group Presentation

My Notes

10. Find each sum or difference of the complex numbers.

a. $(12 - 13i) - (-5 + 4i)$

b. $\left(\frac{1}{2} - i\right) + \left(\frac{5}{2} + 9i\right)$

c. $(\sqrt{2} - 7i) + (2 + i\sqrt{3})$

d. $(8 - 5i) - (3 + 5i) + (-5 + 10i)$

Perform multiplication of complex numbers as you would for multiplication of binomials of the form $a + bx$. The only change in procedure is to replace i^2 with -1.

EXAMPLE 3

Multiply Binomials	**Multiply Complex Numbers**
$(2 + 3x)(4 - 5x)$	$(2 + 3i)(4 - 5i)$
$2(4) + 2(-5x) + 3x(4) + 3x(-5x)$	$2(4) + 2(-5i) + 3i(4) + 3i(-5i)$
$8 - 10x + 12x - 15x^2$	$8 - 10i + 12i - 15i^2$
$8 + 2x - 15x^2$	$8 + 2i - 15i^2$
	Now substitute -1 for i^2.
	$8 + 2i - 15i^2 = 8 + 2i - 15(-1)$
	$= 23 + 2i$

TRY THESE D

Multiply the complex numbers.

a. $(6 + 5i)(4 - 7i)$

b. $(2 - 3i)(3 - 2i)$

c. $(5 + i)(5 + i)$

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Look for a Pattern, Group Presentation

My Notes

MATH TIP

Since $\sqrt{-1} = i$, the powers of i can be evaluated as follows:

$i^1 = i$

$i^2 = -1$

$i^3 = i^2 \cdot i = -1i = -i$

$i^4 = i^2 \cdot i^2 = (-1)^2 = 1$

Since $i^4 = 1$, further powers repeat the pattern shown above.

$i^5 = i^4 \cdot i = i$

$i^6 = i^4 \cdot i^2 = i^2 = -1$

$i^7 = i^4 \cdot i^3 = i^3 = -i$

$i^8 = i^4 \cdot i^4 = i^4 = 1$

ACADEMIC VOCABULARY

complex conjugate

11. Use Example 3 and your knowledge of operations of real numbers to write a general formula for the multiplication of two complex numbers.

$(a + bi) \cdot (c + di) =$

12. Use operations of complex numbers to verify that the two solutions that Cardano found, $x = 5 + \sqrt{-15}$ and $x = 5 - \sqrt{-15}$, have a sum of 10 and a product of 40.

The **complex conjugate** of a complex number $a + bi$ is defined as $a - bi$. For example, the complex conjugate of $2 + 3i$ is $2 - 3i$.

13. A special property of multiplication of complex numbers occurs when a number is multiplied by its conjugate. Multiply each number by its conjugates then describe the product when a number is multiplied by its conjugate.

a. $2 - 9i$

b. $-5 + 2i$

14. Write an expression to complete the general formula for the product of a complex number and its complex conjugate.

$(a + bi)(a - bi) =$

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Group Presentation, Interactive Word Wall, Marking the Text

My Notes

EXAMPLE 4

Divide $\frac{4-5i}{2+3i}$.

Step 1: *Multiply the numerator and denominator by the complex conjugate.* $\frac{4-5i}{2+3i} = \frac{4-5i}{2+3i} \cdot \frac{(2-3i)}{(2-3i)}$

Step 2: *Simplify and substitute −1 for* i^2. $= \frac{8-22i+15i^2}{4-6i+6i-9i^2}$

$= \frac{8-22i-15}{4+9}$

Step 3: *Simplify and write in the form* a + bi. $= \frac{-7-22i}{13} = -\frac{7}{13} - \frac{22}{13}i$

TRY THESE E

a. In Example 4, why is the quotient $-\frac{7}{13} - \frac{22}{13}i$ equivalent to the original expression $\frac{4-5i}{2+3i}$?

Divide the complex numbers. Write your answers on notebook paper. Show your work.

b. $\frac{5i}{2+3i}$ **c.** $\frac{5+2i}{3-4i}$ **d.** $\frac{1-i}{\sqrt{3}+4i}$

15. Use Example 4 and your knowledge of operations of real numbers to write a general formula for the division of two complex numbers.

$$\frac{(a+bi)}{(c+di)} =$$

Complex numbers in the form $a + bi$ can be represented geometrically as points in the *complex plane*. The **complex plane** is a rectangular grid, similar to the Cartesian Plane, with the horizontal axis representing the real part a of a complex number and the vertical axis representing the imaginary part b of a complex number. The point (a, b) on the complex plane represents the complex number $a + bi$.

TECHNOLOGY TIP

Many graphing calculators have the capability to perform operations on complex numbers.

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Create Representations

My Notes

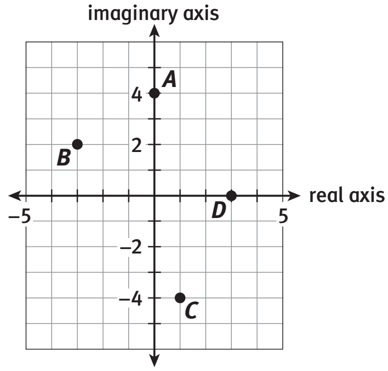
EXAMPLE 5

Point A represents $0 + 4i$.

Point B represents $-3 + 2i$.

Point C represents $1 - 4i$.

Point D represents $3 + 0i$.



TRY THESE F

a. Graph $2 + 3i$ and $-3 - 4i$ on the complex plane above.

Graph each complex number on a complex plane grid.

b. $2 + 5i$ **c.** $4 - 3i$ **d.** $-1 + 3i$ **e.** $-2i$ **f.** -5

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

1. Write each expression in terms of i.

a. $\sqrt{-49}$ **b.** $\sqrt{-13}$

c. $3 + \sqrt{-8}$ **d.** $5 - \sqrt{-36}$

2. These quadratic equations have solutions that are complex. Use the quadratic formula to solve the equations.

a. $x^2 + 3x + 4 = 0$ **b.** $2x^2 - 6x + 9 = 0$

3. Find each sum or difference.

a. $(6 - 5i) + (-2 + 6i)$

b. $(4 + i) + (-4 + i)$

c. $(5 - 3i) - (3 - 5i)$

d. $(-3 + 8i) - \left(\frac{3}{2} + \frac{1}{2}i\right)$

4. Multiply. Write each product in the form $a + bi$.

a. $(2 + 9i)(3 - i)$ **b.** $(-5 + 8i)(2 - i)$

c. $(8 + 15i)(8 - 15i)$

5. Divide. Write each quotient in the form $a + bi$.

a. $\frac{1 + 4i}{4 - i}$ **b.** $\frac{-2 + 5i}{3 - 4i}$ **c.** $\frac{7 - 3i}{i}$

6. Draw the complex plane. Then graph each complex number on the plane.

a. $6i$ **b.** $3 + 4i$ **c.** $-2 - 5i$

d. $4 - i$ **e.** $-3 + 2i$

7. **MATHEMATICAL REFLECTION** Why are complex number solutions of quadratic equations conjugates?

Solving $ax^2 + bx + c = 0$
Deriving the Quadratic Formula

ACTIVITY 3.4

SUGGESTED LEARNING STRATEGIES: Marking the Text, Group Presentation, Activating Prior Knowledge, Quickwrite

My Notes

Recall solving quadratic equations of the form $ax^2 + c = 0$. To solve equations of this type, isolate x^2 and take the square root of both sides of the equation.

EXAMPLE 1

Solve $5x^2 - 23 = 0$ for x.

		$5x^2 - 23 = 0$
Step 1:	*Add 23 to both sides.*	$5x^2 = 23$
Step 2:	*Divide by 5.*	$\frac{5x^2}{5} = \frac{23}{5}$
Step 3:	*Simplify to isolate* x^2.	$x^2 = \frac{23}{5}$
Step 4:	*Take the square root of both sides.*	$x = \pm\frac{\sqrt{23}}{\sqrt{5}}$
Step 5:	*Rationalize the denominator.*	$x = \pm\frac{\sqrt{23}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}$
Step 6:	*Simplify.*	$x = \pm\frac{\sqrt{115}}{5}$

Solution: $x = \pm\frac{\sqrt{115}}{5}$

MATH TIP

When taking the square root of both sides of an equation, remember to include both positive and negative roots. For example,

$x^2 = 4$

$x = \pm\sqrt{4}$

$x = \pm 2$

TRY THESE A

Solve for x. Write your answers on notebook paper. Show your work.

a. $9x^2 - 49 = 0$

b. $25x^2 - 7 = 0$

c. $5x^2 - 16 = 0$

d. $4x^2 + 15 = 0$

MATH TIP

To *rationalize a denominator* that contains a radical, multiply the numerator and denominator by the radical.

Example:

$\frac{7}{\sqrt{3}} = \frac{7}{\sqrt{3}} \cdot \sqrt{3} = \frac{7\sqrt{3}}{3}$

1. Compare and contrast the solutions to the equations in the Try These A section above.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Group Presentation, Activating Prior Knowledge, Quickwrite

My Notes

CONNECT TO AP

In calculus, *rationalizing a numerator* is a skill used to evaluate certain types of limit expressions.

To solve the equation $2(x - 3)^2 - 5 = 0$, you can use a similar process.

EXAMPLE 2

Solve $2(x - 3)^2 - 5 = 0$ for x.

$$2(x-3)^2 - 5 = 0$$

Step 1: *Add 5 to both sides.* $2(x-3)^2 = 5$

Step 2: *Divide by 2.* $(x-3)^2 = \frac{5}{2}$

Step 3: *Take the square root of both sides.* $x - 3 = \pm\frac{\sqrt{5}}{\sqrt{2}}$

Step 4: *Rationalize the denominator and solve for x.* $x - 3 = \pm\frac{\sqrt{10}}{2}$

Solution: $x = 3 \pm \frac{\sqrt{10}}{2}$

TRY THESE B

Solve for x. Write your answers in the My Notes space. Show your work.

a. $4(x + 5)^2 - 49 = 0$

b. $3(x - 2)^2 - 16 = 0$

c. $5(x + 1)^2 - 8 = 0$

d. $4(x + 7)^2 + 25 = 0$

2. Describe the differences in the equations in the Try These B section above.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Activating Prior Knowledge, Create Representations, Group Presentation

My Notes

ACADEMIC VOCABULARY

Completing the square

The standard form of a quadratic equation is $ax^2 + bx + c = 0$. You can solve equations written in standard form by **completing the square**.

EXAMPLE 3

Solve $2x^2 + 12x + 5 = 0$ by completing the square.

$$2x^2 + 12x + 5 = 0$$

Step 1: *Divide both sides by the leading coefficient and simplify.*

$$\frac{2x^2}{2} + \frac{12x}{2} + \frac{5}{2} = \frac{0}{2}$$

$$x^2 + 6x + \frac{5}{2} = 0$$

Step 2: *Isolate the variable terms on the left side.*

$$x^2 + 6x = -\frac{5}{2}$$

Step 3: *Divide the coefficient of the linear term by* 2[6 ÷ 2 = 3], *square the result* [$3^2 = 9$], *and add it* [9] *to both sides. This completes the square.*

$$x^2 + 6x + \boxed{} = -\frac{5}{2} + \boxed{}$$

$$x^2 + 6x + \boxed{9} = -\frac{5}{2} + \boxed{9}$$

Step 4: *Factor the perfect square trinomial on the left side into two binomials.*

$$(x + 3)^2 = \frac{13}{2}$$

Step 5: *Take the square root of both sides of the equation.*

$$x + 3 = \pm\sqrt{\frac{13}{2}}$$

Step 6: *Rationalize the denominator and solve for* x.

$$x + 3 = \pm\frac{\sqrt{13}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \pm\frac{\sqrt{26}}{2}$$

Solution: $x = -3 \pm \frac{\sqrt{26}}{2}$

MATH TIP

You can factor a perfect square trinomial $x^2 + 2xy + y^2$ as $(x + y)^2$.

TRY THESE C

Solve for x by completing the square.

a. $4x^2 + 16x - 5 = 0$

b. $5x^2 - 30x - 3 = 0$

c. $2x^2 - 6x - 1 = 0$

d. $2x^2 - 4x + 7 = 0$

Solving $ax^2 + bx + c = 0$

Deriving the Quadratic Formula

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Create Representations

My Notes

Previously you learned that solutions to the general quadratic equation $ax^2 + bx + c = 0$ can be found using the **Quadratic Formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } a \neq 0$$

You can *derive* the quadratic formula by completing the square on the general quadratic equation.

3. Derive the quadratic formula by completing the square for the equation $ax^2 + bx + c = 0$. (Use Example 3 as a model.)

SUGGESTED LEARNING STRATEGIES: Create Representations, Activating Prior Knowledge, Look for a Pattern, Group Presentation

My Notes

4. Solve $2x^2 - 5x + 3 = 0$ by completing the square. Then verify that the solution is correct by solving the same equation by using the Quadratic Formula.

5. Solve each equation by using the Quadratic Formula. For each equation, write the number of solutions. Tell whether the solutions are real or complex, and, if real, whether the solutions are rational or irrational.

a. $4x^2 + 5x - 6 = 0$

solutions:

number of solutions:

real or complex:

rational or irrational:

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Look for a Pattern, Group Presentation

My Notes

5. (*continued*)

b. $4x^2 + 5x - 2 = 0$

solutions:

number of solutions:

real or complex:

rational or irrational:

c. $4x^2 + 4x + 1 = 0$

solutions:

number of solutions:

real or complex:

rational or irrational:

d. $4x^2 + 4x + 5 = 0$

solutions:

number of solutions:

real or complex:

rational or irrational:

6. What patterns can you identify from your responses to Item 5?

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Note-taking

The **discriminant** of a quadratic equation $ax^2 + bx + c = 0$ is defined as the expression $b^2 - 4ac$. The value of the discriminant determines the *nature of the solutions* of a quadratic equation in the following manner.

Discriminant	Nature of Solutions
$b^2 - 4ac > 0$ and $b^2 - 4ac$ is a perfect square	Two real, rational solutions
$b^2 - 4ac > 0$ and $b^2 - 4ac$ is *not* a perfect square	Two real, irrational solutions
$b^2 - 4ac = 0$	One real, rational solution (a double root)
$b^2 - 4ac < 0$	Two complex conjugate solutions

7. Compute the value of the discriminant for each equation in Item 5 to verify your answers.

a. $4x^2 + 5x - 6 = 0$

b. $4x^2 + 5x - 2 = 0$

c. $4x^2 + 4x + 1 = 0$

d. $4x^2 + 4x + 5 = 0$

My Notes

ACADEMIC VOCABULARY

discriminant

ACADEMIC VOCABULARY

A solution to an equation is also called a **root** of the equation. The roots of a quadratic equation $ax^2 + bx + c = 0$ represent the **zeros** (or x-intercepts) of the quadratic function $y = ax^2 + bx + c$.

MATH TIP

If the values of a, b, and c are integers and the discriminant $b^2 - 4ac$ is a perfect square, then the quadratic expression $ax^2 + bx + c$ is factorable over the integers.

SUGGESTED LEARNING STRATEGIES: Identify a Subtask

My Notes

8. For each equation below, compute the value of the discriminant and describe the solutions without solving.

a. $2x^2 + 5x + 12 = 0$

b. $3x^2 - 11x + 4 = 0$

c. $5x^2 + 3x - 2 = 0$

d. $4x^2 - 12x + 9 = 0$

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper. Show your work.

1. Solve each quadratic equation by taking the square root of both sides of the equation. Identify the solutions as rational, irrational, or complex conjugates.

a. $9x^2 - 64 = 0$

b. $5x^2 - 12 = 0$

c. $16(x + 2)^2 - 25 = 0$

d. $2(x - 3)^2 - 15 = 0$

e. $4x^2 + 49 = 0$

f. $3(x - 1)^2 + 10 = 0$

2. Solve by completing the square.

a. $x^2 - 4x - 12 = 0$

b. $2x^2 - 5x - 3 = 0$

c. $x^2 + 6x - 2 = 0$

d. $3x^2 + 9x + 2 = 0$

e. $x^2 - x + 5 = 0$

f. $5x^2 + 2x + 3 = 0$

3. For each equation, evaluate the discriminant and determine the nature of the solutions. Then solve each equation using the Quadratic Formula to verify the nature of the roots.

a. $x^2 + 5x - 6 = 0$

b. $2x^2 - 7x - 15 = 0$

c. $x^2 - 8x + 16 = 0$

d. $5x^2 - 4x + 2 = 0$

e. $2x^2 + 9x + 20 = 0$

f. $3x^2 - 5x - 1 = 0$

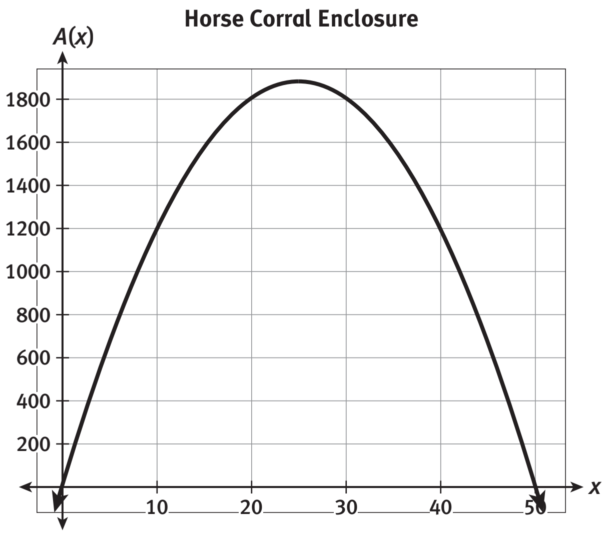
4. MATHEMATICAL REFLECTION What is the discriminant? Why does the value of the discriminant affect the solutions of a quadratic equation as shown in the table on page 171?

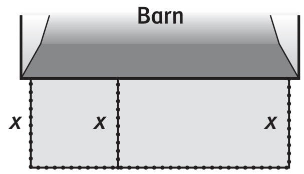
Applications of Quadratic Functions

NO HORSING AROUND

Embedded Assessment 1

Use after Activity 3.4.





Corral Diagram

1. Kun-cha has 150 feet of fencing to make a corral for her horses. The barn will be one side of the partitioned rectangular enclosure, as shown in the diagram above. The graph illustrates the function that represents the area that could be enclosed.

a. Write a function, $A(x)$, that represents the area that can be enclosed by the corral.

b. What information does the graph provide about the function?

c. Which ordered pair indicates the maximum area possible for the corral? Explain what each coordinate tells about the problem.

d. What values of x will give a total area of 1000 ft^2? 2000 ft^2?

2. Tim is the punter for the Bitterroot Springs Mustangs football team. He wrote a function $h(t)$ that he thinks will give the height of a football in terms of t, the number of seconds after he kicks the ball. Use two different methods to find and describe the roots for $h(t) = 16t^2 + 8t + 1$. Show your work. Is Tim's function correct? Why or why not?

3. Tim has been studying complex numbers and quadratic equations. Mrs. Pinto, his teacher, gave the class a quiz. Demonstrate your understanding of the material by responding to each item below.

a. Write a quadratic equation that has two solutions, $x = 2 + 5i$ and $x = 2 - 5i$.

b. Solve $3x^2 + 2x - 8 = 0$, using an algebraic method.

c. Rewrite $\dfrac{4 + i}{3 - 2i}$ in the form $a + bi$, where a and b are rational numbers.

Applications of Quadratic Functions

NO HORSING AROUND

	Exemplary	Proficient	Emerging
Math Knowledge #1c, d; 2, 3b, 3c	The student: • Gives the correct ordered pair for the maximum area. (1c) • Gives the values of x that will give the areas of 1000ft^2 and 2000ft^2. (1d) • Uses two methods to find the roots for $h(t)$ and describes the roots correctly.(2) • Solves the equation using an algebraic method. (3b) • Divides the complex numbers correctly. (3c)	The student: • Gives one correct coordinate of the ordered pair. • Gives only one correct value. • Uses only one method to find the roots and describes the roots correctly. • Solves the equation by a method other than the algebraic method. • Uses the correct method to divide the numbers, but makes a computational error.	The student: • Gives two incorrect coordinates. • Gives two incorrect values. • Tries, but is not able to find the roots. • Tries, but is not able to solve the equation. • Tries, but does not find the correct quotient of the numbers.
Problem Solving #1b, 2, 3a	The student: • States enough information so that the graph could be reproduced. (1b) • States correctly whether or not Tim's function is correct. (2) • Writes a correct quadratic equation. (3a)	The student: • States some correct aspects of the graph, but not enough so that the graph could be reproduced. • Writes a quadratic equation that is partially correct.	The student: • States only one correct aspect of the graph. • Makes an incorrect decision with respect to Tim's function. • Writes a quadratic equation that has no correct terms.
Representations #1a	The student writes the function that correctly describes the graph. (1a)	The student writes a function that has some, but not all, correct terms.	The student writes a function that has no correct terms.
Communication #1c, 2	The student: • Gives a correct and complete explanation for the meaning of both coordinates. (1c) • Gives a correct reason for the validity of Tim's function. (2)	The student: • Gives a correct explanation for one, but not both, coordinates. • Gives a partially correct reason for the validity of Tim's function.	The student: • Gives an incorrect explanation for each of the coordinates. • Gives an incorrect reason for the validity of Tim's function.

Graphing Quadratics and Quadratic Inequalities Calendar Art

ACTIVITY
3.5

SUGGESTED LEARNING STRATEGIES: Marking the Text, Summarize/Paraphrase/Retell, Activating Prior Knowledge, Create Representations

My Notes

Ms. Picasso, sponsor for her school's art club, sells calendars featuring student artwork to raise money for art supplies. A local print shop sponsors the calendar sale and donates the printing and supplies. From past experience, Ms. Picasso knows that she can sell 150 calendars for $3.00 each. She considers raising the price to try to increase the profit that the club can earn from the sale. However, she realizes that by raising the price, the team will sell fewer than 150 calendars.

1. If Ms. Picasso raises the price of the calendar by x dollars, write an expression for the price of one calendar.

2. In previous years, Ms. Picasso found that as the price increased, the number of calendars sold decreased. The table below shows that relationship between the price increase and the number of calendars sold. Use the data to write an expression for the number of calendars sold in terms of x, the price increase.

Increase in price, x	Number of calendars sold
0.00	150
0.40	140
0.80	130
1.20	120

3. Write a function for $A(x)$, the amount of money raised selling calendars when the price is increased x dollars.

MATH TERMS

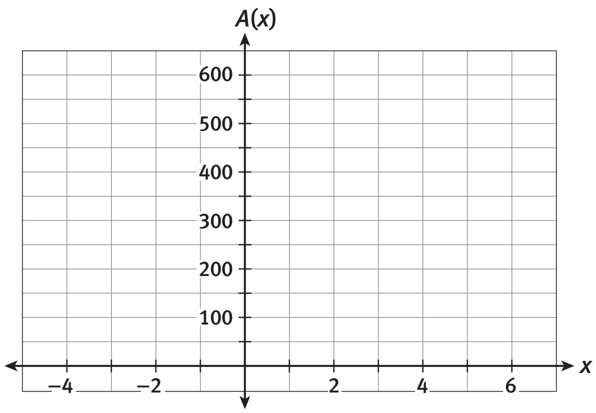
A **quadratic function** in *standard form* is written as $f(x) = ax^2 + bx + c$. Recall that the graph of a quadratic function is a *parabola*.

4. Write your function $A(x)$ in standard form. Identify the constants a, b, and c.

SUGGESTED LEARNING STRATEGIES: Create Representations, Quickwrite, Self/Peer Revision, Marking the Text

My Notes

5. Graph $A(x)$ on the coordinate grid.



6. What is the maximum amount of money that can be earned? What is the increase in price of a calendar that will yield that maximum amount of money?

7. What feature of the graph gives the information that you used to answer Item 6?

The point that represents the maximum value of $A(x)$ is the **vertex** of this parabola. The x-coordinate of the vertex of the graph of $f(x) = ax^2 + bx + c$ can be found using the formula $x = -\frac{b}{2a}$.

8. Use this formula to find the x-coordinate of the vertex of $A(x)$.

MATH TIP

Substitute the x-coordinate of the vertex into the quadratic equation to find the y-coordinate of the vertex.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Create Representations, Quickwrite, Activating Prior Knowledge

My Notes

An **intercept** occurs at the point of intersection of a graph and one of the axes. The ***x*-intercept** is the value n when $f(n) = 0$. The ***y*-intercept** is the value of $f(0)$. Use the graph that you made in Item 5 for Items 9 and 10.

9. What is the y-intercept of the graph of $A(x)$? What is the significance of that point in terms the calendar problem?

10. What are the x-intercepts of the graph of $A(x)$? What is the significance of each point in terms of the calendar problem?

11. The x-intercepts of the graph of $f(x) = ax^2 + bx + c$ can be found by solving the equation $ax^2 + bx + c = 0$. Solve the equation $A(x) = 0$ to verify the x-intercepts of the graph.

12. What is the average of the x-intercepts in Item 10? How does this relate to the symmetry of a parabola?

MATH TERMS

The vertical line $x = -\frac{b}{2a}$ is the **axis of symmetry** for the graph of the function $f(x) = ax^2 + bx + c$.

Graphing Quadratics and Quadratic Inequalities

Calendar Art

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Create Representations, Group Presentation

My Notes

EXAMPLE 1

For the quadratic function $f(x) = 2x^2 - 9x + 4$, identify the vertex, the y-intercept, x-intercept(s), and the axis of symmetry. Graph the function.

$a = 2, b = -9, c = 4$ — Identify a, b, and c.

Vertex

$-\frac{(-9)}{2(2)} = \frac{9}{4}; f\left(\frac{9}{4}\right) = -\frac{49}{8}$ — Use $-\frac{b}{2a}$ to find the x-coordinate of the vertex.

vertex: $\left(\frac{9}{4}, -\frac{49}{8}\right)$ — Then use $f\left(-\frac{b}{2a}\right)$ to find the y-coordinate.

y-Intercept

$f(0) = 4$, so y-intercept is 4. — Evaluate $f(x)$ at $x = 0$.

x-Intercepts

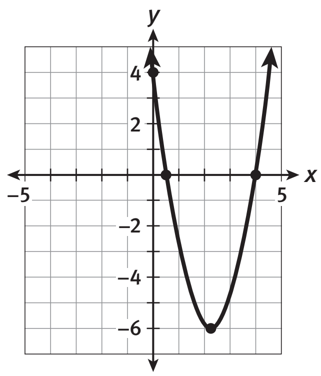
$2x^2 - 9x + 4 = 0$ — Let $f(x) = 0$.

$x = \frac{1}{2}$ and $x = 4$ are solutions, so x-intercepts are $x = \frac{1}{2}$ and 4. — Then solve for x by factoring or by using the quadratic formula.

Axis of Symmetry

$x = \frac{9}{4}$ — The x-coordinate of the vertex, $x = -\frac{b}{2a}$.

Graph



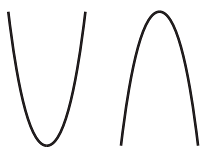
Graph the points identified above: vertex, y-intercept, x-intercepts.

Then draw the smooth curve of a parabola through the points.

The vertex represents a minimum of the function. The minimum is $-\frac{49}{8}$.

MATH TIP

The graph of the function $f(x) = ax^2 + bx + c$ will open upwards if $a > 0$ and will open downwards if $a < 0$.



If the parabola opens up, then the y-coordinate of the vertex is a *minimum* of the function. If it opens down, the y-coordinate of the vertex is a *maximum*.

TRY THESE A

For each quadratic function, identify the vertex, the y-intercept, the x-intercept(s), and the axis of symmetry. Then graph the function and classify the y-coordinate of the vertex as a maximum or minimum. Write your answers on grid paper. Show your work.

a. $f(x) = x^2 - 4x - 5$

b. $f(x) = -3x^2 + 8x + 16$

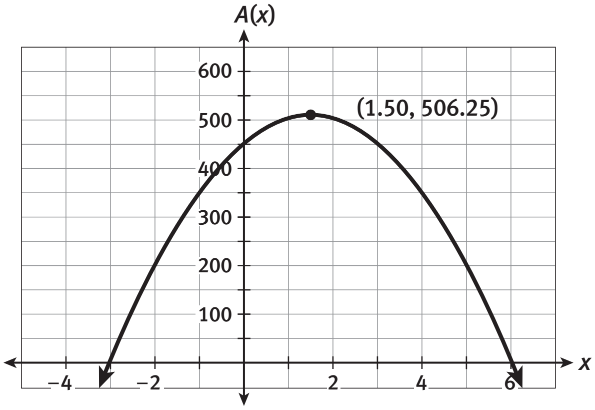
c. $f(x) = 2x^2 + 8x + 3$

d. $f(x) = -x^2 + 4x - 7$

SUGGESTED LEARNING STRATEGIES: Identify a Subtask, Create Representations, Quickwrite

My Notes

Consider the calendar fund-raising function from Item 2, $A(x) = -25x^2 + 75x + 450$, whose graph is below.



MATH TIP

Recall that quadratic equations may be solved by algebraic methods such as factoring or the quadratic formula.

An equation can be solved on a graphing calculator by entering each side of the equation as a function, graphing both functions, and finding the points of intersection. The *x*-coordinates of the intersection points are the solutions.

13. Suppose that Ms. Picasso raises \$450 in the calendar sale. By how much did she increase the price? Explain your answer graphically and algebraically.

14. Suppose Ms. Picasso wants to raise \$600. Describe why this is not possible, both graphically and algebraically.

15. In Item 4, you found that the maximum amount of money that could be raised was \$506.25. Explain both graphically and algebraically why this is true for only one possible price increase.

SUGGESTED LEARNING STRATEGIES: Identify a Subtask, Create Representations, Quickwrite, Activating Prior Knowledge

My Notes

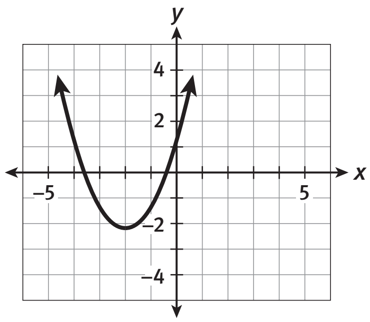
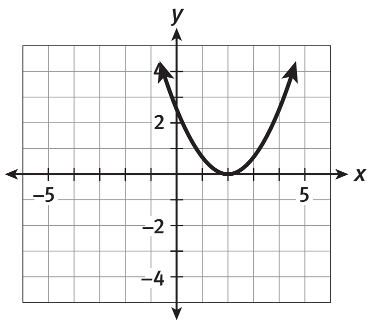
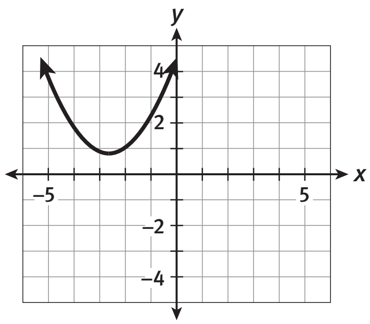
16. What price increase would yield \$500 in the calendar sale? Explain how you determined your solution.

The discriminant of a quadratic equation $ax^2 + bx + c = 0$ can determine not only the nature of the solutions of the equation, but also the number and type of x-intercepts of its related function $f(x) = ax^2 + bx + c$.

<table>
<tr><th>Discriminant of
$ax^2 + bx + c = 0$</th><th>Solutions and
x-intercepts</th><th>Sample Graph of
$f(x) = ax^2 + bx + c$</th></tr>
<tr><td>$b^2 - 4ac > 0$

If $b^2 - 4ac$ is:
• a perfect square
• not a perfect square</td><td>• Two real solutions
• Two x-intercepts

roots are rational

roots are irrational</td><td>
</td></tr>
<tr><td>$b^2 - 4ac = 0$</td><td>• One real, rational solution (a double root)
• One x-intercept</td><td>
</td></tr>
<tr><td>$b^2 - 4ac < 0$</td><td>• Two complex conjugate solutions
• No x-intercepts</td><td>
</td></tr>
</table>

MATH TIP

The x-intercepts of a quadratic function $y = ax^2 + bx + c$ are the *zeros of the function*. The solutions of a quadratic equation $ax^2 + bx + c = 0$ are the *roots of the equation*.

SUGGESTED LEARNING STRATEGIES: Create Representations, Marking the Text, Activating Prior Knowledge

My Notes

TRY THESE **B**

For each equation, find the value of the discriminant and describe the nature of the solutions. Then graph the related function and find the x-intercepts. Write your answers in the My Notes space. Show your work.

a. $4x^2 + 12x + 9 = 0$

b. $2x^2 + x + 5 = 0$

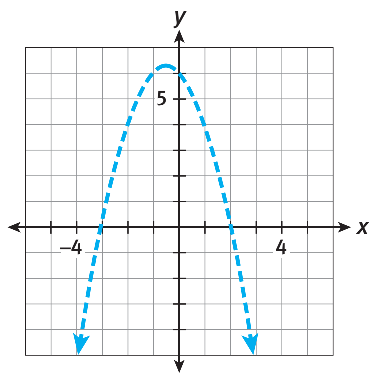
c. $2x^2 + x - 10 = 0$

d. $x^2 + 3x + 1 = 0$

The solutions to **quadratic inequalities** of the form $y > ax^2 + bx + c$ or $y < ax^2 + bx + c$ can be most easily described using a graph. An important part of solving these inequalities is graphing the related quadratic functions.

EXAMPLE 2

Solve $y > -x^2 - x + 6$.



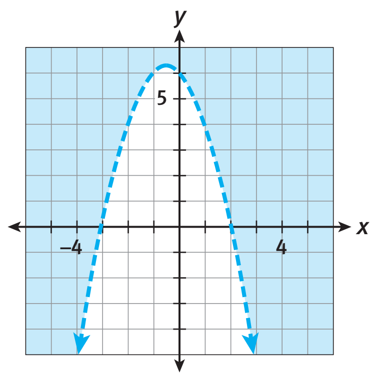
Graph the related quadratic function $y = -x^2 - x + 6$.

If the inequality symbol is $>$ or $<$, use a *dotted curve*.
If the symbol is $\geq$ or $\leq$, then use a *solid curve*.

This curve divides the plane into two regions.

Test (0, 0) in
$y > -x^2 - x + 6$.
$0 > -0^2 - 0 + 6$
$0 > 6$ is a false statement.

Choose a point on the plane, but not on the curve, to test.
(0, 0) is an easy point to use, if possible.



If the statement is true, shade the region that contains the point. If it is false, shade the other region.

The shaded region represents all solutions to the quadratic inequality.

My Notes

TRY THESE C

Solve each inequality by graphing.

a. $y \geq x^2 + 4x - 5$

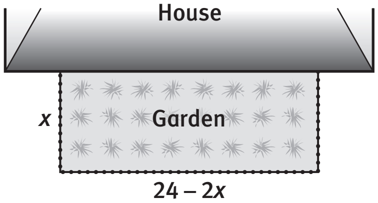
b. $y > 2x^2 - 5x - 12$

c. $y \leq -3x^2 + 8x + 3$

CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

Mr. Picasso would like to create a small vegetable garden adjacent to his house. He has 24 ft of fencing to put around three sides of the garden. The function that describes the area in terms of the width of the garden is $G(x) = -2x^2 + 24x$.



1. Explain why the function $G(x)$ is appropriate for the area of the garden and graph $G(x)$.
2. Write the x- and y-intercepts of $G(x)$ and interpret them in terms of the situation.
3. What is the maximum area for the garden? What are the dimensions of the garden that yield that maximum area?

For each function, identify the vertex, y-intercept, x-intercept(s), and axis of symmetry. Graph the function and identify the maximum or minimum of the function.

4. $f(x) = -x^2 + x + 12$
5. $g(x) = 2x^2 - 11x + 15$

Find all values of x such that $f(x) = 3x^2 - 12x + 16$ equals each value below. How do the number and type of solutions relate to the graph of $f(x)$?

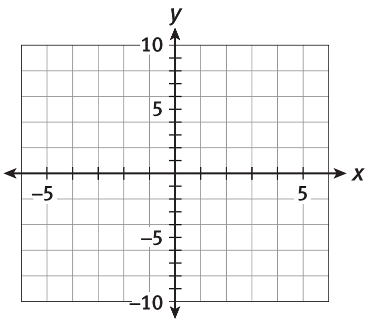
6. $f(x) = 7$
7. $f(x) = 4$
8. $f(x) = 1$
9. Graph the inequality $y \leq x^2 + 4x + 7$.
10. MATHEMATICAL REFLECTION Describe the relationship between solving a quadratic equation and graphing the related quadratic function.

Transformations of $y = x^2$
Parent Parabola

ACTIVITY **3.6**

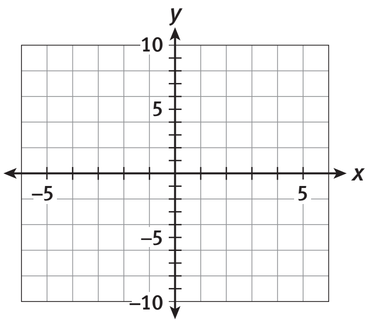
SUGGESTED LEARNING STRATEGIES: Marking the Text, Interactive Word Wall, Create Representations, Quickwrite

1. Graph the **parent quadratic function**, $f(x) = x^2$, on the coordinate grid below. Include the points that have x-values $-2, -1, 0, 1,$ and 2.



The points on the parent function graph that have x-values $-2, -1, 0, 1,$ and 2 are *key points* that can be used when graphing any quadratic function as a **transformation** of the parent quadratic function.

2. Graph $f(x) = x^2$ on the coordinate grid below. Then graph $g(x) = x^2 - 3$ and $h(x) = x^2 + 2$.



3. Identify and describe the transformations of the graph of $f(x) = x^2$ that result in the graphs of $g(x)$ and $h(x)$.

My Notes

MATH TERMS

A *parent function* is the simplest function of a particular type. For example, the parent linear function is $f(x) = x$. The parent absolute-value function is $f(x) = |x|$.

MATH TERMS

A *transformation* of a graph of a parent function is a change in the position, size or shape of the graph.

Transformations of $y = x^2$

Parent Parabola

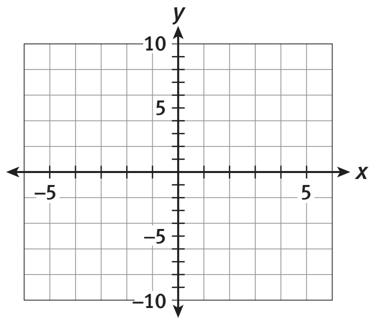
SUGGESTED LEARNING STRATEGIES: Create Representations, Quickwrite, Group Presentation

My Notes

MATH TERMS

Translations are transformations that change the location of a graph but maintain the original shape of a graph. For this reason, they are known as *rigid transformations*.

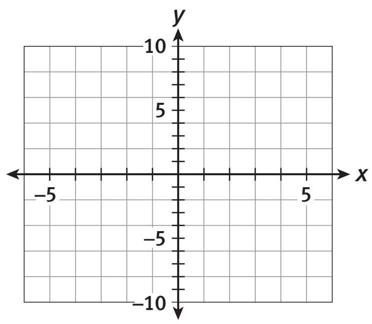
4. Graph $f(x) = x^2$ on the coordinate grid below. Then graph $g(x) = (x - 2)^2$ and $h(x) = (x + 3)^2$.



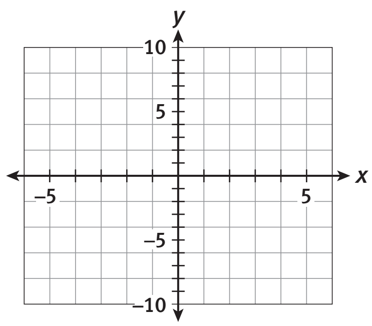
5. Identify and describe the transformations of the graph of $f(x) = x^2$ that result in the graphs of $g(x)$ and $h(x)$.

6. Describe each function as a transformation of $f(x) = x^2$. Then use that information to graph each function on the coordinate grid.

a. $a(x) = (x - 1)^2$



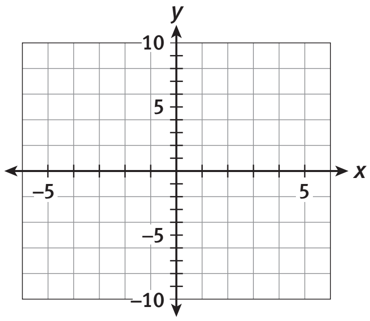
b. $w(x) = x^2 + 4$



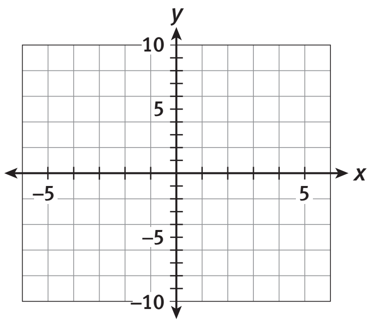
SUGGESTED LEARNING STRATEGIES: Create Representations, Group Presentation, Quickwrite

My Notes

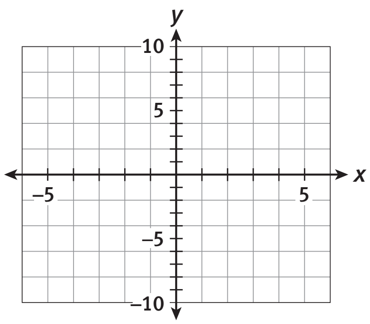
c. $d(x) = (x + 3)^2 - 5$



d. $j(x) = (x - 1)^2 + 2$



7. Graph $f(x) = x^2$ on the coordinate grid below. Then graph $g(x) = 4x^2$ and $h(x) = \frac{1}{4}x^2$.



8. Identify and describe the transformations of the graph of $f(x) = x^2$ that result in the graphs of $g(x)$ and $h(x)$.

MATH TERMS

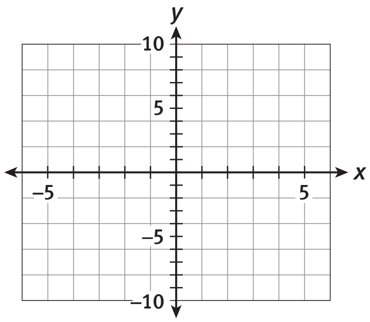
Unlike a rigid transformation, a *vertical stretch* or *vertical shrink* will change the shape of the graph.

SUGGESTED LEARNING STRATEGIES: Create Representations, Group Presentation, Quickwrite

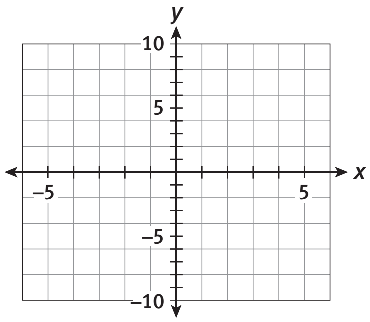
My Notes

9. Describe each function as a transformation of $f(x) = x^2$. Then use the transformations to graph each function.

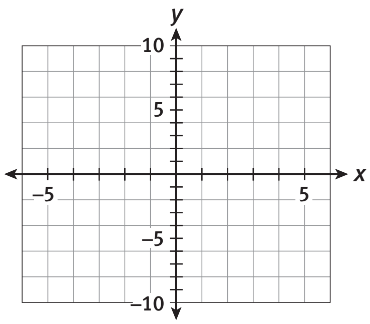
a. $f(x) = 2x^2$



b. $f(x) = \frac{1}{3}x^2$



10. Graph $f(x) = x^2$ on the coordinate grid below. Then graph $g(x) = -x^2$.



11. Identify and describe the transformation of the graph of $f(x) = x^2$ that results in the graph of $g(x)$.

MATH TERMS

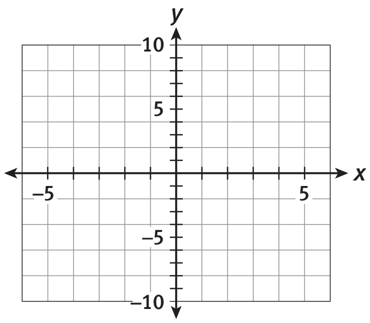
Reflections over axes do not change the shape of the graph, so they are also rigid transformations.

SUGGESTED LEARNING STRATEGIES: Identify a Subtask, Create Representations, Group Presentation

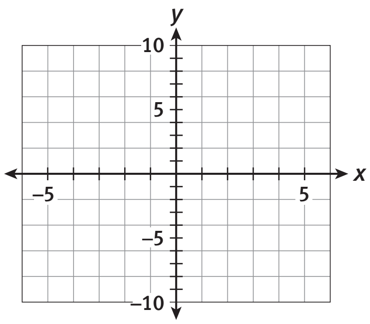
My Notes

12. Describe the transformation(s) of each function from the parent function. Then graph without the use of a graphing calculator.

a. $f(x) = -2x^2$

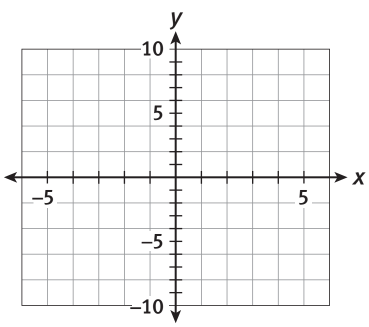


b. $f(x) = -\frac{1}{3}x^2$



13. Multiple transformations can be represented in the same function. Describe the transformations from the parent function. Then graph the function, using your knowledge of transformations only.

a. $f(x) = -4(x + 3)^2 + 2$



MATH TIP

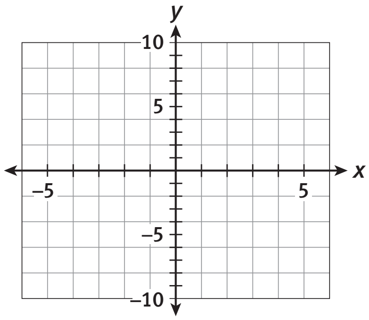
When graphing multiple transformations, you may find it easier to first perform reflections and stretches or shrinks, and then translate.

SUGGESTED LEARNING STRATEGIES: Identify a Subtask, Create Representations, Group Presentation

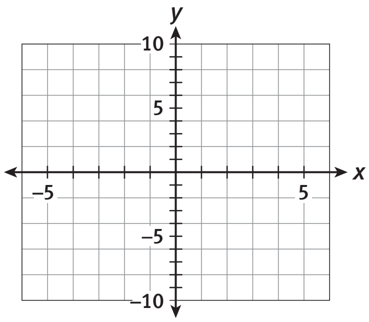
My Notes

13. (*continued*)

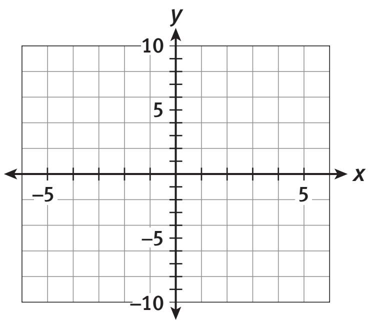
b. $f(x) = 2(x - 4)^2 - 3$



c. $f(x) = 2(x + 1)^2 - 4$



d. $f(x) = -(x - 3)^2 + 5$



SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Marking the Text, Create Representations, Quickwrite, Group Presentation

My Notes

MATH TERMS

The **vertex form** of a quadratic function is $f(x) = a(x - h)^2 + k$, where the vertex of the function is (h, k). Notice that the transformations of $f(x) = x^2$ are apparent when the function is in vertex form.

A quadratic function in standard form, $f(x) = ax^2 + bx + c$, can be changed into **vertex form** by completing the square.

EXAMPLE 1

Write $f(x) = 3x^2 - 12x + 7$ in vertex form.

Step 1: *Factor the leading coefficient from the quadratic and linear terms.* $f(x) = 3(x^2 - 4x) + 7$

Step 2: *Complete the square by taking half the linear coefficient [0.5(−4) = −2], squaring it [(−2)² = 4], and then adding it inside the parentheses.* $f(x) = 3(x^2 - 4x + \underset{+\,4}{\nearrow}\quad) + 7$

Step 3: *To maintain the value of the expression, multiply the leading coefficient [3] by the number added inside the parentheses [4]. Then subtract that product [12].*
$f(x) = 3(x^2 - 4x + 4) - 3(4) + 7$
$f(x) = 3(x^2 - 4x + 4) - 12 + 7$

Step 4: *Write the trinomial inside the parentheses as a perfect square. The function is in vertex form.* $f(x) = 3(x - 2)^2 - 5$

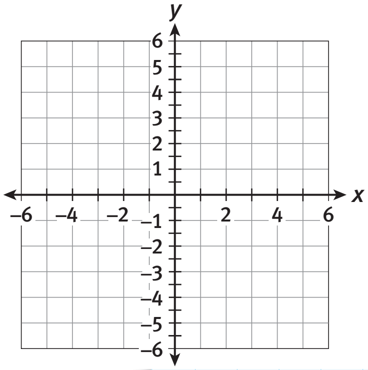
TRY THESE A

Write each quadratic function in vertex form. Write your answers in the My Notes space. Show your work.

a. $f(x) = 5x^2 + 40x - 3$ **b.** $g(x) = -4x^2 - 12x + 1$

14. Write the each function in vertex form. Then describe the transformation(s) from the parent function and graph without the use of a graphing calculator.

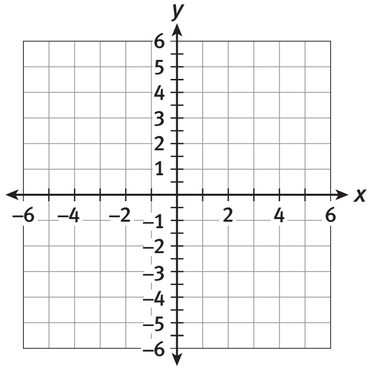
a. $f(x) = -2x^2 + 4x + 3$



SUGGESTED LEARNING STRATEGIES: Create Representations, Quickwrite, Group Presentation

My Notes

b. $g(x) = \frac{1}{2}x^2 + 3x + \frac{3}{2}$



CHECK YOUR UNDERSTANDING

Write your answers on notebook paper or grid paper. Show your work.

1. Identify the transformations to the parent function $f(x) = x^2$ evident in each function.

a. $g(x) = 12x^2$ **b.** $h(x) = (x + 15)^2$

c. $m(x) = x^2 - 47$ **d.** $n(x) = -\frac{1}{100}x^2$

e. $p(x) = 8\pi(x - 40)^2 + 0.5$

2. Write a quadratic function that represents the transformations described below.

a. translated four units to left and 3 units up

b. reflected over the x-axis, stretched vertically by a factor of 5

c. shrunk vertically by a factor of $\frac{1}{4}$, translated 2 units right and ten units down

3. Graph each function using transformations.

a. $f(x) = (x + 3)^2 - 5$

b. $f(x) = 2x^2 - 4$

c. $f(x) = -3(x + 2)^2 + 5$

d. $f(x) = \frac{1}{2}(x - 1)^2 - 3$

4. Write each quadratic function in vertex form.

a. $f(x) = 5x^2 - 20x + 31$

b. $f(x) = -2x^2 - 12x + 5$

5. **MATHEMATICAL REFLECTION** What are the advantages of recognizing a function as a transformation of a parent graph before graphing that function?

Graphs of Quadratic Functions

THE GREEN MONSTER

Embedded Assessment 2
Use after Activity 3.6.

During a Boston Red Sox baseball game at Fenway Park, the opposing team hit a home run over the left field wall. An unhappy Red Sox fan caught the ball and threw it back onto the field. The height of the ball, $h(t)$ in feet, t seconds after the fan threw the baseball, is given by the function $h(t) = -16t^2 + 32t + 48$.

HISTORICAL NOTE
The left field wall in Fenway Park is called The Green Monster, a reference to its unusual height.

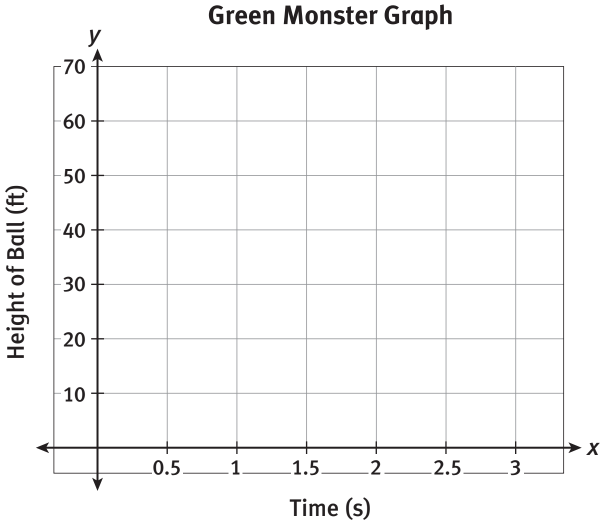
1. Graph the equation on a coordinate grid.

2. Find each measurement value described below. Then tell how each value relates to the graph.

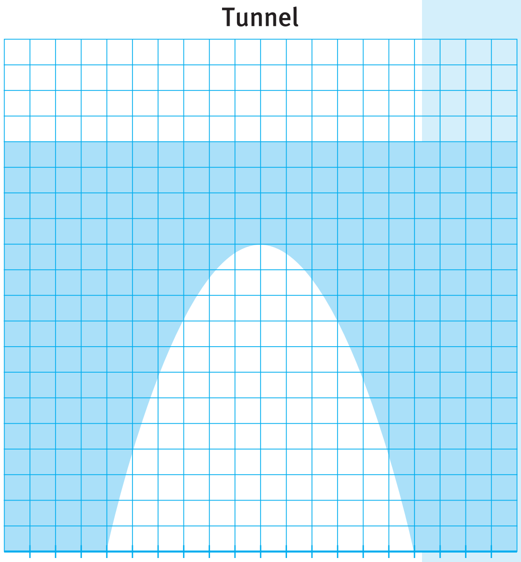
a. At what height was the fan when he threw the ball?

b. What was the maximum height of the ball after the fan threw it?

c. When did the ball hit the field?



Paul's trucking company hauls oversized loads. The trucks travel through a tunnel with an opening in the shape of a parabola that is 12 meters wide and 6 meters high.



3. Find a function $T(x)$ for the shape of the tunnel. Let the road in the diagram on the right represent the x-axis. Draw the y-axis.

4. The height of a loaded truck is between 4.5 and 5 meters, depending on how the cargo is arranged. How wide can the trucks be that haul the oversize loads? Explain how you determined your answer.

5. Consider the quadratic equation $y = 3x^2 + 6x$.

a. Rewrite the equation in the form $y = a(x - h)^2 + k$.

b. Describe the transformations of the parent function $y = x^2$ that are necessary to create the graph of $y = 3x^2 + 6x$.

Graphs of Quadratic Functions

THE GREEN MONSTER

	Exemplary	Proficient	Emerging
Math Knowledge #2a, b, c; 3, 5a	The student: • Correctly states the height when the ball was thrown, the maximum height the ball reached, and the time when the ball hit the field. (2a,b,c) • Draws the y-axis in the correct place. (3) • Rewrites the equation correctly. (5a)	The student: • Correctly states only one of the heights and the gives the correct time. • Finds correct values for only two of *a*, *h*, and *k*.	The student: • Correctly states only one of the two heights and the time. • Draws the *y*-axis in the wrong place. • Finds correct values for only one of *a*, *h*, and *k*.
Problem Solving #3, 4, 5b	The student: • Finds a correct function for the shape of the tunnel. (3) • Determines the correct width for the trucks. (4) • Determines the correct transformations. (5b)	The student: • Finds a partially correct function for the shape of the tunnel. • Determines only two of the transformations correctly.	The student: • Gives an incorrect function for the shape of the tunnel. • Does not determine the correct width for the trucks. • Determines at least one of the transformations.
Representations #1	The student graphs the equation correctly. (1)	The student draws a partially correct graph.	The student draws an incorrect graph.
Communication #2a, b, c; 4, 5b	The student: • States correctly how the three values relate to the graph. (2a, b, c) • Gives a complete explanation of how the widths were determined. (4) • Describes the transformations correctly. (5b)	The student: • States correctly how two of the values relate to the graph. • Gives an incomplete explanation of how the widths were determined. • Describes only two of the transformations.	The student: • States correctly how at least one of the values relates to the graph. • Gives no explanation of how the widths were determined. • Describes fewer than two transformations.

ACTIVITY 3.1

A rectangle has perimeter 40 cm. Use this information for 1–5.

1. Write the dimensions and areas of three rectangles that fit this description.
2. Let the length of one side be x. Then write a function $A(x)$ that represents that area of the rectangle.
3. Graph the function $A(x)$ on a graphing calculator. Then sketch the graph on grid paper, labeling the axes and using appropriate scale.
4. An area of 96 cm^2 is possible. Use $A(x)$ to demonstrate this fact algebraically and graphically.
5. An area of 120 cm^2 is not possible. Use $A(x)$ to demonstrate this fact algebraically and graphically.

ACTIVITY 3.2

6. Factor each quadratic expression.
   a. $2x^2 - 3x - 27$ **b.** $4x^2 - 121$
   c. $6x^2 + 11x - 10$ **d.** $3x^2 + 7x + 4$
   e. $5x^2 - 42x - 27$ **f.** $4x^2 - 4x - 35$
   g. $36x^2 - 100$ **h.** $12x^2 + 60x + 75$
7. Solve each quadratic equation by factoring.
   a. $2x^2 - 5x - 12 = 0$
   b. $3x^2 + 7x = -2$
   c. $4x^2 - 20x + 25 = 0$
   d. $27x^2 - 12 = 0$
   e. $6x^2 - 4 = 5x$
   f. $30x^2 + 5x - 10 = 0$
8. For each set of solutions, write a quadratic equation in standard form.
   a. $x = 5, x = -8$ **b.** $x = \frac{2}{3}, x = 4$
   c. $x = -\frac{7}{5}, x = \frac{1}{2}$ **d.** $x = 6$, a double root
9. Solve each quadratic inequality.
   a. $x^2 - 3x - 4 \leq 0$ **b.** $3x^2 - 7x - 6 > 0$

ACTIVITY 3.3

10. Write each expression in terms of i.
    a. $\sqrt{-64}$
    b. $\sqrt{-31}$
    c. $-7 + \sqrt{-12}$
    d. $5 - \sqrt{-50}$
11. These quadratic equations have solutions that are complex. Use the quadratic formula to solve each equation.
    a. $x^2 + 5x + 9 = 0$
    b. $2x^2 - 4x + 5 = 0$
12. Evaluate each expression.
    a. $(5 - 6i) + (-3 + 9i)$
    b. $(2 + 5i) + (-5 + 3i)$
    c. $(9 - 2i) - (1 + 6i)$
    d. $(-5 + 4i) - \left(\frac{7}{3} + \frac{1}{6}i\right)$
13. Evaluate each multiplication expression. Write each product in the form $a + bi$.
    a. $(1 + 4i)(5 - 2i)$
    b. $(-2 + 3i)(3 - 2i)$
    c. $(7 + 24i)(7 - 24i)$
14. Evaluate each division expression. Write each quotient in the form $a + bi$.
    a. $\frac{3 + 2i}{5 - 2i}$
    b. $\frac{-1 + i}{5 - 2i}$
    c. $\frac{10 - 2i}{5i}$
15. Draw the complex plane on grid paper. Then graph each complex number on the plane.
    a. $-4i$
    b. $6 + 2i$
    c. $-3 - 4i$
    d. $3 - 5i$
    e. $-2 + 5i$

Practice

ACTIVITY 3.4

Solve each equation, using any method that you choose. Classify the solutions as rational, irrational, or complex conjugates.

16. $(x + 3)^2 - 25 = 0$

17. $2x^2 - 9x + 5 = 0$

18. $4x^2 - 33 = 0$

19. $3x^2 + x - 14 = 0$

20. $3x^2 + 25 = 0$

21. $2(x - 7)^2 + 27 = 0$

For each equation, find the value of the discriminant and describe the nature of the solutions.

22. $2x^2 + 3x + 4 = 0$

23. $9x^2 + 30x + 25 = 0$

24. $6x^2 - 7x - 20 = 0$

25. $5x^2 + 12x - 7 = 0$

26. Write a formula that represents the solutions of a quadratic equation of the form $mx^2 + nx + p = 0$. Explain how you arrived at your formula.

ACTIVITY 3.5

Mr. Gonzales would like to create a playground in his back yard. He has 20 ft of fencing to enclose the play area. He determines that the function that describes the area in terms of the width of the playground is $f(x) = -x^2 + 10x$.

27. Explain why the function $f(x)$ is appropriate for the area of the playground. Then graph $f(x)$.

28. Write the x- and y-intercepts of $f(x)$ and interpret them in terms of the playground problem.

29. What is the maximum area for the playground? What are the dimensions of the playground with the maximum area?

For each function, identify the vertex, the y-intercept, the x-intercept(s), and axis of symmetry. Graph the function and identify the maximum or minimum of the function.

30. $f(x) = -x^2 + 4x + 5$

31. $f(x) = 2x^2 - 12x + 13$

32. $f(x) = -3x^2 + 12x - 9$

For each equation, find the value of the discriminant and describe the nature of the solutions. Then graph the related function and find the x-intercepts.

33. $2x^2 - 5x - 3 = 0$

34. $3x^2 + x + 2 = 0$

35. $4x^2 + 4x + 1 = 0$

36. $2x^2 + 6x + 3 = 0$

Graph each quadratic inequality.

37. $y < x^2 + 7x + 10$

38. $y \geq 2x^2 + 4x - 1$

39. $y > x^2 - 6x + 9$

ACTIVITY 3.6

For each function, identify all transformations of the parent function $f(x) = x^2$. Then graph the function.

40. $f(x) = (x + 2)^2 + 3$

41. $f(x) = (x - 3)^2 - 4$

42. $f(x) = \frac{1}{2}(x - 3)^2$

43. $f(x) = -2(x + 3)^2 + 1$

44. $f(x) = -3(x + 2)^2 - 5$

Write a quadratic function that represents each transformation of the parent function $f(x) = x^2$.

45. translate 4 units left and 8 units down

46. reflect over the x-axis, translate 5 units to the right

47. shrink vertically by a factor of $\frac{1}{3}$, translate 6 units up

48. stretch vertically by a factor of $\frac{3}{2}$, reflect over the x-axis, translate 1 unit right and 7 units up.

Write each quadratic function in vertex form.

49. $f(x) = x^2 - 14x + 36$

50. $f(x) = -4x^2 + 24x + 15$

51. $f(x) = 9x^2 + 18x - 14$

Reflection

An important aspect of growing as a learner is to take the time to reflect on your learning. It is important to think about where you started, what you have accomplished, what helped you learn, and how you will apply your new knowledge in the future. Use notebook paper to record your thinking on the following topics and to identify evidence of your learning.

Essential Questions

1. Review the mathematical concepts and your work in this unit before you write thoughtful responses to the questions below. Support your responses with specific examples from concepts and activities in the unit.

1 Why are some solutions of quadratic equations meaningful in real-life applications while others are not?

2 How do graphic, symbolic, and numeric methods of solving quadratic equations compare to one another?

Academic Vocabulary

2. Look at the following academic vocabulary words:

- completing the square
- complex conjugate
- complex number
- discriminant
- imaginary number
- root (zero)

Choose three words and explain your understanding of each word and why each is important in your study of math.

Self-Evaluation

3. Look through the activities and Embedded Assessments in this unit. Use a table similar to the one below to list three major concepts in this unit and to rate your understanding of each.

Unit Concepts	Is Your Understanding Strong (S) or Weak (W)?
Concept 1	
Concept 2	
Concept 3	

a. What will you do to address each weakness?

b. What strategies or class activities were particularly helpful in learning the concepts you identified as strengths? Give examples to explain.

4. How do the concepts you learned in this unit relate to other math concepts and to the use of mathematics in the real world?

Math Standards Review

Unit 3

1. Which quadratic equation below has $i\sqrt{2}$ **and** $-\,i\sqrt{2}$ as solutions?

A. $x^2 + \sqrt{2} = 0$

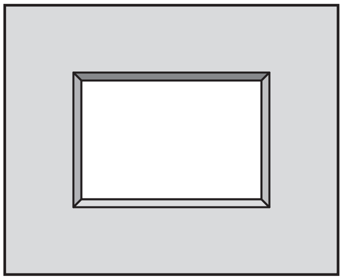
B. $x^2 + 2 = 0$

C. $x^2 + 4 = 0$

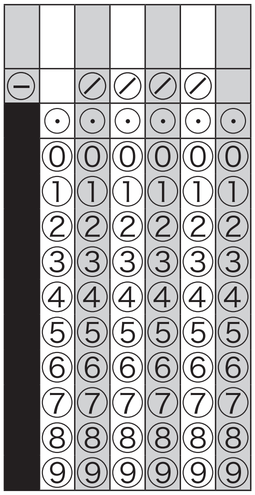
D. $x^2 - 4 = 0$

1. Ⓐ Ⓑ Ⓒ Ⓓ

2. The area of a rectangular frame is 48 square feet. The length of the frame is 2 feet more than the width of the frame. What is the width of the frame in feet?

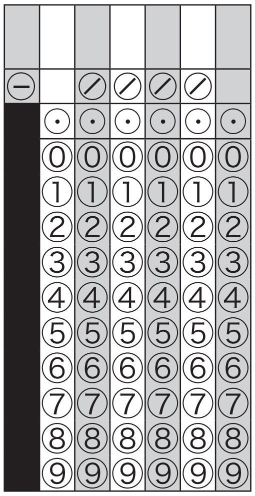


2.



3. The graph of the function $f(x) = (x - 4)^2 + 9$ can be obtained by translating what parent function $f(x) = x^2$ ___?___ units to the right?

3.



Read
Solve
Explain

4. Given the parabola: $y = x^2 + 2x - 2$

Find the axis of symmetry, vertex, domain, range, and x-intercepts.

Answer and Explain