# QUADRATICS

*Candidates should able to:*

- carry out the process of completing the square for a quadratic polynomial $ax^2 + bx + c$, and use this form, e.g. to locate the vertex of the graph of $y = ax^2 + bx + c$ or to sketch the graph;
- find the discriminant of a quadratic polynomial $ax^2 + bx + c$ and use the discriminant, e.g. to determine the number of real roots of the equation $ax^2 + bx + c = 0$;
- solve quadratic equations, and linear and quadratic inequalities, in one unknown;
- solve by substitution a pair of simultaneous equations of which one is linear and one is quadratic;
- recognise and solve equations in *x* which are quadratic in some function of *x*, e.g. $x^4 + 5x^2 + 4 = 0$.

## 1. Quadratic Equations

Quadratic Equations $ax^2 + bx + c = 0, a \neq 0$, can be solved by factorization, using the quadratic formula or by completing the square.

A. By Factorisation

Zero factor property or zero factor principle

$$If\ PQ = 0 \Rightarrow either\ P = 0\ and/or\ Q = 0$$

This method will ONLY work if the product of the two variables equal to zero. For instance, consider this equation $xy = 8$. We could have $x = 2\ and\ y = 4\ or\ x = 1\ and\ y = 8, etc$. So better be extra careful when use this zero factor principle.

Take note:

When solving $x^2 - 4x = 0$, DO NOT cancel one of the $x$, you will lose one solution. Instead you factorise the common term(s) out first.
Please try e.g. 2 in other note for more practice.

B. By Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Before using the quadratic formula, make sure you ARRANGE the equation into $ax^2 + bx + c = 0$.
Please refer to e.g. 3 in the other note for more information on Quadratic Formula.

Extra knowledge: Giving your answers in exact values (or surd form)
When solving quadratic equations using quadratic formula, you can leave your answer either in decimals, fractions or exact values (surd form).
Please see e.g. 3 of the other note.

C. By Completing the Square Method
Refer to Completing square method of the other notes.

Another way to solve a quadratic equation is to rewrite it into the completed square form, then take square root on both sides.

e.g 1: Solve the following equations using the completing the square method.

(a) $x^2 + 6x - 1 = 0$

$$x^2 + 6x = 1$$
$$x^2 + 6x + (\ ) = 1 + (\ )$$
$$x^2 + 6x + (\frac{6}{2})^2 = 1 + (\frac{6}{2})^2$$
$$x^2 + 6x + 3^2 = 1 + 3^2$$
$$(x + 3)^2 = 10$$
$$\sqrt{(x + 3)^2} = \pm\sqrt{10}$$
$$(x + 3) = \pm\sqrt{10}$$
$$x = -3 \pm \sqrt{10}$$

(b) $2x^2 + 8x + 4 = 0$

If the coefficient of $x^2$is more than 1, you need to change it to 1 by dividing both sides by the coefficient of $x^2$ before applying the completing the square method.

$$\frac{2x^2 + 8x + 4}{2} = \frac{0}{2}$$
$$x^2 + \frac{8}{2}x + \frac{4}{2} = 0$$
$$x^2 + 4x = -2$$
$$x^2 + 4x + \left(\frac{4}{2}\right)^2 = -2 + \left(\frac{4}{2}\right)^2$$
$$(x + 2)^2 = 2$$
$$x + 2 = \pm\sqrt{2}$$
$$x = -2 \pm \sqrt{2}$$

(c) $x^2 - x = 3$
(d) $x^2 + 3x = 10$
(e) $x^2 - x - 30 = 0$
(f) $2x^2 = 6 - 10x$
(g) $3x^2 = 12x - 6$
(h) $2x^2 - 6x + 1 = 0$
(i) $3x^2 - 6x - 1 = 0$

## 2. Review of Simultaneous Equations

Equations that are allowed to solve together at the same time are called simultaneous equations. Normally, the solutions for simultaneous equations come in a pair of values (either in term of whole numbers, fractions, or decimal numbers).

Simultaneous equations of two linear equations can be solved using two methods, namely by Elimination method and Substitution method.

A. Elimination Method

In the Elimination method for solving simultaneous equations, two linear equations are simplified by adding them or subtracting them. This eliminates one of the variables so that the other variable can be found.

e.g. 2: Use the Elimination Method to solve each of the following pairs of simultaneous equations.

(a) $x - y = -1$
$2x - y = 0$

$$2(x - y = -1)$$

$$\begin{array}{r} 2x - 2y = -2 \\ -\quad 2x - y = 0 \\ \hline 0 - y = -2 \\ \therefore y = 2 \end{array}$$

Substitute $y = 2$ into either one of the original equations. We will get

$$x - (2) = -1$$
$$x = -1 + 2$$
$$\therefore x = 1$$

(b) $3x - y = 9,\ 4x - y = -14$
(c) $5x + 4y = 4,\ 3x + 4y = 8$
(d) $3x - y = -2,\ x - 3y = 10$

B. Substitution Method

Another method of solving simultaneous equations is by using substitution method. This method involves setting a variable of one of the equations as the subject, then plugging that value into another equation and solve for the answer of that variable.

e.g. 3: Use the Substitution Method to solve each of the following pairs of simultaneous equations.

(a) $2x + y = 1$
$x + 2y = 8$

$$2x + y = 1$$
$$y = 1 - 2x$$

Substitute $y = 1 - 2x$ into $x + 2y = 8$.
You will get

$$x + 2(1 - 2x) = 8$$
$$x + 2 - 4x = 8$$
$$-3x = 8 - 2$$
$$\frac{-3x}{-3} = \frac{6}{-3}$$
$$\therefore x = -2$$

Substitute $x = 2$ into the newly formed equation $y = 1 - 2x$. You will get

$$y = 1 - 2(-2)$$
$$\therefore y = 5$$

(b) $2x + y = 4,\ x - 5y = 2$
(c) $y - 5x = 7,\ 2y + x = 3$
(d) $2x - y = 4,\ x + 3y = 18$
(e) $x = 11 - 2y,\ x = 3y - 4$

### 2.1 Simultaneous Equations – 1 linear and 1 quadratic equation

In early section, you have solved simultaneous equations where both equations are *linear*. In this section, we extend this to solving simultaneous equations where one equation is linear and the other is quadratic. This will normally give you a quadratic equation to solve.

e.g. 4: Solve the following pairs of simultaneous equations.

(a) $y = x^2 - 1$
$y = 5 - x$

Substitute $y = 5 - x$ into $y = x^2 - 1$, you will get

$$5 - x = x^2 - 1$$
$$0 = x^2 - 1 - 5 + x$$
$$0 = x^2 + x - 6$$

From here, you can use any of the three methods you have learned to solve quadratic equations.

$$0 = (x + 3)(x - 2)$$
$$x + 3 = 0 \text{ or } x - 2 = 0$$
$$\therefore x = -3 \text{ or } x = 2$$

Then, substitute $x = -3$ and $x = 2$ into the linear equation. You will get

$$When\ x = -3, \quad y = 5 - (-3) = 8$$
$$When\ x = 2, \quad y = 5 - 2 = 3$$

The solutions are $x = -3, y = 8$ and $x = 2, y = 3$.

(b) $y = 3x^2 - 4,\ \ y = 2x + 3$
(c) $x + 2y = 2,\ \ x^2 + 8y = 8$

**Homework:**
Solve the following simultaneous equations.

(a) $y = x^2 + 5x,\ \ y = 2x + 10$
(b) $y = 2x^2 + x - 3,\ \ y = 3x + 1$
(c) $y = 2x^2 + 12x - 4,\ \ y = 2x - 1$
(d) $4x + y = 19,\ \ 5x^2 + 8y = 10$

## 3. Equations which reduce to quadratic equations

Sometimes, you will come across equations which are not quadratic, but which can be changed into quadratic equations, usually by making the right substitution.

e.g. 5: Solve the following equations.

(a) $t^4 - 13t^2 + 36 = 0$.
This is called a quadratic equation because it has a $t^4$ term. Then, you change the $t^4 = (t^2)^2$, you will get $(t^2)^2 - 13t^2 + 36 = 0$.

Now, if you let $x = t^2$, the original equation becomes $x^2 - 13x + 36 = 0$, which is a quadratic equation in $x$.

$$x^2 - 13x + 36 = 0$$
$$(x-4)(x-9) = 0$$
$$\therefore x = 4 \text{ or } x = 9$$

Now recall that $x = t^2$, so $t^2 = 4$ *or* $t^2 = 9$. This gives $t = \pm 2$ *or* $t = \pm 3$.

(b) $\sqrt{x} = 6 - x$
There are two methods in solving equations involving square root. Namely, either by **letting $y = \sqrt{x}$** or **squaring both sides of the equation**.

Method 1: Letting $y = \sqrt{x}$
By letting $y = \sqrt{x}$, the original equation becomes

$$y = 6 - y^2$$
$$y^2 + y - 6 = 0$$
$$(y+3)(y-2) = 0$$
$$\therefore y = 2 \text{ or } y = -3$$

Now, recall that $y = \sqrt{x}$, you will get

$$2 = \sqrt{x} \text{ or } -3 = \sqrt{x}$$
$$x = 4 \text{ or } x = 9$$

Method 2: Squaring both sides of the equation

$$\left(\sqrt{x}\right)^2 = (6-x)^2$$
$$x = 36 - 12x + x^2$$
$$0 = 36 - 13x + x^2$$

Check your answers with the original equation.
When $x = 4$,
$$\sqrt{4} = 4 - 2$$
$$2 = 2$$
So $x = 4, y = 2$ satisfy the equation.

When $x = 9$,
$$\sqrt{9} = 6 - 9$$
$$3 \neq -3$$
Since $x = 9, y = -3$ does not satisfy the equation, it is rejected.

(c) $x^4 - 25x^2 + 144 = 0$
(d) $x + 7\sqrt{x} + 10 = 0$
(e) $x^6 - 7x^3 - 8 = 0$



**Homework:**
1. Do Exercise 4C Question 3(a), 3(b), 3(f) Page 61.
2. Solve the following equations

(a) $x + 7\sqrt{x} + 6 = 0$
(b) $\sqrt{8x+1} = x + 2$

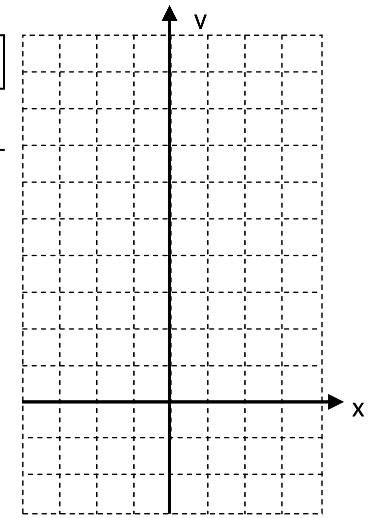
## 4. Quadratic Functions

### $f(x) = ax^2 + bx + c$

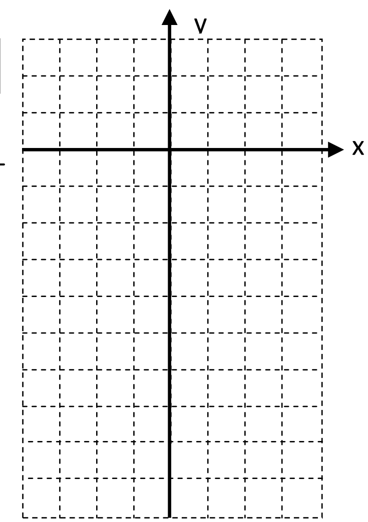
The graph of a quadratic function $f(x) = ax^2 + bx + c, a \neq 0$, has a characteristic shape. It is a curve called a parabola whose line of symmetry is parallel to the y-axis. Note that the highest power of an unknown of a quadratic function is 2.

Sketch the graph of $y = x^2$ and $y = -x^2$.

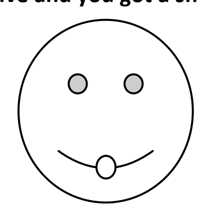
$y = x^2$



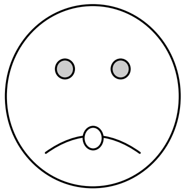

$y = -x^2$




If $a > 0$, the parabola has a minimum value.
**You feel positive and you got a smiling face.**



If $a < 0$, the parabola has a maximum value.
**You feel negative so you got a sad face.**



## 4.1 Characteristics of Quadratic functions

The general shape of a parabola is the shape of a "pointy" letter "u", or a slightly rounded letter, "v".

Looking at the graphs of $y = x^2$ and $y = -x^2$, we can see that the origin is the lowest and highest point of each graph respectively. The minimum point or maximum point of any quadratic functions graph is called the **vertex** or **the turning point**.

If you draw a vertical line through the vertex, it will split the parabola in half so that either side of the vertical line is symmetric with respect to the other side. This vertical line is called the **line of symmetry** or **axis of symmetry**. The line of symmetry is always a vertical line of the form $x = n$, where $n$ is a real number. For the graphs drawn above, their line of symmetry is the vertical line $x = 0$.

## 4.2 Maximum and Minimum values of Quadratic functions

For any quadratic equation $y = ax^2 + bx + c$, the **minimum value** is the y-coordinate of a vertex when $\boldsymbol{a} > 0$. Meanwhile the **maximum value** is the y-coordinate of the vertex when $\boldsymbol{a} < 0$.

To calculate minimum or maximum value, first we need to calculate the x-coordinate of the max/min using this formula $= \frac{-b}{2a}$. Then, we will substitute it back to the original equation to get our minimum or maximum value.

e.g. 6: Find the minimum or maximum value of the following quadratic functions.

(a) $y = x^2 - 4x + 5$

$$a = 1, b = -4, c = 5$$

Since a is positive, the curve has a minimum turning point. This implies that it has a minimum value of quadratic function. Substitute a and b into the formula, you will get

$$x = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2$$

When $x = 2$,

$$y = (2)^2 - 4(2) + 5 = 1$$

So the minimum turning point = (2, 1).
Minimum value = 1.

(b) $y = -x^2 + 2x - 2$
(c) $y = 2x^2 + 5x + 3$
(d) $y = -3x^2 - 6x + 9$
(e) $y = 5 - x - 2x^2$

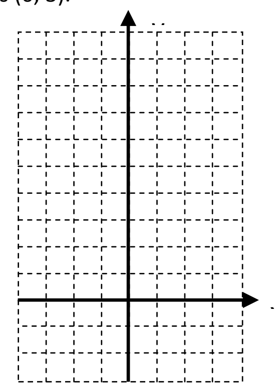
## 4.3 Domain and Range of Quadratic functions

**Domain** is what you can put **IN** a function while **Range** is what you can get **OUT** of a function. In other words, your possible x-values are your domain, and your possible y-values are your range.

$$Domain = x's \;\; \& \;\; Range = y's$$

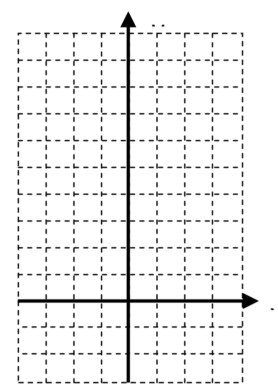
For instance, given 4 points with coordinates of $\{(3, -4), (-2, 3), (1, 2), (-4, -4)\}$
D: { }
R: { }

Another example, say given the quadratic function, $y = x^2 + 3$. It has a minimum point as $a > 0$. The vertex is (0, 3).



Domain: Everything in the function, "all read numbers". $\{x \mid x \in \mathbb{R}\}$
Range: Everything above the vertex. $\{y \mid y \geq 3\}$.

Just remember that in any quadratic functions, Domain is always all real numbers.
For e.g. 6 (a), we found out that the minimum turning point = (2, 1) and its minimum value is 1.



The range is all values above the minimum value.
So, its range = $\{y \mid y \quad 1\}$

**Practice:** Sketch and find the range of all remaining quadratic functions of example 6.

| Function | Vertex | Nature | Range | Symmetry | Sketch |
|---|---|---|---|---|---|
| $y = x^2 + 1$ | | | | | |
| $y = -x^2 + 1$ | | | | | |
| $y = x^2 - 4x + 4$ | | | | | |
| $y = x^2 - 4x + 5$ | | | | | |
| $y = -x^2 - 4x + 3$ | | | | | |
| $y = -2 + 2x - x^2$ | | | | | |
| $y = -4x^2 + 12x - 9$ | | | | | |

## 4.4 Express $ax^2 + bx + c$ to $a(x-h)^2 + k$

Recall that the method of completing the square is used to solve quadratic equations $ax^2 + bx + c = 0$. This method can also be used to find the maximum or minimum value of any quadratic functions. (You can use method in Section 4.2 to find the maximum and minimum value as well).

In order to use completing the square method, you need to express standard quadratic function $y = ax^2 + bx + c$ to $y = a(x-h)^2 + k$.

e.g. 7: Express the following in completed square form and hence, find their vertex.

(a) $x^2 + 2x + 2$

First, we let $y = x^2 + 2x + 2$.

$$y - 2 = x^2 + 2x$$
$$y - 2 + \left(\frac{2}{2}\right)^2 = x^2 + 2x + \left(\frac{2}{2}\right)^2$$
$$y - 1 = \left(x + \frac{2}{2}\right)^2$$
$$y = (x+1)^2 + 1$$

To find the vertex, we put

$$x + 1 = 0$$
$$x = -1$$
$$y = 1$$

$$\therefore Vertex\ or\ Minimum\ turning\ point = (-1, 1)$$

(b) $2x^2 + 5x - 3$

Let $y = 2x^2 + 5x - 3$

$$y + 3 = 2x^2 + 5x$$
$$y + 3 = 2\left(x^2 + \frac{5}{2}x\right)$$
$$y + 3 + 2\left(\frac{\frac{5}{2}}{2}\right)^2 = 2\left[x^2 + \frac{5}{2}x + \left(\frac{\frac{5}{2}}{2}\right)^2\right]$$
$$y + \frac{49}{8} = 2\left[x + \left(\frac{\frac{5}{2}}{2}\right)\right]^2$$
$$y = 2\left(x + 1\frac{1}{4}\right)^2 - 6\frac{1}{8}$$
$$\therefore Vertex = \left(-1\frac{1}{4}, -6\frac{1}{8}\right)$$

**Homework:**

Do the following Past Year Exam question.

Question 4, 19, 23, 29

## 4.5 Sketch of the Quadratic Curve

To sketch a graph, we need to know:

- the shape of the curve maximum ∩ or minimum ∪
- the position of the turning point or the vertex of the curve.
- the line of symmetry that divides the curve into two equal parts
- point at which it cuts the y-axis (this is given by $y = 0$)
- point(s) at which it cuts the x-axis or the positions of the roots.

e.g. 8: Sketch the graph of the following quadratic functions.

(a) $y = x^2 - 6x + 8$

The shape of the curve is ∪ as $a = 1$ (positive).

To find the position of the vertex, we convert the quadratic function into a completed square form.

$$y - 8 = x^2 - 6x$$
$$y - 8 + \left(\frac{6}{2}\right)^2 = x^2 - 6x + \left(\frac{6}{2}\right)^2$$
$$y - 8 + 9 = \left(x - \frac{6}{2}\right)^2$$
$$y + 1 = (x - 3)^2$$
$$y = (x - 3)^2 - 1$$

So the turning point = $(3, -1)$.

The line of symmetry ⇒ $x = 3$.

To find the point at which it cuts the y-axis, we substitute $x = 0$ into $y = (x - 3)^2 - 1$. We will get

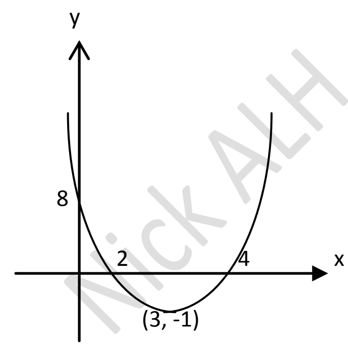
$$y = (0 - 3)^2 - 1$$
$$y = 9 - 1 = 8$$

So the coordinate of the point that cut the y-axis = $(0, 8)$.

To find the point at which it cuts the y-axis, we substitute $y = 0$ into $y = (x - 3)^2 - 1$. We will get

$$0 = (x - 3)^2 - 1$$
$$1 = (x - 3)^2$$
$$\pm\sqrt{1} = x - 3$$
$$x = 3 \pm \sqrt{1}$$
$$\therefore x = 3 + 1 = 4 \;\; or \; x = 3 - 1 = 2$$

So the coordinate of the points that cut the x-axis = $(4, 0)$ and $(2, 0)$.





(b) $y = 2x^2 + 4x - 16$

The shape of the curve is ∪ as $a = 2$ (positive).

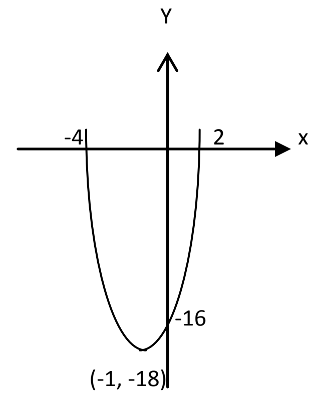
$$y = 2x^2 + 4x - 16$$
$$y + 16 = 2(x^2 + 2x)$$
$$y + 16 + 2\left(\frac{2}{2}\right)^2 = 2\left[x^2 + 2x + \left(\frac{2}{2}\right)^2\right]$$
$$y + 18 = 2(x + 1)^2$$
$$y = 2(x + 1)^2 - 18$$

Vertex = $(-1, -18)$

Cut the y-axis, $x = 0, y = -16 \Longrightarrow (0, -16)$

Cut the x-axis, $y = 0, x = -1 \pm 3 = -4 \; or \; 2$

$$\Longrightarrow (-4, 0) and \; (2, 0)$$




In the case when $y = ax^2 + bx + c$ can be easily factorised to $(x - \alpha)(x - \beta)$, the graph of $y = (x - \alpha)(x - \beta)$ can be sketched using the x-intercepts $\alpha$ and $\beta$. This is called sketching of quadratic functions in factorised form.

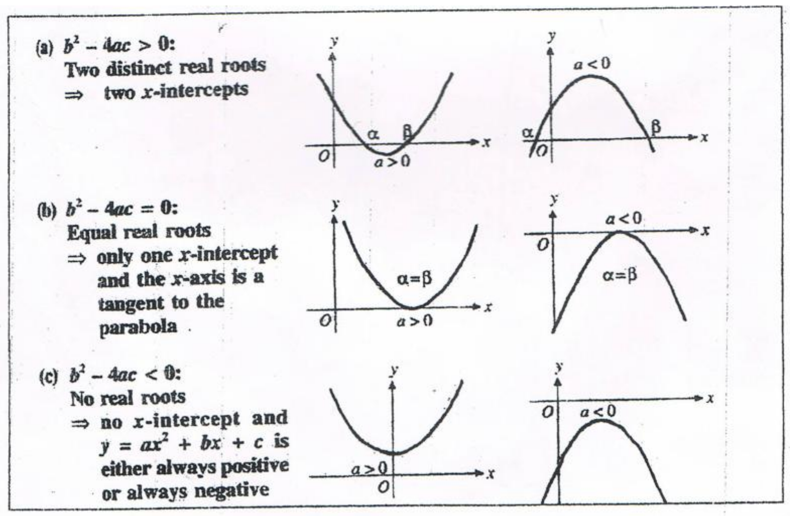
(c) $y = 2x(x - 2)$

(d) $y = x^2 - 5x - 6$

## 5. The discriminant and Roots

For any quadratic equation $ax^2 + bx + c = 0$, the roots (or solutions) are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The expression $b^2 - 4ac$ is called the **discriminant, D,** where $\boldsymbol{D = b^2 - 4ac}$. It allows us to discriminate among the possible types of roots. It also tells us the position of the curve relative to the x-axis.

There are three possible cases for the solution of the equation $ax^2 + bx + c = 0$. Namely, $b^2 - 4ac > 0, b^2 - 4ac = 0 \; and \; b^2 - 4ac < 0$.




Addition information to take note:

If $b^2 - 4ac \geq 0$, the roots are real (i.e. either 2 real distinct root or a repeated root).
If $b^2 - 4ac$ is a perfect square, then the roots are integers or fractions (i.e. rational).
If $b^2 - 4ac$ is a surd, then the roots are irrational.

e.g. 9: What can you deduce from the values of the discriminants of the quadratics in the following equations?

(a) $2x^2 - 3x - 4 = 0$

$$b^2 - 4ac = (-3)^2 - 4(2)(-4) = 41$$
$$\therefore b^2 - 4ac > 0$$

So, the roots are real, distinct and irrational.

(b) $2x^2 - 3x - 5 = 0$
(c) $2x^2 - 4x + 5 = 0$
(d) $2x^2 - 4x + 2 = 0$

e.g. 10: Find the number of roots in each of the following:
(a) $x^2 + 11x - 2 = 0$
(b) $7x^2 - 11x - 5 = 0$
(c) $5x^2 - 3x + 1 = 0$

e.g. 11: Prove that $(k-2)x^2 + 2x - k = 0$ has real roots for any value of k.

$$a = (k-2), b = 2, c = -k$$

$For\ real\ roots, b^2 - 4ac \geq 0.$

$$b^2 - 4ac \geq 0$$
$$(2)^2 - 4(k-2)(-k) \geq 0$$
$$4 - 4(-k^2 + 2k) \geq 0$$
$$4 + 4k^2 - 8k \geq 0$$
$$4k^2 - 8k + 4 \geq 0$$
$$(2k-2)^2 \geq 0$$

Since $(2k-2)^2$ is always positive, therefore the roots are real for any value of k (shown).

e.g. 12: Find the range of values (sometimes, set of values) of k for which the following equations have **no real roots**:

(a) $3kx^2 + 6x - 1 = 0$ $\quad Ans: k < -3$
(b) $2x^2 - 4x + k + 3 = 0$ $\quad Ans: k > -1$
(c) $(2k-3)x^2 + 4x = 5$ $\quad Ans: k < \frac{11}{10}$
(d) $2x^2 - 5x = kx^2 + x - 3$ $\quad Ans: k < -1$

For e.g. 12(a),

$$a = 3k, \qquad b = 6, \qquad c = -1$$

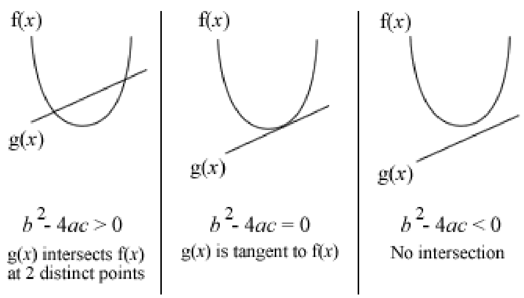
$For\ no\ real\ roots, b^2 - 4ac < 0.$

$$(6)^2 - 4(3k)(-1) < 0$$
$$36 + 12k < 0$$
$$12k < -36$$
$$k < -3$$

**Homework:**
1) Do Exercise 4B Qn 2 (a) – (e) and 5 (a) – (e). Page 58
2) If $ax^2 - 8x + 2 = 0$ has a repeated root, find the value of $a$. $\quad [Ans: a = 8]$
3) The equation $(p+3)x^2 + 2px + p = 1$ has real roots. Find the range of values of $p$. $\quad [Ans: p \leq \frac{3}{2}]$

## 6. Intersection of line and curve leading to quadratic equation

When a line and a curve are drawn on the same graph, the line may intersect, touch or not intersect the curve. To find out, we first solve the equation of the line and the curve simultaneously and then find out what type of roots the resulting equation has.




e.g. 13: Find the range of values of *m* for which the line $y = 2x + 1$ intersects the curve $y = x^2 + 6x + m$.

In order to know which values of *m*, solve the equations simultaneously. We will get

$$2x + 1 = x^2 + 6x + m$$
$$0 = x^2 + 6x - 2x + m - 1$$
$$x^2 + 4x + (m - 1) = 0$$

$$\therefore a = 1, \quad b = 4, \quad c = m - 1$$

Since the line intersects the curve, we will use $b^2 - 4ac > 0$. We will get

$$(4)^2 - 4(1)(m - 1) > 0$$
$$16 - 4(m - 1) > 0$$
$$16 - 4m + 4 > 0$$
$$20 > 4m$$
$$\frac{20}{4} > m$$
$$\therefore m < 5$$

e.g. 14: Find the value of *k* for which the line $y = x + k$ is a tangent to the curve $y^2 = 4x + 8$. $[Ans: k = 3]$

**Homework:**

1) Find the range of values of *m* for which the line $y = mx + 12$ intersects the curve $x^2 + xy = 12$ at two distinct points. $[Ans: m > -4]$
2) Find the range of values of *t* for which the line $y = x + t$ does not intersect with the curve $y^2 = 4x$. $[Ans: t > 1]$
3) Find the values of *r* for which the line $y = r(1 - 2x)$ is a tangent to the curve $y = x^2 + 2$. $[Ans: r = -2 \text{ or } 1]$
4) Past Year Exam, Question 16 and Question 18.

## 7. Review of Inequalities

$>$ *is Greater than*
$<$ *is less than*
$\geq$ *is greater than or equal to*
$\leq$ *is less than or equal to*



The symbols $>$ and $<$ are called strict inequalities and the symbols $\geq$ and $\leq$ are called weak inequalities.

$a > b$ and $b < a$ are equivalent expressions; and $a \geq b$ and $b \leq a$ are also equivalent expressions.

- Adding or subtracting the same number on both sides of an inequality will not change the inequality.
  $i.e.,$ *For any number* $c$,
  *if* $a > b$, *then* $a \pm c > b \pm c$.
- Multiplying or dividing both sides of an inequality by a positive number will not change the inequality.
  $i.e.,$ *For any positive number* $c$,
  *if* $a > b$, *then* $a \times c > b \times c$ and $\frac{a}{c} > \frac{b}{c}$
- Multiplying or dividing both sides of an inequality by a negative number will reverse the inequality.
  $i.e.,$ *For any negative number* $c$,
  *if* $a > b$, *then* $a \times c < b \times c$ and $\frac{a}{c} < \frac{b}{c}$

e.g. 15: Solve the following inequality

(a) $3x + 10 > 10x - 11$

$$3x - 10x > -11 - 10$$
$$-7x > -21$$
$$x < \frac{-21}{-7}$$
$$x < 3$$

(b) $\frac{1}{3}(4x + 3) - 3(2x - 4) \geq 20$ $\quad [Ans: x \leq -\frac{3}{2}]$

(c) $-5 < 2x + 3 \leq 7$

| | |
|---|---|
| $-5 < 2x + 3$<br>$-8 < 2x$<br>$-4 < x$ | $2x + 3 \leq 7$<br>$2x \leq 4$<br>$x \leq 2$ |

$$\therefore -4 < x \leq 2$$

### 7.1 Modulus Sign

The modulus (or absolute value) $|x|$ of a real number *a* is the numerical value of *a* without regard to its sign. So for example, $|3| = 3$, $|-7| = 7$, $|-5| = 5$ etc.

If $b > 0$, two useful properties concerning inequalities are:

- $|a| \leq b \Leftrightarrow -b \leq a \leq b$
- $|a| \geq b \Leftrightarrow a \leq -b$ *or* $b \leq a$

e.g. 16: Solve the following:

(a) $|2x - 3| \leq 7$ $\quad [Ans: -2 \leq x \leq 5]$
(b) $|3x + 1| > 8$ $\quad [Ans: x > \frac{7}{3}, \; x < -3]$

**Homework:** Ex 5A Pg 68 Qn 4 (g),(h), 5(h),(i), 6(a),(g)

## 7.2 Quadratic Inequalities

An inequality with a quadratic expression in one variable on one side and zero on the other is called quadratic inequality in one variable.

There are various methods of solving a quadratic inequality.

e.g. 17: Solve the following inequalities:

(a) $4x + x^2 < 21$

First, you rearrange the inequality into $ax^2 + bx + c < 0$ format.

$$4x + x^2 < 21$$
$$x^2 + 4x - 21 < 0$$
$$(x+7)(x-3) < 0$$

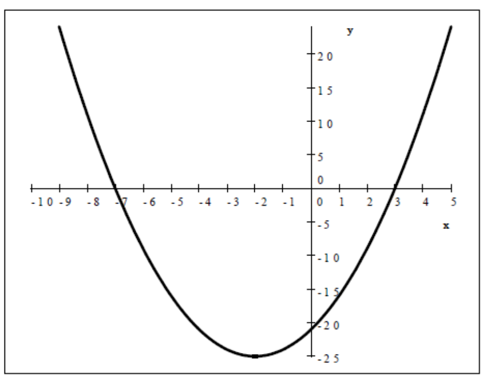
The left-hand side is a quadratic expression. If you solve it, you will get

$$x = -7, \qquad x = 3$$

This two points are the x-intercepts of the L.H.S. quadratic expression.

**Method 1:**

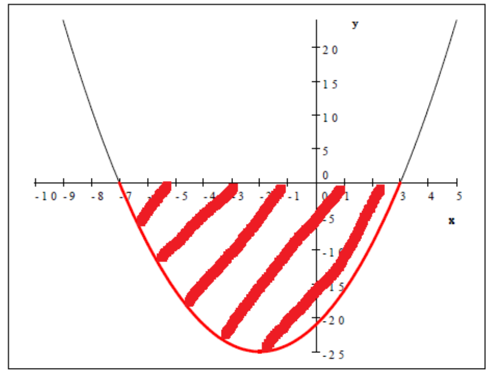
If we plot the graph of this expression, we get this picture



The graph of $y = (x+7)(x-3)$

Now, consider the inequality to be solved:

$(x+7)(x-3) < 0$. We need to find the x-values for which this inequality is negative. In other words, the x-coordinates of all points on the parabola that lie below the x-axis.



The x-coordinate of these points range from -7 to 3. Thus the solution is $-7 < x < 3$.

**Method 2:**

By making use of a number line.

In the number line, a small empty circle denote that the point $x = -7$ and $x = 3$ is not included. If they are included, then a black dot will be used.

Since the inequality is $< 0$, we will choose the negative region of the number line. So the solution is $-7 < x < 3$.

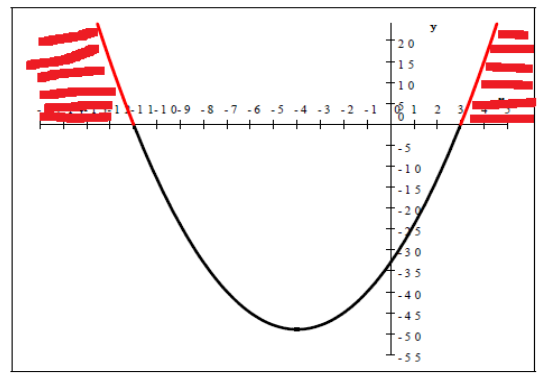
(b) $33 - x^2 \leq 8x$

$$33 - x^2 \leq 8x$$
$$0 \leq x^2 + 8x - 33$$

Say for instance, you are struggling with factorization. You can use quadratic formula to solve the expression on the Right Hand Side.

$$x = \frac{-8 \pm \sqrt{(8)^2 - 4(1)(-33)}}{2(1)}$$
$$x = \frac{-8 \pm \sqrt{196}}{2}$$
$$x = \frac{-8+14}{2} = 3 \text{ or } x = \frac{-8-14}{2} = -11$$

So your two x-intercept will be $x = 3$ and $x = -11$.



So your solution is $x \leq -11$ *or* $x \geq 3$.

(c) $(x+1)(5-x) \leq 0$ $\quad [Ans: x \leq -1, x \geq 5]$

e.g. 18: For what range of values of $p$ if the equation $y = (p+1)x^2 + 4px + 9$ does not intersect the x-axis?

$[Ans: -\frac{4}{3} < x < 3]$

**Homework:** Ex 5B Pg 71 Qn 4 (a),(d),(g), 2(a),(d),(g) 4(a),(d),(k)

**Self Practice:** Past Year Exam Questions – Will discuss during next week afternoon block 2-3pm.