

QUANTUM EFFECTS OF SCALAR FIELDS IN BLACK HOLE AND  
COSMOLOGICAL SPACETIMES

BY

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## Abstract

Using a background field calculation, an analysis is presented of the effects of free quantum scalar fields upon the geometry of a cosmological spacetime containing a big rip singularity. The spacetime investigated is that of a spatially flat FRW metric in which the scale factor increases as a power-law with a divergence at some future time  $t = t_{\text{rip}}$ . Results of analytical and numerical computations of the expectation value of the stress tensor for massive and massless minimally and conformally coupled quantized scalar fields are presented. It is shown that, for physically realistic spacetimes, and for reasonable choices of the state of the quantum field, there is no evidence that quantum effects become important prior to the point where the scalar curvature reaches the Planck scale.

An explicit computation of the noise kernel for the conformally invariant scalar field in a thermal state is shown in Minkowski and Schwarzschild spacetimes for non-null separations of the spacetime points. For Minkowski spacetime, this result is exact; for Schwarzschild, this is an approximate result valid at short distances. Additionally, an exact expression for the noise kernel for the conformal vacuum valid for an arbitrary separation of points is given for all conformally flat spacetimes. This expression is evaluated in the static de Sitter coordinates, and an analysis of the behavior in the region near the cosmological horizon is given. A comparison is made between the noise kernel in Schwarzschild spacetime and the noise kernel in static de Sitter spacetime in the near horizon region. Within the context of static de Sitter space, an investigation is made into the validity of the quasi-local expansion near the cosmological horizon.

## Chapter 1: Introduction

Quantum field theory and Einstein's theory of general relativity are two of the most successful theories of the 20th century. Simple calculations show that, on short distance scales or when the curvature of the spacetime geometry is large, quantum effects are large and should play an important role in the evolution of spacetime. To completely characterize these effects, one would need a full quantum mechanical description of gravity. Unfortunately, naive efforts at combining the two theories fail, and while more sophisticated modern approaches show promise we do not yet know which, if any, is correct [1]. However, a rough hierarchy of low energy approximations has been developed which should be largely independent of the details of the full quantum gravity theory. These approximations have been used successfully to study effects such as black hole evaporation and particle production in the early Universe [2].

At the bottom of the hierarchy is the simplest approach, quantum field theory in curved space [2]. Within a given background spacetime, one can construct a quantum field theory by solving the mode equations of a classical field theory and promoting the coefficients of the mode functions to quantum operators as is done in flat space [2]. In general, one finds two types of quantum effects: particle production, and vacuum polarization effects such as the Casimir effect [3]. In most cases, there is no unique way to disentangle these effects, but both particle production and vacuum polarization are encapsulated in the stress-energy tensor of the quantum field [2].

One question which may be addressed using a background field calculation is whether the inclusion of quantum effects would lead to a significant alteration of the spacetime geometry. To answer this question, one can compute the stress-energy



tensor<sup>1</sup>  $\langle \hat{T}_{ab} \rangle$  for a quantum field and compare it with the classical stress-energy tensor  $T_{ab}^{\text{cl}}$  which appears in Einstein's field equations,

$$G_{ab} = 8\pi T_{ab}^{\text{cl}} . \quad (1.1)$$

If the quantum stress-energy tensor is small compared to the classical stress-energy tensor then quantum effects should not significantly alter the spacetime geometry.<sup>2</sup>

A more quantitative analysis of the effects on the spacetime geometry requires solving the Einstein equations in the presence of a quantum field. This is the subject of the next level in the hierarchy, semiclassical gravity, which approximates the gravitational effects induced by a quantum field by including the expectation value of the stress-energy tensor for that field as an additional source,

$$G_{ab} = 8\pi \left( T_{ab}^{\text{cl}} + \langle \hat{T}_{ab} \rangle \right) . \quad (1.2)$$

This additional term may be thought of as a one-loop correction to the classical Einstein equations [2]. In principle, solutions to the semiclassical backreaction equations consist of spacetimes for which Eq. (1.2) is self-consistent to the order of approximation to which one is working. It should be noted that  $\langle \hat{T}_{ab} \rangle$  includes contributions from all quantum fields, including gravitons. However, if we assume that the matter contribution to stress-energy tensor is due to the presence of a large number  $N$  of identical fields then, to leading order in an expansion in powers of  $1/N$ , the contribution from the graviton field can be neglected [4].

One limitation of semiclassical gravity is that it ignores the effects of fluctuations of the quantum stress-energy tensor away from its mean value. This means that we

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<sup>1</sup>For quantum fields, what we mean by the term “stress-energy tensor” is the expectation value of the stress-energy tensor operator.

<sup>2</sup>Of course, in some cases, such as black hole evaporation, there can be significant changes to the geometry even if quantum effects are small if one waits long enough.

expect the semiclassical approximation not to hold in cases where these fluctuations are in some sense large. Furthermore, statistical physics tells us that even when small, fluctuations can be important drivers behind the dynamics of a system; for instance, phase changes are often fluctuation driven even when the energy of a given fluctuation is small in comparison with the total energy of the system [5].

Attempts to account for the effects of these fluctuations gave rise to the third level in the hierarchy of approximation schemes. A few approaches at this level have been developed, all of which involve two-point correlation functions of the stress-energy tensor [6–11]. One approach is stochastic gravity, [12] a theory which seeks to modify the semiclassical equations by incorporating a technique used in statistical physics. Using a derivation of the semiclassical Einstein equations based on an effective action approach, Calzetta and Hu [9] showed that there are certain imaginary terms at quadratic order in the effective action which don't contribute to the semiclassical equations. Extending work by Feynman and Vernon [13], they showed that these terms (collectively called the noise kernel) could be treated classically by an ansatz in which they were replaced by a Gaussian stochastic tensor field,  $\xi_{ab}$ , with mean and covariance given by,

$$\begin{aligned}\langle \xi_{ab}(x) \rangle_s &= 0 \\ \langle \xi_{ab}(x) \xi_{c'd'}(x') \rangle_s &= N_{abc'd'}(x, x') \\ N_{abc'd'}(x, x') &= \langle \hat{T}_{ab}(x) \hat{T}_{c'd'}(x') \rangle - \langle \hat{T}_{ab}(x) \rangle \langle \hat{T}_{c'd'}(x') \rangle .\end{aligned}\tag{1.3}$$

Here  $N_{abc'd'}(x, x')$  is the noise kernel, and  $\langle \dots \rangle_s$  means the stochastic expectation value. This led to the derivation by Martin and Verdaguer [10] of the Einstein-Langevin equation for stochastic gravity,

$$G_{ab} = 8\pi(\langle \hat{T}_{ab} \rangle + \xi_{ab}) .\tag{1.4}$$

The first step in characterizing the effects of fluctuations within the framework

of stochastic gravity is to compute the noise kernel for a quantum field in a given background geometry. In principle, this background geometry should be a solution of the semiclassical Einstein equations given in Eq. (1.2). Often this is not practical, since in many cases (such as for a Schwarzschild black hole) it can be difficult to find a solution to the semiclassical equations. However, for spacetimes in which the quantum effects are expected to be small, the solution to the classical Einstein's equations can be treated as a lowest order approximation to a solution of the semiclassical equations. Thus, it is still possible to use such spacetimes as the background geometry appearing in the Einstein-Langevin equation for stochastic gravity.

Each of these methods have been used to study quantum effects in a variety of spacetimes. Here, two of these methods are used to investigate quantum effects in cosmological and black hole spacetimes.

First, the background field method is used to study the effects of free quantum fields on spacetimes containing big rip singularities. A big rip singularity is a type of future curvature singularity present in certain cosmological spacetimes in which the Universe expands by an infinite amount in a finite amount of proper time [15]. This expansion is driven by a form of dark energy known as phantom energy whose energy density increases as the Universe expands and diverges at the singularity [16]. Near the final time, the curvature becomes large and eventually dominates over all other forces and unbinds all bound systems [15]. However, under conditions of such rapid expansion, it is expected that quantum effects such as particle creation and vacuum polarization will become important and might serve to moderate or remove the singularity. By calculating the energy density of a quantum field at times close to the final time, the question of when or if the energy density of the quantum field becomes comparable to that of the phantom fluid can be determined.

Second, since any detailed investigation into quantum effects using stochastic

gravity requires knowledge of the noise kernel, explicit expressions for the noise kernel are computed for several spacetimes of interest. The spacetimes investigated are Minkowski space, which is the flat spacetime of special relativity, Schwarzschild spacetime, which is a vacuum solution of Einstein's equations containing a static, non-spinning black hole, and de Sitter space, which is a homogeneous and isotropic cosmological spacetime undergoing exponential expansion.

In Chapter 2, the results of a background field calculation are presented for several types of quantum scalar fields in a class of spacetimes containing big rip singularities. Using the renormalization method of adiabatic regularization, the renormalized energy density for each quantum field considered was computed via numerical and/or analytical methods and was compared with the classical energy density of the phantom field at times near the final time. It is shown that, for physically reasonable spacetimes<sup>3</sup>, the energy density of each quantum field considered remains small in comparison with the classical energy density until well after the time at which the scalar curvature reaches the Planck scale (at which point the semiclassical approximation breaks down). At the end of Chapter 2, a brief summary is given of a comment by Haro and Amoros [17] published in response to the paper reproduced in this chapter. The full text of our reply is included.

In Chapter 3, a calculation of the noise kernel using the general expression derived by Phillips and Hu [citephillips01](#) is presented for the conformally invariant scalar field for a thermal state in Minkowski and Schwarzschild spacetimes when the points are space-like or time-like separated. While the calculation of the noise kernel in Minkowski space is exact, the calculation of the noise kernel in Schwarzschild spacetime involves an approximation scheme based on a method [Page \[18\]](#) found for the stress-energy tensor of the conformally invariant scalar field in this spacetime.

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<sup>3</sup>By physically reasonable spacetimes, we mean spacetimes consistent with observations of the cosmic microwave background radiation. This is discussed further in Chapter 2.

In Chapter 4, a computation of the noise kernel for cosmological spacetimes is presented. The result of an exact computation of the noise kernel for the conformally invariant scalar field in the conformal vacuum which is valid for any conformally flat spacetime is shown. In contrast to the computations shown in Chapter 3, this expression is valid for an arbitrary separation of the points, including null separation. The noise kernel is computed explicitly in the static de Sitter coordinates, and the behavior of the noise kernel near the cosmological horizon is investigated. In addition, the expression for the noise kernel is expanded in terms of the coordinate separation using a scheme similar to the approximation scheme used for Schwarzschild spacetime in Chapter 3. This expansion is compared with the exact expression for the noise kernel in the static de Sitter coordinates and the range of validity of the expansion is discussed.

In the Appendix, additional information is given regarding the calculations in Chapter 2 and 3 which was not included in the journal articles reproduced in those chapters.

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## Chapter 2: Effects of quantized scalar fields in cosmological spacetimes with big rip singularities

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# Effects of quantized scalar fields in cosmological spacetimes with big rip singularities

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## Abstract

Effects of quantized free scalar fields in cosmological spacetimes with Big Rip singularities are investigated. The energy densities for these fields are computed at late times when the expansion is very rapid. For the massless minimally coupled field it is shown that an attractor state exists in the sense that, for a large class of states, the energy density of the field asymptotically approaches the energy density it would have if it was in the attractor state. Results of numerical computations of the energy density for the massless minimally coupled field and for massive fields with minimal and conformal couplings to the scalar curvature are presented. For the massive fields the energy density is seen to always asymptotically approach that of the corresponding massless field. The question of whether the energy densities of quantized fields can be large enough for backreaction effects to remove the Big Rip singularity is addressed.

## 1 Introduction

Surveys of type Ia supernovae and detailed mappings of the cosmic microwave background provide strong evidence that the universe is accelerating [1]. To explain this acceleration within the framework of Einstein's theory of General Relativity requires

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the existence of some form of “dark energy” which has positive energy density and negative pressure [2].

One common model for dark energy [2] is to treat it as a pervasive, homogenous perfect fluid with equation of state  $p = w\rho$ . Cosmic acceleration demands that  $w < -1/3$ , and observations from the Wilkinson Microwave Anisotropy Probe in conjunction with supernova surveys and baryon acoustic oscillations measurements place the current value at  $w = -0.999^{+0.057}_{-0.056}$  [3]. Although this is consistent with the effective equation of state for a cosmological constant,  $w = -1$ , we cannot rule out the possibility that our Universe contains “phantom energy”, for which  $w < -1$ .

If  $w$  is a constant and  $w < -1$  then general relativity predicts that as the universe expands the phantom energy density increases with the result that in a finite amount of proper time the phantom energy density will become infinite and the universe will expand by an infinite amount. All bound objects, from clusters of galaxies to atomic nuclei, will become unbound as the Universe approaches this future singularity, aptly called the “big rip” [4].

A simple model of a spacetime with a big rip singularity can be obtained by considering a spatially flat Robertson-Walker spacetime, with metric

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \quad (1.1)$$

At late times the phantom energy density is much larger than the energy density of classical matter and radiation and the solution to the classical Einstein equations is

$$a(t) \approx a_1(t_r - t)^{-\sigma}, \quad (1.2)$$

with

$$\sigma = -2/(3 + 3w) > 0. \quad (1.3)$$

Here  $a_1$  is a constant and  $t_r$  is the time that the big rip singularity occurs. The phantom energy density is

$$\rho_{ph} = \frac{3}{8\pi}\sigma^2(t_r - t)^{-2}. \quad (1.4)$$

Note that both the scale factor and the phantom energy density become infinite at  $t = t_r$ .

In addition to the singularity in the model described above, other classes of models containing somewhat milder phantomlike future singularities have been identified. Barrow [5] constructed a class of models called “sudden singularities” in which  $w$  is allowed to vary as  $w \propto (t_r - t)^{\alpha-1}$  for  $0 < \alpha < 1$ . In these models, the pressure and scalar curvature diverge at time  $t_r$  but the energy density and scale factor remain finite. Other types of singular behavior were found in [6–8]. Nojiri, Odintsov and Tsujikawa [9] came up with a general classification scheme for future singularities. The strongest singularities are classified as type I and big rip singularities fall into this class. The sudden singularities are examples of type II singularities. Two other classes, type III and type IV were also identified. For type III singularities both the energy density  $\rho$  and the pressure  $p$  diverge at the time  $t_r$  but the scale factor  $a$  remains finite. For type IV singularities, the scale factor remains finite, the energy density  $\rho$  and the pressure  $p$  go to zero at the time  $t_r$ , but divergences in higher derivatives of  $H = \dot{a}/a$  occur. Other classification schemes for cosmological singularities have been given in [10–12].

At times close to  $t_r$  in each of the above cases it is possible that quantum effects could become large and that the backreaction of such effects could moderate or remove the final singularity. One way to investigate whether this would occur is to compute the energy density for the quantum fields in the background geometry of a spacetime with a final singularity. Then a comparison can be made between the phantom energy

density and the energy density of the quantized fields. A second way is to solve the semiclassical backreaction equations to directly see what effects the quantum fields have.

Nojiri and Odintsov [7,13] studied the backreaction of conformally invariant scalar fields in the cases of sudden singularities and big rip singularities and found that quantum effects could delay, weaken or possibly remove the singularity at late times. In Ref. [9] Nojiri, Odintsov and Tsujikawa used a model for the dark energy with an adjustable equation of state to find examples of spacetimes with type I, II, and III singularities. They then solved the semiclassical backreaction equations and found that the singularities were usually either moderated or removed by quantum effects.

Calderón and Hiscock [14] investigated the effects of conformally invariant scalar, spinor, and vector fields on big rip singularities by computing the stress-energy of the quantized fields in spacetimes with constant values of  $w$ . Their results depend on the value of  $w$  and on the values of the renormalization parameters for the fields. For values of  $w$  that are realistic for our universe they found that quantum effects serve to strengthen the singularity. Calderón [15] made a similar computation in spacetimes with sudden singularities and found that whether the singularity is strengthened or weakened depends on the sign of one of the renormalization parameters.

Barrow, Batista, Fabris and Houndjo [16] considered models with sudden singularities when a massless, minimally coupled scalar field is present. They found that in the limit  $t \rightarrow t_r$ , the energy density of this field remains small in comparison with the phantom energy density. Thus quantum effects are never important in this case. Batista, Fabris and Houndjo [17] investigated the effects of particle production when a massless minimally coupled scalar field is present in spacetimes where  $w$  is a constant. To do so they used a state for which Bunch and Davies [18] had previously computed the stress-energy tensor. They found that the energy density of the created

particles never dominates over the phantom energy density.

Pavlov [19] computed both the number density of created particles and the stress-energy tensor for a conformally coupled massive scalar field for the case in which  $w = -5/3$ . It was found that backreaction effects are not important for masses much smaller than the Planck mass and times which are early enough that the time until the big rip occurs is greater than the Planck time.

In this paper we compute the energy densities of both massless and massive scalar fields with conformal and minimal couplings to the scalar curvature in spacetimes with big rip singularities in which the parameter  $w$  is a constant. While our calculations are for scalar fields, it is worth noting that both massive and massless conformally coupled scalar fields can be used to model spin 1/2 and spin 1 fields, and in homogeneous and isotropic spacetimes the massless minimally coupled scalar field can serve as a model for gravitons [20, 21].

For conformally invariant fields the natural choice of vacuum state in homogeneous and isotropic spacetimes is the conformal vacuum [22]. For all other fields there is usually no natural choice. However, it is possible to define a class of states called adiabatic vacuum states which, when the universe is expanding slowly, can serve as reasonable vacuum states [22]. They can be obtained using a WKB approximation for the mode functions, and they are specified by the order of the WKB approximation. It has been shown that the renormalized stress-energy tensor for a quantum field is always finite if a fourth order or higher adiabatic vacuum state is chosen.<sup>1</sup>

Here we compute the renormalized energy densities of the quantum fields in fourth order or higher adiabatic states and investigate their behavior as the universe expands. One focus is on the differences that occur for the same field in different states. We

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<sup>1</sup>In this paper we generalize the definition of an n'th order adiabatic vacuum state to include all states whose high momentum modes are specified by an n'th order WKB approximation but whose other modes can be specified in any way.

find in all cases considered that the asymptotic behavior of the energy density is always the same for fields with the same coupling to the scalar curvature, regardless of whether they are massless or massive and regardless of what states the fields are in. Fields with minimal coupling to the scalar curvature have a different asymptotic behavior than those with conformal coupling.

We also address the question of whether and under what conditions the energy density of the quantized fields becomes comparable to the phantom energy density. We find that for fields in realistic states for which the energy density of the quantized fields is small compared to that of the phantom energy density at early times, and for spacetimes with realistic values of  $w$ , there is no evidence that quantum effects become large enough to significantly affect the expansion of the spacetime until the spacetime curvature is of the order of the Planck scale or larger, at which point the semiclassical approximation breaks down.

In Section 2 the quantization of a scalar field in a spatially flat Robertson-Walker spacetime is reviewed along with a method of constructing adiabatic states. In Section 3, the energy density for massless scalar fields with conformal and minimal coupling to the scalar curvature is discussed and a comparison is made with the phantom energy density. For the massless minimally coupled scalar field a proof is given that one particular state serves as an attractor state in the sense that for a large class of states, the energy density of the field asymptotically approaches the energy density it would have if it was in the attractor state. In Section 4 numerical calculations of the energy density for massive scalar fields with conformal and minimal coupling to the scalar curvature are discussed. A comparison is made with both the phantom energy density and the energy density of the corresponding massless scalar field. Our main results are summarized and discussed in Section 5. Throughout units are used such that  $\hbar = c = G = 1$  and our sign conventions are those of Misner, Thorne, and

Wheeler [23].

## 2 Quantization in Spatially Flat Robertson Walker Spacetimes

We consider a scalar field  $\phi$  obeying the wave equation

$$(\square - m^2 - \xi R)\phi = 0, \quad (2.1)$$

where  $m$  is the mass of the field and  $\xi$  is its coupling to the scalar curvature  $R$ . It is convenient for our calculations to use the conformal time variable

$$\eta = \int_{t_r}^t \frac{d\bar{t}}{a(\bar{t})}. \quad (2.2)$$

We expand the field in terms of modes in the usual way [22]

$$\phi(\mathbf{x}, \eta) = \frac{1}{a(\eta)} \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \left[ a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \psi_{\mathbf{k}}(\eta) + a_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}} \psi_{\mathbf{k}}^*(\eta) \right], \quad (2.3)$$

with the creation and annihilation operators satisfying the commutation relations

$$\begin{aligned} [a_{\mathbf{k}}, a_{\mathbf{k}'}] &= [a_{\mathbf{k}}^\dagger, a_{\mathbf{k}'}^\dagger] = 0 \\ [a_{\mathbf{k}}, a_{\mathbf{k}'}^\dagger] &= \delta(\mathbf{k} - \mathbf{k}'). \end{aligned} \quad (2.4)$$

The time dependent part of the mode function satisfies the equation

$$\psi_{\mathbf{k}}'' + \left[ k^2 + m^2 a^2 + 6 \left( \xi - \frac{1}{6} \right) \frac{a''}{a} \right] \psi_{\mathbf{k}} = 0, \quad (2.5)$$

with a constraint given by the Wronskian condition

$$\psi_{\mathbf{k}} \psi_{\mathbf{k}}^{*'} - \psi_{\mathbf{k}}^* \psi_{\mathbf{k}}' = i. \quad (2.6)$$

Throughout primes denote derivatives with respect to the conformal time  $\eta$ .

We restrict our attention to states for which the stress-energy tensor is homogeneous and isotropic. Thus the stress-energy tensor is uniquely specified by the energy density  $\langle\rho\rangle_r = -\langle T\rangle$  and the trace  $\langle T\rangle$ . The unrenormalized energy density is [24]

$$\begin{aligned}\langle\rho\rangle_u &= \frac{1}{4\pi^2 a^4} \int_0^\infty dk k^2 \left\{ |\psi'_k|^2 + \left[ k^2 + m^2 a^2 - 6 \left( \xi - \frac{1}{6} \right) \frac{a'^2}{a^2} \right] |\psi_k|^2 \right. \\ &\quad \left. + 6 \left( \xi - \frac{1}{6} \right) \frac{a'}{a} (\psi_k \psi_k^{*'} + \psi_k^* \psi_k') \right\}.\end{aligned}\quad (2.7)$$

Renormalization is to be accomplished through the use of adiabatic regularization [25–28]. Following the prescription given in [29] the renormalized energy density is

$$\langle\rho\rangle_r = \langle\rho\rangle_u - \langle\rho\rangle_d + \langle\rho\rangle_{an} \quad (2.8a)$$

with<sup>2</sup>

$$\begin{aligned}\langle\rho\rangle_d &= \frac{1}{4\pi^2 a^4} \int_0^\infty dk k^2 \left\{ k + \frac{1}{k} \left[ \frac{m^2 a^2}{2} - \left( \xi - \frac{1}{6} \right) \frac{3 a'^2}{a^2} \right] \right\} \\ &\quad + \frac{1}{4\pi^2 a^4} \int_\lambda^\infty dk k^2 \left\{ \frac{1}{k^3} \left[ -\frac{m^4 a^4}{8} - \left( \xi - \frac{1}{6} \right) \frac{3 m^2 a'^2}{2} \right. \right. \\ &\quad \left. \left. + \left( \xi - \frac{1}{6} \right)^2 {}^{(1)}H_0^0 \frac{a^4}{4} \right] \right\},\end{aligned}\quad (2.8b)$$

$$\begin{aligned}\langle\rho\rangle_{an} &= \frac{1}{2880\pi^2} \left( -\frac{1}{6} {}^{(1)}H_0^0 + {}^{(3)}H_0^0 \right) + \frac{m^2}{288\pi^2} G_0^0 \\ &\quad - \frac{m^4}{64\pi^2} \left[ \frac{1}{2} + \log \left( \frac{\mu^2 a^2}{4\lambda^2} \right) \right] + \left( \xi - \frac{1}{6} \right) \left\{ \frac{{}^{(1)}H_0^0}{288\pi^2} \right. \\ &\quad \left. + \frac{m^2}{16\pi^2} G_0^0 \left[ 3 + \log \left( \frac{\mu^2 a^2}{4\lambda^2} \right) \right] \right\} \\ &\quad + \left( \xi - \frac{1}{6} \right)^2 \left\{ \frac{{}^{(1)}H_0^0}{32\pi^2} \left[ 2 + \log \left( \frac{\mu^2 a^2}{4\lambda^2} \right) \right] - \frac{9}{4\pi^2} \frac{a'^2 a''}{a^7} \right\},\end{aligned}\quad (2.8c)$$

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<sup>2</sup>Note that Eq.(9a) of Ref. [29] has a misprint. The term on the third line which is proportional to  $m^2$  should be multiplied by a factor of  $(\xi - 1/6)$ .



and

$${}^{(1)}H_0^0 = -\frac{36a''a'}{a^6} + \frac{72a''a'^2}{a^7} + \frac{18a''^2}{a^6} \quad (2.8d)$$

$${}^{(3)}H_0^0 = \frac{3a'^4}{a^8} \quad (2.8e)$$

$$G_0^0 = -\frac{3a'^2}{a^4}. \quad (2.8f)$$

Note that the value of  $\langle \rho \rangle_r$  is independent of the arbitrary cutoff  $\lambda$ . For a massive field  $\mu = m$  whereas for a massless field  $\mu$  is an arbitrary constant. The renormalization procedure for the trace  $\langle T \rangle$  is given in [29] and follows similar lines. For the purposes of this paper, we are concerned only with the energy density.

## 2.1 WKB Approximation and Adiabatic States

A formal solution to the time dependent part of the mode equation can be given using a WKB expansion. First make the change of variables

$$\psi_k(\eta) = \frac{1}{\sqrt{2W(\eta)}} \exp \left[ -i \int_{\eta_0}^{\eta} W(\bar{\eta}) d\bar{\eta} \right] \quad (2.9)$$

where  $\eta_0$  is some arbitrary constant. Substituting into Eq. (2.5) gives

$$W = \left[ k^2 + m^2 a^2 + 6 \left( \xi - \frac{1}{6} \right) \frac{a''}{a} - \frac{1}{2} \left( \frac{W''}{W} - \frac{3W'^2}{2W^2} \right) \right]^{\frac{1}{2}}. \quad (2.10)$$

Approximate solutions to the above equation may be obtained using an iterative scheme, where each successive iteration is given by

$$W_k^{(n)} = \left( k^2 + m^2 a^2 + 6 \left( \xi - \frac{1}{6} \right) \frac{a''}{a} - \frac{1}{2} \left[ \frac{W_k^{(n-2)''}}{W_k^{(n-2)}} - \frac{3}{2} \left( \frac{W_k^{(n-2)'}}{W_k^{(n-2)}} \right)^2 \right] \right)^{\frac{1}{2}}, \quad (2.11)$$

with

$$W_k^{(0)} \equiv (k^2 + m^2 a^2)^{\frac{1}{2}}. \quad (2.12)$$

Taken in conjunction with Eq. (2.9), this can be used to define an  $n$ 'th-order WKB approximation to the mode equation,

$$\psi_k^{(n)}(\eta) = \frac{1}{\sqrt{2W_k^{(n)}(\eta)}} \exp \left[ -i \int_{\eta_0}^{\eta} W_k^{(n)}(\bar{\eta}) d\bar{\eta} \right]. \quad (2.13)$$

In general, this approximation is valid when  $k$  is large and/or the scale factor varies slowly with respect to  $\eta$ . In these cases it will differ from the true solution by terms of higher order than  $n$ .

Exact solutions of the mode equation can be specified by fixing the values of  $\psi_k$  and  $\psi'_k$  at some initial time  $\eta_0$  in such a way that the Wronskian condition (2.6) is satisfied. One way to do this is to use an  $n$ 'th order WKB approximation to generate these values. For example if (2.13) is used then

$$\begin{aligned} \psi_k(\eta_0) &= \psi_k^{(n)}(\eta_0) = \frac{1}{\sqrt{2W_k^{(n)}(\eta_0)}} \\ \psi'_k(\eta_0) &= \psi_k'^{(n)}(\eta_0) = -i \sqrt{\frac{W_k^{(n)}(\eta_0)}{2}}. \end{aligned} \quad (2.14)$$

Note that, while it is acceptable to include derivatives of  $W_k^{(n)}$  in the expression for  $\psi'_k$ , it is not necessary because such terms are of order  $n + 1$ .

If Eq. (2.14) is substituted into Eq. (2.7) and then the quantity  $\langle T \rangle_r$  is computed, it is found that there are state dependent ultraviolet divergences in general for zeroth order and second order adiabatic states. There are no such state dependent ultraviolet divergences for fourth or higher order adiabatic states. This is true for all times

because in a Robertson-Walker spacetime an  $n$ 'th order adiabatic state always remains an  $n$ 'th order adiabatic state [22].

In what follows we are interested in investigating a large class of states. Thus as mentioned in a footnote in the introduction we generalize the definition of an  $n$ 'th order adiabatic vacuum state to include all states whose high momentum modes are specified in the above way by an  $n$ 'th order WKB approximation but whose other modes can be specified in any way. However, we shall not consider states in this paper which result in an infrared divergence of the energy density.

### 3 Massless Scalar Fields

#### 3.1 The Massless Conformally-Coupled Scalar Field

Setting  $\xi = 1/6$  and  $m = 0$  in Eq. (2.5) gives the mode equation for the conformally coupled scalar field

$$\psi_k'' + k^2 \psi_k = 0 \quad (3.1)$$

which has solutions

$$\psi_k = \alpha \frac{e^{-ik\eta}}{\sqrt{2k}} + \beta \frac{e^{ik\eta}}{\sqrt{2k}}. \quad (3.2)$$

The conformal vacuum is specified by the state with  $\alpha = 1$  and  $\beta = 0$ . Following the renormalization procedure given in Eq. (2.8a), we note that the contribution from the modes  $\langle \rho \rangle_u$  is exactly canceled by the counter-terms in  $\langle \rho \rangle_d$ . Thus, the renormalized energy density is given by

$$\langle \rho \rangle_r = \langle \rho \rangle_{an} = \frac{1}{2880\pi^2} \left( -\frac{1}{6} {}^{(1)}H_0^0 + {}^{(3)}H_0^0 \right). \quad (3.3)$$

To determine the point at which quantum effects become important one can compare the energy density of the quantum field with the phantom energy density. This

can be done analytically by using Eqs. (3.9) and (3.10) to write Eq. (3.3) in terms of the scalar curvature as

$$\langle \rho \rangle_r = \frac{1}{34560\pi^2} \frac{27w^2 + 18w - 5}{(1 - 3w)^2} R^2. \quad (3.4)$$

The phantom energy density in terms of the scalar curvature is given by

$$\rho_{ph} = \frac{1}{8\pi(1 - 3w)} R. \quad (3.5)$$

Since  $R$  diverges at the time  $t_r$  it is clear that eventually the energy density of the conformally invariant field becomes comparable to and then larger than the phantom energy density. The two densities are equal at the time when

$$R = 4320\pi \frac{1 - 3w}{27w^2 + 18w - 5}. \quad (3.6)$$

For the semiclassical approximation to be valid we need the spacetime curvature to be much less than the Planck scale or  $R \ll 1$ . From Eq.(3.6) one finds the two densities are equal when  $R = 1$  if

$$w = -\frac{1}{9} \left( 3 + 2160\pi + 2\sqrt{6 + 6480\pi + 1166400\pi^2} \right) \approx -1500. \quad (3.7)$$

Thus for any physically realistic value of  $w$  the energy density of the scalar field will remain small compared to the phantom energy density until the scalar curvature is above the Planck scale. For example, for  $w = -1.25$ , the energy density of the scalar field equals the phantom energy density at  $R \approx 4400$ .

## 3.2 The Massless Minimally-Coupled Scalar Field

Setting  $m = \xi = 0$  in (2.5) gives the mode equation for the massless minimally coupled scalar field,

$$\psi_k'' + \left( k^2 - \frac{a''}{a} \right) \psi_k = 0. \quad (3.8)$$

In terms of the conformal time  $\eta$  the scale factor is

$$a(\eta) = a_0(-\eta)^{-\gamma}. \quad (3.9)$$

with

$$\gamma = -\frac{2}{1+3w} > 0 \quad (3.10)$$

and  $a_0$  a positive constant.

The general solution to this equation is given in terms of Hankel functions,

$$\psi_k(\eta) = \frac{\sqrt{-\pi\eta}}{2} \left( \alpha H_{\gamma+(1/2)}^{(1)}(-k\eta) + \beta H_{\gamma+(1/2)}^{(2)}(-k\eta) \right). \quad (3.11)$$

Substitution into the Wronskian condition (2.6) gives the constraint

$$|\alpha|^2 - |\beta|^2 = 1. \quad (3.12)$$

A special case is the solution with  $\alpha = 1$  and  $\beta = 0$ . The stress-energy tensor was computed analytically for the state specified by this solution by Bunch and Davies [18]. Therefore, in what follows, we will refer to this state as the Bunch-Davies state. Bunch and Davies found that<sup>3</sup>

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<sup>3</sup>In the process of our calculation, we discovered a misprint in Eq. (3.36) of Ref. [18] which is repeated in Eq. (7.52) of Ref. [22]. The final term in both equations should be multiplied by a factor of two.

$$\begin{aligned}
\langle \rho \rangle_r^{BD} &= \frac{1}{2880\pi^2} \left( -\frac{1}{6} {}^{(1)}H_0^0 + {}^{(3)}H_0^0 \right) \\
&\quad - \frac{1}{1152\pi^2} {}^{(1)}H_0^0 \left[ \log \left( \frac{R}{\mu^2} \right) + \psi(2 + \gamma) + \psi(1 - \gamma) + \frac{4}{3} \right] \\
&\quad + \frac{1}{13824\pi^2} [-24\Box R + 24RR_0^0 + 3R^2] - \frac{R}{96\pi^2 a^2 \eta^2} \\
&= -\frac{1}{69120\pi^2} \frac{351w^2 + 54w - 65}{(3w - 1)^2} R^2 \\
&\quad + \frac{1}{256\pi^2} \frac{(w + 1)}{(3w - 1)} R^2 \left[ \log \left( \frac{R}{\mu^2} \right) + \psi \left( \frac{3 + 3w}{1 + 3w} \right) \right. \\
&\quad \left. + \psi \left( \frac{6w}{1 + 3w} \right) + \frac{4}{3} \right]. \tag{3.13}
\end{aligned}$$

The mass scale  $\mu$  in the above expression is an arbitrary constant and  $\psi$  is the digamma function.

As in the case for the massless conformally coupled scalar field, one can compare Eq. (3.13) with Eq. (3.5) and see that eventually the energy density of the scalar field will become comparable to and then greater than the phantom energy density. A graphical analysis shows that for  $w \gtrsim -140$  the two energy densities become comparable at a time when  $R \gtrsim 1$ . For example, if  $w = -1.25$  and  $\mu = 1$ , the energy density of the scalar field equals the phantom energy density at  $R \approx 180$ .

The Bunch Davies state turns out to be an attractor state in the sense that the energy density for all homogeneous and isotropic fourth order or higher adiabatic states approaches  $\langle \rho \rangle_r^{BD}$ . This type of behavior was found in Ref. [30] in de Sitter space for scalar fields in the Bunch-Davies state and we use the same type of argument here as was used in that paper to establish it. First note that since the renormalization counter terms are the same for any choice of  $\alpha$  and  $\beta$  in Eq. (3.11), the renormalized

energy density can be written in terms of the energy density of the Bunch-Davies state plus remainder terms. By substituting Eq. (3.11) into Eq. (2.7), setting  $m = \xi = 0$ , and using the constraint given in Eq. (3.12), one finds that for this class of states the energy density can be written as

$$\langle \rho \rangle_r = \langle \rho \rangle_r^{BD} + I(\eta), \quad (3.14)$$

with

$$\begin{aligned} I(\eta) = & \frac{(-\eta)}{8\pi a^4} \int_0^\infty dk k^4 \left\{ |\beta|^2 \left[ \left| H_{\gamma-\frac{1}{2}}^{(1)}(-k\eta) \right|^2 + \left| H_{\gamma+\frac{1}{2}}^{(1)}(-k\eta) \right|^2 \right] \right. \\ & \left. + \text{Re} \left( \alpha \beta^* \left[ \left( H_{\gamma-\frac{1}{2}}^{(1)}(-k\eta) \right)^2 + \left( H_{\gamma+\frac{1}{2}}^{(1)}(-k\eta) \right)^2 \right] \right) \right\}. \end{aligned} \quad (3.15)$$

The integral in the above expression can be split into three pieces,

$$I(\eta) = I_1(\eta) + I_2(\eta) + I_3(\eta)$$

$$I_1(\eta) = \frac{(-\eta)}{8\pi a^4} \int_0^\lambda dk k^4 \{ \dots \} \quad (3.16a)$$

$$I_2(\eta) = \frac{(-\eta)}{8\pi a^4} \int_\lambda^{-Z/\eta} dk k^4 \{ \dots \} \quad (3.16b)$$

$$I_3(\eta) = \frac{(-\eta)}{8\pi a^4} \int_{-Z/\eta}^\infty dk k^4 \{ \dots \} \quad (3.16c)$$

where  $\lambda$  and  $Z$  are positive constants. To have a fourth order adiabatic state,  $|\beta|$  must fall off faster than  $c_1/k^4$  for large enough values of  $k$  and any positive constant  $c_1$ , thus  $\lambda$  is chosen such that  $|\beta| < 1/k^4$  when  $k \geq \lambda$ . The constant  $Z$  is chosen such that  $0 < Z \ll 1$ .

To evaluate the first integral, we use the series expansion

$$H_\nu^{(1)}(x) = -i \frac{\Gamma(\nu)}{\pi} \left( \frac{x}{2} \right)^{-\nu} + O(x^{-\nu+2}) + O(x^\nu). \quad (3.17)$$

This is valid when  $x = -k\eta \leq Z$ . We note that at late enough times  $Z > -\lambda\eta$  for any choice of  $Z$  and  $\lambda$ . Using the expansion (3.17) in Eq. (3.15), one finds to leading order that

$$I_1(\eta) = \frac{2^{2\gamma} [\Gamma(\gamma + \frac{1}{2})]^2}{4\pi^3 a_0^4} (-\eta)^{2\gamma} \int_0^\lambda dk \{ k^{3-2\gamma} [|\beta|^2 - \text{Re}(\alpha\beta^*)] + O[(-k\eta)^2] + O[(-k\eta)^{2\gamma+1}] \} . \quad (3.18)$$

For states with no infrared divergences the integral is finite, so we find that  $I_1(\eta) \rightarrow 0$  as  $(-\eta) \rightarrow 0$ .

For the second integral, due to the time dependence in the upper limit cutoff, we find that all orders of the expansion given in Eq. (3.17) contribute, so a different treatment is necessary. Using the fact that for  $k \geq \lambda$  we have  $|\beta|^2 < 1/k^8$  and  $|\alpha\beta^*| < c_2/k^4$  for some positive constant  $c_2$ , we can put the following upper bound on the integral in Eq. (3.16b),

$$|I_2(\eta)| < \frac{(-\eta)}{8\pi a^4} \int_\lambda^{-Z/\eta} dk \left\{ \left( \frac{1}{k^4} + c_2 \right) \left[ \left( J_{\gamma+\frac{1}{2}}(-k\eta) \right)^2 + \left( J_{\gamma-\frac{1}{2}}(-k\eta) \right)^2 + \left( Y_{\gamma+\frac{1}{2}}(-k\eta) \right)^2 + \left( Y_{\gamma-\frac{1}{2}}(-k\eta) \right)^2 \right] + 2c_2 \left[ \left| J_{\gamma+\frac{1}{2}}(-k\eta) Y_{\gamma+\frac{1}{2}}(-k\eta) \right| + \left| J_{\gamma-\frac{1}{2}}(-k\eta) Y_{\gamma-\frac{1}{2}}(-k\eta) \right| \right] \right\} . \quad (3.19)$$

Assuming  $\gamma \neq \frac{1}{2}$ , one can use the standard series expansions for Bessel functions to show that the terms in the integrand are all of the form

$$(-\eta)^{4\gamma+1} \int_\lambda^{-Z/\eta} dk \left( \frac{\tilde{c}_1}{k^4} + \tilde{c}_2 \right) \sum_{n,n'=0}^{\infty} \frac{(-1)^{n+n'}}{n!n'} \frac{(-k\eta)^{(\alpha_1+\alpha_2)p+2n+2n'}}{\Gamma(\alpha_1 p + n + 1)\Gamma(\alpha_2 p + n' + 1)} \quad (3.20)$$

where  $\alpha_1$  and  $\alpha_2$  independently take on the values  $\pm 1$ ,  $p$  takes on the values  $\gamma \pm \frac{1}{2}$ ,



and  $\tilde{c}_1$  and  $\tilde{c}_2$  are constants. After integration, Eq. (3.20) becomes

$$\begin{aligned}
& (-\eta)^{4\gamma} \sum_{n,n'=0}^{\infty} \frac{(-1)^{n+n'}}{n!n'} \frac{1}{\Gamma(\alpha_1 p + n + 1)\Gamma(\alpha_2 p + n' + 1)} \\
& \times \left[ \left( \frac{(-\eta)^4 \tilde{c}_1}{Z^4((\alpha_1 + \alpha_2)p + 2n + 2n' - 3)} + \frac{\tilde{c}_2}{(\alpha_1 + \alpha_2)p + 2n + 2n' + 1} \right) Z^{(\alpha_1 + \alpha_2)p + 2n + 2n' + 1} \right. \\
& - \left( \frac{\tilde{c}_1}{\lambda^4((\alpha_1 + \alpha_2)p + 2n + 2n' - 3)} \right. \\
& \left. \left. + \frac{\tilde{c}_2}{(\alpha_1 + \alpha_2)p + 2n + 2n' + 1} \right) (-\lambda\eta)^{(\alpha_1 + \alpha_2)p + 2n + 2n' + 1} \right] \quad (3.21)
\end{aligned}$$

Note that the above series converges. The smallest power of  $(-\eta)$  is  $(-\eta)^{2\gamma}$  and occurs when  $\alpha_1 = \alpha_2 = -1$  and  $p = \gamma + \frac{1}{2}$ . It comes from the  $n = n' = 0$  term. Thus,  $I_2(\eta) \rightarrow 0$  as  $(-\eta) \rightarrow 0$ .

In the case  $\gamma = \frac{1}{2}$ , we must use the appropriate series expansions for  $Y_0(-k\eta)$  and  $Y_1(-k\eta)$ , so Eq. (3.21) will contain different terms. However the argument is similar and the result that  $I_2(\eta) \rightarrow 0$  as  $(-\eta) \rightarrow 0$  is the same.

For the third integral, using  $|\beta|^2 < 1/k^8$  and  $|\alpha\beta^*| < c_2/k^4$ , and changing variables to  $x = -k\eta$  gives the bound

$$\begin{aligned}
|I_3(\eta)| & \leq \frac{(-\eta)^{4\gamma+4}}{8\pi a_0^4} \int_Z^\infty dx \frac{1}{x^4} \left[ \left| H_{\gamma-\frac{1}{2}}^{(1)}(x) \right|^2 + \left| H_{\gamma+\frac{1}{2}}^{(1)}(x) \right|^2 \right] \\
& + \frac{c_2(-\eta)^{4\gamma}}{8\pi a_0^4} \int_Z^\infty dx \left| \left( H_{\gamma-\frac{1}{2}}^{(1)}(x) \right)^2 + \left( H_{\gamma+\frac{1}{2}}^{(1)}(x) \right)^2 \right|. \quad (3.22)
\end{aligned}$$

By noting that  $\left| H_\nu^{(1)}(x) \right| \sim 1/\sqrt{x}$  at large  $x$ , it is clear that these integrals are finite. Further, each integral is independent of  $(-\eta)$ , so the contribution from this piece goes like some constant times  $(-\eta)^{4\gamma}$ . Thus,  $I(\eta) \rightarrow 0$  as  $(-\eta) \rightarrow 0$  and this completes our proof that  $\langle \rho \rangle_r^{BD}$  is an attractor state.

To demonstrate this behavior numerically, we compare the energy density from Eq. (3.13) which occurs for  $\alpha = 1$  and  $\beta = 0$  to the renormalized energy density of a different fourth order adiabatic state. One way to specify a fourth order adiabatic state is to use a fourth order WKB approximation as discussed in Section 2.1. However, for the spacetimes we are considering, using a fourth order WKB approximation in Eqs. (2.13) and (2.14) in the limit  $k \rightarrow 0$  results in an infrared divergence of  $\langle \rho \rangle$ . To avoid such a divergence, we use a zeroth order WKB approximation for the modes with  $0 \leq k \leq k_0$ , for some positive constant  $k_0$ . According to the generalized definition we give in Section I, these are still fourth order adiabatic states. A given state is specified by the value of  $k_0$  and the time  $\eta_0$  at which the matching in Eq. (2.14) is done.

As shown in Figure 2.1, the energy density of the state specified by choosing  $k_0 = 0.2$  and  $\eta_0 = -100$ , approaches that of the Bunch Davies state as expected. A number of other numerical calculations were done for various values of  $k_0$  and  $\eta_0$  with the same qualitative result. Note that for all the numerical calculations that were done for this paper,  $w = -1.25$  and  $a_0 \approx 22.6$ .

It is of course possible to construct states for which, at times close to  $\eta_0$ , the energy density is large compared to the phantom energy density. However, there is no reason to expect that the energy density of quantized fields at early times should be comparable to that of the phantom energy density since that is not the case today. Thus our numerical results provide evidence that if the initial energy density is small compared to the phantom energy density at times close to  $\eta_0$  then, for realistic values of  $w$ , it will remain small during the period when the semiclassical approximation should be valid.

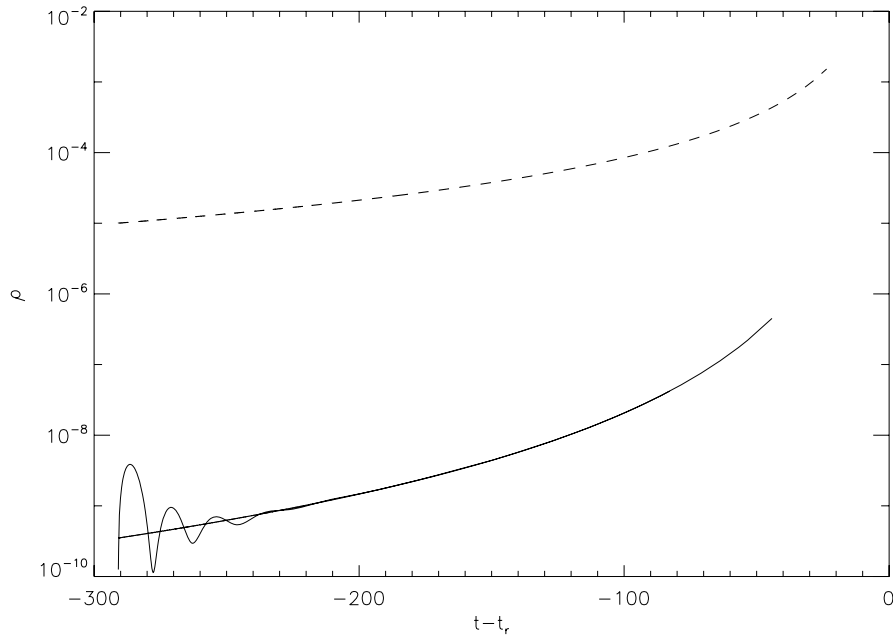


Figure 2.1: The dashed line represents the phantom energy density  $\rho_{ph}$  in the case that  $w = -1.25$ . The two solid lines represent the energy density  $\langle \rho \rangle_r$  for two states of the massless minimally coupled scalar field. The central line is the Bunch Davies attractor state. The oscillating line is a fourth order adiabatic state with  $k_0 = 0.2$  and  $\eta_0 = -100$ . For both states the mass scale  $\mu = 1$  has been used.

## 4 Massive Scalar Fields

### 4.1 Massive Conformally Coupled Scalar Fields

Setting  $\xi = 1/6$  in Eq. (2.5) gives the mode equation for a massive conformally coupled scalar field,

$$\psi_k'' + (k^2 + m^2 a^2) \psi_k = 0. \quad (4.1)$$

In this case, we do not have an analytic solution for the mode equation and there is no obvious choice for the vacuum state. Instead, we choose our initial state using the method outlined near the end of Section 3.2 and compute solutions to the mode equation and the energy density numerically. What we find is that there is an initial contribution to the energy density from the modes of the field in addition to the contribution from Eq. (3.3); however, as the Universe expands this contribution redshifts away and the energy density approaches the energy density of the massless conformally coupled scalar field. An example is shown in Figure 2.2 for a field with mass  $m = 0.005$  in a state specified by  $k_0 = 0.1$  and  $\eta_0 = -100$ . We have tested this for several different initial states and different masses and found consistent behavior in all cases.

As with the massless minimally coupled scalar field, it is possible to construct states with a large initial contribution to the energy density and these states can be ruled out by the same argument as given in Section 3.2. Our numerical results provide evidence that if the energy density of the quantized field is small compared to the phantom energy density at times close to  $\eta_0$ , then, for realistic values of  $w$ , it will remain small during the period when the semiclassical approximation should be valid.

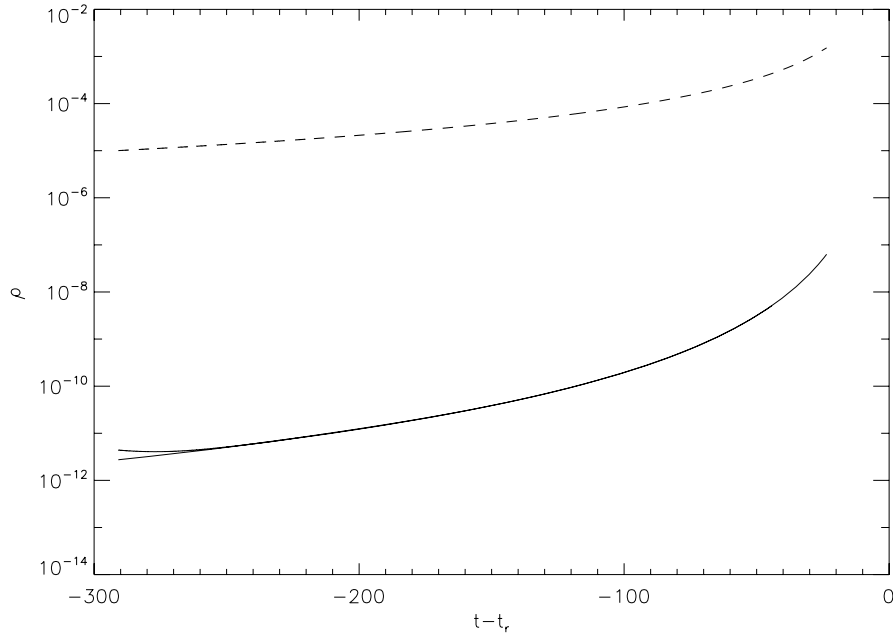


Figure 2.2: The dashed line represents the phantom energy density  $\rho_{\text{ph}}$  in the case that  $w = -1.25$ . The upper solid line is  $\langle \rho \rangle_r$  for a massive ( $m = 0.005$ ) conformally coupled scalar field in the fourth order adiabatic state with  $k_0 = 0.1$  and  $\eta_0 = -100$ ; the lower one is  $\langle \rho \rangle_r$  for the massless conformally coupled scalar field.

## 4.2 Massive Minimally-Coupled Scalar Fields

The last case we consider is that of massive, minimally coupled scalar fields, which obey the mode equation

$$\psi_k'' + \left( k^2 + m^2 a^2 - \frac{a''}{a} \right) \psi_k = 0. \quad (4.2)$$

Again, we choose our initial state using the method outlined near the end of Section 3.2 and evolve the modes forward in time numerically.

Qualitatively, the behavior here is similar to the behavior found for the massless, minimally coupled field in a fourth order adiabatic vacuum state. At late times, the contribution from the mass terms is small compared to other terms and the energy density approaches that of the massless minimally coupled scalar field in the Bunch-Davies vacuum state. We note that there is an arbitrary mass scale  $\mu$  present in the renormalized energy density (3.13) for the massless scalar field; the convergence shown in Figure 2.3 occurs when  $\mu$  is set equal to the mass of the massive field. At very late times the value of  $\mu$  does not affect the leading order behavior of  $\langle \rho \rangle$  for the massless minimally coupled scalar field and thus, for any value of  $\mu$ , the energy density of the massive field asymptotically approaches that of the massless field in all cases considered.

## 5 Summary and Conclusions

We have computed the energy densities of both massless and massive quantized scalar fields with conformal and minimal coupling to the scalar curvature in spacetimes with big rip singularities. We have restricted attention to states which result in a stress-energy tensor which is homogeneous and isotropic and free of ultraviolet and infrared divergences. For the numerical computations we have further restricted attention to

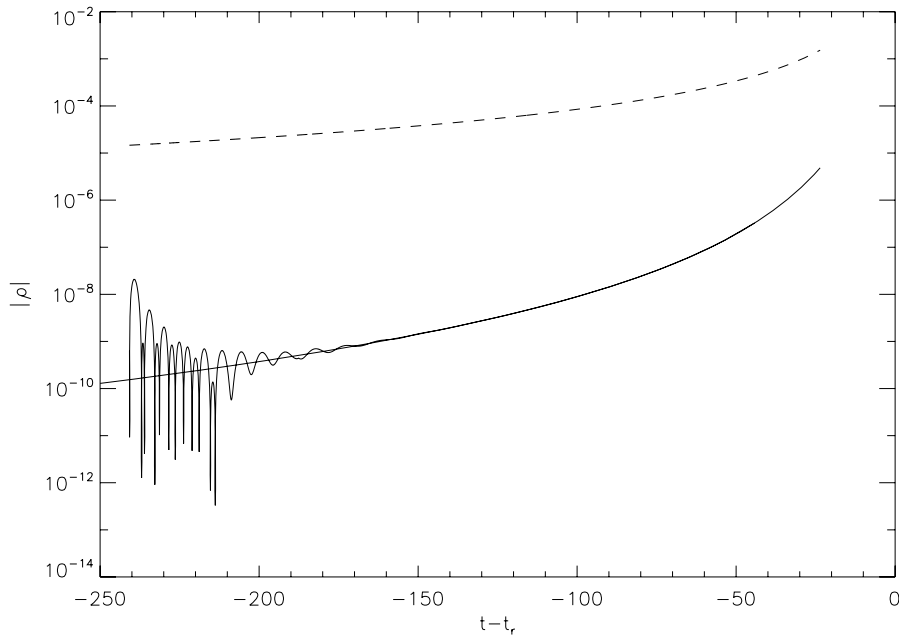


Figure 2.3: The dashed line represents the phantom energy density  $\rho_{\text{ph}}$  in the case that  $w = -1.25$ . The central solid line is the energy density  $\langle \rho \rangle_r$  for a massless minimally coupled scalar field in the Bunch Davies state with mass scale  $\mu = 0.001$ . The oscillating line is the energy density for a massive ( $m = 0.001$ ) minimally coupled scalar field in the fourth order adiabatic state with  $k_0 = 1$  and  $\eta_0 = -50$ . Note that the quantity  $|\rho|$  rather than  $\rho$  has been plotted since the energy density of the massive field is negative at  $\eta = \eta_0$ , oscillates between positive and negative values for a period of time, and then becomes positive definite at late times. Note that, except at the initial time, the times where the curve is nearly vertical when it reaches a lower limit are times at which the energy density goes through zero, which is  $-\infty$  on the scale of this plot. Thus the curve should really extend down to  $-\infty$  at these points.

states for which, near the initial time of the calculation, the energy density of the quantum field is much less than that of the phantom field.

For the massless minimally coupled scalar field we have shown that the energy density for the field in any fourth order or higher adiabatic state for which the stress-energy tensor is homogeneous, isotropic, and free of infrared divergences, always asymptotically approaches the energy density which this field has in the Bunch-Davies state. In this sense the Bunch-Davies state is an attractor state.

For massive minimally coupled scalar fields numerical computations have been made of the energy density for different fourth order adiabatic states and in every case considered the energy density approaches that of the Bunch-Davies state for the massless minimally coupled scalar field at late times. For conformally coupled massive scalar fields numerical computations have also been made of the energy density for different fourth order adiabatic states. In each case considered the energy density asymptotically approaches that of the massless conformally coupled scalar field in the conformal vacuum state. Thus it appears that the asymptotic behavior of the energy density of a quantized scalar field in a spacetime with a big rip singularity depends only upon the coupling of the field to the scalar curvature and not upon the mass of the field or which state it is in, at least within the class of states we are considering.

Analytic expressions for the energy densities of both the massless conformally coupled scalar field in the conformal vacuum state and the massless minimally coupled scalar field in the Bunch-Davies state, in spacetimes with big rip singularities have been previously obtained [18,22] and are shown in Eqs. (3.3) and (3.13). To investigate the question of whether backreaction effects are important in these cases, the energy density of the scalar field can be compared to the phantom energy density to see if there is any time at which they are equal or at least comparable. Then one can determine whether the semiclassical approximation is likely to be valid at this time



by evaluating the scalar curvature and seeing whether or not it is well below the Planck scale. We have done this and find that for the conformally coupled field the two energy densities are equal at the point when the scalar curvature is at the Planck scale if  $w \sim -1500$ . For the minimally coupled scalar field this combination occurs if  $w \sim -140$ . Thus for  $w \ll -1500$  for the conformally coupled field and  $w \ll -140$  for the minimally coupled field, one expects that backreaction effects may be important at times when the scalar curvature is well below the Planck scale. However, the values of  $w$  which satisfy these constraints are completely ruled out by cosmological observations.

Another way in which the energy density of a quantum field can be comparable to the phantom energy density at scales well below the Planck scale is to construct a state for which this is true. There is no doubt that such states exist. The analytic and numerical evidence we have is that over long periods of time the energy density of a conformally or minimally coupled scalar field in such a state would decrease and at late enough times become comparable to that of a massless field in the conformal or Bunch-Davies vacuum state respectively. More importantly one can ask whether any such states exist which are realistic for the universe that we live in. This seems unlikely because today is certainly an early time compared to  $t_r$  if our universe does have a big rip singularity in its future; however the energy densities of quantized fields today are much less than that of the dark energy. Thus it would be necessary for the energy density of a quantum field to be small compared to the phantom energy density today and then to grow fast enough to become comparable to it well before the Planck scale is reached. This type of behavior seems highly artificial, particularly since it is not what happens for a massless scalar field in the conformal or Bunch-Davies vacuum states. Thus we find no evidence which would lead us to believe that backreaction effects due to quantum fields would remove a big rip singularity in our

universe if, indeed, such a singularity lies in our future.

## **Acknowledgements**

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## Comment by Haro and Amoros

In [2], Haro and Amoros published a comment to the journal article reproduced above. In this comment, they displayed the results of a numerical computation of the effects of the conformally invariant quantum scalar field in a Friedmann-Robertson-Walker geometry driven by a phantom fluid with equation of state  $p = w\rho$ . This is identical to the class of spacetimes considered in our treatment. However, in contrast to our calculation, they solved the semiclassical backreaction equations (1.2) for the quantum field, and found solutions in which the presence of the quantum field does significantly alter the geometry. Since they solve the semiclassical backreaction equations, in principle, their calculation should supercede our own; however, we disagree with their results.

Our reply to their comment was published in Physical Review D **84**, 048502 (2011), and is reproduced below, in its entirety.

### Reply to “Comment on ‘Effects of quantized scalar fields in cosmological spacetimes with big rip singularities’ ”

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In a previous paper [1] we computed the energy density for massive and massless quantized scalar fields with arbitrary curvature coupling in spacetimes containing a big rip singularity. We found no evidence that backreaction effects due to quantum fields would remove a big rip singularity in our universe if it has one. In their comment on our work, Haro and Amoros [2] solve the semiclassical backreaction equations

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directly for the case of a conformally invariant scalar field. They find the opposite conclusion.

It seems obvious that the results of a backreaction calculation should supersede those of a background field calculation. However, this is only true if the backreaction calculation yields a physically acceptable solution. In this response we argue that the solutions found by Haro and Amoros are not physically acceptable during the times when the backreaction effects due to the quantum fields are important. The relevant solutions they find follow the classical expansion for some period of time and then deviate from it significantly. From the plots they display, it is clear that once significant deviations from the classical solution occur these deviations manifest on time scales which are less than the Planck time, independent of the amount of time the system remains close to the classical solution. Such rapid variation in time violates the semiclassical approximation and therefore these solutions are not physically acceptable once they deviate significantly from the classical solution. Thus, in our view, the solutions found by Haro and Amoros do not provide evidence that backreaction effects in the context of semiclassical gravity can remove a big rip singularity in our universe.

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### **Chapter 3: Noise kernel for a quantum field in Schwarzschild spacetime under the Gaussian approximation**

The following manuscript was published in Physical Review D. **85**, 044037 (2012). Stylistic variations are due to the requirements of the journal. The initial groundwork for the calculation was provided by A. Eftekharzadeh, A. Roura, and B. L. Hu. J. D. Bates and P. R. Anderson developed the computational scheme used for computing the noise kernel in Schwarzschild spacetime based on prior work done by A. Eftekharzadeh. J. D. Bates wrote the Mathematica code and performed all computations and checks of the noise kernel in Schwarzschild and Minkowski spacetime. A. Roura provided the proofs for the relation describing the conformal transformation of the noise kernel and wrote the Appendix. All five authors contributed to the preparation of the manuscript. J. D. Bates' contribution included providing formatting for the equations for the noise kernel in Sec. 4, and verifying all equations.



# Noise kernel for a quantum field in Schwarzschild spacetime under the Gaussian approximation

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## Abstract

A method is given to compute an approximation to the noise kernel, defined as the symmetrized connected 2-point function of the stress tensor, for the conformally invariant scalar field in any spacetime conformal to an ultra-static spacetime for the case in which the field is in a thermal state at an arbitrary temperature. The most useful applications of the method are flat space where the approximation is exact and Schwarzschild spacetime where the approximation is better than it is in most other spacetimes. The two points are assumed to be separated in a timelike or spacelike direction. The method involves the use of a Gaussian approximation which is of the same type as that used by Page [D. N. Page, Phys. Rev. D **25**, 1499 (1982).] to compute an approximate form of the stress tensor for this field in Schwarzschild

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spacetime. All components of the noise kernel have been computed exactly for hot flat space and one component is explicitly displayed. Several components have also been computed for Schwarzschild spacetime and again one component is explicitly displayed.

## 1 Introduction

Studies of the fluctuations in the stress tensors of quantum fields are playing an increasingly important role in investigations of quantum effects in curved spacetimes [1–8] and semiclassical gravity [9]. In various forms they have provided criteria for and tests of the validity of semiclassical gravity [10–14]. They are relevant for the generation of cosmological perturbations during inflation [15–19] as well as the fluctuation and backreaction problem in black hole dynamics [20,21]. They also provide a possible pathway for one to connect semiclassical to quantum gravity (see, e.g., Ref. [22]). One important theory in which this is done systematically from first principles is stochastic semiclassical gravity [23–28], or stochastic gravity in short, which takes into account fluctuations of the gravitational field that are induced by the quantum matter fields.

In stochastic gravity the induced fluctuations of the gravitational field can be computed using the Einstein-Langevin equation [24,29]

$$G_{ab}^{(1)}[g+h] = 8\pi(\langle\hat{T}_{ab}^{(1)}[g+h]\rangle + \xi_{ab}[g]) . \quad (1.1)$$

Here the superscript (1) means that only terms linear in the metric perturbation  $h_{ab}$  around the background geometry  $g_{ab}$  should be kept, and  $g_{ab}$  is a solution to the semiclassical Einstein equation [9,12]

$$G_{ab}[g] = 8\pi\langle\hat{T}_{ab}[g]\rangle . \quad (1.2)$$

Here  $\langle \dots \rangle$  denotes the quantum expectation value with respect to a normalized state of the matter field [more generally,  $\langle \dots \rangle = \text{Tr}(\hat{\rho} \dots)$  for a mixed state] and  $\hat{T}_{ab}$  is the stress tensor operator of the field<sup>1</sup>. The tensor  $\xi_{ab}$  is a Gaussian stochastic source with vanishing mean which accounts for the stress tensor fluctuations and is completely characterized by its correlation function [27, 28]:

$$\begin{aligned}\langle \xi_{ab}(x) \rangle_s &= 0 \\ \langle \xi_{ab}(x) \xi_{c'd'}(x') \rangle_s &= N_{abc'd'}(x, x'),\end{aligned}\tag{1.3}$$

where  $\langle \dots \rangle_s$  refers to the stochastic average over the realizations of the Gaussian source and the noise kernel  $N_{abc'd'}(x, x')$  is given by the symmetrized connected 2-point function of the stress tensor operator for the quantum matter fields evaluated in the background geometry  $g_{ab}$ :

$$N_{abc'd'} = \frac{1}{2} \langle \{ \hat{t}_{ab}(x), \hat{t}_{c'd'}(x') \} \rangle\tag{1.4}$$

$$\hat{t}_{ab}(x) \equiv \hat{T}_{ab}(x) - \langle \hat{T}_{ab}(x) \rangle.\tag{1.5}$$

Thus, the noise kernel plays a central role in stochastic gravity, similarly to the expectation value of the stress tensor in semiclassical gravity.

One can solve Eq. (1.1) using the retarded Green function for the operator acting on  $h_{ab}$  to obtain [11]

$$h_{ab}(x) = h_{ab}^{(h)}(x) + 8\pi \int d^4y' \sqrt{-g(y')} G_{abc'd'}^{(\text{ret})}(x, y') \xi^{c'd'}(y'),\tag{1.6}$$

where  $h_{ab}^{(h)}$  is a homogeneous solution to Eq. (1.1) which contains all the information on the initial conditions. The resulting two-point function depends directly on the

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<sup>1</sup>The stress tensor expectation value in Eqs. (1.1)-(1.2) is the renormalized one, which is the result of regularizing and subtracting the divergent terms by introducing appropriate local counterterms (up to quadratic order in the curvature) in the bare gravitational action [30]. Any finite contributions from those counterterms other than the Einstein tensor have been absorbed in the renormalized expectation value of the stress tensor.

noise kernel:

$$\begin{aligned} \langle \langle h_{ab}(x)h_{c'd'}(x') \rangle_s \rangle_{\text{i.c.}} &= \langle h_{ab}^{(h)}(x)h_{c'd'}^{(h)}(x') \rangle_{\text{i.c.}} \\ &+ (8\pi)^2 \int d^4y' d^4y \sqrt{g(y')g(y)} G_{abe'f'}^{(\text{ret})}(x, y') N^{e'f'gh}(y', y) G_{c'd'gh}^{(\text{ret})}(x', y), \end{aligned} \quad (1.7)$$

where  $\langle \dots \rangle_{\text{i.c.}}$  denotes the average over the initial conditions weighed by an appropriate distribution characterizing the initial quantum state of the metric perturbations. It should be emphasized that although obtained by solving an equation involving classical stochastic processes, the result for the stochastic correlation function obtained in Eq. (1.7) coincides with the result that would be obtained from a purely quantum field theoretical calculation where the metric is perturbatively quantized around the background  $g_{ab}$ . More precisely, if one considers a large number  $N$  of identical fields, the stochastic correlation function coincides with the quantum correlation function  $\langle \{ \hat{h}_{ab}(x), \hat{h}_{c'd'}(x') \} \rangle$  to leading order in  $1/N$  [11, 31]. The noise kernel is the crucial ingredient in the contribution to the metric fluctuations *induced* by the quantum fluctuations of the matter fields, which corresponds to the second term on the right-hand side of Eq. (1.7).

As pointed out by Hu and Roura [20] using the black-hole quantum backreaction and fluctuation problems as examples, a consistent study of the horizon fluctuations requires a detailed knowledge of the stress tensor 2-point function and, therefore, the noise kernel. That is because, in contrast with the averaged energy flux, the existence of a direct correlation assumed in earlier studies between the fluctuations of the energy flux crossing the horizon and those far from it is simply invalid. The need for the noise kernel of a quantum field near a black hole horizon has been pronounced earlier in order to study the effect of Hawking radiation emitted by a black hole on its evolution as well as the metric fluctuations driven by the quantum field (the “backreaction and fluctuation” problem [32]). For example, Sinha, Raval and Hu [33]

have outlined a program for such a study, which is the stochastic gravity upgrade (via the Einstein-Langevin equation) of those carried out for the mean field in semiclassical gravity (through the semiclassical Einstein equation) by York [34, 35] and by York and his collaborators [36]. Note that strictly speaking the retarded propagator and the noise kernel in Eq. (1.7) should not be computed in the Schwarzschild background but a slightly corrected one (still static and spherically symmetric) which takes into account the backreaction of the quantum matter fields on the mean geometry via the semiclassical Einstein equation [34]. However, one can consider an expansion in powers of  $1/M^2$ ; for the Hartle-Hawking state the difference between calculations employing the Schwarzschild background or the semiclassically corrected one would be of order  $1/M^2$  or higher. Since our approach, which fits naturally within the framework of perturbative quantum gravity regarded as a low-energy effective theory [37], is only valid for black holes with a Schwarzschild radius much larger than the Planck length ( $M \gg 1$ ), those corrections of order  $1/M^2$  will be very small.

An expression for the noise kernel for free fields in a general curved spacetime in terms of the corresponding Wightman function was obtained a decade ago [25, 38, 39]. Since then this general result has been employed to obtain the noise kernel in Minkowski [26], de Sitter [17, 38, 40, 41] and anti-de Sitter [40, 42] spacetimes, as well as hot flat space and Schwarzschild spacetime in the coincidence limit [43]. In this paper we compute expressions for the noise kernel in hot flat space and Schwarzschild spacetime using the same Gaussian approximation for the Wightman function of the quantum matter field that was used in Ref. [43]. The difference is that there the coincident limit was considered and certain terms had to be subtracted. Here we do not take the coincidence limit and no subtraction of divergent terms is necessary. In contrast to the stress tensor expectation value, which is computed in the limit that the points come together, if one wishes to solve the equations of stochastic semiclassical

gravity, it is necessary to have an expression for the noise kernel when the points are separated. This can be seen explicitly in Eqs. (1.3) and (1.7).

Specifically, we calculate an exact expression for the noise kernel of a conformally invariant scalar field in Minkowski space in a thermal state at an arbitrary temperature  $T$ . We also compute approximate expressions for the noise kernel in both the optical Schwarzschild and Schwarzschild spacetimes when the field is in a thermal state at an arbitrary temperature  $T$ . For the latter case when the temperature is that associated with the black hole, the field is in the Hartle-Hawking state, which is the relevant one if one wants to study the metric fluctuations of a black hole in (possibly unstable) equilibrium. In all cases the calculations are done with the points separated (and non-null related). Because we do not attempt to take the limit in which the points come together (or are null related) the results are finite without the need for any subtraction.

To compute the noise kernel we need an expression for the Wightman function,  $G^+(x, x') = \langle \phi(x)\phi(x') \rangle$ . We begin by working in the Euclidean sector of the optical Schwarzschild spacetime, which is ultra-static and conformal to Schwarzschild. We use the same Gaussian approximation for the Euclidean Green function when the field is in a thermal state that Page [44] used for his computation of the stress tensor in Schwarzschild spacetime. As he points out, this approximation corresponds to taking the first term in the DeWitt-Schwinger expansion for the Euclidean Green function. In most spacetimes that would not be sufficient to generate an approximation to the stress tensor which could be renormalized correctly. However, in the optical Schwarzschild spacetime (and for any other ultra-static metric conformal to an Einstein metric in general) the second and third terms in the DeWitt-Schwinger expansion vanish identically, so that the approximation is much better than it would usually be. In the flat space limit this expression is exact.

Having obtained an expression for the Euclidean Green function, we analytically continue it to the Lorentzian sector and take the real part of the result to obtain an expression for the Wightman function when the points are non-null separated. In the optical Schwarzschild spacetime, by substitution into the equation satisfied by the Wightman function, we show that it is valid through  $O[(x - x')^2]$  as expected<sup>2</sup>. This expression is again exact in the flat space limit.

For hot flat space, we next compute the necessary derivatives of the Wightman function and substitute into Eq. (2.3) to obtain an exact expression for the noise kernel. For the optical Schwarzschild spacetime we take a different approach. We expand the approximate part of the Wightman function in powers of  $(x - x')$ , compute the derivatives and substitute the results into Eq. (2.3). The result is valid through  $O[(x - x')^{-4}]$ , while the leading terms are  $O[(x - x')^{-8}]$ . Finally, we conformally transform the results to Schwarzschild spacetime to obtain an expression for the noise kernel that is valid to the same order there. This is done explicitly for two cases of interest. One is the case when the point separation is only in the time direction and the product of the temperature and the point separation is not assumed to be small. The second is for a general spacelike or timelike separation of the points when the product of the point separation and the temperature is assumed to be small.

All nonzero components of the noise kernel have been computed in hot flat space for a non-null separation of the points and the conservation laws given in Eq. (A1) and the partial traces given in Eq. (A2) have been checked. Several components of the noise kernel have also been computed in Schwarzschild spacetime, but for the sake of brevity only one component is explicitly displayed. We have obtained results

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<sup>2</sup>The dimensionless quantity which should be much smaller than one so that this expansion provides a valid approximation corresponds to the square of the geodesic distance,  $2\sigma = O[(x - x')^2]$ , divided by the square of the typical curvature radius scale. In Schwarzschild spacetime the latter is characterized near the horizon by the Schwarzschild radius  $R_S = 2M$ , but more accurately by the inverse fourth root of the curvature invariant  $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim M^2/r^6$  far from it.

for separations along the time direction but without assuming the product of the temperature and the time difference to be small, as well as for arbitrary separations but assuming that the product of the temperature and the separation is small. As discussed in Sec. 4, various checks of our results have been made using the partial traces and conservation laws. Furthermore, the result which is not restricted to small values of the temperature times the time difference is shown to agree with previously computed results in two different flat space limits.

In Sec. 2 we review the form of the noise kernel for a conformally invariant scalar field in a general spacetime and discuss its properties including its change under conformal transformations which enables us to obtain the noise kernel in Schwarzschild spacetime from the result for the optical spacetime. In Sec. 3 we present the relationship between the Wightman and Euclidean Green functions, the relevant parts of the formalism for the DeWitt-Schwinger expansion [45], and its use in the Gaussian approximation derived by Page [44] for the Euclidean Green function in the optical Schwarzschild spacetime. We show that the resulting expression for the Wightman function for any temperature is valid through  $O[(x - x')^2]$ . In Sec. 4 a method for obtaining the Wightman function in the Gaussian approximation is given. The computation of the noise kernel using this Wightman function is described and one component of the noise kernel in Schwarzschild spacetime is explicitly displayed. The flat space limit for this component is discussed. Sec. 5 contains a summary and discussion of our main results. In the Appendix we provide two proofs of the simple rescaling under conformal transformations of the noise kernel for a conformally invariant scalar field. Throughout we use units such that  $\hbar = c = G = k_B = 1$  and the sign conventions of Misner, Thorne, and Wheeler [46].



## 2 Noise kernel for the conformally invariant scalar field

In this section we review the general properties of the noise kernel for the conformally invariant scalar field in an arbitrary spacetime. The definition of the noise kernel for any quantized matter field is given in Eq. (1.4).

The classical stress tensor for the conformally invariant scalar field is

$$T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi + \xi (g_{ab} \square - \nabla_a \nabla_b + G_{ab}) \phi^2, \quad (2.1)$$

with  $\xi = (D - 2)/4(D - 1)$ , which becomes  $\xi = 1/6$  in  $D = 4$  spacetime dimensions. Note that since the stress tensor is symmetric, the noise kernel as defined in Eq. (1.4) is also symmetric under exchange of the indices  $a$  and  $b$ , or  $c'$  and  $d'$ . To compute the noise kernel one promotes the field  $\phi(x)$  in Eq. (2.1) to an operator in the Heisenberg picture while treating  $g_{ab}$  as a classical background metric. The result is then substituted into Eq. (1.4).

Given a Gaussian state for the quantum matter field, one can express the noise kernel in terms of products of two Wightman functions by applying Wick's theorem.<sup>3</sup> The Wightman function is defined as

$$G^+(x, x') = \langle \phi(x) \phi(x') \rangle. \quad (2.2)$$

The result for a scalar field with arbitrary mass and curvature coupling in a general spacetime has been obtained in Refs. [25, 39]. For the conformally invariant scalar field in a general spacetime the noise kernel is [39]

$$N_{abc'd'} = \text{Re} \left\{ \bar{K}_{abc'd'} + g_{ab} \bar{K}_{c'd'} + g_{c'd'} \bar{K}'_{ab} + g_{ab} g_{c'd'} \bar{K} \right\} \quad (2.3)$$

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<sup>3</sup>Gaussian states include for instance the usual vacua, thermal and coherent states. In general most states will not be Gaussian and Wick's theorem will not apply: it will not be possible to write the 4-point function of the field in terms of 2-point functions and expectation values of the field. That is for example the case for all eigenstates of the particle number operators other than the corresponding vacuum.

with<sup>4</sup>

$$\begin{aligned}
9\bar{K}_{abc'd'} &= 4 (G_{;c'b} G_{;d'a} + G_{;c'a} G_{;d'b}) + G_{;c'd'} G_{;ab} + G G_{;abc'd'} \\
&\quad - 2 (G_{;b} G_{;c'd'a} + G_{;a} G_{;c'bd'} + G_{;d'} G_{;abc'} + G_{;c'} G_{;abd'}) \\
&\quad + 2 (G_{;a} G_{;b} R_{c'd'} + G_{;c'} G_{;d'} R_{ab}) \\
&\quad - (G_{;ab} R_{c'd'} + G_{;c'd'} R_{ab}) G + \frac{1}{2} R_{c'd'} R_{ab} G^2
\end{aligned} \tag{2.4a}$$

$$\begin{aligned}
36\bar{K}'_{ab} &= 8 \left( -G_{;p'b} G_{;p'a} + G_{;b} G_{;p'a} + G_{;a} G_{;p'b} \right) \\
&\quad + 4 \left( G_{;p'} G_{;abp'} - G_{;p'} G_{;ab} - G G_{;abp'} \right) \\
&\quad - 2 R' (2 G_{;a} G_{;b} - G G_{;ab}) \\
&\quad - 2 \left( G_{;p'} G_{;p'} - 2 G G_{;p'} \right) R_{ab} - R' R_{ab} G^2
\end{aligned} \tag{2.4b}$$

$$\begin{aligned}
36\bar{K} &= 2 G_{;p'q} G_{;p'q} + 4 \left( G_{;p'} G_{;q} + G G_{;p'q} \right) \\
&\quad - 4 \left( G_{;p} G_{;q'} + G_{;p'} G_{;q} \right) \\
&\quad + R G_{;p'} G_{;p'} + R' G_{;p} G_{;p} \\
&\quad - 2 \left( R G_{;p'} + R' G_{;p} \right) G + \frac{1}{2} R R' G^2 .
\end{aligned} \tag{2.4c}$$

Note that the superscript  $+$  on  $G^+$  has been omitted in the above equations for notational simplicity. Primes on indices denote tensor indices at the point  $x'$  and unprimed ones denote indices at the point  $x$ .  $R_{ab}$  is the Ricci tensor evaluated at the point  $x$ ,  $R_{c'd'}$  is the Ricci tensor evaluated at the point  $x'$ ,  $R$  is the scalar curvature evaluated at the point  $x$ , and  $R'$  is the scalar curvature evaluated at the point  $x'$ .

The definition (1.4) of the noise kernel immediately implies that it is symmetric under interchange of the two spacetime points and the corresponding pairs of indices

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<sup>4</sup>Notice that these equations have two slight but crucial differences with the equations of Ref. [39]. The sign of the last term of the equation for  $N_{abc'd'}$  and also the sign of the term  $G G_{;abp'}$  have been corrected.

so that

$$N_{abc'd'}(x, x') = N_{c'd'ab}(x', x) . \quad (2.5)$$

The noise kernel obeys other important properties as well. These have been proven in Refs. [24,25], so we just state them here. The first property, which is clear from (1.4), is that the following conservation laws must hold:

$$\nabla^a N_{abc'd'} = \nabla^b N_{abc'd'} = \nabla^{c'} N_{abc'd'} = \nabla^{d'} N_{abc'd'} = 0 . \quad (2.6)$$

The second property which must be satisfied because the field is conformally invariant is that the partial traces must vanish, that is

$$N^a{}_{ac'd'} = N_{ab}{}^{c'}{}_{c'} = 0 . \quad (2.7)$$

A third important property is that the noise kernel is positive semidefinite, namely

$$\int d^4x \sqrt{-g(x)} \int d^4x' \sqrt{-g(x')} f^{ab}(x) N_{abc'd'}(x, x') f^{c'd'}(x') \geq 0, \quad (2.8)$$

for any real tensor field  $f^{ab}(x)$ .

Finally the noise kernel for the conformally invariant field has a simple scaling behavior under conformal transformations. In the Appendix two proofs are given which show that under a conformal transformation between two conformally related  $D$ -dimensional spacetimes with metrics of the form  $\tilde{g}_{ab} = \Omega^2(x) g_{ab}$  and conformally related states, the noise kernel transforms as

$$\tilde{N}_{abc'd'}(x, x') = \Omega^{2-D}(x) \Omega^{2-D}(x') N_{abc'd'}(x, x') . \quad (2.9)$$

### 3 Gaussian approximation in the optical Schwarzschild spacetime

As discussed in the introduction, we want to compute the noise kernel in a background Schwarzschild spacetime for the conformally invariant scalar field when the

points are separated. For an arbitrary separation it would be necessary to do this numerically. However, if the separation is small then it is possible to use approximation methods to compute the Wightman function analytically and from that the noise kernel. For a conformally invariant field a significant simplification is possible because the Green function and the resulting noise kernel can be computed in the optical Schwarzschild spacetime (which is conformal to Schwarzschild spacetime) and then conformally transformed to Schwarzschild spacetime. A similar calculation was done by Page [44] for the stress tensor expectation value of a conformally invariant scalar field. He first calculated the Euclidean Green function for the field in a thermal state using a Gaussian approximation. Then the stress tensor was computed and conformally transformed to Schwarzschild spacetime. We shall use Page's approximation for the Euclidean Green function to obtain an approximation for the Wightman Green function and then compute the noise kernel using that approximation.

### 3.1 Gaussian approximation for the Wightman Green function

In order to use Page's approximation we must relate the Euclidean Green function in a static spacetime to the Wightman function. To do so we begin by noting that the Wightman function can be written in terms of two other Green functions [47], the Hadamard function  $G^{(1)}(x, x')$  and the Pauli-Jordan function  $G(x, x')$ :

$$G^+(x, x') = \frac{1}{2} [G^{(1)}(x, x') + iG(x, x')] \quad (3.1a)$$

$$G^{(1)}(x, x') \equiv \langle \{\phi(x), \phi(x')\} \rangle \quad (3.1b)$$

$$iG(x, x') \equiv \langle [\phi(x), \phi(x')] \rangle . \quad (3.1c)$$

As discussed in the Introduction, we restrict our attention in this paper to spacelike and timelike separations of the points. In general  $G(x, x') = 0$  for spacelike separa-

tions. Furthermore, in the optical Schwarzschild spacetime,  $G(x, x') = O[(x - x')^4]$  for timelike separations of the points. To see this consider the general form of the Hadamard expansion for  $G(x, x')$  which is [48]

$$G(x, x') = \frac{u(x, x')}{4\pi} \delta(-\sigma) - \frac{v(x, x')}{8\pi} \theta(-\sigma), \quad (3.2)$$

with  $\sigma(x, x')$  defined to be one-half the square of the proper distance along the shortest geodesic connecting the two points. In [49] it was shown that in Schwarzschild spacetime  $v(x, x') = O[(x - x')^4]$ . Since the Green function in the optical spacetime can be obtained from that in Schwarzschild spacetime by a simple conformal transformation, the same must be true of the quantity  $v(x, x')$ . Thus, so long as we work only to  $O[(x - x')^2]$  and restrict our attention to points which are either spacelike or timelike separated, then in the optical Schwarzschild spacetime

$$G^+(x, x') = \frac{1}{2}G^{(1)}(x, x') + O[(x - x')^4]. \quad (3.3)$$

The Hadamard Green function can be computed using the Euclidean Green function in the following way. First, define the Euclidean time as

$$\tau \equiv it. \quad (3.4)$$

Then the metric in a static spacetime takes the form

$$ds^2 = g_{\tau\tau}(\vec{x})d\tau^2 + g_{ij}(\vec{x})dx^i dx^j. \quad (3.5)$$

The Euclidean Green function obeys the equation

$$(\square - \frac{1}{6}R)G_E(x, x') = -\frac{\delta(x - x')}{\sqrt{g(x)}}. \quad (3.6)$$

Because the spacetime is static  $G_E$  will be a function of  $(\Delta\tau)^2 = (\tau - \tau')^2$ . It is possible to obtain the Feynman Green function  $G_F(x, x')$  by making the following

transformation [50]:

$$(\Delta\tau)^2 \rightarrow -(\Delta t)^2 + i\epsilon , \quad (3.7)$$

under which

$$G_E(x, x') \rightarrow iG_F(x, x') . \quad (3.8)$$

Writing the Feynman Green function in terms of the Hadamard and Jordan functions [47],

$$G_F(x, x') = -\frac{1}{2}i G^{(1)}(x, x') + \frac{1}{2} [\theta(t - t') - \theta(t' - t)] G(x, x') , \quad (3.9)$$

one finds that Eq. (3.3) becomes

$$G^+(x, x') = -\text{Im} G_F(x, x') + O[(x - x')^4] . \quad (3.10)$$

As mentioned above, Page made use of the DeWitt-Schwinger expansion to obtain his approximation for the Euclidean Green function. Before displaying his approximation it is useful to discuss two quantities which appear in that expansion. For a more complete discussion see Ref. [45]. The fundamental quantity out of which everything is built is Synge's world function  $\sigma(x, x')$ , which is defined to be one-half the square of the proper distance between the two points  $x$  and  $x'$  along the shortest geodesic connecting them. It satisfies the relationship

$$\sigma(x, x') = \frac{1}{2} g_{ab}(x) \sigma^{;a}(x, x') \sigma^{;b}(x, x') , \quad (3.11)$$

and it is traditional to use the notation

$$\sigma^a \equiv \sigma^{;a} . \quad (3.12)$$

As shown in Ref. [45], it is possible to expand biscalars, bivectors, and bitensors in powers of  $\sigma^a$  in an arbitrary spacetime about a given point  $x$ . Then the coefficients in that expansion are evaluated at the point  $x$ . For example

$$\sigma_{;ab}(x, x') = g_{ab}(x) - \frac{1}{3} R_{acbd}(x) \sigma^c(x, x') \sigma^d(x, x') + \dots . \quad (3.13)$$

Examination of this expansion shows that to zeroth order in  $\sigma^a$

$$\sigma_{;abc} = 0 . \quad (3.14)$$

The second quantity we shall need is

$$U(x, x') \equiv \Delta^{1/2}(x, x') , \quad (3.15a)$$

$$\Delta(x, x') \equiv -\frac{1}{\sqrt{-g(x)}\sqrt{-g(x')}} \det(-\sigma_{;ab'}) . \quad (3.15b)$$

Note that covariant derivatives at the point  $x'$  commute with covariant derivatives at the point  $x$ . Two important properties of  $U(x, x')$  are

$$U(x, x) = 1 \quad (3.16a)$$

$$(\ln U)_{;a} \sigma^a = 2 - \frac{1}{2} \square \sigma . \quad (3.16b)$$

One can also expand  $U$  in powers of  $\sigma^a$  with the result that [45]

$$\begin{aligned} U(x, x') &= 1 + \frac{1}{12} R_{ab} \sigma^a \sigma^b - \frac{1}{24} R_{ab;c} \sigma^a \sigma^b \sigma^c, \\ &+ \frac{1}{1440} (18 R_{ab;cd} + 5 R_{ab} R_{cd} + 4 R_{paqb} R_c^q d^p) \sigma^a \sigma^b \sigma^c \sigma^d \\ &+ O[(\sigma^a)^5]. \end{aligned} \quad (3.17)$$

The above definitions, properties and expansions apply to arbitrary spacetimes. Given any static metric, one can always transform it to an ultra-static one, called the *optical metric*, by a conformal transformation. This kind of metric is of the form

$$ds^2 = -dt^2 + g_{ij}(\vec{x}) dx^i dx^j , \quad (3.18)$$

with the metric functions  $g_{ij}$  independent of the time  $t$ . In this case Synge's world function is given by

$$\sigma(x, x') = \frac{1}{2} (-\Delta t^2 + \mathbf{r}^2) , \quad (3.19)$$

with

$$\Delta t \equiv t - t', \quad \mathbf{r} \equiv \sqrt{2} {}^{(3)}\sigma. \quad (3.20)$$

The quantity  ${}^{(3)}\sigma$  is the part of  $\sigma$  that depends on the spatial coordinates. Note that we use  $\mathbf{r}$  (with bold roman font) to denote the quantity in Eq.(3.20) while  $r$  (with normal italic font) denotes the radial coordinate. Some useful properties of  $\mathbf{r}$  are

$$\nabla_i \mathbf{r} = \frac{{}^3\sigma_i}{\mathbf{r}}, \quad (3.21a)$$

$$\nabla^2 \mathbf{r} = \frac{{}^3\sigma_i^i - 1}{\mathbf{r}}, \quad (3.21b)$$

$$\nabla_{i\mathbf{r}} \nabla^i \mathbf{r} = \frac{{}^3\sigma_i {}^3\sigma^i}{2 {}^3\sigma} = 1, \quad (3.21c)$$

$$2 ({}^{(3)}\Delta^{1/2})_{,i} \nabla^i \mathbf{r} = \left( \frac{3}{\mathbf{r}} - \nabla^2 \mathbf{r} \right) {}^{(3)}\Delta^{1/2}, \quad (3.21d)$$

where  $\nabla^2 = \nabla^i \nabla_i$ . Note that from Eqs. (3.15b) and (3.19) one can easily see that for an ultra-static spacetime the Van-Vleck determinant  ${}^{(3)}\Delta$  for the spatial metric  $g_{ij}$  coincides with the Van-Vleck determinant  $\Delta$  for the full spacetime. (The advantage of using  ${}^{(3)}\Delta$  rather than  $\Delta$  is that, although noncovariant, it is valid for arbitrary time separations, and one only needs to expand in powers of  $\mathbf{r}$ .)

The optical Schwarzschild metric

$$ds^2 = -dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)^2} dr^2 + \frac{r^2}{1 - \frac{2M}{r}} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (3.22)$$

is of the form (3.18) and is conformally related to the Schwarzschild metric in standard coordinates with a conformal factor

$$\Omega^2 = \left(1 - \frac{2M}{r}\right). \quad (3.23)$$



For this metric Page [44] used a Gaussian approximation to obtain an expression for the Euclidean Green function in a thermal state at the temperature

$$T = \frac{\kappa}{2\pi} . \quad (3.24)$$

This expression is valid for any temperature; however, if

$$\kappa = \frac{1}{4M} , \quad (3.25)$$

then the temperature is that of the black hole in the Schwarzschild spacetime which is conformal to the optical metric (3.22). In this case the state of the field corresponds to the Hartle-Hawking state, which is regular on the horizon. Page found that [44]

$$G_E(\Delta\tau, \vec{x}, \vec{x}') = \frac{\kappa \sinh(\kappa \mathbf{r})}{8\pi^2 \mathbf{r} [\cosh(\kappa \mathbf{r}) - \cos(\kappa \Delta\tau)]} U(\Delta\tau, \vec{x}, \vec{x}') . \quad (3.26)$$

Analytically continuing to the Lorentzian sector using the prescriptions (3.7) and (3.8), and substituting the result into Eq. (3.10) gives

$$G^+(\Delta t, \vec{x}, \vec{x}') = \frac{\kappa \sinh \kappa \mathbf{r}}{8\pi^2 \mathbf{r} [\cosh(\kappa \mathbf{r}) - \cosh(\kappa \Delta t)]} U(\Delta t, \vec{x}, \vec{x}') . \quad (3.27)$$

To determine the accuracy of this approximation we can substitute the above expression into the equation satisfied by  $G^+$  which is

$$\square G^+(x, x') - \frac{R}{6} G^+(x, x') = 0 . \quad (3.28)$$

The accuracy of the Gaussian approximation in the optical Schwarzschild metric will be determined by the lowest order in  $(x - x')$  at which Eq. (3.28) is not satisfied. Applying the differential operator for the metric (3.22) and using Eqs. (3.21c)-(3.21d), one finds after some calculation that

$$\left( \square - \frac{1}{6} R \right) G^+(x, x') = \frac{\kappa \sinh(\kappa \mathbf{r})}{\mathbf{r} [\cosh(\kappa \mathbf{r}) - \cosh(\kappa \Delta t)]} \left( \square - \frac{1}{6} R \right) U(x, x') . \quad (3.29)$$

If Eq. (3.17) is substituted into Eq. (3.29) and Eqs. (3.13)-(3.14) are used, then one finds

$$\left(\square - \frac{R}{6}\right) U(x, x') = Q_0 + Q_p \sigma^p + Q_{pq} \sigma^p \sigma^q + \dots \quad (3.30)$$

with

$$Q_0 = 0, \quad (3.31a)$$

$$Q_a \sigma^a = \sigma^a G_a{}^b{}_{;b} = 0, \quad (3.31b)$$

$$Q_{ab} \sigma^a \sigma^b = \frac{1}{360} \left( 9R_{;ab} + 9R_{ab;c}{}^c - 24R_{ac; b}{}^c - 12R_{ac} R_b{}^c \right. \\ \left. + 6R^{cd} R_{cadb} + 4R_{acde} R_b{}^{edc} + 4R_{acde} R_b{}^{cde} \right) \sigma^a \sigma^b. \quad (3.31c)$$

Here  $G_{ab}$  is the Einstein tensor. For the optical Schwarzschild metric (3.22),  $Q_{ab} \sigma^a \sigma^b = 0$ . Thus, Eq. (3.30) is zero to  $O[(x - x')^2]$ . Since  $\square$  is a second order derivative operator, this means that the Gaussian approximation for  $G^+(x, x')$ , whose leading order behavior is  $G^+(x, x') \sim (x - x')^{-2}$ , is accurate through  $O[(x - x')^2]$ . Note that this approximation is valid for arbitrary temperature since Eq. (3.29) holds for arbitrary values of  $\kappa$ .

It is important to emphasize that the order of accuracy obtained here is for the Schwarzschild optical metric (3.22). Because the Gaussian approximation is equivalent to the lowest order term in the DeWitt-Schwinger expansion, it is only guaranteed to be accurate to leading order in  $x - x'$  in a general spacetime.

## 3.2 Order of validity of the noise kernel

In Sec. 2 an expression for the noise kernel is given in terms of covariant derivatives of the Wightman function. In each term there is a product of Wightman functions and varying numbers of covariant derivatives. The accuracy of the Gaussian approximation for the Wightman function can be used to estimate the order of accuracy of

the noise kernel. First, recall that the leading order of the Wightman function goes like  $(x - x')^{-2}$ . Since there is a maximum of four derivatives acting on a product of Wightman functions, one expects that at leading order the noise kernel will go like  $(x - x')^{-8}$ . Since the Gaussian approximation to the Wightman function in the optical Schwarzschild spacetime is accurate through terms of order  $(x - x')^2$ , it is clear from Eq. (2.4) that our expression for the noise kernel should be accurate up to and including terms of order  $(x - x')^{-4}$ .

## 4 Computation of the Noise Kernel

In this section we discuss the computation of the noise kernel in two different but related cases. In both the field is in a thermal state at an arbitrary temperature  $T$  and the points are separated in a non-null direction. The first case considered is flat space where the calculation of the noise kernel is exact. In the second case an approximation to the noise kernel is computed for the optical Schwarzschild metric (3.22). The result is then conformally transformed to Schwarzschild spacetime using Eq. (2.9).

### 4.1 Hot Flat Space

In flat space the function  $U(x, x')$  is exactly equal to one. Examination of Eq. (3.29) then shows that the expression for  $G^+(x, x')$  in Eq. (3.27) is exact so long as the points are separated in a non-null direction. This expression can be substituted into Eqs. (2.3) and (2.4) to obtain an exact expression for the noise kernel. Here the quantity  $\mathbf{r}$  takes on the following simple form in Cartesian coordinates and components:

$$\mathbf{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}. \quad (4.1)$$

The only subtlety which one must be aware of is that the point separation must be arbitrary before the derivatives are computed. Once they are computed, then any

point separation that one desires can be used.

All components of the noise kernel have been calculated when the points are separated in a non-null direction. Both conservation and the vanishing of the partial traces have been checked. Due to the length of many of the components, we just display one of them here:

$$\begin{aligned}
N_{ttt'v'} = & \frac{\kappa^2 \sinh^2(\kappa \mathbf{r})}{192\pi^4 \mathbf{r}^6 (\cosh(\kappa \Delta t) - \cosh(\kappa \mathbf{r}))^2} \\
& + \frac{\kappa^3 \sinh(\kappa \mathbf{r})}{96\pi^4 \mathbf{r}^5 (\cosh(\kappa \Delta t) - \cosh(\kappa \mathbf{r}))^3} [1 - \cosh(\kappa \Delta t) \cosh(\kappa \mathbf{r})] \\
& + \frac{\kappa^4}{192\pi^4 \mathbf{r}^4 (\cosh(\kappa \Delta t) - \cosh(\kappa \mathbf{r}))^4} [2 - 2 \cosh(\kappa \Delta t) \cosh(\kappa \mathbf{r}) \\
& - \cosh^2(\kappa \mathbf{r}) + \cosh^2(\kappa \Delta t) \cosh(2\kappa \mathbf{r})] \\
& + \frac{\kappa^5 \sinh(\kappa \mathbf{r})}{288\pi^4 \mathbf{r}^3 (\cosh(\kappa \Delta t) - \cosh(\kappa \mathbf{r}))^4} [2 \cosh(\kappa \Delta t) - \cosh(2\kappa \Delta t) \cosh(\kappa \mathbf{r}) \\
& - \cosh(\kappa \Delta t) \cosh^2(\kappa \mathbf{r})] \\
& - \frac{\kappa^6}{576\pi^4 \mathbf{r}^2 (\cosh(\kappa \Delta t) - \cosh(\kappa \mathbf{r}))^6} [12 - 6 \cosh^2(\kappa \Delta t) \\
& + \cosh^4(\kappa \Delta t) - 12 \cosh(\kappa \Delta t) \cosh(\kappa \mathbf{r}) + \cosh^3(\kappa \Delta t) \cosh(\kappa \mathbf{r}) \\
& - 18 \cosh^2(\kappa \mathbf{r}) + 12 \cosh^2(\kappa \Delta t) \cosh^2 - 2 \cosh^4(\kappa \Delta t) \cosh^2(\kappa \mathbf{r})(\kappa \mathbf{r}) \\
& + 17 \cosh(\kappa \Delta t) \cosh^3(\kappa \mathbf{r}) + 3 \cosh^3(\kappa \Delta t) \cosh^3(\kappa \mathbf{r}) - \cosh^4(\kappa \mathbf{r}) \\
& - 6 \cosh(2\kappa \Delta t) \cosh^4(\kappa \mathbf{r}) - \cosh(\kappa \Delta t) \cosh^5(\kappa \mathbf{r})] . \tag{4.2}
\end{aligned}$$

## 4.2 Schwarzschild Spacetime

The calculation of the noise kernel in the optical Schwarzschild metric (3.22) proceeds in the same way as the flat space calculation in Sec. 4.1. That is, one simply substitutes the expression (3.27) for  $G^+(x, x')$ , which is now approximate rather than exact, into Eqs. (2.3)-(2.4) and computes the various derivatives and curvature tensors, again with an arbitrary separation of the points. After the derivatives are computed, the specific separation of the points which is of interest may be taken. Because the expression for  $G^+(x, x')$  is approximate, one must expand the result in powers of  $(x - x')$  and, as discussed at the end of Sec. 3, the result should be truncated at order  $(x - x')^{-4}$ .

The expansion in powers of  $(x - x')$  can be consistently done for all contributions to the noise kernel and the results of such an expansion are shown below in Sec. 4.2. However, if the separation of the points is only in the time direction, then it is possible to obtain a result valid for arbitrary values of  $\kappa$  (and, hence, of  $\kappa\Delta t$ ). This can be achieved by treating exactly the prefactor in Eq. (3.27), which contains all the dependence on  $\kappa$ , while expanding the quantity  $U(x, x')$  and its derivatives in powers of  $(x - x')$ . There are two reasons why this works. The first is that, as can be seen from Eq. (3.29), what is keeping  $G^+(x, x')$  in Eq. (3.27) from being exact is the fact that for the optical Schwarzschild metric  $\square_x U(x, x')$  is not exactly zero but only vanishes to order  $(x - x')^2$ . So in some sense the function that multiplies  $U$  in Eq. (3.27) can be treated as exact. Secondly, although exact analytic expressions for the function  $\mathbf{r}$  and its derivatives in terms of simple functions are not known for an arbitrary splitting of the points, in the limit that the points are separated only in the time direction such expressions are known. Therefore, it should be possible to treat these terms exactly when the final point separation is in the time direction. It is worth noting that it is consistent to have a quasi-local expansion in which  $\Delta t$  is in an appropriate sense small (as specified in footnote 2) and yet to have  $\kappa|\Delta t| \gtrsim 1$ .

The reason is that the scale over which the geometry varies significantly in the optical metric is  $O(M)$ . For the Hartle-Hawking state  $\kappa = 1/(4M)$  and the validity of the quasi-local expansion would start to break down for a  $\Delta t$  such that  $\kappa|\Delta t| \sim 1$ , but the Hartle-Hawking state is a very low temperature state for a macroscopic black hole. Therefore, one can have temperatures which are well below the Planck temperature and still have  $\kappa\Delta t \gg 1$ . Furthermore, even when  $\kappa|\Delta t| \ll 1$  it is sometimes useful to have the exact dependence on  $\kappa\Delta t$ . An example illustrating this point is the Rindler limit of Schwarzschild spacetime with the field in the Hartle-Hawking state, which is discussed in Sec. 4.2.

To compute the noise kernel using the above method it is necessary to find expansions for both  $\mathbf{r}$  and  $U(x, x')$  in powers of  $(x - x')$ . For the former it is easier to work with the quantity  $\sigma(x, x')$  which is one-half the square of the proper distance between  $x$  and  $x'$  along the shortest geodesic that connects them. The relationship between  $\sigma$  and  $\mathbf{r}$  is given by Eq. (3.19). Furthermore, since the metric (3.22) is also spherically symmetric,  $\sigma$  can only depend on the angular quantity

$$\cos(\gamma) \equiv \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi'), \quad (4.3)$$

where  $\gamma$  is the angle between  $\vec{x}$  and  $\vec{x}'$ . It turns out to be convenient to write  $\sigma$  in terms of the quantity

$$\eta \equiv \cos \gamma - 1. \quad (4.4)$$

Then for points that are sufficiently close together one can use the expansion

$$\sigma(x, x') = -\frac{1}{2}(t - t')^2 + \sum_{j,k} w_{jk}(r) \eta^j (r - r')^k, \quad (4.5)$$

with the sums over  $j$  and  $k$  starting at  $j = 0$  and  $k = 0$  respectively. For the

metric (3.22), Eq. (3.11) has the explicit form

$$\sigma = \frac{1}{2} \left[ - \left( \frac{\partial \sigma}{\partial t} \right)^2 + \left( 1 - \frac{2M}{r} \right)^2 \left( \frac{\partial \sigma}{\partial r} \right)^2 - \frac{1}{r^2} \left( 1 - \frac{2M}{r} \right) \left( \frac{\partial \sigma}{\partial \eta} \right)^2 (2\eta + \eta^2) \right]. \quad (4.6)$$

Substituting the expansion (4.5) into Eq. (4.6) and equating powers of  $(x^a - x^{a'})$ , one finds that

$$\begin{aligned} \sigma(x, x') = & -\frac{(\Delta t)^2}{2} + \frac{(\Delta r)^2}{2f^2} - \frac{r^2 \eta}{f} + (\Delta r)^3 \left( \frac{1}{2rf^3} - \frac{1}{2rf^2} \right) \\ & + \eta \Delta r \left( \frac{3r}{2f} - \frac{r}{2f^2} \right) + O[(x - x')^4], \end{aligned} \quad (4.7)$$

with  $\Delta r \equiv r - r'$  and

$$f \equiv 1 - \frac{2M}{r}. \quad (4.8)$$

Because of the form of Eq. (3.16b) an expansion for  $U(x, x')$  can be found by writing

$$\ln U(x, x') = \sum_{j,k} u_{jk}(r) \eta^j (r - r')^k. \quad (4.9)$$

It can be seen from Eqs. (3.19) and (3.15) that for a spacetime with metric (3.22)  $U$  is time independent. If Eq. (4.9) is substituted into Eq. (3.16b), Eqs. (3.16a) and (4.7) are used, and the result is expanded in powers of  $(x - x')$ , then one finds that

$$U(x, x') = 1 + \frac{(\Delta r)^2}{8r^2} \left( 1 - \frac{2}{3f} - \frac{1}{3f^2} \right) + \eta \left( \frac{f}{4} - \frac{1}{3} + \frac{1}{12f} \right) + O[(x - x')^3]. \quad (4.10)$$

Since the leading order of the prefactor multiplying  $U$  in Eq. (3.27) is  $(x - x')^{-2}$ , we need to calculate  $U(x, x')$  through  $O[(x - x')^4]$ . That way we can obtain the Wightman function through  $O[(x - x')^2]$ , which is consistent with the order to which the Gaussian approximation was shown to be valid in Sec. 3. One can compute

$U(x, x')$  to the required order by substituting the expansions (4.7) for  $\sigma$  and (4.10) for  $U$  into Eq. (3.16b). To obtain a final expression for  $U$  valid through  $O[(x - x')^4]$  it is necessary to have the expansion for  $\sigma$  containing terms through  $O[(x - x')^6]$ . [As an illustration, we have shown the results through quadratic order in Eqs. (4.7) and (4.10).]

Using Eq. (3.19) an expansion for the quantity  $\mathbf{r}$  can be obtained from the expansion for  $\sigma(x, x')$ . This along with the expansions for  $U(x, x')$  can be substituted into Eq. (3.27) to obtain an expansion for the Wightman function  $G^+(x, x')$ . That in turn can then be substituted into the expressions (2.4) for the noise kernel and the derivatives can be computed. As discussed in Sec. 3.2, one should keep terms through  $O[(x - x')^{-4}]$  since this is the highest order for which the Gaussian approximation for the noise kernel is valid. To obtain the noise kernel for Schwarzschild spacetime one then uses the conformal transformation (2.9) with  $\Omega^2(x) = 1 - 2M/r$ . Finally, the coordinate  $r'$  is written as  $r' = r - (r - r')$  in order to expand the resulting expression in powers of  $(r - r')$  through quartic order.

### Arbitrary Temperature

Following the method described above and using the exact expression for the  $\kappa$ -dependent prefactor in Eq. (3.27), we have computed several components of the noise kernel for points split in the time direction when  $\kappa\Delta t$  is not assumed to be small.



The result for the  $N_t^{t t'}$  component is

$$\begin{aligned}
N_t^{t t'} = & \frac{1}{1728\pi^4 f^4} \left[ \kappa^8 \frac{(2 \cosh^2(\kappa\Delta t) - \cosh(\kappa\Delta t) + 26)}{(\cosh(\kappa\Delta t) - 1)^4} \right. \\
& + \frac{\kappa^6}{4r^2} \frac{(1-f)^2(1-4\cosh(\kappa\Delta t))}{(\cosh(\kappa\Delta t) - 1)^3} \\
& \left. + \frac{\kappa^4}{8r^4} \frac{(1-f)^2(1-2f+3f^2)}{(\cosh(\kappa\Delta t) - 1)^2} \right]. \tag{4.11}
\end{aligned}$$

There are two ways to take the flat space limit of this result. One possibility is to take  $f \rightarrow 1$  in Eq. (4.11). This limit reduces to the corresponding expression for the exact noise kernel in flat space at arbitrary temperature, whose spatial coincidence limit can be obtained from Eq. (4.2) by taking the limit  $\mathbf{r} \rightarrow 0$ .

A second possibility for obtaining the flat space limit is to realize that the geometry of the near-horizon region is that of Rindler spacetime and in the limit of large Schwarzschild radius this holds for an arbitrarily large region. Indeed, if one introduces the new coordinates

$$\begin{aligned}
\xi &= 4M\sqrt{r/2M-1}, \\
T &= \frac{t}{4M}, \tag{4.12}
\end{aligned}$$

in the near-horizon region, characterized by  $|r/2M-1| \ll 1$ , the standard Schwarzschild metric reduces to

$$ds^2 \approx -\xi^2 dT^2 + d\xi^2 + dx_\perp^2, \tag{4.13}$$

where  $dx_\perp^2 = 4M^2(d\theta^2 + \sin^2\theta d\phi^2)$  becomes the metric of a Euclidean plane (say, tangent to  $\theta = \phi = 0$ ) when  $M \rightarrow \infty$ . In the new coordinates, the near-horizon condition corresponds to  $\xi \ll 4M$ . Therefore, in the limit  $M \rightarrow \infty$  one recovers the

full Minkowski spacetime in Rindler coordinates. Rewriting Eq. (4.11) in terms of  $\xi$  and  $T$ , and taking the limit  $M \rightarrow \infty$ , we get

$$N_{T T'}^{T T'} = \frac{1}{4\pi^4} \frac{1}{[2\xi \sinh(\Delta T/2)]^8}. \quad (4.14)$$

This result agrees with the flat space calculation of the noise kernel in Ref. [26] if one takes into account that our definition of the noise kernel is four times their definition. In order to do the comparison one needs to consider Eqs. (3.10), (4.9) and (4.12) in Ref. [26] and rewrite their results in terms of Rindler coordinates by using their relation to inertial coordinates ( $x^0 = \xi \sinh T$ ,  $x^1 = \xi \cosh T$ ); in particular, this implies that for pairs of points with equal  $\xi$  the Minkowski interval is given by  $(x - x')^2 = 4\xi^2 \sinh^2(\Delta T/2)$ . It also requires transforming the tensor components accordingly using the Jacobian of the coordinate transformation.

It should be noted that in order to get the Minkowski vacuum in this limit, one needs to consider the Hartle-Hawking state, which is regular on the horizon of the black hole. In that case  $\kappa$  is tied to  $M$  as given by Eq. (3.25), so that  $\kappa\Delta t = \Delta T$ . It is, therefore, important to have an expression valid for arbitrarily large  $\kappa\Delta t$ , because this guarantees that the exact Rindler result is obtained, rather than an approximate expansion valid only up to some order for small  $\Delta T$ . (The condition for the validity of the quasi-local expansion is  $\sigma/R_S^2 \approx 8(\xi/4M)^2 \sinh^2(\Delta T/2) \ll 1$ , which is fulfilled for any values of  $\xi$  and  $\Delta T$  as  $M \rightarrow \infty$ .)

The two different ways of obtaining the flat space limit described above provide (partially independent) nontrivial checks of our result. We have also checked one of the partial traces which involve the  $N_t^{t t'}$  component and have shown that it vanishes to the appropriate order. Finally, we have partially checked one of the conservation conditions by initially considering an arbitrary separation of the points, computing the relevant derivatives, and then taking the limit that the separation is only in

the time direction. It was only shown that the conservation condition is satisfied to the second highest order used in the computation. In terms of the expansion of  $U(x, x')$  discussed above this corresponds to order  $(x - x')^2$ . We are confident that the conservation condition is also satisfied to the highest order used in the computation,  $O[(x - x')^4]$ , but due to the large number of terms involved, we have not shown this explicitly.

### Moderate and low temperature

If one is interested in point separations such that  $\kappa\Delta t \ll 1$  and  $\kappa\mathbf{r} \ll 1$ , then it is useful to expand the Wightman function in powers of  $(x - x')$  before substituting it into the expressions (2.4). The general expression (3.26) for  $G^+(x, x')$  in the Gaussian approximation can be expanded as [43]

$$G^+(x, x') = \frac{1}{8\pi^2} \left[ \frac{1}{\sigma} + \frac{\kappa^2}{6} - \frac{\kappa^4}{180} (2(\Delta t)^2 + \sigma) \right. \\ \left. + O[(x - x')^4] \right] U(x, x'). \quad (4.15)$$

Since  $U(x, x') = 1 + O[(x - x')^2]$ , the terms within the square brackets in Eq. (4.15) have been kept through  $O[(x - x')^2]$ , which is consistent with the order to which the Gaussian approximation was shown to be valid in Sec. 3. We use the same expansion through  $O[(x - x')^4]$  of both  $U(x, x')$  and the conformal factor  $\Omega^2(x')$ , described in the general discussion on Schwarzschild of this subsection.

Using this approach we have computed several components of the noise kernel when  $\kappa\Delta t \ll 1$  and  $\kappa\mathbf{r} \ll 1$  through  $O[(x - x')^{-4}]$ . The resulting expressions for an arbitrary separation of the points are too long to display in full here. If the points are

separated along the time direction we get the following result for the  $N_t{}^{t'}$  component:

$$N_t{}^{t'} = \frac{1}{4\pi^4 f^4} \left[ \frac{1}{\Delta t^8} - \frac{(1-f)^2}{72r^2 \Delta t^6} + \frac{(1-f)^2(1-2f+3f^2)}{864r^4 \Delta t^4} \right. \\ \left. - 5\kappa^2 \left( \frac{1}{18\Delta t^6} + \frac{(1-f)^2}{864r^2 \Delta t^4} \right) + \kappa^4 \frac{17}{270\Delta t^4} \right], \quad (4.16)$$

which agrees with the expansion of Eq. (4.11) through  $O[(\kappa\Delta t)^{-4}]$ . If the points are separated along the radial direction, then we find

$$N_t{}^{t'} = \frac{1}{2\pi^4} \left[ \frac{f^4}{2\Delta r^8} - \frac{(1-f)f^3}{r\Delta r^7} + \frac{(1-f)(89-169f)f^2}{144r^2 \Delta r^6} \right. \\ \left. + \frac{(1-f)(1577-292f-441f^2)f}{432r^3 \Delta r^5} \right. \\ \left. - \frac{(1-f)(240199-383185f+98655f^2+18315f^3)}{25920r^4 \Delta r^4} \right. \\ \left. + \kappa^2 \left( \frac{f^2}{36\Delta r^6} - \frac{(1-f)f}{36r\Delta r^5} + \frac{(1-f)(7-39f)}{1728r^2 \Delta r^4} \right) \right. \\ \left. - \frac{\kappa^4}{270\Delta r^4} \right]. \quad (4.17)$$

Note that the limit  $\kappa \rightarrow 0$  of Eqs. (4.16)-(4.17) corresponds to the Boulware vacuum and the limit  $f \rightarrow 1$  corresponds to the flat space limit. The latter coincides through  $O[(\kappa\Delta t)^{-4}]$  and  $O[(\kappa\Delta r)^{-4}]$ , respectively, with the exact result in Eq. (4.2) for the appropriate splitting of the points. Moreover, this coincidence is exact for zero temperature.

As discussed in Sec. 2, the noise kernel has two properties which can be used to check our calculations. One of these, given in Eq. (A1), is that the noise kernel

should be separately conserved at the points  $x$  and  $x'$ . The other, given in Eq. (A2), is the vanishing of the partial traces. Enough components have been computed when  $\kappa|\Delta t| \ll 1$  and  $\kappa\mathbf{r} \ll 1$  that we have been able to check all of the partial traces and all of the conservation conditions which involve the component  $N_t^{t'}$ . In each case they are satisfied to the order to which our computations are valid:  $O[(x - x')^{-4}]$  for the partial traces and  $O[(x - x')^{-5}]$  for the conservation conditions.

## 5 Discussion

Using Page's approximation for the Euclidean Green function of a conformally invariant scalar field in the optical Schwarzschild spacetime, which is conformal to the static region of Schwarzschild spacetime, we have computed an expression for the Wightman function when the field is in a thermal state at an arbitrary temperature. For the case that the temperature is equal to  $(8\pi M)^{-1}$  and one conformally transforms to Schwarzschild spacetime this corresponds to the Hartle-Hawking state. This expression is exact for flat space and is valid through order  $(x - x')^2$  in the optical Schwarzschild spacetime. From this expression for the Wightman function we have calculated the exact noise kernel in flat space and several components of an approximate one in Schwarzschild spacetime. The latter is obtained by conformally transforming the noise kernel in the optical Schwarzschild spacetime to Schwarzschild spacetime. We have shown that, unlike for the case of the stress tensor expectation value, this transformation is trivial. In both the flat space and Schwarzschild cases we have restricted our attention to point separations which are either spacelike or timelike and we do not consider the limit in which the points come together.

For Schwarzschild spacetime we have considered two separate but related approximations for the noise kernel. The first one is valid for small separations (compared to the typical curvature radius scale) and arbitrary temperature. Note that although

the Hartle-Hawking state corresponds to a specific temperature, given by Eqs. (3.24)-(3.25), our results also apply to any other temperature since we have kept  $\kappa$  arbitrary in all our expressions. The states for those other values of the temperature are singular on the horizon (e.g. the expectation value of the stress tensor diverges there), but can sometimes be of interest (e.g. the Boulware vacuum corresponds to the particular case of  $T = 0$ ). The second approximation corresponds to the additional restriction that the separation of the points is much smaller than the inverse temperature and thus works for points that are extremely close together and/or temperatures that are very low. We have computed several components of the noise kernel for both approximations.

The component  $N_t{}^t{}_{t'}{}^{t'}$  is displayed for both flat space (4.2) and Schwarzschild spacetime. In Schwarzschild spacetime it has been computed when the point separation is only in the time direction and the product of the temperature and the point separation is not assumed to be small (4.11). It has also been computed for an arbitrary spacelike or timelike separation of the points when the product of the temperature and point separation is small. In this case, because of its length the expression is shown only for a point separation purely in the time direction (4.16) and for a point separation purely in the radial direction (4.17).

We have performed several nontrivial checks to verify our results. In both the hot flat space case and in Schwarzschild spacetime we have performed checks using both the conservation and partial trace properties given in Eqs. (A1) and (A2). For hot flat space these properties are satisfied exactly. For each check in Schwarzschild spacetime, where our expression for the noise kernel is approximate, we have shown that the relevant quantities vanish up to the expected order. Furthermore, as an additional check of the result (4.11) for Schwarzschild spacetime when the separation is in the time direction and the product of the temperature and the time separation is

not assumed to be small, we have considered two different ways of obtaining the flat space limit of our result. Firstly, one can compare with the hot flat space result (4.2) by taking the limit  $M \rightarrow 0$ . Secondly, one can compare with Eq. (4.14) for the Minkowski vacuum in Rindler coordinates by taking the limit  $M \rightarrow \infty$  near the horizon.

There are several more or less immediate generalizations of our work. First, although the noise kernel corresponds to the expectation value of the anticommutator of the stress tensor, our results are also valid for other orderings of the stress tensor operator (in fact for any 2-point function of the stress tensor). That is always true for spacelike separated points because the commutator of any local operator (such as the stress tensor) vanishes as a consequence of the microcausality condition. Moreover, since for the conformally invariant scalar field in Schwarzschild the commutator of the field,  $iG(x, x')$ , also vanishes for timelike separated points up to the order to which we are working, the previous statement also holds for timelike separations in our case.<sup>5</sup> Second, since the Gaussian approximation is valid for any ultra-static spacetime which is conformal to an Einstein metric (a solution of the Einstein equation in vacuum, with or without cosmological constant) [44], one can straightforwardly extend our calculation to all those cases by taking the general expression for the Wightman function under the Gaussian approximation, given by Eq. (3.27), and substituting it into the general expression for the noise kernel given in Eqs. (2.3) and (2.4).

One of the most interesting uses of the noise kernel is to investigate the effects of quantum fluctuations near the horizon of the black hole. For instance, there have been claims in the literature that the size of the horizon could exhibit fluctuations induced by the vacuum fluctuations of the matter fields which are much larger than

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<sup>5</sup>In general one would need to use the appropriate prescription when analytically continuing the Euclidean Green function to obtain the Wightman function for timelike separated points in the Lorentzian case, and use expressions analogous to Eqs. (3.24)-(3.25) but without symmetrizing with respect to the two points. One can explicitly see how this is done in Ref. [17].

the Planck scale (even for relatively short timescales of the order of the Schwarzschild radius, i.e. much shorter than the evaporation time) [51–54]. So far all these studies have been based on semi-qualitative arguments. However, one should in principle be able to address this issue by computing the quantum correlation function of the metric perturbations, including the effects of loops of matter fields, with the method outlined in the introduction. As a matter of fact, part of the information on the corresponding induced curvature fluctuations is already directly available from our results. Indeed, at one loop the correlator of the Ricci tensor (or, equivalently, the Einstein tensor) is gauge invariant<sup>6</sup> and it is immediately given by the stress tensor correlator [17]. Unlike the Riemann tensor, the Ricci tensor does not entirely characterize the local geometry. In order to get the full information about the quantum fluctuations of the geometry at this order, one needs to use Eq. (1.7) or a related one. In that case, the noise kernel for arbitrary pairs of points is a crucial ingredient. Strictly speaking it is important that the noise kernel, although divergent in the coincidence limit, is a well-defined distribution. Our result for separate points does not completely characterize such a distribution since it does not specify the appropriate integration prescription in the coincidence limit. This can, nevertheless, be obtained using the method in Appendix C of Ref. [20] (see also Ref. [19] for cosmological examples).

It is worthwhile to discuss briefly how the present paper is related to an earlier study of the noise kernel in Schwarzschild spacetime [43], which also considered a conformal scalar field and made use of Page’s Gaussian approximation. The main interest there was evaluating the noise kernel in the coincidence limit. In order to get a finite result, the Hadamard elementary solution was subtracted from the Wightman function before evaluating the noise kernel. Since the Hadamard elementary solution coincides with the  $\kappa = 0$  expression for the Gaussian approximation through order

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<sup>6</sup>This quantity is gauge invariant because the Ricci tensor of the Schwarzschild background vanishes, as does its Lie derivative with respect to an arbitrary vector.



$(x - x')^2$ , which is the order through which the approximation is valid for the optical Schwarzschild spacetime, their subtracted Wightman function will also be valid through that order. The fact that they found a non-vanishing trace for their noise kernel is also compatible with our results because, as we have reasoned, the noise kernel should only be valid through order  $(x - x')^{-4}$  when the Gaussian approximation for the Wightman function is employed (and through order  $(x - x')^{-2}$  when using the subtracted Wightman function, whose leading term is  $O(1)$  rather than  $O[(x - x')^{-2}]$ ). Instead, one would need an expression for the noise kernel accurate through order  $(x - x')^0$  or higher to get a vanishing trace in the coincidence limit. In contrast, for the reasons given in the introduction, here we consider the unsubtracted noise kernel, which is indispensable to obtain the quantum correlation function for the metric perturbations as the subtracted one would lead for instance to a vanishing result –and no fluctuations– for the Minkowski vacuum. Furthermore, in this way one can still get useful and accurate information for the terms of order  $(x - x')^{-8}$  through  $(x - x')^{-4}$ , which dominate at small separations.

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# A Appendix: Noise kernel and conformal transformations

In this appendix we derive the result for the rescaling of the noise kernel under conformal transformations. We provide two alternative proofs based respectively on the use of quantum operators and on functional methods.

First, we start by showing how the classical stress tensor of a conformally invariant scalar field rescales under a conformal transformation  $g_{ab} \rightarrow \tilde{g}_{ab} = \Omega^2(x) g_{ab}$ . The key point is that the classical action of the field,  $S[\phi, g]$ , remains invariant (up to surface terms) if one rescales appropriately the field:  $\phi \rightarrow \tilde{\phi} = \Omega^{(2-D)/2} \phi$ . Taking that into account, one easily gets the result from the definition of the stress tensor as a functional derivative of the classical action:

$$\tilde{T}_{ab} = \frac{2\tilde{g}_{ac}\tilde{g}_{bd}}{\sqrt{-\tilde{g}}} \frac{\delta S[\tilde{\phi}, \tilde{g}]}{\delta \tilde{g}_{cd}} = \frac{2\tilde{g}_{ac}\tilde{g}_{bd}}{\sqrt{-\tilde{g}}} \frac{\delta S[\phi, g]}{\delta \tilde{g}_{cd}} = \Omega^{2-D} \frac{2g_{ac}g_{bd}}{\sqrt{-g}} \frac{\delta S[\phi, g]}{\delta g_{cd}} = \Omega^{2-D} T_{ab}. \quad (\text{A1})$$

## 1 Proof based on quantum operators

A possible way of proving Eq. (2.9) is by promoting the classical field  $\phi$  in Eq. (A1) to an operator in the Heisenberg picture. The operator  $\hat{T}_{ab}(x)$  would be divergent because it involves products of the field operator at the same point. However, in order to calculate the noise kernel what one actually needs to consider is  $\hat{t}_{ab}(x) = \hat{T}_{ab}(x) - \langle \hat{T}_{ab}(x) \rangle$  and this object is UV finite, i.e., its matrix elements  $\langle \Phi | \hat{t}_{ab}(x) | \Psi \rangle$  for two arbitrary states  $|\Psi\rangle$  and  $|\Phi\rangle$  (not necessarily orthogonal) are UV finite because Wald's axioms [47] guarantee that  $\langle \Phi | \hat{T}_{ab}(x) | \Psi \rangle$  and  $\langle \Phi | \Psi \rangle \langle \hat{T}_{ab}(x) \rangle$  have the same UV divergences and they cancel out. Therefore, one can proceed as follows. One starts by introducing a UV regulator (it is useful to consider dimensional regularization since it is compatible with the conformal symmetry for scalar and fermionic fields, but this is not indispensable since we will remove the regulator at the end without having

performed any subtraction of non-invariant counterterms). One can next apply the operator version of Eq. (A1) to the operators  $\hat{t}_{ab}(x)$  appearing in Eq. (1.4) defining the noise kernel. Since all UV divergences cancel out, as argued above, we can then safely remove the regulator and are finally left with Eq. (2.9).

## 2 Proof based on functional methods

An alternative way of proving Eq. (2.9) is by analyzing how the closed-time-path (CTP) effective action  $\Gamma[g, g']$  changes under conformal transformations. This effective action results from treating  $g_{ab}$  and  $g'_{ab}$  as external background metrics and integrating out the quantum scalar field within the CTP formalism [55]:

$$e^{i\Gamma[g, g']} = \int \mathcal{D}\varphi_f \mathcal{D}\varphi_i \mathcal{D}\varphi'_i \rho[\varphi_i, \varphi'_i] \int_{\varphi_i}^{\varphi_f} \mathcal{D}\phi e^{iS_g[g] + iS[\phi, g]} \int_{\varphi'_i}^{\varphi_f} \mathcal{D}\phi' e^{-iS_g[g'] - iS[\phi', g']}, \quad (\text{A2})$$

where  $\rho[\varphi_i, \varphi'_i]$  is the density matrix functional for the initial state of the field <sup>7</sup> (in particular one has  $\rho[\varphi_i, \varphi'_i] = \Psi[\varphi_i] \Psi^*[\varphi'_i]$  for a pure initial state with wave functional  $\Psi[\varphi_i] = \langle \varphi_i | \Psi \rangle$  in the Schrödinger picture),  $S_g[g]$  is the gravitational action including local counterterms,  $S[\phi, g]$  is the action for the scalar field, and the two background metrics are also taken to coincide at the same final time at which the final configuration of the scalar field for the two branches are identified and integrated over. The fields  $\varphi_f$  on the one hand and  $\{\varphi_i, \varphi'_i\}$  on the other, correspond to the values of the field restricted respectively to the final and initial Cauchy surfaces, and their functional integrals are over all possible configurations of the field on those surfaces. Integrating out the scalar field gives rise to UV divergences, but they can be dealt with by renormalizing the cosmological constant and the gravitational coupling constant as

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<sup>7</sup>Under appropriate conditions it is also possible to consider asymptotic initial states. For instance, given a static spacetime, a generalization to the CTP case of the usual  $i\epsilon$  prescription involving a small Wick rotation in time selects the ground state of the Hamiltonian associated with the time-translation invariance as the initial state.

well as introducing local counterterms quadratic in the curvature in the bare gravitational action  $S_g[g]$ , so that the total CTP effective action is finite. After functionally differentiating and identifying the two background metrics, one gets the renormalized expectation value of the stress tensor operator together with the contributions from the gravitational action [25, 56]:

$$g_{ac} g_{bd} \frac{2}{\sqrt{-g}} \frac{\delta \Gamma[g, g']}{\delta g_{cd}} \Big|_{g'=g} = -\frac{1}{8\pi G} (G_{ab} + \Lambda g_{ab}) + \langle \hat{T}_{ab} \rangle_{\text{ren}}, \quad (\text{A3})$$

where the contribution from the counterterms quadratic in the curvature has been absorbed in  $\langle \hat{T}_{ab} \rangle_{\text{ren}}$ . The renormalized gravitational coupling and cosmological constants,  $G$  and  $\Lambda$ , depend on the renormalization scale, but the expectation value also depends on it in such a way that the total expression is renormalization-group invariant since that is the case for the effective action. The equation that one obtains by equating the right-hand side of Eq. (A3) to zero governs the dynamics of the mean field geometry in semiclassical gravity, including the back-reaction effects of the quantum matter fields.

On the other hand, the noise kernel can be obtained by functionally differentiating twice the imaginary part of the CTP effective action:

$$N_{abc'd'}(x, x') = g_{ae}(x) g_{bf}(x) g_{c'g'}(x') g_{d'h'}(x') \frac{4}{\sqrt{g(x)g(x')}} \frac{\delta^2 \text{Im} \Gamma[g, g']}{\delta g_{e'f'}(x) \delta g_{g'h'}(x')} \Big|_{g'=g}. \quad (\text{A4})$$

It is well-known that the imaginary part of the effective action does not contribute to the equations of motion for expectation values derived within the CTP formalism, like Eq. (A3), which are real. Furthermore, one can easily see from Eq. (A2) that, since it is real, the gravitational action (whose contribution can be factored out of the path integral) only contributes to the real part of the effective action. In particular this means that the counterterms and the renormalization process have no effect on

the noise kernel, which will be a key observation in order to prove Eq. (2.9). Indeed, let us start with Eq. (A2) for the conformally related metric and scalar field,  $\tilde{g}_{ab}$  and  $\tilde{\phi}$ , and assume that we use dimensional regularization:

$$\begin{aligned}
e^{i\Gamma[\tilde{g},\tilde{g}']} &= e^{iS_{\mathbb{g}}[\tilde{g}]-iS_{\mathbb{g}}[\tilde{g}']} \int \mathcal{D}\tilde{\varphi}_{\text{f}} \mathcal{D}\tilde{\varphi}_{\text{i}} \mathcal{D}\tilde{\varphi}'_{\text{i}} \tilde{\rho}[\tilde{\varphi}_{\text{i}}, \tilde{\varphi}'_{\text{i}}] \int_{\tilde{\varphi}_{\text{i}}}^{\tilde{\varphi}_{\text{f}}} \mathcal{D}\tilde{\phi} e^{iS[\tilde{\phi},\tilde{g}]} \int_{\tilde{\varphi}'_{\text{i}}}^{\tilde{\varphi}'_{\text{f}}} \mathcal{D}\tilde{\phi}' e^{-iS[\tilde{\phi}',\tilde{g}']} \\
&= e^{i(S_{\mathbb{g}}[\tilde{g}]-S_{\mathbb{g}}[g])-i(S_{\mathbb{g}}[\tilde{g}']-S_{\mathbb{g}}[g'])} \int \mathcal{D}\varphi_{\text{f}} \mathcal{D}\varphi_{\text{i}} \mathcal{D}\varphi'_{\text{i}} \rho[\varphi_{\text{i}}, \varphi'_{\text{i}}] \int_{\varphi_{\text{i}}}^{\varphi_{\text{f}}} \mathcal{D}\phi \left| \frac{\mathcal{D}\tilde{\phi}}{\mathcal{D}\phi} \right| e^{iS_{\mathbb{g}}[g]+iS[\phi,g]} \\
&\quad \times \int_{\varphi'_{\text{i}}}^{\varphi'_{\text{f}}} \mathcal{D}\phi' \left| \frac{\mathcal{D}\tilde{\phi}'}{\mathcal{D}\phi'} \right| e^{-iS_{\mathbb{g}}[g']-iS[\phi',g']}, \tag{A5}
\end{aligned}$$

where we have taken into account in the second equality the fact that dimensional regularization is compatible with the invariance of the classical action  $S[\tilde{\phi}, \tilde{g}]$  under conformal transformations (since it is invariant in arbitrary dimensions). We also considered that the initial states of the scalar field are related by

$$\begin{aligned}
\tilde{\rho}[\tilde{\varphi}_{\text{i}}(x), \tilde{\varphi}'_{\text{i}}(x')] &= \Omega_{\text{i}}^{(D-2)/4}(x) \Omega_{\text{i}}^{(D-2)/4}(x') \rho[\varphi_{\text{i}}(x), \varphi'_{\text{i}}(x')] \\
&= \Omega_{\text{i}}^{(D-2)/4}(x) \Omega_{\text{i}}^{(D-2)/4}(x') \rho\left[\Omega_{\text{i}}^{-1}(x)\tilde{\varphi}_{\text{i}}(x), \Omega_{\text{i}}^{-1}(x)\tilde{\varphi}'_{\text{i}}(x')\right], \tag{A6}
\end{aligned}$$

where the conformal factor  $\Omega_{\text{i}}^2$  is restricted to the initial Cauchy surface and so are the points  $\{x, x'\}$  in this equation. (This relation between the initial states is the choice compatible with conformal invariance after one takes into account the relation between  $\tilde{\phi}$  and  $\phi$ , and the prefactor is determined by requiring that the state remains normalized.) The logarithm of the functional Jacobian  $|\mathcal{D}\tilde{\phi}/\mathcal{D}\phi|$  is divergent but formally zero in dimensional regularization<sup>8</sup>, so that we can take  $|\mathcal{D}\tilde{\phi}/\mathcal{D}\phi| = 1$  in

<sup>8</sup>This can be seen by taking Eq. (18) in Ref. [57] and using dimensional regularization [47] to evaluate the trace of the heat kernel appearing there. Any possible dependence left on the conformal factor evaluated at the initial or final Cauchy surfaces would correspond to a prefactor on the right-hand side of Eq. (A5), and would not contribute to the noise kernel (or the expectation value of the stress tensor) at any intermediate time since it involves functionally differentiating the logarithm of that expression with respect to the metric at such intermediate times.

both path integrals on the right-hand side of Eq. (A5). Taking all this into account, we are left with

$$\Gamma[\tilde{g}, \tilde{g}'] = \Gamma[g, g'] + (S_g[\tilde{g}] - S_g[g]) - (S_g[\tilde{g}'] - S_g[g']), \quad (\text{A7})$$

where the last two pairs of terms on the right-hand side correspond to the difference between the bare gravitational actions of the two conformally related metrics in dimensional regularization; note that whereas each bare action is separately divergent, the difference  $S_g[\tilde{g}] - S_g[g]$  is finite. When working in dimensional regularization, conformally invariant fields only exhibit divergences associated with counterterms quadratic in the curvature. These terms lead to the standard result for the trace anomaly of the stress tensor when one takes the functional derivative of Eq. (A7) with respect to the conformal factor, which can be shown to be equivalent to the trace of Eq. (A3).

The key aspect for our purposes is that the extra terms on the right-hand side of Eq. (A7) only change the real part of the CTP effective action, as already mentioned above, so that the imaginary part remains invariant under conformal transformations. Starting with Eq. (A4) for the metric  $\tilde{g}_{ab}$  and taking into account the invariance of the imaginary part of the CTP effective action under conformal transformations, one immediately obtains

$$\tilde{N}_{abc'd'}(x, x') = \Omega^{2-D}(x) \Omega^{2-D}(x') N_{abc'd'}(x, x'), \quad (\text{A8})$$

in agreement with Eq. (2.9). Note that we have employed dimensional regularization in our argument for simplicity, but one would reach the same conclusion if other regularization schemes had been used. In those cases one would get in general a contribution to the analog of Eq. (A7) from the change of the functional measure, but it would only affect the real part of the effective action<sup>9</sup> and one could still apply

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<sup>9</sup>See Ref. [57], where the calculations are performed in Euclidean time, and analytically continue the result to Lorentzian time.

exactly the same argument as before.

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## Chapter 4: Exact noise kernel in static de Sitter space and other conformally flat spacetimes

### 1 Introduction

One of the major areas of interest within stochastic gravity involves investigating the behavior of quantum fluctuations of the stress-energy tensor in the region near the event horizon of a black hole. Such an investigation requires an expression for the noise kernel which is valid on the horizon.

In Chapter 3, we computed an approximate expression for the noise kernel for the conformally invariant scalar field in Schwarzschild spacetime using a quasi-local expansion. Although we intuitively expect this approximation to be valid for separations of the points on the order of the mass scale in the region near the horizon, it is not known if the approximation remains valid when either of the points is at the horizon. In addition, without an exact expression with which to compare, it is difficult to ascertain the exact range of point separations for which the approximation remains valid.

In light of these difficulties, we would like to find an alternative way to investigate the behavior of quantum fluctuations near a horizon and to test the validity of the quasi-local approximation used in Chapter 3. Fortunately, de Sitter space offers us a way to accomplish both of these tasks.

de Sitter space is a cosmological spacetime which is a vacuum solution to the Einstein equations when a positive cosmological constant is present. In addition, de Sitter space is maximally symmetric, meaning that it possesses the same degree of symmetry as Minkowski space [1]. One representation of de Sitter space is given by

the metric

$$ds^2 = -dt^2 + \alpha^2 \cosh^2(t/\alpha)[d\chi^2 + \sin^2 \chi d\Omega^2] . \quad (1.1)$$

This coordinatization covers the entire manifold when  $-\infty < t < \infty$  and  $0 \leq \chi < \pi$ . Spatial sections in this metric represent closed hypersurfaces which contract and then re-expand in time.

A more common representation is that of the cosmological coordinates<sup>1</sup>, for which the metric is

$$ds^2 = -dt^2 + e^{t/\alpha}(dx^2 + dy^2 + dz^2) . \quad (1.2)$$

This exponential expansion makes de Sitter space of interest to cosmologists, since the Universe may have undergone a period of approximately exponential expansion during inflation. Furthermore, astronomical observations consistent with the presence of a cosmological constant indicate that de Sitter space may be the future of our Universe as well [2].

For our purposes, there are two primary advantages to working in de Sitter space. The first is that, like all Friedman-Robertson-Walker spacetimes, it is conformally flat. This is easily demonstrated by defining

$$-\eta \equiv \alpha e^{-t/\alpha} , \quad 0 \leq (-\eta) < \infty . \quad (1.3)$$

The line element becomes

$$ds^2 = \frac{\alpha^2}{(-\eta)^2}(-d\eta^2 + dx^2 + dy^2 + dz^2) , \quad (1.4)$$

and the conformal factor is

$$\Omega(x) = \frac{\alpha}{(-\eta)} . \quad (1.5)$$

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<sup>1</sup>Note that this coordinate chart covers only half of de Sitter space; the other half is described by an exponentially contracting Universe.

The conformal flatness of de Sitter space allows us to compute an exact expression for the noise kernel for the conformally invariant scalar field using the expression for the Wightman function for Minkowski space together with the relation given in equation (2.9) of Chapter 3 for the conformal transformation of the noise kernel.

The second advantage of de Sitter space is that there is a coordinate system for which the metric has a form similar to Schwarzschild spacetime. This coordinate system is defined by

$$\begin{aligned}
x &\equiv \frac{e^{-T/\alpha}}{\sqrt{B}} \rho \sin \theta \cos \phi , \\
y &\equiv \frac{e^{-T/\alpha}}{\sqrt{B}} \rho \sin \theta \sin \phi , \\
z &\equiv \frac{e^{-T/\alpha}}{\sqrt{B}} \rho \cos \theta , \\
-\eta &\equiv \alpha \frac{e^{-T/\alpha}}{\sqrt{B}} .
\end{aligned} \tag{1.6}$$

The line element is

$$ds^2 = -BdT^2 + \frac{d\rho^2}{B} + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2 , \tag{1.7}$$

where  $B = 1 - \frac{\rho^2}{\alpha^2}$ . For an observer situated at the origin,  $B = 0$  is a cosmological horizon which marks the boundary of his observable universe. It is this coordinate system which interests us the most, since it provides an opportunity to study the noise kernel near the horizon and compare its behavior with that found for the approximate noise kernel in Schwarzschild spacetime.

Our investigation extends the results of several calculations of the noise kernel in flat space and in de Sitter space over the past decade. Early work in flat space was done by Martin and Verdageur [3, 4], who computed a Fourier transformation of

the noise kernel in Minkowski spacetime for massless and massive scalar fields with arbitrary coupling to the scalar curvature when these fields were in the Minkowski vacuum. Additionally, they used the Einstein-Langevin equation to compute correlation functions for the Einstein tensor and the perturbed metric tensor. Phillips and Hu [5] computed the noise kernel for massless scalar fields in a thermal state in flat space in the limit that the two points come together. In [6], Hu and Roura used the technique of smearing to investigate divergences in the noise kernel for the conformally invariant scalar field in flat space.

An early calculation in de Sitter space was given by Roura and Verdageur [7]. They computed an explicit expression for the two point correlation function for the stress-energy tensor for massive and massless minimally coupled scalar fields in the Euclidean vacuum state. Pèrez-Nadal, Roura, and Verdaguer [8] used the symmetry properties of de Sitter space to compute this same quantity for massive and massless minimally coupled scalar fields in a large class of states in  $n$ -dimensional de Sitter space.

Here, we use the Wightman function for the conformally invariant scalar field in the Minkowski vacuum to compute an exact expression for the noise kernel for the conformally invariant scalar field in the conformal vacuum state for any conformally flat spacetime by means of the conformal transformation given in equation (2.9) of Chapter 3. In principle, this computation could have been accomplished by conformally transforming the result obtained by Martin and Verdaguer [4]. In practice, we found it easier to compute the noise kernel directly using the general expression for the noise kernel in terms of the Wightman function derived by Phillips and Hu [9] and shown in equation (2.3) of Chapter 3.

We specialize this result to de Sitter spacetime and compute an explicit expression for the noise kernel in the static coordinate system. This expression is used to investi-



gate the behavior of the noise kernel in the region near the cosmological horizon. In addition, we investigate the likely range of validity of the quasi-local expansion of the noise kernel for Schwarzschild spacetime given in Chapter 3. This is accomplished by computing a similar expansion for the noise kernel in static de Sitter space in terms of inverse powers of the coordinate separation and truncating the series at the same order used for Schwarzschild spacetime. We compute the relative error between the exact expression and the truncated series when the coordinates are split in the time and radial directions, and investigate the range of separations for which that error remains small.

In Sec. 2, we present the results of our computation of the noise kernel for conformally flat spacetimes. We show an exact expression for all components of the noise kernel when the points are spacelike or timelike separated. We also show one component of the exact expression for arbitrary separation of the points (including the delta-function contribution present for null separations). In Sec. 3, we show the result of this expression when it is computed in the static de Sitter coordinates, and investigate its behavior as one or both of the points approaches the cosmological horizon. We show the results of the expansion of the noise kernel in inverse powers of the coordinate separation, and investigate the range of validity of this expansion. The results of these calculations are discussed in Sec. 4.

The conventions used throughout this chapter are those of Misner, Thorne, and Wheeler [10]. Units are chosen such that  $c = \hbar = G = 1$ .

## 2 The noise kernel in conformally flat spacetimes

As shown in Eq. (2.9) of Chapter 3, under a conformal transformation of the metric,

$$\tilde{g}_{ab}(x) = \Omega(x)^2 g_{ab}(x) , \tag{2.1}$$

the noise kernel for a conformally invariant scalar field transforms as

$$\tilde{N}_{abc'd'}(x, x') = \Omega(x)^{-2} N_{abc'd'}(x, x') \Omega(x')^{-2} . \quad (2.2)$$

The Wightman function for the Minkowski vacuum, in Cartesian coordinates, including the contribution from the delta functions, is given by

$$\begin{aligned} G^+(x, x') &= -\frac{1}{4\pi^2} \frac{1}{(\Delta t + i\epsilon)^2 - \mathbf{r}^2} , \\ &= -\frac{1}{4\pi^2} \frac{1}{\Delta t^2 - \mathbf{r}^2} - \frac{i}{8\pi\mathbf{r}} [\delta(\Delta t - \mathbf{r}) - \delta(\Delta t + \mathbf{r})] , \end{aligned} \quad (2.3)$$

where we have used

$$\Delta t = (t - t') , \quad (2.4a)$$

$$\mathbf{r} = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} , \quad (2.4b)$$

for the spatial and time separation of the two points. Using equation (2.3) of Chapter 3, we have computed an exact expression for the noise kernel for the conformally invariant scalar field in this state. Applying the above conformal transformation gives us an exact expression for the noise kernel for this field in the conformal vacuum state for any conformally flat spacetime which is valid for an arbitrary separations of points. When the points are separated in a timelike or spacelike direction, the result is

$$\begin{aligned} N_{abc'd'}(x, x') &= \Omega(x)^{-2} \Omega(x')^{-2} \left[ \frac{\sigma_a \sigma_b \sigma_{c'} \sigma_{d'}}{48\pi^4 \sigma^6} - \frac{\sigma_{(a} \sigma_{b)(c'} \sigma_{d')}}{24\pi^4 \sigma^5} \right. \\ &\quad \left. + \frac{4\sigma_{a(c'} \sigma_{d')b} - \eta_{ab} \eta_{c'd'}}{192\pi^4 \sigma^4} \right] . \end{aligned} \quad (2.5)$$

Here  $\eta_{ab}$  is the Minkowski metric, (...) indicates symmetrization of the indices, and

$$\sigma = \frac{1}{2}(-\Delta t^2 + \mathbf{r}^2), \quad (2.6a)$$

$$\sigma_a = \eta_{ab}(x^b - x'^b), \quad (2.6b)$$

$$\sigma_{c'} = -\eta_{c'd'}(x^d - x'^d), \quad (2.6c)$$

$$\sigma_{ac'} = -\eta_{ab}\delta_{c'}^b. \quad (2.6d)$$

where  $\delta_{c'}^b$  is the Kronecker delta.

When the points are split in the null direction, there is an additional contribution from the delta functions in equation (2.3). Although we have computed explicit expressions for each individual component when the points are null-separated, we have not been able to find a general expression for all components in a compact form such as that of Eq. (2.5). The expression for one of the components is given by

$$\begin{aligned} N_{tt't'}(x, x') = & \Omega(x)^{-2}\Omega(x')^{-2} \left\{ \frac{3\mathbf{r}^4 + 10\mathbf{r}^2(\Delta t)^2 + 3(\Delta t)^4}{12\pi^4 ((\Delta t)^2 - \mathbf{r}^2)^6} \right. \\ & - \frac{1}{192\pi^2\mathbf{r}^6} [\delta(\Delta t - \mathbf{r}) - \delta(\Delta t + \mathbf{r})]^2 \\ & + \frac{1}{96\pi^2\mathbf{r}^5} [\delta(\Delta t - \mathbf{r}) - \delta(\Delta t + \mathbf{r})] [\delta'(\Delta t - \mathbf{r}) - \delta'(\Delta t + \mathbf{r})] \\ & - \frac{1}{96\pi^2\mathbf{r}^4} [[\delta'(\Delta t - \mathbf{r})]^2 + [\delta'(\Delta t + \mathbf{r})]^2] \\ & + \frac{1}{288\pi^2\mathbf{r}^3} \{3 [\delta'(\Delta t - \mathbf{r}) + \delta'(\Delta t + \mathbf{r})] [\delta''(\Delta t - \mathbf{r}) + \delta''(\Delta t + \mathbf{r})] \\ & - [\delta(\Delta t - \mathbf{r}) - \delta(\Delta t + \mathbf{r})] [\delta'''(\Delta t - \mathbf{r}) - \delta'''(\Delta t + \mathbf{r})]\} \\ & + \frac{1}{576\pi^2\mathbf{r}^2} \{8\delta'''(\Delta t - \mathbf{r})\delta'(\Delta t - \mathbf{r}) + 4\delta'''(\Delta t - \mathbf{r})\delta'(\Delta t + \mathbf{r}) \\ & + 4\delta'''(\Delta t + \mathbf{r})\delta'(\Delta t - \mathbf{r}) + 8\delta'''(\Delta t + \mathbf{r})\delta'(\Delta t + \mathbf{r}) - 9[\delta''(\Delta t - \mathbf{r})]^2 \\ & + 6\delta''(\Delta t + \mathbf{r})\delta''(\Delta t - \mathbf{r}) - 9[\delta''(\Delta t + \mathbf{r})]^2 \\ & \left. - [\delta(\Delta t - \mathbf{r}) - \delta(\Delta t + \mathbf{r})] [\delta''''(\Delta t - \mathbf{r}) - \delta''''(\Delta t + \mathbf{r})]\} \right\}. \quad (2.7) \end{aligned}$$

Upon inspection, it is apparent that each delta-function term is of quadratic order, which means that any integral over the noise kernel will contain a  $\delta(0)$  divergence. This holds true for all components, not merely the  $N_{tt't'}$  component. The exact meaning of this divergence in the context of the noise kernel is under investigation.

### 3 Static de Sitter space

We obtain the noise kernel for the conformally invariant field in the conformal vacuum in de Sitter space by taking Eq. (2.7) and letting

$$\begin{aligned}
t &\rightarrow -\eta \\
t' &\rightarrow -\eta' \\
\Omega(x) &= \frac{\alpha}{(-\eta)} \\
\Omega(x') &= \frac{\alpha}{(-\eta')} .
\end{aligned} \tag{3.1}$$

From the definitions given in equation (1.6), we transform this expression to the static coordinates using the relation

$$N_{abc'd'}(x, x') = \frac{\partial x^A}{\partial x^a} \frac{\partial x^B}{\partial x^b} \frac{\partial x^{C'}}{\partial x^{c'}} \frac{\partial x^{D'}}{\partial x^{d'}} N_{ABC'D'}(x, x') . \tag{3.2}$$

Here we have used uppercase letters to represent indicies of the comoving coordinates and lowercase to represent indicies of the static coordinates.

To avoid coordinate singularities, we express the noise kernel in terms of an orthonormal frame at each of the two points. We do this by introducing orthonormal basis vectors at each point which satisfy

$$(e_{\hat{a}})^c (e_{\hat{b}})_c = \eta_{\hat{a}\hat{b}} , \tag{3.3}$$

$$(e^{\hat{a}})_c (e_{\hat{a}})_d = g_{cd} . \tag{3.4}$$

Here  $\eta_{\hat{a}\hat{b}}$  is the Minkowski metric. The components of a vector may be written in the orthonormal basis as

$$A_{\hat{a}} = (e_{\hat{a}})^a A_a . \quad (3.5)$$

Similarly, the noise kernel in this basis is

$$N_{\hat{a}\hat{b}\hat{c}\hat{d}}(x, x') = (e_{\hat{a}})^a (e_{\hat{b}})^b (e_{\hat{c}})^{c'} (e_{\hat{d}})^{d'} N_{abc'd'}(x, x') . \quad (3.6)$$

For the static de Sitter coordinates, we choose basis vectors such that

$$\begin{aligned} (e_{\hat{T}})^T &= \sqrt{-g^{TT}} \\ (e_{\hat{\rho}})^\rho &= \sqrt{g^{\rho\rho}} \\ (e_{\hat{\theta}})^\theta &= \sqrt{g^{\theta\theta}} \\ (e_{\hat{\phi}})^\phi &= \sqrt{g^{\phi\phi}} . \end{aligned} \quad (3.7)$$

All other components are zero.

In general, the expressions resulting from this procedure are quite long. Although we have computed every component for the noise kernel in the static coordinates, for the sake of brevity we present only two of them:

$$\begin{aligned} N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x, x') &= (BB')^{-1} N_{TTTT'}(x, x') \\ &= \frac{1}{12\pi^4 \left[ \alpha^2 \left( \sqrt{BB'} \tau - 2 \right) + 2\rho\rho' \cos(\gamma) \right]^6} \\ &\quad \times \left\{ \alpha^4 \left[ -12\sqrt{BB'} \tau + BB' (\tau^2 + 14) \right. \right. \\ &\quad \left. \left. - (2B + 2B' - 3) (\tau^2 - 1) \right] \right. \\ &\quad \left. + 4\alpha^2 \rho\rho' \cos(\gamma) \left( 3\sqrt{BB'} \tau - 2 (\tau^2 - 1) \right) \right. \\ &\quad \left. + 2\rho^2 \rho'^2 (\tau^2 - 1) \cos(2\gamma) \right\} \end{aligned} \quad (3.8a)$$

$$\begin{aligned}
N_{\hat{T}\hat{\rho}\hat{T}'\hat{\rho}'}(x, x') &= N_{T\rho T'\rho'}(x, x') \\
&= \frac{\alpha^2}{6\pi^4 \left[ \alpha^2 \left( \sqrt{BB'} \tau - 2 \right) + 2\rho\rho' \cos(\gamma) \right]^6} \\
&\quad \times \left\{ \alpha^2 \cos(\gamma) \left[ -4\sqrt{BB'} \tau + BB' (\tau^2 + 4) \right. \right. \\
&\quad \left. \left. - (2B - 2B' + 2) (\tau^2 - 2) \right] \right. \\
&\quad \left. + \rho\rho' (\cos(2\gamma) + 3) \left( \sqrt{BB'} \tau - \tau^2 + 2 \right) \right\}, \quad (3.8b)
\end{aligned}$$

where

$$\tau \equiv 2 \cosh(\Delta T/\alpha), \quad (3.8c)$$

$$\cos \gamma \equiv \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi'), \quad (3.8d)$$

and  $B' = 1 - \rho'^2/\alpha^2$ . Note that we have neglected the delta function contributions in the above expressions. This is due to the fact that the initial computations of the delta function contributions each contained approximately 2700 terms and we have not yet found a means by which to simplify the expressions into displayable forms. The expression shown in Eq. (3.8) is valid only for non-null separations of the points.

### 3.1 Behavior near the horizon

When either of the two points approaches the cosmological horizon, we see that  $\rho \rightarrow \alpha$  and  $B \rightarrow 0$  (or  $\rho' \rightarrow \alpha$  and  $B' \rightarrow 0$ , respectively). However, note that at the horizon  $T$  is a null coordinate and is infinite. Therefore,  $\Delta T$  is an ill-defined quantity when one or both points are on the horizon. Nonetheless, if we fix the value of  $\Delta T$ , then inspection of Eq. (3.8) shows that both of the components displayed therein remain bounded when either point is arbitrarily close to the horizon, so long as the points

are spacelike or timelike separated.<sup>2</sup>

For example, if we take  $\rho'$  to be arbitrarily close to the horizon, then

$$N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x, x') \approx -\frac{\tau^2 - 1}{384\pi^4\alpha^4(\alpha - \rho \cos(\gamma))^6} \times [\alpha^2(B - 3) + 4\alpha\rho \cos(\gamma) - \rho^2 \cos(2\gamma)] \quad (3.9a)$$

$$N_{\hat{T}\hat{\rho}\hat{T}'\hat{\rho}'}(x, x') \approx -\frac{\tau^2 - 2}{384\pi^4\alpha^3(\alpha - \rho \cos(\gamma))^6} \times [(2B - 4)\alpha \cos(\gamma) + \rho(\cos(2\gamma) + 3)] . \quad (3.9b)$$

When  $\rho = \rho'$  and both points are near the horizon, we have

$$N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x, x') \approx \frac{\tau^2 - 1}{3072\pi^4\alpha^8} \csc^8\left(\frac{\gamma}{2}\right) . \quad (3.10a)$$

and

$$N_{\hat{T}\hat{\rho}\hat{T}'\hat{\rho}'} \approx -\frac{\tau^2 - 2}{3072\pi^4\alpha^8} \csc^8\left(\frac{\gamma}{2}\right) . \quad (3.10b)$$

Therefore, the noise kernel is bounded when either or both of the points are near the horizon so long as the separation in the  $T$  coordinate is fixed and either  $\gamma \neq 0$ ,  $\rho \neq \rho'$ , or both. For  $\gamma = 0$  and  $\rho = \rho'$ , the above expression is not bounded as the two points approach the horizon. This is expected since, on the cosmological horizon,  $T$  is a null coordinate and  $\Delta T$  is ill-defined.

### 3.2 Comparison with Schwarzschild spacetime

In Chapter 3, we computed an approximate expression for the noise kernel for the conformally invariant scalar field in Schwarzschild spacetime. We used equations (4.7) and (4.10) of Chapter 3 to generate an expansion for the Wightman function in terms

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<sup>2</sup>We find this behavior to be true for all components of the noise kernel when expressed in the orthonormal frame.

of powers of  $\Delta t$ ,  $\Delta r$ , and  $\eta$ . By inserting this expansion into equation (2.3) of Chapter 3, we obtained an approximate expression for the  $N_{t t'}$  component of noise kernel in terms of  $\Delta t$ ,  $\Delta r$ , and  $\eta$  valid for small separations of the points. For the cases where the points are split only in the time direction or the radial direction, that expression was given in Eqs. (4.16) and (4.17) of Chapter 3, respectively.

As discussed in Sec. 1, we are interested in comparing the noise kernel obtained for Schwarzschild spacetime with that obtained for de Sitter space in the static coordinates. We do this by generating an expansion for the noise kernel in the static de Sitter coordinates that is equivalent to the quasi-local expansion used for the Schwarzschild case. In principle, the proper way to do this is to use a quasi-local expansion for the Wightman function in the static de Sitter coordinates and recompute the noise kernel using that expression. However, since we are primarily interested in comparing our results with equations (4.16) and (4.17) of Chapter 3, we can generate equivalent approximations by expanding Eq. (3.8a) in powers of  $1/\Delta T$ ,  $1/\Delta\rho$ , and  $\eta \equiv \cos\gamma - 1$  and truncating the series at the appropriate order.

For our investigation, we consider the  $N_{\hat{T}\hat{T}'\hat{T}'}$  component and begin by splitting in the time and radial directions. The result is

$$[N_{\hat{T}\hat{T}'\hat{T}'}]_{\Delta\rho=\gamma=0} = \frac{1}{4\pi^4\alpha^8 B^4(\tau-2)^4}, \quad (3.11a)$$

$$[N_{\hat{T}\hat{T}'\hat{T}'}]_{\Delta T=\gamma=0} = -\frac{B+B'-2BB'+2(\sqrt{BB'}-1)\tilde{B}}{64\pi^4\alpha^8(\sqrt{BB'}-\tilde{B})^6}. \quad (3.11b)$$

Here,  $\tilde{B} \equiv 1 - \rho\rho'/\alpha^2$ .

Expanding the  $N_{\hat{T}\hat{T}'\hat{T}'}$  component in powers of  $1/\Delta T$  and  $1/\Delta\rho$  and truncating



the series at order  $O[(x - x')^{-4}]$ , we find

$$\begin{aligned}
[N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x, x')]_{\text{series}} &= \frac{1}{4\pi^4 B^4} \left[ \frac{1}{\Delta T^8} - \frac{1}{3\alpha^2 \Delta T^6} + \frac{7}{120\alpha^4 \Delta T^4} \right] \\
&+ O[\Delta T^{-2}], \tag{3.12a}
\end{aligned}$$

$$\begin{aligned}
[N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x, x')]_{\text{series}} &= \frac{1}{\pi^4} \left[ \frac{B^4}{4\Delta\rho^8} - \frac{(B-1)B^3}{\rho\Delta\rho^7} + \frac{(5-6B)B^2}{4\alpha^2\Delta\rho^6} \right. \\
&+ \frac{(2B^2-3B+1)B}{2\alpha^2\rho\Delta\rho^5} + \left. \frac{8B^2-8B+1}{32\alpha^4\Delta\rho^4} \right] \\
&+ O[\Delta\rho^{-3}]. \tag{3.12b}
\end{aligned}$$

In Chapter 3, we considered thermal states in which  $\kappa \equiv T/2\pi$  was left arbitrary. However, the natural choice of state in Schwarzschild spacetime is the Hartle-Hawking state [11], for which  $\kappa = 1/4M$ , since only in this state does the stress-energy tensor remain finite at the horizon; for all other thermal states, including the zero-temperature Boulware state [12], the stress-energy tensor diverges badly at  $r = 2M$ . Comparing the above expressions with Eqs. (4.16) and (4.17) of Chapter 3 (and setting  $\kappa = 1/4M$ ) we see that although the coefficients at each order are different the general form of the expansion is the same, with  $B$  taking the role of  $f$  and  $\alpha$  taking the role of the mass  $M$ .

In footnote 2 of Chapter 3, we noted that we expected the quasi-local expansion to be valid when the geodesic distance was small in comparison to the mass scale of the black hole in the region near the horizon. For de Sitter space, the corresponding distance scale is given by the Hubble parameter  $\alpha$ . To investigate the range of validity of the expressions above, we compute the relative error between the exact expression for the noise kernel and the truncated series expansions,

$$\left| \frac{[N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x, x')]_{\text{series}}}{N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x, x')} - 1 \right|. \tag{3.13}$$

For the purposes of our investigation, we define the region of validity to be the region within which the relative error is less than 10%.

When the points are separated in time, we find that the error between the truncated series and the exact expression for static de Sitter space is approximately 10% when  $|\Delta T| \approx 1.5\alpha$  (see Fig. 4.1). This result is independent of the distance between the two points and the horizon, as can be seen by inspection of Eqs. (3.11a) and (3.12a). For radial separation, the situation is somewhat different. Unlike in Schwarzschild spacetime, where we are interested in points outside the event horizon, in the static de Sitter case we are interested in the region inside the cosmological horizon; thus, the radial separation can never be larger than  $\alpha$  without one point crossing the horizon or the origin. What we find is that the error remains small so long as both points are sufficiently far from the horizon. As either point nears the cosmological horizon, we find that region of validity scales roughly linearly with the distance from the horizon. Figures 4.2 - 4.6 illustrate this behavior.

In contrast with the cases of time and radial separation, this type of analysis fails when the points are separated in only the angular direction. The reason for this is that when we expand Eq. (3.8a) in powers of  $1/\eta$  with  $\Delta T = \Delta\rho = 0$ , we find that the resulting series contains only terms up to order  $O[(x - x')^{-4}]$ . Explicitly, we find

$$\begin{aligned} [N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x, x')]_{\text{series}} &= -\frac{1}{128\pi^4\eta^6\rho^8} - \frac{1}{64\pi^4\eta^5\rho^8} + \frac{1}{128\pi^4\eta^4\rho^8} \\ &= N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}(x, x'). \end{aligned} \tag{3.14}$$

However, we suspect that this behavior is a coincidence due to the symmetries present in de Sitter space and that this result will likely not hold for Schwarzschild spacetime.

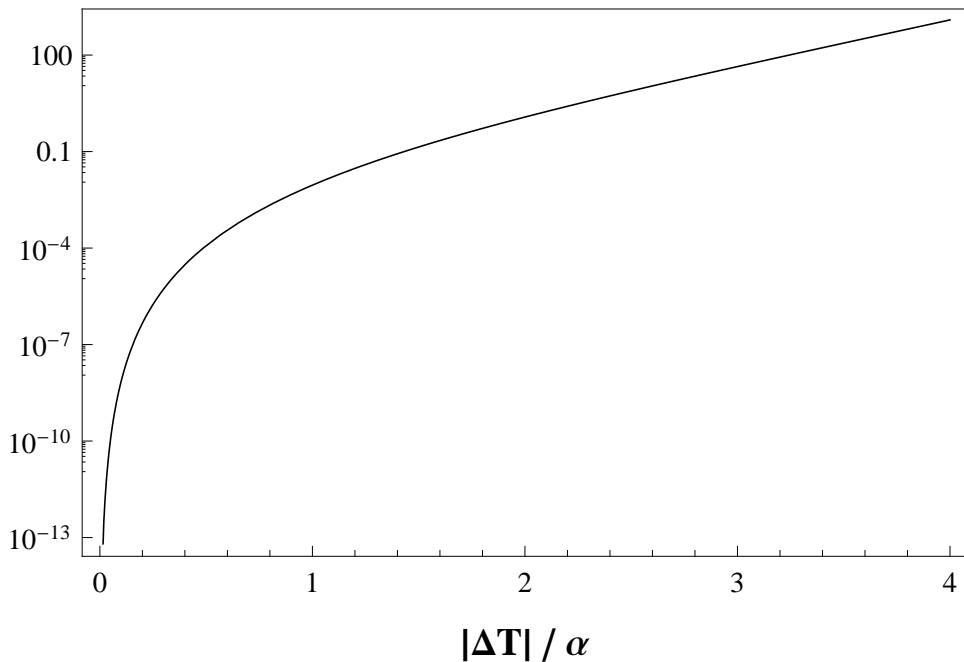


Figure 4.1: This figure shows the relative error in Eq. (3.13) due to separation in  $T$ , with  $\Delta\rho = \gamma = 0$ . This error is independent of the distance from either point to the cosmological horizon.

## 4 Discussion

Using equations (2.3) and (2.9) of Chapter 3 along with the expression given in Eq. (2.3) for the Wightman function for the conformally invariant scalar field in the Minkowski vacuum, we have computed an explicit expression for the noise kernel for this field in the conformal vacuum state in any conformally flat spacetime. A general form of this expression is shown in Eq. (2.5) for all components of the noise kernel when the points are time-like or space-like separated. Due to the length of the expressions involved, only the  $N_{tt't'}$  component is shown in Eq. (2.7) for the case when the points are separated in an arbitrary direction.

In addition, we computed an exact expression for the noise kernel for the conformally invariant scalar field in de Sitter space in the static coordinates (Eq. (1.7)) by means of a coordinate transformation from the expression given in Eq. (2.5) for

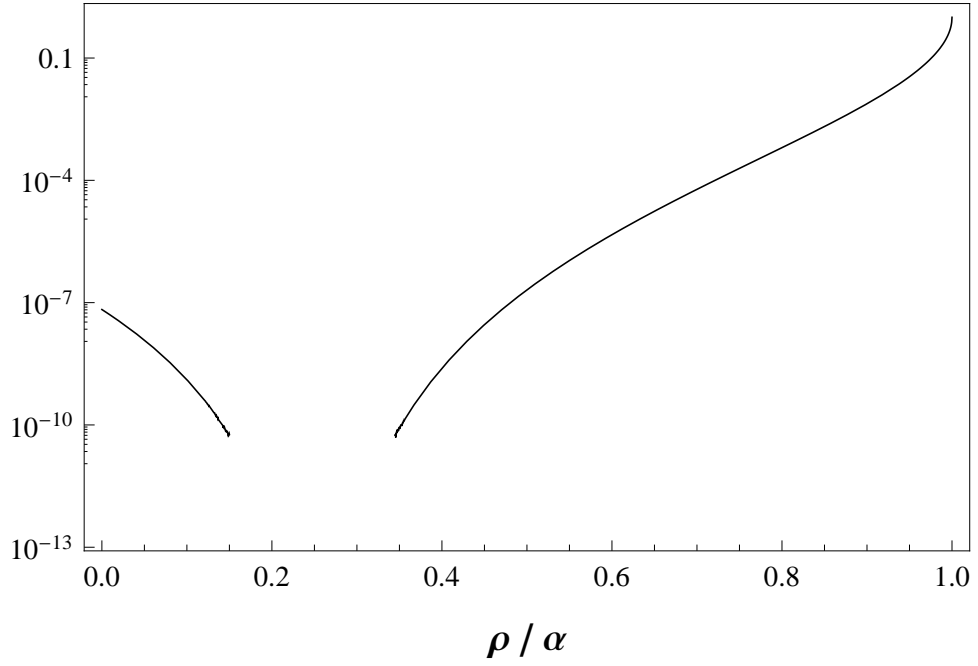


Figure 4.2: This figure shows the relative error in Eq. (3.13) due to changes in  $\rho$  with  $\rho' = 0.25\alpha$  and  $\Delta T = \gamma = 0$ . On this scale,  $\rho/\alpha = 1$  marks the cosmological horizon. The region around  $\rho = \rho'$  has been excised due to limitations on the numerical precision of the computation; however, the relative error in this region is smaller than  $10^{-10}$ .

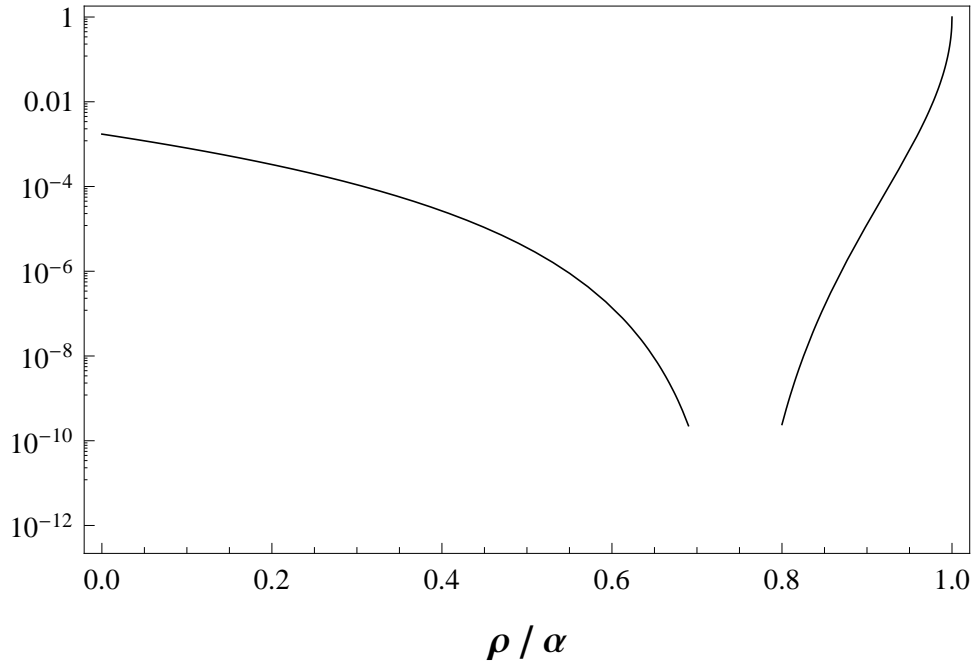


Figure 4.3: This figure shows the relative error in Eq. (3.13) due to changes in  $\rho$  with  $\rho' = 0.75\alpha$  and  $\Delta T = \gamma = 0$ . On this scale,  $\rho/\alpha = 1$  marks the cosmological horizon. The region around  $\rho = \rho'$  has been excised due to limitations on the numerical precision of the computation; however, the relative error in this region is smaller than  $10^{-10}$ .

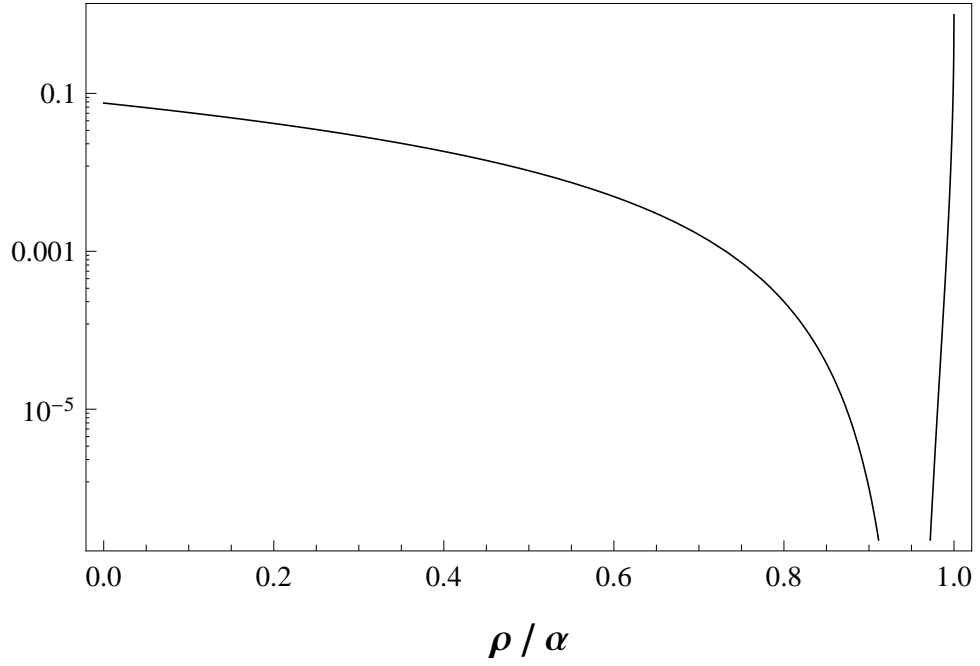


Figure 4.4: This figure shows the relative error in Eq. (3.13) due to changes in  $\rho$  with  $\rho' = 0.95\alpha$  and  $\Delta T = \gamma = 0$ . On this scale,  $\rho/\alpha = 1$  marks the cosmological horizon. The region around  $\rho = \rho'$  has been excised due to limitations on the numerical precision of the computation; however, the relative error in this region is smaller than  $10^{-7}$ .

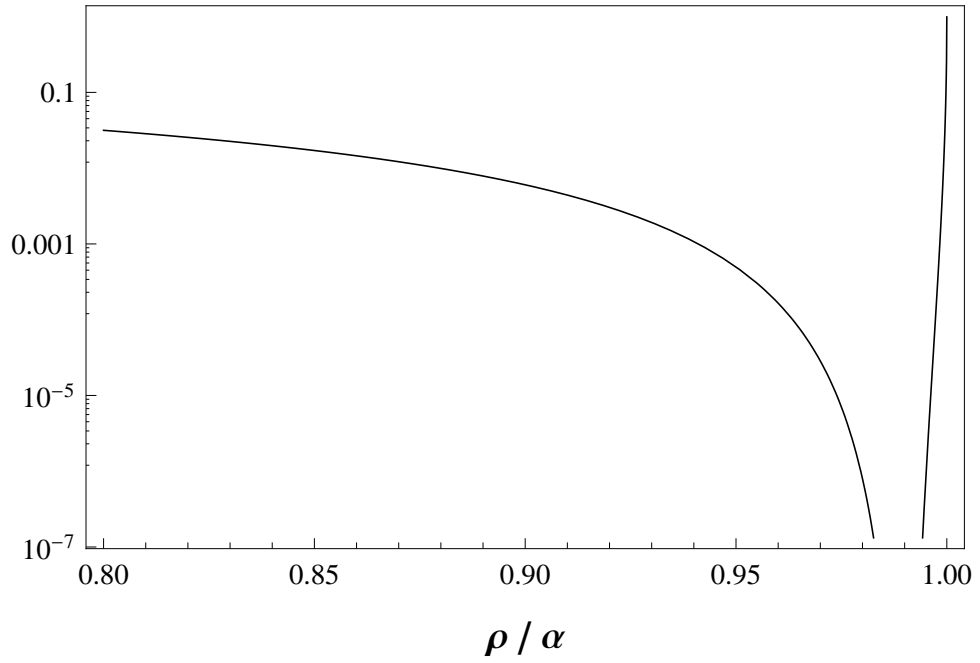


Figure 4.5: This figure shows the relative error in Eq. (3.13) due to changes in  $\rho$  with  $\rho' = 0.99\alpha$  and  $\Delta T = \gamma = 0$ . On this scale,  $\rho/\alpha = 1$  marks the cosmological horizon. The region around  $\rho = \rho'$  has been excised due to limitations on the numerical precision of the computation; however, the relative error in this region is smaller than  $10^{-7}$ .

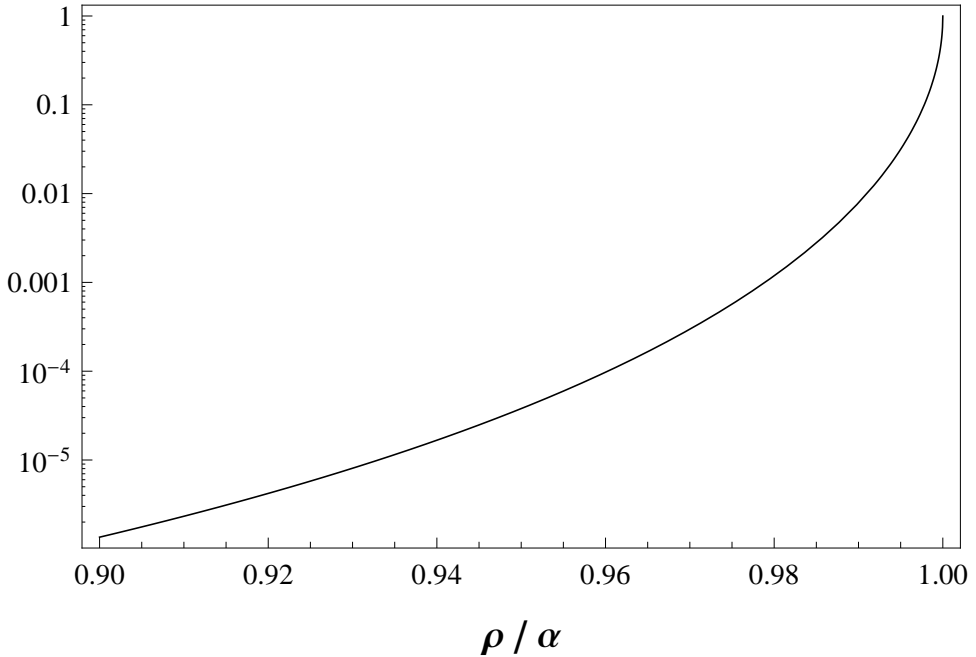


Figure 4.6: This figure shows the relative error in Eq. (3.13) due to fixed radial separations of  $\Delta\rho = 0.1\alpha$  as both points near the horizon, with  $\Delta T = \gamma = 0$ . On this scale,  $\rho/\alpha = 1$  marks the cosmological horizon.

the noise kernel in the spatially flat de Sitter coordinates (Eq. (1.4)). Two of these components are displayed in Eq. (3.8); however, while we have computed all components in the case of an arbitrary separation of the points, we do not display the delta function terms due to the extreme length of the expressions involved.

We have investigated the behavior of the noise kernel in the case that either or both of the points approach the cosmological horizon and we have found that all components of the noise kernel remained finite so long as the points are not null-separated. However, since  $T$  is a null coordinate at the horizon,  $\Delta T$  is ill defined when one or both points are on the horizon; thus, we are restricted to cases where the points are arbitrarily close to but not on the horizon.

The metric for the static de Sitter coordinates is similar in form to the metric for Schwarzschild spacetime, with the event horizon replaced by the cosmological horizon for observers at the origin. Thus, it is expected that the behavior of the noise



kernel near the cosmological horizon in the static de Sitter coordinates, where the exact expression is known, should be similar to the behavior of the noise kernel near the event horizon in Schwarzschild spacetime where we have only an approximate solution.

To investigate the likely validity of the approximate expression for the noise kernel in Schwarzschild spacetime given in Chapter 3 in Eqs. (4.16) and (4.17), we obtained a similar expression in the static de Sitter coordinates by expanding the expression for the  $N_{\hat{T}\hat{T}\hat{T}'\hat{T}'}$  component in Eq. (3.8a) in powers of  $\Delta T$  and  $\Delta\rho$  and investigated the validity of this expansion. As expected, we found that the leading order behaviors of both expansions are identical, and that the remaining orders are similar in form, although the coefficients are different.

To test the validity of the quasi-local approximation, we compared the expansion with the exact result in the static de Sitter coordinates. In general, we found that the quasi-local approximation remains valid for time separations smaller than the Hubble distance  $\alpha$ , and found an error of approximately 10% when  $\Delta T \approx 1.5\alpha$ . For radial separations, we are restricted to separations no greater than  $\alpha$ . We found that the relative error remains small so long as both points are sufficiently far from the cosmological horizon, but goes to infinity as either point approaches the horizon. When both points are near the horizon, we find that the region of validity scales roughly linearly with the distance between the point nearest the horizon and the horizon itself. Finally, for angular separations, we find that the expression generated by the expansion procedure is equal to the exact expression if all three terms are kept; however, we suspect that this result is an artifact of the symmetry of de Sitter space and will not hold in Schwarzschild spacetime.

From these results, we expect the quasi-local approximation used in Chapter 3 to be valid when the separation of the points is less than the mass scale, so long as neither

point is too near the horizon. In addition, we expect for radial separations that the range of validity should scale roughly linearly with distance from the horizon. Finally, if one or both points are on the horizon, we expect the quasi-local approximation to be invalid; however, based on our results in static de Sitter space we expect that the exact noise kernel for Schwarzschild spacetime will remain finite on the horizon so long as the two points are not null-separated.

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## Appendix A: Supplemental information

### 1 Checks on the big rip

The differential equations solver used in the numerical computations presented in Chapter 2 was ODEPACK for Fortran 90 which was developed at Lawrence Livermore National Laboratory [1].

Two checks of the numerical computations were made for the case of the massless minimally coupled field. First, we showed that when the Bunch Davies state was used to set the initial conditions of the field, the output from the numerical computation for the stress tensor at each time step matched the exact solution given in Eq. (3.13) when the mass scale  $\mu$  was chosen appropriately. For the second check, we used Mathematica to numerically evolve individual modes of the field, and showed that for each time step the output from Mathematica matched the output from our code for that mode.

In addition to the plots presented in Chapter 2, a number of other cases were investigated. For both the minimally and conformally coupled fields, the stress energy tensor was computed for a range of masses from 0 to well beyond the Planck mass in spacetimes corresponding to  $w = -1.05$ ,  $w = -1.25$ ,  $w = -1.5$ , and  $w = -2$ . For these computations, the initial states of the field were varied by altering the transition between the 0th-order WKB approximation used for the low-frequency modes and the 4th-order WKB approximation used for the high-frequency modes. We found that the qualitative behavior in each of these cases to be identical to that displayed in Chapter 2.

## 2 Checks on the noise kernel

All computations of the noise kernel in Chapters 3 and 4 were made using the symbolic manipulation software Mathematica. To check the expressions generated, we made use of two properties of the noise kernel. As shown in Chapter 3, the noise kernel obeys the conservation equations

$$\nabla^a N_{abc'd'} = \nabla^b N_{abc'd'} = \nabla^{c'} N_{abc'd'} = \nabla^{d'} N_{abc'd'} = 0 \quad (\text{A1})$$

and all partial traces of the noise kernel vanish,

$$N^a{}_{ac'd'} = N_{ab}{}^{c'}{}_{c'} = 0. \quad (\text{A2})$$

For the expression for the noise kernel in hot flat space (Eq. (4.2) of Chapter 3), the general expression for conformally flat spacetimes (Eqs. (2.5) and (2.7) of Chapter 4), and the explicit expression for static de Sitter space (Eq. (3.8) of Chapter 4), we found that both conditions hold exactly for all components.

For the approximate expression for the noise kernel Schwarzschild spacetime, these conditions were checked for a small number of components on an order by order basis. For the arbitrary  $\kappa$  expression (Eq. (4.11) of Chapter 3), the  $N^t{}_{t'}$ ,  $N^r{}_{r'}$ ,  $N^t{}_{r'}$ ,  $N^\theta{}_{\theta'}$ ,  $N^t{}_{\theta'}$ ,  $N^\phi{}_{\phi'}$ , and  $N^t{}_{\phi'}$  components of the noise kernel were computed to demonstrate that the partial traces

$$\begin{aligned} N^a{}_{a t'}(x, x') &= 0 \\ N^t{}_{t c'}(x, x') &= 0, \end{aligned} \quad (\text{A3})$$

were satisfied up to order  $O[(x - x')^{-4}]$ . For conservation, we computed the  $N^t{}_{t'}$ ,  $N^r{}_{t'}$ ,  $N^\theta{}_{t'}$ , and  $N^\phi{}_{t'}$  components to show

$$\nabla_a N^a{}_{t t'}(x, x') = 0; \quad (\text{A4})$$

however, due to the length of the expressions involved, we were only able to show that the equation was satisfied up to order  $O[(x - x')^{-6}]$ .

For the low temperature case (Eqs. (4.16), and (4.17) of Chapter 3), the same components were computed, and we were able to show all three checks were satisfied up to order  $O[(x - x')^{-4}]$ .

These checks were performed on the Wake Forest University DEAC Cluster with support from an IBM SUR grant and the Wake Forest University IS Department.

## Bibliography

- [1] A. C. Hindmarsh, [https://computation.llnl.gov/casc/odepack/odepack\\_home.html](https://computation.llnl.gov/casc/odepack/odepack_home.html)

# Curriculum Vitae

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## Degrees

B.S., Sam Houston State University, December 2005, Bachelor of Science in Physics.

Ph.D., Wake Forest University, May 2012, Doctor of Philosophy in Physics.

Advisor: Paul R. Anderson

Thesis: *Quantum Effects of Scalar Fields in Black Hole and Cosmological Spacetimes*

## Experience

Graduate Teaching Assistant

Wake Forest University

August 2006-Present

## Publications

A. Eftekharzadeh, J. D. Bates, A. Roura, P. R. Anderson, and B. L. Hu, *Noise kernel for a quantum field in Schwarzschild spacetime under the Gaussian approximation*. Phys. Rev. D **85**, 044037 (2012).



J. D. Bates and P. R. Anderson, *Effects of quantized scalar fields in cosmological spacetimes with big rip singularities*. Phys. Rev. D **82**, 024018 (2010).

J. D. Bates and P. R. Anderson, *Reply to “Comment on ‘Effects of quantized scalar fields in cosmological spacetimes with big rip singularities’ ”* Phys. Rev. D **84**, 048502 (2011).

### **Grants**

East Asia and Pacific Summer Institutes (Taiwan) - NSF Research and Travel Grant (June 26-August 19 2011).

Third Quantum Gravity and Geometry School - APS Travel Award (February 28-March 13 2011).

Pan-American Advanced Studies Institute (PASI) School on Quantum Gravity - APS Travel Award (June 23-July 3, 2010).

### **Awards**

13th Eastern Gravity Meeting: Best Student Talk (May 2010).

### **Teaching Experience**

Phys. 109, *Introduction to Astronomy (Lab)*, Wake Forest University.

One semester introductory lab course in astronomy, including: telescope observations of the night sky, motions of the planets and stars, parallax, spectroscopy, lenses and magnification, stellar evolution, galaxy classification, Hubble’s Law.

## Professional Activities

Colloquium, Wake Forest University, NC, September 14 2011; invited talk, *The Noise Kernel: Characterization of the Quantum Fluctuations of Matter in Curved Spacetime*.

14th Eastern Gravity Meeting, Princeton University, Princeton, NJ, June 2011; contributed talk, *Calculation of the Noise Kernel in Schwarzschild Spacetime*.

Third Quantum Gravity and Geometry School, Zakopane, Poland, February 28-March 13 2011.

Pan-American Advanced Studies Institute (PASI) School on Quantum Gravity, National Autonomous University of Mexico, Morelia, Mexico, June 23-July 3, 2010.

13th Eastern Gravity Meeting, North Carolina State University, Raleigh, NC, May 2010; contributed talk, *Effects of Quantized Scalar Fields in Cosmological Spacetimes with Big Rip Singularities*.

April Meeting of the American Physical Society, Washington, D.C., February 2010; contributed talk, *Effects of Particle Production and Vacuum Polarization on Big Rip Singularities*.

11th Eastern Gravity Meeting, Pennsylvania State University, PA, May 2008.