Quantum Electrodynamics

Physics 217 2013, Quantum Field Theory

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Nov. 2013

Fermions add new issues. Before considering full-fledged QED, we consider a slightly simpler theory, the "Yukawa theory", in which we have a scalar field coupled to a fermion. We will take for the lagrangian:

$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu \phi)^2 - m_s^2 \phi^2\right) + (\bar{\psi}(i \not{\partial} - m_f)\psi) + y\phi\bar{\psi}\psi. \qquad (1)$$

In the limit $y \to 0$, this theory describes a free fermion and a free scalar. The interaction Hamiltonian is:

$$\mathcal{H} = -y\phi\bar{\psi}\psi \qquad (2)$$

We want to study Green's functions and S-matrices in this theory. For example, we can calculate the Green's function:

$$G_{\alpha\beta}(x,y) = \frac{T\langle\Omega|\psi_\alpha(x)\bar{\psi}_\beta(y)|\Omega\rangle}{\langle\Omega|\Omega\rangle}. \tag{3}$$

In the interaction picture this becomes (we also note that G is translationally invariant – why?)

$$G_{\alpha\beta}(x-y) = \frac{T\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)e^{-i\int d^4z\mathcal{H}(z)}|0\rangle}{\langle 0|0\rangle}. \tag{4}$$

Now we want to expand the exponent in powers of the interaction Hamiltonian. As for scalars, we would like to use Wick's theorem. In fact, we need to be a little more careful about how we define the time-ordered products. Recall that for the fermion propagator, we defined

$$T\psi(x)\bar{\psi}(y) = \theta(x^0 - y^0)\psi(x)\bar{\psi}(y) - \theta(x^0 - y^0)\bar{\psi}(y)\psi(x). \quad (5)$$

For time ordered products of more fields, we generalize this by switching signs in each term if we have to permute an odd number of fermion fields. We can write this symbolically as:

$$T\psi(x_1)\dots\psi(x_n) = \sum_{perm} (-1)^{perm} \prod_{i>j} \theta(x^0_{i^P} - x^0_{j^P})\psi(x_{1^P})\dots\psi(x_{n^P}). \quad (6)$$

(The ψ's denote either ψ's or $\bar{\psi}$'s.)

Defining normal products similarly, Wick's theorem holds as for scalars. So we have, for G, to second order in y (noting that disconnected diagrams cancel as for scalars)

$$G_{\alpha\beta} = S_F(x - y) + \tag{7}$$

$$(-iy)^2 \int d^4 z_1 d^4 z_2 T\langle 0|\psi_\alpha(z)\bar{\psi}_\beta(y)\phi(z_1)\bar{\psi}_\gamma(z_1)\psi_\gamma(z_1)\phi(z_2)\bar{\psi}_\delta(z_2)\psi_\delta(z_2)|0\rangle$$

The first term is the free propagator; the second is, by Wick's theorem:

$$(-iy)^2 \int d^4 z_1 d^4 z_2 S_F(x - z_1)_{\alpha\gamma} S_F(z_1 - z_2)_{\gamma\delta} S_F(z_2 - y)_{\delta\beta} D_F(z_1 = z_2). \tag{8}$$

The S's are multiplied as matrices (the product has indices α, β). Note that the indices flow like coordinates. This has a natural diagrammatic interpretation.

Momentum Space

If we Fourier transform with respect to p, this becomes
Exercise: Check this

$$G(p) = \frac{i}{\not{p} - m_f}(-iy)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{\not{q} + \not{p} - m_f} \frac{i}{\not{q} - m_f} \frac{i}{q^2 - m_s^2} \frac{i}{\not{p} - m_f}. \quad (9)$$

We can consider more complicated examples, but from this exercise we can infer the Feynman rules:

S-Matrix for the Yukawa Theory

Let's modify the theory by including a second species of fermion; we will call the fermions “electrons" and “muons", and denote them by the letters $e(x)$ and $\mu(x)$. For the lagrangian we now take:

$$\mathcal{L} = \frac{1}{2}\left((\partial_\mu\phi)^2 - m_s^2\phi^2\right) + (\bar{e}(i\,\partial\!\!\!/ - m_e)e) + (\bar{\mu}(i\,\partial\!\!\!/ - m_\mu)\mu) \quad (10)$$

$$+y_e\phi\bar{e}\,e + y_\mu\phi\bar{\mu}\,\mu.$$

In the limit $y_i \to 0$, this theory describes two types of free fermions and a free scalar. The interaction Hamiltonian is:

$$\mathcal{H} = -(y_e\phi\bar{e}\,e + y_\mu\phi\bar{\mu}\,\mu). \quad (11)$$

The Scattering States

We'll consider the case of an incoming electron and an incoming muon, and a final electron and a final muon.

$$|i\rangle = \sqrt{2E(k)2E(p)}a^\dagger_\mu(k,\tilde{s})a^\dagger_e(p,s)|0\rangle \quad (12)$$

$$|f\rangle = \sqrt{2E(k')2E(p')}a^\dagger_\mu(k',\tilde{s}')a^\dagger_e(p,s')|0\rangle \quad (13)$$

Contractions with External Fermions

Working to second order in the interaction will generate the Feynman diagram below. We want to study

$$\frac{(-i)^2}{2!}\langle k' p'| \int d^4z_1 d^4z_2 y_e \bar{e}(z_1)e(z_1)\phi(z_1)\bar{\mu}(z_2)\mu(z_2)\phi(z_2)|k\ p\rangle. \tag{14}$$

In the case of scalars we contracted the fields in the vertices with the external states. This meant simply that we commuted the positive frequency parts through the creation operators on the right, and the negative frequency parts through the destruction operators on the left. We were left with simply $e^{-ip\cdot x}$ factors, which, after the integration over the location of the vertices, gave momentum conservation at each vertex.

For fermions, we have

$$\psi_\alpha^+ = \sum_s \int \frac{d^3p}{2\pi^3\sqrt{2E(p)}} a(p,s) u_\alpha(p,s) e^{-ip\cdot x} \tag{15}$$

$$\psi_\alpha^- = \sum_s \int \frac{d^3p}{2\pi^3\sqrt{2E(p)}} b^\dagger(p,s) v_\alpha(p,s) e^{iip\cdot x} \tag{16}$$

$$\bar{\psi}_\alpha^+ = \sum_s \int \frac{d^3p}{2\pi^3\sqrt{2E(p)}} b(p,s) \bar{v}_\alpha(p,s) e^{-ip\cdot x} \tag{17}$$

$$\bar{\psi}_\alpha^- = \sum_s \int \frac{d^3p}{2\pi^3\sqrt{2E(p)}} a^\dagger(p,s) \bar{u}_\alpha(p,s) e^{ip\cdot x} \tag{18}$$

Here the contractions give, in addition to canceling the $\sqrt{2E}(2\pi)^3$ factors, and the $e^{\pm ip\cdot x}$ leading to momentum conservation at the vertices, a factor of:

1. $u_\alpha(p, s)$ for each initial state fermion
2. $\bar{v}_\alpha(p, s)$ for each initial state anti-fermion
3. $\bar{u}_\alpha(p, s)$ for each final state fermion
4. $v_\alpha(p, s)$ for each final state anti-fermion.

Note to avoid cluttering the equations too much, we have not always put a subscript e, μ, so, as appropriate:

$E(p) = \sqrt{\vec{p}^2 + m_\mu^2}$, or $E(p) = \sqrt{\vec{p}^2 + m_e^2}$ Similarly, $u(p, s)$ satisfies $(\not{p} - m_e)u(p, s) = 0$ or $(\not{p} - m_\mu)u(p, s) = 0$

So, by analogy with our scalar field studies, we can read off the scattering amplitude:

$$\mathcal{M}(e(p)+\mu(k) \to e(p')+\mu(k')) = -y_e y_\mu \bar{u}(p', s')u(p, s)\bar{u}(k', \tilde{s}')u(k, \tilde{s}) \times \frac{i}{(p'-p)^2 - m_s^2} \quad (19)$$

To construct the cross section, we need to take the absolute square of this expression. In many experimental situations, we are not sensitive to the polarizations of the incoming and outgoing fermions (i.e. the beams are unpolarized and we do not measure the spins of the final state particles). In this case we can *average* over initial spins and sum over final spins. So we compute:

$$\frac{1}{4} \sum_{s,s',\tilde{s},\tilde{s}'} |\mathcal{M}|^2 \quad (20)$$

This summing over spins will allow us to use the polarization sums for spinors we developed in the homework. Let's examine what happens when we square the spinor terms. Consider those for the electron:

$$\sum_{s,s'} \bar{u}_\alpha(p',s')u_\alpha(p,s)\bar{u}^*_\beta(p',s')u^*_\beta(p,s) \tag{21}$$

Rearranging:

$$= \sum_{s,s'} \bar{u}_\alpha(p',s')u_\alpha(p,s)\bar{u}^*_\beta(p',s')u^*_\beta(p,s)\gamma^0_{\beta,\gamma}u_\gamma(p',s')$$

Using our spin sums:

$$-\mathrm{Tr}\left((\not{p}' + m_e)(\not{p} + m_e)\right) .$$

Make sure you understand the trace here!

The traces can readily be evaluated, using the identities:

1. $\mathrm{Tr}(1) = 4$.
2. $\mathrm{Tr}(\not{a}\,\not{b}) = 4a \cdot b$.

The last identity follows from

$$\mathrm{Tr}(\not{a}\,\not{b}) = a_\mu b_\nu \mathrm{Tr}(\gamma^\mu \gamma^\nu) \tag{22}$$

and

$$\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4. \tag{23}$$

So we have, for the spin-averaged amplitude:

$$|\mathcal{M}|^2_{avg} = 4y_e^2 y_\mu^2 \frac{1}{(t - m_s^2)^2}(p \cdot p' + m_e^2)(k \cdot k' + m_\mu^2). \qquad (24)$$

To obtain the cross section, we can proceed exactly as for scalars, Alternatively we will consider an incident electron and positron, and an outgoing $\mu^+\mu^-$.

$$\mathcal{M}(e^-(p)+e^+(p') \to \mu^-(k)+\mu^+(k')) = -y_e y_\mu \bar{v}(p',s')u(p,s)\bar{u}(k,\tilde{s})v(k',\tilde{s}') \times \frac{i}{(p'+p)^2 - m_s^2} \quad (25)$$

We can spin average and sum as before. We again obtain traces:

$$|\mathcal{M}|^2_{avg} = \frac{1}{4}\mathrm{Tr}\,((\not{p}+m_e)(-\not{p}'+m_e))\,\mathrm{Tr}\left((\not{k}+m_\mu)(-\not{k}'+m_\mu)\right) \times \frac{y_e^2 y_\mu^2}{(s-m_s^2)^2} \quad (26)$$

We can evaluate the traces as before, and obtain the cross section.

$$|\mathcal{M}|^2_{avg} = 4y_e^2 y_\mu^2 \frac{1}{(s-m_s^2)^2}(-p\cdot p' + m_e^2)(-k\cdot k' + m_\mu^2). \quad (27)$$

We can rewrite the invariants in terms of s, t and u.

Quantum Electrodynamics – At Last!

We know how to deal with initial and final state fermions. We also know how to deal with virtual photons (photons internal to the Feynman diagrams). The photon vertex is very similar to the scalar vertex in the Yukawa theory. Now, however, we get a factor of e (the electron charge), and, instead of $\delta_{\alpha\beta}$, we obtain a factor $\gamma^{\mu}_{\alpha\beta}$, where the μ is the index on the photon. So, for example, for $e^+e^- \to \mu^+\mu^-$, we obtain the amplitude:

$$\mathcal{M}(e^-(p) + e^+(p') \to \mu^-(k) + \mu^+(k')) = -e^2\bar{v}(p', s')\gamma^{\mu}u(p, s) \times \bar{u}(k, \tilde{s})\gamma^{\nu}v(k', \tilde{s}')\frac{-ig_{\mu\nu}}{(p' + p)^2 - m_s^2} \tag{28}$$

We can spin average and sum as before. We again obtain traces:

$$|\mathcal{M}|^2_{avg} = \frac{1}{4}\mathrm{Tr}\left((\not{p} + m_e)\gamma^\mu(-\not{p}' + m_e)\gamma^\nu\right)\mathrm{Tr}\left((\not{k} + m_\mu)\gamma_\mu(-\not{k}' + m_\mu)\right. \tag{29}$$

$$\times \frac{e^2}{(s - m_s^2)^2}$$

Now we need another trace identity:

$$\mathrm{Tr}(\not{a}\,\not{b}\,\not{c}\,\not{d}) = 4(a \cdot b\; c \cdot d - a \cdot c\; b \cdot d + a \cdot d\; b \cdot c). \tag{30}$$

Derive! Hint: anticommute $\not{a}$ through the various terms.

This computation is particularly simple in the high energy limit (center of mass energy) where we can neglect the masses of the electron and muon. Then the product is simply

$$\frac{1}{4}\sum |\mathcal{M}|^2 = \frac{e^4}{s^2} 4\left(p'^{\mu} p^{\rho} - p \cdot p' g^{\mu\rho} + p'^{\rho} p^{\mu}\right)\left(k'_{\mu} k_{\rho} - k \cdot k' g_{\mu\rho} + k'_{\rho} k_{\mu}\right) \tag{31}$$

$$= \frac{8e^4}{s^2}\left(p' \cdot k' \; p \cdot k + p' \cdot k \; p \cdot k'\right).$$

In the center of mass frame, this is

$$\frac{8e^4p^4}{s^2}\left[(1-\cos\theta)^2+(1+\cos\theta)^2\right] \tag{32}$$

$$= \frac{16e^4p^4}{s^2}[1+\cos^2\theta].$$

The cross section is then, noting $|v_1 - v_2| = 2$:

$$d\sigma = \frac{8e^4 p^4}{2s^2} \int \frac{d^3p' d^3k'}{16p^4(2\pi^6)} (2\pi)^4 \delta(p+p'-k-k')[1+\cos^2\theta]. \quad (33)$$

The d^3k' integral is done trivially with the momentum delta-function; the energy delta function just gives a factor of $1/2$. So we are left with

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{16\pi^2 E_{cm}^2}(1+\cos^2\theta). \quad (34)$$

$$\sigma_{tot} = \frac{4\pi\alpha^2}{3E_{cm}^2} \quad (35)$$

Processes with external photons

To have a sensible notion of creation of states, we saw that we need to work in a gauge like Coulomb gauge. Here we'll have the photon field, $\vec{A}$, with positive and negative frequency parts:

$$A_i^+ + A_i^- = \sum_{pol} \int \frac{d^3p}{(2\pi)^3\sqrt{E(p)}} \left(a(p,\lambda)\epsilon_i(p,\lambda)e^{-ip\cdot x} + a^\dagger(p,\lambda)\epsilon_i^*(p,\lambda)e^{ip\cdot x} \right) . \tag{36}$$

Contracting will give us an ϵ_i. When we square, if we average and sum as we did for spins, we will have factors like $P_{ij}(\vec{p})$.

We would like to convert this into something more relativistic-looking. Crucial is that the photon couples to a conserved current. When we square the amplitude, this means that we have

$$P_{ij}J^{ij}. \tag{37}$$

But $\frac{1}{k^0}k_i J^{i\nu} = -J^{0\nu}$, so

$$(\delta_{ij} - \frac{k^i k^j}{k_0^2})J^{ij} = J^{ii} - J^{00} = -J^{\mu}_{\mu}. \tag{38}$$

So in effect we can write

$$\sum_{pol} \epsilon^{\mu}\epsilon^{\nu} = -g^{\mu\nu}. \tag{39}$$

This is the analog of our spin sums for fermions.

With this, we are ready to go on to Compton scattering, pair annihilation of electrons and positrons to photons, and similar processes.

Compton Scattering

This is slightly more complicated than the electron-muon scattering examples because we now have two diagrams and interference. At this point we are adept at writing down the scattering amplitudes upon examination of the diagrams. Calling the initial electron and photon momenta p and k, and the final momenta p' and k', and the initial and final photon polarizations $\epsilon(k)$ and $\epsilon(k')$, we have

$$\mathcal{M} = (-ie)^2 \tag{40}$$

$$\times \bar{u}(p') \left[\gamma^\mu i \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2} \gamma^\nu - \gamma^\nu i \frac{\not{p} - \not{k}' + m}{(p-k')^2 - m^2} \gamma^\mu \right] u(p) \epsilon(k)^\mu \epsilon(k')^\nu .$$

Before squaring, it is useful to simplify this expression. In the denominators, we can use $k^2 = k'^2 = 0$, $p^2 = p'^2 = m^2$, while in the numerators we can anticommute the $\not{p}$ factors with the γ matrices and use $\not{p} u(p) = m u(p)$, to write:

$$\mathcal{M} = (-ie)^2 \bar{u}(p') \left[\frac{2p^\nu \gamma^\mu + \gamma^\mu \not{k} \gamma^\nu}{2p \cdot k} + \frac{2p^\mu \gamma^\nu + \gamma^\nu \not{k}' \gamma^\mu}{(-2p \cdot k')} \right] u(p) \epsilon(k)^\mu \epsilon(k')^\nu . \tag{41}$$

Now we consider the spin-averaged and spin-summed expression. As in the simple cases we have considered up to now, the effect of taking the absolute square leads us to simple expressions. It is important to introduce additional dummy indices for the sums over the polarization vectors, and to use the rule for the sum over polarizations we have derived above. This gives

$$\frac{1}{4}\sum_{s,s';\lambda,\lambda'}|M|^2 = \frac{A}{(p\cdot k)^2} + \frac{B+C}{(p\cdot k)(-p\cdot k')} + \frac{D}{p\cdot (k')^2.} \tag{42}$$

Here

$$A = \mathrm{Tr}\left[(\not p' + m)(2p^\nu\gamma^\mu + \gamma^\mu \not k\gamma^\nu)(\not p + m)(2p_\nu\gamma_\mu + \gamma_\mu \not k\gamma_\nu)\right] \tag{43}$$

$$B = \mathrm{Tr}\left[(\not p' + m)(2p^\nu\gamma^\mu + \gamma^\mu \not k\gamma^\nu)(\not p + m)(2p_m u\gamma_\nu - \gamma_\nu \not k\gamma_\mu)\right] \tag{44}$$

C and D are quite similar; in fact, it is a simple exercise to show that $B = C$, $A(k) = D(k')$.

Here we have traces of up to eight gamma matrices, but the identities we have proven are adequate to evaluate all of them. Consider, for example, A. The term with 8 γ matrices simplifies immediately due to the following identity:

$$\gamma^\nu \not{a} \gamma_\nu = -2 \not{a} \tag{45}$$

which follows from

$$a_\rho \gamma^\nu \gamma^\rho \gamma_\nu = a_\rho (2\gamma^\nu g^\rho_\nu - \gamma^\nu \gamma_\nu \gamma^\rho). \tag{46}$$

Then (the braces below indicate traces)

$$[\not{p}' \gamma^\mu \not{k} \gamma^\nu \not{p} \gamma_\nu \not{k} \gamma_\mu] = [-2 \not{p}' \gamma^\mu \not{k} \not{p} \not{k} \gamma_\mu] \tag{47}$$

$$= [4 \not{p}' \not{k} \not{p} \not{k}]$$

which can be evaluated using our earlier identities.

One finds

$$A = 16(4m^4 - 2m^2 p \cdot p' + 4m^2 p \cdot k - 2m^2 p' \cdot k + 2p \cdot kp' \cdot k). \quad (48)$$

and working through all four terms:

$$\frac{1}{4} \sum |\mathcal{M}|^2 \quad (49)$$

$$= 2e^4 \left[\frac{p \cdot k'}{p \cdot k} + \frac{p \cdot k}{p \cdot k'} + 2m^2 \left(\frac{1}{p \cdot k} - \frac{1}{p \cdot k'} \right) + m^4 \left(\frac{1}{p \cdot k} - \frac{1}{p \cdot k'} \right)^2 \right].$$

Now we can work out the cross section in various frames.

It is helpful to be methodical and to write the four vectors in detail

$$k = (\omega, \omega \hat{z}) \quad p = (m, \vec{0}) \quad k' = (\omega', \omega' \sin\theta, 0, \omega' \cos\theta).$$

(Note that this defines the z axis).
Then four momentum conservation allows us to solve for ω':

$$p'^2 = m^2 = (p + k - k')^2 = p^2 + 2p \cdot (k - k') - 2k \cdot k'. \tag{50}$$

Evaluating the invariants in terms of the lab frame ω, ω', and θ, this is:

$$0 = 2m(\omega - \omega') - 2\omega\omega'(1 - \cos\theta). \tag{51}$$

Solving for ω':

$$\omega' = \frac{\omega}{1 + \frac{\omega}{m}(1 - \cos\theta)}. \tag{52}$$

The invariants appearing in $|\mathcal{M}|^2$ are simple:

$$p \cdot k = m\omega ; p \cdot k' = m\omega' . \tag{53}$$

Finally, we need to evaluate the phase space integral. For this we need an expression for $E_{p'}$. Starting with

$$\vec{p}' = \vec{k} - \vec{k}' \Rightarrow \vec{p}'^2 = \omega^2 + \omega'^2 - 2\omega\omega' \cos\theta \tag{54}$$

we have

$$E_{p'} = \sqrt{m^2 + \omega^2 + \omega'^2 - 2\omega\omega' \cos\theta} \tag{55}$$

In the energy conserving δ function, we have $f = \omega' + E_{p'} - \omega - m$, so

$$\frac{\partial f}{\partial \omega'} = \frac{m + \omega - \omega \cos\theta}{E_{p'}} \tag{56}$$

Klein-Nishina Expression for the Compton Cross Section

So

$$d\sigma = \frac{d^3k' d^3p'}{(2\pi)^6(2\omega')2E'_p}(2\pi)^4\delta^{(4)}(k'+p'-k-p)|\mathcal{M}|^2 \qquad (57)$$

$$= \frac{\omega'^2 d\omega' d\Omega_{k'}}{(2\pi)^2 4\omega' E_{p'}}\delta(\omega'+E'_p-\omega-m)|\mathcal{M}|^2.$$

Using our expressions above

$$d\sigma = \frac{1}{8\pi}\int \frac{d\cos\theta}{2m2\omega}\frac{\omega'}{m(1+\frac{\omega}{m}(1-\cos\theta))}\frac{1}{4}\sum|\mathcal{M}|^2. \qquad (58)$$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]. \tag{59}$$

At low frequencies:

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}(1 + \cos^2\theta). \tag{60}$$

$$\sigma_{tot} = \frac{8\pi\alpha^2}{3m^2}. \tag{61}$$

This is the same as the Thompson formula we derive in E and M.

The High Energy Limit

Here we will uncover an interesting feature. Work in the center of mass frame; take

$$k = (\omega, 0, 0, \omega);\ p = (E, -\omega\hat{z})\ p' = (E, -\omega \sin\sin\theta, 0, -\omega\cos\theta).$$

We have all of the ingredients we need to compute the cross section:

$$\frac{d\sigma}{d\cos\theta} \approx \frac{2\pi\alpha^2}{2m^2 + s(1+\cos\theta)} \tag{62}$$

where $s = m^2 + 2p\cdot k$. The total cross section is:

$$\sigma_{tot} \approx \frac{2\pi\alpha^2}{s}\ln(s/m^2). \tag{63}$$

Colinear Singularities

Why the singularity as $m \to 0$. Should be able to see a problem if set $m = 0$ from the start problem comes when $2p \cdot k$ or $2p \cdot k' = 0$. Corresponds to p along k or k'. The precise form of the singularity requires understanding the behavior of the spinors $u(p)$ (see Peskin and Schroder).

Radiative Corrections

So far, we have considered “tree" diagrams; now we consider diagrams with loops (in particular, this means diagrams which have an internal momentum integration). To perform these integrals, we will need to develop certain tools. We'll do this in the course of three examples which are particularly important at one loop (e^2 corrections to the leading order result). We'll also have to face the problem of ultraviolet divergences (high momentum). This will lead us to the problem of renormalization. Finally, we will face infrared divergences – these are associated with real physics.

Fermion Self Energy

This one is the easiest in some ways. We will see that there is a correction to the fermion mass, and an overall constant correction to the propagator. We might expect a linearly divergent correction to the mass. This follows by analogy to Lorentz's calculation of the self-energy of the electron, and also from dimensional analysis. But let's check. We'll work with the fermion Green's function. We'll drop the external lines; the result is called the "one-particle irreducible graph." We'll call it $-i\Sigma$, the "fermion self energy".

$$-i\Sigma(p) = (-i)^2(i)(-i)\int \frac{d^4k}{(2\pi)^4}\gamma^\mu \frac{\not{p}+\not{k}+m}{(p-k)^2-m^2}\gamma_\mu \frac{1}{k^2}. \quad (64)$$

Doing this integral would seem to require introducing angles in the four dimensional space. But there is a much better trick, introduced by Feynman.

Start with the simple identity:

$$\frac{1}{AB} = \int_0^1 \frac{dx}{(Ax + B(1-x))^2}. \tag{65}$$

While we're at it, we can write a generalization we will need later.

$$\frac{1}{ABC} = \int d\alpha_1 d\alpha_2 d\alpha_3 \delta(1 - \sum \alpha_i) \frac{1}{A\alpha_1 + B\alpha_2 + C\alpha_3}. \tag{66}$$

I'll leave you to prove this second one; the generalization to more denominator factors should be clear. This is known as the "Feynman parameter trick."

Let's apply it to $\Sigma(p)$:

$$-i\Sigma(p) = -\int \frac{d^4k}{(2\pi)^4} \int d\alpha \frac{\not{p} + \not{k} + m}{[p^2\alpha - 2p\cdot k\alpha + k^2]^2} \tag{67}$$

Now we can make one further simplification – and this is the critical one. The change of variables $k \to k - \alpha p$ gets rid of the nasty cross term in the denominator, leaving us with

$$-i\Sigma(p) = -\int \frac{d^4k}{(2\pi)^4} \int d\alpha \frac{\not{p}(1-\alpha) + \not{k} + m}{[k^2 + p^2\alpha(1-\alpha)]^2}. \tag{68}$$

$i\epsilon$'s and the *Wick rotation*

So far I have been sloppy about the $i\epsilon$ factors in the denominator. In fact, at this stage, we have a $+i\epsilon$ in the denominator factor. We can deal with this by doing the k^0 integral first as a contour integral, locating the poles, etc. But we can be more clever. Note that we can rotate the contour 90^o, avoiding both poles, *provided p^2 is space-like*. Then $k^{0\,2} \to -k^{0\,2}$, and

$$k^{0\,2} - \vec{k}^2 \to -k_E^2 \tag{69}$$

i.e. we can reduce the integral to a Euclidean integral. We may be interested eventually in p^2 time-like (e.g. $p^2 = m^2$), but often we can deal with this by analytic continuation, as we will see (we will also understand some of the analytic properties of Feynman diagrams as functions of momenta).

So we now have

$$\Sigma(p) = \int \frac{d^4k}{(2\pi)^4} \int d\alpha \frac{\not{p}(1-\alpha) + \not{k} + m}{[k^2 + \alpha(1-\alpha)(-p)^2]^2}. \qquad (70)$$

The integral is now an ordinary integral without peculiar singularities anywhere (remember $p^2 < 0$).

First simplification: the integral over the $\not{k}$ term in the numerator is odd, so gives zero.
First complication: the integral is not well-defined. It diverges (logarithmically) for large k.
This latter problem we might deal with as follows. We don't know about physics at arbitrarily large $|k|$. So we might just include all of the physics we know, and cut off the integrals above that. Call that scale Λ; we might think of this as 100 TeV, or perhaps something larger. This is, in fact, what we will do. But we would like a way to cut off the integral so that the result is simple and Lorentz invariant.

To start, we construct an integral table in arbitrary numbers of dimensions. We will evaluate:

$$\int \frac{d^d k}{(k^2 + \Delta)^n} \tag{71}$$

We can try to write this in terms of a d dimensional solid angle and an integral over powers of k. To figure out the d dimensional solid angle, we can use a trick familiar from integrals over Gaussian's.

$$\int d\Omega_d \int_0^\infty dk k^{d-1} e^{-k^2} = \left[\int_{-\infty}^{\infty} dk e^{-k^2}\right]^d = \sqrt{\pi}^{\,d}. \tag{72}$$

The integral on the left hand side can be converted into an integral familiar from Γ functions by simply substituting $u = k^2$. The k integral becomes

$$\int_0^\infty du u^{d/2-1} e^{-u} = \Gamma(d/2). \tag{73}$$

In this way one obtains

$$\int d\Omega_d \frac{1}{2}\Gamma(d/2). \tag{74}$$

So we have for the d dimensional solid angle:

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{75}$$

To construct the integral of interest we need:

$$\int_0^\infty dk \frac{k^{d-1}}{(k^2+\Delta)^2} = \frac{1}{2}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \int_0^1 dx x^{1-d/2}(1-x)^{\frac{d}{2}-1}. \tag{76}$$

The integral here is a standard integral, and the final result is an integral table:

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} . \qquad (77)$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2 + \Delta)^n} = \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)} \left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} . \qquad (78)$$

't Hooft observed that these expressions make sense for any number of dimensions, *including non-integers*. Consider $d = 4 - \epsilon$. The integral

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 + m^2)^2} \approx \frac{1}{16\pi^2} \int_m^\infty \frac{dk}{k^{1+\epsilon}} \propto \frac{1}{\epsilon}. \qquad (79)$$

So we have

$$\frac{1}{\epsilon} \sim \log(\Lambda/m). \qquad (80)$$

So rather than cutting off integrals in the ultraviolet at some momentum scale, we can use this *dimensional regularization* to *define* ill-defined Feynman integrals.

This allows us to put together a result for Σ:

$$\Sigma \approx \frac{e^2}{16\pi^2}\Gamma(\epsilon/2)\int_0^1 d\alpha(\not{p}(1-\alpha)+m) \tag{81}$$

$$= \frac{2}{\epsilon}\frac{e^2}{16\pi^2}\frac{1}{2}(\not{p}-m)+\frac{3}{2}m).$$

We can use this to correct the propagator. Calling

$$\Sigma(p) = (1 - Z^{-1}) + \delta m \tag{82}$$

we have

$$S_F(p) = i\frac{Z}{\not{p}-(m+\delta m)}. \tag{83}$$

So we see that there is a shift in the normalization of the propagator, and also of the physical mass. This correction to the mass is logarithmically divergent (Weiskopf).

Physical Meaning of Wave Function and Mass Renormalization: The Spectral Representation

We will focus on scalar field theories, to avoid writing lots of indices, and consider, in the interacting theory, the Green's function:

$$G(x-y) = T\langle\Omega|\phi(x)\phi(y)|\Omega\rangle. \tag{84}$$

Let's consider one particular time ordering, $x^0 > y^0$, and introduce a complete set of states. which we take to be energy eigenvalues. These states can be labeled by their total energy-momentum, p, and some other quantum numbers, n. In other words:

$$G(x-y) = \int \frac{d^3p}{2E(p)}\langle\Omega|\phi(x)|n,p\rangle\langle n,p|\phi(y)|\Omega\rangle \tag{85}$$

Now we use translation invariance to rewrite this as:

$$G(x-y) = \int \frac{d^3}{2E(p)}|\langle\Omega|\phi(0)|p,n\rangle|^2 e^{-ip\cdot(x-y)}. \tag{86}$$

Now we separate off states of definite mass $(\sqrt{E(p)^2 - p^2} = M^2)$. We define

$$\rho(M^2) = \delta(p^2 - M^2)|\langle p, n|\phi(0)|\Omega\rangle|^2. \tag{87}$$

Then we have, including the other time ordering, and noting the connection to the free propagator:

$$G(x - y) = \int dM^2 \rho(M^2) D_F(x - y; M). \tag{88}$$

One can immediately Fourier transform this expression. In simple field theories, $\rho(M^2)$ includes a δ function (at the mass of the meson) and a continuum (e.g. starting at $9M^2$ in the case of the ϕ^4 theory).

One writes:

$$\rho(M^2) = Z\delta(M^2 - m^2) + f_{cont}(M^2). \tag{89}$$

m^2 is the actual mass of the physical state, by our construction.

This is known as the “spectral representation", or the "Kallen-Lehman representation".
We can identify the Z and δm we have computed with the quantities here (to the order we have worked).

Renormalization

In δm, we have our first real example of the *renormalization* of a parameter. The observed, physical mass is $m + \delta m$. δm is “infinite" (depends on the cutoff, $1\epsilon \sim \log(\Lambda)$, but we only care about the observable quantity, the physical mass, and this is a parameter of the theory in any case (not something we can predict). We will see that the electromagnetic coupling is also renormalized.

For this, we consider the corrections to the photon propagator, focussing on the one loop expressions. This is equivalent to

$$T\langle\Omega|j^\mu(x)j^\nu(y)|\Omega\rangle \equiv -i\Pi^{\mu\nu}(q). \tag{90}$$

(after Fourier transform).

The first thing to note is that since this involves conserved currents, we have

$$q_\mu\Pi^{\mu\nu} = 0. \tag{91}$$

We can write

$$\Pi^{\mu\nu} = (g^{\mu\nu} q^2 - q^\mu q^\nu)\Pi(q^2) \tag{92}$$

This makes the calculation easier; it is not hard to read off the $q^\mu q^\nu$ piece.

To simplify the analysis, we will take $q^2 \gg m^2$. Then writing down the full diagram, it is not hard to pull out the $q^\mu q^\nu$ pieces.

Introducing Feynman parameters, combining denominators, and shifting the k integral in the usual way:

$$-i\Pi^{\mu\nu} = -e^2 \int \frac{d^4k}{(2\pi)^4} d\alpha \frac{\mathrm{Tr}\left(\left(\not{k} + \not{q}(1-\alpha) + m\right)\gamma^\mu \left(\not{k} - \not{q}\alpha + m\right)\gamma^\nu\right)}{\left[k^2 + q^2\alpha(1-\alpha) - m^2\right]^2}. \quad (93)$$

The $q^\mu q^\nu$ part can only arise from the numerator piece involving

$$\mathrm{Tr}(\not{q}(1-\alpha)\gamma^\mu \not{q}\alpha\gamma^\nu) = 8q^\mu q^\nu \alpha(1-\alpha). \quad (94)$$

So we have

$$\Pi(q^2) = -e^2 \int \frac{d^4k}{(2\pi)^4} d\alpha \frac{8\alpha(1-\alpha)}{[k^2 + q^2\alpha(1-\alpha) - m^2]^2}. \tag{95}$$

Now we consider $q^2 < 0$ (this is natural here, e.g. for scattering in the field o f a nucleus). Then we can use our integral table to obtain (for $q^2 \ll m^2$):

$$\Pi = \frac{4}{3}\frac{e^2}{16\pi^2}\left(\frac{2}{\epsilon}\right)(1 - 2\epsilon/2\log(\mu^2/m^2)). \tag{96}$$

Renormalization of the electric charge (e):

Consider Coulomb scattering. The amplitude is now proportional to:

$$\frac{e^2}{q^2}\left(1-\frac{4}{3}\frac{e^2}{16\pi^2}\frac{2}{\epsilon}(1-2\epsilon/2\log(\mu^2/m^2))\right) \tag{97}$$

Since what we call the electron charge* is the coefficient of $\frac{e^2}{q^2}$, we can simply define the “renormalized" charge

$$e_R^2 = e^2\left(1-\frac{4}{3}\frac{e^2}{16\pi^2}\frac{2}{\epsilon}(1-2\epsilon/2\log(\mu^2/m^2))\right) \tag{98}$$

The Vertex

Finally, we need to think about one more diagram at one loop, the *vertex*. This introduces some new features. We will see another infinity, which cancels against the infinite wave function renormalization in the fermion self-energy. Then we will encounter an infrared divergence, which we will have to explain (we can't "renormalize" away). Finally, we will see a correction to the electron magnetic moment from its Dirac value, the famous $g - 2$.

$$\Gamma^\mu(p, p', q) = -e^3 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\gamma^\nu(\not{p}' + \not{k} + m)\gamma^\mu(\not{p} + \not{k} + m)\gamma_\nu u(p)}{[(p' + k)^2 - m^2][(p + k)^2 - m^2]k^2}. \tag{99}$$

Let's look for the ultraviolet divergent part of the vertex. This comes from the term with most factors of k in the numerator, So we have:

$$-e^3 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\gamma^\nu \not{k}\gamma^\mu \not{k}\gamma_\nu u(p)}{[k^2 + 2p' \cdot k\alpha_1 + 2p \cdot k\alpha_2 - m^2\alpha_{12}]^3}. \quad (100)$$

Under the integral,

$$k_\rho k_\sigma \gamma^\rho \gamma^\mu \gamma^\sigma \to -\frac{2}{4} k^2 \gamma^\mu. \quad (101)$$

The k integral can be done shifting as usual, and using our integral table. The result is

$$-ie\bar{u}(p')\gamma^\mu u(p) \left(\frac{e^2}{16\pi^2} \frac{2}{\epsilon} \right) \quad (102)$$

This is also a renormalization of the electron charge, but it cancels against a similar infinity from the self energy (*Ward identity*).

Infrared Divergences

Now we consider the behavior of the vertex at low (virtual) photon momentum. We can neglect factors of k in the numerator, and use the Dirac equation to write as:

$$\Gamma^\mu = -e^3 \int \frac{d^4k}{(2\pi)^4} \frac{(4p \cdot p')\gamma^\mu}{k^2(2p \cdot k)(2p' \cdot k)}. \tag{103}$$

Doing the k^0 integral (restoring the $i\epsilon$ gives:

$$-e^3 \frac{2\pi i}{(2\pi)^4} \int \frac{d^3k}{2|k|} (\frac{(4p \cdot p' - 2m^2)\gamma^\mu}{(2p \cdot k)(2p' \cdot k)}. \tag{104}$$

Cancelation by soft photon emission

Consider the interference of the tree graph and one loop vertex correction, and compare with the interference diagram involving photon emission before and after the virtual photon exchange. The latter is

$$\bar{u}(p')\gamma^\nu \frac{\not{p}' + \not{k} + m}{2p' \cdot k} \gamma^\mu u(p) \bar{u}(p) \frac{\gamma^\nu (\not{p} - \not{k} + m)\gamma^\rho}{-2p \cdot k} u(p). \qquad (105)$$

For low k, these have the same form up to a sign.

Now no experiment can resolve photons of arbitrarily small $\vec{k}$. So we can introduce an energy resolution, E_r. The actual divergence then cancels between the two diagrams, and we are left with a result proportional to $\log(E_r/E)$, where E is a typical energy scale in the process. For high energies, we also get a log of the mass, as we saw in Compton scattering, from the integral over angles, The result is known as a “Sudakov double logarithm". There are actually such logs in every order of perturbation theory, and it is possible to add up these large terms (they exponentiate; see Peskin and Schroeder).

The magnetic moment

For the magnetic moment, we look for a coupling of the form

$$F_2(q^2)q_\mu\sigma^{\mu\nu} \tag{106}$$

Taking q to have spatial components, and taking $\mu = j, \nu = k$, this is

$$\partial_i A^j \sigma^k = \vec{\sigma} \cdot \vec{B}. \tag{107}$$

It is not hard to isolate this coupling.

Starting with our expression for the vertex:

$$-i\Gamma^\mu(p,p',q) = -e^3 \int \frac{d^4k}{(2\pi)^4} \tag{108}$$

$$\times \frac{\bar{u}(p')\gamma^\nu(\not{p}'+\not{k}+m)\gamma^\mu(\not{p}+\not{k}+m)\gamma_\nu u(p)}{[(p'+k)^2-m^2][(p+k)^2-m^2]k^2}.$$

introduce Feynman parameters:

$$\Gamma^\mu(p,p',q) = -2e^3 \int \frac{d^4k}{(2\pi)^4} d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\alpha_{123}) \tag{109}$$

$$\frac{\bar{u}(p')\gamma^\nu(\not{p}'(1-\alpha_{12}+\not{k}+\not{q}\alpha_1+m)\gamma^\mu(\not{p}+\not{k}+m)\gamma_\nu u(p)}{[k^2-(p\alpha_1+p'\alpha_2)^2]^3}.$$

Terms in the numerator containing only k cannot contribute to F_2 (they can contribute to F_1, the coefficient of γ^μ). We can rewrite the numerator using the Dirac equation:

$$\bar{u}(p')(2p'^\nu(1-\alpha_{12})+\gamma^\nu m\alpha_{12}+\gamma^\rho \not{q}\alpha_1)\gamma^\mu(2p^\rho(1-\alpha_{12})+m\gamma^\rho\alpha_{12}-\not{q}\gamma^\rho\alpha_2)u(p). \quad (110)$$

There are nine terms in the product. Many don't contribute to F_2. E.g.

1. $4p \cdot p'(1-\alpha_{12})^2\gamma^\mu$: F_1 only.
2. $$2m\gamma^\mu \not{p}'\alpha_{12}(1-\alpha_{12})_2 m \not{p}\alpha_{12}(1-\alpha_{12})\gamma^\mu$$
$$= 2m\gamma^\mu(\not{p}+\not{q})\alpha_3(1-\alpha_3) + 2m(\not{p}'-\not{q})\gamma^\mu\alpha_3(1-\alpha_3)$$
$$= 2m\alpha_3(1-\alpha_3)[\gamma^\mu, \not{q}] + F_1 \text{ term.}$$

Combining the rest, the result is:

$$\Gamma^\mu = \cdots - 2e^3 \int \frac{d^4k d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\alpha_{123}) m\alpha_3(1-\alpha_3)[\gamma_\mu, \not{q}]}{(2\pi)^4[k^2 - m^2(1-\alpha_3)]^3}. \tag{111}$$

(In the denominator we have set $q^2 = 0$.) The k integral yields:

$$\Gamma^\mu = -ie^3 \int \frac{d\alpha_1 d\alpha_2 d\alpha_3 \delta(1-\alpha_{123}) 4\alpha_3(1-\alpha_3)}{16\pi^2} \frac{iq_\nu \sigma^{\nu\mu}}{2m(1-\alpha_3)^2} \tag{112}$$

$$= -i\frac{\alpha}{\pi} \frac{iq_\nu \sigma^{\nu\mu}}{2m}.$$

This yields

$$\frac{g-2}{2} = \frac{\alpha}{2\pi} = 0.0011614 \tag{113}$$

vs. measured 0.0011596.

g-2 fast

$$-i\Gamma'^{\mu} = -2e^2 \int \frac{d^4k}{(2\pi)^4} d\alpha_1 d\alpha_2 d\alpha_3 \, \delta(1-\Sigma\alpha_i)$$

$$\times \bar{u}(p')\gamma^{\rho}(\not{k}+\not{p}'(1-\alpha_{12})+\not{q}\alpha_1+m)\gamma^{\mu}$$

$$\frac{\times(\not{k}+\not{p}(1-\alpha_{12})-\not{q}\alpha_3+m)\gamma_{\rho}u(p)}{[k^2-(p\alpha_1+p'\alpha_2)^2]^3}$$

Now terms in numerator containing only k contribute only to F_1. Rewrite numerator using Dirac eqn:

$$\bar{u}(p')\left(2p^{\rho'}(1-\alpha_{12})+\gamma^{\rho}m\alpha_{12}+\gamma^{\rho}\not{q}\alpha_1\right)\gamma^{\mu}$$

$$(2p^{\rho}(1-\alpha_{12})+m\gamma^{\rho}\alpha_{12}-\not{q}\gamma^{\rho}\alpha_2)u(p)$$

Nine terms in product; many don't contribute to F_2

① $4p\cdot p'(1-\alpha_{12})^2\gamma^{\mu}$: F_1 only

Result is:

$$\Gamma^\mu_{(2)} = -2e^2 \int \frac{d\alpha_1 d\alpha_2 d\alpha_3\, \delta(1-\alpha_{123})}{(2\pi)^4} \frac{m\alpha_3(1-\alpha_3)[\gamma_\mu, \not{q}]}{[k^2+m^2(1-\alpha_3)]^3}$$

(in denominator, we have set $q^2=0$)

$$= -e^2 \int \frac{d\alpha_1 d\alpha_2 d\alpha_3\, \delta(1-\Sigma\alpha_i)}{16\pi^2} 4\alpha_3(1-\alpha_3) \frac{i q_\nu \sigma^{\nu\mu}}{2m(1-\alpha_3)^2}$$

$$= \frac{-e^2}{16\pi^2} \int_0^1 d\alpha_3 \int_0^{1-\alpha_3} d\alpha_2 \frac{\alpha_3}{(1-\alpha_3)} \frac{i q_\nu \sigma^{\nu\mu}}{2m}$$

$$= \frac{\alpha}{4\pi} \frac{i q_\nu \sigma^{\nu\mu}}{2m}$$

$$\frac{g-2}{2} = \frac{\alpha}{2\pi} = 0.0011614 \quad \text{vs. measured } 0.0011596$$