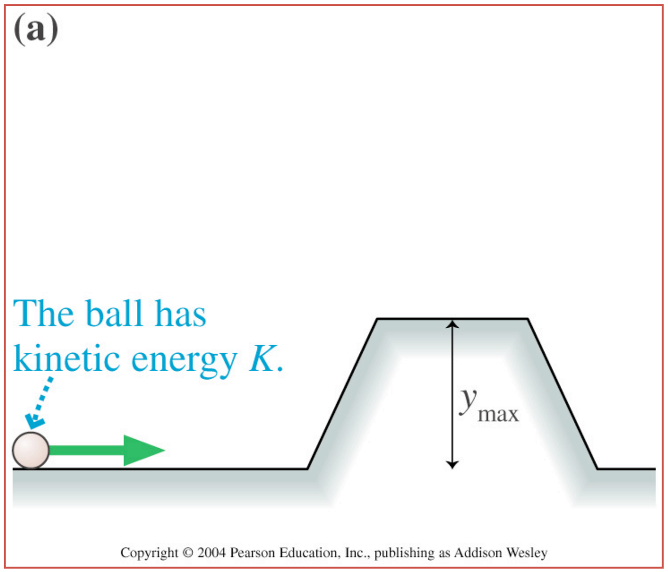
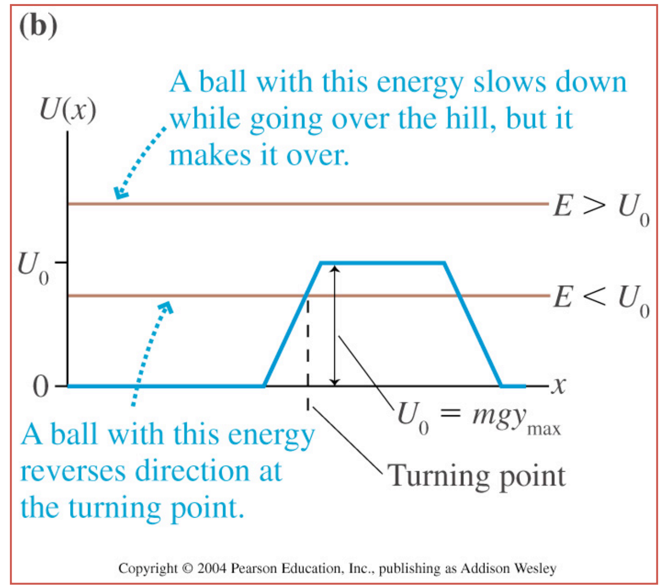
Quantum Mechanical Tunneling

The square barrier:

Behaviour of a classical ball rolling towards a hill (potential barrier):





If the ball has energy E less than the potential energy barrier (U=mgy), then it will not get over the hill.

The other side of the hill is a **classically forbidden** region.

Quantum Mechanical Tunneling

The square barrier:

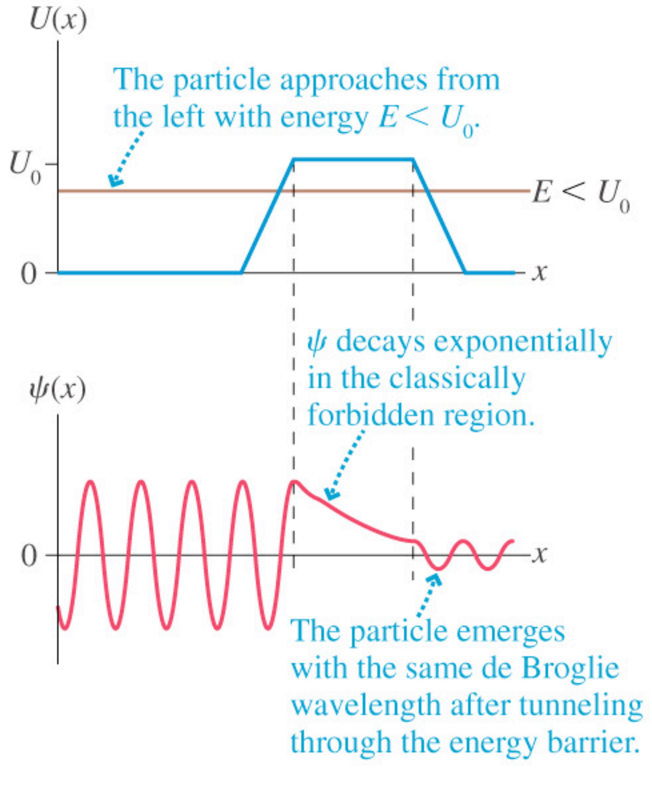
Behaviour of a quantum particle at a potential barrier

Solving the TISE for the square barrier problem yields a peculiar result:

If the quantum particle has energy E less than the potential energy barrier U, there is still a non-zero probability of finding the particle classically forbidden region !

This phenomenon is called **tunneling**.

To see how this works let us solve the TISE…

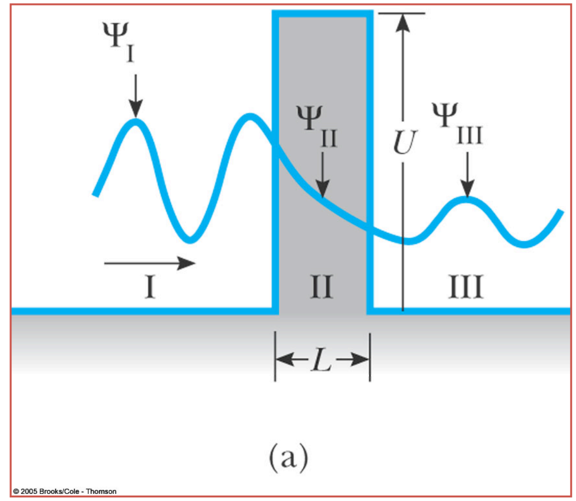


Quantum Mechanical Tunneling

The square barrier:

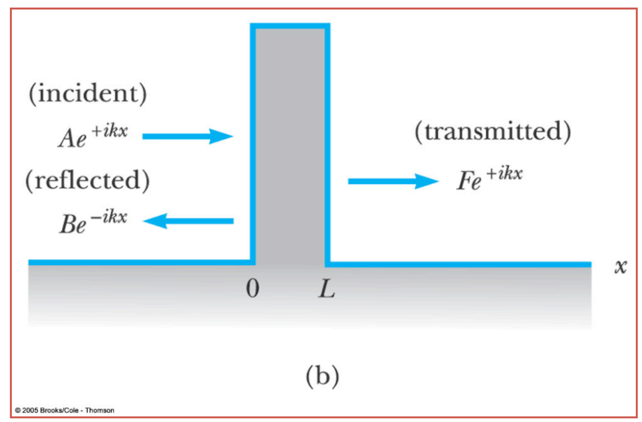
Behaviour of a quantum particle at a potential barrier

To the left of the barrier (region I), U=0
Solutions are free particle plane waves:



$$\phi(x) = Ae^{ikx} + Be^{-ikx}, \quad k = \frac{\sqrt{2mE}}{\hbar}$$

The first term is the incident wave moving to the **right**
The second term is the reflected wave moving to the **left**.



Reflection coefficient: $$R = \frac{|\phi_{reflected}|^2}{|\phi_{incident}|^2} = \frac{|B|^2}{|A|^2}$$

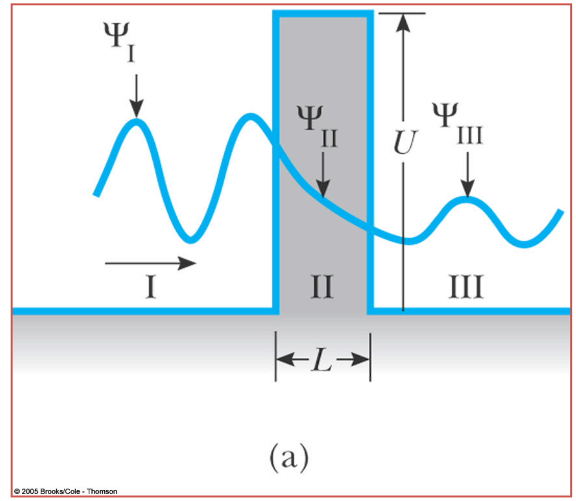
Quantum Mechanical Tunneling

The square barrier:

Behaviour of a quantum particle at a potential barrier

To the right of the barrier (region III), U=0. Solutions are free particle plane waves:

$$\phi(x) = Fe^{ikx}, \quad k = \frac{\sqrt{2mE}}{\hbar}$$

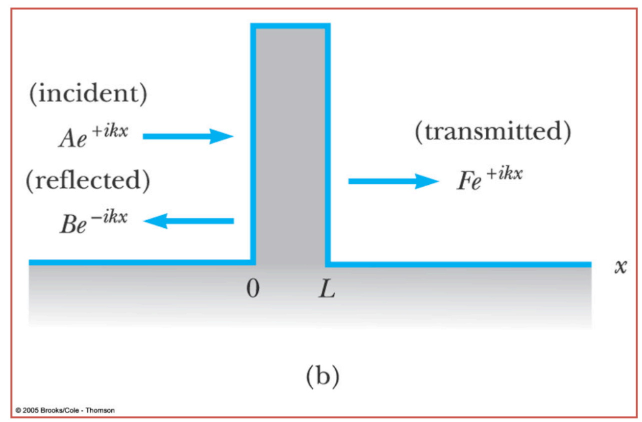


This is the transmitted wave moving to the **right**

Transmission coefficient:

$$T = \frac{|\phi_{transmitted}|^2}{|\phi_{incident}|^2} = \frac{|F|^2}{|A|^2}$$

$$T + R = 1$$



Quantum Mechanical Tunneling

The square barrier:

Behaviour of a quantum particle at a potential barrier

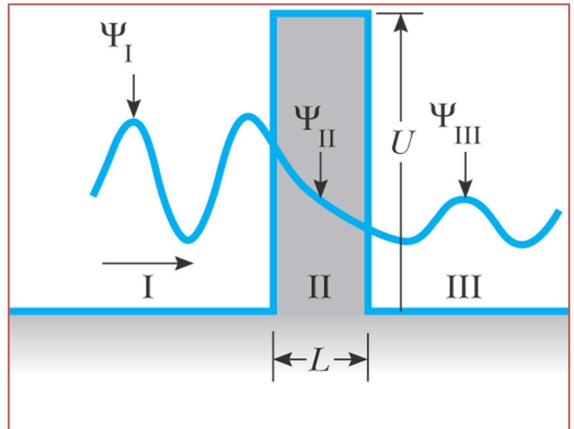
In the barrier region (region II), the TISE is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\phi(x) = (E-U)\phi(x)$$

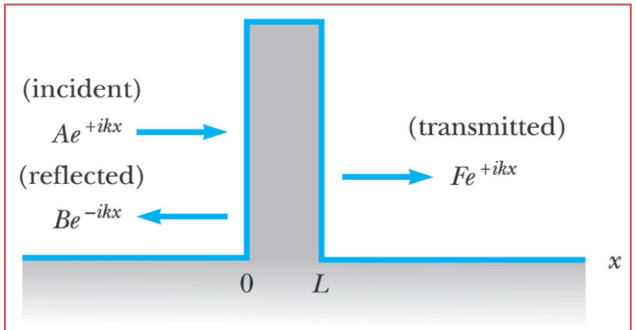
Solutions are

$$\phi(x) = Ce^{-\alpha x} + De^{\alpha x}$$

$$\alpha = \frac{\sqrt{2m(U-E)}}{\hbar}$$



(a)



(b)

Quantum Mechanical Tunneling

The square barrier:

Behaviour of a quantum particle at a potential barrier

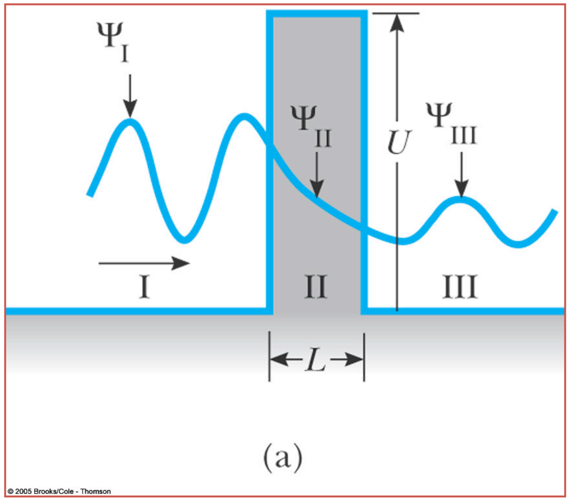
At x=0, region I wave function = region II wave function:

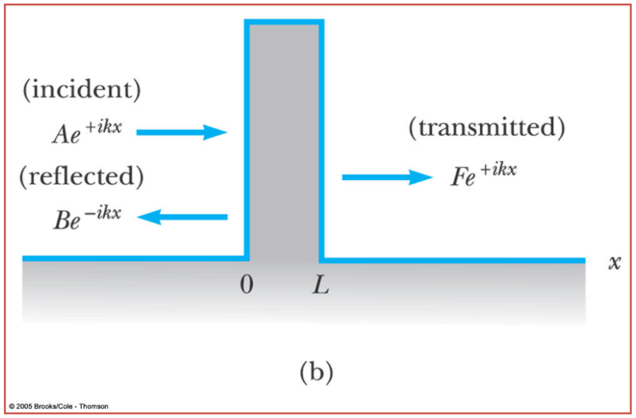
$$Ae^{ikx} + Be^{-ikx} = Ce^{-\alpha x} + De^{\alpha x}$$

$$A + B = C + D$$

At x=L, region II wave function = region III wave function:

$$Ce^{-\alpha L} + De^{\alpha L} = Fe^{ikL}$$





Quantum Mechanical Tunneling

The square barrier:

Behaviour of a quantum particle at a potential barrier

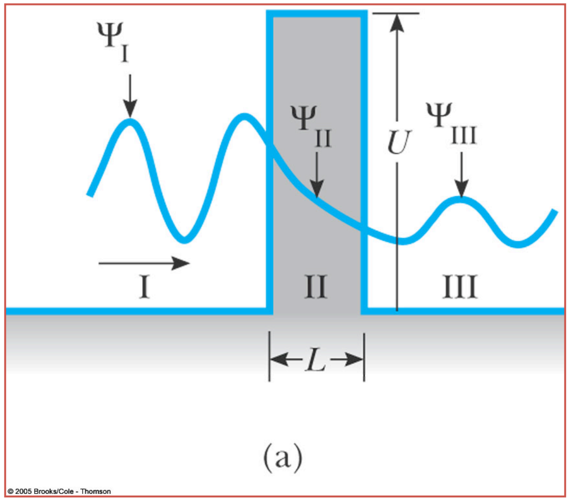
At x=0, dφ/dx in region I = dφ/dx in region II:

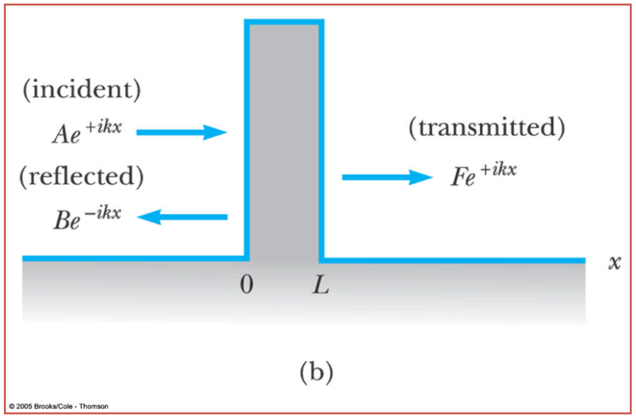
$$ikAe^{ikx} - ikBe^{-ikx} = -\alpha Ce^{-\alpha x} + \alpha De^{\alpha x}$$

$$ikA - ikB = -\alpha C + \alpha D$$

At x=L, dφ/dx in region II = dφ/dx in region III :

$$-\alpha Ce^{-\alpha L} + \alpha De^{\alpha L} = ikFe^{ikL}$$





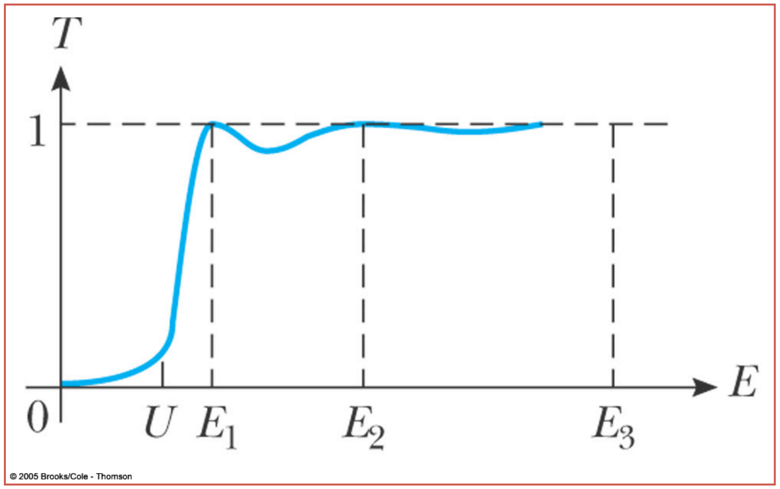
Quantum Mechanical Tunneling

The square barrier:

Behaviour of a quantum particle at a potential barrier

Solving the 4 equations, we get

$$T = \frac{1}{1 + \frac{1}{4}\left[\frac{U^2}{E(U-E)}\right]\left[\frac{e^{\alpha L} + e^{-\alpha L}}{2}\right]^2}$$



For low energies and wide barriers,

$$T \approx e^{-\alpha L}$$

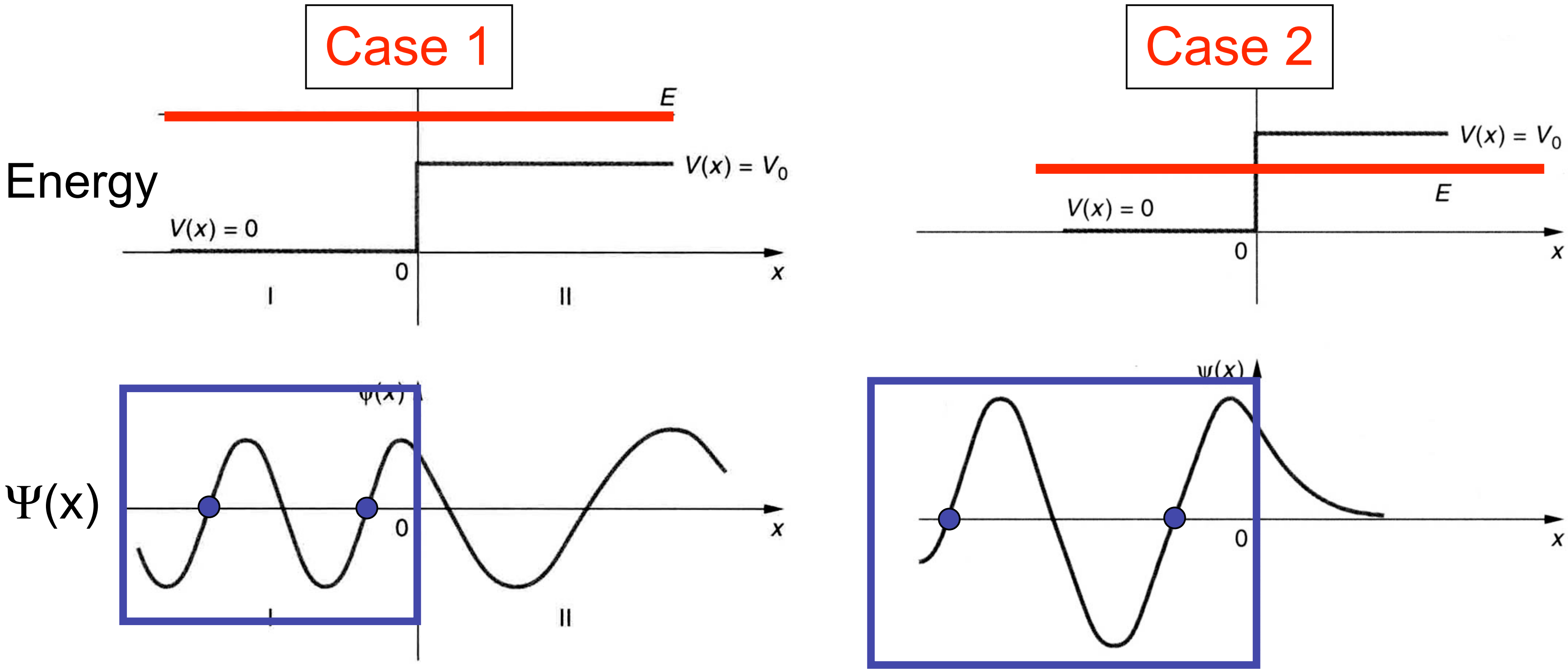
For some energies, T=1, so the wave function is fully transmitted **(transmission resonances).**
This occurs due to wave interference, so that the reflected wave function is completely suppressed.

Quantum Mechanical Tunneling

The step barrier:

To the left of the barrier (region I), U=0.
Solutions are free particle plane waves:

$$\phi(x) = Ae^{ikx} + Be^{-ikx}, \quad k = \frac{\sqrt{2mE}}{\hbar}$$



Quantum Mechanical Tunneling

The step barrier:

Inside Step:
$U = V_o$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\phi(x) = (E - V_0)\phi(x)$$

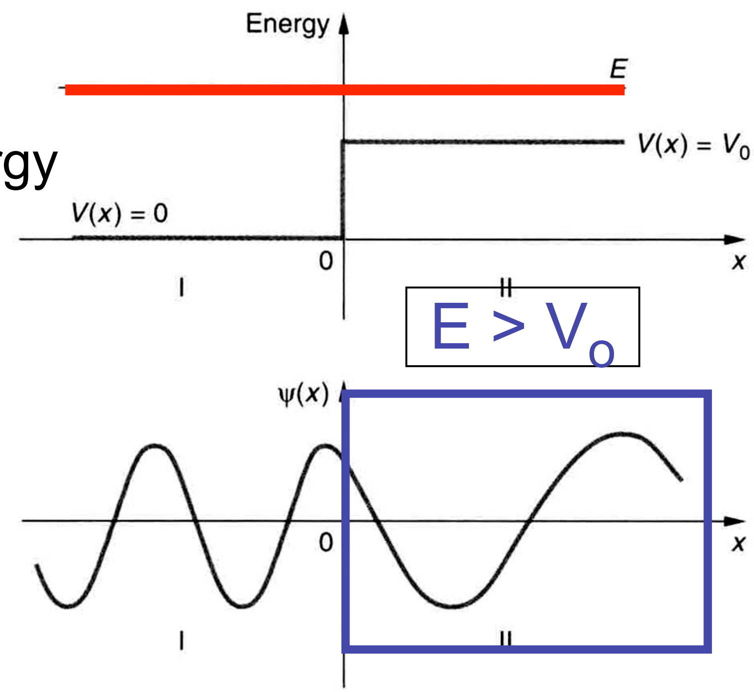
$$k_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

Case 1

Ψ(x) is oscillatory for $E > V_o$

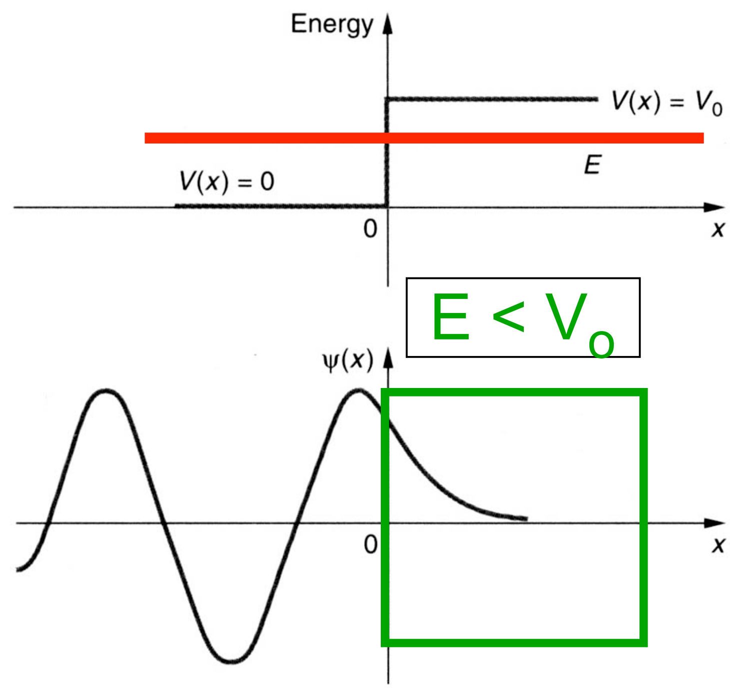
Energy

Ψ(x)



Case 2

Ψ(x) is decaying for $E < V_o$



Quantum Mechanical Tunneling

The step barrier:

$$R = \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2$$

$$T = \frac{4k_1 k_2}{(k_1 + k_2)^2}$$

$$k_1 = \frac{\sqrt{2mE}}{\hbar}$$

$$k_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

R(reflection) + T(transmission) = 1

Reflection occurs at a barrier ($R \neq 0$), regardless if it is step-down or step-up.
R depends on the wave vector difference (k_1 - k_2) (or energy difference), but not on which is larger.
Classically, $R = 0$ for energy E larger than potential barrier (V_o).

Quantum Mechanical Tunneling

The step barrier:

A free particle of mass m, wave number k_1 , and energy $E = 2V_o$ is traveling to the right. At x = 0, the potential jumps from zero to $-V_o$ and remains at this value for positive x. Find the wavenumber k_2 in the region $x > 0$ in terms of k_1 and V_o. Find the reflection and transmission coefficients R and T.

$$k_1 = \frac{\sqrt{2mE}}{\hbar} = \frac{\sqrt{2m(2V_o)}}{\hbar} = \frac{\sqrt{4mV_o}}{\hbar} \quad and$$

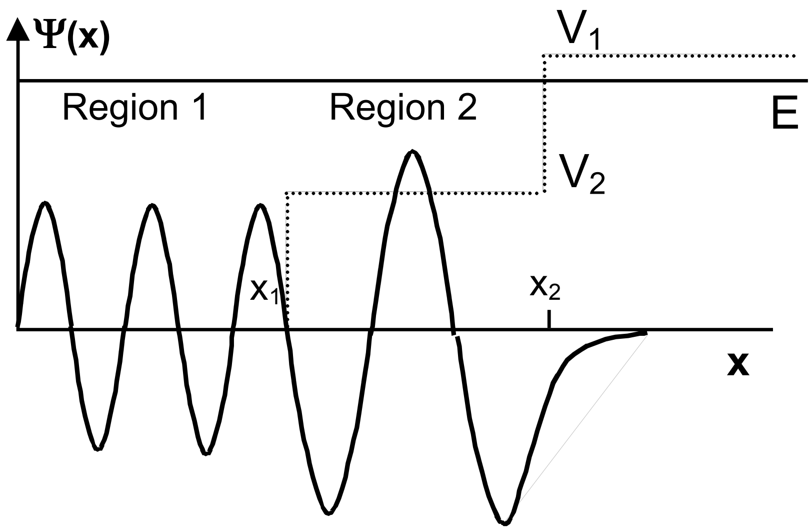
$$k_2 = \frac{\sqrt{2m|V-E|}}{\hbar} = \frac{\sqrt{2m|-V_o - 2V_o|}}{\hbar} = \frac{\sqrt{2m(3V_o)}}{\hbar} = \boxed{\frac{\sqrt{6mV_o}}{\hbar} \quad or \quad \sqrt{\frac{3}{2}}\, k_1}$$

$$R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 = \left(\frac{k_1 - \sqrt{\frac{3}{2}}k_1}{k_1 + \sqrt{\frac{3}{2}}k_1}\right)^2 = \left(\frac{-0.225}{2.225}\right)^2 = \boxed{0.0102} \quad \text{(1\% reflected)}$$

$$T = 1 - R = 1 - 0.0102 = 0.99 \quad \text{(99\% transmitted)}$$

Quantum Mechanical Tunneling

Sketch the **wave function** $\psi(x)$ corresponding to a particle with energy E in the potential well shown below. Explain how and why the wavelengths and amplitudes of $\psi(x)$ are **different** in regions 1 and 2.



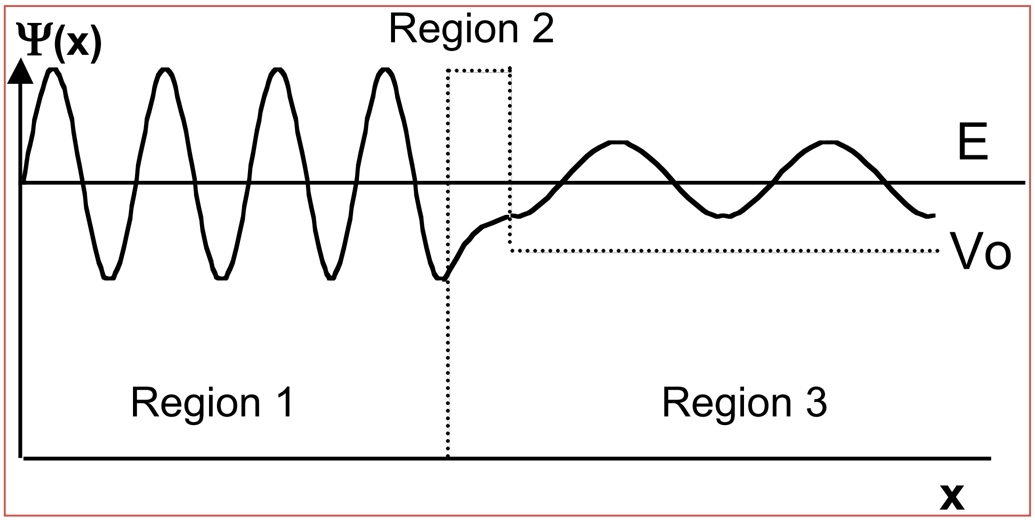
$\psi(x)$ **oscillates inside** the potential well because $E > V(x)$, and **decays exponentially outside** the well because $E < V(x)$.

The **frequency** of $\psi(x)$ is **higher in Region 1** vs. Region 2 because the kinetic energy is higher [$E_k = E - V(x)$].

The **amplitude** of $\psi(x)$ is **lower in Region 1** because its higher E_k gives a higher velocity, and the particle therefore spends less time in that region.

Quantum Mechanical Tunneling

Sketch the wave function $\psi(x)$ corresponding to a particle with energy E in the potential shown below. Explain how and why the wavelengths and amplitudes of $\psi(x)$ are different in regions 1 and 3.



$\psi(x)$ **oscillates** in **regions 1 and 3** because $E > V(x)$, and **decays exponentially** in **region 2** because $E < V(x)$.

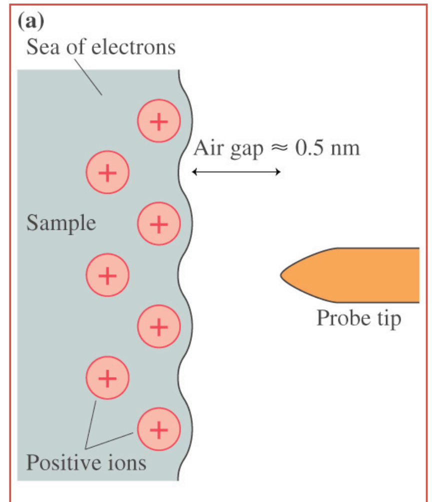
Frequency of $\psi(x)$ is **higher in Region 1** vs. 3 because kinetic energy is higher there.

Amplitude of $\psi(x)$ in Regions 1 and 3 depends on the initial location of the wave packet. If we assume a bound particle in Region 1, then the amplitude is higher there and decays into Region 3 (case shown above).

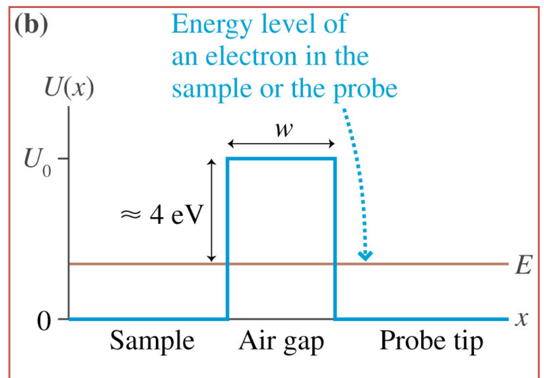
Quantum Mechanical Tunneling

The scanning tunneling microscope:

Scanning-tunneling microscopes allow us to see objects at the atomic level.



- A small air gap between the probe and the sample acts as a potential barrier.
- Energy of an electron is less than the energy of a free electron by an amount equal to the work function.
- Electrons can tunnel through the barrier to create a current in the probe.
- The current is highly sensitive to the thickness of the air gap.
- As the probe is scanned across the sample, the surface structure is mapped by the change in the tunneling current.

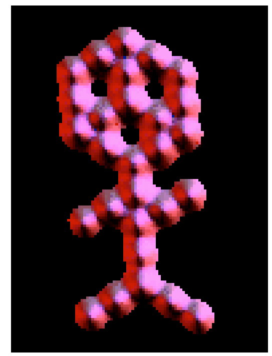


Quantum Mechanical Tunneling

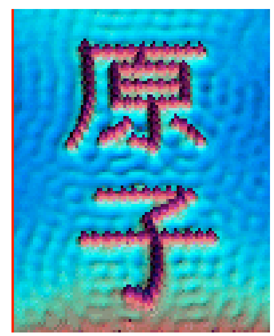
The scanning tunneling microscope:

Scanning-tunneling microscopes allow us to see objects at the atomic level.

- A small air gap between the probe and the sample acts as a potential barrier.
- Energy of an electron is less than the energy of a free electron by an amount equal to the work function.
- Electrons can tunnel through the barrier to create a current in the probe.
- The current is highly sensitive to the thickness of the air gap.
- As the probe is scanned across the sample, the surface structure is mapped by the change in the tunneling current.



Carbon Monoxide on Platinum



Iron on Copper

[www.almaden.ibm.com/vis/**stm**/](http://www.almaden.ibm.com/vis/stm/)

Quantum Mechanical Tunneling

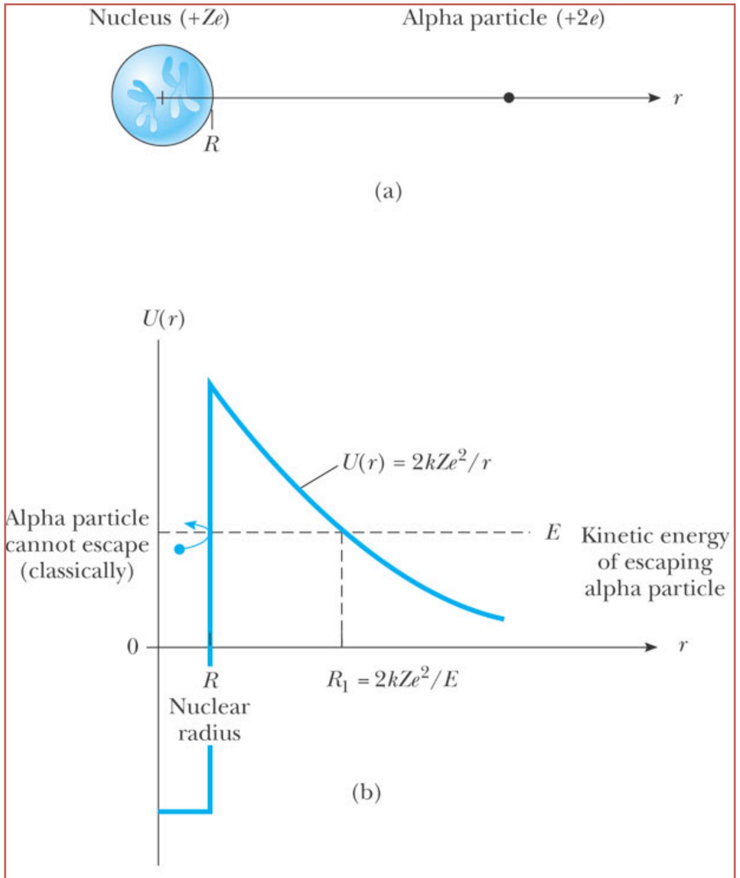
Decay of radioactive elements:

Emission of α particles (helium nucleii) in the decay of radioactive elements is an example of tunneling

- α particles are confined in the nucleus modeled as a square well
- α particles can eventually tunnel through the Coulomb potential barrier.
- Tunneling rate is very sensitive to small changes in energy, accounting for the wide range of decay times:

$$T = e^{8\sqrt{\frac{ZR}{r_0}} - 4\pi Z\sqrt{\frac{E_0}{E}}},$$

$$r_0 \approx 7.25\, fm,\ E_0 = 0.0993 MeV$$



Quantum Mechanical Tunneling

Decay of radioactive elements:

Emission of α particles (helium nucleii) in the decay of radioactive elements is an example of tunneling

- Transmission probability:

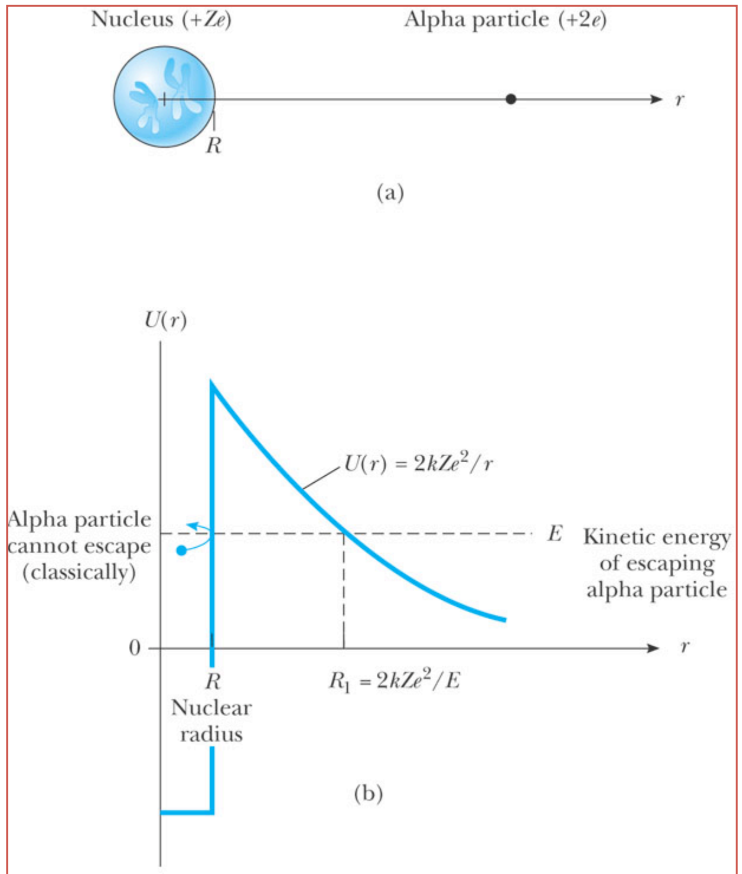
$$T = e^{8\sqrt{\frac{ZR}{r_0}} - 4\pi Z\sqrt{\frac{E_0}{E}}},$$

$$r_0 \approx 7.25\, fm,\ E_0 = 0.0993 MeV$$

- Transmission rate λ = frequency of collisions with the barrier x T

$$\lambda = fT \approx 10^{21} e^{8\sqrt{\frac{ZR}{r_0}} - 4\pi Z\sqrt{\frac{E_0}{E}}},$$

- Half life: $$t_{1/2} = \frac{0.693}{\lambda}$$



Quantum Mechanical Tunneling

Other applications of quantum mechanical tunneling:

- Tunneling diodes (used in digital chips in computers)
- Explanation of ammonia inversion (see text)
- Theory of black hole decay
- …..