

# Quantum symmetric Kac-Moody pairs

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# Symmetric pairs for quantum groups

$\text{char}(k) = 0, k = \bar{k}$

$\mathfrak{g}$  semisimple Lie algebra / $k$ ,  $\theta : \mathfrak{g} \xrightarrow{\cong} \mathfrak{g}$ ,  $\theta^2 = \text{Id}$ ,  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ .

Generalizations:

$U_q(\mathfrak{g})$  quantum envel. algebra.

$\mathfrak{g}$  symmetrizable Kac-Moody alg.

P1. Construct  $\theta_q : U_q(\mathfrak{g}) \rightarrow U_q(\mathfrak{g})$

$\dim(\theta(\mathfrak{b}^+) \cap \mathfrak{b}^-) < \infty$  (1st kind)

P2. Construct  $U'_q(\mathfrak{k}) \subset U_q(\mathfrak{g})$

$\dim(\theta(\mathfrak{b}^+) \cap \mathfrak{b}^+) < \infty$  (2nd kind)

Solved by M. Noumi, T. Sugitani,  
M. Dijkhuizen, G. Letzter

classified by V. Kac, S. Wang '92.

## Main result of this talk

Let  $\mathfrak{g}$  be a symmetrizable Kac-Moody algebra and  $\theta$  an involutive automorphism of the second kind. Then both  $\theta_q$  and  $U'_q(\mathfrak{k})$  can be constructed. There exists a rich structure theory.

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# Examples of quantum symmetric Kac-Moody pairs

- $q$ -Onsager algebra:  $\widehat{\mathfrak{sl}}_2$  with Chevalley involution
  - Baseilhac, Belliard: factorisable scattering on the half line.
  - Terwilliger: polynomial association schemes.
- twisted  $q$ -Yangians:  $\dim(\mathfrak{g}) < \infty$ .  
Consider  $\widehat{\mathfrak{g}}' = \mathfrak{g} \otimes k[t, t^{-1}] \oplus kc$  with involution

$$\widehat{\theta} : \widehat{\mathfrak{g}}' \rightarrow \widehat{\mathfrak{g}}', \quad \widehat{\theta}(x \otimes t^n) = \theta(x) \otimes t^{-n}, \quad \widehat{\theta}(c) = -c.$$

$\widehat{\mathfrak{g}}' = \mathfrak{k}' \oplus \mathfrak{p}'$ . Construction of  $U'_q(\mathfrak{k}') \subset U_q(\widehat{\mathfrak{g}}')$

A. Molev, E. Ragoucy, P. Sorba '03:  $\theta$  of type  $AI$ ,  $All$

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- quantized generalized intersection matrix algebras  
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$A = (a_{ij})_{i,j \in I}$  symmetric,  $\mathfrak{g} = \mathfrak{g}(A)$ ,  $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$

$$U_q(\mathfrak{g}') = k(q)\langle E_i, F_i, K_i, K_i^{-1} \mid i \in I \rangle / \text{relations}$$

with the following relations (for all  $i, j \in I$ ):

- (1)  $K_i K_i^{-1} = 1 = K_i^{-1} K_i, \quad K_i K_j = K_j K_i$
- (2)  $K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j$
- (3)  $E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}$
- (4)  $F_{ij}(E_i, E_j) = 0 = F_{ij}(F_i, F_j) \quad \text{if } i \neq j$

where  $F_{ij}(x, y) = \sum_{n=1}^{1-a_{ij}} (-1)^n \begin{bmatrix} 1-a_{ij} \\ n \end{bmatrix}_q x^{1-a_{ij}-n} y x^n$   
 with  $\begin{bmatrix} N \\ n \end{bmatrix}_q = \frac{[N]_q [N-1]_q \dots [N-n+1]_q}{[1]_q [2]_q \dots [n]_q}, \quad [m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}.$

Hopf algebra with coproduct:  $\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$   
 $\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$   
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 $\Delta(K_i) = K_i \otimes K_i$

# Right coideal subalgebras

Observe: Let  $x \in \mathfrak{g}'$ .

- ~~~  $kx \subset \mathfrak{g}'$  (commutative) Lie subalgebra of  $\mathfrak{g}'$ .
- ~~~  $k\langle x \rangle$  Hopf subalgebra of  $U_q(\mathfrak{g}')$ .

But  $k\langle F_i \rangle$  is not a Hopf subalgebra of  $U_q(\mathfrak{g}')$ .

## Definition

A subalgebra  $C \subseteq U_q(\mathfrak{g}')$  is called a **right coideal subalgebra** if

$$\Delta(C) \subseteq C \otimes U_q(\mathfrak{g}').$$

General principle: Quantum group analogs of Lie subalgebras of  $\mathfrak{g}'$  appear as right (or left) coideal subalgebras of  $U_q(\mathfrak{g}')$ .

Example:  $k\langle F_i \rangle$  is RCSA of  $U_q(\mathfrak{g}')$ .

# Admissible pairs

A generalized Cartan matrix,  $\mathfrak{g} = \mathfrak{g}(A)$

## Definition

Let  $X \subset I$  be of finite type and  $\tau \in \text{Aut}(A, X)$ .

The pair  $(X, \tau)$  is called **admissible** if

- ①  $\tau^2 = \text{Id}$
- ②  $\tau|_X = -w_X$
- ③ If  $j \in I \setminus X$  and  $\tau(j) = j$  then  $\alpha_j(\rho_X^\vee) \in \mathbb{Z}$ .

Notation:  $\text{Aut}(A, X)$ : diagram automorphisms which fix  $X$ .

$w_X$ : longest element in the parabolic subgroup  $W_X \subset W$ .

$\rho_X^\vee$ : half sum of positive coroots for  $\mathfrak{m}_X$ .

( $\mathfrak{m}_X$ : semisimple Lie subalgebra corresponding to  $X$ .)

	$d$	$d^+$	$m(\lambda_i)$	$m(2\lambda_i)$
$AII$			1	0
$AIH$			4	0
$AIH$			2 (for $i < p$ ) $(2 \leq i \leq \frac{l}{2})$	0
$AIH$			$2(l-p+1)$ $(\text{for } i = p)$	1
$AIH$			2 (for $i < l'$ ) $(l=2l'+1, l' \geq 2)$	0
$AIH$			$\frac{1}{2}(l-l')$	0
$AIIV$			$2(l-1)$	1
$BII$			0	0
$BII$			$\frac{1}{2}(l-p)$ $2(l-p)+1$ $(l=p)$	0
$CII$			$2l-1$	0
$CII$			1	0
$CI$			0	0
$CI$			$\frac{4}{3}(l-p)$ $4(l-2p)$ $(l=p)$	0
$CI$			0	0
$CI$			$\frac{4}{3}(l-p)$ $(l=p)$	0
$DI$			0	0
$DI$			$\frac{1}{2}(l-p)$ $2(l-p)$ $(l=p)$	0
$DI$			0	0

$DII$			$2(l-1)$	0
$DIII$			$\frac{4}{3}(l-p)$ $(l=2l'+1, l' \geq 2)$	0
$EI$			$1(i=l)$	0
$EII$			$1(i=1, 2)$ $2(i=3, 4)$	0
$EIII$			$6(i=1)$ $8(i=2)$	0
$EIV$			8	0
$EV$			1	0
$EVII$			$1(i=1, 2)$ $4(i=3, 4)$	0
$EVIII$			$1(i=1)$ $8(i=2, 3)$	0
$EIX$			1	0
$PI$			1	0
$FII$			8	7
$G$			1	0

# Involutive automorphisms of the second kind

$Br(W)$ : Braid group associated to  $W$ .

Action:  $\text{Ad} : Br(W) \rightarrow Aut(\mathfrak{g})$

$$s_i \mapsto \text{Ad}(s_i) = \exp(\text{ad}(e_i)) \exp(\text{ad}(-f_i)) \exp(\text{ad}(e_i))$$

Theorem (Kac, Wang '92)

The following map is a bijection:

$$\left\{ \begin{array}{c} \text{admissible} \\ \text{pairs} \end{array} \right\} \Big/ Aut(A) \rightarrow \left\{ \begin{array}{c} \text{involutive automorphisms} \\ \text{of } \mathfrak{g} \text{ of the second kind} \end{array} \right\} \Big/ Aut(\mathfrak{g})$$
$$(X, \tau) \mapsto \theta(X, \tau) := \text{Ad}(s(X, \tau)) \circ \text{Ad}(w_X) \circ \tau \circ \omega.$$

Notation:

- $\omega$  Chevalley involution:

$$\omega(e_i) = -f_i, \omega(f_i) = -e_i, \omega(h) = -h \quad \forall i \in I, h \in \mathfrak{h}.$$

- $s(X, \tau) \in \text{Hom}(Q, k^\times)$

$$s(X, \tau)(\alpha_j) = \begin{cases} 1 & \text{if } j \in X \text{ or } \tau(j) = j \\ i^{\alpha_j(2\rho_X^\vee)} & \text{if } j \notin X \text{ and } \tau(j) > j \\ (-i)^{\alpha_j(2\rho_X^\vee)} & \text{if } j \notin X \text{ and } \tau(j) < j \end{cases}$$

# Lusztig's braid group action

$$Br(W) \rightarrow Aut(U_q(\mathfrak{g}')), \quad s_i \mapsto T_i, \quad w \mapsto T_w$$

where

$$T_i(E_i) = -F_i K_i, \quad T_i(F_i) = -K_i^{-1} E_i, \quad T_i(K_j) = K_j K_i^{-a_{ij}}$$

and for  $j \neq i$  one has

$$T_i(E_j) = \frac{\text{ad}(E_i^{-a_{ij}})(E_j)}{[-a_{ij}]_q [-a_{ij}-1]_q \dots [1]_q}, \quad T_i(F_j) = -q^{-a_{ij}} \omega(T_i(E_j)).$$

Notation:

- $\omega$  Chevalley involution for  $U_q(\mathfrak{g}')$ :  
 $\omega(E_i) = -F_i, \omega(F_i) = -E_i, \omega(K_i) = K_i^{-1} \quad \forall i \in I.$
- $\text{ad}(E_i)(x) = E_i x - K_i x K_i^{-1} E_i$  (adjoint action).

# Quantum involution

$(X, \tau)$  admissible pair,  $\theta = \theta(X, \tau)$

Notation:  $\mathcal{M}_X = k(q)\langle E_i, F_i, K_i^{\pm 1} \mid i \in X \rangle$

$K_h = \prod_{i \in I} K_i^{m_i}$  for  $h = \sum_{i \in I} m_i h_i \in Q^\vee$

Goal: Define algebra automorphism

$$\theta_q = \theta_q(X, \tau) : U_q(\mathfrak{g}') \rightarrow U_q(\mathfrak{g}')$$

such that (1)  $\theta_q(K_h) = K_{\theta(h)}$   $\forall h \in Q^\vee$

(2)  $\theta_q|_{\mathcal{M}_X} = \text{Id}_{\mathcal{M}_X}$

(3)  $\theta_q \xrightarrow{q \rightarrow 1} \theta$

- Things we **do not** ask for:
  - $\theta_q$  Hopf algebra automorphism
  - $(\theta_q)^2 = \text{Id}_{U_q(\mathfrak{g}')}$
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# Quantum involution II

Define algebra homomorphism

$$\begin{aligned}\Psi : U_q(\mathfrak{g}') &\rightarrow U_q(\mathfrak{g}') \\ \Psi(E_i) = E_i K_i, \quad \Psi(F_i) = K_i^{-1} F_i, \quad \Psi(K_i) &= K_i.\end{aligned}$$

Then

$$\theta_q = \text{Ad}(s(X, \tau)) \circ T_X \circ \tau \circ \omega \quad \text{with } T_X = T_{w_X} \circ \Psi$$

satisfies (1), (2), and (3). Moreover,  $T_X$  commutes with  $\tau$ .

# Quantum symmetric pair coideal subalgebras

$(X, \tau)$  admissible pair,  $\theta = \theta(X, \tau)$ ,  $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$

- The Lie algebra  $\mathfrak{k}'$  is generated by
  - $h$  for all  $h \in \mathfrak{h}'$  with  $\theta(h) = h$
  - $\mathfrak{m}_X$  (semisimple Lie subalgebra corresp. to  $X$ )
  - $f_i + \theta(f_i)$  for all  $i \in I \setminus X$ .
- Notation:  $K_h = \prod_{i \in I} K_i^{n_i}$  if  $h = \sum_{i \in I} n_i h_i \in Q^\vee$ .

Define  $U'_q(\mathfrak{k}')$  to be the subalgebra of  $U_q(\mathfrak{g}')$  generated by

- $K_h$  for all  $h \in Q^\vee$  with  $\theta(h) = h$
- $\mathcal{M}_X$
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# Quantum symmetric pair coideal subalgebras

$(X, \tau)$  admissible pair,  $\theta = \theta(X, \tau)$ ,  $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$

- The Lie algebra  $\mathfrak{k}'$  is generated by
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# Specialization

Define  $\mathcal{A} = k[q, q^{-1}]_{(q-1)}$  and  $(K_i; 0)_q = \frac{K_i - 1}{q - 1}$ .

$$\mathcal{U}'_{\mathcal{A}} := \mathcal{A} \left\langle E_i, F_i, K_i^{\pm 1}, (K_i; 0)_q \mid i \in I \right\rangle \quad \text{'$\mathcal{A}$-form'}$$

$$\mathcal{U}'_1 := k \otimes_{\mathcal{A}} \mathcal{U}'_{\mathcal{A}}$$

## Proposition ('well known')

There exists an isomorphism of algebras  $\mathcal{U}'_1 \rightarrow U(\mathfrak{g}')$  such that

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$$\overline{\theta_q(X, \tau)} = \theta(X, \tau) \text{ and } \overline{U'_q(\mathfrak{k}')} = U(\mathfrak{k}').$$

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# Generalization and classification

Let  $\underline{c} = (c_i)_{i \in I \setminus X}$ ,  $\underline{s} = (s_i)_{i \in I \setminus X} \in \mathcal{A}^{I \setminus X}$  such that  $c_i(1) = 1$ .

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Properties of  $C = U'_q(\mathfrak{k}')_{\underline{c}, \underline{s}}$  (for ‘good’  $\underline{c}$ ,  $\underline{s}$ ):

- (1)  $C$  is a right coideal subalgebra of  $U_q(\mathfrak{g}')$ .
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- (3) Maximality condition: Let  $V \subset U_q(\mathfrak{g}')$  be a subspace such that
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## Theorem (Letzter '02)

Let  $\mathfrak{g}$  be of finite type. If  $C \subset U_q(\mathfrak{g})$  is a subalgebra satisfying (1), (2), and (3) above. Then  $C = U'_q(\mathfrak{k})_{\underline{c}, \underline{s}}$  for some  $\underline{c}, \underline{s}$  as above.

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# Generators and relations

Let  $U_\theta^0 = k(q)\langle K_h \mid h \in Q^\vee, \theta(h) = h \rangle$  and  $\mathcal{M}_X^+ = k(q)\langle E_i \mid i \in X \rangle$ .

## Theorem

The algebra  $U'_q(\underline{\mathfrak{k}'})_{\underline{c}}$  is generated over  $\mathcal{M}_X^+ U_\theta^0$  by elements  $B_i$  for all  $i \in I$  and relations

$$K_h B_i = q^{-\alpha_i(h)} B_i K_h \quad \text{for all } K_h \in U_\theta^0, i \in I \quad (1)$$

$$E_i B_j - B_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}} \quad \text{for all } i \in X, j \in I \quad (2)$$

$$F_{ij}(B_i, B_j) = C_{ij}(\underline{c}) \quad (3)$$

for some  $C_{ij}(\underline{c}) \in \sum_{|J| < 1 - a_{ij}} \mathcal{M}_X^+ U_\theta^0 B_J$ .

Here, for any multi-index  $J = (j_1, j_2, \dots, j_n) \in I^n$  we define

$$B_J = B_{j_1} B_{j_2} \dots B_{j_n} \quad \text{and} \quad |J| = n$$

with  $B_i = F_i$  if  $i \in X$ .

# Problems and outlook

- Complete the classification of subalgebras  $C$  satisfying (1), (2), and (3) up to Hopf algebra automorphism of  $U_q(\mathfrak{g}')$  in the Kac-Moody case.  
Relate the result to Poisson geometry for the corresponding symmetric space in the finite case.
- Perform the program of harmonic analysis/special functions in the non-finite case.
- Dream something up !