

QUASI-OPTIMAL RATES OF CONVERGENCE FOR THE GENERALIZED FINITE ELEMENT METHOD IN POLYGONAL DOMAINS

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ABSTRACT. We consider a mixed-boundary-value/interface problem for the elliptic operator $P = -\sum_{i,j} \partial_i(a_{ij}\partial_j u) = f$ on a polygonal domain $\Omega \subset \mathbb{R}^2$ with straight sides. We endowed the boundary of Ω partially with Dirichlet boundary conditions $u = 0$ on $\partial_D\Omega$, and partially with Neumann boundary conditions $\sum_{i,j} \nu_i a_{ij} \partial_j u = 0$ on $\partial_N\Omega$. The coefficients a_{ij} are piecewise smooth with jump discontinuities across the interface Γ , which is allowed to have singularities and cross the boundary of Ω . In particular, we consider “triple-junctions” and even “multiple junctions.” Our main result is to construct a sequence of Generalized Finite Element spaces S_n that yield “ h^m -quasi-optimal rates of convergence,” $m \geq 1$, for the Galerkin approximations $u_n \in S_n$ of the solution u . More precisely, we prove that $\|u - u_n\| \leq C \dim(S_n)^{-m/2} \|f\|_{H^{m-1}(\Omega)}$, where C depends on the data for the problem, but not on f , u , or n . and $\dim(S_n) \rightarrow \infty$. Our construction is quite general and depends on a choice of a good sequence of approximation spaces S'_n on a certain subdomain W that is at some distance to the vertices. In case the spaces S'_n are Generalized Finite Element spaces, then the resulting spaces S_n are also Generalized Finite Element spaces.

INTRODUCTION

The purpose of this work is to present a general construction of finite-dimensional approximation spaces S_n that yields quasi-optimal rates of convergence for the Galerkin approximation of the solution to an elliptic equation in a polygonal domain, when mixed Dirichlet-Neumann conditions are given at the boundary. The coefficients of the equation are piecewise smooth, but may have *jump discontinuities* across the union of a finite number of closed polygonal lines, which we call the *interface*. The interface may intersect the boundary of the polygonal domain.

The construction of the Galerkin spaces S_n employs a sequence of local spaces S'_n with good approximation properties given on a subset of Ω at a positive distance from the singular points of the domain and the interface. Once S'_n are chosen, grading towards the vertices and suitable partitions of unity are employed to define the Galerkin spaces on the whole domain. Therefore, the construction of the spaces S_n falls into the category of Generalized Finite Element Methods (GFEM) and do not require any particular meshing of the domain in advance.

We next describe the problem and the geometric set-up more precisely. Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with straight sides (we will call it a *straight polygonal domain*.) We assume that $\bar{\Omega} = \cup_{k=1}^K \bar{\Omega}_k$, where Ω_k are disjoint straight polygonal

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domains. The set $\Gamma := \partial\Omega \setminus \cup_{j=1}^K \partial\Omega_k$, that is, the part of the boundary of some Ω_k that is not contained in the boundary of Ω , will be called *the interface*. We then consider the following boundary value problem:

$$(0.1) \quad \begin{cases} -\operatorname{div}(A\nabla u) & = f & \text{in } \Omega \\ \nu \cdot A \cdot \nabla u & = 0 & \text{on } \partial_N \Omega \\ u & = 0 & \text{on } \partial_D \Omega, \end{cases}$$

where $\partial_N \Omega := \partial\Omega \setminus \partial_D \Omega$ is a decomposition $\partial\Omega$ into two disjoint sets, with $\partial_D \Omega$ a finite union of closed straight segments, and ν is the unit outer normal to Ω , defined everywhere except at the vertices. We assume that the differential operator $P := -\operatorname{div} A \nabla = \sum_{i,j} \partial_i a_{ij} \partial_j$ is *uniformly strongly elliptic* and that its coefficients a_{ij} are piecewise smooth, but may jump across the interface Γ . For this reason, we will refer to the Problem (0.1) as a *mixed boundary value/interface problem on Ω* .

Mixed boundary value/interface problems often appear in engineering and physics. It is well-known that if Ω is convex, $f \in L^2(\Omega)$, and the coefficient matrix $A = [a_{ij}]$ is smooth on $\bar{\Omega}$ (so there is no interface), then the solution u of (0.1) is in $H^2(\Omega)$, and we can get quasi-optimal rates of convergence for the standard Finite Element Method (FEM) with piecewise linear polynomials and quasi-uniform meshes. When Ω is not convex and the boundary has singularities or the matrix A is discontinuous, on the other hand, then u does not belong to $H^2(\Omega)$ and we may obtain decreased rates of convergence of the Finite Element approximations of u on quasi-uniform meshes. Here and throughout the paper, we denote by $H^m(\Omega)$, $m \in \mathbb{Z}_+$, the standard L^2 -based Sobolev spaces.

Finding efficient methods to treat mixed boundary value problem on straight polygons using Generalized Finite Element Method is part of the general problem of numerically treating singularities. If the coefficients a_{ij} are smooth on each subdomain Ω_j , then singularities arise only at the vertices of the domain Ω , at the points where the boundary conditions change, and at the singular points of the interface or where the interface touches the boundary. Additional singularities will arise if some of the coefficients a_{ij} or the data f are singular at some other points. In this paper, however, we shall assume that our coefficients are piecewise smooth and that data is regular, i.e., $f \in H^{m-1}(\Omega)$, $m \geq 1$.

The structure of corner singularities in two dimensional space is well known by the works [17, 19] and many others. (See, for instance, [4, 10, 16, 21, 19, 26] for more information about singularities that are especially relevant to this paper.) Singularities in the solution in the neighborhood of a corner are determined by the spectrum of the resulting pencil of elliptic operators obtain through the Mellin Transform [19, 20].

The FEMs and GFEMs are examples of Galerkin-based numerical methods, a concept we briefly recall. It is based on the weak formulation of problem (0.1), which is discussed in Section 1. Suppose we are given a sequence of finite-dimensional spaces $S_n \subset H^1(\Omega)$ such that all the functions $\psi \in S_n$ satisfy the essential (i.e., Dirichlet) boundary conditions of Equation (0.1) on $\partial_D \Omega$. For the simplicity of the presentation, we shall assume that $\partial_D \Omega$ is not empty. That is, we do not consider the *pure Neumann problem* explicitly. To consider also the pure Neumann problem, all that one needs to do in practice is to restrict to functions $v \in S_n$ with zero mean. We define, as usual, the Galerkin approximation $u_n \in S_n$ of the variational solution

u of Problem (0.1), to be the exact solution of the projected problem:

$$(0.2) \quad B(u_n, v_n) := \sum_{ij} \int_{\Omega} a_{ij} \partial_i u_n \partial_j v_n = (f, v_n), \quad \text{for all } v_n \in S_n \subset H_D^1(\Omega),$$

where $H_D^1(\Omega) := \{f \in H^1(\Omega), f = 0 \text{ on } \partial_D \Omega\}$ and the bilinear form $B(u, v)$ is given in (1.5). We want a h^m -quasi-optimal rate of convergence, that is, we want to have the following error estimate for all n

$$\|u - u_n\|_{H^1(\Omega)} \leq C \dim(S_n)^{-m/2} \|f\|_{H^{m-1}},$$

where C is independent of f and n . Up to the value of C this is the ideal rate that can be obtained if $u \in H^{m+1}(\Omega)$ and quasi-uniform meshes are used in the Finite Element Method. In this case, if h is the typical size of an element, then $\dim(S_n) \sim h^{-2}$ hence the name. However, in general, $u \notin H^{m+1}(\Omega)$. If Ω is concave, even $u \notin H^2(\Omega)$ in general for the standard Poisson problem. In fact, as mentioned, singularities may lower the rate of convergence of the Finite Element solutions of the discrete problem when using quasi-uniform meshes.

Our approach to the optimal rate of convergence is based on the weighted Sobolev spaces for mixed boundary value and interface problems on polygonal domains, obtained by two of the authors among others [6, 7, 5, 21], and a grading toward each singular point. The weight is here the distance to the singular set. Our main result is to construct a sequence S_n of the Generalized Finite Element spaces that yields quasi-optimal rates of convergence. We are not assuming $u \in H^{m+1}(\Omega)$ and we can relax the condition $f \in H^{m-1}(\Omega)$ to $f \in \hat{H}^{m-1}(\Omega) := \sum H^{m-1}(\Omega_j)$ if an interface is present.

As we mentioned above, we use some auxiliary, “good approximation spaces” S'_n , defined on an auxiliary, but fixed domain W away from the vertices. Together with grading towards the singular point and partitions of unity, the spaces S'_n lead to the construction of the Galerkin spaces S_n that then yield our desired h^m -quasi-optimal rates of convergence. Many choices for the spaces S'_n exist [1, 8, 9, 21], and their definition is, for the most, part very well known, so we do not recall them here.

We notice, however, that if the sequence S'_n is a sequence of GFEM spaces, then S_n will also be a sequence of GFEM spaces. However, if S'_n consists of FEM spaces, then S_n will not consist of FEM spaces, in general. In fact, we mention two examples that satisfy the required approximation condition (see (2.5)). One example is that of FEM spaces consisting of piecewise linear elements on a sequence of appropriately graded meshes, where the grading is determined by the strength of the singularities of the solution at the corner. When interfaces are present, the triangles in the mesh must be aligned with the interface and the construction gives rise to FEM spaces S_n (see [21] for a thorough discussion.) The other example is that of non-conforming GFEM spaces based on partitions of unity and piecewise polynomials. The fact that the sides of each Ω_k are straight allows us to implement the boundary conditions and the transmission conditions at the interface exactly (see (1.6) later in the paper). A more general construction of GFEM spaces for curvilinear smooth domains with smooth interfaces is given in [22]. Our main motivation was to construct a sequence of GFEM spaces that achieves h^m -quasi-optimal rates of convergence, and we achieved this. See [1, 2, 11, 12, 13, 14, 15, 18, 23, 24] for more on the definition of GFEM spaces and their applications.

The paper is organized as follows. In Section 1, we discuss the problem and the geometric set up in more details. In Section 2 we construct our sequence S_n of approximation spaces. Finally, in Section 3, using regularity in weighted Sobolev spaces, we prove that the sequence S_n yields h^m quasi-optimal rates of convergence.

1. MIXED-BOUNDARY-VALUE/INTERFACE PROBLEMS ON POLYGONAL DOMAINS

We begin by discussing in more detail problem (0.1) and the geometry of the domain. We recall the set up of the problem from the introduction. We assume that $\bar{\Omega} = \cup_{k=1}^K \bar{\Omega}_k$, where Ω_k are disjoint straight polygonal domains. The set $\Gamma := \partial\Omega \setminus \cup_{j=1}^K \partial\Omega_k$, that is, the part of the boundary of some Ω_k that is not contained in the boundary of Ω , will be called the *interface*, as before.

We let P denote the divergence form operator:

$$(1.1) \quad Pu := -\operatorname{div}(A\nabla u) = -\sum_{i,j=1}^2 \partial_j(a_{ij}\partial_i u), \quad A = [a_{ij}],$$

where the coefficients are symmetric $a_{ij} = a_{ji}$ and satisfy $a_{ij} \in C^\infty(\bar{\Omega}_k)$ for all $k = 1, \dots, K$, but may jump across the interface Γ . We will also assume that P is *uniformly strongly elliptic*; that is, there exists a constant $\gamma > 0$ such that

$$(1.2) \quad \sum_{i,j=1}^2 a_{ij}(x)\xi_i\xi_j \geq \gamma|\xi|^2, \quad \text{for all } x \in \bar{\Omega} \text{ and } \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

We give the domain Ω a singular structure that is not entirely based on geometry, instead it is adapted to problem (0.1). We denote by \mathcal{V} the set of *singular points* of (0.1) as the collections of points that are either vertices of some Ω_k or points where the boundary conditions change. In particular, any point where the interface Γ meets the boundary of Ω or Γ is not smooth are included. We do not exclude the so-called “triple junctions,” which are points where three or more subdomains meet. At all these points the solution may have singularities. We will call all the singular points in \mathcal{V} vertices, regardless whether these are true geometrical vertices of Ω or artificial vertices where the boundary conditions changes or the interface is not smooth. The set \mathcal{V} endows Ω with a polygonal structure, which is unique only if problem (0.1) is specified. We denote by ℓ_{\min} the minimum distance between points in \mathcal{V} . We will use ℓ_{\min} to construct weighted Sobolev spaces later on in the paper.

A simple example of a domain with polygonal structure is provided by the L-shape domain of Figure 2. In this case, no interface is present, but there exists a vertex with an interior angle greater than π . We will call such a vertex a *re-entrant vertex*. It is well known that the regularity of the solution is decreased by the presence of re-entrant vertices and finite-element approximations based on uniform meshes may not achieve optimal rates of convergence. One of the advantages of the GFEM is that it is not based on a mesh.

To specify the polygonal structure on Ω , we will also need to assume that

$$\partial\Omega = \partial_D\Omega \cup \partial_N\Omega,$$

with $\partial_D\Omega \neq \emptyset$ a union of finitely many closed segments and $\partial_N\Omega := \partial\Omega \setminus \partial_D\Omega$. In particular, $\partial_D\Omega$ has positive measure and $\partial_N\Omega$ is an open subset of the boundary

of Ω . Let us denote by D_ν the co-normal derivative associated to the operator P :

$$(1.3) \quad D_\nu u := \sum_{i,j} \nu_i a_{ij} \partial_j u_\pm = \nu \cdot A \cdot \nabla u.$$

where ν is the outer unit normal vector to Ω . We will impose homogeneous Dirichlet boundary conditions $u \equiv 0$ on $\partial_D \Omega$, a union of closed sides of Ω , and homogeneous Neumann conditions $D_\nu \equiv 0$ on $\partial_N \Omega$, a union of open sides of Ω , in trace sense. Non-homogeneous conditions can be treated as well. We let then

$$(1.4) \quad H_D^1(\Omega) := \{f \in H^1(\Omega), f = 0 \text{ on } \partial_D \Omega\}.$$

We will consider problem (0.1) always in variational form, which is well defined and yields a unique solution in $H_D^1(\Omega)$ by the Lax-Milgram theorem. We introduce the bilinear form on $H_D^1(\Omega)$:

$$(1.5) \quad B(u, v) := \int_{\Omega} (A \nabla u) \cdot \nabla v \, dx.$$

Then, the weak form of (0.1) reads:

$$B(u, v) = (f, v), \quad \forall v \in H_D^1(\Omega).$$

When the coefficients a_{ij} have jump discontinuities at the interface Γ , the weak formulation implies that any solution of problem (0.1) must satisfy matching and jump conditions, the so-called *transmission conditions*, at the interface Γ :

$$(1.6) \quad u_+ = u_-, \quad D_{\nu_+} u = D_{\nu_-} u.$$

Above, we label the limits u_+ , u_- of u at each side of the interface, and denote the respective conormal derivatives by D_{ν_+} and D_{ν_-} , $D_{\nu_\pm} = \sum_{i,j} \nu_i a_{ij\pm} \partial_j u$. These limits should be intended in trace sense on each Ω_k or a.e. in a non-tangential approach to $\partial\Omega_k$. We think of the two sides of Γ as given by the boundaries of the various Ω_k . The labeling \pm is only for notational convenience and plays no role. It refers to an arbitrary labeling of the ‘‘two sides’’ of the interface Γ .

2. CONSTRUCTION OF THE APPROXIMATION SPACES S_n

In this section, we shall present the construction of a sequence $S_n \subset H_D^1(\Omega)$ of finite-dimensional approximation spaces that satisfy $\dim(S_n) \sim 2^{2n}$ and yield *quasi-optimal rates of convergence* for the Galerkin approximation of the mixed-boundary-value/interface problem (0.1). Given two sequences of (a_n) and (b_n) of positive numbers, we shall write $a_n \sim b_n$ if both sequences a_n/b_n and b_n/a_n are bounded.

We recall that by Galerkin approximation of the solution to (0.1), we mean a sequence of functions $u_n \in S_n \subset H_D^1(\Omega)$ such that u_n is the variational solution of (0.1) when the space of test function is restricted to S_n , that is, (0.2) holds.

Definition 2.1. Let us fix $m \in \mathbb{N} := \{1, 2, 3, \dots\}$. We say that a sequence $S_n \subset H^1(\Omega)$ of finite dimensional approximation spaces yields *h^m -quasi-optimal rates of convergence* for the Galerkin approximation $u_n \in S_n$ for the solution u of the mixed-boundary-value/interface problem (0.1), with data $f \in H^{m-1}(\Omega)$, if there exists C independent of $n \in \mathbb{N}$ and of $f \in H^{m-1}(\Omega)$ such that u_n satisfies

$$(2.1) \quad \|u - u_n\|_{H^1(\Omega)} \leq C \dim(S_n)^{-m/2} \|f\|_{H^{m-1}(\Omega)}.$$

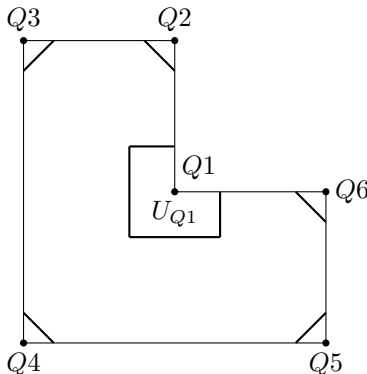


FIGURE 1. A L-Shape domain with 6 vertices. The subdomain U_Q at each corner Q of Ω .

The integer $m \geq 1$ is order of the approximation and will be kept fixed throughout this paper.

Next we proceed to give explicit construction of the approximation spaces S_n in several steps. For each vertex Q in \mathcal{V} , we will choose a small neighborhood U_Q of Q in Ω such that for each $P \in U_Q$, the open segment PQ is completely contained in U_Q (so U_Q is star-shaped with respect to Q). Often it is possible to choose U_Q to be a small triangle in Ω with Q one of the vertices. In general, however, U_Q need not be convex, as in the case of vertex $Q1$ in Figure 2. If there are interfaces, we apply this construction to each subdomain Ω_k .

The construction we present is quite general and will be in terms of an auxiliary family of finite-dimensional subspaces $S'_n \subset H^1(W)$ (with W defined below in Equation (2.4)), exploiting grading towards each of the singular points. The grading at each $Q \in \mathcal{V}$ will be defined in terms of a parameter κ_Q associated to Q , which is determined by the strengths of the singularities of solutions at Q . For instance, for the Poisson problem $\Delta u = f$ with Dirichlet boundary conditions, if the angle at Q is α , we choose $0 < \kappa_Q < 2^{-\frac{m\alpha}{\pi}}$, where m is the order of the approximation. For simplicity, we shall assume that *the parameter κ_Q is the same for all Q , $\kappa_Q = \kappa$* , and hence we will drop the dependence on Q from κ . While this choice is not optimal, the case of a uniform κ_Q is easily achieved by replacing the parameters κ_Q with the minimum of their values. Our algorithm can easily be extended to the case of a vertex-dependent κ_Q .

Below, we will call a vertex of Ω the sides of which are both endowed with Neumann boundary conditions a *Neumann-Neumann* or NN vertex.

2.1. The case of no interfaces and no Neumann-Neumann corners. As indicated above, we first must choose for each singular point $Q \in \mathcal{V}$ a neighborhood U_Q of Q such and $U_P \cap U_Q = \emptyset$ for $P \neq Q$, $P, Q \in \mathcal{V}$. Here we can choose this U_Q such that the distance $\rho(x)$ from any point $x \in U_Q$ to Q satisfies $\rho(x) \leq \ell_{\min}/4$. We then extend ρ to a smooth function on Ω that is comparable to the Euclidean distance in the complement of $\cup_{Q \in \mathcal{V}} U_Q$ in Ω . We then note that $\rho(x) > 0$ for $x \notin \cup_{Q \in \mathcal{V}} U_Q$.

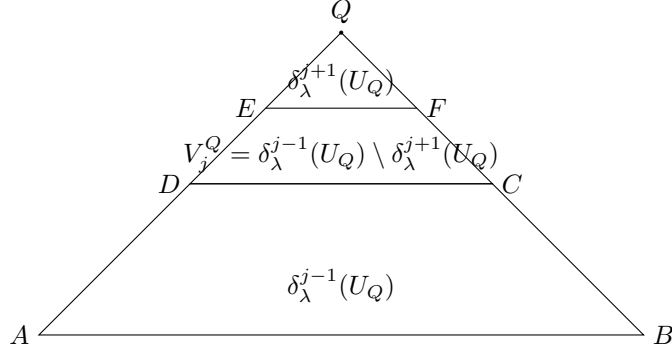


FIGURE 2. The subdomain $V_j = V_j^Q = \delta_\lambda^{j-1}(U_Q) \setminus \delta_\lambda^{j+1}(U_Q)$ near the vertex Q , $V_j \cap V_{j+1} \neq \emptyset$

Next, we denote by δ_λ^Q the dilation with ratio $\lambda > 0$ and center Q . We can assume that all the subdomains $U_Q \subset \Omega$ are dilation invariant, in the sense that

$$(2.2) \quad \delta_\lambda(U_Q) := \delta_\lambda^Q(U_Q) \subset U_Q, \quad \lambda \in (0, 1).$$

Given $\kappa > 0$ to be determined later, we introduce the domains V_j^Q

$$(2.3) \quad V_j^Q := \delta_\kappa^{j-1}(U_Q) \setminus \delta_\kappa^{j+1}(U_Q),$$

$j = 1, 2, \dots$, obtained by repeated applications of the dilation operator δ_λ with $\lambda = \kappa$. In particular, $U_Q = \cup_{j=1}^{\infty} V_j^Q$. We also take the remainder set W of the form

$$(2.4) \quad W := \Omega \setminus (\cup_Q \delta_\kappa^2(U_Q)),$$

so that for example $V_1^Q \subset W$ for all $Q \in \mathcal{V}$.

We choose now a sequence of finite dimensional spaces $S'_j \subset H^1(W)$ $j = 0, \dots, n, \dots$, with $\dim(S'_j) \sim 2^{2j}$ that have good approximation properties in the sense that

$$(2.5) \quad \inf_{v \in S'_j} \|u - v\|_{H^s} \leq C \dim(S'_j)^{-(t-s)/2} \|u\|_{H^t} \leq C 2^{-j(t-s)} \|u\|_{H^t}, \quad 0 \leq s < t,$$

for a constant $C > 0$ that is independent of j or $u \in H^t(W)$. Examples of spaces S'_n satisfying the condition of Equation (2.5) can be found in many works, e.g. [1, 8, 9, 21]. A typical example is that of spaces of continuous piecewise polynomials of degree m on a quasi-uniform sequence of meshes. Another example is that of non-conforming GFEM spaces based on partitions of unity and piecewise polynomials (see [22] for the case of smooth domains with smooth interfaces.)

In either the classical case of Finite Element spaces defined using piecewise polynomials on a quasi-uniform sequence of meshes, or the case of Generalized Finite Element spaces defined using a partition of unity subordinated to a suitable sequence of coverings, the typical size of the elements or of the covering patches, h_j , is of order 2^{-j} , $h_j \sim 2^{-j}$. Then the factor $2^{-j(t-s)}$ in (2.5) can be replaced by the more familiar factor h_j^{t-s} . In this paper, we shall use the above approximation property (2.5) only in the variational space H^1 of Ω or Ω_k , that is, for $s = 1$.

In order to introduce the global approximation spaces $S_n \subset H^1(\Omega)$, we shall need the following construction. First, we choose smooth enough functions $\eta_j^Q \geq 0$,

$j = 1, 2, \dots$, around each vertex Q such that η_j^Q has support in V_j^Q and the sequence of functions η_j^Q is compatible with dilations in the following sense:

$$\eta_{j+1}^Q(\delta_\kappa^j(x)) = \eta_1^Q(x), \quad \text{supp}(\eta_{j+1}^Q) \subset V_{j+1}^Q.$$

We complete the set of functions η_j^Q with $\eta_0 = 1 - \sum_{Q,j} \eta_j^Q$ on W . Then, we can assume that $\{\eta_j^Q\}_{j=0}^\infty$ is an infinite partition of unity on Ω . The function η_0 will have a large ‘‘flap top,’’ being equal to one on almost all of Ω_k except in a neighborhood of each vertex.

Next, we introduce the notation for each $j \geq 1$,

$$\tilde{\eta}_j := \sum_Q \eta_j^Q,$$

where $Q \in \mathcal{V}$ ranges over all singular points of Ω , and

$$\tilde{\eta}_k S'_j := \{\tilde{\eta}_k \phi, \phi \in S'_j\}.$$

We observe that any function $\phi \in \tilde{\eta}_1 S'_{n-j}$ has support in the union of the disjoint open sets V_1^Q over all vertices $Q \in \mathcal{V}$. We can therefore write $\phi = \sum_Q \phi_Q$, where each ϕ_Q has support in V_1^Q . We then set:

$$\delta_\kappa^j(\phi) = \sum_Q \phi_Q \circ (\delta_\kappa^Q)^{-j},$$

noticing that $\phi_Q \circ (\delta_\kappa^Q)^{-j}$ has support in V_{j+1}^Q . Finally we define inductively:

$$(2.6) \quad \begin{aligned} S_0 &= \eta_0 S'_0, \\ S_n &:= \eta_0 S'_n + \sum_{j=1}^n \tilde{\eta}_j \delta_\kappa^{j-1}(S'_{n-j}) = \eta_0 S'_n + \sum_{j=1}^n \delta_\kappa^{j-1}(\tilde{\eta}_1 S'_{n-j}). \end{aligned}$$

We continue with some remarks on the definition and properties of the spaces S_n , before considering the more complex case when interface or Neumann-Neumann vertices are present. The approximation spaces will be denoted \tilde{S}_n there to distinguish these special case.

Remark 2.2. Since the support of none of the functions η_j contain any of the points of \mathcal{V} , the functions in S_n are equal to 0 in a neighborhood of each $Q \in \mathcal{V}$. This property, however, does not hold at the NN vertices and at the non-smooth points of the interface. Close to these points the approximation functions will be non-zero constant instead to account for the non-trivial kernel of the differentially operator locally near these points.

Remark 2.3. The definition of the approximation spaces S_n is quite general. However, if the initial spaces $\{S'_j\}_{j=0}^n$ are local GFEM spaces, then S_n are also GFEM spaces.

Remark 2.4. The condition that $\sum_{j=1}^n \eta_j^Q = 1$ on all V_j^Q , $j = 2, 3, \dots$ can be achieved by choosing η_1^Q to be positive with large enough support, then defining the functions η_j^Q to be compatible with dilations, and finally by using a Shepard procedure to make the η_j^Q be a partition of unity on $\cup_{Q,j \geq 2} V_j^Q$.

To close, the basic idea of the construction of the GFEM space S_n is the following. Choose $U = \cup U_Q$ as in Section 2.1 and define $W \subset \Omega$ at some distance to the vertices, as in (2.4), so that the solution u will be smooth on W . In the subdomain W , construct a sequence of spaces $\{S'_j\}_{j=0}^n \subset H^k(W)$, satisfying the approximation property. Then extend S'_n towards the vertices using the dilations δ_λ . Finally, define S_n by glueing the spaces $\delta_\lambda^{j-1}(S'_{n-j})$ using a partition of unity.

In the following section, we prove that the spaces S_n defined by Equation (2.6) yield h^m -quasi-optimal rates of convergence for the Galerkin method on polygons, provided that the initial spaces S'_n satisfy the condition of Equation (2.5).

Next, we introduce the approximation spaces \tilde{S}_n for the general case.

2.2. Construction of the approximation spaces when there are interfaces and Neumann-Neumann vertices. In this case, the approximation spaces will be given by a slight modification of (2.6). The change is in the approximation condition (2.5). Due to jumps in the coefficients, the solution is not in H^{m+1} , not even locally near the interface, although it is still in the variational space $H^1(\Omega)$.

As shown in [21], in case interfaces are present, the approximation property holds in what we call the *broken Sobolev spaces* $\hat{K}_{a+1}^{m+1}(\Omega)$, which we will introduce in the next section as they are used to prove the quasi-optimal rates of convergence.

We content ourselves for now to define the broken spaces away from the vertices, that is, on W . No weight is required here. We refer to [22] for further discussion.

We recall again from the Introduction that $\bar{\Omega} = \cup_{k=1}^K \bar{\Omega}_k$, where Ω_k are disjoint straight polygonal domains. The broken Sobolev spaces $\hat{H}^k(W)$, $W \subset \Omega$ open, are defined as follows

$$\hat{H}^m(W) = \{u : \Omega \rightarrow \mathbb{R}, u|_{W \cap \Omega_k} \in H^m(\Omega_k \cap W), \text{ for all } k\}.$$

As in the case when there were no interfaces present, we choose a sequence of finite dimensional spaces $S'_j \subset \hat{H}^k(W) \cap H^1(W)$ $j = 0, \dots, n, \dots$, $k \geq 1$, with $\dim(S'_j) \sim 2^{2j}$ that have good approximation properties in the sense that

$$(2.7) \quad \inf_{v \in S'_j} \|u - v\|_{H^s} \leq C \dim(S'_j)^{-(t-s)/2} (\|u\|_{\hat{H}^t} + \|u\|_{H^1}), \quad s = 0, 1,$$

for a constant $C > 0$ independent of $u \in \hat{H}^t(W) \cap H^1(W)$ and j .

We remark that the definition of the broken spaces depends on the choice of Ω_k , even though we do not explicitly display this dependence in the notation.

Examples of spaces S'_n satisfying the condition of Equation (2.5) can be constructed using the results in [1, 8, 9]. The most typical example is that of continuous piecewise polynomials of degree m on a quasi-uniform sequence of meshes, provided that the meshes are *aligned* with the interface.

Let us notice that condition (2.7) is not the direct analog of condition (2.5), since we only allow $s \leq 1$ and we require $u \in \hat{H}^t(W) \cap H^1(W)$, the intersection of a broken Sobolev space and a regular Sobolev space. On the other hand, the error is given in a regular Sobolev space for this case as well. The restriction $s \leq 1$ is reasonable since we are only studying second order differential equations. In particular, the form of the approximation condition (2.7) would not be appropriate for higher order equations in the presence of interfaces. In general, much less is known on transmission problems for higher-order operators.

To complete the definition of the approximation spaces \tilde{S}_n in the case of interfaces and Neumann-Neumann corners, we proceed as follows. For each singular point

$P \in \mathcal{V}$, we pick a function $\chi_P \geq 0$ that is equal to 1 in a small neighborhood of P and satisfies $D_\nu \chi_P = 0$ on the *whole* boundary. We may assume that all the supports of the functions χ_P are disjoint and disjoint from the boundary of Ω , unless P is on $\partial\Omega$. We denote by \mathcal{W} the set of $P \in \mathcal{V}$ such that χ_P satisfies the boundary conditions for problem (0.1). It is easy to see that χ_P satisfies the boundary conditions in one of the following mutually exclusive cases: P is a vertex separating two edges endowed with Neumann boundary conditions, P is an interior point of a side endowed with Neumann boundary conditions where the interface touches the boundary, P is a non-smooth point of the interface (this includes "triple junctions"). We then define \mathcal{W}_s be the linear span of the functions χ_P that satisfy the boundary conditions, thus

$$(2.8) \quad \mathcal{W}_s := \left\{ \sum_{P \in \mathcal{W}} a_P \chi_P, a_P \in \mathbb{R} \right\}.$$

Then we change the definition of the space S_n as follows:

$$(2.9) \quad \begin{aligned} \tilde{S}_n &:= S_n + \mathcal{W}_s = \eta_0 S'_n + \sum_{j=1}^n \tilde{\eta}_j \delta_\kappa^{j-1}(S'_{n-j}) + \mathcal{W}_s \\ &= \eta_0 S'_n + \sum_{k=1}^n \delta_\kappa^{j-1}(\tilde{\eta}_1 S'_{n-j}) + \mathcal{W}_s. \end{aligned}$$

The only difference with respect to equation (2.6) is therefore that the S_n 's are complemented with the space \mathcal{W}_s . We remark that we can choose, for each $P \in \mathcal{W}$,

$$\chi_P = \sum_{n \geq 2} \eta_n^P.$$

We will implicitly make this assumption throughout the rest of the paper.

3. OPTIMAL RATES OF CONVERGENCE

This section is devoted to proving our main result, that is, we will prove that h^m -quasi-optimal rates of convergence hold for the Galerkin approximation of the solution in the spaces S_n .

By Céa's Lemma, it will be sufficient to construct an approximation of the solution u for which estimate (2.1) holds. One difficulty is the lack of elliptic regularity for problem (0.1) when singular points are present, so that the H^{m-1} norm of f does not control the H^{m+1} norm of the solution, which has only limited regularity in standard Sobolev spaces even if f and the coefficient of the operator P are smooth on $\bar{\Omega}$. Elliptic regularity is restored if weighted Sobolev spaces are used instead. We now briefly recall the definition and main properties of these spaces. For brevity, we consider only domains in the plane. For the higher-dimensional case and more details, we refer to [5, 6, 7].

3.1. Weighted Sobolev Spaces. We begin by recalling the notion of regularized distance function, which is the basis for the construction of the weight. We have denoted with $\rho : \bar{\Omega} \rightarrow [0, 1]$ a continuous function that is smooth except at the points of \mathcal{V} , satisfies $\rho^{-1}(0) = \mathcal{V}$, and, most importantly, has the property that $\rho(x)$ is the distance from x to \mathcal{V} , whenever x is close to \mathcal{V} .

We define the weighted Sobolev space $\mathcal{K}_a^m(\Omega)$ with $m \in \mathbb{Z}_+$, as follows:

$$(3.1) \quad \mathcal{K}_a^m(\Omega) = \{u : \Omega \rightarrow \mathbb{R}, \rho^{|\alpha|-a} \partial^\alpha u \in L^2(\Omega), \text{ for all } |\alpha| \leq m \},$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index. This space will be endowed with the induced Hilbert space norm. As already stated, for simplicity we restrict to isotropic spaces, that is, we assume a uniform weight for all the vertices, though this is not the optimal choice. In practice, one can easily implement the case of different parameters κ_Q .

We immediately have from the definition that

$$\mathcal{K}_0^0(\Omega) = L^2(\Omega) \quad \text{and} \quad \rho^b \mathcal{K}_a^m(\Omega) = \mathcal{K}_{a+b}^m(\Omega),$$

where $\rho^b \mathcal{K}_a^m = \{\rho^b v, \text{ for any } v \in \mathcal{K}_a^m\}$.

When interfaces are present, that is, when we are given a decomposition $\bar{\Omega} = \cup_k \bar{\Omega}_k$, the needed regularity results will be expressed in terms of the *broken weighted Sobolev spaces*:

$$\hat{\mathcal{K}}_a^m(\Omega) = \{u : \Omega \rightarrow \mathbb{R}, u|_{\Omega_k} \in \mathcal{K}_a^m(\Omega_k), \text{ for all } k\}.$$

If $G \subset \Omega$ is an open subset, we shall use the same function ρ used to define $\mathcal{K}_a^m(\Omega)$ in Equation (3.1) to define

$$(3.2) \quad \mathcal{K}_a^m(G) = \{u : G \rightarrow \mathbb{R}, \rho^{|\alpha|-a} \partial^\alpha u \in L^2(G), \text{ for all } |\alpha| \leq m\}.$$

We shall need the following three lemmas. We refer to [6, 7] for a detailed proof.

Lemma 3.1. *Let $m \in \mathbb{Z}_+$ and $a \in \mathbb{R}$. Let $G \subset \Omega$ be an open subset such that $\rho(x) \leq \lambda$ for $x \in G$. Then, if $u \in \mathcal{K}_a^m(G)$,*

$$\|u\|_{\mathcal{K}_{a'}^{m'}(G)} \leq \lambda^{a-a'} \|u\|_{\mathcal{K}_a^m(G)}$$

for all m', a' such that $m \geq m'$ and $a \geq a'$.

Proof. The proof is a direct verification from the definition of \mathcal{K}_a^m . \square

The following lemma states that the H^m and \mathcal{K}_a^m -norms are equivalent on $H^m(G)$ for any region G for which the function ρ is bounded from below away from zero.

Lemma 3.2. *Let $G \subset \Omega$ be an open proper subset of Ω such that the distance $\rho \geq \gamma$ on G , for some positive constant γ . Then*

$$\|u\|_{H^m(G)} \leq M_1 \|u\|_{\mathcal{K}_a^m(G)} \quad \text{and} \quad \|u\|_{\mathcal{K}_a^m(G)} \leq M_2 \|u\|_{H^m(G)}$$

for any $u \in H^m(G)$, where M_1 and M_2 may depend on γ and m , but not on u .

Proof. The result follows directly from the definition of the H^m and \mathcal{K}_a^m -norms, given that ρ is smooth and bounded away from zero. \square

Lemma 3.3. *Let Ω be a polygonal domain. Assume that there are no Neumann-Neumann corners, no interfaces, and that $\partial_D \Omega \neq \emptyset$. Then the $H^1(\Omega)$ -norm, the \mathcal{K}_1^1 -norm, and the seminorm $|\cdot|_{H^1(\Omega)}$ are equivalent on $H_D^1(\Omega)$. In particular,*

$$\mathcal{K}_1^1(\Omega) \cap \{u = 0, \text{ on } \partial_D \Omega\} = H_D^1(\Omega),$$

We observe that by definition it is always true that $\mathcal{K}_1^1(\Omega) \subset H^1(\Omega)$ with equivalent seminorms.

We shall also need the following well-known dilation invariance property, which is one of the main reasons weighted Sobolev spaces are more convenient than the usual Sobolev spaces in dealing with corner singularities. Recall that $\delta_\lambda^Q : U_Q \rightarrow U_Q$ denotes the dilation with center Q and ratio $\lambda \in [0, 1]$. Here U_Q is the distinguished neighborhood of Q in Ω that we fixed throughout the paper.

Lemma 3.4. *Let $0 < \lambda < 1$ and let $G \subset U_Q$. Set $G_\lambda = \delta_\lambda^Q(G) = \{\delta_\lambda^Q x, x \in G\}$, and set $u_\lambda(x) = u(\delta_\lambda^Q x)$. Then, for any $u \in \mathcal{K}_a^m(G_\lambda)$, we have*

$$\|u_\lambda\|_{\mathcal{K}_a^m(G)} = \lambda^{a-1} \|u\|_{\mathcal{K}_a^m(G_\lambda)}.$$

Next, we state well-posedness and regularity results for problem (0.1) in weighted spaces for certain ranges of the weight. The spaces are augmented by appropriate ‘‘singular functions’’ in the case of Neumann-Neumann vertices and non-smooth interfaces. These results were established in [6, 19, 21], in various degrees of generality. We state the results for the case of problem (0.1), although non-homogeneous boundary data $g_D \in \mathcal{K}_{a+1/2}^{m+1/2}(\partial_D \Omega)$ and $g_N \in \mathcal{K}_{a-1/2}^{m-1/2}(\partial_N \Omega)$ can be considered as well.

Theorem 3.5. *Let $\Omega \in \mathbb{R}^2$ be a bounded polygonal domain and $m \in \mathbb{Z}$. Assume the differential operator P is uniformly strongly elliptic with coefficients that are smooth on $\bar{\Omega}$. Assume in addition that no two adjacent sides of Ω are given Neumann boundary conditions. Then there exists $\eta > 0$ with the following property: for any $|a| < \eta$, the boundary value problem (0.1) has a unique solution $u \in \mathcal{K}_{a+1}^{m+1}(\Omega) \cap H_D^1(\Omega)$ for any $f \in \mathcal{K}_{a-1}^{m-1}(\Omega)$. This solution depends continuously on f .*

When Neumann-Neumann vertices or interfaces are present, well-posedness is achieved instead in the broken Sobolev spaces, augmented with \mathcal{W}_s , provided that $a > 0$. The space \mathcal{W}_s is given in (2.8).

As before, by abuse of notation, we shall denote also by $\|\cdot\|_{\hat{\mathcal{K}}_{a+1}^{m+1}}$ the norm

$$\|u_0 + \sum_P a_P \chi_P\|_{\hat{\mathcal{K}}_{a+1}^{m+1}} := \sum_{k=1}^K \|u_0\|_{\mathcal{K}_{a+1}^{m+1}(\Omega_k)} + \sum_{P \in \mathcal{W}} |a_P|$$

on the space $\hat{\mathcal{K}}_{a+1}^{m+1}(\Omega) + \mathcal{W}_s$. We similarly extend the norm $\|\cdot\|_{\mathcal{K}_{a+1}^1}$ from the space $\mathcal{K}_{a+1}^1(\Omega)$ to $\hat{\mathcal{K}}_{a+1}^1(\Omega) + \mathcal{W}_s$.

Theorem 3.6. *Consider an interface problem $Pu = f$ on the bounded polygonal domain $\Omega \in \mathbb{R}^2$, $\bar{\Omega} = \cup_{k=1}^K \bar{\Omega}_k$ and let $m \in \mathbb{Z}$. Assume the Dirichlet part of the boundary is non-empty. Then there exists $\eta > 0$ with the following property: for any $0 < a < \eta$, the interface/boundary value problem (0.1) has a unique solution $u \in (\hat{\mathcal{K}}_{a+1}^{m+1}(\Omega) \cap \mathcal{K}_{a+1}^1(\Omega) + \mathcal{W}_s) \cap H_D^1(\Omega)$, for any $f \in \hat{\mathcal{K}}_{a-1}^{m-1}(\Omega)$. This solution depends continuously on f :*

$$\|u\|_{\mathcal{K}_{a+1}^1(\Omega)} + \|u\|_{\hat{\mathcal{K}}_{a+1}^{m+1}(\Omega)} \leq C \sum_{k=1}^K \|f\|_{\mathcal{K}_{a-1}^{m-1}(\Omega_k)}.$$

In the case of the pure Neumann problem, uniqueness holds only up to constant functions on $\bar{\Omega}$, assuming Ω is connected. Otherwise the result is similar.

3.2. Approximation away from the vertices. We start by discussing the simpler approximation of the solution u away from the singular points. For simplicity, we shall deal first with the case when there are no interfaces and no Neumann-Neumann vertices, and then indicate what are the changes needed to deal with the case when there are interfaces. So, we assume in this and next subsection that there are no interfaces and there are no Neumann-Neumann vertices.

We recall that we have denoted $V_j^Q := \delta_\lambda^{j-1}(U_Q) \setminus \delta_\lambda^{j+1}(U_Q)$, $j \geq 1$, and $W := \Omega \setminus (\cup_Q \delta_\kappa^2(U_Q))$, so that $V_1^Q \subset W$ for all $Q \in \mathcal{V}$.

Recall that, since the distance function ρ is bounded away from zero on W by construction, $S'_n \subset \mathcal{K}_1^1(W)$. Let us denote by $u_{n,W}$ the $\mathcal{K}_1^1(W)$ projection of u onto the space S'_n . The equivalence of $H^{m+1}(W)$ and $\mathcal{K}_{a+1}^{m+1}(W)$ -norms on functions defined on W , Lemma 3.2, together with Equation (2.5) then gives

$$(3.3) \quad \|u - u_{n,W}\|_{\mathcal{K}_1^1(W)} \leq \inf_{v \in S'_n} \|u - v\|_{\mathcal{K}_1^1(W)} \\ \leq C \inf_{v \in S'_n} \|u - v\|_{H^1(W)} \leq C2^{-nm} \|u\|_{H^{m+1}(W)}.$$

We can then take $u_{n,W}$ as approximation on W . When interfaces are present, we apply the same construction on each Ω_k , $k = 1, \dots, K$, and replace the H^m norm on the right hand side of (3.3) with the norm in the space $\hat{H}^m(\Omega) \cap H^1(\Omega)$.

In order to define the approximation near a vertex Q , we need to give it first on V_1^Q . Then we will use dilations to define the approximation on the set V_j^Q .

Proposition 3.7. *Given any vertex Q of Ω , there exists a constant $C_Q > 0$ with the following property. For any $u \in \mathcal{K}_{a+1}^{m+1}(V_1^Q)$ and any n , there exists $u_{n,Q} \in S'_n$ such that*

$$\|u - u_{n,Q}\|_{\mathcal{K}_1^1(V_1^Q)} \leq C_Q 2^{-nm} \|u\|_{\mathcal{K}_{a+1}^{m+1}(V_1^Q)}.$$

Proof. We define $u_{n,Q} \in S'_n$ to be the $\mathcal{K}_1^1(V_1^Q)$ projection of u onto S'_n . Let E be the extension map

$$E : \mathcal{K}_{a+1}^{m+1}(V_1^Q) \rightarrow \mathcal{K}_{a+1}^{m+1}(W),$$

which is a bounded operator by classical results for Sobolev spaces on Lipschitz domains [25], and $(Eu)_{n,W}$ denotes the $\mathcal{K}_1^1(W)$ projection of Eu onto the space S'_n . Equation (3.3) and Lemma 3.2 then give

$$\|u - u_{Q,n}\|_{\mathcal{K}_1^1(V_1^Q)} \leq \|u - (Eu)_{n,W,E}\|_{\mathcal{K}_1^1(V_1^Q)} \leq \|Eu - (Eu)_{n,W,E}\|_{\mathcal{K}_1^1(W)} \\ \leq C2^{-nm} \|Eu\|_{\mathcal{K}_{a+1}^{m+1}(W)} \leq C2^{-nm} \|u\|_{\mathcal{K}_{a+1}^{m+1}(V_1^Q)},$$

This completes the proof. \square

3.3. Approximation near the vertices. Recall that in this subsection we continue to assume there are no interfaces or NN vertices. We will use grading to define the approximation of u near the vertices of Ω . To do so, we extend Proposition 3.7 to the sets $V_j^Q = \delta_\kappa^{j-1}(V_1^Q)$, given in Equation (2.3), for $j = 2, \dots, n+1$.

Throughout, we fix Q and hence write V_j for V_j^Q and δ_λ for δ_λ^Q for notational ease, but we still indicate the dependence on Q for the approximation of the solution u near Q . It will be convenient to write $S'_n(V_1)$ for the set of restrictions to V_1 of the functions in S'_n , which are functions on the whole W . Recall that the sets V_j are defined by repeated applications of the dilation δ_κ , $V_j = \delta_\kappa^{j-1}(V_1)$, with $\kappa \leq 2^{-m/a}$. Thus we can define

$$S'_n(V_j) = \delta_\kappa^{j-1}(S'_n(V_1)).$$

Proposition 3.8. *Let $C_Q > 0$ be the constant of Proposition 3.7. Then, given any $1 \leq j \leq n$ and any $u \in \mathcal{K}_{a+1}^{m+1}(V_j)$, there exists $u_{Q,n,j} \in S'_{n-j}(V_j) = \delta_\kappa^{j-1}(S'_{n-j}(V_1))$ such that*

$$\|u - u_{Q,n,j}\|_{\mathcal{K}_1^1(V_j)} \leq C_Q 2^{-nm} \|u\|_{\mathcal{K}_{a+1}^{m+1}(V_j)}.$$

Proof. For $j = 1$ the statement has been proved in Proposition 3.7 with $u_{Q,n,1} = u_{Q,n}$. We then define $u_{Q,n,j} \in S'_{n-j}(V_j) = \delta_\kappa^{j-1} S'_{n-j}(V_j)$ as the $\mathcal{K}_1^1(V_j)$ projection of u onto $S'_{n-j}(V_j)$. Using the behavior of the sets and function spaces under dilations, we will reduce the proof to an application of Proposition 3.7. We let $v \in \mathcal{K}_{a+1}^{m+1}(V_1)$ be given by

$$v(x) := u(\delta_\kappa^{j-1}x), \quad x \in V_1,$$

and let $v_Q \in S'_{n-j}(V_1)$ be the $\mathcal{K}_1^1(V_1)$ projection of v . Then $u_{Q,n,j}$ is obtained from v_Q by dilation, and Lemma 3.4 gives that $\|u - u_{Q,n,j}\|_{\mathcal{K}_1^1(V_j)} = \|v - v_Q\|_{\mathcal{K}_1^1(V_1)}$.

Finally Proposition 3.7 implies that

$$\begin{aligned} \|u - u_{Q,n,j}\|_{\mathcal{K}_1^1(V_j)} &= \|v - v_Q\|_{\mathcal{K}_1^1(V_1)} \leq C_Q 2^{-(n-j)m} \|v\|_{\mathcal{K}_{a+1}^{m+1}(V_1)} \\ &\leq C_Q 2^{-(n-j)m} \kappa^{a(j-1)} \|u\|_{\mathcal{K}_{a+1}^{m+1}(V_j)} \leq C_Q 2^{-nm} \|u\|_{\mathcal{K}_{a+1}^{m+1}(V_j)}, \end{aligned}$$

since $\kappa \leq 2^{-m/a}$. This completes the proof. \square

We now prove a similar error estimate for the region

$$\tilde{V}_n = U_Q \setminus \left(\bigcup_{j=1}^{n-1} V_j \right),$$

which is the region closest to the vertex in our grading. We remark that, by construction, S_n consists of functions that are zero on \tilde{V}_j for $j > n$, in case there are no interfaces and no Neumann-Neumann vertices. Otherwise, it consists of functions that are constant on \tilde{V}_j , $j > n$. The error estimate follows from the regularity properties of u in weighted spaces.

Proposition 3.9. *There exists a constant $C_Q > 0$ such that, for any n and any $u \in \mathcal{K}_{a+1}^{m+1}(V_n)$, we have*

$$\|u\|_{\mathcal{K}_1^1(\tilde{V}_n)} \leq C_Q 2^{-nm} \|u\|_{\mathcal{K}_{a+1}^{m+1}(\tilde{V}_n)}.$$

Proof. We first notice that

$$\lambda := \sup_{x \in \tilde{V}_n} \rho(x) \leq C \kappa^n \leq C (2^{-m/a})^n = C 2^{-mn/a},$$

where C is a constant that depends only on the initial mesh refinement. We then use Lemma 3.1 for this value of λ to obtain

$$(3.4) \quad \|u\|_{\mathcal{K}_1^1(V_n)} \leq (C 2^{-mn/a})^a \|u\|_{\mathcal{K}_{1+a}^1(V_n)} \leq C_Q 2^{-nm} \|u\|_{\mathcal{K}_{a+1}^{m+1}(V_n)}.$$

\square

Recall the functions η_k used to define the spaces S_n (Equations 2.2 and 2.6). Given a sufficiently regular function ϕ on Ω , we also denote

$$(3.5) \quad \|\phi\|_k := \max_i \|\rho^k \partial_{x_1}^i \partial_{x_2}^{k-i} \phi\|_{L^\infty(\Omega)}$$

We shall need the following estimate for the norms $\|\cdot\|_k$.

Lemma 3.10. *There exist constants C_m , $m \geq 1$, such that*

- (i) $\|\phi u\|_{\mathcal{K}_a^m} \leq C_m \|\phi\|_m \|u\|_{\mathcal{K}_a^m}$, for any $\phi \in C^m(\Omega)$ and $u \in \mathcal{K}_a^m$,
- (ii) $\|\eta_n^Q\|_m \leq C_m$ for any n ,
- (iii) if $r_{Q,n} := \sum_{k \geq n+1} \eta_n^Q$, then $\|r_{Q,n}\|_m \leq C_m$ for any n .

Proof. Estimate (i) follows by a direct calculation. We only need to consider what happens for functions localized near the vertices. We observe that the dilated function u_λ , $0 < \lambda < 1$, satisfies $\|u_\lambda\|_k = \|u\|_k$ provided that both u and its u_λ have support in a neighborhood U_Q for some $Q \in \mathcal{V}$. Then the bound (ii) follows from the dilation invariance of the family of functions η_k (Equation 2.2) by taking $C_m := \|\eta_1\|_m$. Lastly estimate (iii) is proved in a similar way. \square

The following standard lemma (see for instance [3] will be useful.)

Lemma 3.11. *Given a space X , assume that for any point $x \in X$, at most M of the values $f_k(x)$ of a sequence of functions $f_k \in L^2(X)$, $k = 1, 2, \dots$, are not zero. Then, $\|\sum_k f_k\|^2 \leq M \sum_k \|f_k\|^2$.*

We are now ready to state a global approximation result. Recall that we still assume that there are no interfaces and no Neumann-Neumann vertices in this subsection. Then, the solution of problem (0.1) belongs to \mathcal{K}_{a+1}^{m+1} provided $f \in \mathcal{K}_{a-1}^{m-1}$, so we state the result for functions with this regularity.

Theorem 3.12. *There exists a constant $C > 0$ such that for any n and for any $u \in \mathcal{K}_{a+1}^{m+1}(\Omega) \cap H_D^1(\Omega)$, there exists $u_{I,n} \in S_n$ such that*

$$\|u - u_{I,n}\|_{\mathcal{K}_1^1(\Omega)} \leq C 2^{-nm} \|u\|_{\mathcal{K}_{a+1}^{m+1}(\Omega)}.$$

The constant C may depend on m and a , but not on n and u .

Proof. For each vertex $Q \in \mathcal{V}$, recall that we defined $S'_{n-j}(V_j^Q) = \delta_k^{j-1} S'_{n-j}(V_1^Q)$. Let $u_{Q,n,j} \in S'_{n-j}(V_j^Q)$ be as in Proposition 3.8. Also, let $u_W \in S'_n$ be the $\mathcal{K}_1^1(W)$ projection of u onto S'_n . Then we define

$$u_I = \eta_0 u_W + \sum_Q \sum_{j=1}^n \eta_j^Q u_{Q,n,j} \in S_n,$$

Since $1 = \eta_0 + \sum_Q (r_{Q,n} + \sum_{j=1}^n \eta_j^Q)$ by construction, we have that

$$u - u_{I,n} = \eta_0(u - u_W) + \sum_Q \left(r_{Q,n} u + \sum_{j=1}^n \eta_j^Q (u - u_{Q,n,j}) \right).$$

We next notice that the functions $\eta_0(u - u_W)$, $r_{Q,n} u$, and $\eta_j^Q(u - u_{Q,n,j})$ satisfy the assumption of Lemma 3.11 for $M = 2$ as $1 \leq j \leq n$ and n vary. (We have $M = 2$ because any point belongs to at most two of the sets V_j^Q and W .)

Let C_1 be the constant provided by Lemma 3.10 with $m = 1$, and set $\tilde{C}_1 := \max(C_1, \|\eta_0\|_1)$. Propositions 3.7, 3.8, and 3.9 then imply that

$$\begin{aligned}
& \|u - u_{I,n}\|_{\mathcal{K}_1^1(\Omega)}^2 \\
& \leq 2 \left(\|\eta_0(u - u_W)\|_{\mathcal{K}_1^1(\Omega)}^2 + \sum_Q (\|r_{Q,n}u\|_{\mathcal{K}_1^1(\Omega)}^2 + \sum_{j=1}^n \|\eta_j^Q(u - u_{Q,n,j})\|_{\mathcal{K}_1^1(\Omega)}^2) \right) \\
& \leq \tilde{C}_1 \left(\|u - u_W\|_{\mathcal{K}_1^1(W)}^2 + \sum_Q (\|u\|_{\mathcal{K}_1^1(\tilde{V}_n)}^2 + \sum_{j=1}^n \|u - u_{Q,n,j}\|_{\mathcal{K}_1^1(V_j^Q)}^2) \right) \\
& \leq 2\tilde{C}_1 C_Q 2^{-nm} \left(\|u\|_{\mathcal{K}_{a+1}^{m+1}(W)}^2 + \sum_Q (\|u\|_{\mathcal{K}_{a+1}^{m+1}(\tilde{V}_n)}^2 + \sum_{j=1}^n \|u\|_{\mathcal{K}_{a+1}^{m+1}(V_j^Q)}^2) \right) \\
& \leq 4\tilde{C}_1 C_Q 2^{-nm} \|u\|_{\mathcal{K}_{a+1}^{m+1}(\Omega)}^2.
\end{aligned}$$

The proof is complete. \square

3.4. The case of interfaces and Neumann-Neumann vertices. When interfaces and Neumann-Neumann vertices are present, regularity for the solution u to problem (0.1) must be measured in the space $\hat{\mathcal{K}}_{a+1}^{m+1}(\Omega) + \mathcal{W}_s$. We also need to use Equation (2.7) instead of Equation (2.5) and the definition of S_n spaces Equation (2.9) instead of Equation (2.6). We consequently state an approximation result for this case. The proof is identical to that of Theorem 3.12.

Theorem 3.13. *There exists a constant $C > 0$ such that for any n and for any $u \in (\hat{\mathcal{K}}_{a+1}^{m+1}(\Omega) + \mathcal{W}_s) \cap H_D^1(\Omega)$, there exists $u_{I,n} \in \tilde{S}_n := S_n + \mathcal{W}_s$ on Ω such that*

$$\|u - u_{I,n}\|_{H^1(\Omega)} \leq C 2^{-nm} \|u\|_{\hat{\mathcal{K}}_{a+1}^{m+1}(\Omega)}.$$

3.5. Optimal rates of convergence. Now we are ready to prove our main result: the quasi-optimal rates of convergence, stated in (2.1), for the Galerkin approximations of the mixed-boundary-value/interface problem (0.1). Let η be the constant of Theorem 3.6. As before, by abuse of notation, we shall denote also by $\|\cdot\|_{\hat{\mathcal{K}}_{a+1}^{m+1}}$ the norm on the space $\hat{\mathcal{K}}_{a+1}^{m+1}(\Omega) + \mathcal{W}_s$.

Theorem 3.14. *Let $m \geq 1$ and $a \in (0, \eta)$. Then there exists a constant $C > 0$ such that for any n and for any $f \in \hat{\mathcal{K}}_{a-1}^{m-1}(\Omega)$, the solution $u \in \hat{\mathcal{K}}_{a+1}^{m+1}(\Omega) + \mathcal{W}_s$ of (0.1) and its Galerkin approximation $u_n \in \tilde{S}_n = S_n + \mathcal{W}_s$ satisfy*

$$\|u - u_n\|_{H^1(\Omega)} \leq C 2^{-nm} \|f\|_{\mathcal{K}_{a-1}^{m-1}(\Omega)},$$

where C may depend on m and a , but not on n and f .

Proof. This result is an immediate consequence of Theorems 3.6 and 3.13 and of Céa's Lemma. \square

Estimate (2.1) then follows easily from the fact that if $f \in H^{m-1}(\Omega)$, then $f \in K_{a-1}^{m-1}$ for the given range of weight a :

$$\|u - u_n\|_{H^1(\Omega)} \leq C \dim(S_n)^{-m/2} \|f\|_{H^{m-1}(\Omega)},$$

recalling that $\dim(S_n) \sim 2^{2n}$.

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